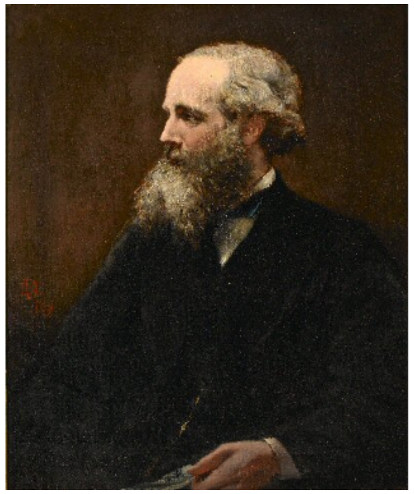
Electrodynamics

⚡ The Helmholtz Theorem
⚡ Boundary Conditions
⚡ The Electrostatic Potential
⚡ Work and Energy

Maxwell's Equations

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial \mathbf{t}} = \mathbf{0}$$

$$\nabla \cdot \mathbf{E} = \frac{\mathbf{1}}{\epsilon_{\mathbf{0}}} \rho$$



$$\nabla \cdot \mathbf{B} = \mathbf{0}$$

$$\nabla \times \mathbf{B} - \epsilon_{\mathbf{0}} \, \mu_{\mathbf{0}} \frac{\partial \, \mathbf{E}}{\partial \, \mathbf{t}} = \mu_{\mathbf{0}} \, \mathbf{J}$$

$$\frac{\partial \, \rho}{\partial \, t} + \nabla \cdot \mathbf{J} = \mathbf{0}$$

$$\mathbf{F} = q \, (\mathbf{F} + \mathbf{v} \times \mathbf{B})$$

Gauss's Law & Electrostatics

- The most basic of Maxwell's Equations of Electrostatics is **Gauss's Law**:

$$\vec{\nabla} \cdot \vec{E} = -\nabla^2 \phi = \frac{1}{\epsilon_0} \rho(\vec{x}) \quad , \quad \text{with } \vec{E} = -\vec{\nabla} \phi \quad .$$

- We saw during last class (and *you know* this since kindergarten!) that the solution is:

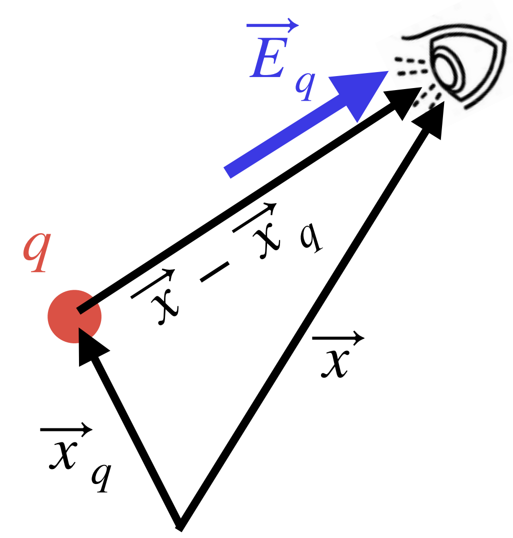
$$\phi(\vec{x}) = \frac{1}{4\pi \epsilon_0} \int d^3x' \, \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|} \quad .$$

- Furthermore, we can consider the charge density (a ***distribution***), to be a sum of point charges:

$$\rho(\vec{x}) = \sum_i q_i \, \delta(\vec{x} - \vec{x}_i)$$

- For a point charge, the electric field is given by:

$$\vec{E}_q = -\vec{\nabla} \phi_q(\vec{x}) = -\vec{\nabla} \left(\frac{1}{4\pi \epsilon_0} \frac{q}{|\vec{x} - \vec{x}_q|} \right) = \frac{q}{4\pi \epsilon_0} \frac{\vec{x} - \vec{x}_q}{|\vec{x} - \vec{x}_q|}$$



- Now, how can we be sure that the solution above is **unique**?

Helmholtz Theorem

- Put another way: given **only** the equation

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} \rho(\vec{x}) \quad ,$$

can we find a **unique solution**?

- No! We can add to $\vec{E}$ any field $\vec{H}$ which solves the **homogeneous equation**, $\vec{\nabla} \cdot \vec{H} = 0$, i.e.,

$$\vec{\nabla} \cdot \left(\vec{E} + \vec{H} \right) = \frac{1}{\epsilon_0} \rho(\vec{x}) + 0 \ .$$

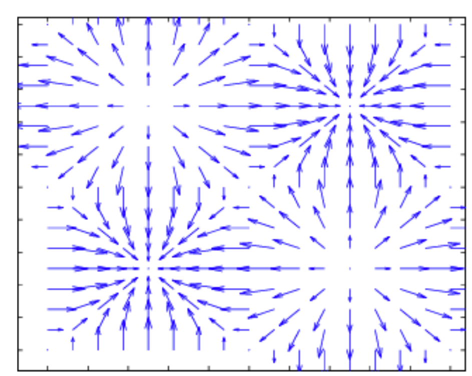
- However, if we specify not only the **divergence** of a field, but also its **curl** ("rotational"), then the equations yield a unique solution.

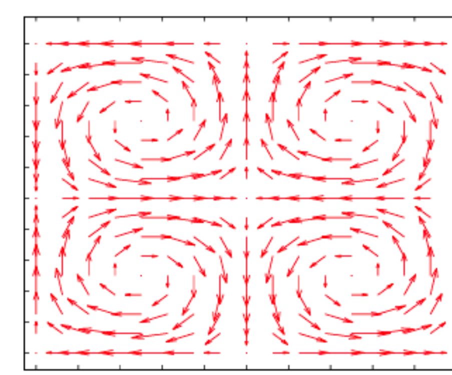
- In other words, for any field $\vec{F}$, once we specify:

$$\vec{\nabla} \cdot \vec{F} = S_{\rm div}$$

$$\vec{\nabla} \times \vec{F} = \vec{S}_{\rm rot} \ ,$$

then there is a **unique solution for** $\vec{F}$. This is the **Helmholtz Theorem**.





Helmholtz Theorem

- We will show that, given the pair of equations:

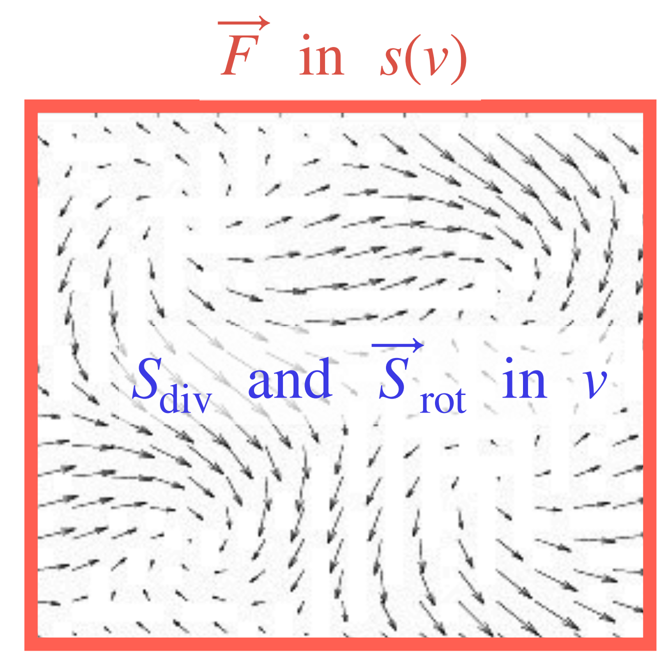
$$\vec{\nabla} \cdot \vec{F} = S_{\text{div}} \quad \text{and} \quad \vec{\nabla} \times \vec{F} = \vec{S}_{\text{rot}} \quad ,$$

then the **unique solution** for the vector field $\vec{F}$ is given by:

$$\vec{F} = -\vec{\nabla}\Psi + \vec{\nabla} \times \vec{R} \quad , \quad \text{where}$$

$$\Psi = \frac{1}{4\pi}\int_v d^3x' \frac{S_{\text{div}}(\vec{x}')}{|\vec{x} - \vec{x}'|} - \frac{1}{4\pi}\oint_{s(v)} d\vec{s}' \cdot \frac{\vec{F}(\vec{x}')}{|\vec{x} - \vec{x}'|} \quad , \quad \text{and}$$

$$\vec{R} = \frac{1}{4\pi}\int_v d^3x' \frac{\vec{S}_{\text{rot}}(\vec{x}')}{|\vec{x} - \vec{x}'|} - \frac{1}{4\pi}\oint_{s(v)} d\vec{s}' \times \frac{\vec{F}(\vec{x}')}{|\vec{x} - \vec{x}'|}$$



- So, we have to specify not only the **sources** of the divergence and curl **on the "bulk"** (volume), but also the **behavior of the field itself** ($\vec{F}$) on the **boundary** (the surface of the volume)!

