

Analysis of off-axis solenoid fields using the magnetic scalar potential: An application to a Zeeman-slower for cold atoms

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In a region free of currents, magnetostatics can be described by the Laplace equation of a scalar magnetic potential, and one can apply the same methods commonly used in electrostatics. Here, we show how to calculate the general vector field inside a real (finite) solenoid, using only the magnitude of the field along the symmetry axis. Our method does not require integration or knowledge of the current distribution and is presented through practical examples, including a nonuniform finite solenoid used to produce cold atomic beams via laser cooling. These examples allow educators to discuss the nontrivial calculation of fields off-axis using concepts familiar to most students, while offering the opportunity to introduce themes of current modern research. © 2015 American Association of Physics Teachers.

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I. INTRODUCTION

Magnetic fields produced by solenoids and axially symmetric coils are ubiquitous, and the ability to calculate them is an integral part of training in physics. Time constraints, however, tend to focus the attention of most introductory electromagnetism (EM) courses on the analytical solution of only a few highly symmetrical cases, such as the field along the axis of a circular coil or inside an infinite solenoid.^{1–5} Nevertheless, many applications require at least an estimate of the full vector field in regions away from the axis,^{6–8} which involve mathematical tools often not discussed at the introductory level. On the other hand, most EM courses already dedicate a fair amount of time teaching students to identify and solve electrostatic problems using the Laplace equation. In some cases, the same methods can be applied to magnetostatic problems, sometimes leading to useful insights.

Sadly, most students do not appreciate the similarities between the two classes of problems⁹ due to a limited exposure to practical examples involving the magnetic potential. We feel that the ability to make use of the magnetic potential is useful,^{10,11} particularly because scalar potentials are generally more intuitive and easier to visualize. Besides, a unified treatment could be pedagogically relevant in generalizing the discussion of the multipole expansions.^{12–14} Therefore, the primary goal here is to present a couple of pedagogical examples illustrating the application of the magnetic potential method to real solenoids.

In addition, these examples also offer the opportunity to discuss in the classroom axisymmetric fields evaluated off-axis, without the need to introduce the formalism of elliptic integrals. Although other methods for finding off-axis magnetic fields have been mentioned earlier in the literature,^{6,7} to our knowledge, it has not been presented from such a simple and intuitive viewpoint.

As further motivation, we have chosen an example that brings a real and practical application from the cutting edge of research into the classroom: a nonuniform solenoid used in many research laboratories to produce beams of slow (cold) atoms. This solenoid, known as a Zeeman-slower,^{15–17} is used in conjunction with appropriately prepared laser beams to

slow down and cool neutral atoms, from hundreds of Kelvin to milliKelvin temperatures, by combining the action of radiation pressure with the Zeeman effect. This device is one of the key developments in the area of laser cooling,^{18,19} and one of the enabling technologies leading to the 1997 Nobel prize in Physics.²⁰ The techniques for laser cooling and trapping of atoms have produced many dramatic advancements in our understanding of quantum physics,²¹ including the achievement of Bose-Einstein condensation, which was recognized with another Nobel prize²⁰ in 2001. In both cases, magnetic fields were an important part of experimental design and data interpretation. Educators can use the solenoid discussed here, as well as the references herein, to introduce and discuss some of these modern developments in quantum physics, making the subject more interesting to students.

II. REVIEWING SOME BASIC CONCEPTS

We begin here by recalling the fundamental equation of magnetostatics: $\nabla \times \vec{H} = \vec{J}$, where \vec{H} is the magnetic field and \vec{J} the current density. Typically, \vec{H} is related to the magnetic induction field \vec{B} by some constitutive relation expressing the properties of a particular material. For linear and isotropic materials with magnetic permeability μ , $\vec{B} = \mu\vec{H}$ and in a current-free region $\nabla \times \vec{B} = 0$, implying that $\vec{B} = -\nabla\phi_M$. Since Maxwell's equations also state that $\nabla \cdot \vec{B} = 0$, this results in $\nabla^2\phi_M = 0$, which is Laplace's equation for the magnetic (scalar) potential ϕ_M , in any current-free region.

