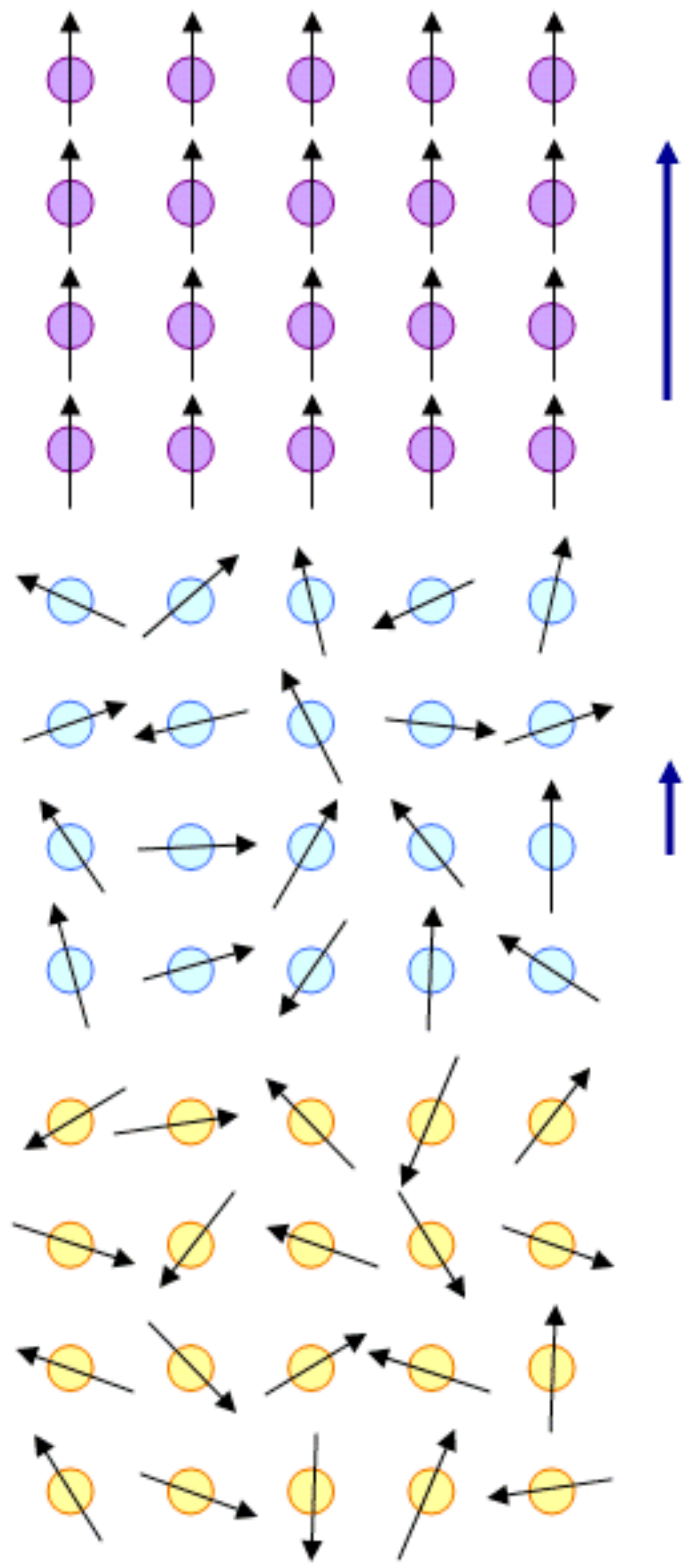
Dielectric media

⚡Polarizable media
⚡Induced dipoles
⚡Maxwell equations in dielectric media
⚡Boundary conditions



Polarization

- All materials are made of atoms and molecules that, when subjected to an external electric field, have their **charge distributions deformed** by the field.
- The polarization of a medium is measured by the **density of electric dipoles** in that medium:

$$\vec{P} = \frac{d\vec{p}}{dV}$$, where $\vec{p} = q\,\vec{d}$ denote the individual (microscopic) dipoles.

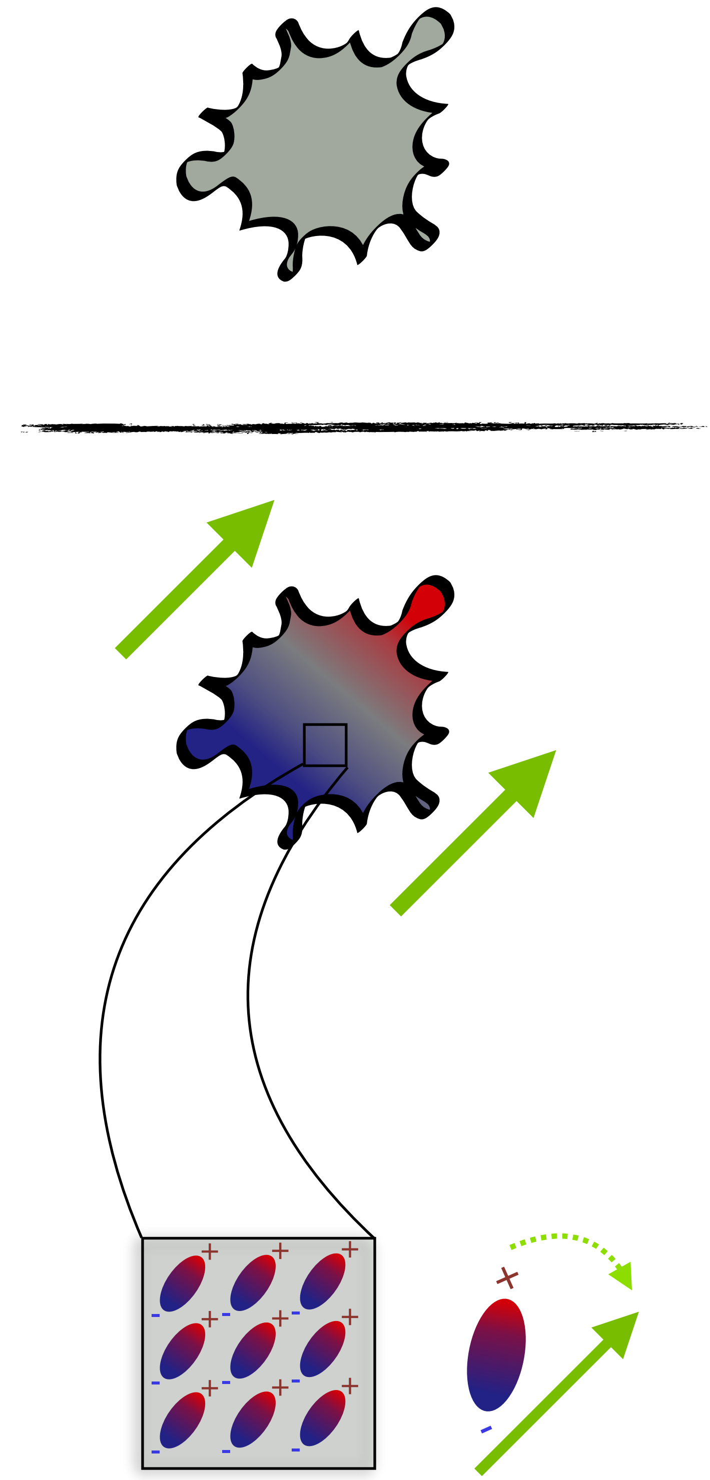
- Let's assume, for simplicity, that our medium is homogeneous — i.e., all atoms (or molecules) are the same, so they respond in the same way to an external electric field.
- In the presence of an external field, an electric dipole is subjected to a **torque**:

