6.1 State of stress $\sigma_x = 50$, $\sigma_y = 10$, and $\tau_{xy} = -15$ MPa, with other components zero due to free surface. Find:
(a) σ_1, σ_2, τ_3, rotations, (b) σ_{max}, τ_{max}.

$$\sigma_1, \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_1, \sigma_2 = \sigma_{\tau 3} \pm \tau_3$$

$$\sigma_{\tau 3} = \frac{50 + 10}{2} = 30 \text{ MPa}$$

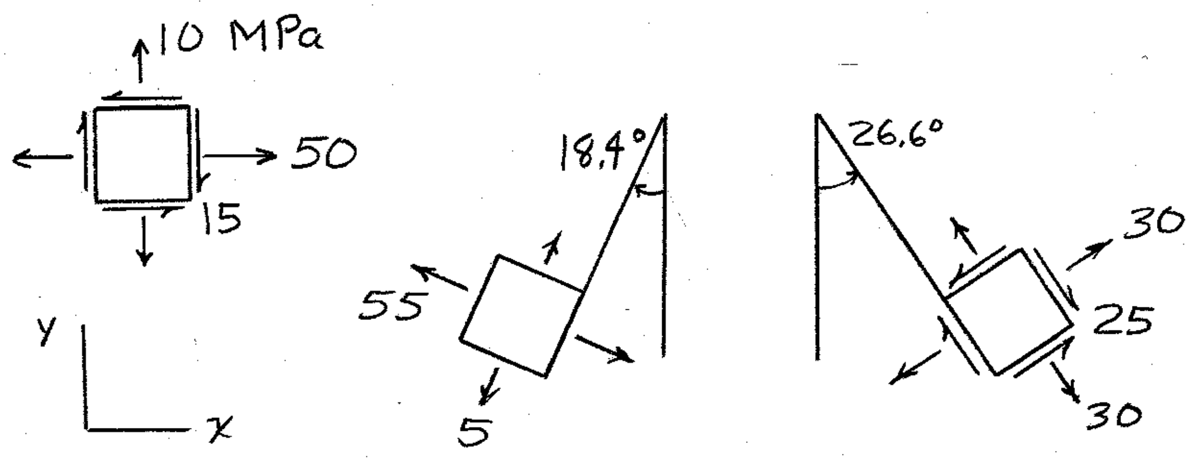
$$\tau_3 = \sqrt{\left(\frac{50-10}{2}\right)^2 + (-15)^2} = 25 \text{ MPa} \quad ◀$$

$$\sigma_1, \sigma_2 = 30 \pm 25 = 55,\ 5 \text{ MPa} \quad ◀$$

$$\tan 2\theta_n = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2(-15)}{50-10} = -0.750$$

$$\theta_n = -18.4^\circ = 18.4^\circ \text{ CW} \quad ◀$$

$$|\theta_n - \theta_s| = 45^\circ, \quad \theta_s = 26.6^\circ \text{ CCW} \quad ◀$$



(6.1, p.2)

Directions for sketches above are obtained as follows: The direction for the larger of σ_1, σ_2 is more nearly aligned with that of the larger of σ_x, σ_y. The positive shear diagonal for τ_3 is parallel to the larger of σ_1, σ_2.

(b) $\sigma_3 = 0$

$$\sigma_{max} = MAX(\sigma_1, \sigma_2, \sigma_3) = MAX(55, 5, 0)$$

$$\sigma_{max} = 55 \text{ MPa} \quad \blacktriangleleft$$

$$\tau_{max} = MAX(\tau_1, \tau_2, \tau_3)$$

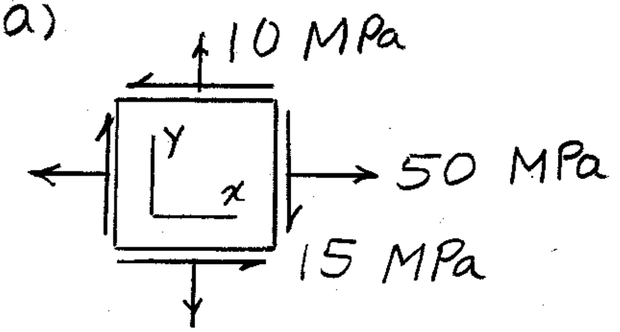
$$\tau_{max} = MAX\left(\frac{|\sigma_2 - \sigma_3|}{2}, \frac{|\sigma_1 - \sigma_3|}{2}, \frac{|\sigma_1 - \sigma_2|}{2}\right)$$

$$\tau_{max} = MAX(2.5, 27.5, 25) = 27.5 \text{ MPa} \quad \blacktriangleleft$$

Note that in this case τ_3 is not τ_{max}.

(6.1, p. 3) Second solution by Mohr's circle.

(a)

