

PROMOTIONS AND INCENTIVES

Michael Gibbs*
University of Chicago

Abstract: This paper studies properties of promotion-based incentive schemes. Two general promotion rules, tournaments and standards, are compared. These differ in many ways, but are shown to have *virtually identical* empirical predictions about the structure of compensation in hierarchies. A relatively general characterization of multi-person tournaments is presented. Several empirically testable hypotheses are developed, especially on the relationship between the promotion probability and optimal rewards from promotion.

Michael Gibbs
Graduate School of Business
University of Chicago
Chicago, IL 60637
773-834-4351
Mike.Gibbs@gsb.uchicago.edu

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I. INTRODUCTION

Promotions appear to be the most important form of pay for performance in most organizations, especially in hierarchical, white-collar firms. They are the primary means by which workers can increase their long-run compensation (McCue 1992; Lazear 1992). They are usually given to the best performers (Medoff & Abraham 1980, 1981; Gibbs 1993). Therefore, promotions should generate substantial motivation in many settings. Moreover, there often does not seem to be strong pay for performance within jobs, which only increases the apparent importance of promotions for organizational incentives (Hedström 1987).

It follows that in order to analyze the systems firms use to motivate employees, it is essential to fully understand the incentive effects of promotions. Promotions have generated a great deal of theoretical interest, especially in the context of tournament models (Lazear & Rosen 1981). However, this literature is still incomplete and somewhat stylized, making it difficult to link the theories to empirical work. This paper fills in some of the gaps, answering some remaining theoretical questions and making the theory more amenable to empirical testing.

Three particular gaps in existing career-based incentive theories are analyzed. The first is the role of the promotion rule chosen by the firm. There are two extreme models of how the firm selects employees for advancement. The first is to run a contest or tournament. The second is to set a quota or absolute performance standard. These rules have different properties with respect to performance measurement, incentives for sabotage and collusion, enforceability of the incentive contract, risk sharing, and the firm's ability to control the quantity and quality of promotees. Despite this, I show that in terms of incentive properties and the patterns we expect to

see in firm-level compensation data, these two seemingly disparate rules are almost identical. This surprising result means that the implications of tournament models for compensation patterns are more generally applicable in empirical work than has been recognized.

The second contribution is to extend tournament models to multi-person settings. Most tournament models are based on simple two-person comparisons, or cases where a larger group of contestants compete for a single prize or to avoid a single penalty. Such models abstract considerably from the more realistic setting in which a group of employees competes for more than one promotion slot. I derive a simple general way to analyze such cases, and characterize incentives as the number of prizes and number of contestants vary.

The third issue addressed builds on the first two. I derive empirically testable implications about the relationship between the firm's hierarchical structure and incentives. A key to this is how incentives vary with promotion rates. It is *not* true that increasing the likelihood of winning always increases incentives. Instead, maximal incentives occur for intermediate promotion rates, and lower incentives occur for lower *and* higher promotion rates. This is true for both tournaments and standards. Because promotion rates vary systematically with hierarchical structure, this yields predictions about incentive structures in hierarchies.

Section II briefly discusses why promotions may provide important incentives. In Section III, some properties of standards and tournaments are compared. Sections IV-V model the incentive properties of both rules. Section VI discusses implications of the theory for compensation in hierarchies, and concludes the paper.

II. PROMOTIONS AS INCENTIVES

It is worth considering why firms might use promotions for incentives. Individualistic schemes, especially ones that do not depend on job assignments, allow more flexibility in providing incentives. In other words, promotions are often used to achieve two goals simultaneously that in principle might be separated: putting employees in the right jobs, and generating motivation. Thus, it is not immediately obvious why promotions should be used as incentives.

An important reason that promotions are sources of incentives is worker reputation, or "career concerns" (Fama 1980; Holmström 1983; MacLeod & Malcomson 1988; Gibbons & Murphy 1991). Suppose that a

worker's ability is not publicly known, and the labor market pays wages based assessments of ability (reputation). If effort and ability are substitutes at producing output or ability signals, then the worker has an incentive to improve reputation by increasing effort. Where does reputation come from? Often it is the worker's history of positions or promotions which provides the greatest evidence on productivity and potential (Waldman 1984). Thus promotions can play a key role in incentives, even when firms do not intend them to.

A second reason that promotions may be important motivators is that they can be self-enforcing incentive schemes (Malcomson 1984). Assume, for example, that the firm attaches wages to jobs, not to individuals, and fills slots by promoting the best performers. If this is done, then the firm's wage bill is fixed, regardless of who is promoted. In order for this system to provide incentives, the firm needs to credibly promote good performers rather than poor performers. Because the wage bill is fixed, the firm has no reason not to; thus, the incentive contract is self-enforcing. This may give a contracting advantage over other incentive mechanisms.

III. CHOICE OF PROMOTION RULE: STANDARD OR TOURNAMENT

In choosing who to promote, the firm faces a tradeoff between "filling slots" and staffing jobs with people of appropriate ability. Along this spectrum, there are two extremes. One is to promote all whose performance meets a fixed standard. The other is to promote a fixed number of workers; i.e., to use a tournament. We have already seen that filling fixed slots (using a tournament) may provide self-enforcement of the incentive contract. This property does not hold for standards. However, if performance is verifiable than standards are also enforceable incentive contracts. In this section I outline other properties of these two rules.

Quality of Promotees: A tournament promotes the best performers regardless of how poorly they perform, whereas a standard promotes only those whose performance is high enough. Where there is more uncertainty in the ability distribution of employees, a standard yields better control over the quality of promotees. In more certain environments, the firm will have a better idea of what type of employees a promotion contest yields, and this difference will be less important.

Quantity of Promotees: A tournament fixes exactly the quantity of promotees, whereas under a standard this is random. Further, when the firm uses a standard, changes in the reward (perhaps because of external labor

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market factors) or in the toughness of the standard (perhaps because of new technology) change the number who earn promotion. Tournaments are more impervious to such influences. The extent to which this is an advantage depends on the structure of the firm. If the firm has a specific position to fill, unless it hires from outside (the value of this option depends on the degree of firm-specific human capital), it must run a tournament. More generally, the more costly is the loss in coordination and control from deviating from a given number of employees in each position, the more a tournament has the advantage.

Performance Measurement: Tournaments differ from standards in the measure used to decide promotions: employee evaluations are a function of the performance of co-workers. The properties of relative performance evaluation have been discussed elsewhere (e.g., Holmström 1982; Lazear 1989; Gibbons & Murphy 1990). Briefly, relative evaluation filters out measurement error common to all workers, but adds individualistic errors of other workers to an employee's performance measure; the measure may or may not have less noise than an individualistic measure. Tournaments may also cause uncooperative behavior, sabotage, collusion, and efforts by employees to change the set of workers which they are compared against. When a standard is used, employees will have no incentive to collude or sabotage. They will have incentives to affect the choice of the output standard where possible.