Helmholtz Theorem: proof

- In order to prove this theorem, let's start by recalling that:

$$\vec{\nabla} \frac{1}{|\vec{x} - \vec{x}'|} = -\frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|} \quad , \text{ and that}$$

$$\vec{\nabla} \cdot \left(\vec{\nabla} \frac{1}{|\vec{x} - \vec{x}'|} \right) = -4\pi\, \delta(\vec{x} - \vec{x}')$$

- Now let's rewrite the field itself as:

$$F(\vec{x}) = \int_v d^3x'\, \vec{F}(\vec{x}')\, \delta(\vec{x} - \vec{x}') = -\frac{1}{4\pi} \int_v d^3x'\, \vec{F}(\vec{x}')\, \vec{\nabla}_x^2 \frac{1}{|\vec{x} - \vec{x}'|}$$

$$= -\frac{1}{4\pi} \vec{\nabla}_x^2 \int_v d^3x'\, \frac{\vec{F}(\vec{x}')}{|\vec{x} - \vec{x}'|}$$

- But we also know that $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2 \vec{A}$, so we can rewrite the field as:

$$\vec{F}(\vec{x}) = -\frac{1}{4\pi} \vec{\nabla}_x \Bigg(\underbrace{\vec{\nabla}_x \cdot \int_v d^3x'\, \frac{\vec{F}(\vec{x}')}{|\vec{x} - \vec{x}'|}}_{①} \Bigg) + \frac{1}{4\pi} \vec{\nabla}_x \times \Bigg(\underbrace{\vec{\nabla}_x \times \int_v d^3x'\, \frac{\vec{F}(\vec{x}')}{|\vec{x} - \vec{x}'|}}_{②} \Bigg)$$

Helmholtz Theorem: proof

- The first term can be "massaged" to appear more graceful, as follows:

$$① = \int_v d^3x' \, \vec{\nabla}_x \cdot \frac{\vec{F}(\vec{x}')}{|\vec{x} - \vec{x}'|} = \int_v d^3x' \, \vec{F}(\vec{x}') \cdot \vec{\nabla}_x \frac{1}{|\vec{x} - \vec{x}'|} = \int_v d^3x' \, \vec{F}(\vec{x}') \cdot \left(-\vec{\nabla}_{x'}\right) \frac{1}{|\vec{x} - \vec{x}'|}$$

$$= -\int_v d^3x' \, \vec{\nabla}_{x'} \cdot \left[\frac{\vec{F}(\vec{x}')}{|\vec{x} - \vec{x}'|}\right] + \int_v d^3x' \, \left[\vec{\nabla}_{x'} \cdot \vec{F}(\vec{x}')\right] \frac{1}{|\vec{x} - \vec{x}'|}$$

$$= -\oint_{s(v)} d\vec{s}' \cdot \left[\frac{\vec{F}(\vec{x}')}{|\vec{x} - \vec{x}'|}\right] + \int_v d^3x' \, \left[S_{\text{div}}(\vec{x}')\right] \frac{1}{|\vec{x} - \vec{x}'|} \quad = \; 4\pi \, \Psi \qquad !!!$$

- In a completely analogous way (YOU work it out!) we have that:

$$② = \int_v d^3x' \, \vec{\nabla}_x \times \frac{\vec{F}(\vec{x}')}{|\vec{x} - \vec{x}'|}$$

$$= -\oint_{s(v)} d\vec{s}' \times \left[\frac{\vec{F}(\vec{x}')}{|\vec{x} - \vec{x}'|}\right] + \int_v d^3x' \, \left[\vec{S}_{\text{rot}}(\vec{x}')\right] \frac{1}{|\vec{x} - \vec{x}'|} \quad = 4\pi \, \vec{R} \qquad !!!$$

Helmholtz Theorem: proof

- Therefore, we arrive at the final result that:

$$\vec{F}(\vec{x}) = -\frac{1}{4\pi}\vec{\nabla}_x ① + \frac{1}{4\pi}\vec{\nabla}_x \times ②$$

$$\Rightarrow \vec{F}(\vec{x}) = -\vec{\nabla}_x \Psi + \vec{\nabla}_x \times \vec{R}$$

- Some familiar cases are the **electrostatic field** with boundary conditions such that $\vec{E} \to 0$ as $x \to \infty$. We then have that $S_{\rm div} \to \rho/\epsilon_0$, $\vec{S}_{\rm rot} \to 0$, and $\Psi \to \phi$, with the result that:

$$\vec{E}(\vec{x}) = -\vec{\nabla}_x \phi = -\vec{\nabla}_x \left[\frac{1}{4\pi\,\epsilon_0} \int d^3x' \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|} \right]$$

- For the **magnetostatic field** with boundary conditions such that $\vec{B} \to 0$ as $x \to \infty$, we have that $S_{\rm div} \to 0$, $\vec{S}_{\rm rot} \to \mu_0 \vec{J}$ and $\vec{R} \to \vec{A}$, with the result that:

$$\vec{B}(\vec{x}) = \vec{\nabla}_x \times \vec{A} = \vec{\nabla}_x \times \left[\frac{\mu_0}{4\pi} \int d^3x' \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} \right]$$

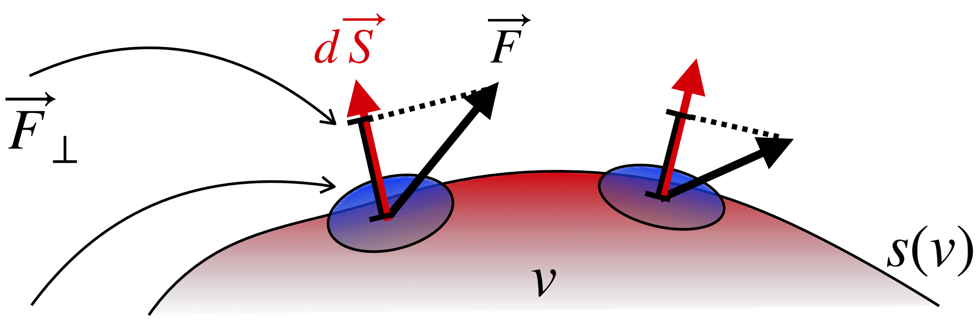
Boundary conditions

- The Helmholtz theorem highlights the need to consider carefully the **boundary conditions** of our fields:

$$\vec{F} = -\vec{\nabla}\Psi + \vec{\nabla}\times\vec{R} \quad , \quad \text{with}$$

$$\Psi = \frac{1}{4\pi}\int_v d^3x' \frac{S_{\rm div}(\vec{x}')}{|\vec{x}-\vec{x}'|} - \frac{1}{4\pi}\oint_{s(v)} d\vec{s}' \cdot \frac{\vec{F}(\vec{x}')}{|\vec{x}-\vec{x}'|} \quad \Longleftarrow \quad \vec{F}_\perp$$

$$\vec{R} = \frac{1}{4\pi}\int_v d^3x' \frac{\vec{S}_{\rm rot}(\vec{x}')}{|\vec{x}-\vec{x}'|} - \frac{1}{4\pi}\oint_{s(v)} d\vec{s}' \times \frac{\vec{F}(\vec{x}')}{|\vec{x}-\vec{x}'|} \quad \Longleftarrow \quad \vec{F}_{||}$$