Although Laplace's equation is only typically valid in a region free of charges or currents, they are allowed to exist on or outside a surface S surrounding that region. The solutions of Laplace's equation present three important properties: superposition, smoothness, and uniqueness. The property of superposition results from the fact that Laplace's equation is a linear equation. Smoothness implies that no solution in a region V of space, bounded by a surface S , can present either a maximum or a minimum within V (extreme values can occur only at the surface S). The third property is the one most relevant to us here; it states³ that if one finds a solution ϕ_M in a region of space consistent with the

prescribed boundary conditions, then that solution is unique up to an additive constant. Therefore, it does not matter which particular method is used to find the solution. Once an appropriate solution is found, we know it is the *only* solution.

Despite the obvious similarities between the electrostatic and magnetic potentials, there are indeed reasons why the analogy can only be taken so far^{9,12} and why it is not widely explored further in textbooks. The first one arises whenever $\vec{J} \neq 0$, in which case it is not trivial to write a relation between ϕ_M and \vec{J} . The second complication occurs due to the fact that the scalar potential is generally a multiply valued function, requiring a prescription specifying where it can be used. However, as shown by Bronzan,¹² these complications can be overcome, permitting one to exploit the advantages of a magnetic scalar potential.

III. THE MAGNETIC FIELD OF A FINITE UNIFORM SOLENOID

Let us start by considering axisymmetric fields produced by a solenoid (see Fig. 1). Because of the axisymmetric symmetry of this problem, the field on the z -axis can only depend on $z = r \cos \theta$ and must point in the $\pm z$ -direction. The magnetic potential can be found using as boundary condition the magnitude of the field along the z -axis, $B_z(z)$, which is readily available through simple summation formulas over the approximately circular coils forming the solenoid or by direct measurement along the axis.

For generality and convenience, we describe the problem using spherical coordinates, where the solution of the axisymmetric scalar potential ϕ_M can be written in the form

$$\phi_M(r, \theta) = \sum_{\ell=0}^{\infty} \left(a_{\ell} r^{\ell} + \frac{b_{\ell}}{r^{\ell+1}} \right) P_{\ell}(\cos \theta), \quad (1)$$

where a_{ℓ} and b_{ℓ} are the coefficients to be determined, and P_{ℓ} represents a Legendre polynomial of order ℓ . Because we are mainly interested in the values of the field inside the solenoid, we set $b_{\ell} = 0$ to avoid a singularity at $r = 0$. As a result, the potential takes the simpler form

$$\phi_M(r, \theta) = a_0 + a_1 r P_1(\cos \theta) + a_2 r^2 P_2(\cos \theta) + \dots, \quad (2)$$

$$\phi_M(r, \theta) = -\frac{\alpha L r \cos \theta}{(R^2 + L^2/4)^{1/2}} + \frac{3}{2} \left(\frac{5}{3} \cos^3 \theta - \cos \theta \right) \times \left[\frac{L\alpha}{2(R^2 + L^2/4)^{3/2}} - \frac{L^3\alpha}{8(R^2 + L^2/4)^{5/2}} \right] r^3 + \dots \quad (9)$$

