3 Discrete Network Location Models

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3.1 Introduction

Undoubtedly, humans have been analyzing the effectiveness of locational decisions since they inhabited their first cave. We use the term “facility” here in its broadest sense. That is, it is meant to include entities such as air and maritime ports, factories, warehouses, retail outlets, schools, hospitals, day-care centers, bus stops, subway stations, electronic switching centers, computer concentrators and terminals, rain gages, emergency warning sirens, and satellites, to name but a few that have been analyzed in the research literature.

The ubiquity of locational decision-making has led to a strong interest in location analysis and modeling within the operations research and management science community. The long and voluminous history of location research results from several factors. First, location decisions are frequently made at all levels of human organization from individuals and households to firms, government agencies and even international agencies. Second, such decisions are often strategic in nature. That is, they involve large sums of capital resources and their economic effects are long term. In the private sector they have a major influence on the ability of a firm to compete in the market place. In the public sector they influence the efficiency by which jurisdictions provide public services and the ability of these jurisdictions to attract households and other economic activity. Third, they frequently impose economic externalities. Such externalities include pollution, congestion, and economic development, among others.

Fourth, location models are often extremely difficult to solve, at least optimally. Even the most basic models are computationally intractable for large problem instances. In fact, the computational complexity of location models is a major reason that the widespread interest in formulating and implementing such models did not occur until the advent of high-speed digital computers. Finally, location models are application specific. That is, their structural form (the objectives, constraints and variables) is determined by the particular location problem under study. Consequently, there does not exist a general location model that is appropriate for all potential or existing applications.

3.1.1 Examples of facility location decision contexts

Much of the literature on facility location modeling has not been directed to specific applications (i.e., case studies). Rather, it has been directed to formulating new models and modifications to existing models which have many potential applications, and to developing efficient solution techniques for existing or newly formulated models.

There are several causes for this bias away from reporting specific applications in the literature. First, applications frequently employ existing models and solution techniques. Consequently, they are not viewed as scientific advances by the research community, but rather as applications of existing technology. Second, specific applications are frequently analyzed by consultants and planners; two professions which are rarely motivated to publish in research journals. Third, private sector advances in location modeling are often viewed as proprietary because they give the firm a competitive advantage; consequently, they are not shared with the larger community. In spite of this bias, there are still many articles which relate directly to a specific application or application area.

Table 3.1 lists some of the applications that have appeared in the literature. It is not intended to be exhaustive, but rather to demonstrate their diversity. A single citation is given for each application. This is done because of space limitations and is not meant to imply that the citation given is either the only or the most important article addressing the topic. In fact, for several of these applications, such as emergency medical services (EMS) siting, there exists a rather extensive literature.

Undoubtedly, the multitude of applications is a major reason for the multidisciplinary interest in location modeling. In addition to the more traditional applications listed in Table 3.1 there have also been less obvious ones. Some of these are listed in Table 3.2. The reader is referred to Eiselt (1992) for a review of facility location applications.

3.2 Basic Facility Location Models

In this section we present eight basic facility location models: set covering, maximal covering, p-center, p-dispersion, p-median, fixed charge, hub, and maxisum. In all of these models, the underlying network is given, as are the locations of the demands to be served by the facilities and the locations of existing facilities (if pertinent.) The general problem is to locate new facilities to optimize some objective. Distance or some measure more or less functionally related to distance (e.g., travel time or cost, demand satisfaction) is fundamental to such problems. Consequently, we have classified them according to their consideration of distance. The first four are based on maximum distance and the second four are based on total (or average) distance.

Table 3.1. Applications of Facility Location Models

Application	Citation
Airline hubs	O'Kelly, 1987
Airports	Saatcioglu, 1982
Auto Emission testing stations	Swersey and Thakur, 1995
Blood bank	Price and Turcotte, 1986
Brewery depots	Gelders, et al., 1987
Bus stops	Gleason, 1975
Bus garages	Maze et al., 1981
Coal handling facilities	Osleeb and Ratick, 1983
Computer concentrators	Pirkul, 1987
Computer service centers	Ghosh and Craig, 1986
Day-Care Centers	Holmes, et al., 1972
Electric power generating plants	Cohon, et al., 1980
Emergency medical services	ReVelle, et al., 1977
Emergency equipment for oil spills	Belardo et al., 1984
Essential air services	Flynn and Ratick, 1988
Fast-food restaurants	Min, 1987
Fire stations	Schilling et al., 1980
Forest harvesting sites	Hodgson, et al., 1987
Franchise outlets	Pirkul, et al., 1987
Hazardous waste disposal sites	ReVelle et al., 1991
Grain subterminals	Hilger, et al., 1977
Public swimming pools	Goodchild and Booth, 1980
Railroad sidings	Higgins, et al., 1997
Rain gauges	Hogan, 1990
Regional health facilities	Abernathy and Hershey, 1972
Rural health workers	Bennett, et al., 1982
Satellite homing stations	Helme and Magnanti, 1989
Satellite orbits	Drezner, 1988
Schools	Tewari and Sidheswar, 1987
Social Service Centers	Patel, 1979
Solar power system design	Birge and Malyshko, 1985
Solid waste collection	Marks and Liebman, 1971
Telecommunication switching centers	Hakimi, 1965
Truck terminals	Love, et al., 1985
Vehicle Inspection Stations	Hodgson, et al., 1996
Warehouses	Kuehn and Hamburger, 1963

Table 3.2. Additional Applications of Facility Location Models

Application	Citation
Apparel sizing	Tryfos, 1986
Archaeological settlement analysis	Bell and Church, 1985
Chip manufacturing	Cho and Sarrafzadeh, 1994
Data base management	Pirkul, 1986
Flexible Manufacturing System tool selection	Daskin, et al., 1990
Ingot size selection	Vasko, et al., 1988
Location of bank accounts	Cornuejols, et al., 1977
Medical diagnosis	Reggia, et al., 1983
Metallurgical grade assignment	Vasko, et al., 1989
Placement of dampers	Kincaid and Berger, 1993
Political party platform	Ginsberg, et al., 1987
Product positioning in feature space	Gavish et al., 1983
Product procurement and standardization	Watson, 1996
Production lot sizing	Van Oudheusden and Singh, 1988
Vehicle routing	Bramel and Simchi-Levi, 1995

3.2.1 Maximum Distance Models

In some locations problems, a maximum distance exists *a priori*. For example, in many school districts elementary school students within a mile of their school must walk to school. Public transportation must be provided for those not within this maximum distance. In the private sector, some businesses guarantee service within a pre-determined time (e.g., 20 minute pizza delivery). In the former case, a school district might want to locate schools to minimize the number of students who must be bussed at public expense. In the latter example, a pizza chain might want to locate its outlets to maximize the number of potential customers within 20 minutes of one of the outlets.

In the facility location literature, *a priori* maximum distances such as these are known as "covering" distances. Demand within the covering distance of its closest facility is considered "covered." An underlying assumption of this measure of maximum distance is that demand is fully satisfied if the nearest facility is within the coverage distance and is not satisfied if the closest facility is beyond that distance. That is, being closer to a facility than the maximum distance does not improve satisfaction.