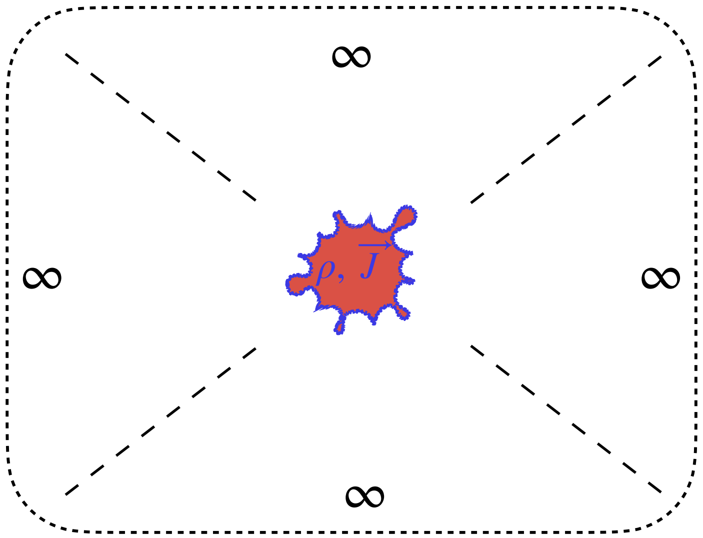
- Notice that we can always add constant to the "potentials": $\Psi \to \Psi + c \, , \, \vec{R} \to \vec{R} + \vec{c}$

- Therefore, the component **perpendicular to the surface** (and parallel to $d\vec{S}$) is needed to specify the **divergence component** Ψ ($\to \phi$);

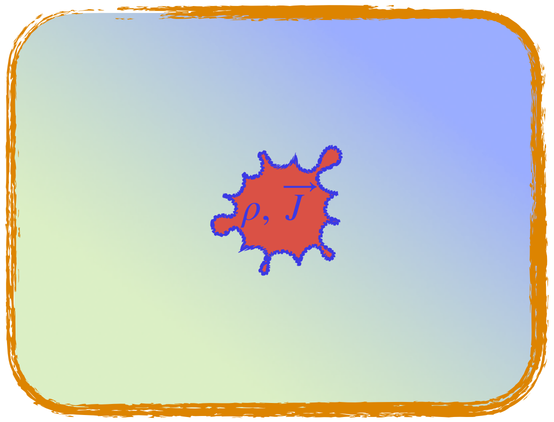
- The component **parallel to the surface** (perpendicular to $d\vec{S}$) is needed to specify the **curl component** $\vec{R}$ ($\to \vec{A}$) .

Boundary conditions

- The simplest situation is one where we want to find the electromagnetic fields in around sources which are in ***complete isolation***. We then have $v \to \mathbb{R}^3$, and the boundary is pushed off to infinitely far away from the sources, so we naturally take $\overrightarrow{F} \to 0$ at $|\overrightarrow{x} - \overrightarrow{x}'| \to \infty$.



- Another common situation is when we place sources inside and/or in the vicinity of **conductors, insulators or magnetic materials**, which **constrains** the electromagnetic fields on the surfaces of those materials.



The electrostatic potential

- When charges are static and there are no currents ($\vec{J} = 0$), the Maxwell equations reduce to:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \quad , \quad \vec{\nabla} \times \vec{E} = 0$$

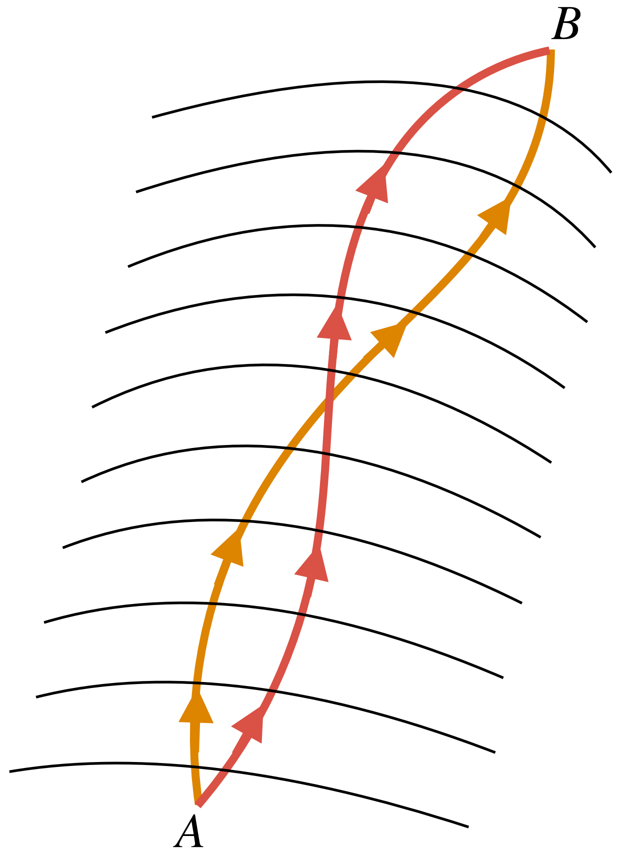
- In the absence of non-trivial boundary conditions, the solution is then simply:

$$\vec{E}(\vec{x}) = - \vec{\nabla} \phi \quad ,$$

$$\phi(\vec{x}) = \frac{1}{4\pi\,\epsilon_0} \int d^3x' \, \frac{\rho(\vec{x}\,')}{|\,\vec{x} - \vec{x}\,'|}$$

- In this case the electrostatic field is the gradient of a scalar function (ϕ), and one of the properties of the gradient is that, along any path, the integration of the gradient gives back the scalar function itself, i.e.:

$$\int_A^B d\vec{l} \cdot \vec{\nabla} \phi = \phi(B) - \phi(A) \quad ,$$ the potential difference between A and B.



The electrostatic potential

- Evidently, this property is connected with the notion of **work** in an electric field. Let's recall the Lorentz force:

$$\vec{F}_L = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

- Therefore, the **work done by the electric field** on a **point charge** q is:

$$W_{AB} = \int_A^B d\vec{l} \cdot (q\,\vec{E}) = -\,q \int_A^B d\vec{l} \cdot \vec{\nabla}\phi = = -\,q\,\Delta\phi_{AB}$$

$$\Rightarrow \quad W_{AB} = q\,\left(\phi_A - \phi_B\right)$$

- The work needed to move that charge arbitrarily away (*to infinity and beyond*!) is then given by:

$$W_{A\to\infty} = q\,\phi_A$$

- This means that we can associate this **energy at any given point** with a **potential energy**:

$$W_{A\to\infty} \;\to\; U(\vec{x}_A) \quad\Longrightarrow\quad \vec{F}(\vec{x}_A) = -\left[\vec{\nabla}\,U\right]_{\vec{x}_A} = -\,q\left[\vec{\nabla}\,\phi\right]_{\vec{x}_A} = q\,E(\vec{x}_A)$$

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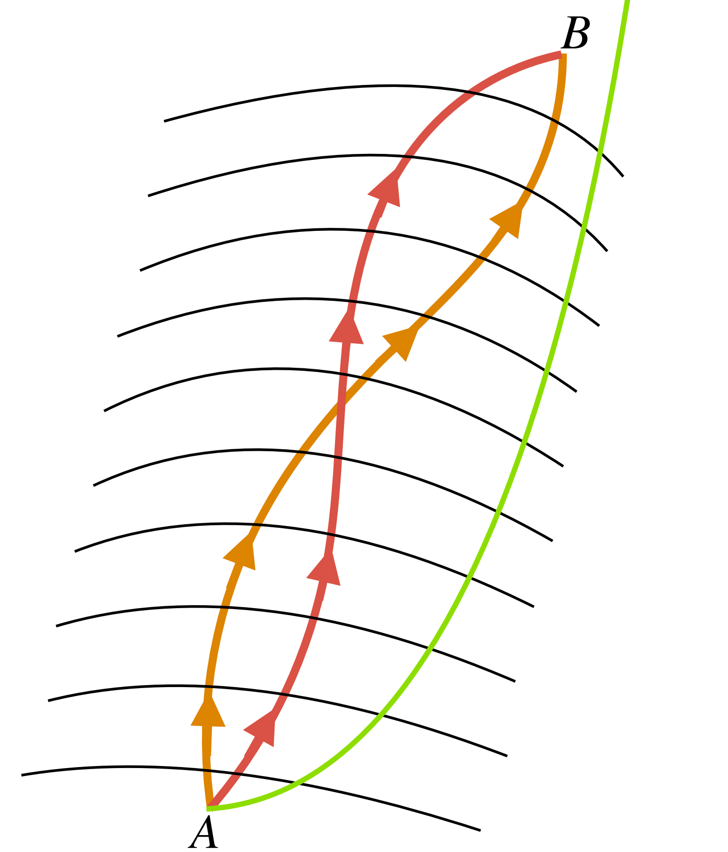