Risk Sharing: Standards cause the wage bill to fluctuate with employee and thus firm performance. This provides implicit risk sharing between workers and the firm's owners. With tournaments, such risk sharing is more complicated. If a tournament is used to make the incentive scheme self-enforcing, then the structure of *relative wages* across jobs must be held constant. Conditional on this, though, it is still possible for the firm to vary the total wage bill with firm performance, so that this form of risk sharing can be preserved. However, such risk sharing is automatic in the case of standards, but not in the case of tournaments. Finally, under both rules the total wage bill can fluctuate with exogenous variables, such as inflation, and so provide risk sharing over these as well. Neither system does so automatically.

Budget Systems: Finally, it is worth pointing out that the theory of incentives based on meeting a standard is more generally applicable than in the context of promotions. Any case in which a reward is offered if output exceeds a given target is similar, especially when the reward is fixed in size. Thus, for example, budget or quota

systems that reward employees with a bonus if the budget is met, without changing job assignments, are examples of standards.

Thus the major difference between the two rules is that standards give better control over quality, and tournaments give better control over quantity. The costs and benefits of these determine the extent to which a firm uses either. In reality, a firm might use a standard except when the number of promotees becomes too low or too high. Note, however, that if the firm does not have perfect flexibility in the number promoted (as is often the case), then actions taken by one worker affect the chances of all others eligible for promotion. This is precisely the definition of a tournament. Therefore, many promotions unavoidably have a tournament-like aspect to them. It remains to examine how these rules differ with respect to provision of incentives and optimal compensation systems. This is the topic of the next two sections.

IV. A MODEL OF THE INCENTIVES FROM PROMOTION

I now consider a simple model in which the firm chooses the rewards that accompany promotion in order to maximize incentives. This is done taking the structure of the hierarchy, which ultimately determines the promotion rate, as fixed. The primary reason for this is to simplify the exposition. However, it is likely that in many circumstances it will not be optimal to alter the structure of the firm radically in order to optimize incentives. The firm has the alternative of substituting within-job pay for performance when promotion-based incentives are not very large. Furthermore, changing the hierarchical structure of the firm can be very costly in terms of production efficiency, since it changes spans of control and the effectiveness of monitoring, delegation, and communication.

Assume that workers maximize expected income less the disutility of effort, $EU = EI - C(e)$, with $C' > 0$, $C'' \geq 0$. The firm offers a wage w_0 . Incentives come from the potential to earn promotion, with accompanying increase in income Δ .¹ Promotion depends on a performance measure, $q_i = e_i + \pi_i$, where e_i equals employee i 's effort. The measurement error, $\pi_i = \varepsilon_i + \eta$, consists of individual-specific luck or measurement error ε_i , and an error term η that is common to all employees. Without loss of generality, assume $E\varepsilon = E\eta = E\pi = 0$. Although the firm uses promotions to sort employees by ability, workers do not know their relative ability initially (otherwise

bonding contracts might be feasible), and so are ex ante identical. Think of ability as part of an employee's luck draw ε_i . Introducing ex ante heterogeneity substantially complicates the model with some interesting results; however, the general conclusions remain unchanged (Gibbs 1991).

If promotion is based on a standard, then the probability of promotion is a function of the employee's effort, e . If promotion is based on a tournament, then the probability also depends on the efforts of all competitors, denoted by the vector $\underline{e}$. Therefore, the promotion probability is written as $p = p(e, \underline{e})$, where $\underline{e}$ affects p only in a tournament.

Assuming that an interior optimum exists, the first-order condition for optimal effort is:

$$C'(e^*) = \frac{\partial p}{\partial e^*} \Delta.$$

The second-order condition is satisfied as long as the probability function does not exhibit too-large increasing returns in effort:²

$$-C''(e^*) + \frac{\partial^2 p}{\partial e^2} \Delta \leq 0.$$

Equation (1) shows that effort depends on the marginal incentive effects of the promotion rule, $\partial p / \partial e$, and on the prize Δ . Call $\partial p / \partial e$ the *marginal probability of effort* (MPE).³

The firm offers a contract setting w_0 and Δ , given p , that maximizes employees' expected utilities, subject to a zero expected-profit constraint and to (1). Zero profit implies that $E(q) = EI = e^* = w_0 + p\Delta$. Given p , and since e is a function of p and Δ , w_0 is fully determined once Δ is chosen. The firm's problem reduces to

$$C'(e^*) = I,$$

maximizing $EU = e^* - C(e^*)$. The first-order condition is:

which yields first-best effort. Combining (1)-(2) gives the optimal prize as a function of the MPE:

$$\Delta^* = \frac{I}{\partial p / \partial e^*}.$$

Thus, the prize varies inversely with the marginal incentive effect of the promotion rule. The MPE depends on the promotion probability, and also on whether or not the firm uses a standard or tournament. In order to make predictions about the structure of rewards in promotion systems, it is necessary to characterize p' . This is the goal of the next section.⁴

V. CHARACTERIZING THE MARGINAL PROBABILITY OF EFFORT

Standards

Define the distribution of measurement error π as $H(\pi)$, with density $h(\pi)$, and the standard to beat in order to advance as z . The probability of promotion is:

$$p(e) = p(e + \pi > z) = 1 - H(z - e).$$

The marginal probability of effort (MPE) under this promotion rule is:

$$\frac{\partial p}{\partial e} = h(z - e).$$

Intuitively, for given effort e , a minimum amount of *luck* $\pi > z - e$ is needed to beat the standard. Extra effort lowers the minimum luck needed by ∂e , so promotion is more likely. The change in probability ∂p is the integral under the density of π from $z - e - \partial e$ to $z - e$, or $\partial p / \partial e = h(z - e)$.

Equations (4)-(5) reveal that the probability of promotion p is an *equilibrium outcome* of the standard imposed and the rewards from promotion. The probability depends on optimal effort e^* , which the employee chooses based on the prize and the toughness of the standard. In most of the following discussion on standards, the thought experiment will involve comparative statics on the probability p . To change p the firm must change the standard z . This induces new optimal effort e^* , and the new combination of z and e^* determine the new probability p . It should be understood in what follows that for a fixed prize, choosing the standard z amounts to the same thing as choosing the probability p , and the firm's choice of either determines e^* .

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Equations (1) and (5) indicate that, for fixed prize, incentives are independent of the actions of colleagues, but depend on the standard and the shape of the p.d.f. of measurement error. To illustrate, assume that π is uniformly distributed. In this case, but only this case, incentives are independent of the standard z and probability p . This is true because the MPE is constant (unless $p = 0$ or 1 , when $e^* = \partial p / \partial e = 0$). Intuitively, all values of luck are equally likely, so that a change in effort affects the chance of winning equally for all performance standards and promotion rates.

