1. Mathematical models and the finite element solution. Hierarchical modeling

The objective of this chapter is to introduce a modern approach for the modeling of structures (and in fact any physical system). The hierarchical modeling process is the central concept. We discuss the need for such a process, detail it in a demonstrative problem and examine the important benefits associated with its usage in engineering. Mathematical modeling and finite element methods are introduced as natural ingredients of the hierarchical modeling procedure.

While the discussion in this chapter and in those to follow focuses on the analysis of solids and structures, we also point out that the basic ideas of the hierarchical modeling approach are directly applicable to the analysis of fluid flows, to the solution of multi-physics and multi-scale problems, and in fact to the analysis of any problem encountered in engineering and in the sciences.

1.1 Introductory remarks

Historically, the design of engineering structures has progressed through a number of stages. Engineering structures have been built much before any rational method of analysis was available. Then a structural design was based on trial and error and as experience was gathered for a particular type of structure, practical guidelines were developed which helped new designs.

The development of mathematical methods and experimental approaches have drastically altered and improved structural design methodologies. Nowadays, the design process is aided by several factors: a vast experience with previous designs of similar structures, modern methods of analysis and, of course, still experiments.

It is easy to recognize that structural design is a vast field, and involves many architectural, engineering and societal issues that we encounter in the design of bridges, buildings, airplanes, motor cars, ships, bio-medical devices, any household appliances, power plants, jet engines, and so forth. However, in order to focus the discussion on the issues that we would like to address in this chapter, we assume that a design for a structure has been proposed. Of course, many issues should have been addressed before a structural design is reached. The major considerations are linked to structural safety, operational requirements, construction/manufacturing feasibility and cost. During this design process, the most important challenge that the structural engineer faces is to give reliable answers to the following questions:

- Is the structure safe to be used?
- Will the structure perform in operation as required?

Of course, additional questions arise depending on the type of structure considered. Nevertheless, the questions above highlight the main concern of a structural engineer.

If we consider the broad spectrum of engineering structures, to build prototypes and small-scale models for experiments can be very inefficient. Of course experimental structural analysis is a very important field and physical experiments in the laboratory or on prototype structures constitute usually an important step to assure that a design is safe and functional. However, to reach a satisfactory design it is, in many industries, common practice today, to perform finite element analyses of proposed designs – prior to actually building the structure or a physical (laboratory) model thereof.

1.2 Mathematical models

This book is concerned with methods that permit to predict the behavior of a structure without requiring that the structure has already been built. Even if the structure already exists, such methods do not demand physical experimentation with the structure. It is important to appreciate that to have a method with this characteristic constitutes a major step since it implicitly assumes that we are able to construct an *abstract representation* of a structure which can capture its structural behavior. In other words, for a given structure we need to represent its geometry, materials, the loads acting on it and the governing relations linking these quantities, all of which allows us to simulate its structural behavior. In general, we are interested in predicting the deformations and internal forces of the structure subjected to various loadings and whether it would "break" or not.

The abstract representation of the structure which allows us to make predictions of its behavior is called a *mathematical model* of the structure at hand. We summarize the process described above in Figure 1.1. Note that we intentionally refer to a physical problem in this figure to place mathematical modeling into a wide perspective. Our physical problem will be a structural problem and its modeling constitutes the main objective of this book. However, the discussion that we will undertake is also valid in the broad context of predicting the behavior of other, very general, physical systems, for example involving also fluids and the interactions of structures and fluids.

To further consider this point, the structures that we primarily think of analyzing are of course the traditional structures like bridges, motor cars,

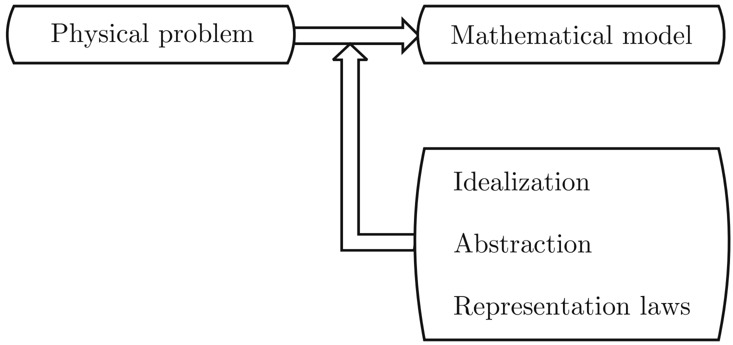


Fig. 1.1. Concepts involved in the construction of a mathematical model

airplanes, that we see around us in everyday life. We are aiming to predict the stiffness and strength characteristics of these structures. However, our discussion is also directly applicable to the analysis of structures of much smaller and much larger scales, see e.g. Bathe, 2005. For example, on the much smaller scale, we may design a new composite material based on micromechanical considerations, or analyze, on the nano-scale, cells moving in the human body. On the much larger scale, geological problems like the eroding of beaches and the construction of tunnels through mountains may be analyzed, or in weather forecasting, the development and movement of storms and their effect on structures may be simulated. It is apparent that all these "structures" can be exposed to important multi-physics events, that may involve fluid flows, and thermal, chemical and electro-magnetic effects. In such cases, the full coupling between very different media, that need to be modeled at different scales, might need to be considered.