A word about units

- Sorry to bother you with such trivialities, but at this point we need to talk about **units**.
- **Coulomb's Law**:

$$\vec{F}_{q_1 q_2} = \frac{1}{4\pi\,\epsilon_0}\, q_1\, q_2\, \frac{\hat{r}}{r^2} \quad , \quad \epsilon_0 = 8.85 \times 10^{-12} \frac{\mathbb{C}^2}{N\,m^2} \quad , \quad 1\mathbb{C} = 6.2 \times 10^{18}\, e$$

- **Electric field**:

$$\vec{F}_q = q\,\vec{E} \quad \Rightarrow \quad [E] = \frac{N}{\mathbb{C}}$$

- **Electrostatic potential**:

$$\vec{E} = -\vec{\nabla}\phi \quad \Rightarrow \quad [\phi] = [E]m = \frac{N\,m}{\mathbb{C}} = \mathrm{V} \quad \text{(Volt)}$$

$$1\ \mathrm{V} = \frac{N\,m}{\mathbb{C}} = \frac{N\,m/s}{\mathbb{C}/s} = \frac{W}{A} \quad \text{(Watts/Ampère)} \quad , \quad \text{and} \quad [E] = \frac{\mathrm{V}}{m}$$

- Example: a spark produces fields of strength $10^3 - 10^4\ \mathrm{V}/m$

Warming up...

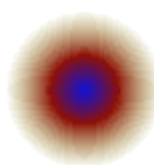
Spherical coordinates:

$$\vec{\nabla}\phi = \hat{r}\frac{\partial\phi}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial\phi}{\partial\theta} + \hat{\theta}\frac{1}{r\sin\varphi}\frac{\partial\phi}{\partial\varphi}$$

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{r^2}\frac{\partial r^2 E_r}{\partial r} + \frac{1}{r\sin\theta}\left[\frac{\partial \sin\theta E_\theta}{\partial\theta} + \frac{\partial E_\varphi}{\partial\varphi}\right]$$

- Let's warm up our brain muscles by working out some very, very simple examples.
- First, let's compute the electric field of a **spherically symmetric** charge density $\rho(r)$.

$$\vec{\nabla}\cdot\vec{E} = -\nabla^2\phi = \frac{\rho(r)}{\epsilon_0} \quad , \quad \rho(r) \ \to$$



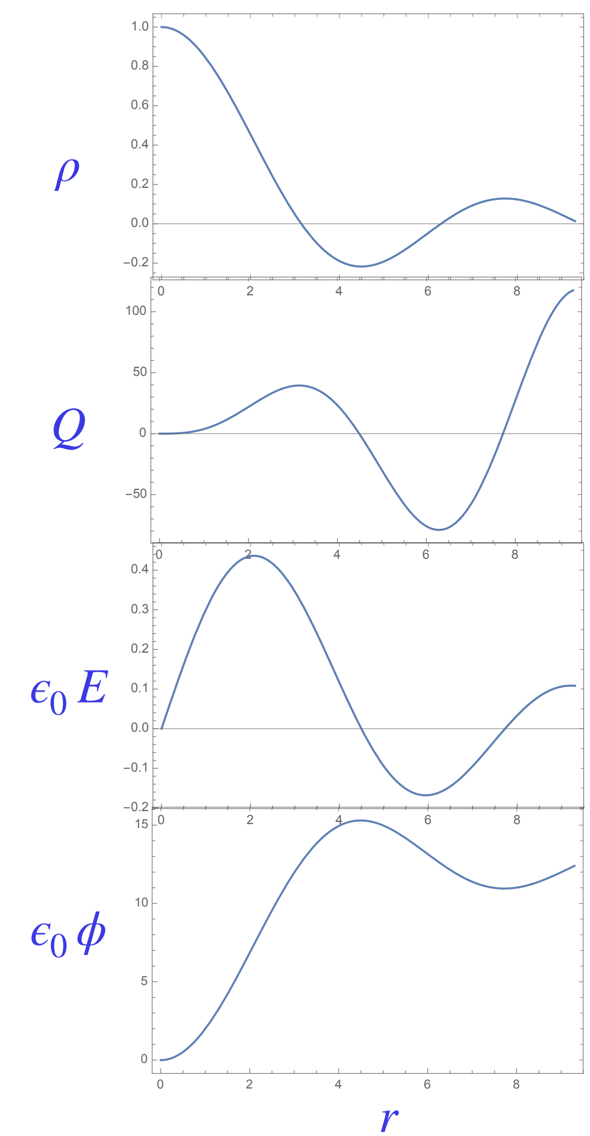
- Using spherical symmetry we can use the integral form of this equation:

$$\oint_r d\vec{S}\cdot\vec{E} = \frac{Q(r)}{\epsilon_0}$$, to derive immediately that:

$$4\pi\, r^2\, E_r(r) = \frac{Q(r)}{\epsilon_0} \quad , \quad \text{where} \quad Q(r) = 4\pi\int_0^r dr'\, r'^2\,\rho(r')$$

- We can also compute the potential, using that, for spherical symmetry, we have

$$-\frac{\partial}{\partial r}\phi(r) = E_r(r) \quad \Longrightarrow \quad \phi(r) = -\frac{1}{4\pi\epsilon_0}\int_0^r dr'\,\frac{Q(r')}{r'^2}$$



Warming up...

Cylindrical coordinates:

$$\vec{\nabla}\phi = \hat{s}\frac{\partial\phi}{\partial s} + \hat{\varphi}\frac{1}{s}\frac{\partial\phi}{\partial\varphi} + \hat{z}\frac{\partial\phi}{\partial z}$$

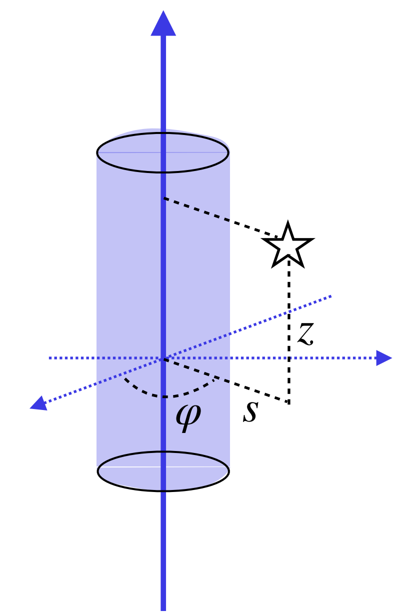
$$\vec{\nabla}\cdot\vec{E} = \frac{1}{s}\frac{\partial\, s\, E_s}{\partial s} + \frac{1}{s}\frac{\partial\, E_\varphi}{\partial\varphi} + \frac{\partial E_z}{\partial z}$$

- Basic example #2: axial (cylindrical) symmetry [notation: $s \to \rho$]:

$$\vec{\nabla}\cdot\vec{E} = -\nabla^2\phi = \frac{\rho(s)}{\epsilon_0}$$

- Using axial symmetry we can try to write the integral form of this equation:

$$\oint_s d\vec{S}\cdot\vec{E} = \frac{Q(s)}{\epsilon_0}$$

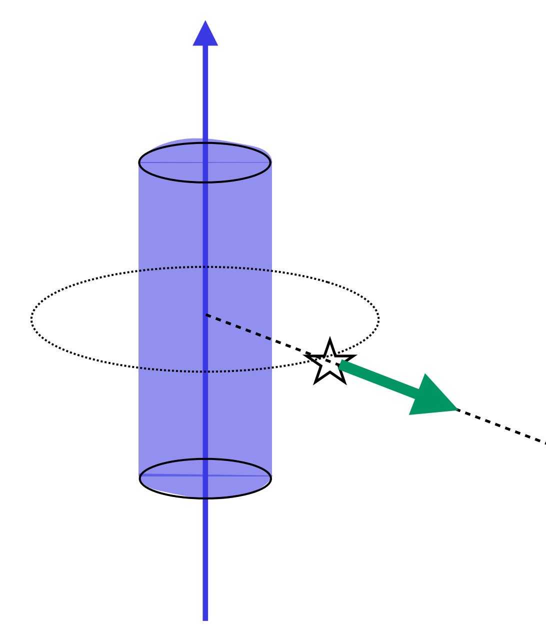


- However, notice that in this case the volume (and the charge) can **diverge**. Nevertheless, the **field should not blow up**. Let's write this for a **very very long cylinder** of height Z, for which we can neglect the contribution from the top and bottom:

$$(2\pi s\, Z)\, E_s = \frac{Z}{\epsilon_0}\int_0^s ds'\, s'\,\rho(s') = \frac{Z}{\epsilon_0}\lambda(s) \quad \Longrightarrow E_s = \frac{1}{\epsilon_0}\frac{\lambda(s)}{2\pi\, s}$$

- The potential can be computed in the same way from:

$$-\frac{\partial}{\partial s}\phi(s) = E_s(s) \quad \Longrightarrow \quad \phi(s) = -\frac{1}{2\pi\epsilon_0}\int_0^s ds'\,\frac{\lambda(s')}{s'}$$



- Notice that unless the linear charge density $\lambda \to 0$ for $s \to \infty$, then $|\phi| \to \infty$!! Why??…

Warming up...