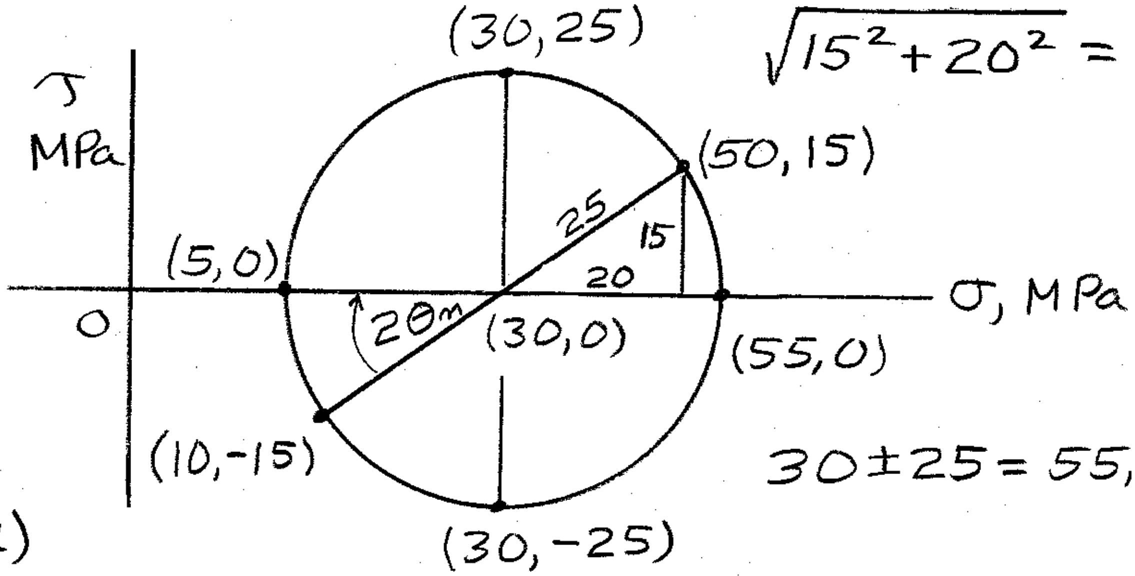


$\tan 2\theta_n = \frac{15}{20}$

$\theta_n = 18.4°$

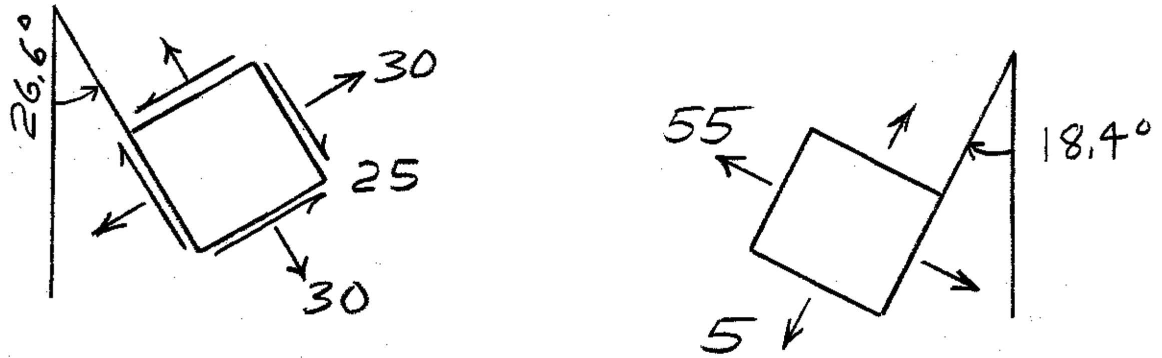
$\theta_s = 45 - \theta_n = 26.6°$

$\sqrt{15^2 + 20^2} = 25$



$30 \pm 25 = 55, 5$

(a)



$\sigma_1, \sigma_2 = 55, 5$ MPa, $\tau_3 = 25$ MPa ◀

(b) Same as above.

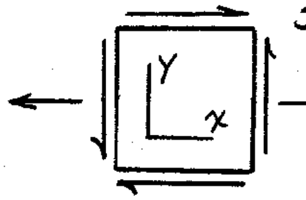
6.2 to 6.8

Solutions for these closely parallel that for Prob. 6.1. Results are given below. More detail for some is given on pages that follow.

Prob.	σ_x	σ_y	τ_{xy}	$\sigma_{\tau 3}$	τ_3	σ_1	σ_2	θ_n deg
6.1	50	10	-15	30.00	25.00	55.00	5.00	-18.43
6.2	60	0	30	30.00	42.43	72.43	-12.43	22.50
6.3	50	100	-60	75.00	65.00	140.00	10.00	33.69
6.4	120	40	-30	80.00	50.00	130.00	30.00	-18.43
6.5	-100	40	-50	-30.00	86.02	56.02	-116.02	17.77
6.6	-30	-84	27	-57.00	38.18	-18.82	-95.18	22.50
6.7	70	-25	30	22.50	56.18	78.68	-33.68	16.14
6.8	50	80	20	65.00	25.00	90.00	40.00	-26.57

Prob.	σ_3	τ_1	τ_2	σ_{max}	τ_{max}
6.1	0	2.50	27.50	55.00	27.50
6.2	0	6.21	36.21	72.43	42.43
6.3	0	5.00	70.00	140.00	70.00
6.4	0	15.00	65.00	130.00	65.00
6.5	0	58.01	28.01	56.02	86.02
6.6	0	47.59	9.41	0.00	47.59
6.7	0	16.84	39.34	78.68	56.18
6.8	0	20.00	45.00	90.00	45.00

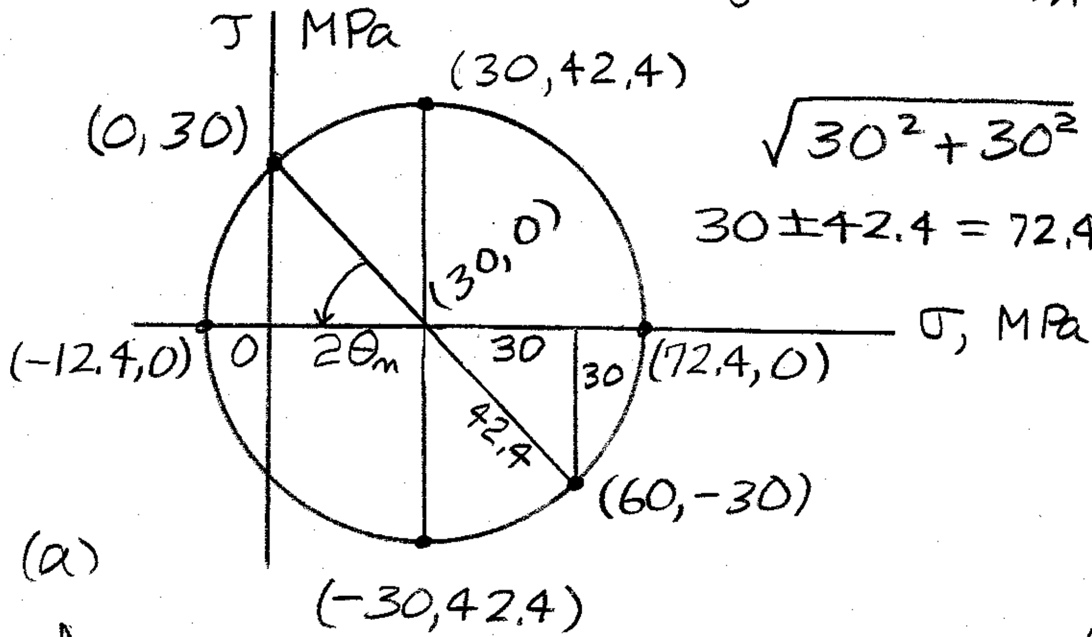
6.2



$$\tan 2\theta_m = \frac{30}{30}$$

$$\theta_m = 22.5^\circ$$

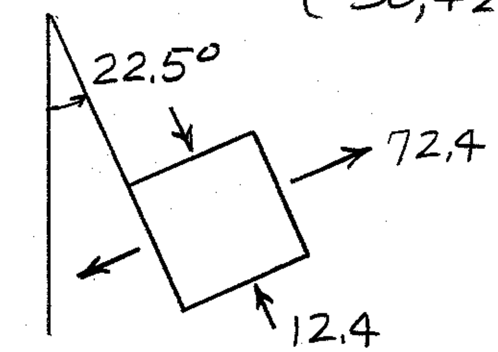
$$\theta_s = 45 - \theta_m = 22.5^\circ$$

