The 3-D elasticity model is of great importance, since it is our highest order hierarchical model assuming linear elastic behavior. Therefore, it provides a conceptual reference model from which other specific and frequently more effective models can be derived. The solutions of these models can then be compared with the solution of the full 3-D model giving good insight into the modeling of physical problems – all within the objective of using hierarchical mathematical modeling in engineering analysis.

Before we close this chapter, we present two illustrative cases of analytical solutions of the 3-D elasticity model which are important for structural analysis: the pure bending of a prismatic bar of rectangular cross-section (given in Example 3.7) and the uniform torsion of a prismatic bar (given in Section 3.6).

Example 3.7

Consider the undeformed configuration of a solid as shown in Figure 3.60.

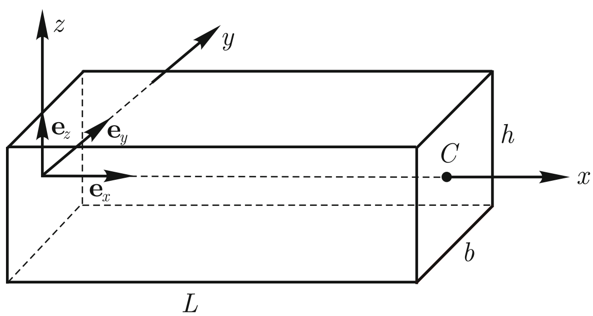

Fig. 3.60. Definition of the solid under study

Let the displacement field be given by

$$u = -\frac{M}{EI}xz \tag{3.119}$$

$$v = \nu\frac{M}{EI}yz \tag{3.120}$$

$$w = \frac{M}{2EI}\left[x^2 + \nu\left(z^2 - y^2\right)\right]. \tag{3.121}$$

where M is a positive constant, E and ν are Young's modulus and Poisson's ratio of the material and I is the moment of inertia about the y axis, $I = \frac{bh^3}{12}$ (see Section 4.2.2).

(i) Find the stress field associated with the given displacement field.

(ii) Identify the problem for which the given displacement field represents the exact solution of the 3-D elasticity model.

Solution

(i) The strains are given as

$$\begin{aligned}
\varepsilon_{xx} &= \frac{\partial u}{\partial x} = -\frac{M}{EI}z \\
\varepsilon_{yy} &= \frac{\partial v}{\partial y} = \nu\frac{M}{EI}z \\
\varepsilon_{zz} &= \frac{\partial w}{\partial z} = \nu\frac{M}{EI}z \\
\varepsilon_{xy} &= \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0 \\
\varepsilon_{xz} &= \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = \frac{1}{2}\left(-\frac{M}{EI}x + \frac{M}{EI}x\right) = 0 \\
\varepsilon_{yz} &= \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = \frac{1}{2}\left(\nu\frac{M}{EI}y - \nu\frac{M}{EI}y\right) = 0.
\end{aligned}$$

The stresses can be obtained using the generalized Hooke's law (see equation (3.112))

$$\tau_{xx} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\left[\varepsilon_{xx} + \frac{\nu}{1-\nu}(\varepsilon_{yy} + \varepsilon_{zz})\right].$$

Noting that for our problem

$$\varepsilon_{yy} = \varepsilon_{zz} = -\nu\varepsilon_{xx} \tag{3.122}$$

we obtain

$$\tau_{xx} = E\varepsilon_{xx} = -\frac{M}{I}z.$$

We also have

$$\begin{aligned}
\tau_{yy} &= \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\left[\varepsilon_{yy} + \frac{\nu}{1-\nu}(\varepsilon_{xx} + \varepsilon_{zz})\right] \\
\tau_{zz} &= \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\left[\varepsilon_{zz} + \frac{\nu}{1-\nu}(\varepsilon_{xx} + \varepsilon_{yy})\right]
\end{aligned}$$

and considering (3.122) we obtain

$$\tau_{yy} = \tau_{zz} = 0.$$

Of course, since $\gamma_{xy} = \gamma_{xz} = \gamma_{yz} = 0$, we have

$$\tau_{xy} = \tau_{xz} = \tau_{yz} = 0.$$

This completes part (i) of the problem.