Finally, using this potential one can calculate the components B_r and B_{θ} of the magnetic field as

$$B_r = -\frac{\partial \phi_M(r, \theta)}{\partial r} = \frac{\alpha L \cos \theta}{(R^2 + L^2/4)^{1/2}} - \frac{3}{2} (5 \cos^3 \theta - 3 \cos \theta) \times \left[\frac{L\alpha}{2(R^2 + L^2/4)^{3/2}} - \frac{L^3\alpha}{8(R^2 + L^2/4)^{5/2}} \right] r^2 + \dots, \quad (10)$$

and

$$B_{\theta} = -\frac{1}{r} \frac{\partial \phi_M(r, \theta)}{\partial \theta} = -\frac{L\alpha \sin \theta}{(R^2 + L^2/4)^{1/2}} - \frac{3}{2} (\sin \theta - 5 \cos^2 \theta \sin \theta) \times \left[\frac{L\alpha}{2(R^2 + L^2/4)^{3/2}} - \frac{L^3\alpha}{8(R^2 + L^2/4)^{5/2}} \right] r^2 + \dots \quad (11)$$

and for points along the z -axis becomes

$$\phi_M(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \dots \quad (3)$$

Equation (3) is a Taylor series about the point $z = 0$, so the coefficients are given by

$$a_{\ell} = \frac{1}{\ell!} \left(\frac{\partial^{\ell} \phi_M}{\partial z^{\ell}} \right)_{z=0}. \quad (4)$$

In this way, the full scalar potential in Eq. (2) becomes analytically determinable, allowing us to evaluate $\vec{B} = -\vec{\nabla} \phi_M$ at any point in space.

As a first example, let us consider the case of a finite solenoid of length L and radius R carrying a uniform current I , as illustrated in Fig. 1. If the solenoid has N turns per unit length, the magnetic field along the z -axis can be calculated by integrating the expression for the axial field of a circular current loop,¹ giving

$$B(z) = \alpha \left(\frac{z_+}{\sqrt{R^2 + z_+^2}} - \frac{z_-}{\sqrt{R^2 + z_-^2}} \right), \quad (5)$$

where $z_{\pm} = z \pm L/2$ and $\alpha = \mu_0 NI/4\pi$ in SI units. Now, since

$$B_z(z) = -\frac{\partial \phi_M}{\partial z}, \quad (6)$$

we can write

$$\phi_M(z) = -\int B(z') dz'. \quad (7)$$

Using Eq. (5) in Eq. (7), we obtain the general form of the potential for the finite solenoid along the axis:

$$\phi_M(z) = -\alpha \left(\sqrt{R^2 + z_+^2} - \sqrt{R^2 + z_-^2} \right). \quad (8)$$

Expanding Eq. (8) in a Taylor series about $z = 0$, we get the various coefficients for $\phi_M(z)$. Then, using these coefficients in Eq. (2), we find the scalar potential valid everywhere inside the solenoid as

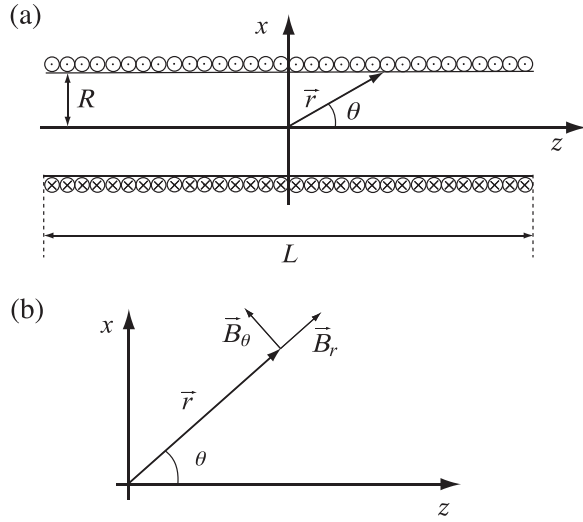


Fig. 1. (a) Schematic representation of a finite and homogenous solenoid. The crosses (dots) represent current flowing into (out of) the page. (b) The relevant coordinates and field directions.