$$\vec{\tau} = \vec{p} \times \vec{E}$$

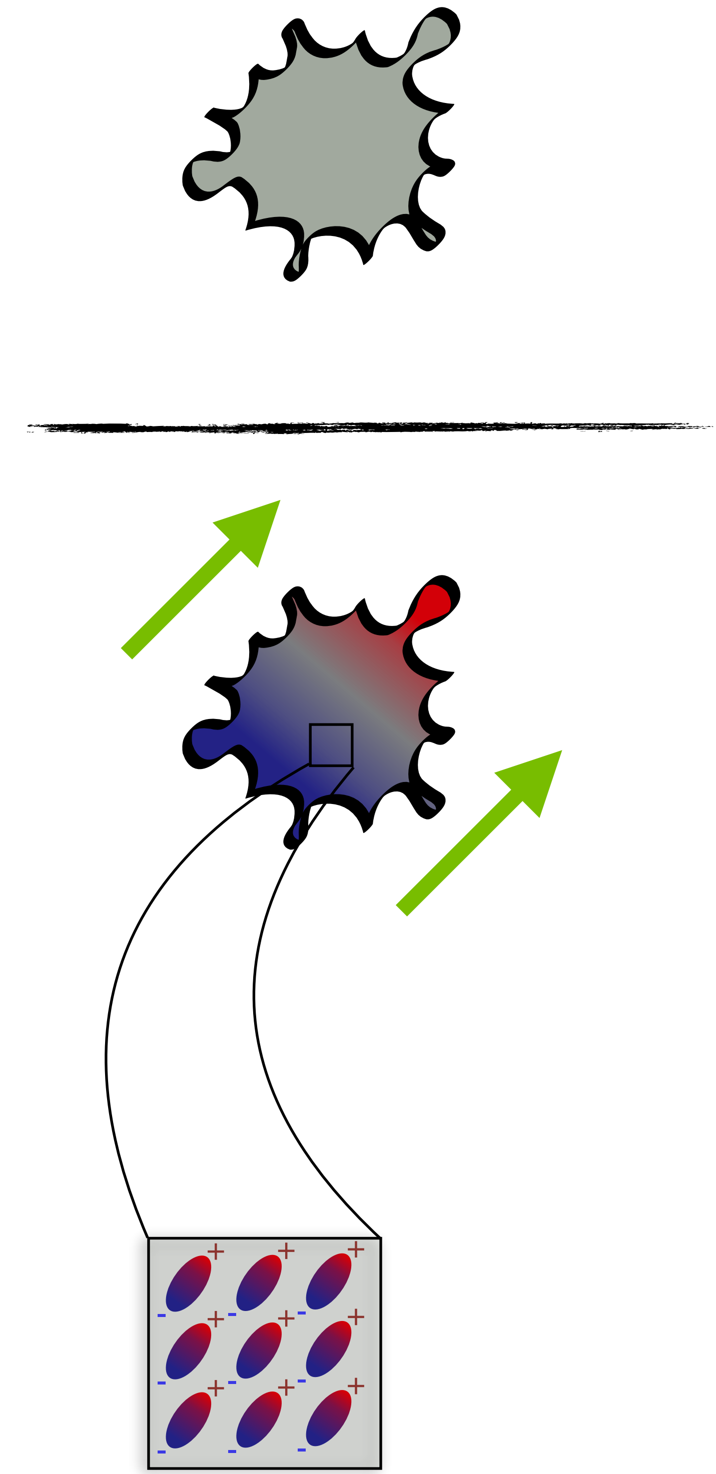
- And the energy of a dipole in the presence of an external electric field is given by:

$$U_p = -\,\vec{p} \cdot \vec{E}$$

- Hence, when a dipole is subjected to an electric field it **tends to align with that field**.
- The nature of the material, which constrains the configurations of its atoms and molecules, determines to what extent this alignment takes place. This is the **polarizability** of the medium



Polarization



- Let's now compute the field which is generated by these induced dipoles.
- The potential at position $\vec{x}$ which is generated by a dipole $d\vec{P}(\vec{x}\,') = \vec{P}\,dV'$ at some position $\vec{x}\,'$, is given by:

$$d\phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \frac{(\vec{x} - \vec{x}\,') \cdot d\vec{P}(\vec{x}\,')}{|\,\vec{x} - \vec{x}\,'|^3}$$

$$\Rightarrow \quad \phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int d^3x' \, \frac{(\vec{x} - \vec{x}\,') \cdot \vec{P}(\vec{x}\,')}{|\,\vec{x} - \vec{x}\,'|^3}$$

- Let's rewrite this result in a more enlightening way. Recall that:

$$\vec{\nabla}_x \frac{1}{|\,\vec{x} - \vec{x}\,'|} = - \frac{\vec{x} - \vec{x}\,'}{|\,\vec{x} - \vec{x}\,'|^3} \quad \leftrightarrow \quad \vec{\nabla}_{x'} \frac{1}{|\,\vec{x} - \vec{x}\,'|} = - \frac{\vec{x}\,' - \vec{x}}{|\,\vec{x} - \vec{x}\,'|^3}$$

- This means we can rewrite the potential above as:

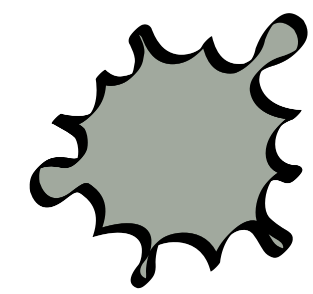
$$\phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int d^3x' \, \vec{P}(\vec{x}\,') \cdot \left(\vec{\nabla}' \frac{1}{|\,\vec{x} - \vec{x}\,'|} \right)$$

- This expression can then be integrated by parts, resulting in:

$$\phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \oint_{S(V)} d^2\vec{S}\,' \cdot \frac{\vec{P}(\vec{x}\,')}{|\,\vec{x} - \vec{x}\,'|} - \frac{1}{4\pi\epsilon_0} \int_V d^3x' \, \frac{\vec{\nabla}\,' \cdot \vec{P}(\vec{x}\,')}{|\,\vec{x} - \vec{x}\,'|} \quad ,$$

where V is the volume of the (homogeneous) dieletric — which is always restricted to a finite, limited region.