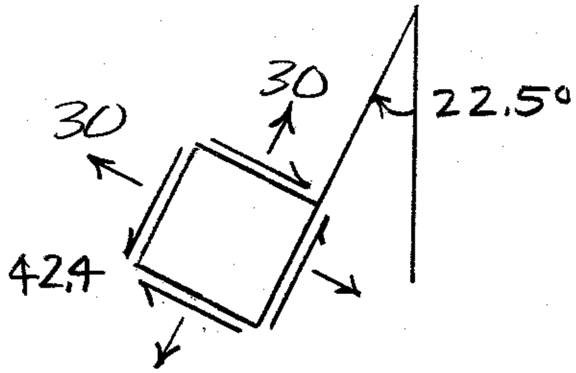


$$\sqrt{30^2 + 30^2} = 42.4$$

$$30 \pm 42.4 = 72.4, -12.4$$

(a)





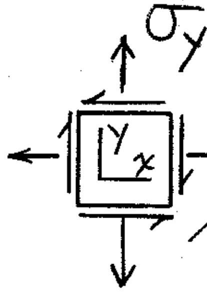
$$\sigma_1, \sigma_2 = 72.4, -12.4 \text{ MPa}, \quad \tau_3 = 42.4 \text{ MPa}$$

(b) $\sigma_3 = 0, \quad \sigma_{max} = 72.4$ MPa

$$\tau_{max} = MAX\left(\frac{|\sigma_2 - \sigma_3|}{2}, \frac{|\sigma_1 - \sigma_3|}{2}, \frac{|\sigma_1 - \sigma_2|}{2}\right)$$

$$\tau_{max} = MAX\,(6.2, 36.2, 42.4) = 42.4 \text{ MPa}$$

6.3



$$\sigma_y = 100\text{ MPa}$$

$$\sigma_x = 50\text{ MPa}$$

$$\tau_{xy} = 60\text{ MPa}$$

(a)

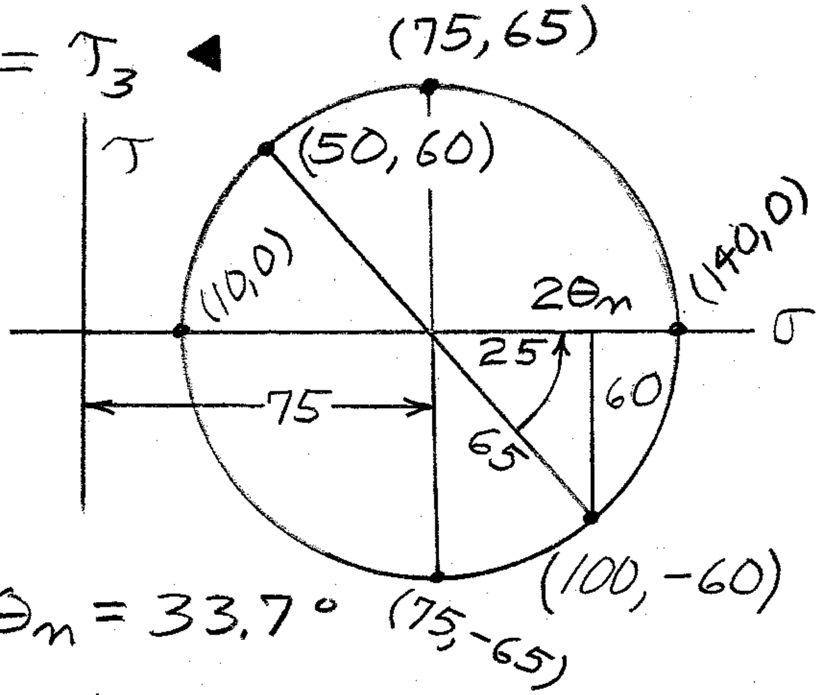
$$\sqrt{60^2 + 25^2} = 65 = \tau_3 \quad ◀$$

$$\sigma_{1,2} = 75 \pm 65$$

$$\sigma_1 = 140\text{ MPa}$$

$$\sigma_2 = 10\text{ MPa} \quad ◀$$

$$\sigma_z = \sigma_3 = 0$$

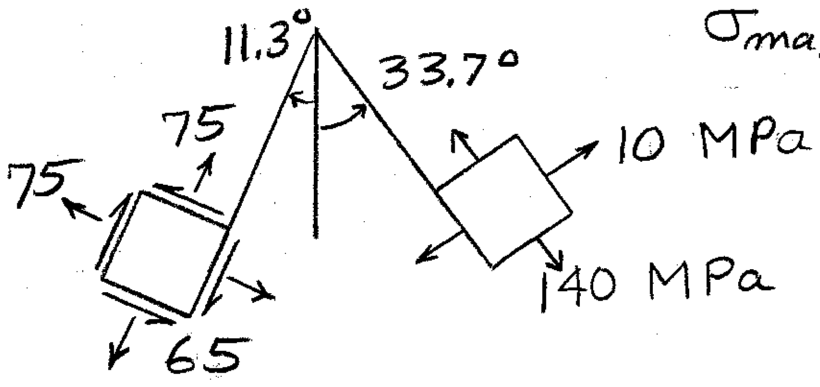


$$\tan 2\theta_n = \frac{60}{25}, \quad \theta_n = 33.7°$$

$$\tau_{max} = MAX\left(\frac{|\sigma_2 - \sigma_3|}{2}, \frac{|\sigma_1 - \sigma_3|}{2}, \frac{|\sigma_1 - \sigma_2|}{2}\right) \quad \text{(b)}$$