A more important example is when π is distributed symmetrically and is unimodal at 0; e.g., a normal distribution. Holding the prize fixed, optimal effort is zero when the outcome has no uncertainty ($p = 0$ or 1 , or $z = +\infty$ or $-\infty$), so that $\partial p / \partial e = 0$. Maximal incentives occur when $e^* = z$, so that $\partial p / \partial e = h(0)$: this is the maximum of the p.d.f. $h(\pi)$. This corresponds to setting the standard so that the chance of promotion is $1/2$, since $p = 1 - H(z - e^*) = 1 - H(0) = 1/2$. In other words, incentives are greatest when promotion is a “coin toss” in equilibrium. Now consider two contests with equal prizes and standards z_1 and z_2 that result in probabilities p_1 and $p_2 = 1 - p_1$, respectively. These give *identical* incentives. To see this, note that if π is symmetric about zero then $p_1 = 1 - H(z_1 - e^*) = 1 - p_2 = H(z_2 - e^*)$, so $z_1 - e^* = -(z_2 - e^*)$. By symmetry, $h(z_1 - e^*) = h(z_2 - e^*)$, or $\partial p_1 / \partial e_1 = \partial p_2 / \partial e_2$. Combined with (1), this implies that $e^* = e^*$. Therefore, for symmetric unimodal errors, holding the prize fixed, incentives rise from 0 as p increases from 0, are largest when $p = 1/2$, and decline symmetrically to zero as p increases to 1. These results are summarized in Proposition 1.⁴

PROPOSITION 1. *Under a standard, the MPE, $\partial p / \partial e$, is independent of the promotion probability p if and only if π is uniformly distributed. If the distribution of π is symmetric unimodal, the MPE equals 0 if $p = 0$ or $p = 1$, has a maximum value of $h(0)$ if $p = 1/2$, and rises symmetrically as p rises from 0 to $1/2$ and as p falls from 1 to $1/2$. In this case, two promotions with probabilities equal to p and $1 - p$ have equal MPEs.*

Now consider an increase in the variance of π (still assumed to be symmetric unimodal). This shifts mass of $h(\pi)$ to both tails, away from $h(0)$. By (5), this decreases incentives when the standard is set so that p is near $1/2$. However, for values of p nearer to 0 or 1, incentives *increase*. When beating the standard is very unlikely (very likely), extra effort only makes a difference under the rare circumstance of large good (bad) luck. If such luck becomes more likely (by an increase in variance), extra effort is more likely to make a difference. The variance

also affects the rate of increase of the MPE as p rises from 0 to $\frac{1}{2}$, and of decrease as p rises from $\frac{1}{2}$ to 1. The smaller the variance, the greater the change in $\partial p / \partial e$ (and incentives) as the promotion rate changes.

For more general distributions of π the analysis is more complicated, but a simple rule of thumb applies. Since $p = 1 - H(z - e^*)$, $z - e^*$ equals the $1 - p^{\text{th}}$ percentile of the distribution of π . Therefore, to see how the MPE (and incentives, for a constant prize) changes as p changes, it is only necessary to look at a plot of $h(\pi)$ that has been flipped around the vertical axis at zero. For example, if $h(\pi)$ is skewed to the right and has a maximum at the 25th percentile, then the MPE will be largest when $p = .75$, and a plot of the MPE versus p will be skewed left.

Multi-person Tournaments

In most tournament models, a worker competes against a single opponent; the one with higher performance is promoted. However, to model promotions with probabilities different from $\frac{1}{2}$, it is necessary to study more general competitions between n contestants for k equal prizes. Assuming that the k prizes are equal is not only natural in the context of promotions, but also makes analysis of multi-person tournaments more tractable than in previous studies (Green & Stokey 1983; Nalebuff & Stiglitz 1983). Derivation of the MPE is more complicated than in the case of a standard, and will be done in several steps.⁵

Worker i 's performance measure is again $q_i = e_i + \varepsilon_i + \eta$. Let $F(\varepsilon)$ be the c.d.f. and $f(\varepsilon)$ be the p.d.f. of ε , respectively. The promotion probability is $p = k/n$. Equilibrium e^* now depends on the actions of co-workers. To win the contest a worker must place k^{th} or higher, which amounts to beating at least $n - k$ co-workers in pair-wise output comparisons. For example, the probability that worker i beats worker g equals $p(q_i > q_g) = p(e_i + \varepsilon_i + \eta > e_g + \varepsilon_g + \eta) = p(\varepsilon_g < e_i - e_g + \varepsilon_i) = F(e_i - e_g + \varepsilon_i)$; note that the common error η cancels out in a tournament. Given identical contestants, a symmetric Nash equilibrium is a natural assumption. In this case, denote the optimal effort for all contestants by e^* . Then the probability that worker i beats any opponent equals $F(e_i - e^* + \varepsilon_i)$, and the probability that i loses to any opponent equals $1 - F(e_i - e^* + \varepsilon_i)$.

To finish *exactly* j^{th} from the top out of a field of n contestants, one must beat $n - j$ opponents and lose to $j - 1$ opponents. The probability of doing so for any *given* partition of competitors into the $n - j$ the worker beats and the $j - 1$ the worker loses to, conditional on i 's luck ε_i , is $F(e_i - e^* + \varepsilon_i)^{n-j} (1 - F(e_i - e^* + \varepsilon_i))^{j-1}$. The number of ways to

choose $n-j$ elements from a collection of $n-1$ is $\binom{n-1}{n-j} = (n-1)!/(n-j)!(j-1)!$, so the probability of placing exactly j^{th}

from the top, conditional on ε_i , is:

$$pr(\text{place } j^{\text{th}} \text{ out of } n | \varepsilon_i) = \binom{n-1}{n-j} F(e_i - e^* + \varepsilon_i)^{n-j} (1 - F(e_i - e^* + \varepsilon_i))^{j-1}.$$

Finally, the probability of promotion conditional on ε_i equals the sum of the conditional probabilities of placing exactly 1^{st} through k^{th} . Integrating out ε_i then gives the unconditional probability of promotion (subscripts are dropped to simplify notation):

$$p(e, \underline{e}^*) = \sum_{j=1}^k \binom{n-1}{n-j} \int F(e - e^* + \varepsilon)^{n-j} (1 - F(e - e^* + \varepsilon))^{j-1} f(\varepsilon) d\varepsilon.$$

It is straightforward to show that $p(e, \underline{e})$ is constant if $p = 0$ or 1 ($k = 0$ or n), and the MPE and incentives are zero.