We note that these results give (approximate) analytical results for the magnetic field everywhere inside the solenoid, with their precision limited only by the number of terms included in the power series expansion. One can test these results by comparing Eqs. (9) and (10) with those presented in Chapter 5 of Ref. 2, where a different method was used to evaluate the field components. In particular, we will show that if one keeps only the first order term in the expansion, the result simplifies to the approximate solution given in problem 5.2. To begin, we recall the relations

$$B_\rho = B_r \sin \theta + B_\theta \cos \theta, \quad (12)$$

$$B_z = B_r \cos \theta - B_\theta \sin \theta, \quad (13)$$

from which we obtain, up to third order,

$$B_\rho \simeq \frac{3\alpha LR^2 r^2}{2(R^2 + L^2/4)^{5/2}} \sin \theta \cos \theta. \quad (14)$$

Finally, using $\rho = r \sin \theta$ and $z = r \cos \theta$ in the limit $R \ll L$, we find

$$B_\rho(\rho, z) \simeq \frac{96\pi NI}{c} \left(\frac{R^2 z \rho}{L^4} \right), \quad (15)$$

which is expressed here in CGS units (with $\alpha = 2\pi NI/c$) to facilitate direct comparison with Ref. 2.

In Fig. 2, we compare the results obtained by keeping the first eight terms in the series expansion (solid curve), corresponding to the 15th order in r , against numerical simulations of the field (dots/circles) using a direct summation over the exact analytical expression (elliptic integrals) for each individual coil of the finite solenoid. In addition, Fig. 2 also shows the good partial agreement obtained using the third-order approximation (dashed curve), extending to distances up to about half the size of the solenoid. Note that, for a real finite system, the disagreement increases rapidly after some point (for $|z| \gtrsim 0.25$ for third order and $|z| \gtrsim 0.5$ for 15th order). The agreement can be improved significantly by

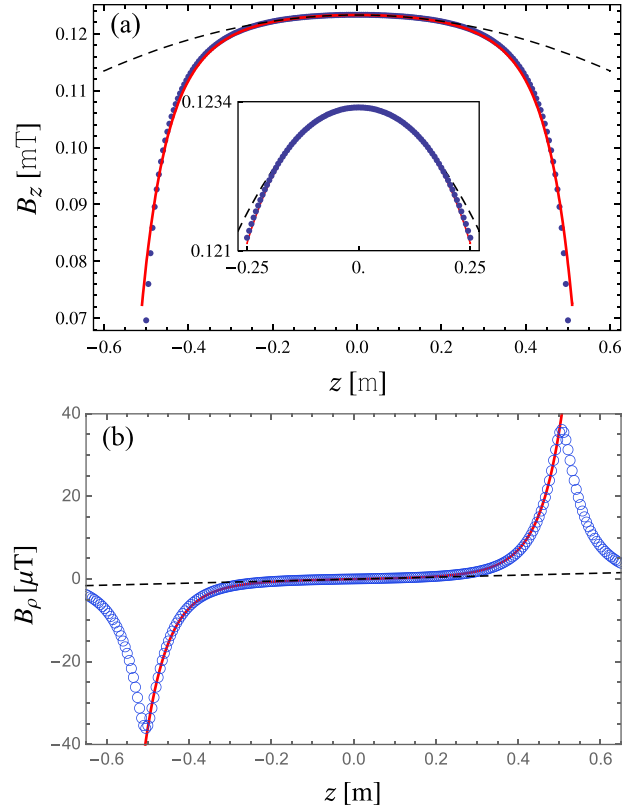


Fig. 2. Comparison of the power series approximation (solid and dashed curves) with a numerical simulation (points/circles) of the exact field (using elliptic integrals) for a uniform solenoid ($L = 1$ m, $R = 10$ cm, and $N = 100$) carrying a current of $I = 1$ A. We plot both the axial field (a) and the transverse field (b) profiles. All results are evaluated off-axis ($\rho = 8$ cm) and the power series results are shown for expansions up to third (dashed) and 15th (solid) orders. The inset in (a) shows the good agreement obtained near $z = 0$, even for the third-order approximation.

including higher-order terms, allowing for a much better approximation near the edges, as shown in Fig. 2. However, due to the simplifications made, the power series approximation still does not contain all the physics of the problem. For instance, it does not accurately describe the field outside the solenoid. Nevertheless, the magnetic potential method presented here still provides a reasonable representation of the internal fields up to the very end of the solenoid.