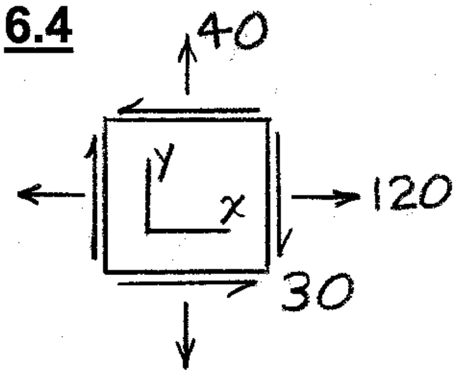
$$\tau_{max} = MAX(5, 70, 65) = 70\text{ MPa} \quad ◀$$

$$\sigma_{max} = 140\text{ MPa} \quad ◀$$



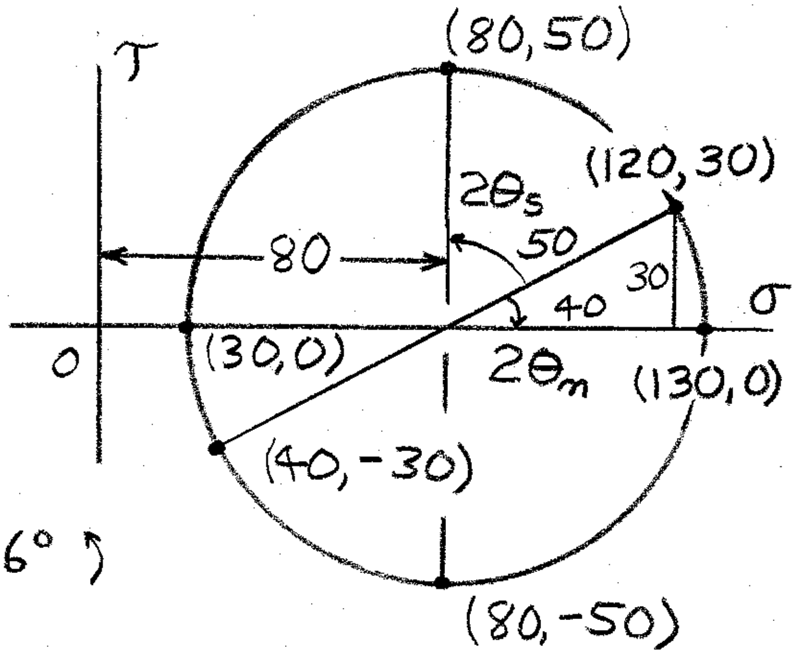
$\tau_3 = 65$ MPa is not τ_{max}

6.4



$\sigma_x = 120,\ \sigma_y = 40$ MPa

$\tau_{xy} = -30$ MPa

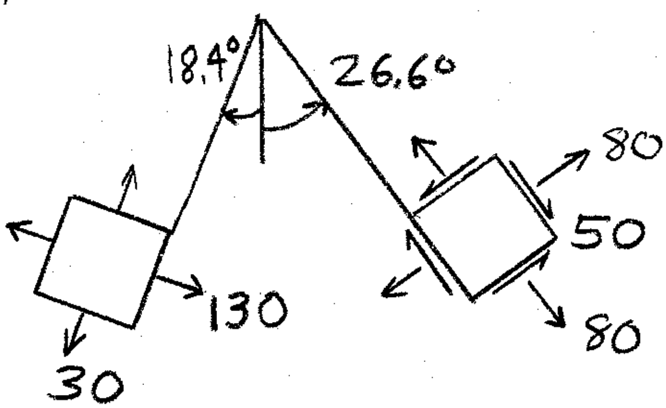


$$\tan 2\theta_n = \frac{30}{40}$$

$\theta_n = 18.4°$

$\theta_s = 45 - 18.4 = 26.6°$

(a)



$\sigma_1, \sigma_2 = 130, 30$ MPa ◀

$\tau_3 = 50$ MPa

(τ_3 is not τ_{max})

$\sigma_3 = 0$

(b) $\sigma_3 = 0$

$$\tau_{max} = MAX\left(\frac{|\sigma_2 - \sigma_3|}{2}, \frac{|\sigma_1 - \sigma_3|}{2}, \frac{|\sigma_1 - \sigma_2|}{2}\right)$$

$\tau_{max} = MAX(15, 65, 50) = 65$ MPa ◀

$\sigma_{max} = 130$ MPa ◀

6.9 Given $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}$ as below, find:
(a) $\sigma_1, \sigma_2, \sigma_3$, and τ_1, τ_2, τ_3; (b) σ_{max}, τ_{max};
(c) directions 1, 2, 3.

Generalized plane stress:

$$\sigma_1, \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sigma_{\tau 3} \pm \tau_3$$

$$\sigma_3 = \sigma_z \,, \qquad \tan 2\theta_n = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

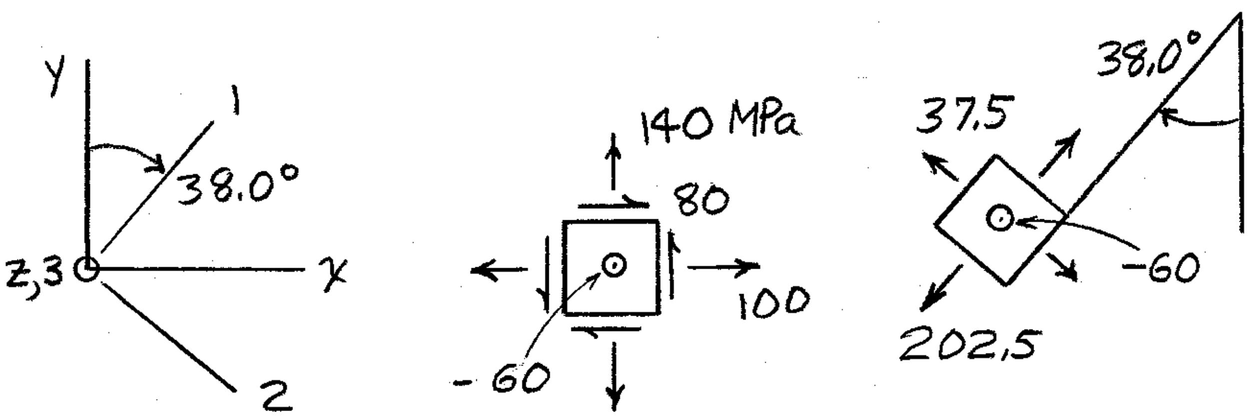
$$\tau_1, \tau_2, \tau_3 = \frac{|\sigma_2 - \sigma_3|}{2}, \frac{|\sigma_1 - \sigma_3|}{2}, \frac{|\sigma_1 - \sigma_2|}{2}$$

$$\sigma_{max} = \mathrm{MAX}(\sigma_1, \sigma_2, \sigma_3), \qquad \tau_{max} = \mathrm{MAX}(\tau_1, \tau_2, \tau_3)$$

Stresses in MPa

σ_x	σ_y	σ_z	τ_{xy}	$\sigma_{\tau 3}$	τ_3	σ_1	σ_2	σ_3
100	140	-60	80	120.00	82.46	202.46	37.54	-60.00

θ_n deg	τ_1	τ_2	σ_{max}	τ_{max}
-37.98	48.77	131.23	202.46	131.23



6.10 Given $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}$ as below, find:
(a) $\sigma_1, \sigma_2, \sigma_3$, and τ_1, τ_2, τ_3; (b) σ_{max}, τ_{max};
(c) directions 1, 2, 3.