To solve for the MPE, differentiate and substitute in the symmetric Nash equilibrium condition $e_i = e^*$:

$$\frac{\partial p}{\partial e} = \sum_{j=1}^k \binom{n-1}{n-j} \int \left((n-j)F(\varepsilon)^{n-j-1} (1 - F(\varepsilon))^{j-1} - (j-1)F(\varepsilon)^{n-j} (1 - F(\varepsilon))^{j-2} \right) f^2(\varepsilon) d\varepsilon.$$

Since $(m-1) \binom{n-1}{n-m} = (n-(m-1)) \binom{n-1}{n-(m-1)}$, and the second expression is zero if $j = 1$, this equals:

$$\frac{\partial p}{\partial e} = \sum_{j=1}^k (n-j) \binom{n-1}{n-j} \int F(\varepsilon)^{n-j-1} (1 - F(\varepsilon))^{j-1} f(\varepsilon)^2 d\varepsilon$$

$$- \sum_{m=2}^k (n-(m-1)) \binom{n-1}{n-(m-1)} \int F(\varepsilon)^{n-m} (1 - F(\varepsilon))^{m-2} f(\varepsilon)^2 d\varepsilon.$$

Letting $j = m-1$, all terms in the second sum cancel terms in the first, leaving the term $j = k$:

$$\frac{\partial p}{\partial e} = (n-k) \binom{n-1}{n-k} \int F(\varepsilon)^{n-k-1} (1 - F(\varepsilon))^{k-1} f(\varepsilon)^2 d\varepsilon.$$

It is now possible to interpret the MPE, $\partial p / \partial e$, in a way analogous to the model of a standard. A contestant is actually trying to beat the $n-k-1^{\text{st}}$ order statistic; i.e., the random variable that is the $n-k-1^{\text{st}}$ score

(from the bottom) among competitors. Call this order statistic, or performance to beat, κ . Subtracting off optimal effort of competitors gives the luck ε required to win, $\lambda \equiv \kappa - e^*$, a random variable. With a symmetric Nash equilibrium effort terms cancel out, and λ is also an order statistic: the $n-k-1^{\text{st}}$ luck (from the bottom) among the $n-1$ competitors. If this random variable has c.d.f. $G(\lambda)$ and p.d.f. $g(\lambda)$, then for given luck ε , the probability of promotion equals:

$$pr(promoted | \varepsilon) = p(e + \varepsilon > \kappa | \varepsilon) = p(e + \varepsilon > e^* + \lambda | \varepsilon) = G(e - e^* + \varepsilon).$$

Integrating over the luck distribution gives the unconditional probability of promotion, $p = \int G(e - e^* + \varepsilon) f(\varepsilon) d\varepsilon$, equation (6). Differentiating and substituting in $e = e^*$ gives equation (7), $\partial p / \partial e = \int g(\varepsilon) f(\varepsilon) d\varepsilon$. The integrand in (7), except for one $f(\varepsilon)$, is an order statistic density. This intuition explains an important feature of (7): the MPE is *independent of equilibrium effort*.⁶ Therefore $\partial p / \partial e$ depends only on k and n . The analysis of the MPE under a standard is a little more complicated, because p and e^* are determined simultaneously in equilibrium in that case.

In present form, analysis of (7) is not trivial, but the equation can be simplified with the change of variables $\xi \equiv F(\varepsilon)$. Since $f d\varepsilon = dF = d\xi$, and $\varepsilon = F^{-1}(\xi)$, (7) can be rewritten as:

$$\frac{\partial p}{\partial e} = \int (n-k) \binom{n-1}{n-k} \xi^{n-k-1} (1-\xi)^{k-1} f(F^{-1}(\xi)) d\xi.$$

where integration is over $[0,1]$. The integrand, except for $f(F^{-1}(\xi))$, has the form of a beta density function $\beta(k, n-k)$ with parameters k and $n-k$. Thus ξ has the effect of a beta-distributed random variable in (8). Under this interpretation, which is used below,

$$\frac{\partial p}{\partial e} = E_{\xi} f(F^{-1}(\xi)).$$

Properties of beta densities are well known, so with knowledge of the behavior of $f(F^{-1}(\xi))$ over $[0,1]$, statements about $\partial p / \partial e$ may be obtained. Some properties of beta densities are useful in providing intuition for the Propositions given below:

1. For $k, n-k > 1$, $\beta(k, n-k)$ is unimodal at $(n-k-1)/(n-2)$, with mean $(n-k)/n = 1-p$. $\beta(1,1)$ is uniform over $[0,1]$.

2. $\beta(n-k, k)$ is a reflection of $\beta(k, n-k)$ about $1/2$. This is true because ξ is a reflection of $1-\xi$ about $1/2$:

$$(n-k) \binom{n-1}{n-k} \xi^{n-k-1} (1-\xi)^{k-1} = (n-(n-k)) \binom{n-1}{n-(n-k)} (1-\xi)^{n-(n-k)-1} \xi^{(n-k)-1}.$$
3. Either of the above properties implies that $\beta(k, n-k)$ is symmetric (about $1/2$) for $k \geq 1$ if and only if $k = 1/2n$, i.e., when exactly half are promoted. $\beta(k, n-k)$ is skewed to the right if $k > 1/2n$, to the left if $k < 1/2n$, and becomes more skewed as k/n deviates from $1/2$ in either direction.
4. Holding k/n fixed, in the limit as n (and k) $\rightarrow \infty$, $\beta(k, n-k)$ approaches a distribution with all of its mass at $(n-k)/n = 1-p$.

These properties allow statements about $\partial p / \partial e$ when $f(\varepsilon)$ is symmetric and unimodal at 0, for then $f(F^{-1}(\xi))$ is symmetric over $[0, 1]$, unimodal at $1/2$, and $f(F^{-1}(1/2)) = f(0)$ (see the appendix). By (9), $\partial p / \partial e$ then amounts to taking the expectation of a symmetric function of a beta-distributed random variable. The following propositions discuss the incentive effects of increasing or decreasing the number of contestants or the number promoted. Together, they summarize how incentives vary with the probability of promotion, and build a comparison of the tournament model to the standard model. Propositions 3 through 6 are proved in the appendix, with intuition given here.

PROPOSITION 2. The MPE, $\partial p / \partial e$, is constant, and therefore independent of the number of contestants n and the number of winners k , if and only if ε is distributed uniformly. For non-uniform f , $\sup\{\partial p / \partial e\} \leq f(0)$.