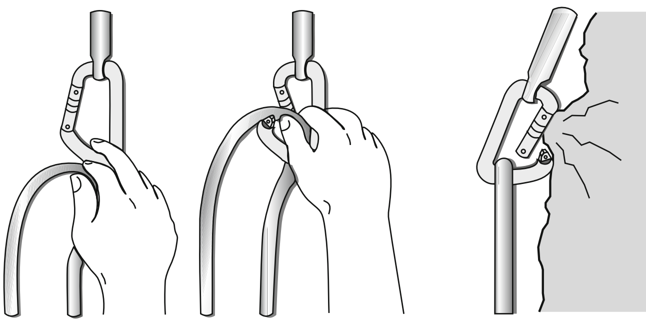
While we do not discuss in this book the analysis of fluid flows, multiphysics and multi-scale problems, it is important to note, though, that the basic concepts of hierarchical modeling – as we expose them in detail in this book – are directly applicable to the analysis of such problems as well. Hence the basic modern approach of analysis presented here is very general and indeed can be followed in any field of analysis.

In order to further develop our discussion of modeling, we now will focus on an example analysis that we use to exemplify the basic concepts of hierarchical modeling.

1.2.1 A demonstrative problem – a carabiner

A photograph of the chosen "structure" is shown in Figure 1.2. It is called a carabiner and it is used in rock climbing. A schematic representation of its usage is shown in Figure 1.3a. From a structural point of view, the carabiner should transfer the load between ropes without "breaking" and without deforming "too much", *i.e.*, the deformations must not jeopardize its performance. Therefore, an important objective for the mathematical modeling of this structure is to determine the magnitude of the load that can safely be transferred by the carabiner.

Fig. 1.2. Photograph of a carabiner



a) b)

Fig. 1.3. Schematic representation of the carabiner usage and of a situation that makes it to work open

Let us focus on the situation when the carabiner works open which could arise as schematically shown in Figure 1.3b. In order to discuss the modeling of the carabiner we need to use some structural mechanics terminology and concepts but do so in an introductory manner. We will revisit this problem in much more detail in Chapter 7.

The task at hand is to construct a mathematical model for the carabiner which allows us to predict the carabiner's structural response. Hence, there are some choices to be made that, although interconnected, can be organized as follows:

- How to represent the loads.
- How to represent the restrictions to motion (the displacement boundary conditions).
- How to represent the actual structure.

These are addressed below.

How to represent the loads

The load is transferred from the lower rope to the carabiner by the contact between the rope and the carabiner. A simplified way of modeling this action is by a concentrated load applied to the carabiner whose magnitude represents the load applied by the rope.

How to represent the restrictions to motion

We note that the motion is not clearly restricted at any specific point of the structure. Actually, the opposing ropes and the carabiner find an equilibrated position. Hence to restrict the motion we may consider the final equilibrated position. At one side the load is imposed as explained above and at the opposite side we prevent the motion. Of course, the reactions associated with restricting the motion equilibrate the applied load, see Figure 1.4.

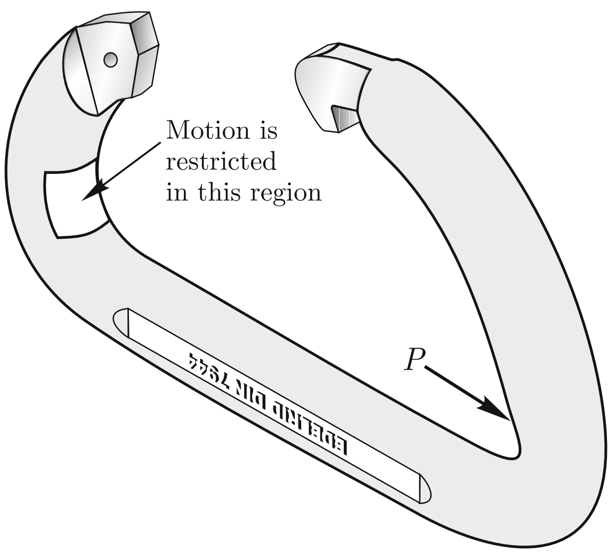


Fig. 1.4. Model representation of loading and boundary conditions

We note that since we are considering the carabiner to be open, Figure 1.4 shows only the part which is structurally relevant.