Cartesian coordinates:

$$\vec{\nabla}\phi = \hat{x}\frac{\partial\phi}{\partial x} + \hat{y}\frac{\partial\phi}{\partial y} + \hat{z}\frac{\partial\phi}{\partial z}$$

$$\vec{\nabla}\cdot\vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

- Basic example #3: planar symmetry/Cartesian coordinates: $\vec{\nabla}\cdot\vec{E} = -\nabla^2\phi = \dfrac{\rho(z)}{\epsilon_0}$
- Using planar symmetry we can again try to write the integral form of this equation:

$$\oint_s d\vec{S}\cdot\vec{E} = \frac{Q(z)}{\epsilon_0}$$

- However, once again the volume (and the charge) can **diverge**, since the area in the plane x-y diverges. Nevertheless, the **field should not blow up at some position.**
- In addition, we should specify what is exactly this closed. Let's assume that we integrate from $z = -\infty$ to some position z. We then have, for some area A in the plane x-y:

$$A\,[E_z(z) - E_z(z\to-\infty)] = \frac{A}{\epsilon_0}\int_{-\infty}^{z} dz'\,\rho(z') = \frac{A}{\epsilon_0}\,\sigma(z)$$

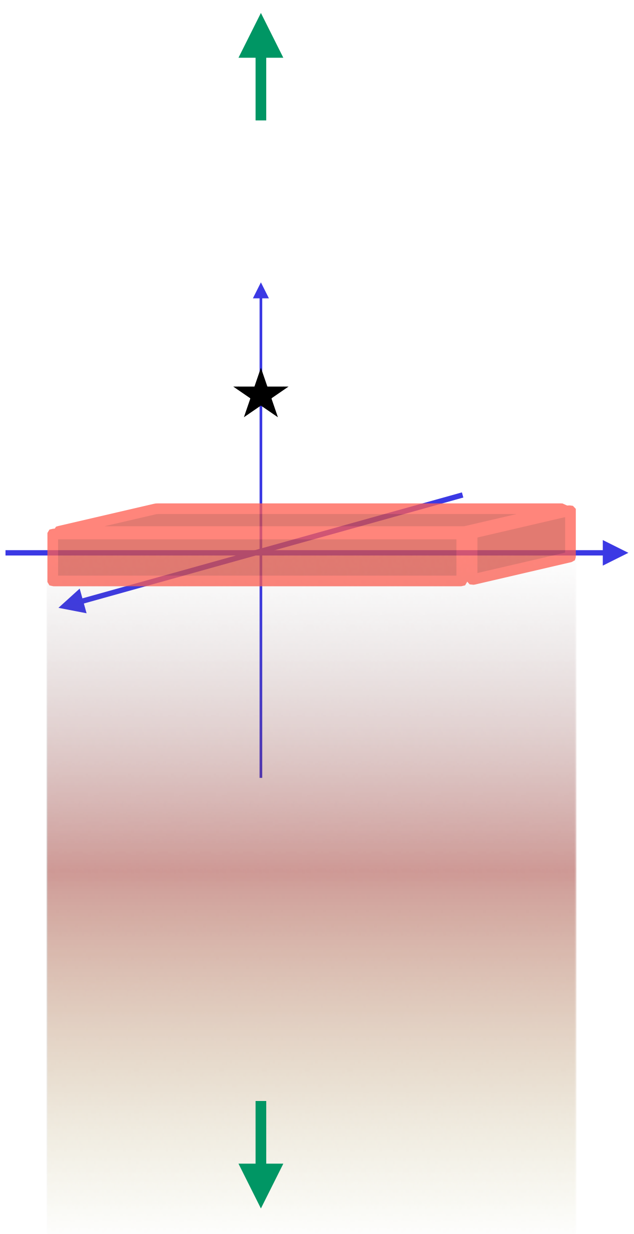
$$\Longrightarrow E_z + E_{z,\infty} = \frac{\sigma(z)}{\epsilon_0}$$, where we defined the field at $z\to-\infty$ in terms of the field at $z\to+\infty$: $E_z(z\to-\infty)\equiv -E_{z,\infty}$

- If the charge is bounded to some region not too far from the plane $z = 0$, then we can determine the value of the field at infinity. Since there is no difference between "up" and "down", we can take $z\to+\infty$ and write:

$$2\,E_{z,\infty} = \frac{\sigma_\infty}{\epsilon_0} \quad \Longrightarrow E_z = \frac{\sigma(z)}{\epsilon_0} - \frac{\sigma_\infty}{2\epsilon_0} = \frac{1}{\epsilon_0}\left[\int_{-\infty}^{z} dz'\rho(z') - \frac{1}{2}\int_{-\infty}^{\infty} dz'\rho(z')\right]$$

- But we can rewrite the first integral as:

$$E_z = \frac{1}{\epsilon_0}\left[\frac{1}{2}\int_{-\infty}^{z} dz'\rho(z') + \frac{1}{2}\int_{-\infty}^{z} dz'\rho(z') - \frac{1}{2}\int_{-\infty}^{\infty} dz'\rho(z')\right] \quad \Longrightarrow E_z = \frac{1}{2\,\epsilon_0}\left[\int_{-\infty}^{z} dz'\rho(z') - \int_{z}^{\infty} dz'\rho(z')\right]$$



Warming up...

- The simplest application would be for a pair of planes of equal and opposite charge — a capacitor . In that case we have, **between** the plates:

$$E_z = \frac{\sigma}{2\epsilon_0} - \left(-\frac{\sigma}{2\epsilon_0} \right) = +\frac{\sigma}{\epsilon_0} \quad ,$$

while **outside** the capacitor we have:

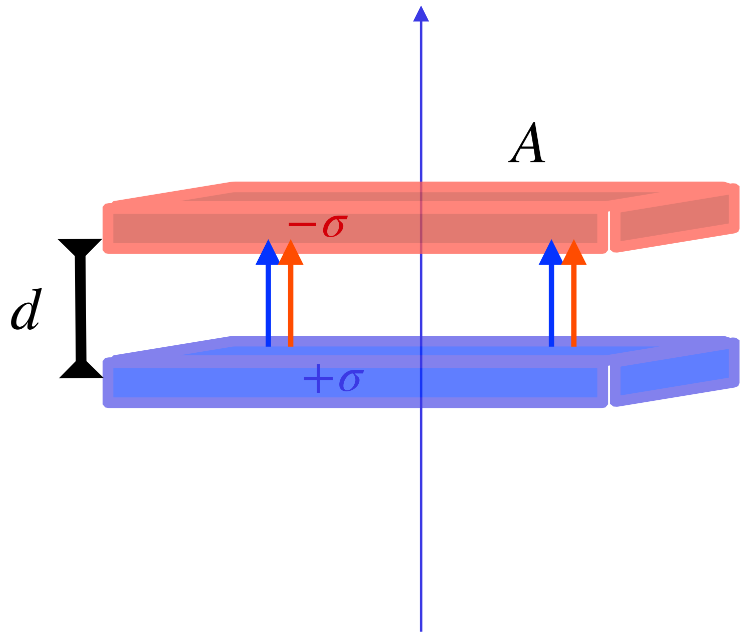
$$E_z = \frac{\sigma}{2\epsilon_0} + \left(-\frac{\sigma}{2\epsilon_0} \right) = 0 \quad .$$