This follows immediately from (7). Since the shape of the density of ξ changes with n and k , $\partial p / \partial e$ depends on n and k in general, with the exception of the uniform case where f can be factored out of (8) ($\beta(k, n-k)$ integrates to 1). Intuitively, when f is uniform, a contestant is equally likely to finish in any position from 1st to n^{th} , and extra effort increases equally the probability of finishing in all positions above the lowest. In this case, $\partial p / \partial e$ cannot depend on which ranking promotion depends on. The second part of the proposition is true from (9) because $f(0) = \max(f)$.⁷ This means that the marginal probability always has an upper bound determined by the distribution of the measurement error ε . For example, if $\varepsilon \sim N(0, \sigma^2)$, then $\partial p / \partial e \leq (2\pi\sigma^2)^{-1/2}$. As σ^2 gets larger, the upper bound on the MPE falls.

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In the familiar two-person tournament, increasing the variance of ε lowers incentives. This result does not generalize so nicely to multi-person contests. Increasing the variance shifts mass away from $\frac{1}{2}$, which definitely lowers $\partial p / \partial \varepsilon$ only when $p = \frac{1}{2}$, since in this case the beta density in (8) is also symmetric about $\frac{1}{2}$. For $p \neq \frac{1}{2}$, increasing the variance could *increase* incentives by increasing $\partial p / \partial \varepsilon$ under particular combinations of k , n , and shifts of the mass of f . This becomes more likely as p diverges from $\frac{1}{2}$ in either direction. In these cases, the distribution of extreme values of luck becomes more important, since as p approaches 0 or 1 it takes more unusual good or bad luck to win or lose. Increasing the variance of ε makes these outcomes more likely, so that marginal effort has more effect on the probability of winning.

The results to this point are identical to those of the standard case. Propositions 3-5 complete the analogy. Proposition 3 shows that the symmetry of the MPE with respect to the promotion rate holds under a tournament as well as under a standard.

PROPOSITION 3. *When f is symmetric unimodal, the MPE, $\partial p / \partial \varepsilon$, is equal in the two tournaments where k out of n are promoted and where $n-k$ out of n are promoted. These are any two tournaments with the same number of contestants and promotion probabilities of p and $1-p$.*

This is easily shown using Property 2 of beta densities. Since f is symmetric about $\frac{1}{2}$, and the density of ξ in the first tournament is a reflection about $\frac{1}{2}$ of the density in the second, the integrals that equal $\partial p / \partial \varepsilon$ in the two cases must be equal. Intuitively, this is a contest among even competitors, and ε is distributed symmetrically. Therefore, the probability of placing j^{th} from the top equals the probability of placing j^{th} from the bottom, and otherwise identical contests with probabilities symmetric about $\frac{1}{2}$ give the same incentives. Consider, for example, two tournaments with the same number of players, but in the first $_$ win, while in the second $_$ win. If the prizes are equal, both games have exactly the same incentives. It is not true in general that contests of *different* size and probabilities p and $1-p$ have equal MPE's. Propositions 4 and 5 summarize the relationship between the promotion rate and the MPE under a tournament, which is similar to this relationship under a standard.

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PROPOSITION 4. *When f is symmetric unimodal, adding a contestant (n to $n+1$), while holding the number of winners fixed (k), decreases the MPE, $\partial p/\partial e$, for $k < \frac{1}{2}n$ (less than $\frac{1}{2}$ promoted), does not change $\partial p/\partial e$ for $k = \frac{1}{2}n$, and increases $\partial p/\partial e$ for $k > \frac{1}{2}n$ (more than $\frac{1}{2}$ promoted).*

For $p < \frac{1}{2}$, increasing n but not k increases the skew of the density of ξ , shifting mass away from $\frac{1}{2}$. Since f decreases away from $\frac{1}{2}$ the result follows. For $p > \frac{1}{2}$, increasing n but not k reduces the skew, with the opposite effect. Intuitively, as the promotion becomes less or more likely (p approaches 0 or 1), more extreme good or bad luck is required to win or lose. Extreme luck is less likely than luck closer to zero, so that marginal effort is less likely to make a difference.

Green and Stokey (1983) show that $\lim_{n \rightarrow \infty} \partial p/\partial e = 0$ if $k = 1$ (and $k/n = p < \frac{1}{2}$), for more general (not necessarily symmetric) densities. Proposition 4 says that for symmetric unimodal f , $\partial p/\partial e$ decreases monotonically to 0 (after $n > 2k$), and the result holds for any sequence of contests with the number of winners held fixed at any value of k .

One surprising illustration of the last two propositions is that the following three contests have equal MPEs: 1 winner/2 contestants; 1 winner/3 contestants; and 2 winners/3 contestants. If the contests have equal prizes, they all have equal incentives.

PROPOSITION 5. *When f is symmetric unimodal, adding one more winner while keeping the number of contestants fixed increases the MPE, $\partial p/\partial e$, for $k < \frac{1}{2}(n-1)$, does not change $\partial p/\partial e$ for $k = \frac{1}{2}(n-1)$, and decreases $\partial p/\partial e$ for $k > \frac{1}{2}(n-1)$ (a special case of which is $k = \frac{1}{2}n$, or $p = \frac{1}{2}$).*

The intuition from Proposition 4 applies here as well. Propositions 3-5 summarize the incentive effects of changing the fraction of winners, holding the prize fixed: incentives rise as p rises from 0 to $\frac{1}{2}$, peak at $p = \frac{1}{2}$, and fall as p rises from $\frac{1}{2}$ to 0. Moreover, this effect on incentives is symmetric with respect to the absolute difference of the promotion rate and $\frac{1}{2}$, $|p - \frac{1}{2}|$. Adding competitors or decreasing promotion slots can *raise* incentives if it brings p closer to $\frac{1}{2}$. This is analogous to the standard case. Proposition 6 illustrates the only substantive difference between incentive properties of standards and tournaments; in tournaments, contest size matters.

PROPOSITION 6. *Increasing the number of contestants n , while keeping the probability k/n fixed, raises the MPE, $\partial p/\partial \varepsilon$, if $p = 1/2$. It generally does so for middle values of p . For extreme values of p near 0 or 1, increasing n tends to decrease the MPE. This economies-to-scale effect is more likely to hold for given p if: (1) $f(\cdot)$ is less peaked (has greater variance); or (2) the contest size is smaller.*

Increasing n with $p = k/n$ fixed increases the peakedness of the density of ξ . For $p = 1/2$, and for p near to $1/2$, this shifts mass of the beta density toward the center, raising $\partial p/\partial \varepsilon$. Intuitively, the greater the number of contestants, the smaller the variance in the order statistic that must be beaten to win promotion. Therefore the change in MPE is much like that from a decrease in the variance of ε . For contests with p further from $1/2$, this means that smaller amounts of luck are more likely to affect who wins. Since these are the more likely values of ε , effort has a greater marginal effect.

Although the proof holds only for $p = 1/2$, numerical analysis (the method is described in the appendix) yielded no examples where increasing tournament size lowers $\partial p/\partial \varepsilon$ when $p \in (1/3, 2/3)$. For contests with $p \notin (1/3, 2/3)$, the opposite tends to be true. The numerical analysis is also the basis for the other claims in Proposition 6.

