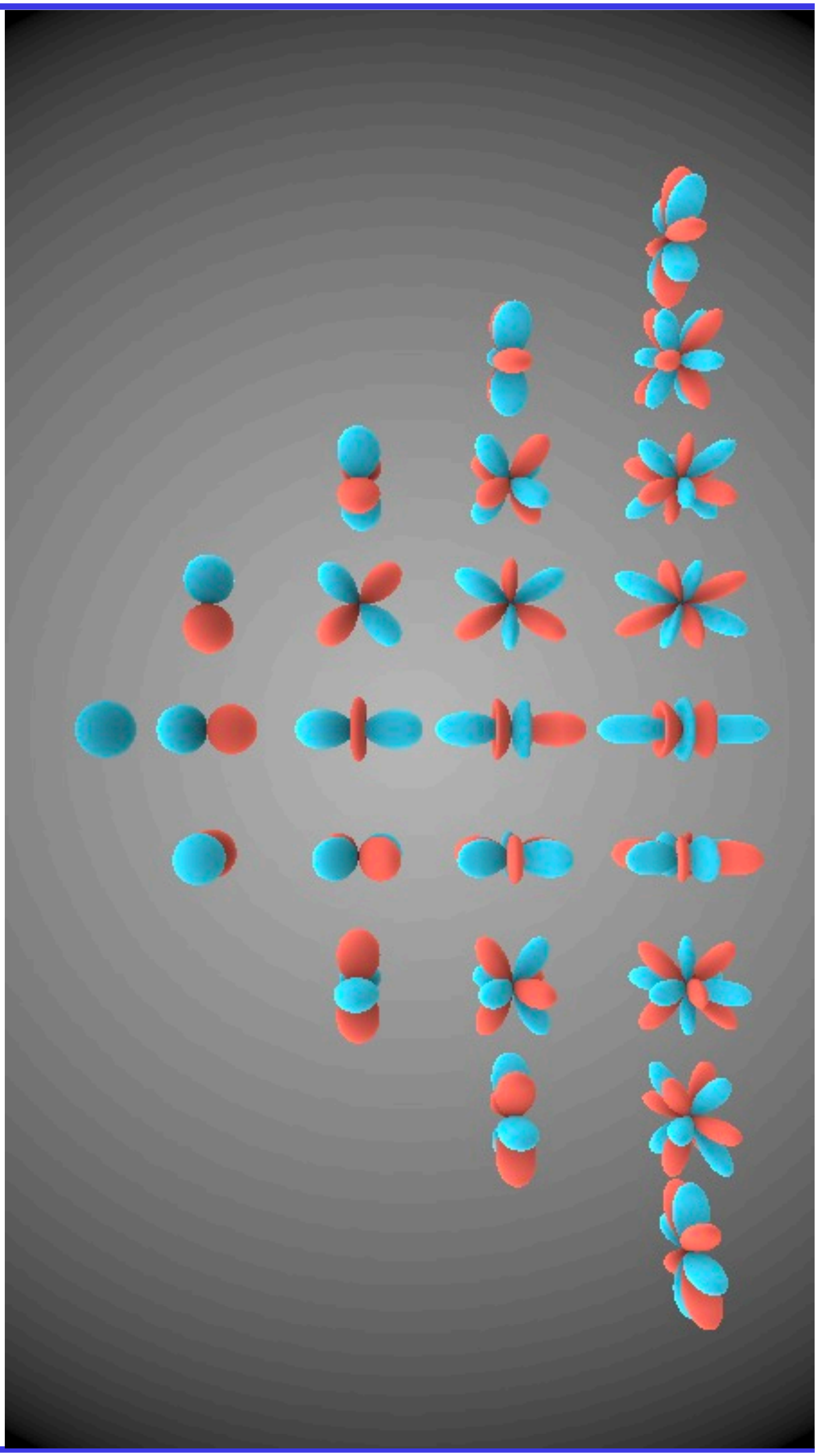
Electrodynamics

⚡Laplace equation in spherical coordinates
⚡Green's functions
⚡Boundary value problems
⚡Laplace equation in cylindrical coordinates



Multipole expansion

- In the last class we saw that (scalar) angular functions can be expressed in terms of a multipole expansion — a series over spherical harmonics:

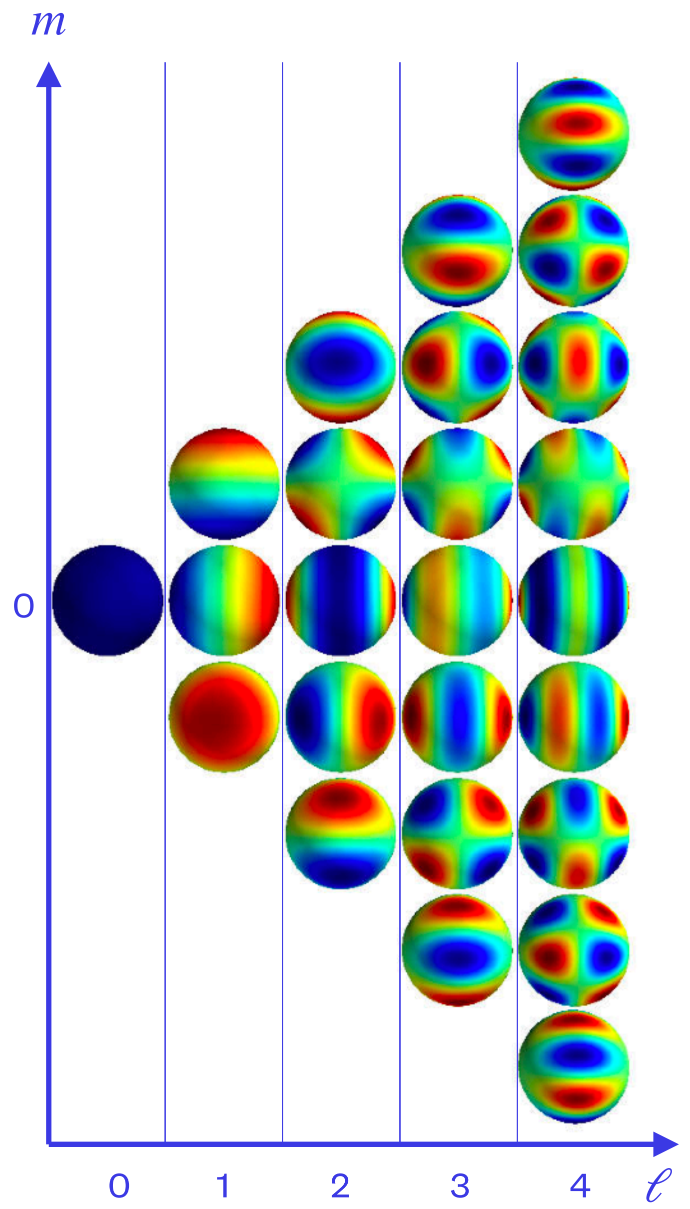
$$f(\theta,\varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} f_{\ell m} \, Y_\ell^{(m)}(\theta,\varphi) \quad , \text{ where:}$$

$$f_{\ell m} = \int d^2\Omega \, f(\theta,\varphi) \, Y_\ell^{(m)*}(\theta,\varphi) \quad .$$

- But in electromagnetism we have not only the angular, but also the radial coordinate. In fact, we also showed that the basic "building block" of electrostatics, the potential of a point charge, can be written in terms of:

$$\frac{1}{|\vec{x} - \vec{x}'|} = \sum_{\ell,m} \frac{4\pi}{2\ell+1} \frac{r_<^\ell}{r_>^{\ell+1}} Y_\ell^{(m)}(\hat{n}) \, Y_\ell^{(m)*}(\hat{n}') \quad .$$

- Now we want to express an arbitrary charge density using a similar expansion.