Generalized plane stress:

$$\sigma_1, \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sigma_{\tau 3} \pm \tau_3$$

$$\sigma_3 = \sigma_z\,, \qquad \tan 2\theta_n = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

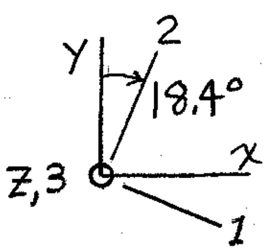
$$\tau_1, \tau_2, \tau_3 = \frac{|\sigma_2 - \sigma_3|}{2}, \frac{|\sigma_1 - \sigma_3|}{2}, \frac{|\sigma_1 - \sigma_2|}{2}$$

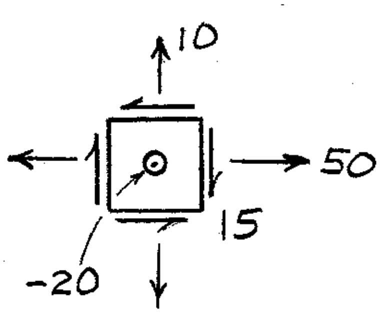
$$\sigma_{max} = \mathrm{MAX}(\sigma_1, \sigma_2, \sigma_3)\,, \qquad \tau_{max} = \mathrm{MAX}(\tau_1, \tau_2, \tau_3)$$

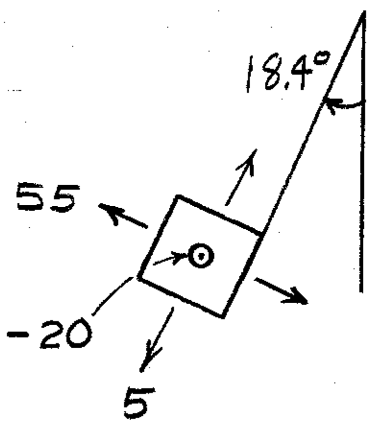
Stresses in MPa

σ_x	σ_y	σ_z	τ_{xy}	$\sigma_{\tau 3}$	τ_3	σ_1	σ_2	σ_3
50	10	-20	-15	30.00	25.00	55.00	5.00	-20.00

θ_n deg	τ_1	τ_2	σ_{max}	τ_{max}
-18.43	12.50	37.50	55.00	37.50







6.11 Given σ_x, σ_y, σ_z, τ_{xy} as below, find: (a) σ_1, σ_2, σ_3, and τ_1, τ_2, τ_3; (b) σ_{max}, τ_{max}; (c) directions 1, 2, 3.

Generalized plane stress:

$$\sigma_1, \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sigma_{\tau 3} \pm \tau_3$$

$$\sigma_3 = \sigma_z\,, \qquad \tan 2\theta_n = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

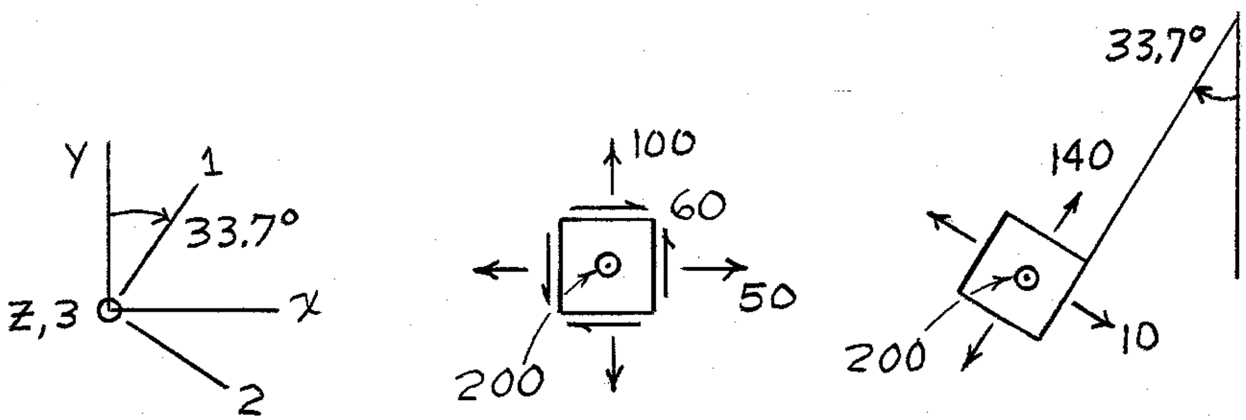
$$\tau_1, \tau_2, \tau_3 = \frac{|\sigma_2 - \sigma_3|}{2}, \frac{|\sigma_1 - \sigma_3|}{2}, \frac{|\sigma_1 - \sigma_2|}{2}$$

$$\sigma_{max} = \mathrm{MAX}(\sigma_1, \sigma_2, \sigma_3)\,, \qquad \tau_{max} = \mathrm{MAX}(\tau_1, \tau_2, \tau_3)$$

Stresses in MPa

σ_x	σ_y	σ_z	τ_{xy}	$\sigma_{\tau 3}$	τ_3	σ_1	σ_2	σ_3
50	100	200	60	75.00	65.00	140.00	10.00	200.00

θ_n deg	τ_1	τ_2	σ_{max}	τ_{max}
-33.69	95.00	30.00	200.00	95.00



6.12 Given $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}$ as below, find: (a) $\sigma_1, \sigma_2, \sigma_3$, and τ_1, τ_2, τ_3; (b) σ_{max}, τ_{max}; (c) directions 1, 2, 3.