(ii) To identify the 3-D problem actually solved, we need to determine which fields of body forces and surface tractions are in equilibrium with this stress field. Considering the differential equilibrium equations and introducing this stress field, we have

$$\begin{aligned}
\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + f_x^B &= 0 \Rightarrow f_x^B = 0 \\
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + f_y^B &= 0 \Rightarrow f_y^B = 0 \\
\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + f_z^B &= 0 \Rightarrow f_z^B = 0.
\end{aligned}$$

Therefore, there should be no body forces. Since the only nonzero stress component is τ_{xx}, the surface tractions should be zero on all four lateral surfaces. Further, for the surface defined by $x = L$

$$\mathbf{Tn} = \mathbf{f}^S \Rightarrow \tau_{xx}\mathbf{e}_x = \mathbf{f}^S \Rightarrow \mathbf{f}^S = -\frac{M}{I} z\mathbf{e}_x$$

and for $x = 0$

$$\mathbf{Tn} = \mathbf{f}^S \Rightarrow \text{-}\tau_{xx}\mathbf{e}_x = \mathbf{f}^S \Rightarrow \mathbf{f}^S = \frac{M}{I} z\mathbf{e}_x.$$

If we reduce the surface tractions at the section given by $x = L$ to its center of gravity we obtain for the force resultant

$$\mathbf{R} = \int_A \mathbf{f}^S \, dA = \int_A -\frac{M}{I} z\mathbf{e}_x \, dA = 0$$

and for the moment resultant with respect to the center of the section, represented by $\mathbf{C}$

$$\begin{aligned}
\mathbf{M_C} &= \int_A (y\mathbf{e}_y + z\mathbf{e}_z) \times \mathbf{f}^S \, dA \\
&= \int_A \left(\frac{M}{I} yz\mathbf{e}_z - \frac{M}{I} z^2\mathbf{e}_y \right) dA = -\frac{M}{I} \int_A z^2 dA \; \mathbf{e}_y = -M \; \mathbf{e}_y.
\end{aligned}$$

Hence the applied surface tractions at $x = L$ correspond to a bending moment, see Figure 3.61. Analogously, for the end section defined by $x = 0$ we obtain $\mathbf{R} = \mathbf{0}$ and $M\mathbf{e}_y$ leading to a self-equilibrated force system.
Note also that the displacement field given in (3.119) – (3.121) satisfies $u = 0$ for $x = 0$, and $u = v = w = 0$ for $x = y = z = 0$.

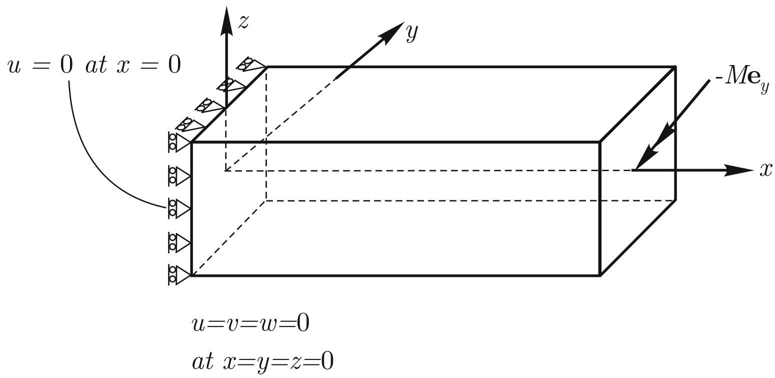

Fig. 3.61. Problem summary. Body forces are zero

We can conclude that the displacements given in (3.119) to (3.121), the derived strains and stresses correspond to the exact solution of the cantilever beam subjected to a pure bending moment at the tip, as summarized in Figure 3.61, as long as the bending moment at $x = L$ is introduced by the surface traction field $\mathbf{f}^S = -\frac{M}{I} z \mathbf{e}_x$ and the displacement restrictions at the "built-in section" $x = 0$ are as shown in Figure 3.61.