Set covering location model The first location covering location problem was the set covering problem (Toregas et al., 1971). Here the objective is to locate the minimum number of facilities required to "cover" all of the demand nodes. To formulate this problem we define the following inputs and sets

I = the set of demand nodes indexed by i
J = The set of candidate facility locations, indexed by j
d_{ij} = distance between demand node i and candidate site j
D_c = distance coverage
$N_i = \{j \mid d_{ij} \leq D_c\}$
= the set of all candidate locations that can cover demand point i

and the following decision variable

$$x_j = \begin{cases} 1 \text{ if we locate at site } j \\ 0 \text{ if not} \end{cases}$$

With this notation, the set covering location problem (SCLP) can be formulated as follows:

$$\text{Minimize} \quad \sum_{j \in J} x_j \tag{3.1}$$

subject to:

$$\sum_{j \in N_i} x_j \geq 1 \ \forall i \in I \tag{3.2}$$

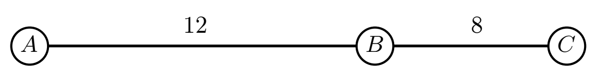
$$x_j \in \{0, 1\} \ \forall j \in J \tag{3.3}$$

The objective function (3.1) minimizes the number of facilities located. Constraint set (3.2) ensures that each demand node is covered by at least one facility. Constraint set (3.3) enforces the yes or no nature of the siting decision. The objective function can be generalized by including site-specified costs as coefficients of the decision variables. In this case, the objective would be to minimize the total fixed cost of the siting configuration rather than the number of facilities sited. Both versions of the set covering problem are NP-hard (Garey and Johnson, 1979). However, the linear programming relaxation of the set covering location problem as formulated above often results in an all-integer solution. Typically, only a few branches in a branch and bound algorithm are needed to obtain an optimal all-integer solution when the LP relaxation is fractional.

A variety of row and column reduction rules have been developed to reduce the size of the problem considerably (see Daskin (1995) for a discussion of such rules). For example, variable x_k can be eliminated from the formulation if $M_k \subset M_j$, where $M_j = \{i|d_{ij} \leq D_c\}$ and $M_k = \{i|d_{ik} \leq D_c\}$. This column reduction is possible because a facility at j would cover all of the demand nodes that a facility at k would cover and possibly additional ones as well; therefore, location j "dominates" location k. Individual constraints, say h, of constraint set (3.2) can be eliminated if there is some covering set, say N_i, such that $N_i \subset N_h$. This row reduction is possible because the constraint in (3.2) for demand node h is redundant. That is, if the coverage constraint for demand node i is satisfied, then the constraint for demand node h is also satisfied.

Formulation (3.1)-(3.3) assumes that the candidate facility sites are located at the nodes of the network. A lower cost facility siting scheme might be possible if the facilities could be located along the arcs of the network as well. This is illustrated by Figure 3.1. If the coverage distance is 10 and facilities can only be located at the nodes, then two facilities are needed: one at node A and one at either node B or C. If we could locate along the arcs as well as at the nodes, then a single facility located ten units to the right of node A would cover all three demand nodes. Church and Meadows (1979) present a method to modify the original network to permit siting along the arcs but still solve the problem using formulation (3.1)-(3.3). This method augments the original network with a finite (albeit potentially large) number of nodes located along the arcs of the network. The inclusion of these additional nodes in set J will result in a solution as good as one permitting locations anywhere along the arcs.

Fig. 3.1. Example network



Maximal covering location problem An underlying assumption of the set covering location problem is that all of the demand nodes must be covered. In essence, there is no budget constraint. However, in many facility planning situations, a budget does exist. For example, many school districts would like to have an elementary school within walking distance of all of its elementary age students. However, satisfying such a requirement may require more schools than the district is prepared to build. The maximal covering location problem (MCLP, Church and ReVelle, 1974) was formulated to address planning situations which have an upper limit on the number of facilities to be sited. The objective of the MCLP is to locate a predetermined number of facilities, p, in such a way as to maximize the demand that is covered. Thus, the MCLP assumes that there may not be enough facilities to cover all of the demand nodes. If not all nodes can be covered, the model seeks the siting scheme that covers the most *demand*.

To formulate the maximal covering problem, we augment the definitions used in the SCLP with:

h_i = demand at node i

p = the number of facilities to locate

$$z_i = \begin{cases} 1 \text{ if demand node } i \text{ is covered} \\ 0 \text{ if not} \end{cases}$$

The maximal covering problem may now be formulated as follows:

$$\text{Maximize} \quad \sum_{i \in I} h_i z_i \tag{3.4}$$

subject to:

$$\sum_{j \in N_i} x_j - z_i \geq 0 \; \forall i \in I \tag{3.5}$$

$$\sum_{j \in J} x_j = p \tag{3.6}$$

$$x_j \in \{0, 1\} \quad \forall j \in J \tag{3.7}$$

$$z_i \in \{0, 1\} \quad \forall i \in I \tag{3.8}$$

The objective function (3.4) maximizes the total demand covered. Constraint set (3.5) ensures that demand at node i is not counted as covered unless we locate at one of the candidate sites that covers node i. Constraint (3.6) limits the number of facilities to be sited. Constraint sets (3.7) and (3.8) reflect the binary nature of the facility siting decisions and demand node coverage, respectively. Interestingly, constraint sets (3.5) and (3.7), allow us to replace constraint set (3.8) with $z_i \leq 1, \; \forall i \in I$, without loss of generality.

As with the SCLP, if the facilities can be sited anywhere along the arcs of the network, the network can be modified as proposed by Church and Meadows (1979) and the problem solved with (3.4)-(3.8). By systematically varying p from 1 to k, where k is the minimum number of facilities required to cover the entire demand, one can use (3.4)-(3.8) to determine the marginal benefits associated with additional facilities.

The maximal covering problem is also NP-hard (Megiddo, Zemel and Hakimi, 1983), but it can generally be solved effectively using heuristics of the sort outlined later. Particularly useful is Lagrangean relaxation embedded within a branch and bound algorithm (Daskin, 1995; Daskin and Owen, 1998; Galväo and ReVelle, 1996). As mentioned earlier, some planning scenarios exist where there is a desired coverage distance and some maximum distance beyond which service is unacceptable. For problems like this, Church and ReVelle (1974) formulated the maximal covering location problem with mandatory closeness constraints.

Given the previous definitions and D_m = the maximum distance that a demand node may be from an opened facility and $M_i = \{j | d_{ij} \leq D_m\}$, the MCLP with mandatory closeness constraints may be formulated by adding the following constraint set to formulation (3.4)–(3.8).

$$\sum_{j \in M_i} x_j \geq 1 \quad \forall i \in I \tag{3.9}$$

The SCLP and MCLP assume that the covering distance, D_c, is a fixed, predetermined standard. This is certainly true in many location planning situations. However, in other situations D_c may be a goal, or target, rather than a fixed standard. For example, in siting facilities such as public libraries and recreational facilities, public agencies may desire to minimize the maximum distance that a citizen is from such a facility for equity reasons (Marsh and Schilling, 1994). Other facilities, such as schools or fire stations, may have a desired distance (e.g., less than 1 mile or 3 minutes travel time) and another distance (e.g., 5 miles or 10 minutes travel time) beyond which service is unacceptable. The following sections address planning situations of this nature.

***p*-center problem** The p-center problem (Hakimi, 1964,1965) addresses the problem of minimizing the maximum distance that demand is from its closet facility given that we are siting a pre-determined number of facilities. There are several possible variations of the basic model. The "vertex" p-center problem restricts the set of candidate facility sites to the nodes of the network while the "absolute" p-center problem permits the facilities to be anywhere along the arcs. Both versions can be either *weighted* or *unweighted*. In the unweighted problem, all demand nodes are treated equally. In the weighted model, the distances between demand nodes and facilities are multiplied by a weight associated with the demand node. For example, this weight might represent a node's importance or, more commonly, the level of its demand.