Polarization



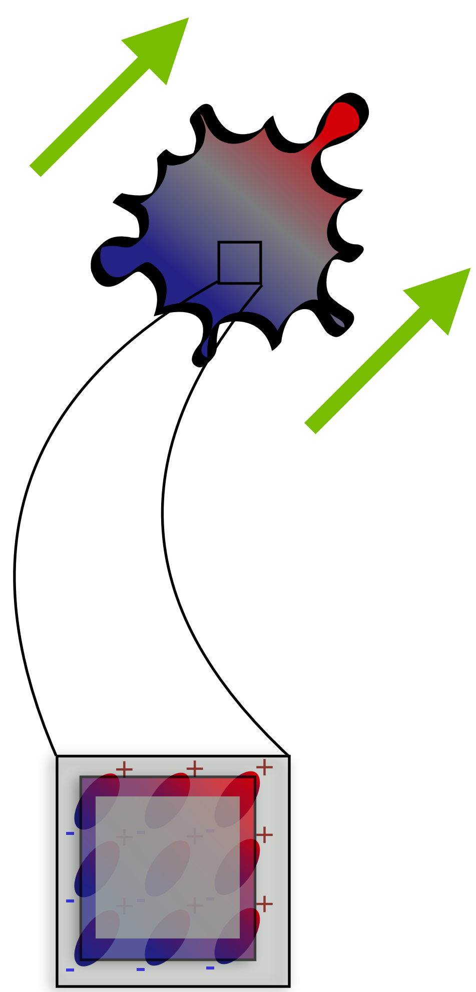
- The expression we just found (which is completely general) is written in terms of a surface integral and a volume integral:

$$\phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\oint_{S(V)} d^2\vec{S}\,'\cdot\frac{\vec{P}(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|} - \frac{1}{4\pi\epsilon_0}\int_V d^3x'\,\frac{\vec{\nabla}\,'\cdot\vec{P}(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|}$$

- These two terms have different physical meanings, in terms of a surface polarization and a volume polarization:

$\sigma_P = \vec{P}\cdot\hat{n}$ and $\rho_P = -\vec{\nabla}\cdot\vec{P}$, which then leads to the expression:

$$\phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\oint_{S(V)} d^2S'\cdot\frac{\sigma_P(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|} + \frac{1}{4\pi\epsilon_0}\int_V d^3x'\,\frac{\rho_P(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|}$$



- Notice that if the polarization is homogeneous, then $\vec{\nabla}\cdot\vec{P} = 0$, and everything can be expressed in terms of the surface term.

- Notice also that, when the polarization is uniform, we find that the electric field of the dipoles inside the dielectric is given by:

$$\vec{E}_p = -\vec{\nabla}\phi(\vec{x}) = -\frac{1}{4\pi\epsilon_0}\vec{P}\cdot\vec{\nabla}\left[\int d^3x'\,\frac{\vec{x}-\vec{x}\,'}{|\vec{x}-\vec{x}\,'|^3}\right] = -(\vec{P}\cdot\vec{\nabla})\,\vec{\mathscr{E}}$$

where $\vec{\mathscr{E}}$ is just like an electric field, except for the "unit charge".

Polarization: example

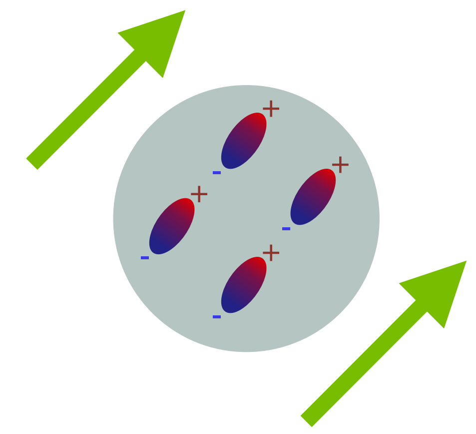
- As an example, take a sphere with a total of Q positive charges and Q negative charges that have a typical distance D between them. **If all the dipoles were aligned**, the polarization of that sphere would be:

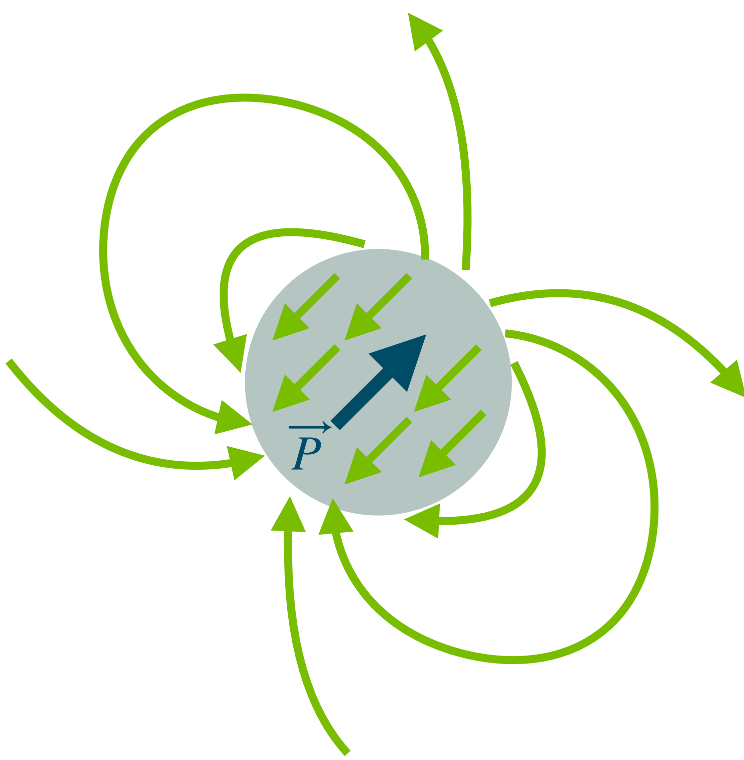
$$\vec{P} = \frac{1}{V}\sum_i \vec{p}_i = \frac{3}{4\pi R^3} Q \vec{D}$$

You may want to make an exercise for any macroscopic object, and the result would something absolutely gigantic — check it out!