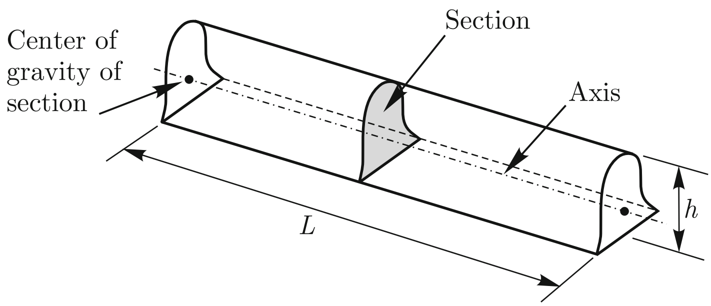


Fig. 1.5. A typical bar obtained by extruding a plane section through a straight line, normal to the plane of the section. The line, or axis, passes through the plane's center of gravity

How to represent the actual structure

In the first instance we may choose to represent the carabiner by an assemblage of straight bars. As we discuss in Chapter 4, the bar model corresponds to a structural theory in which the behavior of long solids is modeled. Figure 1.5 shows a typical bar. The solid of Figure 1.5 is called a bar when $L >> h$ (typically $\frac{1}{100} \leq h/L \leq \frac{1}{10}$). The bar model relates the loading applied on the bar to the displacements and internal forces/stresses of the bar. Of course, in the curved region, we need several straight bar elements to represent, approximately, the geometry. In Figure 1.6 we show a bar model of the carabiner. In this figure only the bar axis is represented and there is a straight bar element between each of two consecutive diamond symbols. The loading is also shown and the restriction to the movement is modeled by preventing the section of the bar at B to either displace or rotate. The solution gives us the displacements, rotations and stresses at each point of the model.

1.2.2 The case for hierarchical modeling

Suppose that we have limit values for the displacements/rotations and for the stresses, referred to as variables, below which we are sure that the carabiner considered in Section 1.2.1 would well function and not break. Then, the question that naturally arises is: "How accurate are the calculated values obtained by solving the model of Figure 1.6 for these variables?" Of course, we do not know the actual values of these variables in the actual physical problem and we need to answer this question based on mathematical modeling *alone*.

The reader might appreciate that several assumptions were made in the definition of the mathematical models. Besides imposing the loading and the displacement boundary conditions in a simplified manner, possibly the strongest assumption is to represent the carabiner as an assemblage of straight bars. Clearly, these assumptions will affect the accuracy of the predictions obtained for the variables when compared to the actual situation in a

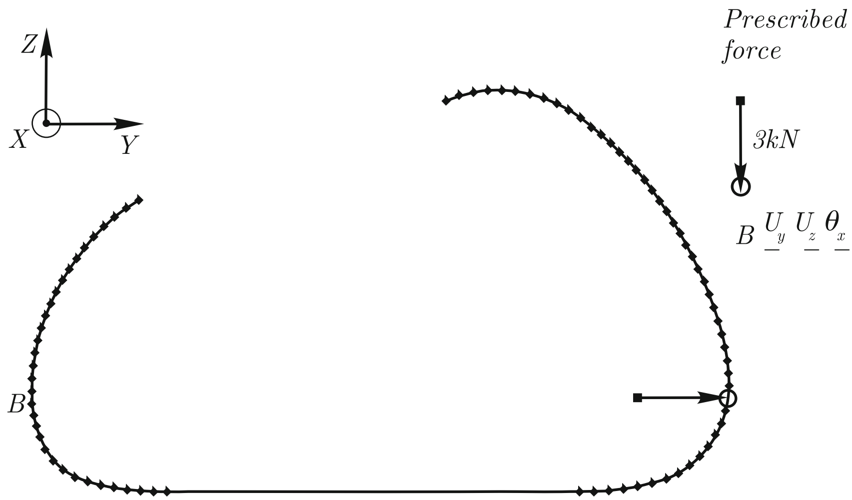


Fig. 1.6. Straight bar model in Y-Z plane. The material of the carabiner is aluminum with $E = 70$ GPa, $\nu = 0.33$

laboratory experiment (or in an actual rock climbing event). It is the purpose of hierarchical modeling to obtain good accuracy in the analysis results, and thus reliable analysis data for the design of the structure.

Let us introduce the hierarchical modeling process using the same demonstrative problem – the carabiner, see also Chapter 1 of Bathe, 1996, and Bathe, Lee and Bucalem, 1990.

1.2.3 Demonstrative hierarchical modeling example – a carabiner

The essence of hierarchical modeling is to set up and solve additional – and more accurate models – of the structure considered. For the carabiner, we can set up another more accurate mathematical model – a second model – by considering, for example, a curved bar structural theory. For such a theory the bar axis does not need to be straight and can be represented by a curve. This second model would be constructed by an assemblage of straight and curved bars. By representing the regions of the carabiner which are actually curved by curved bars we should arrive at a more accurate representation of the structural behavior of the carabiner. Of course, the curved bar structural theory is more complex and, hence, the model solution is more difficult to obtain.