Given our previous definitions and the following decision variables

W = the maximum distance between a demand node and the facility to which it is assigned

$$y_{ij} = \begin{cases} 1 \text{ if demand node } i \text{ is assigned to a facility at node } j \\ 0 \text{ if not} \end{cases}$$

the p-center problem can be formulated as follows:

$$\text{Maximize} \quad W \tag{3.10}$$

subject to:

$$\sum_{j \in J} x_j = p \tag{3.11}$$

$$\sum_{j \in J} y_{ij} = 1 \qquad \forall i \in I \tag{3.12}$$

$$y_{ij} - x_j \leq 0 \qquad \forall i \in I, j \in J \tag{3.13}$$

$$W - \sum_{j \in J} h_i d_{ij} y_{ij} \geq 0 \; \forall i \in I \tag{3.14}$$

$$x_j \in \{0,1\} \qquad \forall j \in J \tag{3.15}$$

$$y_{ij} \in \{0,1\} \qquad \forall i \in I, j \in J \tag{3.16}$$

The objective function (3.10) minimizes the maximum demand-weighted distance between each demand node and its closest open facility. Constraint (3.11) stipulates that p acilities are to be located. Constraint set (3.12) requires that each demand node be assigned to exactly one facility. Constraint set (3.13) restricts demand node assignments only to open facilities. Constraint (3.14) defines the lower bound on the maximum demand-weighted distance, which is being minimized. Constraint set (3.15) established the siting decision variable as binary. Constraint set (3.16) requires the demand at a node to be assigned to one facility only. Constraint set (3.16) can be replaced by $y_{ij} \geq 0 \; \forall i \in I; j \in J$ because constraint set (3.13) guarantees that $y_{ij} \leq 1$. If some y_{ij} are fractional, we can simply assign node i to its closest open facility.

For fixed values of p, the vertex p-center problem can be solved in $O(N^p)$ time since we can enumerate each possible set of candidate locations in this amount of time. Clearly, even for moderate values of N and p, such enumeration is not realistic and more sophisticated approaches are required. For variable values of p, the problem is NP-hard (Garey and Johnson, 1979.)

If integer-valued distances can be assumed, the unweighted vertex or absolute p-center problem is most often solved using a binary search over a range of coverage distances (Handler and Mirchandani, 1979; Handler, 1990). For each coverage distance, a set covering problem is solved. When the solution to the SCP equals p, the minimum associated coverage distance is the solution to the p-center problem. Daskin (2000) has recently shown how the maximal covering model can be used effectively in place of the set covering model as a sub-problem in solving the unweighted vertex p-center problem.

3.2.2 The *p*-dispersion problem

For all of the models discussed above the concern is with the distance between demand and new facilities. Also, an unspoken assumption is that being close to a facility is desirable. The p-dispersion problem (PDP) differs from those problems in two ways (Kuby, 1987). First, it is concerned only with the distance between new facilities. Second, the objective is to maximize the minimum distance between any pair of facilities. Potential applications of the PDP include the siting of military installations where separation makes them more difficult to attack or locating franchise outlets where separation reduces cannibalization among stores.

To formulate this model we require an additional input (M) and a decision variable (D):

M = a large constant (e.g., $\max\limits_{i \in I, j \in J} \{d_{ij}\}$)
D = the minimum separation distance between any pair of facilities

With this notation, the p-dispersion model may be formulated as follows:

$$\text{Maximize} \quad D \tag{3.17}$$

subject to:

$$\sum_{j \in J} x_j = p \tag{3.18}$$

$$D + (M - d_{ij})x_i + (M - d_{ij})x_j \leq 2M - d_{ij} \quad \forall i, j \in J, i < j \tag{3.19}$$

$$x_j \in \{0, 1\} \quad \forall j \in J \tag{3.20}$$

The objective function (3.17) maximizes the distance between the two closest facilities. Constraint (3.18) requires that p facilities are located. Constraint (3.20) is a standard integrality constraint. Constraint (3.19) defines the minimum separation between any pair of open facilities. Note that if either x_i or x_j is zero, the constraint will not be binding. However, if both are equal to 1, then the constraint is equivalent to $D \leq d_{ij}$. Therefore, maximizing D has the effect of forcing the smallest inter-facility distance to be as large as possible.

3.2.3 Total or Average Distance Models

Many facility location planning situations in the public and private sections are concerned with the total travel distance between facilities and demand nodes. An example in the private sector might be the location of production facilities that receive their inputs from established sources by truckload deliveries. In the public sector, one might want to locate a network of service providers such as licensing bureaus in such a way as to minimize the total distance that customers must traverse to reach their closest facility. This approach may be viewed as an "efficiency" objective as opposed to the "equity" objective of minimizing the maximum distance, which was mentioned earlier.

***p*-median problem** One classic model in this area is the p-median model (Hakimi, 1964, 1965) which finds the locations of p facilities to minimize the demand-weighted total distance between demand nodes and the facilities to which they are assigned. This model may be formulated as follows:

$$\text{Minimize} \sum_{i \in I} \sum_{j \in J} h_i d_{ij} y_{ij} \tag{3.21}$$

subject to:

$$\sum_{j \in J} x_j = p \tag{3.22}$$

$$\sum_{j \in J} y_{ij} = 1 \quad \forall i \in I \tag{3.23}$$

$$y_{ij} - x_j \leq 0 \quad \forall i \in I, j \in J \tag{3.24}$$

$$x_j \in \{0, 1\} \quad \forall j \in J \tag{3.25}$$

$$y_{ij} \in \{0, 1\} \quad \forall i \in I, j \in J \tag{3.26}$$

The objective function (3.21) minimizes the demand-weighted total distance traveled. Constraint set (3.22) through (3.24) are identical to (3.11) through (3.13) of the p-center problem. Constraints sets (3.25) and (3.26) are identical to (3.15) and (3.16). Constraint set (3.26) can be eliminated following the same arguments as were used for constraint set (3.16). Toregas and ReVelle (1972) show that this formulation also minimizes the average travel distance between the sited facilities and the demand.

This formulation (3.21–3.26) assumes that the potential facility sites are nodes on the network. Hakimi (1964) proved that relaxing the problem to allow facility locations on the arcs of the network would not reduce total travel cost. Consequently, this formulation will yield an optimal solution, even if the facilities could be located anywhere on an arc. Like the p-center problem, the p-median problem can be solved in polynomial time for fixed values of p, but is NP-hard for variable values of p (Garey and Johnson, 1979).

Fixed Charge Location Problem The p-median problem makes three important assumptions that may not be appropriate for certain siting scenarios. First, it assumes that each potential site has the same fixed costs for locating a facility at it. Secondly, it assumes that the facilities being sited do not have capacities on the demand that they can serve. In the parlance of the literature, it is an "uncapacitated" problem. Finally, it also assumes that one knows, *a priori*, how many facilities should be opened (i.e., p)

The fixed charge location problem (FCLP) relaxes all three of these assumptions. The objective of the FCLP is to minimize total facility and transportation costs. In so doing, it determines the optimal number and locations of facilities, as well as the assignments of demand to a facility. Given the fact that the facilities have capacities, demand may not be assigned to its closest facility, as was the case in the previous models presented in this chapter.