Proposition 6 has an interesting implication: if p is in the middle range, contestants should be pooled into one large contest instead of being segregated into smaller contests with similar promotion rates. For example, suppose that otherwise identical contests with equal probability p , but different numbers of players, exhibit rising incentives (MPE) with contest size up to some size t^* , and diminishing incentives thereafter. For some range of contestants greater than t^* , it may not pay to break the contest into smaller games. Doing so would lower $\partial p/\partial \varepsilon$ to its value in each of the smaller contests, which might be smaller than incentives in the one large contest. For example, if $t^* = 30$, then a contest of size 33 has smaller incentives than one with 30 contestants. However, leaving the 33 players in one contest might still dominate all possible ways to break up the contest, e.g., (30,3), (27,6), (27,3,3), etc., because each smaller contest may also have lower MPE than the one with 30 players. Whether or not to break up the contest depends on the best alternative average incentives that can be generated for that number of players. Nevertheless, for p small or large (close to 0 or 1), it is generally optimal to split contests into as many smaller games as possible.

Incorporating non-symmetric error distributions is straightforward. What is important to incentives is the degree to which the distribution of mass over $[0,1]$ of the beta density is correlated with the distribution of mass of the transformed distribution $f(F^{-1}(\xi))$. As in the symmetric case, the transformation $f(F^{-1}(\xi))$ squeezes the density $f(\varepsilon)$ onto the unit interval, retaining general characteristics such as skewness, number of modes, and shape.

Consider a distribution of ε which is skewed to the right, such as lognormal. $f(F^{-1})$ is also skewed to the right, but over $[0,1]$, and will have a single mode to the left of $\frac{1}{2}$. Suppose that this mode is at ξ_m . A beta density with mode near to ξ_m has similar assignment of mass over the unit interval as $f(F^{-1})$, and thus high incentives relative to other beta densities. This corresponds to a promotion contest with probability in the region of $1-\xi_m$. In this case maximum incentives come at a promotion rate greater than $\frac{1}{2}$. If the distribution of ε is skewed to the left, then the opposite is true; promotions rates lower than $\frac{1}{2}$ tend to have greater incentives. Note that in both cases in which ε is unimodal but skewed in either direction, incentives fall monotonically as p diverges in either direction from the value that maximizes incentives, just as in the symmetric case. For more general non-symmetric error distributions, tournaments again have very similar incentive effects as standards, as long as the distributions of ε and π are similar.

The incentive properties of tournament- or standard-based promotions are virtually identical: both have the same implications for the relationship between p and the MPE, for a change in variance of the performance measure, etc. This should not be surprising, since competing in a tournament amounts to competing against the n - k^{th} order statistic, which is like trying to beat a standard. The difference is that in the tournament the standard is a moving target, depending on the actions of co-workers. The only substantive difference between standards and tournaments is that under a tournament incentives also depend on the size of the group competing.

VI. EMPIRICAL IMPLICATIONS AND CONCLUSIONS

It is often argued that firms use promotion systems to provide incentives for employees. As discussed above, such an assumption is not obvious. For example, external labor market pressures on salaries at each position may limit the ability of the firm to optimize the salary structure for incentives. Nevertheless, the idea that the firm can use the promotion system as an incentive scheme has generated a great deal of interest. It also seems clear empirically that promotions are a very important source of incentives. It therefore is important to consider ways to empirically study how firms manage career-based incentives. The model developed here yields several testable implications for compensation structures in hierarchies.

Tournaments vs. Standards: Tournaments are often criticized for having negative consequences, such as inducing competitive behavior between employees (Dye 1984). Usually, it is argued that firms use standards for promotion decisions instead. However, an important conclusion of this paper is that such criticisms have little impact on the relevance of tournament models for analyzing compensation data. This is because the implications of both models for compensation patterns in hierarchies are almost identical. This means that tournament models are still useful in guiding our ideas about optimal incentive schemes. Further, in cases where they are more tractable than standards, tournaments may be an appropriate modeling option.

Relationship between Promotion Rates and Salary Differentials: The most important prediction of the paper is that if the firm sets salary differentials between hierarchical levels to provide incentives, there will be a positive relationship between differentials and the absolute difference of the promotion rate and $\frac{1}{2}$, $|p - \frac{1}{2}|$. More generally, prizes will be larger for promotion rates closer to 0 or 1. This prediction is in marked contrast to that arising from models of optimal spans of control and hierarchical production functions (Keren & Levhari 1979; Rosen 1982). Such models imply that salary differentials will be strictly negatively related to promotion rates. This is because higher promotion rates imply smaller differences in average ability between those promoted and not promoted. Thus, an incentive perspective yields an implication in direct contrast to a sorting perspective. An empirical finding that salary differentials rise as promotion rates approach one would be strong evidence that the firm uses promotions to provide incentives.

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Relationship between Base Pay and Salary Differentials: Hierarchical production models predict that there are increasing returns to ability with higher position, because decisions of higher-level workers are spread over more subordinates and lower levels, in a form of production externality. In this view, salary spreads between levels should also rise with level, all else equal. Thus, such models predict a positive relationship between base salary levels (which rise with job level) and salary differentials between levels. The incentive model developed here has a quite different implication: there should be *no* relationship between base salary and differences in pay between levels. Thus, evidence of such a link would support hierarchical production models.

Both of the implications discussed above may be tested easily with appropriate data. Such data would measure promotion rates and salary differentials or rewards earned on promotion. Promotion rates are easy to estimate either by measuring spans of control between hierarchical levels, or by using historical data on actual promotions. Promotion rewards may be very closely proxied by the raise earned on promotion, or by average differences in compensation between positions (Gibbs 1993). Potentially, data from a single firm with a sufficient number of positions or hierarchical levels could be used to test this theory. Unfortunately, most firms do not have enough hierarchical levels or job titles for statistically significant tests. A more promising approach would be to use across-firm data on promotion rates or spans of control, and on compensation levels in each job, such as the data used by Leonard (1990) and Lambert, Larcker, & Weigelt (1989).

Variance of Performance Measures: Interactions between promotion rates and the accuracy of performance measures also affect incentives in the model presented above. When p is near $\frac{1}{2}$, greater accuracy leads to greater incentives. When p is farther from $\frac{1}{2}$, the opposite is true; *more* variance can increase incentives. The firm can also potentially change the variance by altering the monitoring intensity. Thus, we predict that for promotion rates farther from $\frac{1}{2}$ in either direction, larger variance in measurement errors will lead to larger prizes. This contrasts with simple piece-rate incentive models, in which variance always reduces incentives. This implication is more difficult to test, but it might be possible to do so with accounting measures. Further, as discussed above, the model of a standard is applicable to budgets as well as promotions. Thus, data from budget-based control systems could be used to test the effect of the interaction between noise and the likelihood of making the budget on rewards associated with the budget.