The second model using curved bars is somewhat natural because there are curved parts in the carabiner. This second model is an example of the general and important fact that for a given physical problem there is not only one mathematical model to represent it. Indeed, there are always a number of mathematical models for the same physical problem. This idea is summarized in Figure 1.7, where it is shown that for a physical problem we can actually consider a sequence of mathematical models. The model number indicates an hierarchical order, *i.e.*, model 2 represents, through its assumptions, more phenomena of the physical problem than model 1. Therefore it leads to more complete predictions. This is always the case as we increase the index number of our model.

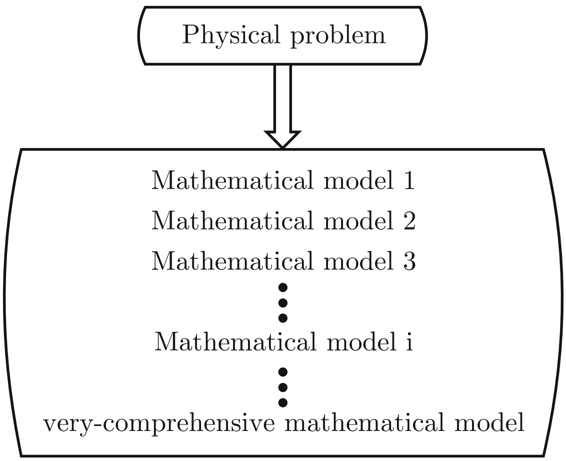


Fig. 1.7. Sequence of mathematical models for a physical problem

We can think – at least conceptually – of the highest order model as a "very-comprehensive mathematical model". This model would be the most precise and complete representation of the physical problem at hand. The very-comprehensive mathematical model is not to be actually solved but to provide a conceptual reference to which we compare the lower-order models.

For the carabiner problem, we have already described models 1 and 2. Let us consider, for illustrative purposes, a few higher-order models in the sequence.

If we evaluate the ratio of the radius of curvature to the thickness of the bars, we recognize that the curved bar model can only approximately represent the true 3-D behavior in the curved regions. Replacing the curved bar model by the 3-D elasticity model leads us to model 3.

A fourth model is motivated by evaluating the order of the displacements of the carabiner. Considering the material of which the carabiner is made (aluminum) and a typical design load (3 kN), deformed and undeformed configurations corresponding to the solution of the assemblage of straight bars are shown to scale in Figure 1.8, *i.e.*, without magnification. In this figure the solid line represents the undeformed bar axis, whereas the dotted line represents the same axis for the deformed configuration. We see that the

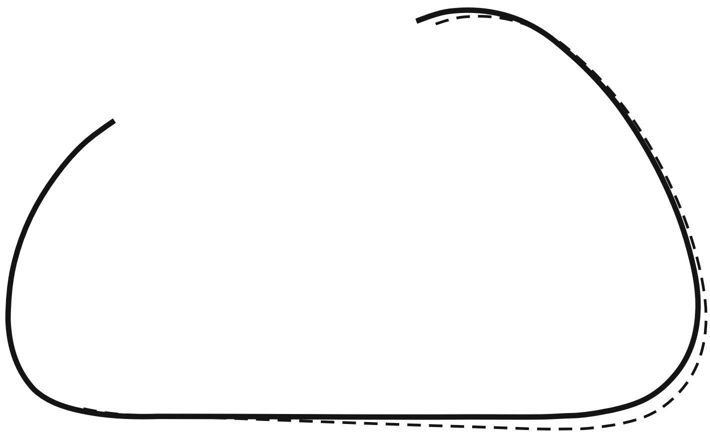


Fig. 1.8. Deformed (dashed line) and undeformed (solid line) configuration for the bar model

displacements are quite significant when compared to the dimensions of the structure. Hence, the modeling would more accurately represent the actual physical situation by considering that the geometry of the structure changes as the load is increased. This additional consideration would give rise to a geometrically nonlinear model (model 4).