- What is going on here is the fact that the dipoles are **not all aligned**! Only when this sphere is subjected to an **external field** $\vec{E}_{ext}$ is that, **on average**, some of those dipoles align juuuust **a little bit**.
- The result is that it acquires a polarization that we can express as:

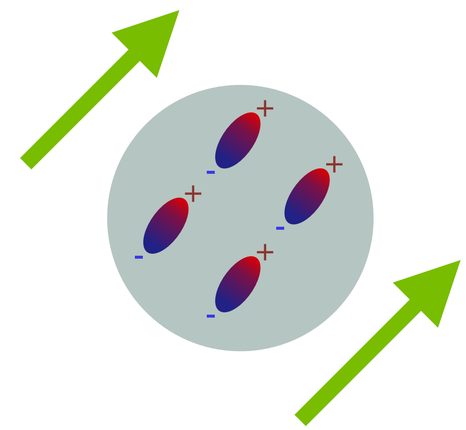
$$\vec{P} = \frac{1}{V}\sum_i \vec{p}_i = \frac{3}{4\pi R^3} Q \vec{d} = \alpha \vec{E}_{ext}$$





Polarization: example

- Now, let's go back to our earlier result and apply this to the sphere of our example:



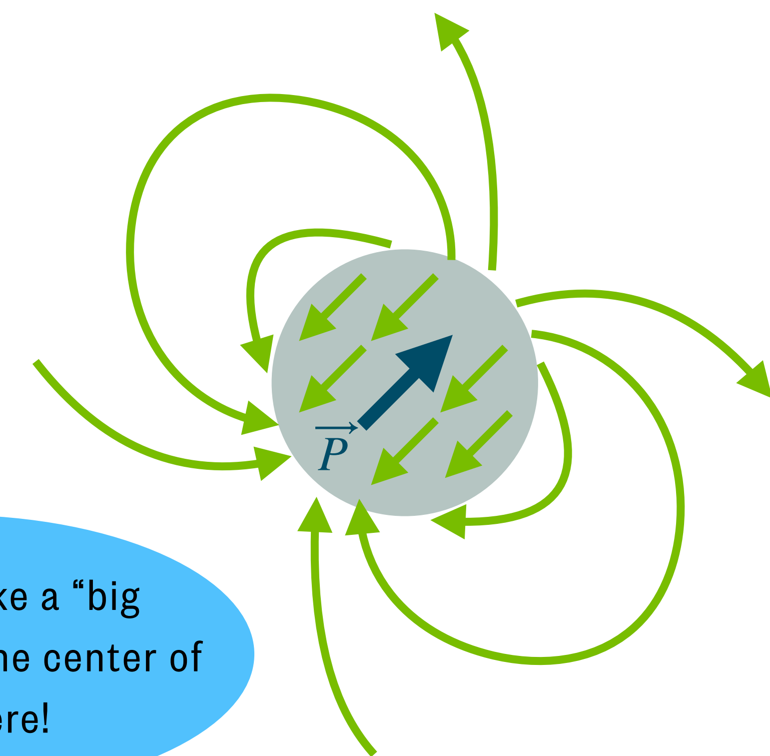
$$\vec{E}_p = -\vec{\nabla}\phi(\vec{x}) = -\frac{1}{4\pi\epsilon_0}\vec{P}\cdot\vec{\nabla}\left[\int d^3x'\,\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}\right] = -(\vec{P}\cdot\vec{\nabla})\,\vec{\mathscr{E}} \quad ,$$

You can use Gauss' law to show that the "field" $\mathscr{E}$ is given by:

$$\vec{\mathscr{E}} = \frac{1}{4\pi\epsilon_0}\int d^3x'\,\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3} = \frac{\vec{x}}{3\epsilon_0}$$ inside the sphere ($r \leq R$), and

$$\vec{\mathscr{E}} = \frac{\vec{x}\,V}{4\pi\epsilon_0\,x^3} = \frac{\vec{x}\,R^3}{3\,\epsilon_0\,x^3}$$ outside the sphere ($r \geq R$)

- Now, use the fact that $\nabla_i x_j = \delta_{ij}$ and that $\nabla_i(x_j/x^3) = \nabla_j(x_i/x^3)$ to show, from the equations above, that:

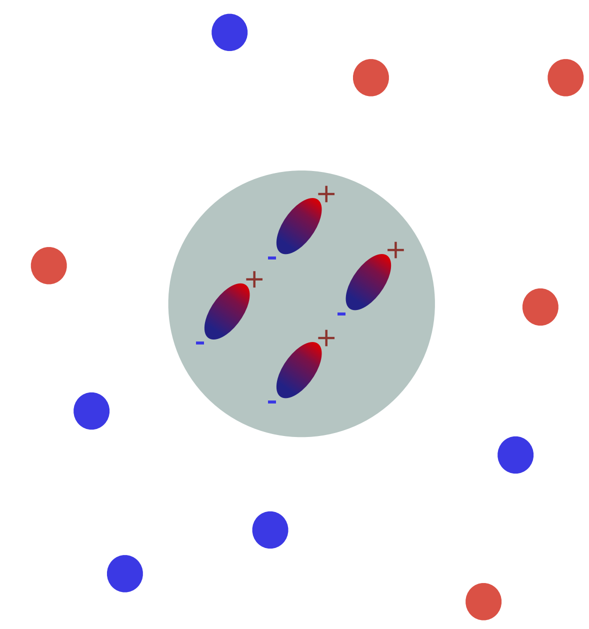


$$\vec{E}_p = -\frac{\vec{P}}{3\epsilon_0}$$ inside the sphere ($r \leq R$), and

$$\vec{E}_p = \frac{V}{4\pi\epsilon_0}\frac{3(\hat{x}\cdot\vec{P})\hat{x}-\vec{P}}{x^3}$$ outside the sphere ($r \geq R$)

This is just like a "big dipole" right at the center of the sphere!

Polarization charges



- The discussion above implies that we can define ***polarization charges***:

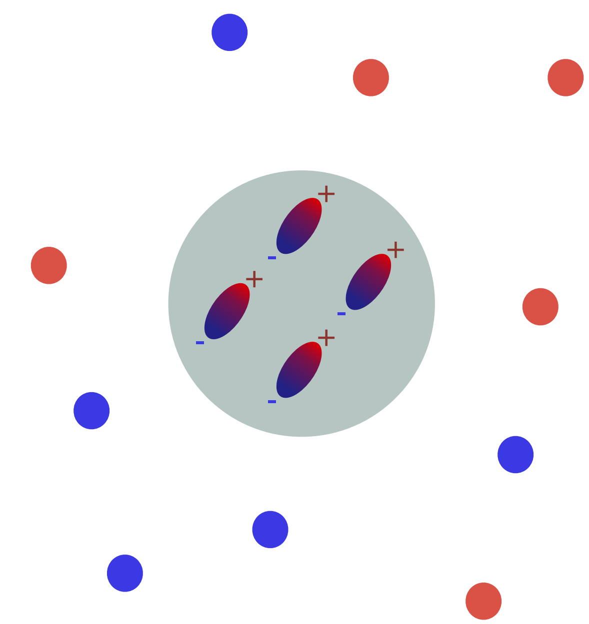
$$\sigma_P = \vec{P} \cdot \hat{n} \quad \text{and} \quad \rho_P = -\vec{\nabla} \cdot \vec{P}$$

- Notice that even for a homogenous dielectric, we cannot simply discard the charge density, since the density and the surface density obey the constraint that:

$$\int d^3x\, \rho_P = \int d^3x \left[-\vec{\nabla} \cdot \vec{P} \right] = -\oint d\vec{S} \cdot \vec{P} = -\oint dS\, \sigma_P$$

So, in some sense the charge density has a **discontinuity** right at the **surface** — where the medium itself is discontinuous (the material stops, and vacuum starts!)