IV. THE ZEEMAN-SLOWER: AN INHOMOGENEOUS FINITE SOLENOID

We now consider a practical problem, familiar to many atomic physics laboratories, which is the design of a solenoid capable of producing a field on the z -axis of the form

$$B(z) = B_b + B_0 \sqrt{1 - \beta z}, \quad (16)$$

where B_b , B_0 , and β are constants. Such a field is suitable for slowing atomic beams using laser light.¹⁵ The field of Eq. (16) causes a spatially varying Zeeman effect that compensates for the changing Doppler shift of the moving atoms, thus keeping them in resonance with the light as they decelerate along the beam path. This technique is called Zeeman slowing,¹⁵ and the shape is chosen to keep the radiation pressure constant, typically with a particle acceleration of $\sim 10^6$ m/s² throughout the Zeeman solenoid¹⁷ (see Fig. 3).

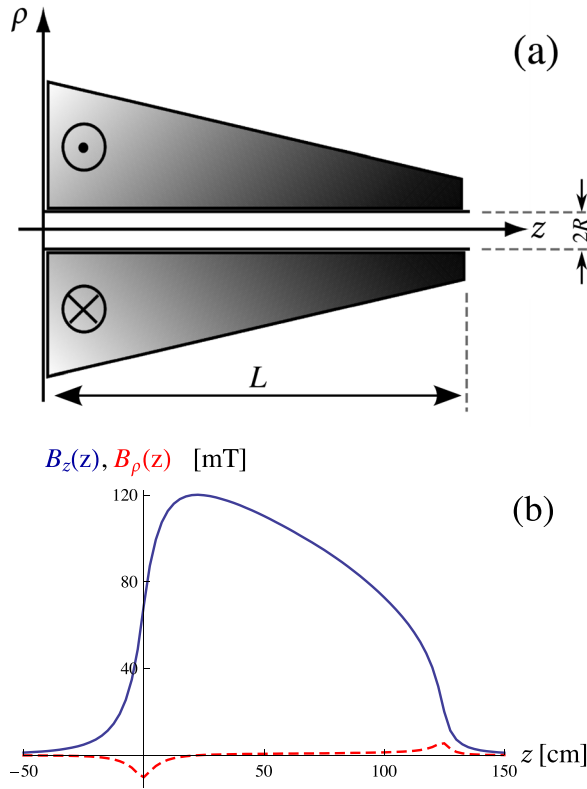


Fig. 3. (a) Sketch of a tapered (triangular shape) solenoid creating an inhomogeneous current distribution to produce the appropriate axial field profile (b) for a Zeeman solenoid, with $L = 125$ cm, $R = 4$ cm, and $B_{\max} \approx 120$ mT. The solid line shows $B (= B_z)$ along the axis ($\rho = 0$), whereas the dashed line represents the transverse field B_ρ at $\rho = 2.5$ cm.

In general, the atomic beam encompasses a certain solid angle as it traverses the solenoid and most atoms follow trajectories that do not lie exactly on the axis. Since the resonance condition with the laser depends on both the magnitude (via detuning) and direction (via polarization) of the magnetic field, the knowledge of the off-axis field is important in understanding how light interacts with atoms at different points inside the solenoid.