In Figure 3.62, we show the deformed and undeformed configuration of the solid. We see that a line parallel to the y axis in the undeformed configuration is deformed into an arc, whose curvature is opposite to the curvature of the deformed axis. This transverse curvature is known as *anticlastic curvature* and it is due to the Poisson effect.

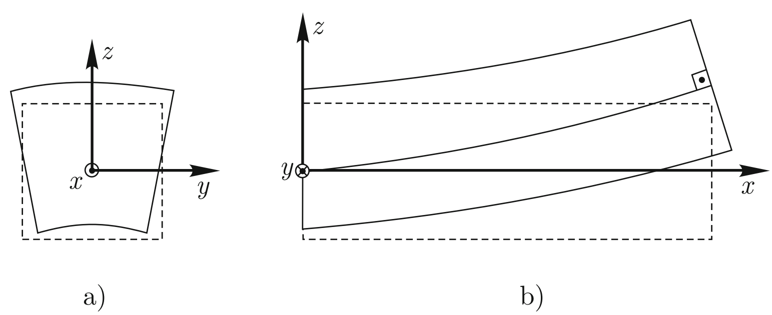

Fig. 3.62. Deformations for selected planes. a) Plane given by $x = 0$; b) Plane given by $y = 0$. The magnitude of the displacements is chosen for visualization purposes; the shown displacements are much larger than those for which the linear model would be adequate

□

3.6 Torsion of a prismatic bar

We present below the exact solution of a 3-D elasticity problem – the Saint Venant torsion solution for a prismatic bar – which is of considerable practical importance. It is also a classical example of the so-called semi-inverse method for deriving solutions. In this method, some assumptions on the functional form of the displacements are made, either motivated by experimental observations or by intuition. These displacement assumptions are the starting point towards obtaining a solution.

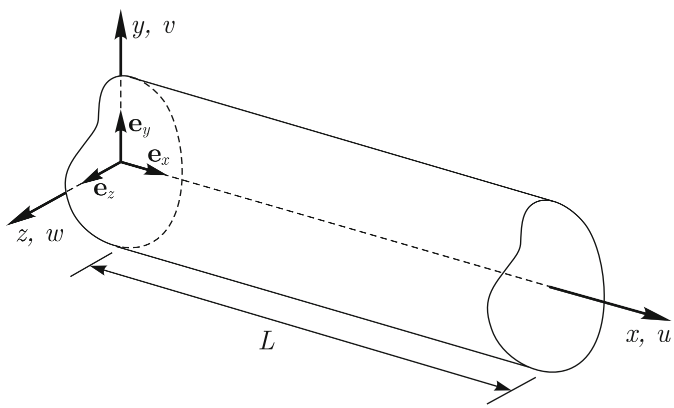

Fig. 3.63. Generic prismatic bar which will be subjected to torsion. Solid section with no holes

Consider a prismatic bar with a generic cross-section as shown in Figure 3.63. Suppose that the bar is subjected to self-equilibrated torsional moments at the end sections, *i.e.*, $\mathbf{M}_t = M_t\mathbf{e}_x$ at $x = L$ and $-\mathbf{M}_t$ at $x = 0$. It is an experimental observation that for a bar subjected to such loading the cross-sections rotate as rigid bodies in their own plane (see Example 3.5 for the in-plane displacements). However, these sections do not remain plane, they display some warping. Motivated by these observations, we will seek a solution of the 3-D elasticity mathematical model of this problem using the following displacement assumptions

$$u = \theta'\psi(y,z) \tag{3.123}$$

$$v = -\theta' xz \tag{3.124}$$

$$w = \theta' xy \tag{3.125}$$

where θ' is the rate of rotation of the cross-sections with respect to the x axis which is assumed to be constant. Hence, denoting by $\theta(x)$ the angle of rotation of a generic section, we have