Polarization charges



- Let's associate the field generated by the polarized dipoles as:

$$\vec{\nabla} \cdot \vec{E}_P = \frac{\rho_P}{\epsilon_0} \quad ,$$

while all the other charges (the "free charges") are the ones giving rise to the field in the absence of the material — i.e., in vacuum:

$$\vec{\nabla} \cdot \vec{E}_f = \frac{\rho_f}{\epsilon_0}$$

- The combined (total) field is therefore:

$$\vec{\nabla} \cdot \vec{E} = \vec{\nabla} \cdot (\vec{E}_f + \vec{E}_P) = \frac{\rho_f + \rho_P}{\epsilon_0}$$

- But we know (above) that $\rho_P = -\vec{\nabla} \cdot \vec{P}$, so we can write:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho_f - \vec{\nabla} \cdot \vec{P}}{\epsilon_0} \quad \Longrightarrow \quad \vec{\nabla} \cdot (\epsilon_0 \vec{E} + \vec{P}) = \rho_f \quad \Longrightarrow \quad \vec{\nabla} \cdot \vec{D} = \rho_f$$

Electric "displacement"

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

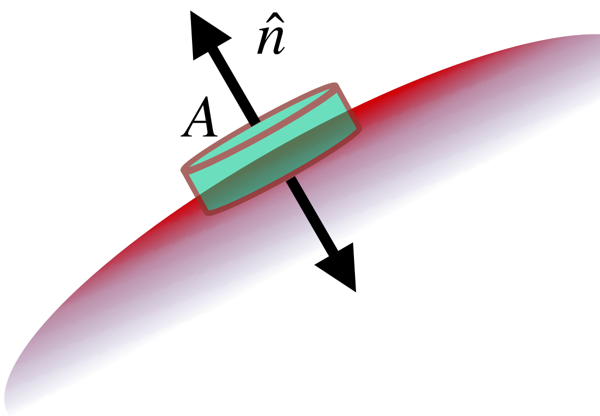
$$\vec{\nabla} \cdot \vec{D} = \rho_f$$

$$(\vec{\nabla} \times \vec{D} = ???\ldots)$$

Boundary conditions

- Let's look at what's happening at the **interface** between a dielectric and vacuum in terms of the "electric displacement" $\vec{D}$ and the electric field $\vec{E}$.

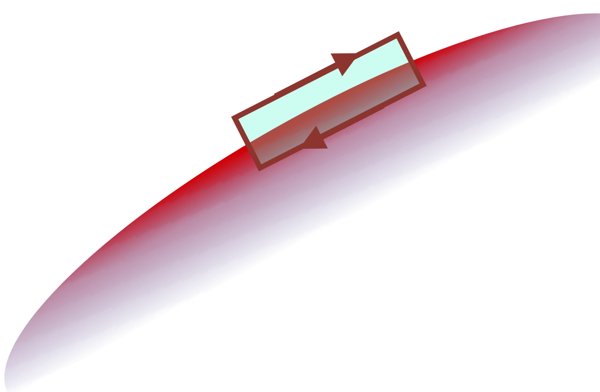
- Applying Gauss' law to a very thin volume that includes an area A of the interface gives us:



$$\int_V d^3x\, \vec{\nabla} \cdot \vec{D} = \oint d\vec{S} \cdot \vec{D} = \int d^3x\, \rho_f$$

$$\Rightarrow \quad A \left(\hat{n} \cdot \vec{D}\right)_{bottom}^{top} = A\, \Delta D_\perp = Q_f = \sigma_f\, A$$

- On the other hand, for the parallel components we apply instead Faraday's law on a similar circuit:



$\vec{\nabla} \times \vec{E} = 0$, and we obtain that

$$\Delta \vec{E}_{||} = 0$$

- Therefore, the **boundary conditions** are, in the most general case:

$\Delta D_\perp = \sigma_f$, and $\Delta \vec{E}_{||} = 0$

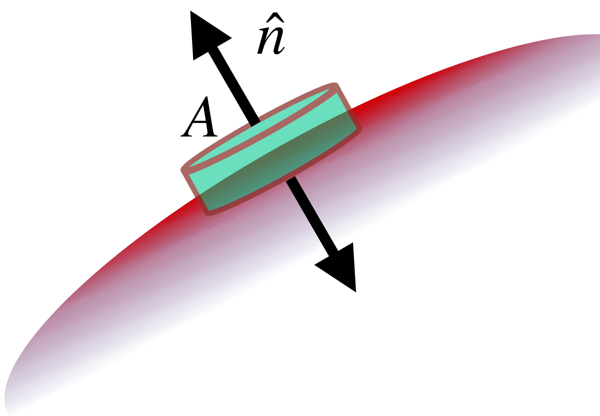
Linear dielectric materials

- Recall that the electric displacement combines the electric field with the polarization caused by the field itself in the material, $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$.

- Of course, the greater the field, the greater is the polarization. But it is often the case that this polarization is not too dramatic, so that:

$$\vec{P} = \chi_E \, \epsilon_0 \, \vec{E} \quad ,$$

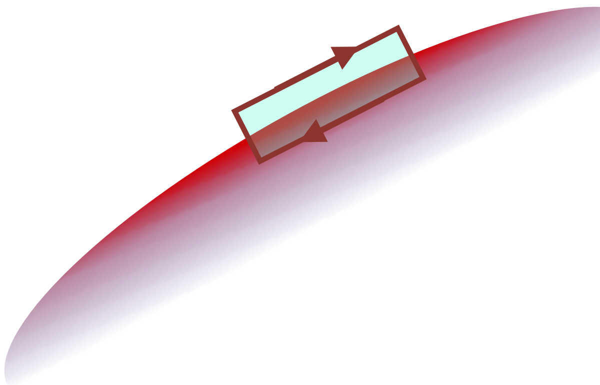
where χ_E is the ***electric susceptibility*** of the material.



- With this approximation we can then write:

$$\vec{D} = \epsilon_0 \vec{E} + \chi_E \, \epsilon_0 \vec{E} = (1 + \chi_E) \, \epsilon_0 \, \vec{E} \;=\; \epsilon \, \vec{E} \quad ,$$

where $\epsilon = (1 + \chi_E)\epsilon_0$ is called the ***relative permittivity*** of the (linear) media.