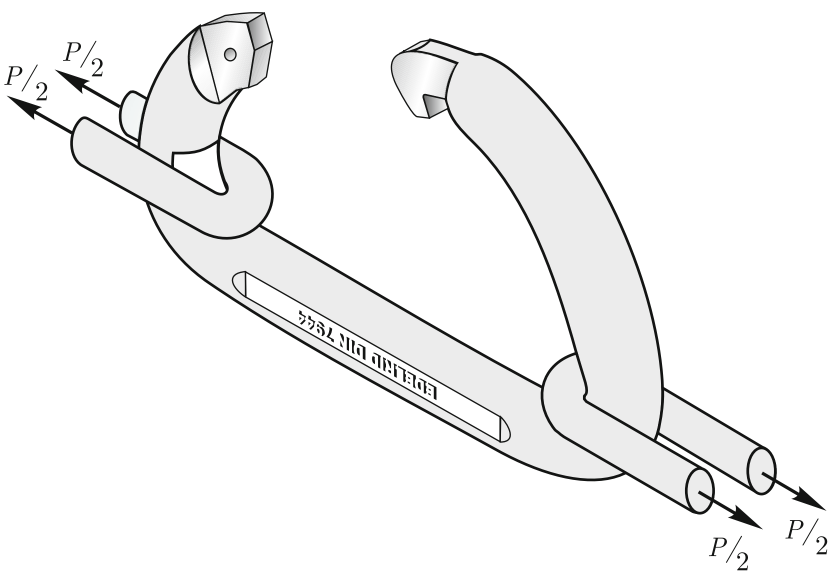


Fig. 1.9. Model of the carabiner with ropes included

We note that the modeling improvements discussed so far lead to a better description of the structural behavior of the carabiner when the models are all subjected to the loading and the displacement boundary conditions in Figure 1.4. Therefore, a fifth model could be established to improve the representation of the loads and boundary conditions. For example, we might consider the ropes as part of the model and represent the contact between the ropes and carabiner surfaces. The load would be introduced at the end of the ropes as schematically shown in Figure 1.9. This model is referred to as model 5 and its solution would be much more difficult (and therefore more expensive) to obtain than those of the former models. We summarize in Figure 1.10 the modeling process for the carabiner in which we show the sequence of models discussed so far.

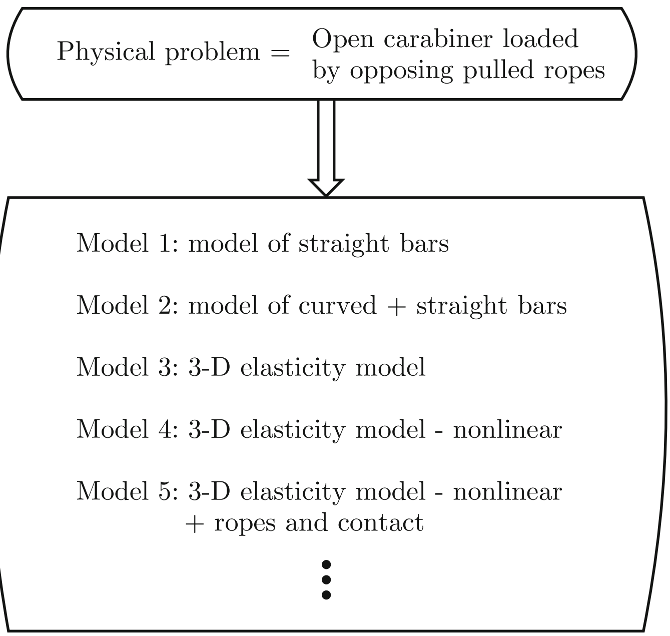


Fig. 1.10. Sequence of hierarchical models for the carabiner

Next we need to return to the question which we posed for model 1: How accurate are the predictions of the model for the variables of interest when compared to those of a laboratory test or a field event?

Suppose that we solve models 1 to 3 for the variables sought. In order to fix ideas, let us consider the vertical displacement at point A of the carabiner, as shown in Figure 1.11, as being representative of the displacement pattern of the structure. Let $\varDelta$ be the required precision for δ. Figure 1.12 shows the solutions for the models.