Given the previous definitions and the following inputs

f_j = fixed cost of locating a facility at candidate site j
C_j = capacity of a facility at candidate site j
α = cost per unit demand per unit distance

the capacitated fixed charge location problem can be formulated as follows (Balinski, 1965):

$$\text{Minimize } \sum_{j \in J} f_j x_j + \alpha \sum_{i \in I} \sum_{j \in J} h_i d_{ij} y_{ij} \tag{3.27}$$

subject to:

$$\sum_{j \in J} y_{ij} = 1 \qquad \forall i \in I \tag{3.28}$$

$$y_{ij} - x_j \leq 0 \qquad \forall i \in I, j \in J \tag{3.29}$$

$$\sum_{j \in J} h_i y_{ij} - C_j x_j \leq 0 \qquad \forall i \in I \tag{3.30}$$

$$x_j \in \{0,1\} \qquad \forall j \in J \tag{3.31}$$

$$y_{ij} \in \{0,1\} \qquad \forall i \in I, j \in J \tag{3.32}$$

The objective function (3.27) minimizes the sum of the fixed facility location costs and the total travel costs for demand to be served. The second set of terms in (3.27) is often referred to as demand-weighted distance. Constraint (3.30) prohibits the total demand assigned to a facility from exceeding the capacity of the facility, C_j. Constraint sets (3.28), (3.29), (3.31) and (3.32) function as similar constraint sets in the previous problems. Relaxing constraint set (3.32) allows demand at a node to be assigned (partially) to multiple facilities. We also note that constraint (3.29) is not needed in this integer programming formulation since constraint set (3.30) will also force demands to be assigned only to open facilities. However, including constraint set (3.29) in the formulation significantly strengthens the linear programming relaxation of the model.

There are several other features of the FCLP that may not be initially apparent. The inclusion of (3.32) as a binary constraint requires that all demand points be singly sourced. That is, all demand at a particular location is assigned to one facility. Note also that, due to the facility capacities, demand may be served by a facility which is not its closest one. If constraint set (3.30) is removed, the model becomes the uncapacitated fixed charge location problem (UFCLP). In this case, each demand can be completely served by its closest facility and (3.32) can be replaced by non-negativity constraints on the assignment variables, y_{ij}.

Hub location problems Many logistics systems such as less-than-truckload carrier networks, airline networks, and inter-modal carriers, employ hub and spoke systems. These systems are designed to utilize larger capacity or faster vehicles or modes over the long-haul portion of an origin to destination delivery. Consequently, these systems reduce average per mile transportation cost or total delivery time. Numerous models (e.g. O'Kelly, 1986a, 1986b; and Campbell, 1990, 1994) have been formulated to locate the hubs and delivery routes of hub and spoke systems. Most of these models attempt to minimize total cost (as a function of distance.) We refer the reader to chapter 12 of this book for a detailed discussion of hub location problems. The basic p-hub location model can be formulated using the following notation inputs

h_{ij} = number of units of flow between nodes i and j
c_{ij} = unit cost of transportation between nodes i and j
α = discount factor for transport between hubs

and the following decision variables

$$x_j = \begin{cases} 1 \text{ if a hub is located at node } j \\ 0 \text{ if not} \end{cases}$$

$$y_{ij} = \begin{cases} 1 \text{ if demands at node } i \text{ are assigned to a hub located at node } j \\ 0 \text{ if not} \end{cases}$$

as follows (O'Kelly, 1987):

$$\text{Minimize} \sum_{i\in N}\sum_{j\in N} h_{ij}\left(\sum_{k\in N} c_{ik}y_{ik} + \sum_{m\in N} c_{jm}y_{jm} + \alpha\sum_{k\in N}\sum_{k\in N} c_{km}y_{ik}y_{jm}\right) \tag{3.33}$$

subject to:

$$\sum_{j\in N} x_j = p \tag{3.34}$$

$$\sum_{j\in J} y_{ij} = 1 \qquad \forall i \in I \tag{3.35}$$

$$y_{ij} - x_j \leq 0 \qquad \forall i \in I, j \in J \tag{3.36}$$

$$x_j \in \{0,1\} \qquad \forall j \in J \tag{3.37}$$

$$y_{ij} \in \{0,1\} \qquad \forall i \in I, j \in J \tag{3.38}$$

The objective function (3.33) minimizes the sum of the cost of moving items between a non-hub node and the hub to which the node is assigned, the cost of moving from the final hub to the destination of the flow, and the inter-hub movement cost which is discounted by a factor of α. The model assumes that the hub portion of the network is a complete graph and therefore flows between any pair of nodes i and j will pass through at most two different hub nodes. Constraints (3.34) through (3.38) are identical to constraints (3.22) through (3.27) for the p-median model above. In particular, constraints (3.35) stipulate that each node should be assigned to exactly one hub. In practical contexts, it may be valuable to relax this constraint and to allow flows from particularly large nodes to be served directly by two or more hub nodes.

Despite the similarity between the constraints of the two models, it is worth noting a number of important differences. First, the demands in the

p-hub location model are node-to-node flows and not simply demands at a particular node. Second, and perhaps most importantly, the objective function is quadratic in the assignment variables. Third, it may not be optimal to assign a node to the nearest hub since the objective function is measured in terms of node-to-node flows and not simply in terms of the cost of accessing the hub system.

Given the difficulties in solving even moderate sized hub location problems optimally, Ernst and Krishnamoorthy (1996) propose heuristic (or approximate) efficient algorithms to attack such problems. Kuby and Gray (1993) analyzed an express air carrier's network with a hub location model. For a more complete review of hub location models, the reader is referred to O'Kelley and Miller (1994) and chapter 12 of this book.

The maxisum location problem The average distance models discussed above assume that locating facilities as close as possible to demands is desirable. For many facilities this is the case. However, for undesirable facilities (e.g., prisons, power plants, and solid waste repositories) at least one objective involves locating facilities far from demand nodes.

The maxisum location problem seeks the locations of p facilities such that the total demand-weighted distance between demand nodes and the facilities to which they are assigned is maximized. This model may be formulated as follows:

$$\text{Maximize} \quad \sum_{i \in I} \sum_{j \in J} h_i d_{ij} y_{ij} \tag{3.39}$$

subject to:

$$\sum_{j \in J} x_j = p \tag{3.40}$$

$$\sum_{j \in J} y_{ij} = 1 \qquad \forall i \in I \tag{3.41}$$

$$y_{ij} - x_j \le 0 \qquad \forall i \in I, j \in J \tag{3.42}$$

$$\sum_{k=1}^{m} y_{i[k]_i} - x_{[m]_i} \ge 0 \; \forall i \in I, m = 1, \ldots, N-1 \tag{3.43}$$

$$x_j \in \{0,1\} \qquad \forall j \in J \tag{3.44}$$

$$y_{ij} \in \{0,1\} \qquad \forall i \in I, j \in J \tag{3.45}$$

This formulation is identical to that of the p-median problem with two notable exceptions. First, the objective (3.39) is to maximize the demand-weighted total distance and not to minimize it. The unfortunate impact of this objective is that it forces demands to be assigned to the most remote facility. Thus, the formulation has been extended with constraint (3.43), which ensures that demands are assigned to the nearest facility. In this constraint, $[k]_i$ is the index of the k^{th} farthest candidate location from demand node i.

Constraint (3.43) then states that if the mth closest facility to demand node i is opened then demand node i must be assigned to that facility or to a closer facility.

3.3 Location-Routing Models

The basic models discussed above assume that the demand is served directly from a facility. This is valid for many siting scenarios such as those involving direct shipments from factories to distribution centers or emergency medical services. However, many facilities such as solid waste collection substations and distribution centers provide collection and/or distribution functions in which demand is served by multiple drop off and/or pickup routes. In such cases, the overall effectiveness of the facility siting scheme depends not only upon the distances from the individual demands but also upon the efficiency of the vehicle routes needed to serve multiple demands. Such problems are referred to as location-routing problems. It is well established that modeling distribution cost as the cost of a simple round trip from a facility to a customer may significantly misrepresent the actual costs and may, as a consequence, result in the selection of sub-optimal facility sites when multi-stop tours are used. (Webb, 1968, Eilon, Watson-Gandy and Christofides, 1971, and Perl and Daskin, 1985) As Perl and Daskin pointed out, location-routing problems involve three inter-related, fundamental decisions: where to locate the facilities, how to *allocate* customers to facilities, and how to *route* the vehicles to serve customers.