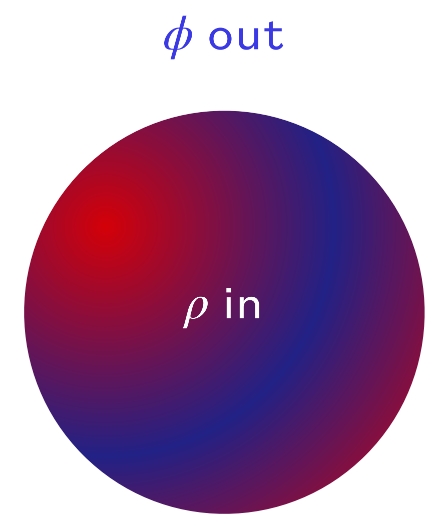
Multipole expansion

- Let's say that we start with the simplest problem: an "out" ("exterior") solution for a charge density that lies all inside a radius R.
- If we think just a little bit about this, we see that we can express the potential as:

$$\phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell,m} A_{\ell m} \frac{1}{r^{\ell+1}} Y_\ell^{(m)}(\hat{n}) \quad \text{, where} \quad r > R \quad \text{and}$$

$$A_{\ell m} = \int d^3r' \, \rho(\vec{r}\,') \, (r')^\ell \, Y_\ell^{(m)*}(\theta', \varphi') \quad , \quad \text{with } r' \leq R \; .$$

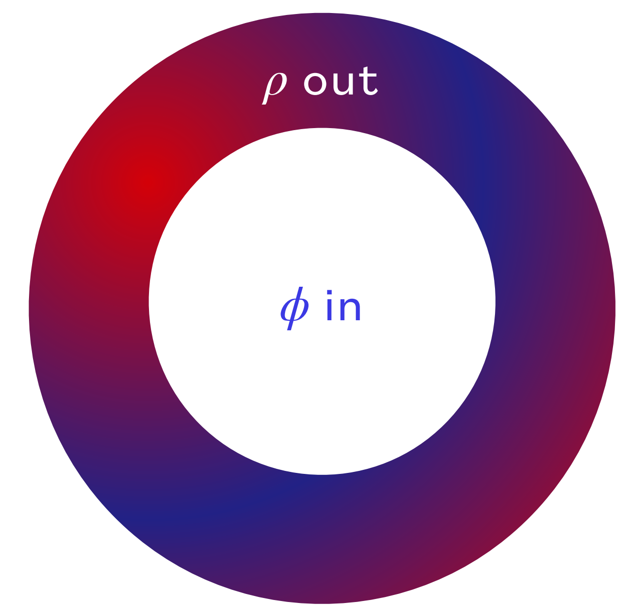
Notice that the *monopole* is $A_{00} = \int d^3r' \rho(\vec{r}\,') = Q$, i.e., the ***total charge***.



- For the "in" type solution, when the charge densities are all outside some radius R, we have the converse:

$$\phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell,m} B_{\ell m} \, r^\ell \, Y_\ell^{(m)}(\hat{n}) \quad \text{, where} \quad r < R \quad \text{and}$$

$$B_{\ell m} = \int d^3r' \, \rho(\vec{r}\,') \, r'^{-\ell-1} \, Y_\ell^{(m)*}(\theta', \varphi') \quad , \quad \text{with } r' \geq R \; .$$



Green's function in spherical coordinates

- The solutions above are often sufficient to solve many problems where we specify a charge density, but they are not enough to tackle most **boundary value problems**.
- A typical boundary value problem specifies the potential on some given radius. This can be an inner radius (for an "out" b.c. problem), or an outer radius (for an "in" b.c. problem). In those cases, it is very useful to have a Green's function that can be adapted to those two situations.
- To be more specific, we are searching for the solution of:

$$\nabla^2 G(\vec{r},\vec{r}\,') \;=\; \delta(\vec{r}-\vec{r}\,') \;=\; \frac{1}{r^2}\delta(r-r')\,\delta(\cos\theta-\cos\theta')\,\delta(\varphi-\varphi')$$

- At this point it is worth recalling the ***completeness relation***, which we obtained in the last class:

$$\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} Y_\ell^{(m)}(\theta,\varphi)Y_\ell^{(m)*}(\theta',\varphi') \;=\; \delta(\cos\theta-\cos\theta')\,\delta(\varphi-\varphi')$$

- Remember also that $\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\nabla^2_\Omega$, with $\nabla^2_\Omega\, Y_\ell^{(m)} = -\,\ell(\ell+1)\,Y_\ell^{(m)}$.
- This gives us a *hint* that we should try a solution to the Green's function of the type:

$$G(\vec{r},\vec{r}\,') \;\to\; \sum_{\ell,m} G_\ell(r,r')\,Y_\ell^{(m)}(\theta,\varphi)Y_\ell^{(m)*}(\theta',\varphi')$$, where the radial part gives us:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial\,G_\ell(r,r')}{\partial r}\right]-\frac{\ell(\ell+1)}{r^2}G_\ell(r,r') = \frac{1}{r^2}\delta(r-r')$$

Green's function in spherical coordinates

- This time, we will proceed to the construction of the Green's function using the solutions to the homogeneous equation.

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial\, G_\ell(r,r')}{\partial r}\right]-\frac{\ell(\ell+1)}{r^2}G_\ell(r,r')=\frac{1}{r^2}\delta(r-r')$$

- But the solutions to the homogeneous equation are precisely the radial functions that we found before:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial\, H_\ell(r)}{\partial r}\right]-\frac{\ell(\ell+1)}{r^2}H_\ell(r)=0$$

$$\Rightarrow \quad H_\ell^{(1)}=r^\ell \quad , \quad H_\ell^{(2)}=r^{-\ell-1}$$

- The Green's function is then constructed as:

$$G_\ell(r,r')=\begin{cases}\frac{H_\ell^{(1)}(r)H_\ell^{(2)}(r')}{W} & r>r' \\ \frac{H_\ell^{(1)}(r')H_\ell^{(2)}(r)}{W} & r'>r\end{cases}=\frac{1}{W}\left[\Theta(r'-r)H_\ell^{(1)}(r)H_\ell^{(2)}(r')+\Theta(r-r')H_\ell^{(2)}(r)H_\ell^{(1)}(r')\right]$$

where $\Theta(x)$ is the Heaviside (step) function, and the denominator is the Wronskian:

$$W=r^2\left[H_\ell^{(2)}H_\ell'^{(1)}-H_\ell^{(1)}H_\ell'^{(2)}\right]=2\ell+1 \quad .$$

Green's function in spherical coordinates

- We thus recover the Green's function for a point charge:

$$G_\ell(r,r') = \frac{1}{2\ell+1}\left[\Theta(r-r')\frac{r'^\ell}{r^{\ell+1}} + \Theta(r'-r)\frac{r^\ell}{r'^{\ell+1}}\right] = \frac{1}{2\ell+1}\frac{r_<^\ell}{r_>^{\ell+1}}$$

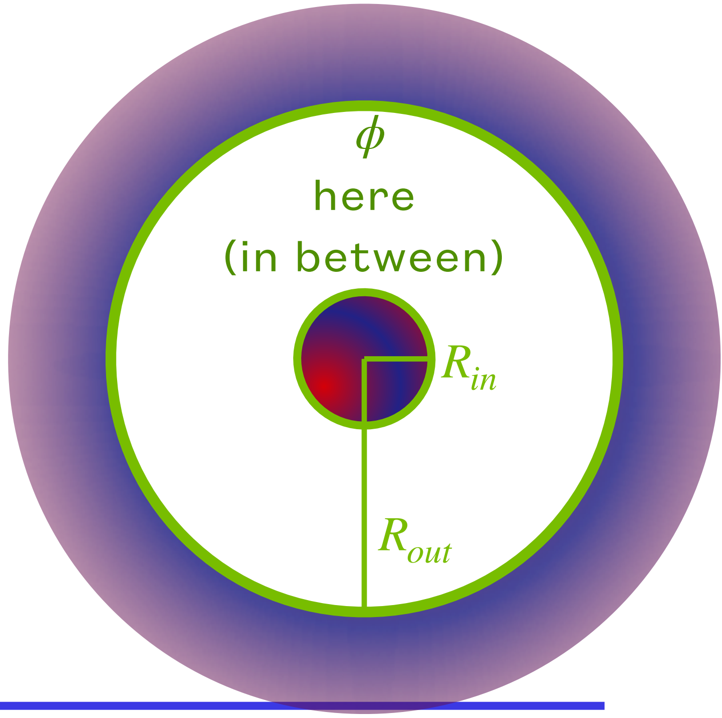
which leads to:

$$G(\vec{r},\vec{r}') = \sum_{\ell,m}\frac{1}{2\ell+1}\frac{r_<^\ell}{r_>^{\ell+1}}Y_\ell^{(m)}(\theta,\varphi)Y_\ell^{(m)*}(\theta',\varphi')$$