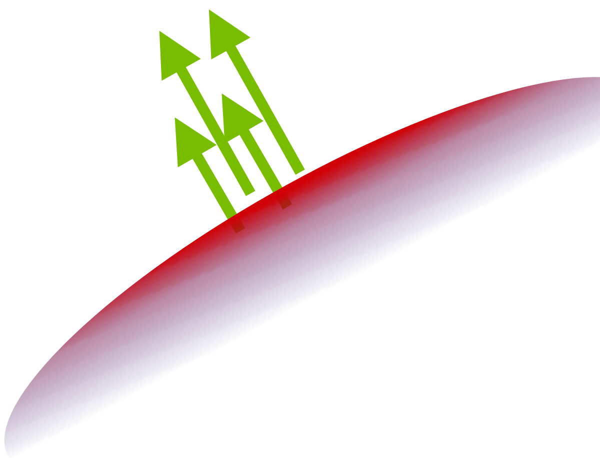
- For linear media the laws of electrostatics become simpler:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho_f}{\epsilon} \quad \text{, and} \qquad \vec{\nabla} \times \vec{E} = 0$$

Linear dielectric materials

- Even for linear media the boundary conditions need to be taken carefully. Take the result that the discontinuity in $\vec{D}$ gives the charge density at the interface:

$$\Delta D_\perp = \sigma_f$$



- If that interface happens to be right at the surface of a dielectric, then:

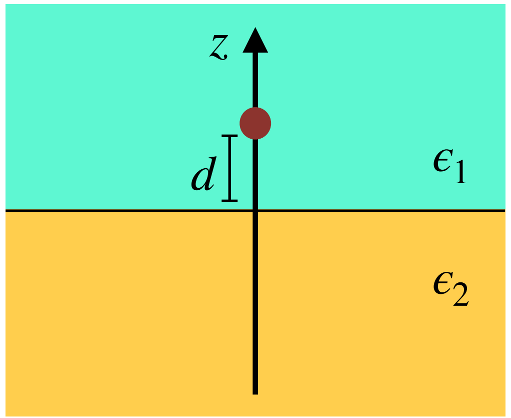
$$\epsilon_0 E_\perp(\text{above}) - \epsilon E_\perp(\text{below}) = \sigma_f$$

- Even if there are no free charges ($\sigma_f = 0$), you will **still** get a discontinuity in the electric field, due to the surface polarization charges:

$$0 = \epsilon_0 E_\perp(\text{above}) - (1 + \chi_E)\epsilon_0 E_\perp(\text{below}) = 0$$

$$\Rightarrow \quad \epsilon_0 \Delta E_\perp(\text{above}) = \chi_E \epsilon_0 E_\perp(\text{below}) = \vec{P} \cdot \hat{n} = \sigma_P$$

Example: point charge near dielectric plane



- Let's consider an interesting example/application of the concepts we just saw. Consider a **point charge** q that is placed at a distance d above the **interface of two dielectric media**.

- In the the **upper** ($z > 0$) half-space we have:

$$\vec{\nabla} \cdot \vec{D} = \epsilon_1 \vec{\nabla} \cdot \vec{E} = \rho_f$$

- In the the **lower** ($z < 0$) half-space we have:

$$\vec{\nabla} \cdot \vec{D} = \epsilon_2 \vec{\nabla} \cdot \vec{E} = \rho_f = 0$$

- Moreover, we have that $\vec{\nabla} \times \vec{E} = 0$ everywhere.

- Recall now that the boundary conditions are:

$$\Delta D_\perp = \sigma_f = 0 \quad \text{and} \quad \Delta \vec{E}_{||} = 0$$

- Therefore, we can express the fields just above and just below the surface as:

$$\lim_{z \to 0^+} \epsilon_1 E_z = \lim_{z \to 0^-} \epsilon_2 E_z \quad , \quad \text{and}$$

$$\lim_{z \to 0^+} E_{x,y} = \lim_{z \to 0^-} E_{x,y} \quad .$$

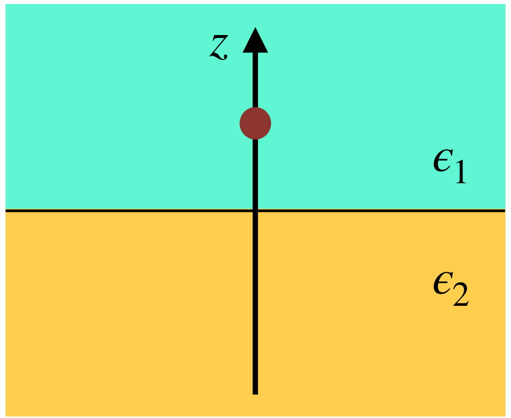
- OK, but how do we go about actually solving this problem? What technique should we use?

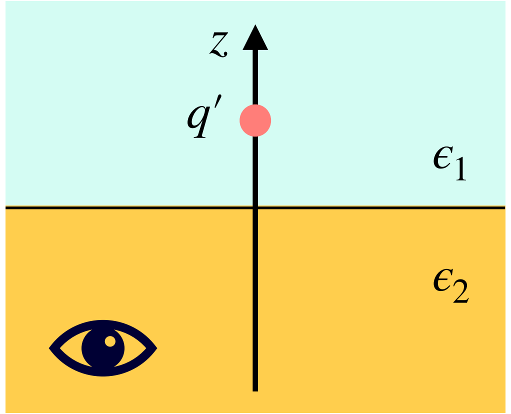
Example: point charge near dielectric plane

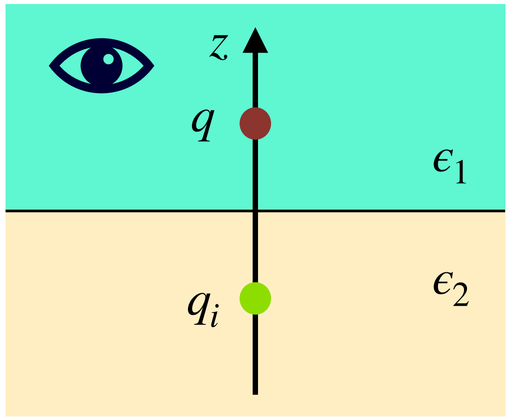
- The best way to deal with this problem is to use the **method of images** — but we need to make some adaptations to it in order to satisfy the boundary conditions.
- The idea is that the dielectric **shields** partially the charge in the upper half-volume, so an observer in the lower half sees a charge q' that is **different:** the actual charge gets **screened**. (For a conductor, the shielding would be perfect, and in the lower half there would be no sign whatsoever of the charge, and the field would vanish.)
- Likewise, the image charge (q_i) in the lower half that is necessary to enforce the boundary condition at the interface is *not* equal and opposite to the real charge of the upper half.
- So, we separate the solutions for the potential in two, and make the ansätze:

$$\phi_> = \frac{1}{4\pi\,\epsilon_1}\left(\frac{q}{|\vec{x} - d\,\hat{z}|} + \frac{q_i}{|\vec{x} + d\,\hat{z}|}\right)$$