Many variations of location-routing problems exist. For example, Laporte, Nobert, and Taillefer (1988) consider three variants of location-routing problems, including (1) capacity constrained vehicle routing problems, (2) cost constrained vehicle routing problems, and (3) cost constrained location-routing problems. The authors examine multi-depot, asymmetrical problems and develop an optimal solution procedure that enables them to solve problems with up to 80 nodes. Berger (1997) formulates a location routing problem for perishable commodities as a variant of a fixed charge facility location problem. Her model is applicable in cases in which the routes are constrained to be short (since the commodity is perishable) and the vehicle does not have to return to the original depot within a time window. Current and Schilling (1994) formulate the median and maximal covering tour problems. These problems determine which nodes should have facilities to serve demand as well as the route (a cycle) connecting the nodes with facilities. The interested reader is referred to Laporte (1988) for a discussion of application, formulations and solution approaches to a wide variety of location-routing problems.

3.4 Facility Location-Network Design Models

It is obvious that the underlying network is important in facility location and location-routing problems. The underlying network was assumed to be given in the models presented in the previous two sections. However, in many location problems, one must determine which arcs should be included in the network as well as where facilities should be located. Examples of such problems include the design of subway or rail systems, electricity distribution systems, and computer networks. The facilities in such systems are stations, transformers, or concentrators and the network arcs are rail lines, power lines, or fiber optic cable. One objective in these problems is to minimize total cost, which includes facility and arc costs.

Current (1988) and Current and Pirkul (1991) develop a model and solution procedure for problems where the desired network is a path and the facilities are “entry locations” (e.g., station) for demand to enter the path (e.g., rail line). The objective is to minimize facility and network costs as well as the cost of arcs needed for demand to reach a facility on the path. The models in Current and Schilling (1989) can be used in a similar fashion when the network design requires a cycle.

Many hub location problems also fall into this category. For example, in studying airline hub and spoke networks, we must simultaneously determine the location of the hubs, the assignment of non-hub airports to the hubs (i.e., the links to be used in the spoke part of the network), and the connectivity of the hubs. Similar problems arise in telecommunication, power transmission, and computer networks. This is especially true when they address problems where the spoke nodes are not necessarily assigned to their nearest hub or when the hub network is not a complete graph. Melkote (1996) proposed a number of novel formulations of the network design/facility location problem and outlined heuristic and optimal solution procedures. In particular, he extended a number of coverage-based models to include network design decisions. Such formulations are particularly appropriate for decisions in developing regions.

3.5 Multiobjective Models

Facility location decisions are often strategic in nature as they frequently involve large capital outlays and long-term planning horizons. In general, they are the least flexible component of a firm’s supply chain or a government’s provisions of services. Factories, distribution centers, libraries and sewage treatment plants have expected life times of 20 to 50 years. They impact not only the providers of the facility but also their users and neighbors. They impact the human resources, finance, accounting, marketing, production, and logistics functions of a firm. Consequently, facility location decisions often involve many stake holders and must consider multiple, often conflicting, objectives.

There are essentially two approaches to multiobjective problems: generating techniques and preference-based techniques (Cohon, 1978). Preference-based techniques employ some method to "rank" the objectives and then find the solution that optimizes this ranking. Ranking may be done by a variety of means from a simple ordinal weighting of the objectives to a complex utility function. In general, generating techniques identify the pareto optimal siting configurations from which the decision makers select the one that they prefer.

The importance of distance and the ways it can be addressed in facility location objectives is demonstrated by the development of models that include multiple distance-related objectives. For example, Halpern (1976) included a maximum distance (center) and an average distance objective (median) for locating a single facility. Schilling, et al. (1980) included several different maximum covering objectives for fire equipment location. Church, et al. (1991) included a maximum distance (covering) objective and an average distance (median) objective.

Other researchers have included new objectives in facility location models. Many of these have appeared in models to locate obnoxious or undesirable facilities such as waste disposal sites, prisons, and power plants (e.g., Ratick and White, 1988; Erkut and Newman, 1989, 1992; and Erkut and Verter, 1995). These models often consider objectives related to risk and equity as well as efficiency.

Multiple objectives have also been considered in location-routing and location-network design models. For example, List et al. (1991) and Current and Ratick (1995) formulated hazardous material facility location-routing models which include objectives related to cost, risk, and the equity of risk imposed resulting from the facilities and the the routing of hazardous materials to them. Current et al (1985) formulated location-network design problem in which one objective minimizes the cost of a path network and the second mazimizes the demand covered by facilities at the nodes of the path. Current, et al. (1987) replaced the second objective with one to minimize access cost for demand to reach a node on the path.

The models mentioned so far in this section are designed primarily to generate pareto optimal solutions to multiobjective facility location problems. That is, they are generating techniques. If preference weights on the objectives are known *a priori*, these models can be used to find the solution which optimizes a convex combination of the objectives based on these weights.

Another preference-based method employs a lexicographical ordering of the objectives. This approach optimizes a primary objective and then, from among the alternate optima for the primary objective, optimizes a secondary (or tertiary) objective. Daskin and Stern (1981) proposed a hierarchical objective approach to locating emergency medical services. Their primary objective is that of the set covering model, to which they append a secondary objective of maximizing a measure of backup coverage. Plane and Hendrick (1977) proposed a hierarchical objective location problem for locating fire stations in Denver. Their primary objective is also to minimize the number of required facilities, while their secondary objective is to maximize the number of existing facilities in the solution. Benedict (1983) outlines a number of other hierarchical objective covering models.

For reviews of multiobjective facility location problems, the reader is referred to Current, et al. (1990), and Erkut and Verter (1995).

3.6 Dynamic Location Models

Since the pioneering work of Manne (1961, 1967), researchers have been interested in dynamic location problems. As Ballou (1968) states: "the effect of the future time dimension cannot be neglected in location analysis." (p. 271) The basic models presented in section 3.2.3 have ignored time; that is, they are static. Dynamic models incorporate time. Current et al. (1998) define two categories of dynamic models: "implicitly" dynamic and "explicitly" dynamic. Implicitly dynamic models are "static" in the sense that all of the facilities are to be opened at one time and remain open over the planning horizon. They are dynamic because they recognize that problem parameters (e.g., demand) may vary over time and attempt to account for these changes in the facility location scheme generated. Examples of implicitly dynamic models include Mirchandani and Odoni (1979), Weaver and Church (1983), Drezner and Wesolowsky (1991), and Drezner (1995), which consider problems where demand and travel times change over time.

Explicitly dynamic models are those designed for problems where the facilities will be opened (and possibly closed) over time. Typically, explicitly dynamic models extend the basic, static models with the addition of temporal subscripts to the facility location and assignment variables and constraints linking these variables over time. Early examples of such problems include Roodman and Schwarz (1975), Wesolowsky and Truscott (1976), Schilling (1980), Van Roy, Erlenkotter (1982), and Campbell (1990). The decision to open and close facilities over time is related to changes in the problem parameters over time. Examples of parameters that might change include demand, travel time/cost, facility availability, fixed and variable costs, profit, and the number of facilities to be opened. Multiobjective approaches to dynamic problems also have been developed (e.g., Schilling, 1980; Gunawardane 1982, and Min 1988).

The interested reader is referred to Current et al. (1998), and Owen and Daskin (1998a) for reviews of dynamic location problems.