But remember that for a point charge q at the position $\vec{r}'$ we have:

$$\phi = \frac{q}{4\pi\epsilon_0}\frac{1}{|\vec{x}-\vec{x}'|} \quad , \text{ where}$$

$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{\ell,m}\frac{4\pi}{2\ell+1}\frac{r_<^\ell}{r_>^{\ell+1}}Y_\ell^{(m)}(\theta,\varphi)Y_\ell^{(m)*}(\theta',\varphi')$$



Green's function in spherical coordinates

- We want to build a Green's functions that satisfy some boundary condition — e.g., we can require it to **vanish** at some inner radius R_{in} and at some outer radius R_{out}, so that we can solve a Dirichlet-type problem.
- Now, recall that we can **add any homogeneous solutions** to the Green functions — exactly the r^ℓ and $r^{-\ell-1}$ functions above! In practice, we simply take the solution above and substitute:

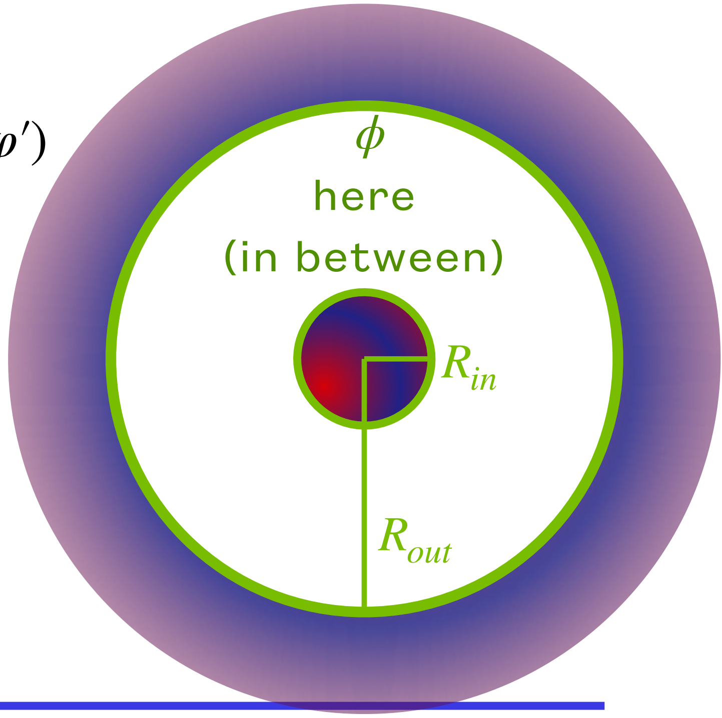
$$r_<^\ell \;\to\; r_<^\ell - \frac{R_{in}^{2\ell+1}}{r_<^{\ell+1}}$$, which vanish for $r_< \to R_{in}$; and $$\frac{1}{r_>^{\ell+1}} \;\to\; \frac{1}{r_>^{\ell+1}} - \frac{r_>^\ell}{R_{out}^{2\ell+1}}$$

. Hence we have:

$$G_{in,out}(\vec{r},\vec{r}\,') \;=\; \sum_{\ell,m} \frac{1}{2\ell+1}\frac{1}{N_\ell}\left(r_<^\ell - \frac{R_{in}^{2\ell+1}}{r_<^{\ell+1}}\right)\left(\frac{1}{r_>^{\ell+1}} - \frac{r_>^\ell}{R_{out}^{2\ell+1}}\right) Y_\ell^{(m)}(\theta,\varphi)Y_\ell^{(m)*}(\theta',\varphi')$$

which is trivial to see vanishes when $r_< \to R_{in}$ and when $r_> \to R_{out}$.

The normalization necessary to obtain $\nabla^2 G = \delta(\vec{r}-\vec{r}\,')$ is $N_\ell = 1 - (R_{in}/R_{out})^{2\ell+1}$ (check!)



Green's function in spherical coordinates

- Let's make a simple application of the Green's function that we obtained above:

$$G_{in,out}(\vec{r},\vec{r}\,') = \sum_{\ell,m} \frac{1}{2\ell+1}\frac{1}{N_\ell}\left(r_<^\ell - \frac{R_{in}^{2\ell+1}}{r_<^{\ell+1}}\right)\left(\frac{1}{r_>^{\ell+1}} - \frac{r_>^\ell}{R_{out}^{2\ell+1}}\right) Y_\ell^{(m)}(\theta,\varphi) Y_\ell^{(m)*}(\theta',\varphi')$$

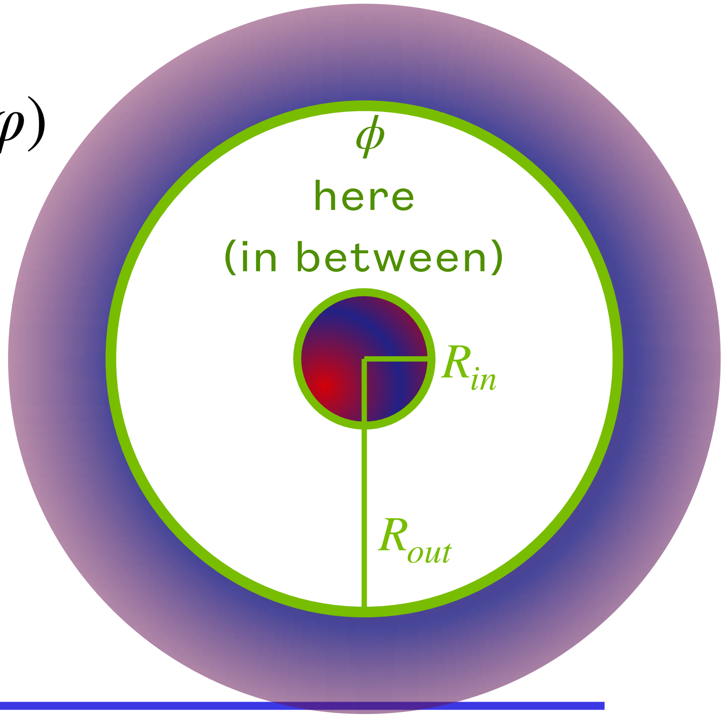
- Let's say that the charge is zero, and that the potential at the boundaries $r = R_{in}$ and $r = R_{out}$ are given by some pure spherical harmonic mode — e.g.:

$$\phi(r=R_{in},\theta,\varphi) = \phi_{in}\, Y_{\ell_1}^{(m_1)}(\theta,\varphi) \quad , \quad \phi(r=R_{out},\theta,\varphi) = \phi_{out}\, Y_{\ell_2}^{(m_2)}(\theta,\varphi)$$

.

- The solution to the Laplace equation is then given by:

$$\phi(\vec{x}) = \oint d^2S' \, \frac{\partial\, G_{in,out}(\vec{r},\vec{r}\,')}{\partial r'}\, \phi(\vec{r}\,') \quad .$$