- Therefore we get the field inside the capacitor:

$$E_z = \frac{\sigma}{\epsilon_0}$$

- The electric potential is, therefore:

$\phi = -\frac{\sigma}{\epsilon_0} z + C$, and the potential difference is $\Delta\phi = \frac{\sigma}{\epsilon_0} d = \frac{q}{A\,\epsilon_0} d$



Capacitance:

$$C = \frac{q}{\Delta\phi} = \frac{A\epsilon_0}{d}$$

Potential and potential energy

- The force of an electric field $\vec{E}$ on a charge q is given by:

$$\vec{F} = q\,\vec{E} \quad \Rightarrow \quad \vec{F} = -\vec{\nabla}\,U = q\,(-\vec{\nabla}\,\varphi) \qquad U = q\,\varphi$$

- Therefore, the potential energy of a small charge that is placed in this capacitor is:

$$dU = dq\,\Delta\phi = dq\,\frac{q}{C}$$

- As we add more and more charges to the capacitor (bringing them in pairs from $\pm\infty$ to each plate), the energy increases such that:

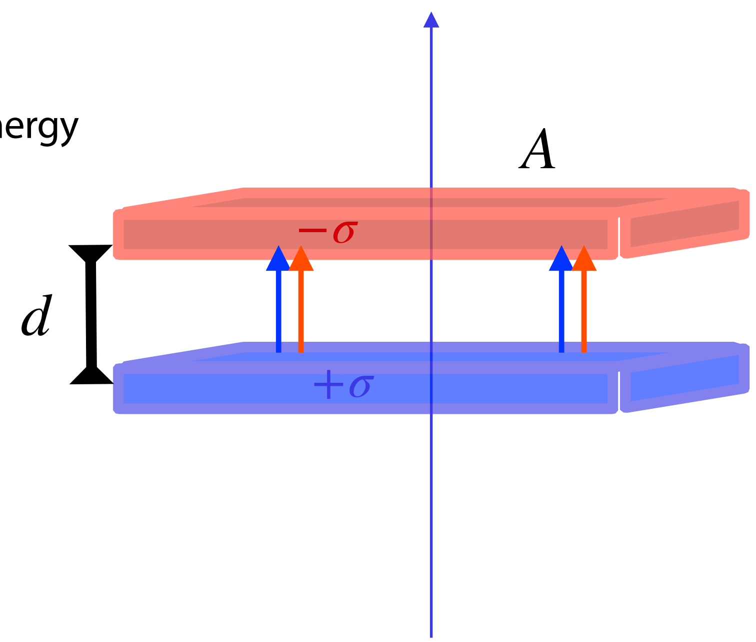
$$U(q) = \int_0^q dq\frac{q}{C} = \frac{1}{2}\frac{q^2}{C}$$

- But $E = q/(A\epsilon_0)$, so $q = E \times A\epsilon_0$, and we can write:

$$U(q) = \frac{1}{2}\frac{E^2\,A^2\epsilon_0^2}{\frac{A\epsilon_0}{d}} = \frac{1}{2}\epsilon_0\,(A\,d)\,E^2 = \frac{1}{2}\epsilon_0\,V\,E^2$$

- This, of course, is simply the statement that the **energy density** in the electric field is given by:

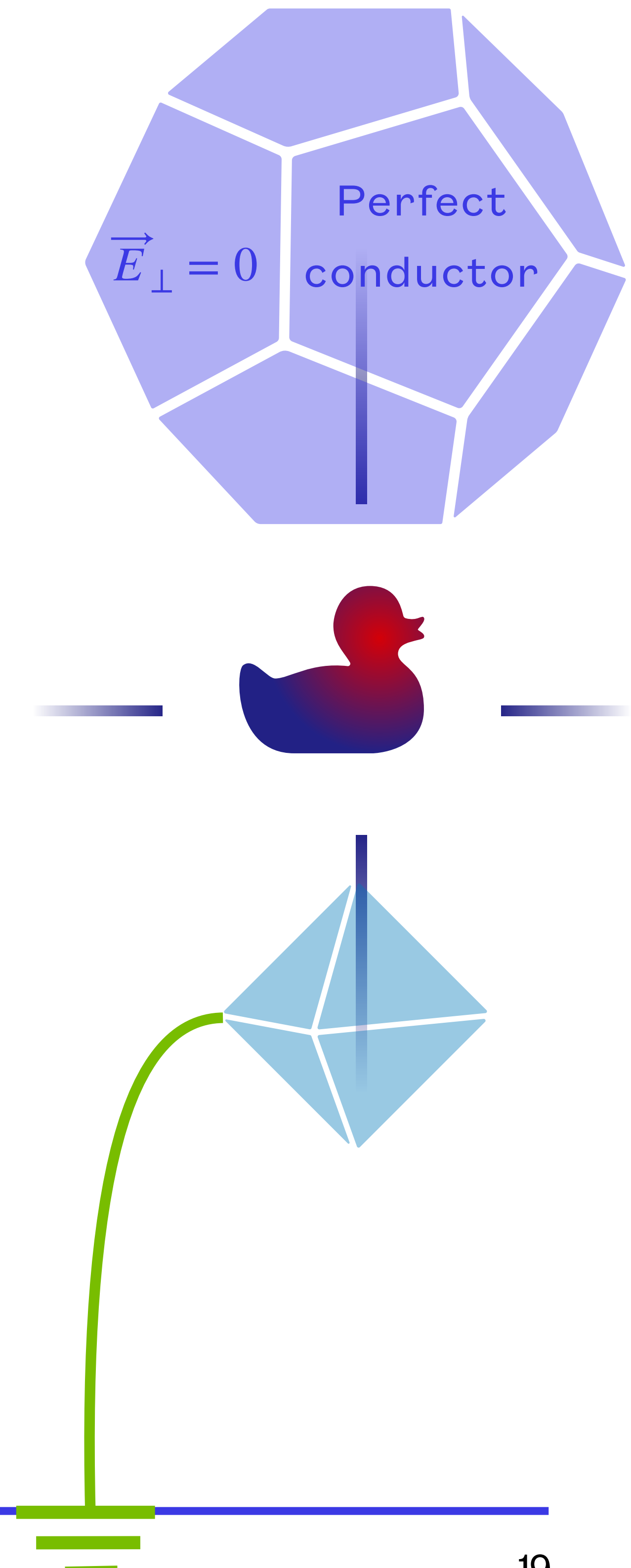
$$\rho_E = \frac{1}{2}\epsilon_0\,E^2$$



Capacitance:

$$C = \frac{q}{\Delta\phi} = \frac{A\epsilon_0}{d}$$

Boundary conditions



- In these examples we have only considered charge distributions which are somehow **bounded** in one or more dimensions, but the space is zero everywhere else, such that **the field drops to zero at infinity**.
- Now let's suppose that we introduce some **non-trivial boundary conditions**, which constrain the field over some surface (or surfaces).
- There are (generically) **two types** of boundary conditions for a scalar field (ϕ) on a given surface:

$$\text{Dirichlet b.c.}: \quad \phi_S = 0 \text{ (or constant)}$$

$$\text{Neumann b.c.}: \quad \left(\vec{\nabla}_\perp \phi\right)_S = 0$$

- Now, how can we accommodate all of this into our formal solution in terms of the Green function?

$$\nabla_x^2 \phi(\vec{x}) = -\frac{\rho(\vec{x})}{\epsilon_0} \quad \Rightarrow \quad \nabla^2 G(\vec{x}, \vec{x}') = \delta(\vec{x} - \vec{x}') \quad \Rightarrow \quad G(\vec{x}, \vec{x}') = -\frac{1}{4\pi} \frac{1}{|\vec{x} - \vec{x}'|}$$

- Therefore, we obtain:

$$\phi(\vec{x}) = \frac{1}{4\pi\,\epsilon_0} \int d^3x' \, \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|}$$

- But what about the boundary conditions?…

Boundary conditions and Green's function

- At issue here is the fact that the Green function is **not unique** — it is *degenerate* with respect to terms which solve the homogeneous equation:

  $\nabla^2 G(\vec{x}, \vec{x}\,') = \delta(\vec{x} - \vec{x}\,') \quad \Rightarrow \quad G \to G + \phi_h$, where

  $\nabla^2 \phi_h(\vec{x}) = 0$ over the *domain of interest*.