3.7 Stochastic Location Models

The basic facility location models presented in section 2 assume that the parameters of the problem are known with certainty. Many of the dynamic models discussed in the previous section assume that the changes over time are known with certainty. However, there is considerable uncertainty in most facility location problems. This is particularly true given the long life spans of most facilities. Demand, travel time, facility costs, and even distance may change. These changes are often random. Uncertain parameters which have been addressed in the literature include demand (e.g., Frank, 1966; Manne, 1961; Carbonne, 1974; Berman, 1985), travel time (e.g., Mirchandani and Odoni, 1979; Mirchandani, 1980; Berman and Odoni, 1982; Weaver and Church, 1983; Berman and LeBlanc, 1984; and Louveaux, 1986) the availability of the facility for service (e.g., Daskin, 1982, 1983; ReVelle and Hogan, 1989a, b; Marianov and ReVelle, 1992, 1996), and the number of facilities to be sited (Current, et al., 1998).

There have been four basic approaches to stochastic facility location problems. The first, approximates the uncertainty via a deterministic surrogate. For example, Bean, et al. (1992) formulated an equivalent deterministic problem by "replacing stochastic demand by its deterministic trend and discounting all costs by a new interest rate that is smaller than the original, in approximate proportion to the uncertainty in the demand." (p. S210) They cite seven articles since Manne (1967) that have used a similar approach.

The second approach develops chance constrained models (Chapman and White, 1974). For example, Daskin (1982, 1983) formulated a probabilistic extension of the maximal covering problem in which facilities are assumed to be busy with probability ρ. If busy, they cannot serve demand. The objective of these models is to maximize the number of demands that are covered by an available (i.e., not busy) facility. ReVelle and Hogan (1989a) formulated a similar model in which they maximized the number of demands that are covered at least b times, where b is the number of coverages needed to ensure that a demand is covered by an available facility with probability β. ReVelle and Hogan (1989b) formulated a set covering version of the problem which minimizes the number of facilities required to ensure that all demands are covered with probability β. Other articles using this approach include Marianov and ReVelle (1992) which incorporated multiple vehicle types housed at the facilities and Marianov and ReVelle (1996) which incorporated a M/G/s-loss queuing model to compute the minimum number of facilities needed to ensure coverage of a node within some minimum probability α. Daskin, Hogan and ReVelle (1988) summarize these and other related models.

The third approach explicitly accounts for the queuing interactions that occur in a spatially distributed queuing system with facilities at multiple locations in a network. Larson (1974) formulated a hypercube queuing model with 2^N states to account for all possible combinations of N facilities being available or unavailable. The resulting model has either 2^N linear equations or N non-linear equations in an approximate model (Larson, 1975), thus making it very difficult to embed in an optimization algorithm. Batta, Dolan and Krishnamurthy (1989), however, used the model to show that the implicit assumption of server independence in Daskin's expected covering model is often violated. Berman, Larson and Chiu (1985) used an M/G/1 queuing model to explore the location of a single facility on a network as a function of the demand intensity when demands could wait for service. At very low and very high demands, they showed that the facility would be located at the 1-median location; for intermediate demand intensities, alternate locations were shown to be optimal. When no queuing was permitted, they showed that the optimal location was always the 1-median.

The fourth approach utilizes scenario planning (van der Heijden, 1994; Vanston et al., 1977). Scenarios represent possible values for parameters that may vary over the planning horizon. One of the first applications of scenario planning to facility location problems was Sheppard (1974) which minimized the expected cost over all scenarios. Ghosh and McLafferty (1982) used scenarios to locate retail stores. Schilling (1982) extended the maximal covering location problem to incorporate scenarios by maximizing the number of covered demands over all possible scenarios in an EMS siting situation. In this model some facilities must be common to all scenarios, while others can be located in a scenario-specific manner. Daskin, Hopp and Medina (1992) demonstrated that this approach can lead to the selection of the worst possible sites under certain conditions. Serra and Marianov (1997) used scenarios to incorporate varying travel times and demand conditions over the course of a day. One of their objectives was to find locations that minimize the maximum average travel time over seven defined scenarios as well as locations that minimize the maximum regret over the scenarios. Carson and Batta (1990) used scenarios to describe demand conditions at different times of the day in an ambulance location problem. Jornsten and Bjorndal (1994) formulated an uncapacitated dynamic fixed charge facility location model using scenario planning to minimize the expected cost across all scenarios and time periods.

In recent years there have been several facility location articles which minimize regret or expected opportunity loss in scenario planning. Generally, regret in these articles is defined as the difference between the optimal solution (once the future is known) and the siting configuration selected when the future is not known. For example, Serra et al. (1996) incorporated a minimax regret approach in which demands vary over the scenarios. Serra and Marianov (1997) included a minimax regret objective for average travel times. Current, et al. (1998) considered the problem where demand, travel cost, and the number of facilities sited may vary over the different scenarios. Averbakh (1997) and Averbakh and Berman (1997a, 1997b, 1997c) have focused on the development of polynomial time algorithms for specially structured instances of these problems.

Daskin, Hesse and ReVelle (1987) extend the minimax regret approach by allowing the analyst or decision-maker to specify a reliability level, α. The model then endogenously picks a subset of the scenarios whose total probability of occurrence exceeds α and whose maximum regret is as small as possible. In other words, the model minimizes the maximum regret over an endogenously determined subset of the scenarios whose total probability is at least α. Clearly, if $\alpha = 1$, the model reduces to the standard minimax regret model. Owen (1998) formulates a variety of related models that do not necessitate the specification of scenario probabilities. These models can be solved effectively using evolutionary algorithms (Owen and Daskin, 1998b).

3.8 Solution Approaches for Location Models

As we have seen from the previous sections, discrete location models are generally constructed as mixed-integer linear programs. However, formulating an appropriate model is only one step in analyzing a location problem. Another (and often larger) challenge is identifying the optimal solution. Typically, the first approach to finding the optimal solution to such problems is to apply one of the well-known algorithms such as branch and bound or cutting planes. While these methods work on at least some instances of most location models, they are typically only useful on small problems. Realistically scaled location models can easily have thousands even hundreds of thousands of constraints and variables. Attempting a solution with these standard optimization methods will quite often consume unacceptable computational resources in terms of both computer memory and time and with no guarantee of success. The reason is that even the most basic location models are classified as NP-Hard (Garey and Johnson, 1979).

As a result, the location analyst must devise other methods to identify optimal solutions and, failing that, at least find very good solutions. A method of the latter type is known as a heuristic, which is an algorithm, that can find good solutions to a decision problem, but will not guarantee finding the optimal solution. In the remainder of this section, we will explore several of the most common solution approaches used by location analysts. Throughout, we will use the p-median model to describe the solution process. However, the methods discussed are generally applicable to a wide variety of location models.

Greedy heuristics When faced with selecting a subset of things (in the case of the p-median, facilities to open) that will optimize some objective, there are numerous tactics or "rules-of-thumb" that quickly suggest themselves. The most common is a sequential approach that begins by evaluating each site individually and selecting the one facility site that yields the greatest impact on the objective. That facility site is then fixed open. The location of the next facility is then identified by enumerating all remaining possible locations and choosing the site that provides the greatest improvement in the objective. Each subsequent facility is located in an identical manner. The method stops when the required number of facilities, p, have been sited.

For obvious reasons, this approach is known as a greedy heuristic. More specifically it is called *Greedy-Add* since there is a reverse approach known as *Greedy-Drop*. Greedy-Drop starts with facilities located at all potential sites, and then removes (drops) the facility that has the least impact on the objective function. We continue to drop facilities until p facilities remain.