Green's function in spherical coordinates

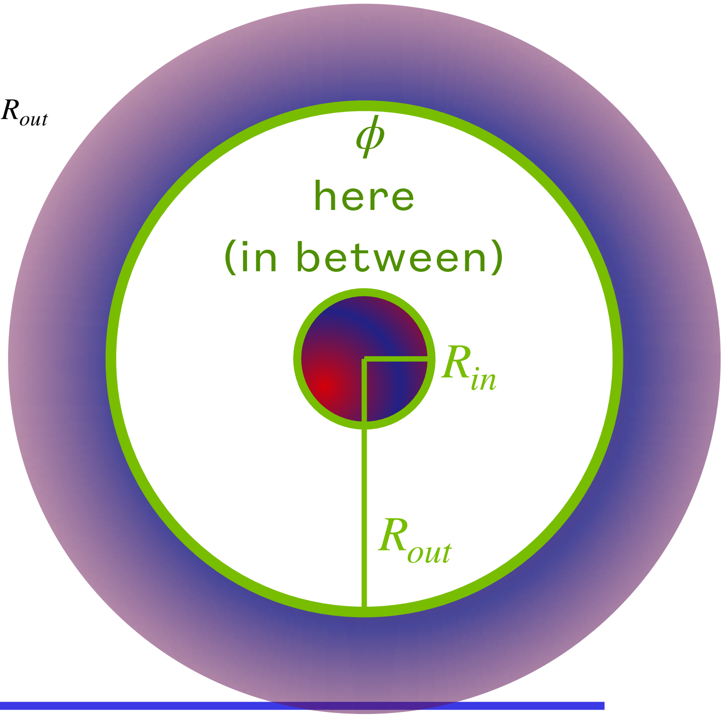
- Notice that the surface has **two disjoint pieces**: the inner one at $r = R_{in}$ and the outer one at $r = R_{out}$. Moreover, for the inner surface we get $r_< = r'$, while for the outer surface we have $r_> = r'$ — but remember that the "outer" surface points inward! Therefore:

$$\left.\frac{\partial G_{in,out}(\vec{r},\vec{r}\,')}{\partial r'}\right|_{r'=R_{in}} = \frac{\partial}{\partial r'}\sum_{\ell,m}\frac{1}{2\ell+1}\frac{1}{N_\ell}\left(r'^\ell - \frac{R_{in}^{2\ell+1}}{r'^{\ell+1}}\right)\left(\frac{1}{r^{\ell+1}} - \frac{r^\ell}{R_{out}^{2\ell+1}}\right) Y_\ell^{(m)}(\theta,\varphi)Y_\ell^{(m)*}(\theta',\varphi')\Bigg|_{r'=R_{in}}$$

$$= \sum_{\ell,m}\frac{1}{2\ell+1}\frac{1}{N_\ell}\left((2\ell+1)R_{in}^{\ell-1}\right)\left(\frac{1}{r^{\ell+1}} - \frac{r^\ell}{R_{out}^{2\ell+1}}\right) Y_\ell^{(m)}(\theta,\varphi)Y_\ell^{(m)*}(\theta',\varphi')$$

$$\left.\frac{\partial G_{in,out}(\vec{r},\vec{r}\,')}{\partial r'}\right|_{r'=R_{out}} = -\frac{\partial}{\partial r'}\sum_{\ell,m}\frac{1}{2\ell+1}\frac{1}{N_\ell}\left(r^\ell - \frac{R_{in}^{2\ell+1}}{r^{\ell+1}}\right)\left(\frac{1}{r'^{\ell+1}} - \frac{r'^\ell}{R_{out}^{2\ell+1}}\right) Y_\ell^{(m)}(\theta,\varphi)Y_\ell^{(m)*}(\theta',\varphi')\Bigg|_{r'=R_{out}}$$

$$= \sum_{\ell,m}\frac{1}{2\ell+1}\frac{1}{N_\ell}\left(r^\ell - \frac{R_{in}^{2\ell+1}}{r^{\ell+1}}\right)\left((2\ell+1)R_{out}^{-\ell-2}\right) Y_\ell^{(m)}(\theta,\varphi)Y_\ell^{(m)*}(\theta',\varphi')$$



Green's function in spherical coordinates

- The full solution is therefore:

$$\phi(\vec{x}) = \int d^2S' \left.\frac{\partial G_{in,out}(\vec{r},\vec{r}')}{\partial r'}\right|_{r'=R_{in}} \phi(\vec{r}'=R_{in}) \;+\; \int d^2S' \left.\frac{\partial G_{in,out}(\vec{r},\vec{r}')}{\partial r'}\right|_{r'=R_{out}} \phi(\vec{r}'=R_{out})$$

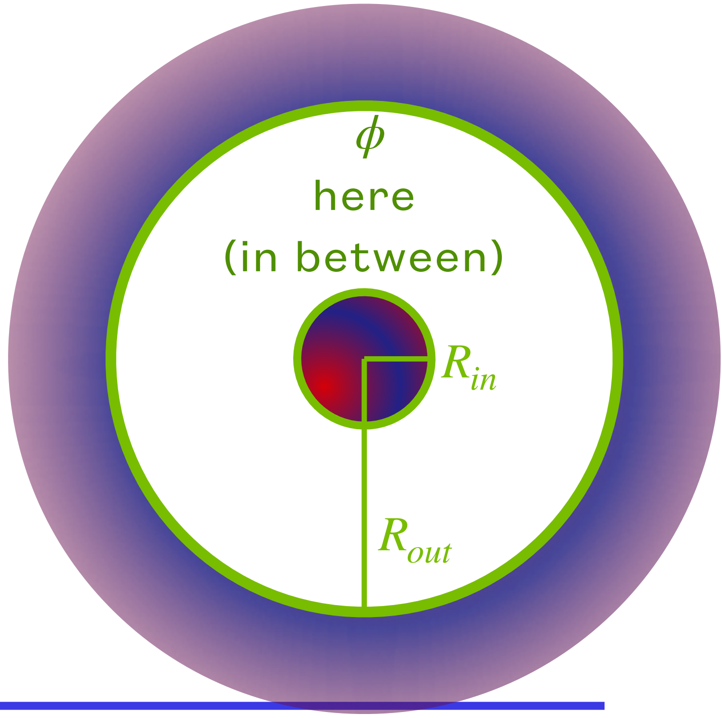
$$= R_{in}^2 \int d^2\Omega' \left.\frac{\partial G_{in,out}(\vec{r},\vec{r}')}{\partial r'}\right|_{r'=R_{in}} \phi(\vec{r}'=R_{in}) \;+\; R_{out}^2 \int d^2\Omega' \left.\frac{\partial G_{in,out}(\vec{r},\vec{r}')}{\partial r'}\right|_{r'=R_{out}} \phi(\vec{r}'=R_{out})$$

- Let's take the first term (the "in" one). We have then:

$$\begin{aligned}
\phi_1(\vec{r}) \;&=\; \int d\Omega' \sum_{\ell,m} \frac{R_{in}^{\ell+1}}{N_\ell}\left(\frac{1}{r^{\ell+1}} - \frac{r^\ell}{R_{out}^{2\ell+1}}\right) Y_\ell^{(m)}(\theta,\varphi) Y_\ell^{(m)*}(\theta',\varphi') \times \phi_{in} Y_{\ell_1}^{(m_1)}(\theta',\varphi') \\
&=\; \phi_{in} \sum_{\ell,m} \frac{R_{in}^{\ell+1}}{N_\ell}\left(\frac{1}{r^{\ell+1}} - \frac{r^\ell}{R_{out}^{2\ell+1}}\right) Y_\ell^{(m)}(\theta,\varphi) \times \delta_{\ell,\ell_1}\delta_{m,m_1} \\
&=\; \phi_{in} \frac{R_{in}^{\ell_1+1}}{N_{\ell_1}}\left(\frac{1}{r^{\ell_1+1}} - \frac{r^{\ell_1}}{R_{out}^{2\ell_1+1}}\right) Y_{\ell_1}^{(m_1)}(\theta,\varphi)
\end{aligned}$$

- In completely similar fashion, the "out" term reads:

$$\phi_2(\vec{r}) \;=\; \phi_{out} \frac{R_{out}^{-\ell_2}}{N_{\ell_2}}\left(r^{\ell_2} - \frac{R_{in}^{2\ell_2+1}}{r^{\ell_2+1}}\right) Y_{\ell_2}^{(m_2)}(\theta,\varphi)$$



Green's function in spherical coordinates

- In this way we have derived the "building blocks" of any kind of solution with in/out boundary conditions in spherical coordinates:

$$\phi(\vec{x}) = \phi_1(\vec{x}) + \phi_2(\vec{x}) \quad , \quad \text{where}$$

$$\phi_1 \;=\; \phi_{in} \frac{R_{in}^{\ell_1+1}}{N_{\ell_1}} \left(\frac{1}{r^{\ell_1+1}} - \frac{r^{\ell_1}}{R_{out}^{2\ell_1+1}} \right) Y_{\ell_1}^{(m_1)}(\theta, \varphi)$$

$$\phi_2 \;=\; \phi_{out} \frac{R_{out}^{-\ell_2}}{N_{\ell_2}} \left(r^{\ell_2} - \frac{R_{in}^{2\ell_2+1}}{r^{\ell_2+1}} \right) Y_{\ell_2}^{(m_2)}(\theta, \varphi)$$