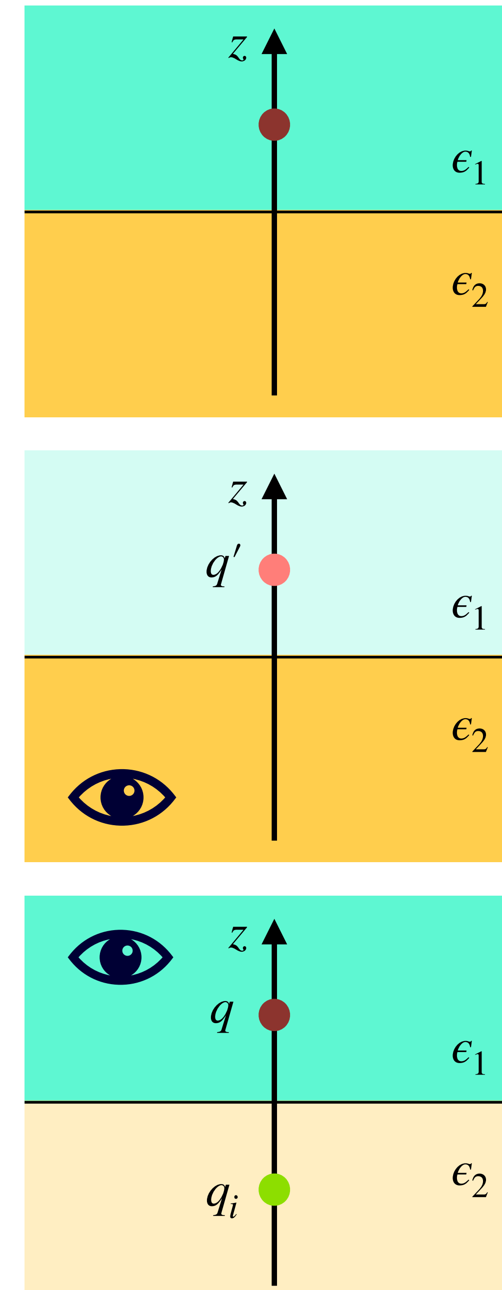
$$\phi_< = \frac{1}{4\pi\,\epsilon_2}\frac{q'}{|\vec{x} - d\,\hat{z}|}$$







Example: point charge near dielectric plane



- Now, let's use these two "trial solutions", and try to satisfy the boundary conditions:

$$\epsilon_1 \frac{d\,\phi_>}{dz} = \epsilon_2 \frac{d\,\phi_<}{dz} \quad , \quad \frac{d\,\phi_>}{dx} = \frac{d\,\phi_<}{dx} \quad , \quad \frac{d\,\phi_>}{dy} = \frac{d\,\phi_<}{dy}$$

- The conditions above lead to the equations (check!):

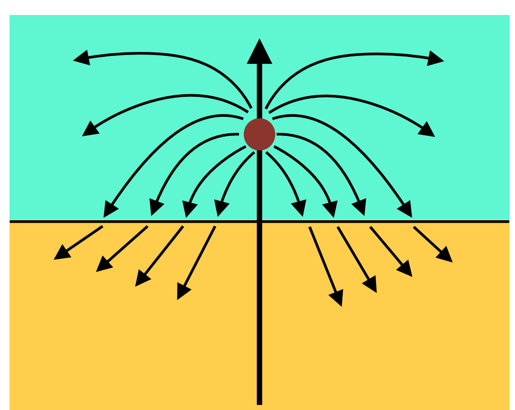
$$q_i = -\frac{\epsilon_2 - \epsilon_1}{\epsilon_1 + \epsilon_2}\, q \quad , \quad \text{and}$$

$$q' = \frac{2\epsilon_2}{\epsilon_1 + \epsilon_2}\, q$$

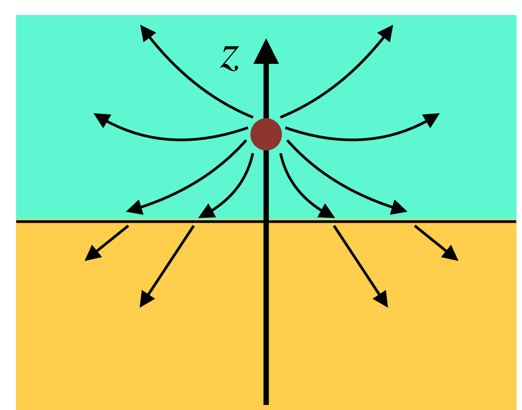
- The limit of vacuum in the upper half and a conductor in the lower half is obtained by taking $\epsilon_2 \gg \epsilon_1$, which leads to:

$q_i \to -q$. (Notice that in this limit q' becomes irrelevant, since $\phi_< \sim q'/\epsilon_2 \ll q/\epsilon_1$!)

- The field has the following behavior:



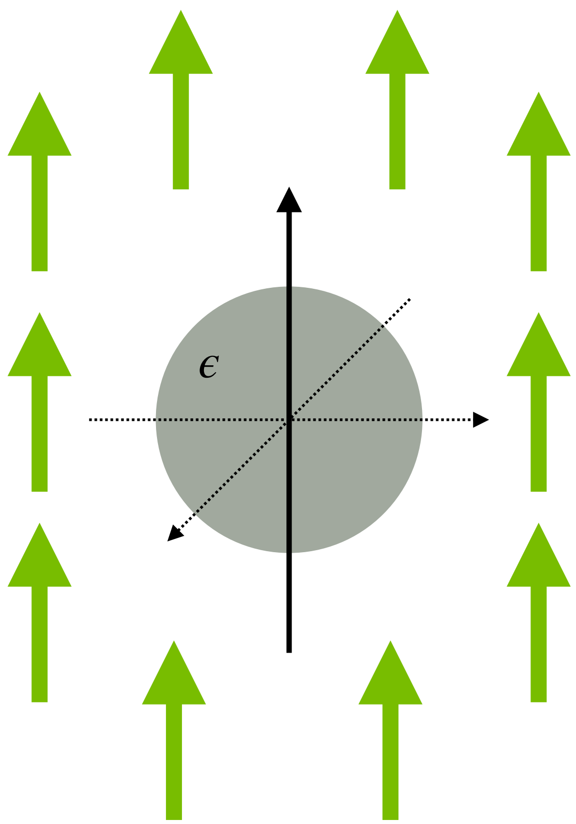
$\epsilon_2 > \epsilon_1$



$\epsilon_2 < \epsilon_1$

Exercise: show that the polarization charges are $\sigma_P = -\dfrac{q}{2\pi}\dfrac{\epsilon_0(\epsilon_2 - \epsilon_1)}{\epsilon_1(\epsilon_2 + \epsilon_1)}\dfrac{2}{(x^2 + y^2 + d^2)^{3/2}}$