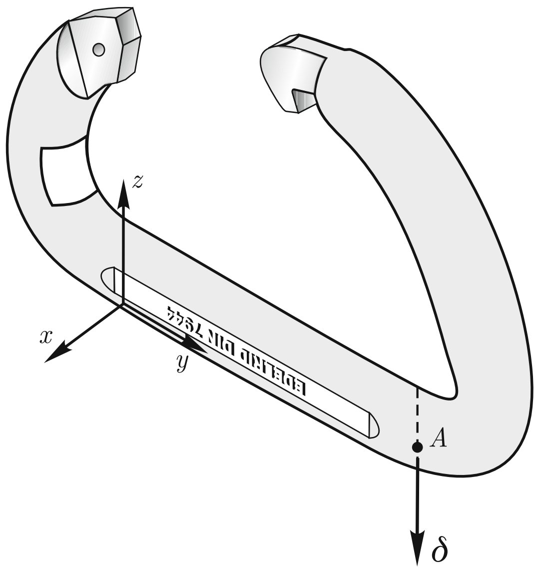


Fig. 1.11. Location of point A for which the vertical displacement is sought

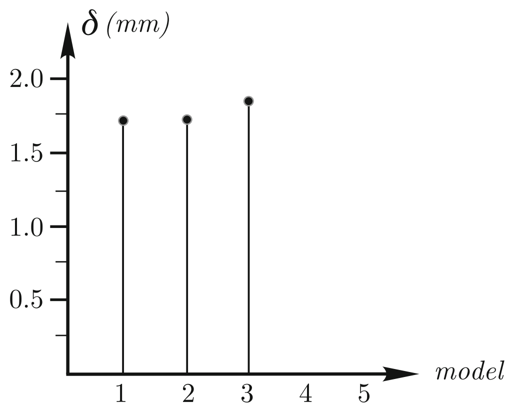


Fig. 1.12. Vertical displacements of point A in millimeters

A reliable model, say model i, for predicting the displacement δ would be a model for which the difference between its prediction δ_i and the displacement δ_a of the actual structure (measured in a laboratory experiment), would be less than Δ in absolute value, that is, $|\delta_i - \delta_a| < \Delta$.

However, since we are restricted to mathematical modeling and do not have the structure to experiment with in a laboratory, we do not have access to δ_a. Our best estimate for δ_a would be the solution of the very-comprehensive mathematical model, which we denote by δ_∞. Therefore, our criterion is $|\delta_i - \delta_\infty| < \Delta$. But also, we do not know δ_∞ since the very-comprehensive mathematical model will never be solved. However, the analyst knows all the phenomena that would be represented in the very-comprehensive mathematical model and that can influence this displacement. By comparison with the phenomena which are already represented in the mathematical model i, the analyst will have to assume whether or not $|\delta_i - \delta_\infty| < \Delta$. To make this decision properly is of major importance and requires knowledge of the structural problem and of the details of the mathematical models already solved as well as of those not yet solved. Of course, the solutions of the lower-order models in the sequence will already have given some insight into the behavior of the structure.

Assume that the analyst is not sure if a particular phenomenon, which is present in the very-comprehensive mathematical model and which is not taken into account in the currently considered model, might have an influence in the prediction of δ by a quantity greater than Δ. Then the analyst would have to conceive and include in the sequence a next model which should incorporate that particular phenomenon.

With the above considerations in mind, let us return to the data presented in Figure 1.12. We observe that the displacement of point A is changing (here increasing) as we proceed from model 1 to model 3. Some additional change in displacement will occur as the model is further refined (refer to models 4 and 5), and it is up to the analyst to decide, by experience or otherwise, if the results of a given model i are sufficiently accurate, that is, whether it can be assumed that $|\delta_i - \delta_\infty| < \Delta$.

Of course if the design requirement is a limit value, *i.e.*, for this example it could be $\delta < \delta_{Limit}$, the same reasoning is valid with the conditions $|\delta_i - \delta_\infty| < \Delta$ (as before) and additionally $\delta_{Limit} - \delta_i > \Delta$ which would allow to conclude that $\delta_\infty < \delta_{Limit}$[1]

From this discussion we can already draw some fundamental observations regarding the hierarchical modeling process:

- The process gives a logical and systematic framework to analyze engineering structures.
- The process relies on the understanding of the analyst of the problem at hand and of the mathematical models available. This understanding is revealed in the choice of the hierarchical models, the conceptual view of the very-comprehensive model and the interpretation of the results.
- The choice of the sequence of models is influenced by what variables are to be predicted and with which precision.

We can now further detail the diagram of Figure 1.7. The resulting diagram is shown in Figure 1.13. We implicitly assumed so far that we are always

[1] In this book we are always considering all the variables to be deterministic since our concern is with the modeling issues. Of course, additional considerations arise for a stochastic description of the variables. By the same argument we are supposing that an applicable safety factor is used when determining Δ or δ_{Limit}