Generalized plane stress;

$$\sigma_1, \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sigma_{\tau 3} \pm \tau_3$$

$$\sigma_3 = \sigma_z , \qquad \tan 2\theta_n = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

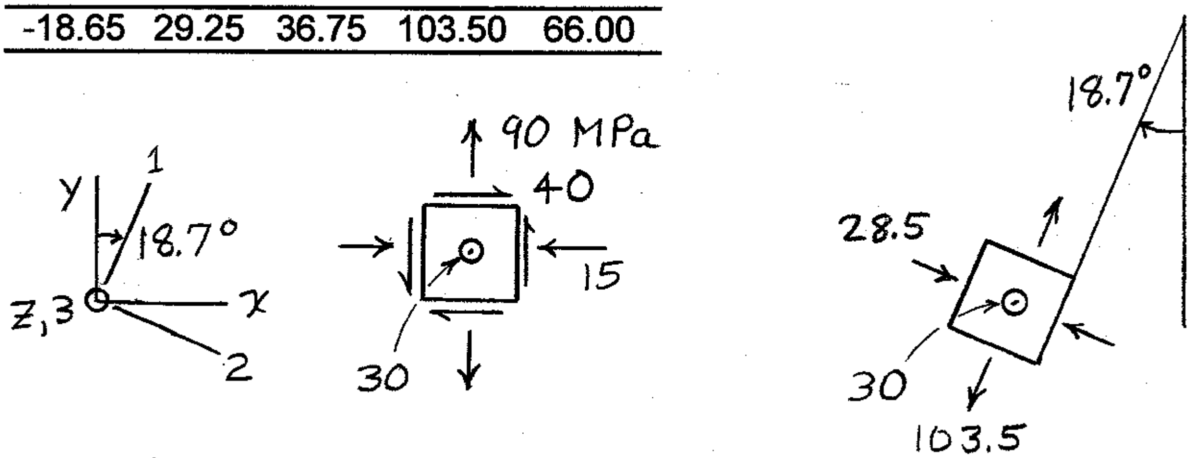
$$\tau_1, \tau_2, \tau_3 = \frac{|\sigma_2 - \sigma_3|}{2}, \frac{|\sigma_1 - \sigma_3|}{2}, \frac{|\sigma_1 - \sigma_2|}{2}$$

$$\sigma_{max} = \mathrm{MAX}(\sigma_1, \sigma_2, \sigma_3), \qquad \tau_{max} = \mathrm{MAX}(\tau_1, \tau_2, \tau_3)$$

Stresses in MPa

σ_x	σ_y	σ_z	τ_{xy}	$\sigma_{\tau 3}$	τ_3	σ_1	σ_2	σ_3
-15	90	30	40	37.50	66.00	103.50	-28.50	30

θ_n deg	τ_1	τ_2	σ_{max}	τ_{max}
-18.65	29.25	36.75	103.50	66.00



6.13 Plane stress, $\sigma_x = -50$, $\sigma_y = 40$, $\tau_{xy} = 0$. (a) Find $\sigma_1, \sigma_2, \sigma_3$, τ_{max}. (b) Show that for $\tau_{xy} = 0$, always $\sigma_1, \sigma_2 = \sigma_x, \sigma_y$.

$$\sigma_1, \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_1, \sigma_2 = \frac{-50+40}{2} \pm \sqrt{\left(\frac{-50-40}{2}\right)^2 + 0^2} = -5 \pm 45$$

$$\sigma_1, \sigma_2 = 40, -50, \quad \sigma_3 = \sigma_z = 0 \quad \text{MPa}$$ ◀

$$\tau_{max} = MAX\left(\frac{|\sigma_2 - \sigma_3|}{2}, \frac{|\sigma_1 - \sigma_3|}{2}, \frac{|\sigma_1 - \sigma_2|}{2}\right) = 45 \text{ MPa}$$ ◀

(b) In general

$$\sigma_1, \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + 0^2}$$

$$\sigma_1, \sigma_2 = \frac{\sigma_x}{2} + \frac{\sigma_y}{2} \pm \left(\frac{\sigma_x}{2} - \frac{\sigma_y}{2}\right) = \sigma_x, \sigma_y$$

$$\tan 2\theta_n = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{0}{\sigma_x - \sigma_y} = 0$$

Hence, the in-plane principal normal stresses are σ_x and σ_y if $\tau_{xy} = 0$, as the magnitudes are the same and the axis rotation is zero. ◀

6.14 For the surface strains on a steel part of Prob. 5.17, estimate σ_{max}, τ_{max}. Stresses calculated from Hooke's Law in Prob. 5.17 are $\sigma_x = -7.58$, $\sigma_y = -163.34$, $\tau_{xy} = 24.59$ MPa; other components zero due to free surface.

$$\sigma_1, \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}, \quad \sigma_3 = \sigma_z = 0$$

$$\sigma_1, \sigma_2 = -85.46 \pm 81.67 = -3.79, -167.13 \text{ MPa}$$

$$\sigma_{max} = 0 \quad ◀$$

$$\tau_{max} = MAX\left(\frac{|\sigma_2 - \sigma_3|}{2}, \frac{|\sigma_1 - \sigma_3|}{2}, \frac{|\sigma_1 - \sigma_2|}{2}\right)$$

$$\tau_{max} = 83.6 \text{ MPa} \quad ◀$$

6.15 For the surface strains on a Ti alloy part of Prob. 5.18, estimate σ_{max}, τ_{max}. Stresses calculated from Hooke's Law in Prob. 5.18 are $\sigma_x = 532.3$, $\sigma_y = 211.4$, and $\tau_{xy} = 31.74$ MPa; other components zero due to free surface.

$$\sigma_1, \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_1, \sigma_2 = 371.8 \pm 163.6 = 535.4,\ 208.2 \text{ MPa}$$

