Equation (3.65) is referred to as the additive decomposition of the displacement gradient $\nabla \mathbf{u}$ into its symmetric part – the infinitesimal strain tensor – and into its skew part $\mathbf{W}$ – the infinitesimal rotation tensor.

Example 3.5

Consider the cylinder shown in Figure 3.32. Suppose that the transverse sections rotate without deformation in the plane yz around the cylinder's axis by an angle $\theta(x)$ with the constant rate of rotation $\frac{d\theta}{dx} = \alpha$. Calculate, assuming infinitesimal displacement conditions:

(i) The displacement field.
(ii) The strain tensor within the cylinder.

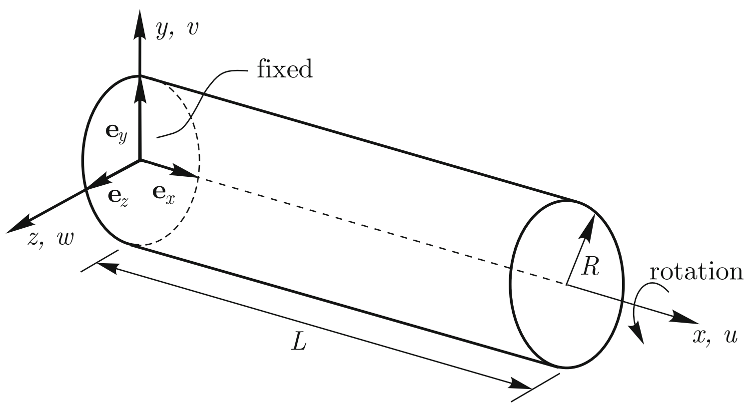

Fig. 3.32. Cylinder under study

Solution

(i) We obtain by integration

$$\theta(x) = \alpha x + C.$$

Since the rotation at $x = 0$ is prevented

$$\theta(0) = 0 \ \Rightarrow \ C = 0$$

and therefore the rotation of a generic section is given by

$$\theta(x) = \alpha x.$$

Since the section rotations are infinitesimal, we can use directly the results derived in Example 3.4. Therefore, considering $x_1 \equiv y$, $x_2 \equiv z$, $x_3 \equiv x$, we obtain

$$
\begin{aligned}
u &= 0 \\
v &= -\theta z = -\alpha x z \\
w &= \theta y = \alpha x y.
\end{aligned}
$$

(ii) The strain components are

$$
\begin{aligned}
\varepsilon_{xx} &= \frac{\partial u}{\partial x} = 0, \quad \varepsilon_{yy} = \frac{\partial v}{\partial y} = 0, \quad \varepsilon_{zz} = \frac{\partial w}{\partial z} = 0 \\
\varepsilon_{xy} &= \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = -\frac{1}{2}\alpha z \\
\varepsilon_{xz} &= \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = \frac{1}{2}\alpha y \\
\varepsilon_{yz} &= \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = \frac{1}{2}\left(-\alpha x + \alpha x\right) = 0.
\end{aligned}
$$

Therefore

$$
\mathbf{E} = \begin{bmatrix} 0 & -\frac{1}{2}\alpha z & \frac{1}{2}\alpha y \\ -\frac{1}{2}\alpha z & 0 & 0 \\ \frac{1}{2}\alpha y & 0 & 0 \end{bmatrix}
$$

and the engineering shear strains are

$$
\begin{aligned}
\gamma_{xy} &= -\alpha z \\
\gamma_{xz} &= \alpha y \\
\gamma_{yz} &= 0.
\end{aligned}
$$

In Figure 3.33 a geometrical interpretation of γ_{xz} is given. Referring to Figure 3.33 we can calculate γ_{xz} for a point of coordinates x, $y = R$, $z = 0$ as the ratio

$$
\gamma_{xz} = \frac{d\theta R}{dx} = \alpha R
$$

which is in accordance with the derived expression.

□

3.3 Stresses

In Section 2.1.3 we introduced the concept of stress, see Figure 2.3. In this figure, a field of forces per unit area – the field of stresses – is acting on the internal surface of the part $\Delta^t V$.