Group Size: One implication of the paper is that it is difficult to distinguish empirically between standards and tournaments. However, in some circumstances it may be possible to do so. If the firm uses a tournament, then the number of employees eligible for promotion will be related to the prize the firm offers. In particular, for promotion rates closer to $\frac{1}{2}$, larger groups will tend to have smaller rewards. Under standards there should be no effect of group size.

Conclusions

Promotion-based incentives appear to be very important in many firms, and have generated substantial interest among agency theorists. However, they have received little empirical study (Lambert, Larcker, & Weigelt 1989; Gibbs 1993) using firm personnel data. Unfortunately, there are few predictions from most promotion models that are easy to test with compensation data. Yet in order to make progress in understanding how organizations motivate employees, moving from theory to evidence is essential. This paper has extended the theory in several ways. I have shown that standards and tournaments are surprisingly similar in empirically testable predictions, and I have also generalized multi-person tournament models. These results make the theory more amenable to testing, and suggest appropriate data to collect to examine the hypothesis that firms alter their compensation to reflect promotion incentives. Many questions remain. For example, we understand little about how the external labor market affects career-based incentive systems. Some of these questions deserve theoretical examination, but many require empirical analysis.

FOOTNOTES

1. For an extension of the model to include short-term incentive compensation through bonuses, see Gibbs (1993).
2. If $p < 1/2$, second-order conditions may not be met due to non-convexity in $p(e)$ (Ferrall 1989).
3. This should not be confused with the statistical term marginal probability.
4. Introducing risk aversion is unlikely to change this finding. To a second-order variance in income is important. This equals $p(1-p)\Delta$, which is symmetric in the difference of p from $1/2$. Thus risk aversion effects will not tend to alter the relationship between optimal prizes and the promotion rate. Risk aversion makes the model intractable, even using Lazear & Rosen's (1981) second-order method that works for simple two-person tournaments. For more complicated tournaments competitors additional terms arise that make even this approach fruitless.
5. See Ferrall (1989) for an independent derivation of some similar results.
6. This property holds for any tournament where the performance measure is additively separable in effort and individualistic measurement error (McLaughlin 1988).
7. To see that $\sup(\partial p/\partial e) = f(0)$, define a sequence of tournaments indexed by t , with equal probabilities $p = tk/tn$. By Property 4 of beta densities, $\lim_{n \rightarrow \infty} \partial p_t/\partial e = f(0)$ if $p = 1/2$, and is smaller for $p \neq 1/2$.

APPENDIX: PROOF OF PROPOSITIONS 3-6

This appendix proves Propositions 3-6. Three results are established first: equation (A2) and Lemmas 1-2.

First, change notation to emphasize the dependence of the MPE on k and n :

$$p'(k, n) \equiv \frac{\partial p}{\partial e} = (n-k) \binom{n-1}{n-k} \int_0^1 \xi^{n-k-1} (1-\xi)^{k-1} f(F^{-1}(\xi)) d\xi.$$

Lemma 1 shows that $f(F^{-1})$ “squeezes” f onto $[0,1]$, shifting the mode to $1/2$ and retaining symmetry.

LEMMA 1. *If a density $f(\varepsilon)$ with c.d.f. $F(\varepsilon)$ is symmetric and unimodal at 0, then $f(F^{-1}(\varepsilon))$ is symmetric over $[0,1]$, and unimodal at $1/2$.*

Proof: Symmetry of $f(\varepsilon)$ about 0 implies that $F^{-1}(1/2) = 0$ and $F^{-1}(1/2+x) = -F^{-1}(1/2-x)$. Therefore $f(F^{-1}(1/2+x)) = f(F^{-1}(1/2-x))$; this establishes symmetry about $1/2$. The slope is:

$$\frac{df(F^{-1}(\xi))}{d\xi} = \frac{f'(F^{-1}(\xi))}{f(F^{-1}(\xi))} \geq 0 \iff F^{-1}(\xi) \leq 0 \iff F^{-1}(\xi) \leq F^{-1}(1/2).$$

Therefore, $f(F^{-1}(\xi))$ is increasing over $(0, 1/2)$, unimodal at $1/2$, and decreasing over $(1/2, 1)$. □

Use Lemma 1 and Property 2 of beta densities to rewrite (A1) as an integral over $[0, 1/2]$:

$$\begin{aligned} p'(k, n) &= (n-k) \binom{n-1}{n-k} \left(\int_0^{1/2} \xi^{n-k-1} (1-\xi)^{k-1} f d\xi + \int_{1/2}^1 \xi^{n-k-1} (1-\xi)^{k-1} f d\xi \right) \\ &= (n-k) \binom{n-1}{n-k} \int_0^{1/2} \left(\xi^{n-k-1} (1-\xi)^{k-1} + \xi^{k-1} (1-\xi)^{n-k-1} \right) f d\xi \\ &= (n-k) \binom{n-1}{n-k} \int_0^{1/2} \xi^{k-1} (1-\xi)^{k-1} \left(\xi^{n-2k} + (1-\xi)^{n-2k} \right) f d\xi. \end{aligned}$$

(A2) “flips” the right half of the beta density part of (A1) to the left side of $1/2$, and adds it to the left half, which is possible because of the symmetry of f . (A2) will be useful in the proofs that follow, for the following reason.

Lemma 1 showed that f is strictly increasing over $(0, 1/2)$. Therefore, any changes in k or n that shift mass of the

transformed beta density in (A2) toward $\frac{1}{2}$, and away from 0, will increase $p'(k,n)$ and vice versa. Lemma 2 makes precise this idea of “shifting” mass to or away from $\frac{1}{2}$. In the remainder of the appendix, all integrals are over $[0, \frac{1}{2}]$ unless stated otherwise.