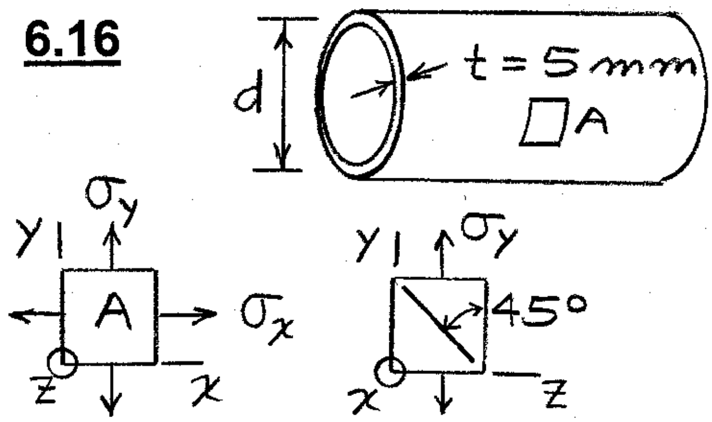
$$\sigma_3 = \sigma_z = 0$$

$$\sigma_{max} = 535.4 \text{ MPa} \quad \blacktriangleleft$$

$$\tau_{max} = \mathrm{MAX}\left(\frac{|\sigma_2 - \sigma_3|}{2}, \frac{|\sigma_1 - \sigma_3|}{2}, \frac{|\sigma_1 - \sigma_2|}{2}\right)$$

$$\tau_{max} = 267.7 \text{ MPa} \quad \blacktriangleleft$$

6.16

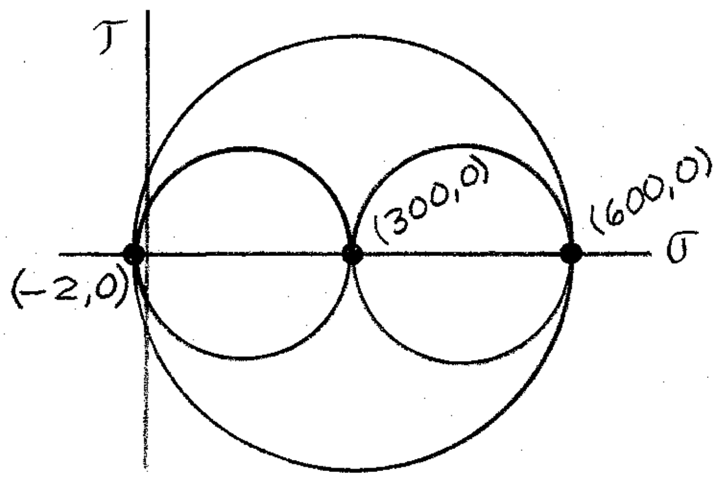


$d = 3\ m$ (inner)
$p = 2\ MPa$
$\sigma_{max},\ \tau_{max} = ?$
What planes?

$$\sigma_x = \frac{p r_1}{2t} = \frac{(2\ MPa)(1500\ mm)}{2(5\ mm)} = 300\ MPa$$

$$\sigma_y = \frac{p r_1}{t} = 600\ MPa$$

$\sigma_z = 0$ (outside), $\sigma_z = -p = -2\ MPa$ (inside)

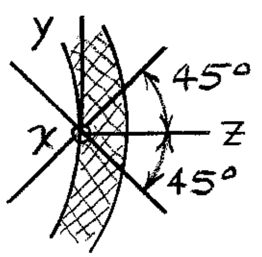


$\sigma_x, \sigma_y, \sigma_z = \sigma_1, \sigma_2, \sigma_3$
Inside gives the largest possible Mohr's circle, and so controls τ_{max}.

$\sigma_{max} = \sigma_y = 600\ MPa$ in hoop (y) dir.

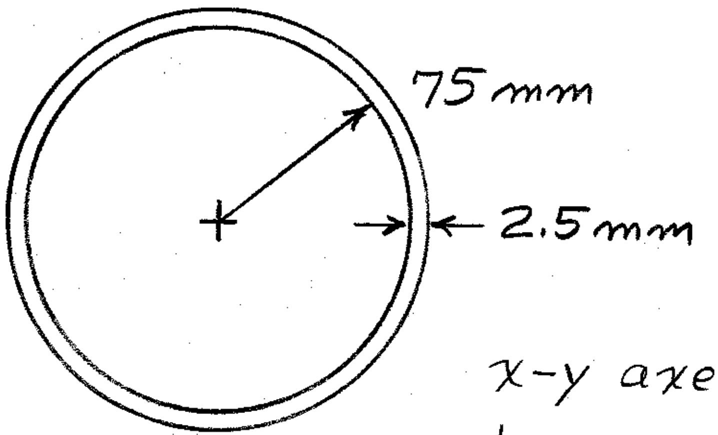
$$\tau_{max} = MAX\left(\frac{|\sigma_2 - \sigma_3|}{2}, \frac{|\sigma_1 - \sigma_3|}{2}, \frac{|\sigma_1 - \sigma_2|}{2}\right)$$

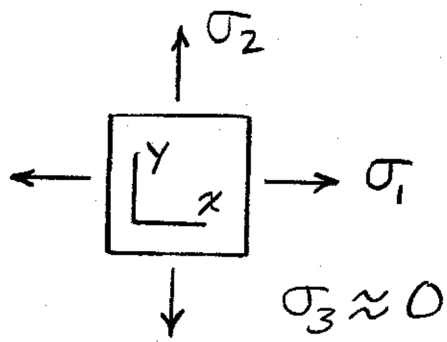
$$\tau_{max} = MAX(301, 151, 150) = 301\ MPa$$



τ_{max} acts on planes rotated 45° about the longitudinal (x) axis.

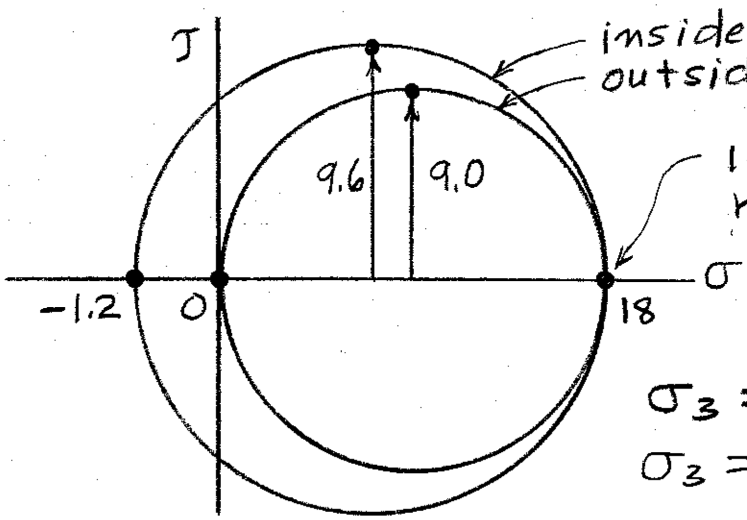
6.17 Spherical vessel, $p = 1.2$ MPa





x–y axes are any ⊥ dir. tangent to surface.
x–y are (1–2) prin. axes.

$$\sigma_1 = \sigma_2 = \frac{pr}{2t} = \frac{(1.2\ \text{MPa})(75\ \text{mm})}{2\,(2.5\ \text{mm})} = 18.0\ \text{MPa}$$



inside
outside } 2–3 and 3–1 planes

1–2 plane Mohr's circle reduces to a point.