- Therefore, we have that in fact the **general solution** for the Green function is:

$$G(\vec{x}, \vec{x}\,') = -\frac{1}{4\pi}\frac{1}{|\vec{x} - \vec{x}\,'|} + \phi_h(\vec{x})$$

- Now, what are these homogeneous solutions? From the equation $\nabla^2 \phi_h = 0$ we get immediately that *some* of those solutions are:

$$\phi_h(\vec{x}) = \phi_0 - \vec{\mathscr{E}}_0 \cdot \vec{x} \quad ,$$

  where ϕ_0 is a constant scalar and $\vec{\mathscr{E}}_0$ is a constant vector .

- In terms of the electric field, these correspond to:

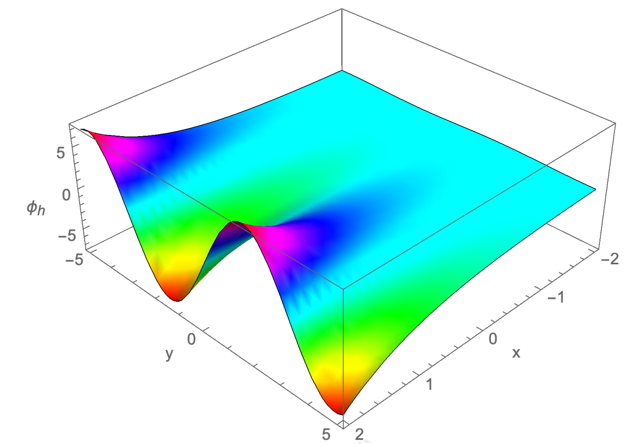
$$\vec{E}_h = -\nabla \phi_h = -\vec{\mathscr{E}}_0 \qquad \text{(notice that } \vec{\nabla} \cdot \vec{E}_h = 0\,!)$$

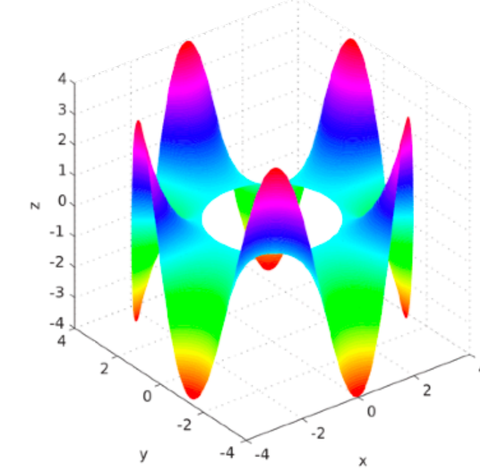
Boundary conditions and Green's function

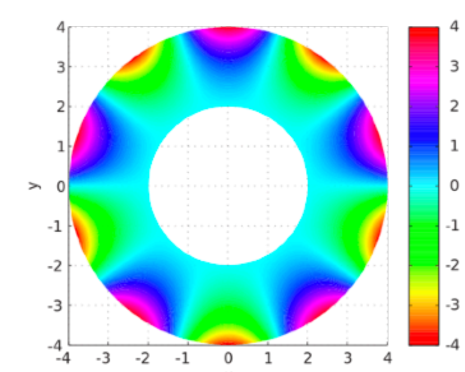
- These solutions are hardly unique. You can in fact construct a whole *functional space* of *harmonic functions*.
- Here is just one example, a simple solution with planar symmetry, valid in $\mathbb{R}^3$:

$$\phi_h \to e^{ax}e^{iay} - e^{ax}e^{-iay} \to e^{ax} \sin ay$$

- There are, of course, infinitely many such solutions — if you have time, it is a fascinating topic!
- **Any combination** of harmonic functions may work in terms of the Green function — but it is up to us to determine which works for each problem, in terms of **satisfying the boundary conditions**.

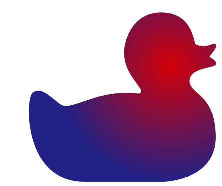






Boundary conditions and Green's function

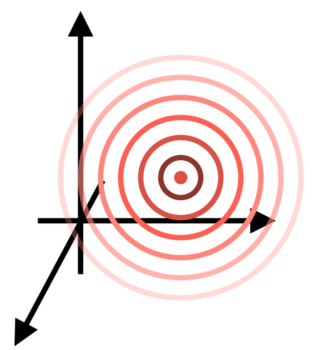
- Now, clearly when we solved for any of the previous problems, we didn't use any of these harmonic functions. **Why is that?**
- Because we have always assumed a *kind* of **boundary condition**: if we move infinitely far away from the source, the field must vanish!



- This is a **physical** condition: we know that the interactions **fall with the distance** — at least for electromagnetism.
- We must become increasingly insentivive to the sources of forces and fields, as we move away from them*. This was in fact a **hidden assumption** that we made earlier!

* There is a caveat: radiation! We will get back to this later.

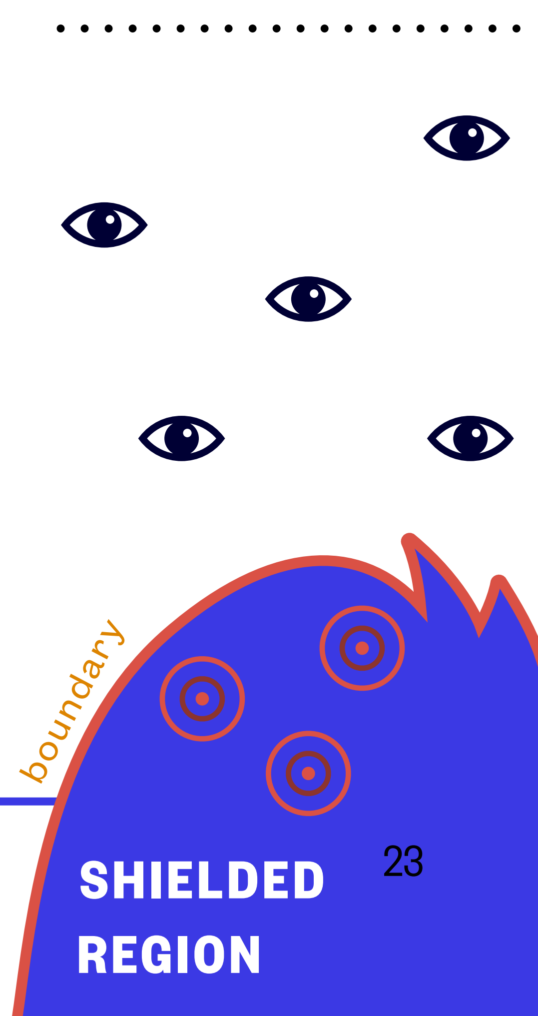
Boundary conditions and Green's function



- The harmonic functions are one way in which we can impose boundary conditions over some surface.
- Another way uses "copies" of the Green function itself.
- A good way of visualizing the basic Green function is to regard it as the **potential of a point charge**:

$$\nabla^2 \phi_q = \nabla^2 \left[\frac{q}{\epsilon_0} \times \frac{1}{4\pi} \frac{1}{|\vec{x} - \vec{x}_q|} \right] = - \frac{q}{\epsilon_0} \delta(\vec{x} - \vec{x}_q) = - \frac{\rho_q(\vec{x})}{\epsilon_0}$$

- Now, suppose that our problem is such that a whole region is **shielded** from us by the **boundary**. Anything that goes inside of that boundary is **irrelevant** if we want to compute the fields on the **outside region** — this is guaranteed by the Helmholtz Theorem.
- This means, in particular, that we can in principle put as many "point charges" inside that shielded region as we want — after all, in that region, $\nabla^2 \phi_q(\in V) = -(q/\epsilon_0)\,\delta(\vec{x} - \vec{x}_q) = 0$
- The only issue is that we must pick those charges, as well as their positions, carefully such that we end up with the boundary conditions that we want to impose.