LEMMA 2. *Let B_1 and B_2 be differentiable density functions over $[0, \frac{1}{2}]$, and f be a strictly increasing, non-negative function over $[0, \frac{1}{2}]$. If $d(B_1/dB_2)d\xi > 0$, then $\int B_1 f d\xi > \int B_2 f d\xi$. If (B_1/B_2) is decreasing, $\int B_1 f d\xi < \int B_2 f d\xi$.*

Proof: If $\int B_1 d\xi = \int B_2 d\xi = 1$, and if B_1/B_2 is monotonically increasing over $(0, \frac{1}{2})$, this ratio begins below 1 at 0, passes through 1 at a unique point ξ_0 , and ends above 1 at $\frac{1}{2}$. Furthermore,

$$\int_0^{\xi_0} (B_1 - B_2) d\xi + \int_{\xi_0}^{1/2} (B_1 - B_2) d\xi = 0$$

since both B_1 and B_2 integrate to 1. Consider:

$$\int_0^{1/2} B_1 f d\xi - \int_0^{1/2} B_2 f d\xi = \int_0^{1/2} (B_1 - B_2) f d\xi$$

$$= \int_0^{\xi_0} (B_1 - B_2) f d\xi + \int_{\xi_0}^{1/2} (B_1 - B_2) f d\xi.$$

In the first integral on the right-most side of (A3) (which is negative) $f \leq f(\xi_0)$, and in the second (which is positive) $f \geq f(\xi_0)$. Therefore,

$$\int_0^{1/2} (B_1 - B_2) f d\xi > f(\xi_0) \left(\int_0^{\xi_0} (B_1 - B_2) d\xi + \int_{\xi_0}^{1/2} (B_1 - B_2) d\xi \right) = 0,$$

which is the desired result. Similarly, if B_1/B_2 is decreasing over $(0, \frac{1}{2})$, then $\int (B_1 - B_2) f d\xi < 0$. $\square$

Lemma 2 is used to prove Propositions 4-6: the ratio of two beta densities (transformed as in (A2)), differing by some change in k or n , will be shown to be monotonic over $(0, \frac{1}{2})$. Since $p'(k,n)$ is of the form $\int B f d\xi$, Lemma 2 then implies p' is larger in one of the contests compared.

Proof of PROPOSITION 3: By Property 2 of beta densities,

$$p'(k, n) = (n - k) \binom{n - 1}{n - k} \int \xi^{n-k-1} (1 - \xi)^{k-1} f d\xi$$

□

$$= k \binom{n - 1}{k} \int \xi^{n-(n-k)-1} (1 - \xi)^{(n-k)-1} f d\xi = p'(n - k, n).$$

Proof of PROPOSITION 4: This proof uses Lemma 2. The comparison is of $p'(k, n)$ to $p'(k, n+1)$. The ratio of densities (using (A2)) is, with $A \equiv ((1 - \xi)/\xi)^{n-2k}$ ($\propto$ means “is proportional to”):

$$\frac{\beta(k, n)}{\beta(k, n+1)} \propto \frac{\xi^{k-1} (1 - \xi)^{k-1} (\xi^{n-2k} + (1 - \xi)^{n-2k})}{\xi^{k-1} (1 - \xi)^{k-1} (\xi^{n+1-2k} + (1 - \xi)^{n+1-2k})}$$

$$= \frac{1 + A}{\xi + (1 - \xi)A} \equiv R;$$

$$\frac{dR}{d\xi} = \frac{A'(\xi + (1 - \xi)A) - (1 + A)(1 - A + (1 - \xi)A')}{(\xi + (1 - \xi)A)^2}.$$

The denominator is positive. The numerator is $(2\xi - 1)A' - (1 + A)(1 - A)$; with $2\xi - 1 \leq 0$, $1 + A > 0$. Further,

$$k \leq \frac{n}{2} \iff A', (1 - A) \leq 0.$$

Therefore R is increasing if $k < \frac{1}{2}n$, constant if $k = \frac{1}{2}n$, and decreasing if $k > \frac{1}{2}n$, over $(0, \frac{1}{2})$; these mean that $p'(k, n)$ is less than, equal to, or greater than $p'(k, n+1)$ in each respective case. □

Proof of PROPOSITION 5: Compare $p'(k, n)$ to $p'(k+1, n)$; the density ratio is:

$$\frac{\beta(k, n)}{\beta(k+1, n)} \propto \frac{\xi^{k-1} (1 - \xi)^{k-1} (\xi^{n-2k} + (1 - \xi)^{n-2k})}{\xi^k (1 - \xi)^k (\xi^{n-2k-2} + (1 - \xi)^{n-2k-2})}$$

$$= \frac{\xi + (1 - \xi)A}{1 - \xi + \xi A} \equiv R;$$

$$\frac{dR}{d\xi} = \frac{(1 - A + (1 - \xi)A')(1 - \xi + \xi A) - (\xi + (1 - \xi)A)(A - 1 + \xi A')}{(1 - \xi + \xi A)^2},$$

with $A \equiv ((1 - \xi)/\xi)^{(n-2k-1)}$. The numerator of this is $(1 - 2\xi)A' + (1 + A)(1 - A)$. It is easy to show that:

$$k = \frac{n-1}{2} - A', (1-A) = 0.$$

which with Lemma 2 establishes the Proposition. $\square$

Proof of PROPOSITION 6: The comparison is $p'(k, 2k)$ and $p'(tk, 2tk)$. Let $t > 1$ represent the contest size; tk and $2tk$ are integers. The probability stays constant at $p = \frac{1}{2}$. The relevant ratio is:

$$\frac{\beta(k, 2k)}{\beta(tk, 2tk)} \propto \frac{\xi^{k-1} (1-\xi)^{k-1}}{\xi^{tk-1} (1-\xi)^{tk-1}} \\ = (\xi(1-\xi))^{-(t-1)k},$$

which is strictly decreasing in ξ . The rest of the proposition is based on numerical analysis (described below).

A Method for Numerical Analysis of the MPE

A simple method for numerical analysis of $\partial p/\partial e$ is now illustrated. Assume $f(F^{-1}(\xi))$ is proportional to a symmetric beta density $\beta(r+1, r+1)$ over $[0, 1]$:

$$f(F^{-1}(\xi)) \propto \xi^r (1-\xi)^r.$$

$f(F^{-1}(\xi))$ is *not* a density, since $\int f d\xi = \int f^2(\varepsilon) d\varepsilon$, which does not generally equal 1. However, for relative comparisons of tournaments with the same density f , $f(F^{-1}(\xi))$ may be parameterized by any function over $[0,1]$ (such as $\beta(r,r)$), if the only interest is in whether p' becomes larger or smaller with changes in k or n . In this case, multiplicative constants in $f(F^{-1}(\xi))$ are irrelevant since they would be equal across all values of p' . With this assumption the MPE is proportional to:

$$\begin{aligned}
 p'(tk, tn) &\propto \int_0^1 \frac{(tn-1)!}{(t(n-k)-1)!(tk-1)!} \xi^{t(n-k)-1} (1-\xi)^{tk-1} \xi^r (1-\xi)^r d\xi \\
 &= \frac{(tn-1)!(r+t(n-k)-1)!(r+tk-1)!}{(t(n-k)-1)!(tk-1)!(2r+tn-1)!},
 \end{aligned}$$

which is easily compared for various values of r , t , k , and n . The general method is useful because beta densities are flexible parameterizations of functions. They can represent many shapes, including monotonically increasing or decreasing, U-shaped, bell-shaped, and uniform, with varying skew and peakedness.

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