Improvement heuristics While both the Greedy-Add and the Greedy-Drop heuristics are effective at identifying a feasible solution with modest computational effort, neither can be relied upon to consistently produce good solutions. Therefore, several different approaches have been developed that begin with a good (or at least feasible) solution and seek to improve upon it. Not surprisingly, these are known as improvement or search heuristics.

One of the earliest improvement heuristics is the neighborhood search algorithm (Maranzana, 1964). In this method, we begin with any feasible solution or specifically a set of p facility sites. Demand nodes are then assigned to their nearest facility. The set of nodes assigned to a facility constitutes a "neighborhood" around that facility. Within each neighborhood, the 1-median problem can be solved optimally by simply evaluating each potential site in the neighborhood and selecting the best. The facilities are then relocated to the optimal 1-median locations within each neighborhood. Then, if any facility sites are relocated, new neighborhoods can be defined and the algorithm is repeated. This cycle continues until there are no further changes in the facility sites or neighborhoods.

The most widely known improvement method was introduced by Teitz and Bart (1968). The basic idea is to move a facility from the location it occupies in the current solution to an unused site. Each unused location is tried in turn and when a move produces a better objective function value, then that relocation is accepted and we have a new (improved) solution. When an improved solution is obtained, the search process is repeated on the new solution. The procedure stops when no better solution can be found via this method. This heuristic is known as an "interchange", "exchange" or "substitution" approach, because it can be thought of as exchanging an open site with one of the unused sites. Although commonly used as an algorithm for the p-median problem, this approach has been found useful in innumerable facility location models.

While seemingly straightforward in concept, the exchange heuristic has a number of alternative approaches that can be used in implementing it. One, of course, is the process described above, where every time an exchange is found that yields a better solution, the search process is restarted and applied to improve this new solution. Alternatively, we could select the best solution after considering all possible moves for a given facility site, or even choose the best after all possible exchanges for all sites are examined. There are many other variations possible, and these often influence the computational speed of the heuristic. The most efficient implementation of the exchange algorithm was presented by Whitaker (1983). His "Fast Interchange" method is described in detail in Hansen and Mladenovic (1997).

One issue in using improvement heuristics is to decide how the initial solution is generated. An obvious choice is to use the result of another heuristic, such as one of the greedy heuristics mentioned earlier. However, since the interchange heuristic is relatively fast, many analysts have applied it to a series of randomly generated solutions, selecting the best solution among all of the local optima found as the one to be implemented.

The variety of local optima identified in repeated applications of an interchange heuristic often have an interesting characteristic the solutions are quite similar in the set of sites chosen for open facilities. Rosing and ReVelle (1997) used this characteristic in developing their Heuristic Concentration approach. In this methodology, the selected sites from a number of locally optimal solutions are used as a reduced set of potential facility sites. This reduced set produces a smaller model that can then be to be solved more efficiently using conventional optimization technology. In most search heuristics, each iteration of the search focuses on a "neighborhood" of the solution it is trying to improve (the current set of open facilities). Rosing and ReVelle, however, treat a neighborhood not as the set of nodes assigned to a facility, but rather as the set of solutions that can be examined around the current solution. For the interchange heuristic, the neighborhood is defined as that set of solutions that can be reached by a single exchange.

In a strategy designed to escape local optima, Hansen and Mladenovic (1997) present a "variable neighborhood" search algorithm for solving the p-median problem. In their approach, they extend the notion of neighborhood with a distance measure. That is, the neighborhood at a distance of k from a current solution is the set of solutions that can be obtained by moving k facilities. The algorithm performs an intensive local search (similar to the interchange algorithm outlined above) on the current solution until it settles in a local optima. It then diversifies the search by randomly selecting a solution from a neighborhood at a distance of k from the current best solution. The process continues, incrementing k, until some exogenously specified maximum value of k is attained. The algorithm compares very well with conventional heuristics as well as the enhancements provided by tabu search.

The problem with many search heuristics is that, instead of yielding the sought-after optimal solution, they become 'stuck' in local optima. More recently, researchers have sought to apply heuristics in a more thoughtful or "intelligent" manner. The strategy is to use what is known as a metaheuristic to guide the application of a core search heuristic. The intent is to help them break out of local optima and explore other regions of the solution space.

One of the earliest metaheuristics is Tabu search (Glover, 1989, 1990; Glover and Laguna, 1997; Hansen, 1986). The basic procedure employs "tabu" restrictions, which inhibit certain moves (exchanges). The tabu restrictions are generally implemented with a short-term memory function to make them time-dependent. That is, after a certain number of iterations, the moves are no longer tabu. There are also "aspiration criteria" which allow very good solutions to overcome any tabu status.

Designing tabu search heuristics involves defining what types of moves to restrict and the nature of the aspiration criteria and short-term memory to utilize. In addition to these features, most tabu search designs include other strategies such as a long-term memory function to diversify the search into other areas of the solution space. Examples in the literature of tabu search applied to location problems include: Hub location (Klincewicz, 1992; Skorin-Kapov and Skorin-Kapov, 1994), the location-routing problem (Tazun and Burke, 1999), multicommodity location (Gendron and Potvin, 1999), and the p-median (Rolland, et al, 1997).

Lagrangean relaxation When using any heuristic, we are trading off savings in solution time against the quality of the solution. While the heuristics discussed above often find good solutions to a variety of location problems and do so relatively quickly, it is difficult to evaluate the tradeoff since we have no way of knowing how far from optimality those solutions are. Without having the optimal value of the objective function available for comparison, we can sometimes approximate the difference between a heuristic's solution and the optimal solution by finding bounds. Worst-case bounds are one type that has been the focus of considerable research. Such bounds are the greatest distance from optimality the heuristic solution will be when solving the theoretically worst of all possible problem instances. Unfortunately, while they are theoretically precise, these bounds are often quite far from optimality (thus providing little insight). Fortunately, the average performance of heuristics is often far better than these worst-case bounds would indicate.

One of the primary attractions of the technique known as Lagrangean relaxation is that it provides both upper and lower bounds on the value of the objective function (Fisher, 1981). That is, we know the optimal objective function value lies between the value of the best feasible solution found and a value that it can be no better than. The difference between the bounds is known as the "gap."

Lagrangean relaxation replaces the original problem with an associated Lagrangean problem whose optimal solution will provide a bound on the objective function of the original problem. This is done by eliminating (i.e. relaxing) one or more of the constraints of the original model and adding these constraints, multiplied by an associated Lagrange multiplier, to the objective function. The idea is to relax constraints that will result in a relaxed problem that, given values of the multipliers, is much easier to solve optimally. The role of these multipliers is to drive the Lagrangean problem toward a solution that satisfies the relaxed constraints. Unfortunately, determining the values of the optimal Lagrangean multipliers is generally very difficult. In essence, the Lagrangean relaxation approach replaces the problem of identifying the optimal values of all of the decision variables with one of finding optimal or good values for the Lagrange multipliers. Most Lagrangean-based heuristics use a search heuristic to identify the optimal multipliers. The most common routine will be discussed below.

A major benefit of Lagrangean-based heuristics is that they generate bounds (i.e. lower bounds on minimization problems and upper bounds on maximization problems) on the value of the optimal solution of the original problem. For discussion purposes, we will limit ourselves to minimization problems like the four basic models presented in section 3.2. For any set of values for the Lagrangean multipliers, the solution to the Lagrangean model is less than or equal to the solution to the original model. Therefore, the Lagrangean solution is a lower bound on the solution to the original problem. However, all lower bounds are not created equal; the higher the bound, the smaller the gap in which the true optimal solution can be found.