In this case, the magnetic potential along the z -axis takes the form

$$\phi_M(z) = - \int B(z') dz' = -B_b z + \frac{2B_0}{3\beta} (1 - \beta z)^{3/2}, \quad (17)$$

where the constant of integration has been suppressed. Following the same steps as in Sec. III and after calculating the derivatives and solving for the coefficients a_ℓ , we obtain the general form of the magnetic potential for the Zeeman solenoid:

$$\begin{aligned} \phi_M(r, \theta) = & \frac{2B_0}{3\beta} - (B_b + B_0)r \cos \theta \\ & + \frac{B_0}{\sqrt{\pi}} \sum_{n=2}^{\infty} \frac{\beta^{n-1} \Gamma\left(n - \frac{1}{2}\right)}{n!(2n-3)} r^n P_n(\cos \theta), \end{aligned} \quad (18)$$

where $\Gamma(n)$ is the gamma function. Calculating the spherical components of the magnetic field then gives

$$\begin{aligned} B_r(r, \theta) = & (B_b + B_0) \cos \theta \\ & - \frac{B_0}{\sqrt{\pi}} \sum_{n=2}^{\infty} \frac{\beta^{n-1} \Gamma\left(n - \frac{1}{2}\right)}{(n-1)!(2n-3)} r^{n-1} P_n(\cos \theta) \end{aligned} \quad (19)$$

and

$$\begin{aligned} B_\theta(r, \theta) = & -(B_b + B_0) \sin \theta \\ & - \frac{B_0}{\sqrt{\pi} \sin \theta} \sum_{n=2}^{\infty} \frac{\beta^{n-1} \Gamma\left(n - \frac{1}{2}\right)}{(n-1)!(2n-3)} r^{n-1} \\ & \times [\cos \theta P_n(\cos \theta) - P_{n-1}(\cos \theta)]. \end{aligned} \quad (20)$$

The transverse and axial components can be obtained from Eq. (13), giving

$$\begin{aligned} B_\rho(\rho, z) = & \frac{-B_0}{\sqrt{\pi}} \sum_{n=2}^{\infty} \frac{\beta^{n-1} \Gamma\left(n - \frac{1}{2}\right)}{(n-1)!(2n-3)} \\ & \times \frac{(\sqrt{z^2 + \rho^2})^n}{\rho} [P_n(\tilde{z}) - \tilde{z} P_{n-1}(\tilde{z})] \end{aligned} \quad (21)$$

and

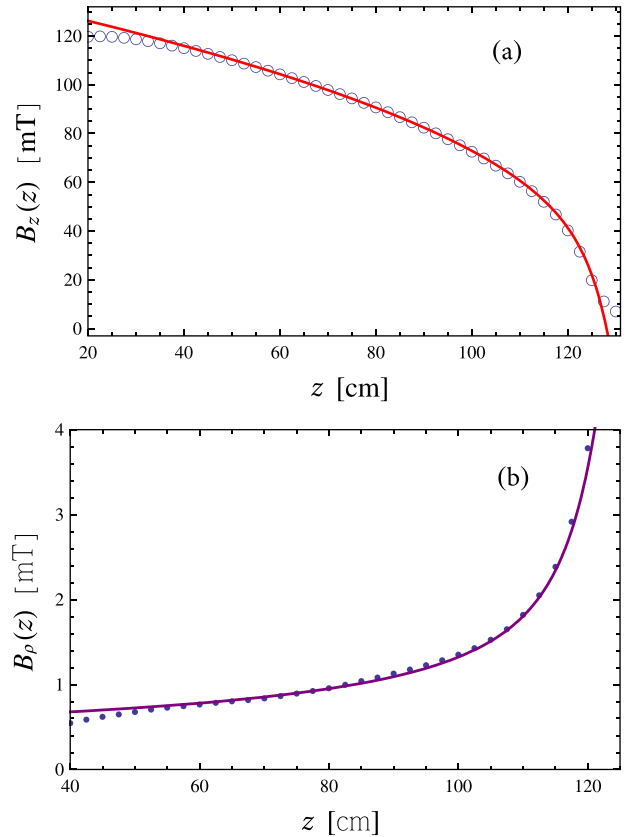


Fig. 4. Comparison for the series solutions of the Zeeman solenoid for B_z with $\rho = 0$, and B_ρ with $\rho = 2.5$ cm. The points represent the actual experimental field and the solid lines are the analytical (series) approximations [Eqs. (21) and (22)] of the model function in Eq. (16).