- You can check that the solution converges to the boundary condition both at $r \to R_{in}$ and $r \to R_{out}$. Using that $N_\ell = 1 - R_{in}^{2\ell+1}/R_{out}^{2\ell+1}$ we get that:

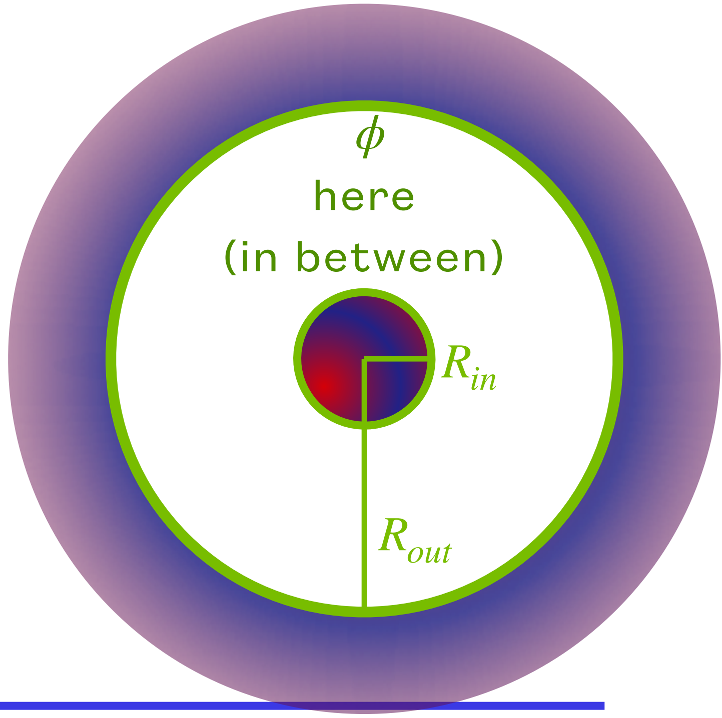
$$\phi_1(r = R_{in}) \;=\; \phi_{in} Y_{\ell_1}^{(m_1)}(\theta, \varphi) \quad , \quad \phi_1(r = R_{out}) \;=\; 0$$

$$\phi_2(r = R_{in}) \;=\; 0 \quad , \quad \phi_2(r = R_{out}) \;=\; \phi_{out} Y_{\ell_2}^{(m_2)}(\theta, \varphi)$$

- Another interesting check is to take $R_{out} \to \infty$, $\phi_{out} = 0$, and $\ell_1 = m_1 = 0$. Then we get:

$$\phi(\vec{x}) = \phi_{in} \frac{R_{in}}{r} \quad ,$$

which is simply the potential of a spherically symmetric charge distribution, $Q \equiv 4\pi\epsilon_0\, \phi_{in}\, R_{in}$!



Laplace equation in cylindrical coordinates

- One last type of problem we want to address is when some kind of symmetry calls for us to use cylindrical coordinates. In that case the Laplace equation reads:

$$\nabla^2 f \; = \; \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2} \; = \; 0$$

- Clearly, the structure of each term gives us hints about the different types of solutions. For φ and z it is more or less obvious what we will have:

$f = R(\rho)\,\Psi(\varphi)\,Z(z)$, with

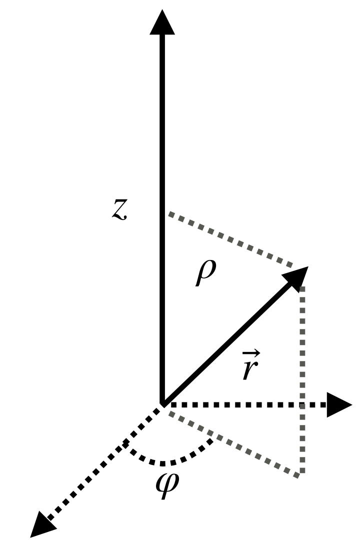
$$\Psi(\varphi) \to e^{\pm i m \varphi} \quad \Leftrightarrow \quad \frac{d^2\Psi}{d\varphi^2} = -\,m^2\,\Psi \qquad \text{(with } m \text{ integer)}$$

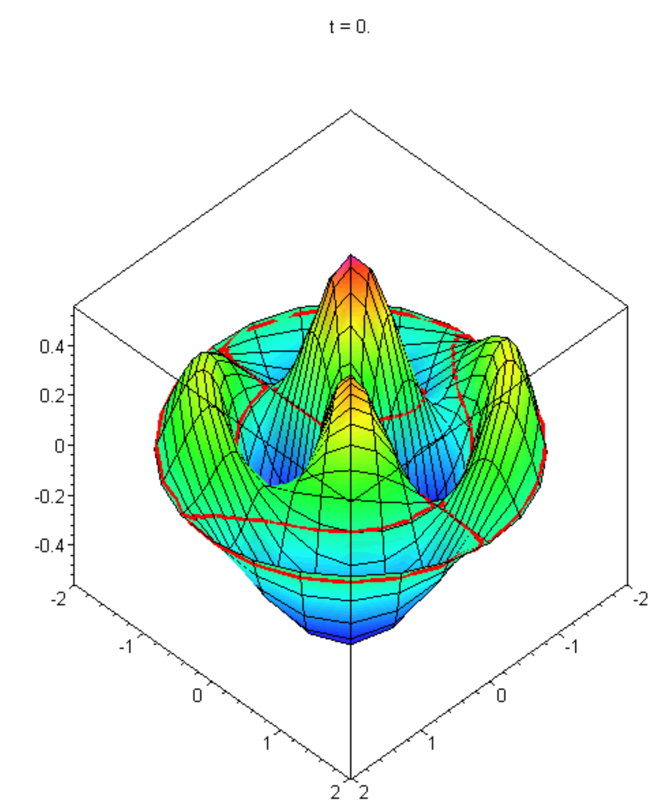
$$Z(z) \to e^{\pm k z} \quad \Leftrightarrow \quad \frac{d^2 Z}{dz^2} = +\,k^2\,Z \qquad \text{(with } k \text{ in units of } 1/\text{length)}$$

- By re-scaling the cylindrical radius $\rho \to k\rho = x$, which in this way become adimensional, we obtain the following radial equation:

$$\frac{d^2 R}{dx^2} + \frac{1}{x}\frac{dR}{dx} + \left(1 - \frac{m^2}{x^2}\right) R \; = \; 0 \quad .$$

This is a very famous equation: the **Bessel equation**. The solutions are… **Bessel functions**!





Laplace equation in cylindrical coordinates

- Notice that m is integer, but k is continous (so is $x = k\rho$, of course!). The Bessel functions are labelled by this integer, m, and we have two types of orthogonal solutions:

  $$J_m(x) \quad \text{and} \quad N_m(x)$$

- The two functions above are like the $\sin x$ and $\cos x$ of the Fourier expansion, and we often combine them in a way similar to the e^{ix} and e^{-ix}, into the so-called **Hankel functions**:

  $$H_m^{(1)}(x) = J_m(x) + iN_m(x) \quad ,$$

  $$H_m^{(2)}(x) = J_m(x) - iN_m(x) = H_m^{(1)*}(x)$$

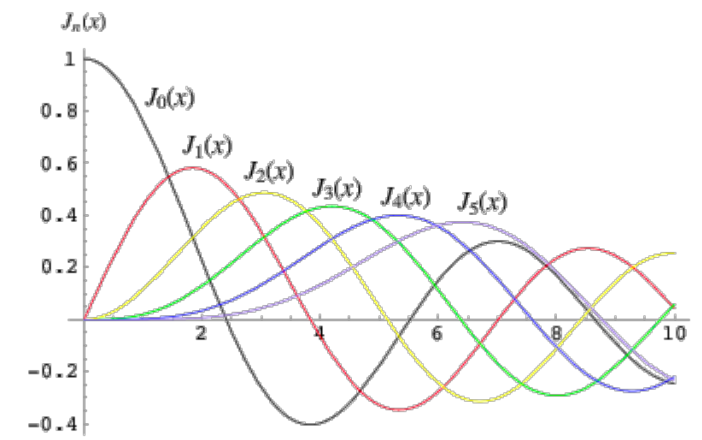
  Sometimes N_m is also called Y_m

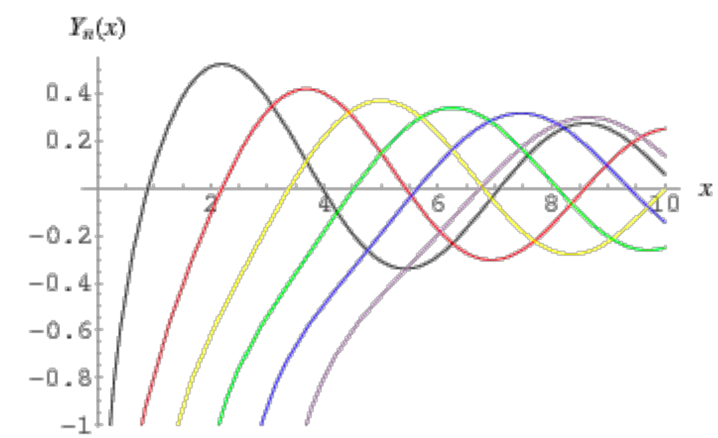
  where in this combination we have $H_{-m}^{(1)}(x) = e^{im\pi}\, H_m^{(1)}(x)$

- This combination is particularly useful when we take the limit $x \gg m$, in which case we have:

  $$\lim_{x \gg m} H_m^{(1)}(x) = \sqrt{\frac{2}{\pi x}}\, e^{i\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right)}$$

- Ok, but in what sense these functions form a complete set? Are they orthogonal? Can they be normalized…?





Laplace equation in cylindrical coordinates

- Yes, yes and YES!
- The Bessel functions are simply magical: not only are they orthogonal and normalized, they carry within themselves incredible relations that retain the “memory" of the geometry of the problems that they express.
- Recursion relations (there are many more!):

$$\frac{2m}{x} R_m(x) = R_{m-1}(x) + R_{m+1}(x)$$

$$\frac{d}{dx} R_m(x) = R_{m-1}(x) - R_{m+1}(x)$$

- The Bessel function of the first kind can also be expressed in terms of the Jacobi-Anger integral:

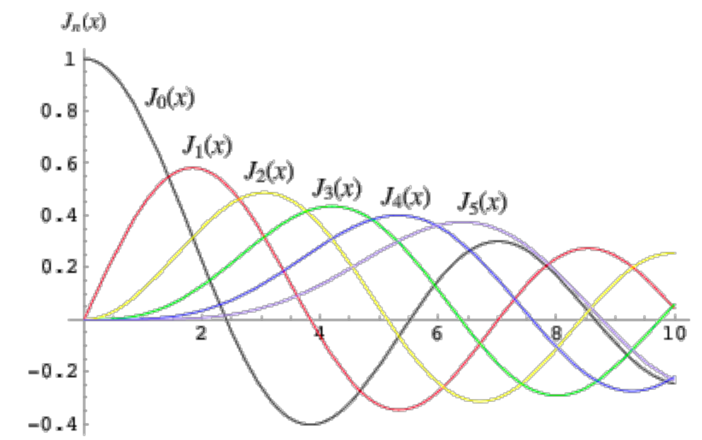
$$e^{ix\cos\varphi} = \sum_{m=-\infty}^{\infty} i^m \, e^{im\varphi} \, J_m(x)$$

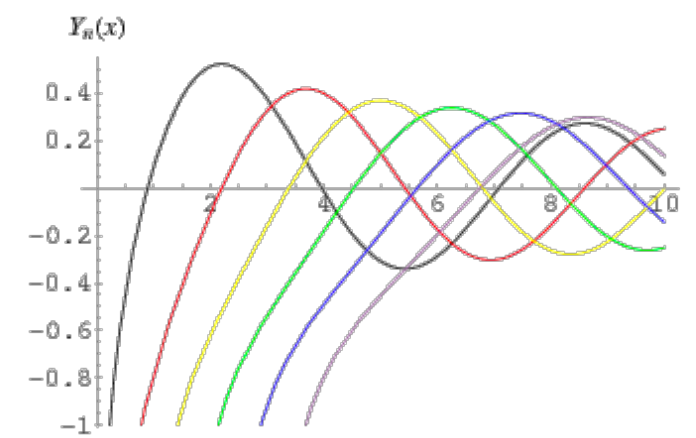
- They obey the **orthogonality** relation:

$$\int_0^\infty \frac{dx}{x} \, J_m(x) \, J_{m'}(x) = \frac{2}{\pi} \frac{\sin\left[\frac{\pi}{2}(m - m')\right]}{m^2 - m'^2} \quad ;$$

- They also obey a **closure** relation:

$$\int_0^\infty dx \, x \, J_m(\alpha x) \, J_m(\beta x) = \frac{1}{\alpha} \delta(\alpha - \beta)$$





Laplace equation in cylindrical coordinates

- Finally, just like the trigonometric functions $\sin$ and $\cos$, the Bessel functions have **roots** (when the function = zero), at intervals which become approximately fixed as the argument $x \to \infty$.
- For each index m we can order the roots of the Bessel function $J_m(x)$ as x_{mi} (i=1,2,3,…) :

$$
\begin{array}{lllll}
m=0 & : \quad x_{01}=2.405 & , \quad x_{02}=5.520 & , \quad x_{03}=8.654 & , \quad \ldots \\
m=1 & : \quad x_{11}=3.832 & , \quad x_{12}=7.016 & , \quad x_{13}=10.173 & , \quad \ldots \\
m=2 & : \quad x_{21}=5.136 & , \quad x_{22}=8.417 & , \quad x_{23}=11.620 & , \quad \ldots
\end{array}
$$

- Now we can use these roots and construct orthogonality relations for a finite domain of x:

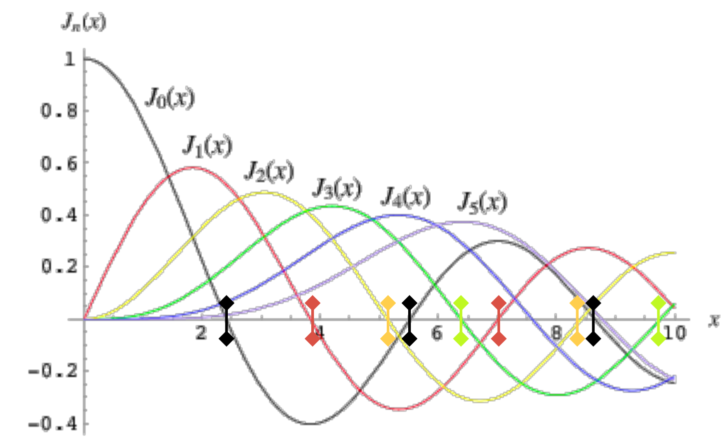
$$\int_0^1 dt\, t\, J_m(t\, x_{mi})\, J_m(t\, x_{mj}) = \frac{\delta_{ij}}{2}\left[J_{m+1}(x_{mi})\right]^2 = \frac{\delta_{ij}}{2}\left[J_m'(x_{mi})\right]^2$$

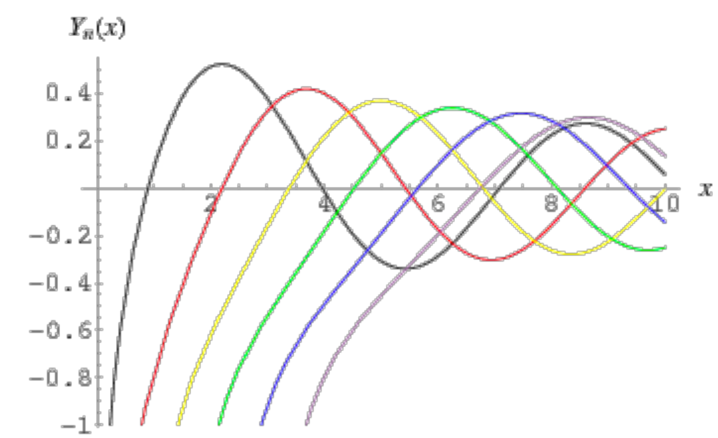
- These relations allow us to decompose not only functions in $\rho \in [0,\infty)$, but also in some interval $0 \leq \rho \leq R$:

$$f(\rho) = \sum_{i=0}^{\infty} f_i^{(m)} J_m\left(\frac{\rho\, x_{mi}}{R}\right) \quad , \quad \text{with}$$

$$f_i^{(m)} = \frac{2}{J_{m+1}^2(x_{mi})} \int_0^1 dt\, t\, J_m\left(t\, x_{mi}\right) f\left(\rho = tR\right)$$

- When we expand any function in terms of Bessel functions this is called a **Fourier-Bessel series**.