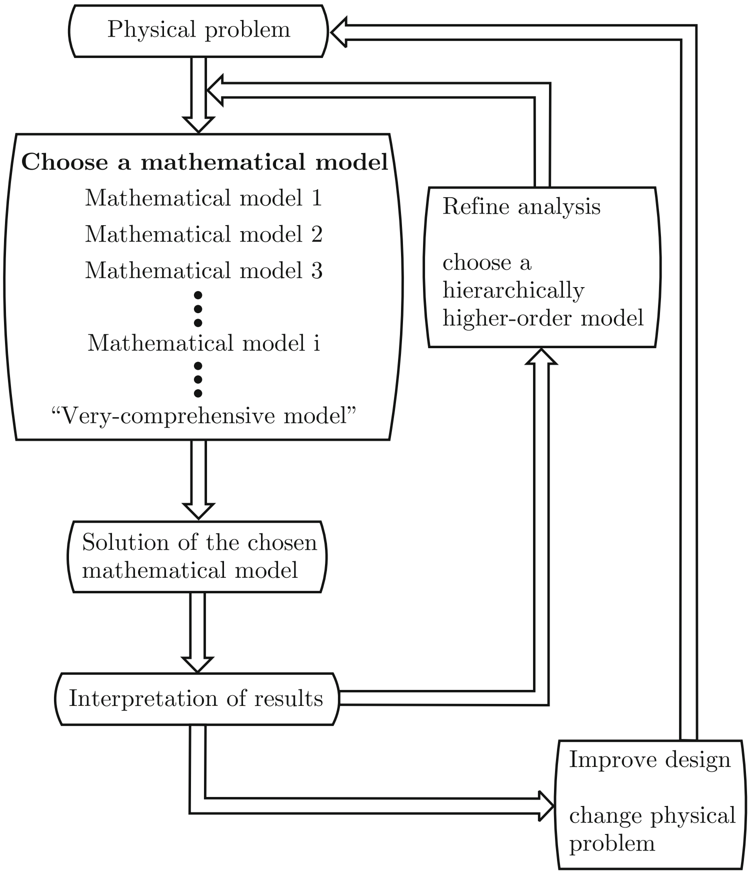


Fig. 1.13. The process of engineering design

able to solve the chosen mathematical model. We defer to later the discussion of how to solve the models when an analytical solution is not readily available.

Referring to Figure 1.13, we note that associated with the "interpretation of results" there is a decision to be made. At this point the analyst has to determine whether to assume that the modeling of this particular physical problem has "converged" or not. In other words, the analyst needs to decide if the variables to be predicted have been found with the required precision measured with respect to the very-comprehensive mathematical model ($|\delta_i - \delta_\infty| < \Delta$ in the terminology of the above discussion).

Every mathematical model in the sequence which gives the variables to be predicted with the required precision is termed a *reliable mathematical model.* As an engineer, when performing the hierarchical modeling procedure, we would like to find among all reliable models the one of lowest order which is therefore termed the *most effective mathematical model*. These two concepts are very important and will be used throughout the book.

In the diagram of Figure 1.13, we show that from the engineering point of view the modeling does not end when the hierarchical modeling process for a particular physical problem has converged. The predictions might suggest design improvements leading, for example, to geometric or material changes to the physical problem. Of course, such changes actually lead to a new physical problem and the hierarchical modeling should be restarted for the new physical problem. We would then restart the analysis process with more understanding based on the earlier hierarchical modeling, allowing us to take significant shortcuts in the choice of models.

1.3 Remarks on the hierarchical modeling process

The discussion presented so far could have been written, with small modifications, two hundred years ago. We should, therefore, point out what changes have occurred that make the hierarchical modeling process so important nowadays.

The key aspect is that due to developments in computational engineering during the last decades we can actually solve many more and much more complex mathematical models. Therefore, the relevance of using a structured approach towards analysis has dramatically increased and the hierarchical modeling provides this approach. In addition, of course, the fact that more complex mathematical models can now be solved has also spurred the development of new models, and regarding structural analysis, in particular, for the nonlinear analysis of solids and structures.

As it is well known, we are able to find analytical solutions only to very low-order models. Therefore, in the past, setting up a sequence of models would not result into significant benefits because the engineering decision of which model to choose was seldom based on whether such model was reliable or not. This decision had to be largely based on whether the next model in the sequence could actually be solved. On the other hand, in today's engineering environments very complex models can be solved using finite element methods. Hence the solution of higher-order models in structural and solid mechanics can now be achieved.

Of course, incredibly bold structures were built long time ago – even without any analysis at all. This was possible by introducing large safety factors, using the experience gained previously with similar structures and by relying on the physical intuition of great builders/designers. Clearly, the modern approach of analysis by no means diminishes the great earlier accomplishments in structural engineering. On the contrary, these accomplishments should stand out as a paradigm of what has been achieved without the finite element analysis tools that we have today. These tools enable us in a very exciting manner to now design even bolder, safer and more economical structures.

The finite element method is the predominant method to solve structural engineering problems. The study of the origin, history and development of the finite element method is *per se* rich and fascinating. We refrain ourselves from exploring in detail these items but we would like to only mention a few facts below, see Bathe, 1996, Bathe, 2009.

The finite element method, as used in engineering, was proposed in the 1950s based largely on intuition by *structural engineers* as an attempt to generalize matrix structural analysis from bars to continuum-like structures. At that time, with the development of digital computers, matrix structural analysis was successfully applied to solve relatively complex bar structures. Airplane wings – which were important structures to be analyzed – require, however, the representation of plate/shell like behavior. These needs led to the first finite element formulations for 2-D continua.

In the early 1970s, the tremendous value of the method to solve real engineering problems was widely realized when actual finite element computer programs became available to engineering analysts, and not just researchers. These tools made it possible that the method could be used in many industries and research centers to solve relatively complex problems of engineering interest. Various mathematical foundations were also identified, and from then onwards, the method developed at an exponential rate.