$$B_z(\rho, z) = (B_b + B_0) - \frac{B_0}{\sqrt{\pi}} \sum_{n=2}^{\infty} \frac{\beta^{n-1} \Gamma\left(n - \frac{1}{2}\right)}{(n-1)!(2n-3)} \times \left(\sqrt{z^2 + \rho^2}\right)^{n-1} P_{n-1}(\tilde{z}), \quad (22)$$

where we have introduced the shorthand $\tilde{z} = z/\sqrt{z^2 + \rho^2}$.

Notice that the transverse component of the magnetic field inside the Zeeman-slower does not depend on B_b . Also, note that $B_\rho = 0$ at $\rho = 0$ (on-axis), as expected, and it can be verified that the on-axis field sums back to the exact expression of Eq. (16). Although caution may be necessary when evaluating the field for $\rho = 0$ ($\theta = 0$), the careful use of L'Hospital's rule ensures finite and correct answers.

Now, to compare these analytical approximations (series solutions) to the actual field, shown in Fig. 4, we will use a different approach. The motivation is to mimic a situation where the current distribution that generates the field may not be known exactly, but the axial field can be measured directly in the laboratory. This could be the case in a real application, where imperfections in the winding pattern often are not considered in the ideal model. From the experimental data, one can then build a mathematical model by using either a fitting function (if the functional form is known or can be easily guessed) or by using an interpolating function, such as a polynomial, to represent the data in a limited region of space. Here, since the approximate functional form of the axial field is known, we will extract the model parameters by numerically fitting the data in Fig. 4, to $B_z(z)$ in Eq. (16), and substituting them into Eqs. (21) and (22). Note that the limitations of this type of modeling may result in some inaccuracies, particularly close to the edges, where fringe effects are important. In any practical situation, one may need to explore different approaches to find a mathematical model that is accurate enough in the region of interest.

After following these steps to model the data, we show in Fig. 4 a comparison between the series solution and the real field. There is a reasonable agreement between the solid lines, representing Eq. (21) and Eq. (22), and the data points. Note that, in contrast to the uniform finite solenoid where a power series was used to approximate the exact solution, here the power series simply approximates our model²² (fitting) function. Therefore, increasing the order²³ of the series expansion only improves the agreement with the model function, which represents the data only over a limited region and does not contain all the information in the problem. This is clearly visible in Fig. 4, where good agreement is found only in the range $z \approx 0.4\text{--}1.2\text{ m}$.

V. CONCLUSION

Using the simple concept of the magnetostatic scalar potential, and only the knowledge of the field along the symmetry axis, we have shown how to determine the magnetic field anywhere inside an inhomogeneous finite solenoid, without explicitly integrating (or even knowing) the current distribution. In cases where the current distribution is known, but the expression for the off-axis field is nontrivial (for instance, given by elliptical integrals), one can still gain some insight by using the method described here. This simple analysis follows from a straightforward analogy with the electrostatic boundary value problem and can be useful in determining

field inhomogeneities in various practical experiments involving solenoids. In the present article, we have used an example from contemporary experiments in atomic physics to demonstrate the method. However, we believe that a simplified version of this discussion (e.g., the uniform finite solenoid) could be used in an undergraduate classroom as a practical example of a calculation of off-axis magnetic fields.