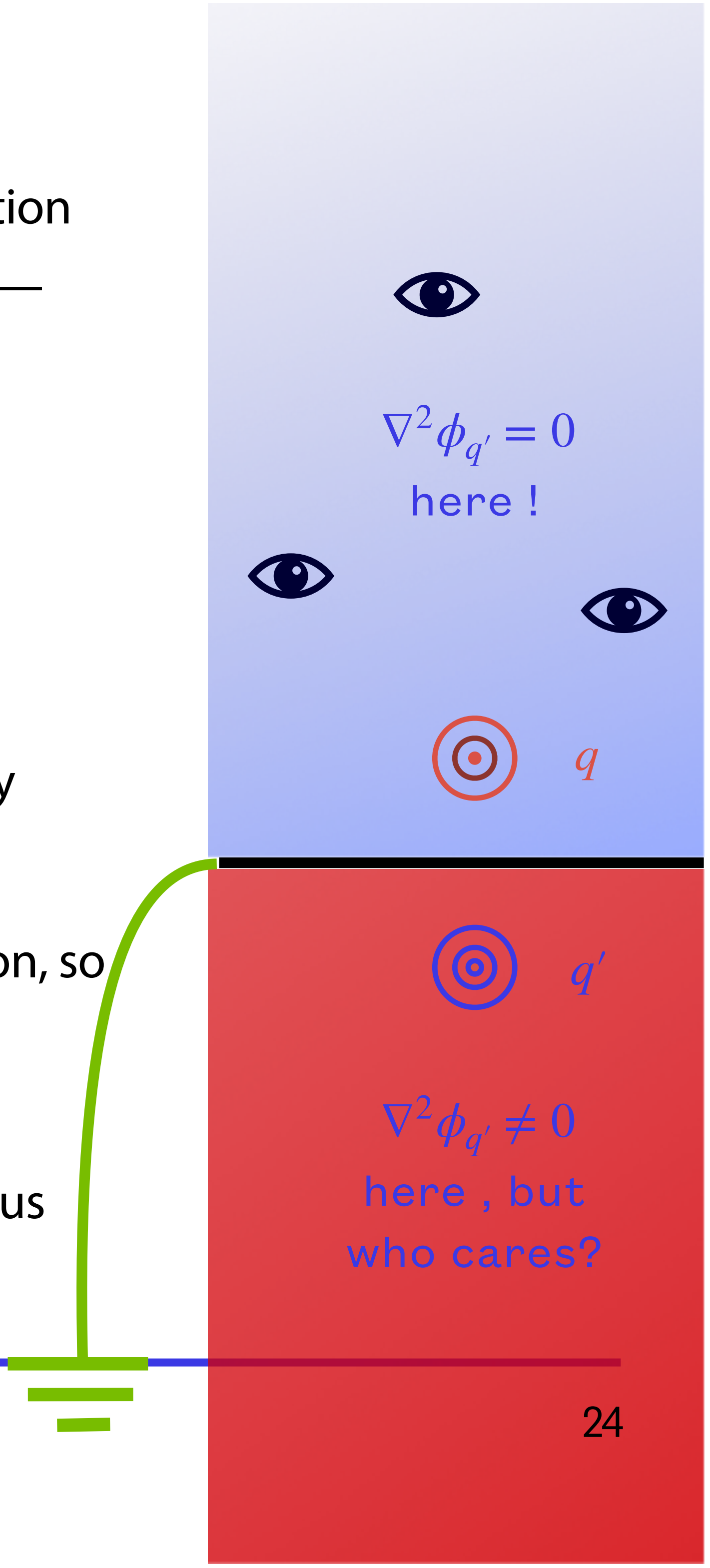


Boundary conditions and Green's function

- A simple application of this idea is the so-called "method of images".
- In its simplest incarnation, the method is applied to the case of a point charge q at a position $\vec{x}_q = \{0,0,Z_0\}$, but in the presence of an infinite plane (at $z = 0$) that is also grounded — hence, $\phi(z=0) = 0$.
- In the absence of the plane, the potential of the charge would be simply:

$$\phi_q = \frac{q}{\epsilon_0} \times \frac{1}{4\pi} \frac{1}{|\vec{x} - \vec{x}_q|}$$

- But the plane changes everything, of course, because the potential above does not satisfy $\phi(z=0) = 0$.
- Now, notice that the charge and the observer are in fact assumed to be in the $z > 0$ region, so anything that happens at $z < 0$ is quite irrelevant: all the physics is determined by what happens in the volume $z > 0$ together with the conditions at the boundary $z = 0$.
- This means that we could add a "homogeneous term" like the one we found in the previous page: $\phi_{q'}$, where the charge q' is located in the region $z < 0$!



Boundary conditions and Green's function

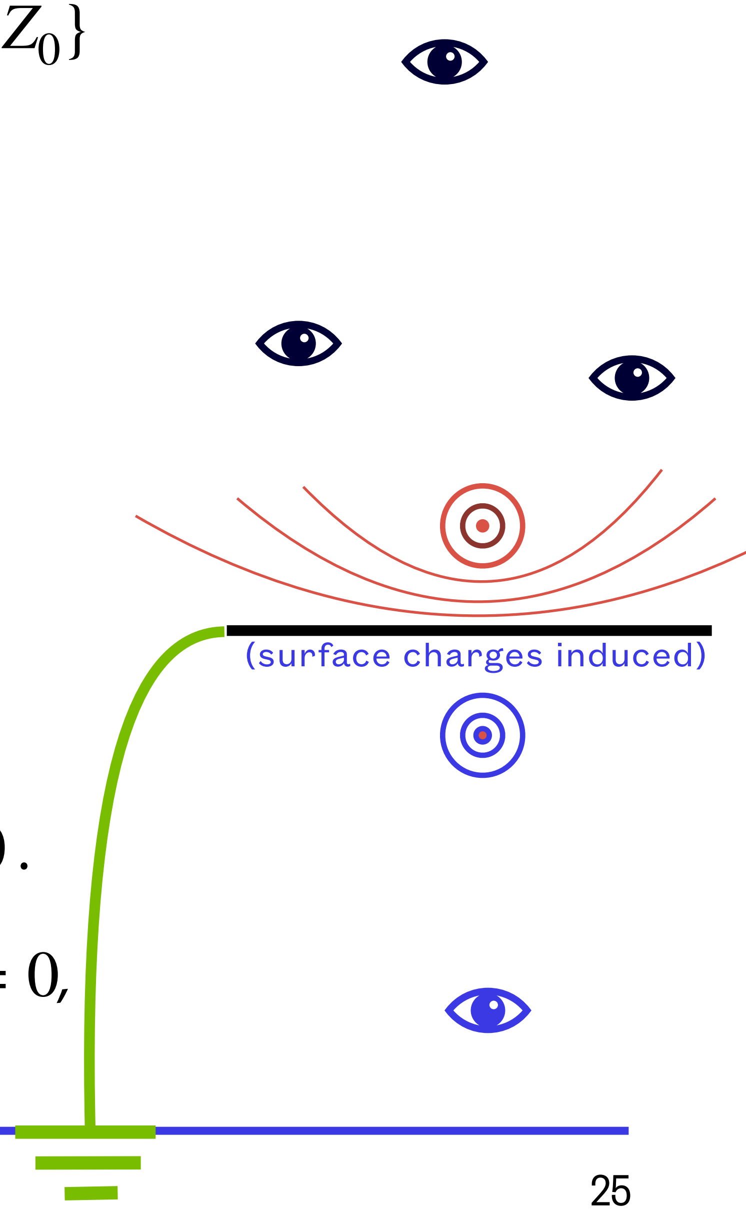
- If we think a bit about this, the "phantom" point charge in the $z < 0$ region will cancel the potential of the "real" charge if it has an equal but opposite charge ($q' = -q$), and if it is placed exactly opposite the real charge: $\vec{x}_{q'} = \{0,0,-Z_0\}$
- Then, the potential of the two charges (the real and the "phantom", or "image" charge), is:

$$\phi = \frac{q}{4\pi\epsilon_0}\frac{1}{|\vec{x} - \vec{x}_q|} - \frac{q}{4\pi\epsilon_0}\frac{1}{|\vec{x} - \vec{x}_{q'}|}$$

$$\phi = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{|\vec{x} - Z_0\,\hat{z}|} - \frac{1}{|\vec{x} + Z_0\,\hat{z}|}\right)$$

- Notice that for $z = 0$ the two terms above cancel out, giving us $\phi(z=0) = 0$.

[A question for you: suppose that we have only a very thin conducting plate at $z = 0$, and that somebody is behind that plate. What would that observer measure?….]



Next class:

- Solutions to the Laplace equations with boundary conditions: Green's functions for Dirichlet and Neumann conditions
- Variational techniques
- Relaxation method
- Jackson, Ch. 1