Laplace equation in cylindrical coordinates

- Now, the solutions we found above correspond to one "branch" of the whole story. That is because we assumed that the eigenvalues of the function $Z(z)$ were always positive:

$$\frac{d^2 Z}{dz^2} = + k^2 Z$$

- This is, of course, because we assumed that the solutions should decay *appropriately* at $z \to \pm \infty$, as $e^{\mp z}$!
- But suppose that we wanted to find some solutions for **boundary value problems given at some lower and upper planes**, $z = z_1$ and $z = z_2$. Then, instead of exponential solutions, what we need are **trigonometric functions**! In that case, what we actually want is to set:

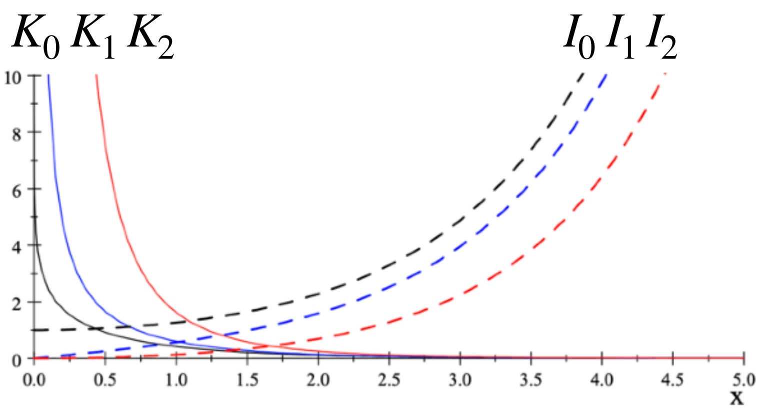
$$\frac{d^2 Z}{dz^2} = - k^2 Z \quad ,$$

which would mean changing $k \to \pm ik$ in the previous Bessel functions:

$$J_m(k\rho) \to J_m(ik\rho) \quad , \quad \text{etc.}$$

- These are called **Modified Bessel Functions** (duh!), and they can be constructed from the original ones:

$$J_m(x) \to I_m(x) = \frac{1}{i^m} J_m(ix) \quad \text{and} \quad N_m(x) \to K_m(x) = \frac{\pi}{2} i^{m+1} H_m^{(1)}(ix)$$

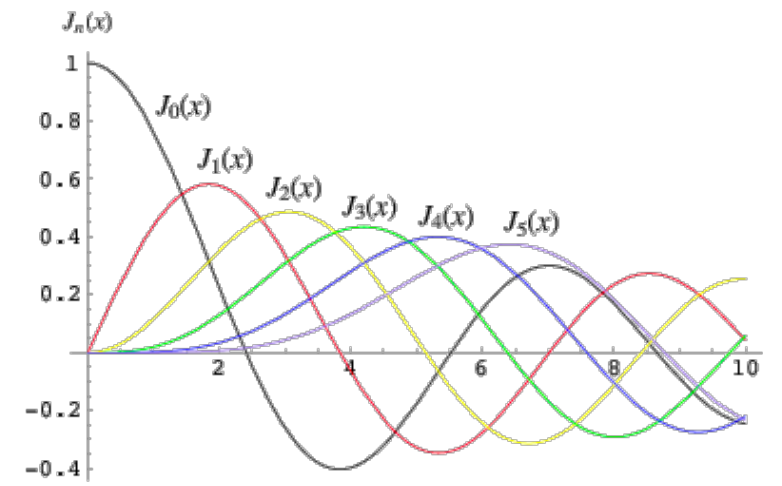


- Now the asymptotic limits for $x \to \infty$ are not trigonometric, but exponential functions:

$$\lim_{x\to\infty} I_m(x) = \frac{1}{\sqrt{2\pi x}} e^x \quad \text{and} \quad \lim_{x\to\infty} K_m(x) = \sqrt{\frac{\pi}{2x}} e^{-x}$$

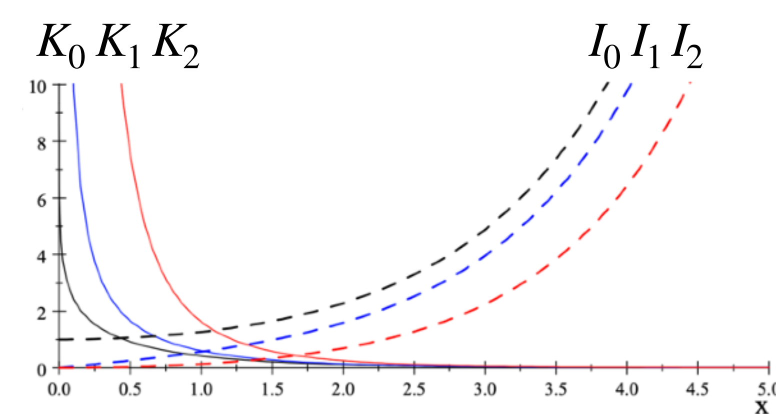
Laplace equation in cylindrical coordinates

- So, to summarize: in a boundary value problem in cylindrical coordinates, we have two situations:



(a) If the problem is **unbounded** in the z direction:

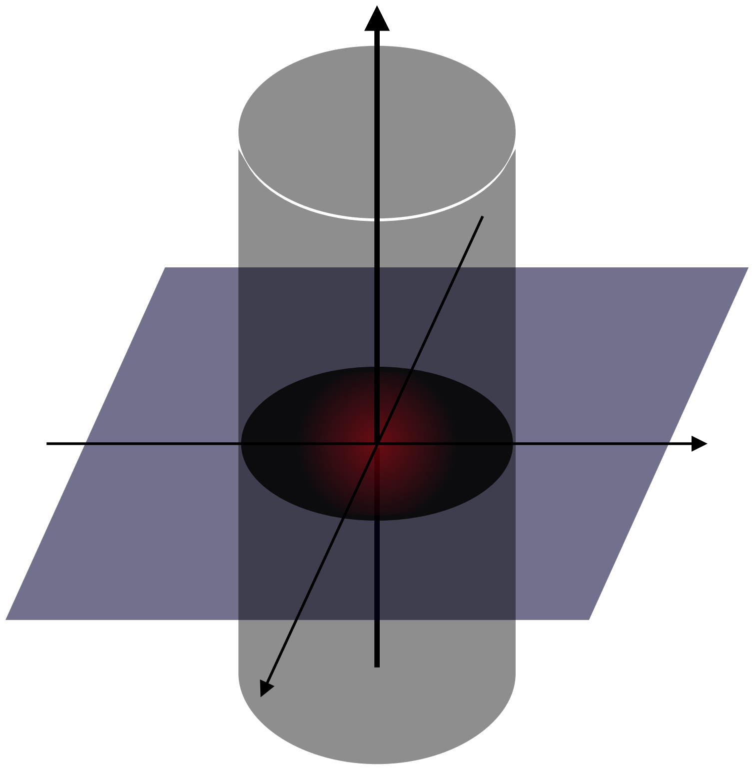
the basis functions are **exponential in** z **(**$e^{\pm kz}$**)** , but **oscillatory in** ρ (the Bessel functions)



(b) If the problem is **bounded** in the z direction:

the basis functions are **oscillatory in** z ($e^{\pm ikz}$) , but "**exponential" in** ρ (the *modified* Bessel functions)