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- ¹D. J. Griffiths, *Introduction to Electrodynamics*, 3rd ed. (Prentice-Hall, Upper Saddle River, NJ, 1999).
- ²J. D. Jackson, *Classical Electrodynamics*, 2nd ed. (J. Wiley, New York, 1974), Chaps. 1, 3 and 5 (and particularly Problem 5.2).
- ³M. A. Heald and J. B. Marion, *Classical Electromagnetic Radiation*, 3rd ed., Chap. 3 has a good discussion of the Laplace equation (Saunders/Harcourt Brace, Fort Worth, TX, 1994).
- ⁴B. B. Dasgupta, "Magnetic field due to a solenoid," *Am. J. Phys.* **52**, 258 (1984).
- ⁵V. Labinac, N. Erceg, and D. Kotnik-Karuza, "Magnetic field of a cylindrical coil," *Am. J. Phys.* **74**, 621–627 (2006).
- ⁶R. H. Jackson, "Off-axis expansion solution of Laplace's equation: Application to accurate and rapid calculation of coil magnetic fields," *IEEE Trans. Electron Devices* **46**, 1050–1062 (1999).
- ⁷J. T. Conway, "Exact solutions for the magnetic fields of axisymmetric solenoids and current distributions," *IEEE Trans. Magn.* **37**, 2977–2988 (2001).
- ⁸M. I. Grivich and D. P. Jackson, "The magnetic field of current-carrying polygons: An application of vector field rotations," *Am. J. Phys.* **68**, 469–474 (2000).
- ⁹M. J. Lahart, "Use of electromagnetic scalar potentials in boundary value problems," *Am. J. Phys.* **72**, 83–91 (2004).
- ¹⁰B. B. Dasgupta, "A novel method of solving electrostatic potential problems," *Am. J. Phys.* **53**, 971–973 (1985).
- ¹¹B. Irvine, M. Kemnetz, A. Gangopadhyaya, and T. Ruubel, "Magnet traveling through a conducting pipe: A variation on the analytical approach," *Am. J. Phys.* **82**, 273–279 (2014).
- ¹²J. B. Bronzan, "The magnetic scalar potential," *Am. J. Phys.* **39**, 1357–1359 (1971).
- ¹³C. G. Gray, "Simplified derivation of the magnetostatic multipole expansion using the scalar potential," *Am. J. Phys.* **46**, 582–583 (1978).
- ¹⁴C. G. Gray, "Magnetic multipole expansion using the scalar potential," *Am. J. Phys.* **47**, 457–459 (1979).
- ¹⁵W. D. Phillips and H. Metcalf, "Laser deceleration of an atomic beam," *Phys. Rev. Lett.* **48**, 596–599 (1982); W. D. Phillips and H. J. Metcalf, "Cooling and trapping atoms," *Sci. Am.* **256**, 50–56 (1987).
- ¹⁶C. J. Dedman, J. Nes, T. M. Hanna, R. G. Dall, K. G. H. Baldwin, and A. G. Truscott, "Optimum design and construction of a Zeeman slower for use with a magneto-optic trap," *Rev. Sci. Instrum.* **75**, 5136–5142 (2004).
- ¹⁷H. J. Metcalf and P. van der Straten, *Laser Cooling and Trapping* (Springer-Verlag, New York, 1999).
- ¹⁸See open article at <http://focus.aps.org/story/v21/st11>; W. D. Phillips, P. L. Gould, and P. D. Lett, "Cooling, stopping and trapping atoms," *Science* **239**, 877–883 (1988); S. Chu, "Laser trapping of neutral particles," *Sci. Am.* **266**, 70–76 (1992).
- ¹⁹S. C. Zilio and V. S. Bagnato, "Radiative forces on neutral atoms—A classical treatment," *Am. J. Phys.* **57**, 471–474 (1989).
- ²⁰For more information visit the official Nobel prize website at <http://www.nobelprize.org/nobel_prizes/physics>.
- ²¹See review articles on special *Nature Insight* issue on Ultracold Matter. K. Southwell, *Nature* **416**, 205–246 (2002).
- ²²The fitting yields $B_b = 27.6\text{ mT}$, $B_0 = 132.8\text{ mT}$, and $\beta = 0.544\text{ m}^{-1}$.
- ²³For the agreement shown in Fig. 4, 70 terms were kept.