The solution to the Lagrangean problem for any given values of the Lagrange multipliers will generally violate one or more of the relaxed constraints. Many Lagrangean based algorithms incorporate additional heuristics to convert these infeasible solutions to feasible ones. In this way, they can produce good solutions to the original model. The best feasible solution among those found by the procedure at any point, represents the upper bound on the value of the true optimal solution. The difference between the upper and lower bounds is referred to as the "gap." If the gap reaches zero (or some minimum value based on the integer properties of the model) then we have found the optimal solution. Otherwise, when the gap gets sufficiently small (e.g. less than 1%), the analyst may stop the procedure and be satisfied that the current best solution is within 1% of optimality.

The primary challenge in applying Lagrangean relaxation is in selecting which constraints to relax. The goal is to end up with a relaxed problem that can be solved very easily, and result in good lower bounds. Since the relaxed model may have to be solved hundreds or thousands of times in the search for the best multiplier values, the ease of solution is critical to the success of the approach. Ideally, the relaxed problem ought to be solvable by inspection or by a simple sorting the objective function coefficients.

An excellent tutorial on the general application of Lagrangean relaxation can be found in Fisher (1985). An exposition of its use in location models is in the text by Daskin (1995). To make our discussion of Lagrangean relaxation more concrete, consider its application to the p-median problem, which follows.

Recall the p-median formulation given above. Suppose we relaxed the constraints (3.22). When these constraints are multiplied by Lagrange multipliers, we obtain the following model:

$$\max_{\lambda} \min_{x,y} L = \sum_{i \in I} \sum_{j \in J} h_i d_{ij} y_{ij} + \sum_{i \in I} \lambda_i (1 - \sum_{j \in J} y_{ij})$$
$$= \sum_{i \in I} \sum_{j \in J} (h_i d_{ij} - \lambda_i) y_{ij} + \sum_{i \in I} \lambda_i \tag{3.46}$$

subject to:

$$\sum_{j \in J} x_j = p \tag{3.47}$$
$$y_{ij} - x_j \leq 0 \qquad \forall i \in I, j \in J \tag{3.48}$$
$$x_j \in \{0,1\} \qquad \forall j \in J \tag{3.49}$$
$$y_{ij} \in \{0,1\} \qquad \forall i \in I, j \in J \tag{3.50}$$

Note that the objective function L in (3.46) is minimized with respect to the original (location and assignment) variables (x_j and y_{ij}, respectively) and is maximized with respect to the Lagrange multipliers (λ_i). The largest value of L over all iterations of the procedure represents a lower bound on the objective function for the original p-median model.

As we discussed earlier, the overall approach is to, iteratively, 1) set values of the multipliers, 2) solve the Lagrangean model and 3) adjust the multipliers. To do this effectively, we must construct a procedure which efficiently solves for x_j and y_{ij} when we are given values for the multipliers. In (3.46), when the λ_i are known, the last term is a constant and can be ignored. The objective is thus seen to be a direct function of the y_{ij}, but, as in many location problems, if we have the values of the facility siting variables (x_j in this case), we can derive the values of the remaining variables. That is, when $x_j = 0$, the y_{ij} must $= 0$, and there is no impact on the objective. When $x_j = 1$, then the y_{ij} can be either $= 0$ or 1. To minimize the first term of the objective function we would like to set any $y_{ij} = 1$, if the associated $(h_i d_{ij} - \lambda_i)$ is < 0. The remaining y_{ij} can be 0. Thus, to see the overall influence on the objective function of setting a particular $x_j = 1$, we calculate V_j, which is defined as $\sum_i \min\{0, h_i, d_{ij} - \lambda_i\}$. If we rank order the V_j values from smallest to largest, we can identify the first p values and set the corresponding x_j variables $= 1$. Then $y_{ij} = 1$, if $x_j = 1$ and $(h_i d_{ij} - \lambda_i)$ is < 0. This procedure yields the minimum objective function value for a given set of λ_i.

The Lagrangean solution found above may not be feasible for the original p-median model since the constraint we relaxed may be violated (demand nodes i may not be allocated to only one facility.) We can, however, develop a feasible solution by simply assigning the demand points to their nearest open facility. The resulting value of the p-median objective function represents an upper bound on the optimal solution. The best of the feasible solutions found over all iterations would also have the best (lowest) upper bound.

The final task is to modify the multipliers based on the solutions just obtained. (See Bazaara and Goode, 1979, for a survey of various methods.)

A common approach is subgradient optimization. Briefly, after any iteration t, the Lagrangean multipliers are updated as follows:

$$\lambda_i^{t+1} = \max\left\{0, \lambda_i^t - T^t\left(\sum_j y_{ij}^t - 1\right)\right\} \quad \forall i$$

where

$$T^t = \Delta\left[\frac{\overline{Z} - Z_l^t}{\sum_i\left(\sum_j y_{ij}^t - 1\right)^2}\right]$$

t = an index of iterations =1,2,3, ...
$\overline{Z}$ = best (smallest) feasible solution value
Z_L^t = Lagrangean value from current iteration (t)

Typically, Δ is initially set to 2. Then, if there has been no improvement in the lower bound over some pre-specified number of iterations (e.g. 200), then Δ is replaced with $\frac{\Delta}{2}$. All λ_i are sometimes then reset to the values used to obtain the current best lower bound. Since the procedure is not guaranteed to terminate at optimality, it is usually stopped after reaching a certain number of iterations (e.g. 1000) or when Δ becomes sufficiently small. If the gap is not sufficiently small at the end of the Lagrangean procedure, the entire process can be embedded in a standard branch and bound algorithm.

For a numerical example of this procedure, the interested reader is directed to Daskin (1995).

3.9 Conclusions

Discrete network location problems have attracted the attention of both researchers and practitioners for several decades due to (1) the practical value of such models in both private and public-sector decision-making contexts, (2) the ever-present need and desire to incorporate increasingly realistic constraints and objectives into the models, (3) the challenges associated with solving the models and (4) the ability of the basic formulations to represent important decision-making issues in contexts far removed from a facility location environment. These four factors continue to this day and are likely to be present for years to come. We anticipate continued research, development and application of the models we have outlined.

Several areas of potential research and development warrant particular attention. Since location decisions are inherently strategic and long-term in nature, they often entail balancing conflicting objectives held by multiple constituents. This suggests four important developmental needs. First, multi-objective modeling will become increasingly important. Since most of the single objective problems are NP-hard, their multi-objective extensions are also NP-hard. Thus, it will be important to develop efficient and effective heuristic algorithms for identifying non-inferior solutions for multi-objective location problems. Methods for evaluating the quality of the solutions attained by such algorithms will need to be developed as well.

Second, long-term location decisions impact and constrain the shorter-term tactical and operational decisions made after facilities are in place. Such decisions include production scheduling and planning, inventory management and vehicle routing. Thus, the development of models that integrate facility location decisions with approximations of such tactical and operational decisions will be another important area for future work. One promising approach to such integrated problems is stochastic programming (Birge and Louveaux, 1999).

Third, as the famous philosopher (and baseball player) Yogi Berra once observed, "It's tough to make predictions, especially about the future." Thus, it is impossible to predict the future conditions under which facilities will operate. Therefore, it is important that models and solution algorithms be developed that account for future uncertainty explicitly and that identify solutions that are robust with respect to this uncertainty (Kouvelis and Yu, 1996).

Finally, public and private sector facilities interact with other parts of the infrastructure (e.g., highways, airports, rail lines, ports, production facilities and equipment). Embedding facility location modeling approaches in more general network design algorithms will also be an important and challenging area for future work. For example, in locating production facilities, managers need to consider not only the location of the plants, but the equipment that will be housed in the plant (ReVelle and Laporte, 1996).

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