Boundary value problems in cylindrical coordinates



- Let's look at the **simplest** possible boundary value problem with cylindrical coordinates.
- Let's say we specify the potential in a thin of radius R disk which is placed at the $z = 0$ plane, inside a grounded cylinder of radius R, in such a way that:

$$\phi(\rho, \varphi) = \phi_0 \, J_n \left(\frac{\rho}{R} \, x_{nj} \right) \, e^{in\varphi} \quad \text{for} \quad \rho \leq R \quad \text{and} \quad \phi = 0 \quad \text{for} \; \rho \geq R \quad ,$$

and

$$\phi(\rho = R) = 0$$

- On the other hand, the general expansion for the potential in cylindrical coordinates is:

$$\phi(\rho, \varphi, z) = \sum_m e^{\pm kz} \, J_m(k\rho) \, e^{im\varphi}$$

- We can immediately see that the boundary conditions at $z = 0$ and $\rho = R$ are automatically satisfied if we set:

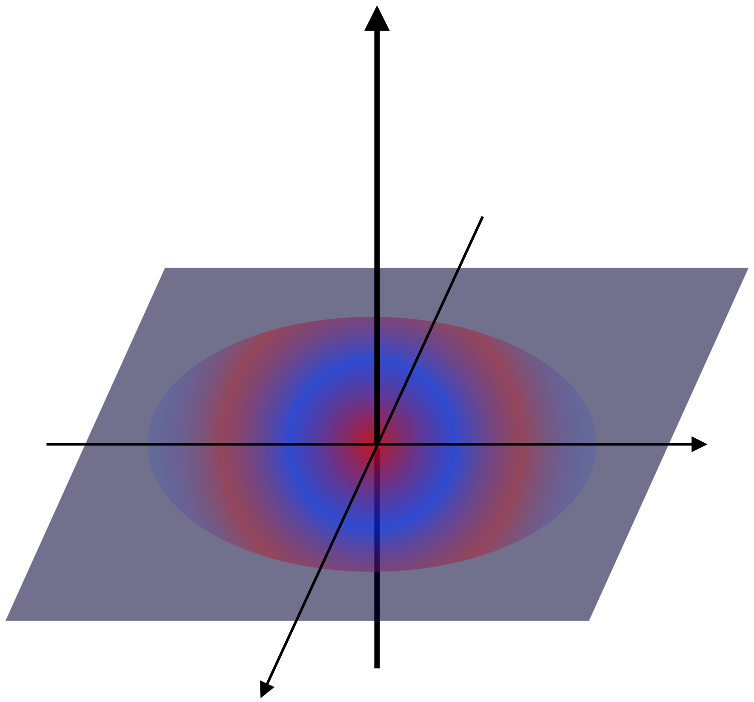
$$m \to n \quad , \quad k \to \frac{x_{nj}}{R}$$

- The solution is then immediate: for the upper/lower half of the volume we have:

$$\phi(\rho, \varphi, z) = \phi_0 \, e^{-kz} \, J_n(k\rho) \, e^{in\varphi} \quad z > 0 \quad , \quad \text{and}$$

$$\phi(\rho, \varphi, z) = \phi_0 e^{+kz} \, J_n(k\rho) \, e^{in\varphi} \quad z < 0 \quad .$$

Boundary value problems in cylindrical coordinates



- Here is another problem in cylindrical coordinates. Suppose that the potential in the $z = 0$ plane is given by:

$$\phi(\rho, \varphi) = \phi_0 \,\mathrm{sinc}\left(\frac{\rho}{R}\right) = \phi_0 \frac{\sin \rho/R}{\rho/R}$$

- In principle we should search for a solution like

$$\phi(\rho, \varphi, z) = \sum_m e^{im\varphi} \int_0^\infty dk\, f(k)\, e^{-kz}\, J_m(k\rho) \qquad (z \geq 0)$$

- Clearly, the problem has azimuthal symmetry, which means only the mode $m = 0$ will appear:

$$\phi(\rho, z) = \int_0^\infty dk\, f(k)\, e^{-kz}\, J_0(k\rho)$$

- In order to obtain the function $f(k)$ we remember the closure relation:

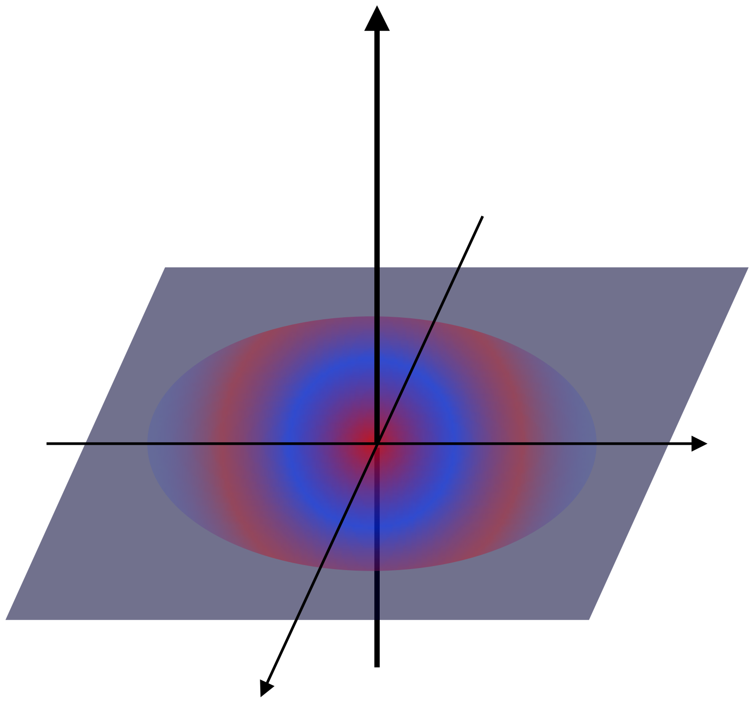
$$\int_0^\infty d\rho\, \rho\, J_m(k\rho)\, J_m(q\rho) = \frac{1}{k}\delta(k-q)$$, so we integrate both sides of the equation above:

$$\int_0^\infty d\rho\, \rho\, J_0(q\rho) \phi(\rho, z) = \int_0^\infty d\rho\, \rho\, J_0(q\rho) \int_0^\infty dk\, f(k)\, e^{-kz}\, J_0(k\rho)$$

$$= \int_0^\infty dk\, f(k)\, e^{-kz} \int_0^\infty d\rho\, \rho\, J_0(q\rho)\, J_0(k\rho)$$

$$= \int_0^\infty dk\, f(k)\, e^{-kz}\, \frac{1}{q}\delta(k-q) \;=\; \frac{f(q)}{q}\, e^{-qz}$$

Boundary value problems in cylindrical coordinates



- Now, the potential on left-hand side of the equation above gives us, at $z = 0$:

$$\phi(\rho, \varphi) = \phi_0 \,\mathrm{sinc}\left(\frac{\rho}{R}\right) = \phi_0 \frac{\sin \rho/R}{\rho/R}$$, so the equation now reads, at $z = 0$:

$$\int_0^\infty d\rho\, \rho\, J_0(q\rho) \times \phi_0 \frac{\sin \rho/R}{\rho/R} = \frac{f(q)}{q} e^0$$

- As it turns out, the integral above is exact, and it yields the result:

$$f(q) = \phi_0 \frac{q R^2}{\sqrt{1 - q^2 R^2}} \quad \text{for} \quad 0 < qR < 1 \quad , \quad \text{and} \quad f(q) = 0 \quad \text{for} \quad qR > 1$$

- Therefore, the potential everywhere in the upper half-volume is:

$$\phi(\rho, z) = \int_0^{1/R} dk\, e^{-kz} J_0(k\rho) \times \phi_0 \frac{k R^2}{\sqrt{1 - k^2 R^2}} = \phi_0 \int_0^1 dx\, e^{-xz/R} J_0\left(x\rho/R\right) \frac{x}{\sqrt{1 - x^2}}$$

- You can in fact check that:

$$\phi(\rho, z = 0) = \phi_0 \int_0^1 dx\; J_0(x\rho/R) \frac{x}{\sqrt{1 - x^2}} = \phi_0 \frac{\sin \rho/R}{\rho/R}$$

- I will leave the expression for $\phi(\rho, z)$ in the integral form, but you can expand that if you want!

Next class:

- Electrostatics in conducting media
- Dielectric media
- Field discontinuities at boundaries of dielectrics
- Jackson, Ch. 4