Example 2: dielectric sphere in external field



- Take a sphere of radius R which is subjected to an external field $\vec{E}_{ext} = E_0 \hat{z}$.
- Inside the sphere we have:

$$\vec{\nabla} \cdot \vec{D}_< = \epsilon \vec{\nabla} \cdot \vec{E}_< = 0 \quad \Rightarrow \quad \nabla^2 \phi_< = 0$$

- And outside the sphere, almost the same:

$$\vec{\nabla} \cdot \vec{D}_> = \epsilon_0 \vec{\nabla} \cdot \vec{E}_> = 0 \quad \Rightarrow \quad \nabla^2 \phi_> = 0$$

- Given the axial symmetry of the problem, we can use the familiar solutions for the Laplace equation:

$$\phi_< = \sum_\ell A_\ell \, r^\ell \, P_\ell(\cos\theta) \quad \text{and}$$

$$\phi_> = -E_0 \, r \, \cos\theta + \sum_\ell B_\ell \, r^{-\ell-1} P_\ell(\cos\theta) \quad ,$$

where the first term satisfies the "boundary conditions" at $r \to \infty$ that $\vec{E} \to \vec{E}_{ext}$.

- The other boundary conditions are now:

$$\Delta D_\perp = 0 \quad \Rightarrow \quad -\epsilon \left.\frac{\partial \phi_<}{\partial r}\right|_{r=R} = -\epsilon_0 \left.\frac{\partial \phi_>}{\partial r}\right|_{r=R} \quad , \text{ and}$$

$$\Delta \vec{E}_{||} = 0 \quad \Rightarrow \quad \left.\frac{\partial \phi_<}{\partial \theta}\right|_{r=R} = \left.\frac{\partial \phi_>}{\partial \theta}\right|_{r=R}$$

Example 2: dielectric sphere in external field

- Substituting the expressions for $\phi_<$ and $\phi_>$ and solving for the boundary conditions, we find that $A_\ell = B_\ell = 0$ for $\ell \neq 1$, and:

$$A_1 = -\frac{\epsilon_0}{\epsilon + 2\epsilon_0} E_0 \quad , \quad \text{and} \quad B_1 = \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} E_0 R^3$$

- Substituting these values back into the expressions for the potential we can compute the electric field:

$$\vec{E}_< = -\vec{\nabla}\phi_< = \frac{3\epsilon_0}{\epsilon + 2\epsilon_0} E_0 \hat{z} \quad , \quad \text{and}$$

$$\vec{E}_> = -\vec{\nabla}\phi_> = E_0 \hat{z} + \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} \frac{E_0 R^3}{r^3} \left(2\cos\theta\, \hat{r} + \sin\theta\, \hat{\theta} \right)$$

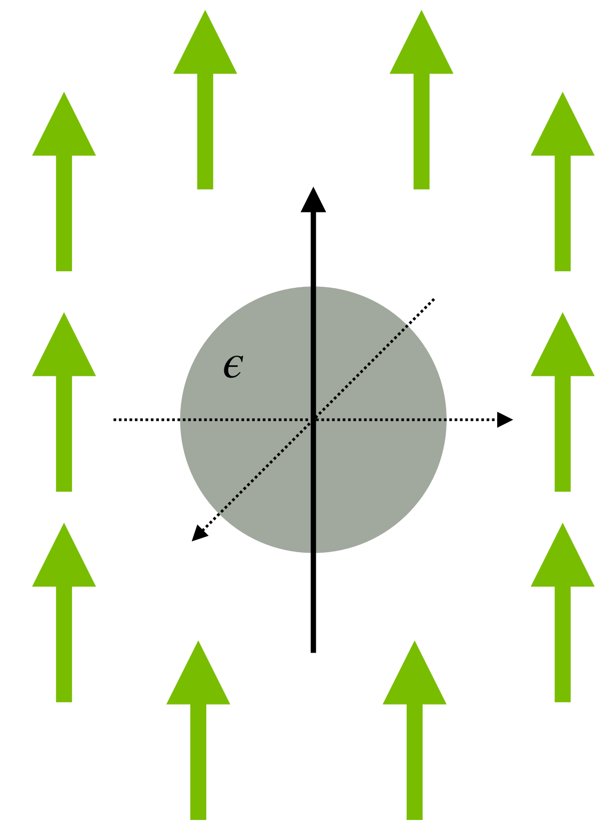
- So, we find two interesting results here:

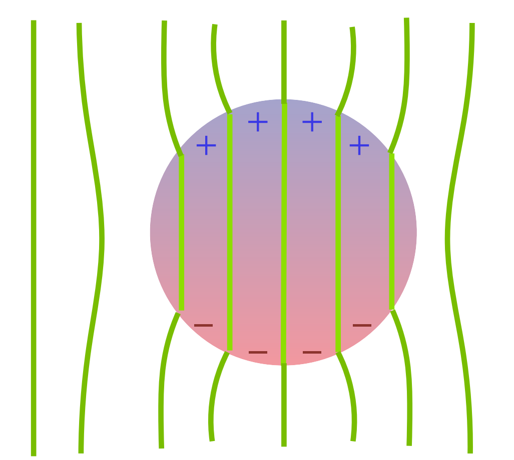
  (a) the field inside the sphere is constant; and

  (b) the field strength just outside the sphere actually increases compared with the external field , by a term which is identical to a dipole at the center of the sphere.

- Both results are somewhat surprising. First, somehow the polarization charges distribute themselves to create a constant electric field inside the sphere, that partially cancels the external field.

- And second, the field outside the sphere actually increases. But think about it: this is expected, since the sphere is now like a big dipole, with the negative and positive charges aligned to the direction of the external field. In other words: the negative charges in the opposite side of the field direction increase the strength of the field in their vicinity; and the positive charges on the side aligned with the direction of the field also increase the field strength there.





Some final remarks

- This is basically all I have to say about dielectric materials.
- However, almost all that I said here is based on a very simple model for the polarization (the Lorentz model) which is… well, **incorrect**! In fact, we cannot simply state that the polarization of a material is just the sum of all its "dipoles", something like:

$$\vec{P} = \int d^3x \frac{d\vec{p}}{dV}$$

- In reality, polarization in atoms and molecules is a much more interesting phenomenon, that often is intrinsically **quantum-mechanical** in nature.
- Moreover, with simple assumptions about the nature of the molecules themselves, it is possible to obtain relationships between properties of materials, like the relative permittivity of a medium (ϵ), given its polarizability.
- For many more details about this topic, I refer you to Zangwill, Ch. 6 .

Next class:

- Magnetostatics
- Biot-Savart and Ampère’s laws
- The vector potential
- Jackson, Ch. 5