The availability of general purpose finite element computer programs brought a myriad of modeling options, which, associated with the ever increasing power and availability of computers, have made it possible to perform today on PCs analyses of very complex nature. Indeed, the widespread availability of powerful finite element resources has drastically altered the potential for analysis of very complex structures, Bathe 2007. The use of a conceptual framework for modeling, which was implied in the past but not explicitly detailed and used due to the scarcity of models which could be solved, is nowadays a requirement – although frequently overlooked.

Referring to the diagram of Figure 1.13 we could now replace the block "solution of the chosen mathematical model" by "solution of the chosen mathematical model by finite element procedures". In this way, considering the present software/hardware resources, we can proceed quite far in the sequence of mathematical models. For the carabiner problem we can easily find the solution of model 5 and beyond. However, we need to be aware of some new issues and concerns that arise, namely:

- Since powerful analysis resources are available, there is a tendency to immediately start the analysis using very complex models. This is actually bad engineering, since complex models usually cloud the basic structural behavior which is best revealed by low-order models. Also, the high-order models usually generate a large amount of results which are difficult and costly to interpret. On the other hand, it is quite different a situation to solve a complex high-order model after having solved lower-order models and considering that a higher-order model is necessary. Then the solution of the lower-order models has given enough understanding of the problem to recognize that for the response sought, and the required precision, the higher-order model is needed. We emphasize that we should not directly solve a high-order model only because the resources are available.

- The finite element solution is an approximation to the exact solution of the mathematical model considered. However, using *reliable finite element procedures*, we can obtain a finite element solution which is as close to the exact solution of the mathematical model as desired. Clearly, the use of reliable finite element procedures is imperative. The issues related to the reliability of finite element procedures will be discussed later in the book.
- The possibility of conceiving complex mathematical models and actually solving them is placing strong demands on the analyst. Besides knowing the formulations and the assumptions contained in the low-order models and being able to solve them analytically, the analyst needs to also be able to construct and solve applicable high-order models and be able to interpret the results obtained. Such demands for expertise are even higher when we consider coupled problems such as the solution of thermo-mechanical problems or fluid-structure interactions.

At this point a word of caution is in order. Of course, it is not reasonable to expect that every person working in structural analysis would have the knowledge and the experience required to tackle all high-order models. However, the above observations do apply for the engineering team as a whole if state-of-the-art modeling is to be used, and they give an idea of the challenges in this activity.

The analyst – the person or the team of people – who will perform the hierarchical modeling process should possess the required abilities to develop and solve hierarchical models. We emphasize that these requirements comprise:

- To know the candidate mathematical models involved: the analyst should be comfortable with the formulations, the assumptions contained in the models, the hypotheses and the basic behavior that can be predicted by the models.
- To know the related finite element procedures. The analyst should be able to set up appropriate finite element representations and guarantee that solutions have been obtained with the required accuracy measured with respect to the exact solution of the mathematical model.

These requirements highlight the basic aspects that we want to present in this book such that an analyst can harvest more of the full spectrum of mathematical models (and, hence, finite element analysis possibilities) currently available. We believe that the process of hierarchical modeling focused upon in this book is an art and the basis of modern analysis in engineering and the sciences; and we hope that this book will be valuable in the practice of this art – rich and full of challenges requiring knowledge and ample imagination.

1.4 Outline of book

We are now in a good position to outline the contents of the book. In the next chapter we discuss some fundamental concepts in structural mechanics which are applicable to all models discussed in the book. Then we apply these concepts to the simplest mathematical model – the one-dimensional truss/bar model and truss structures.

In Chapter 3 we motivate the need for two and three-dimensional theories and formulate the problem of 3-D linear elasticity.

Chapter 4 is dedicated to the formulation of mathematical models for structures. We proceed from linear plane elasticity to simple shells, discussing also bars and plates.

In Chapter 5 we discuss the equivalence between differential and variational formulations. The variational formulation is the basis of the finite element method. The emphasis is on a mechanical and physical approach through the introduction of the principle of virtual work for the mathematical models previously formulated in Chapters 3 and 4.

The finite element method is introduced in Chapter 6, first for the one-dimensional problem and then for two and three-dimensional analyses. The emphasis is placed on basic aspects and on the mechanical properties that the finite element solution satisfies. Effective finite elements which can be recommended for engineering practice are also presented. A brief discussion on locking, which was the major obstacle to arrive at reliable finite elements for structures, is also given.

In Chapter 7 we synthesize what we presented by applying the hierarchical modeling process to selected problems, using the mathematical models of Chapters 3 to 4 and the finite element procedures of Chapter 6.

Finally, in Chapter 8 we briefly address basic issues encountered in non-linear analyses.

All finite element solutions given in this book have been obtained using ADINA, 2008.