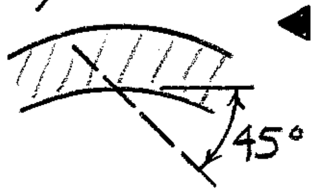
$\sigma_3 = 0$ (outside)
$\sigma_3 = -p = -1.2$ MPa (ins.)

Inside has largest possible Mohr's circle, controls.

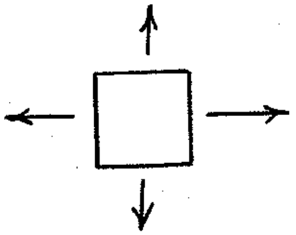
$$\tau_{max} = \text{MAX}\left(\frac{|\sigma_2 - \sigma_3|}{2},\ \frac{|\sigma_1 - \sigma_3|}{2},\ \frac{|\sigma_1 - \sigma_2|}{2}\right)$$

$$\tau_{max} = \text{MAX}(9.6, 9.6, 0) = 9.6\ \text{MPa}$$

on any plane 45° to surface.



6.18 Closed-end pipe, $d_2 = 80$, $t = 2.0$ mm, $p = 10$ MPa, $M = 2.0$ kN.m. Find σ_{max}, τ_{max}.
Use Fig. A.7(a) and A.8.
$r_2 = 40$, $r_1 = 38$, $r_{avg} = 39$ mm



$$\sigma_x = \frac{p r_1}{2t} \pm \frac{M}{\pi r_{avg}^2 t}$$

$$\sigma_y = \sigma_t = \frac{p r_1}{t}$$

Bending stress highest for $y_{avg} = r_{avg}$
$\pm$, T or C side

$\sigma_z = \sigma_r = -p$ (inside), $\sigma_z = \sigma_r = 0$ (outside)

$$\sigma_y = \frac{(10\text{ MPa})(38\text{ mm})}{2\text{ mm}} = 190.0\text{ MPa}$$

$$\sigma_x = 95.0 \pm \frac{2.0 \times 10^6\ \text{N}\cdot\text{mm}}{\pi (39)^2 (2.0)\ \text{mm}^3} = 95.0 \pm 209.3\ \text{MPa}$$

$\sigma_x = 304.3$ (T-side), $\sigma_x = -114.3$ (C-side)

Consider the four combinations of inside or outside wall on tension or compression side of bending axis.

$$\sigma_1, \sigma_2, \sigma_3 = \sigma_x, \sigma_y, \sigma_z \qquad (\text{since } \tau_{xy}, \tau_{yz}, \tau_{zx} = 0)$$

$$\tau_1, \tau_2, \tau_3 = \frac{|\sigma_2 - \sigma_3|}{2}, \frac{|\sigma_1 - \sigma_3|}{2}, \frac{|\sigma_1 - \sigma_2|}{2}$$

$$\sigma_{max} = \text{MAX}(\sigma_1, \sigma_2, \sigma_3), \qquad \tau_{max} = \text{MAX}(\tau_1, \tau_2, \tau_3)$$

(6.18, p, 2)

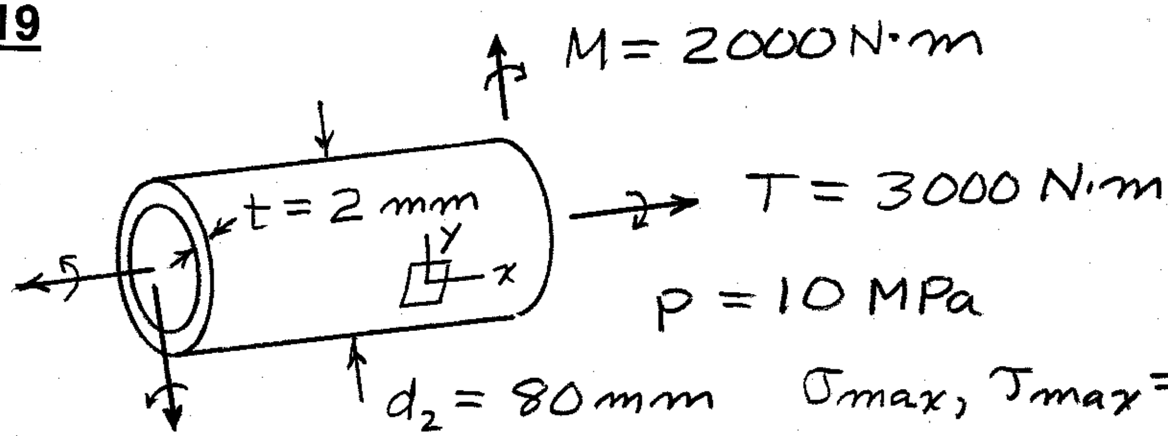
Stresses in MPa

Location	$\sigma_x = \sigma_1$	$\sigma_y = \sigma_2$	$\sigma_z = \sigma_3$	τ_1	τ_2	τ_3	σ_{max}	τ_{max}
outside, T	304.3	190.0	0.0	95.0	152.1	57.1	304.3	152.1
inside, T	304.3	190.0	-10.0	100.0	157.1	57.1	304.3	157.1
outside, C	-114.3	190.0	0.0	95.0	57.1	152.1	190.0	152.1
inside, C	-114.3	190.0	-10.0	100.0	52.1	152.1	190.0	152.1

$\sigma_{max} = 304.3$ MPa on tension side of bending, same for inside or outside. ◀

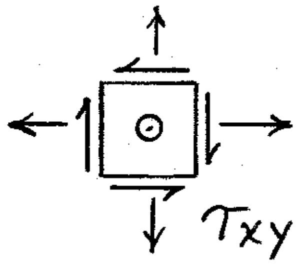
$\tau_{max} = 157.1$ MPa, tension side, inner wall ◀

6.19



$d_2 = 80\ mm \quad \sigma_{max}, \tau_{max} = ?$

$r_2 = 40,\ r_1 = 38,\ r_{avg} = 39\ mm$



$$\sigma_x = \frac{p r_1}{2t} \pm \frac{M r_{avg}}{I}, \quad I = \pi r_{avg}^3 t$$

$\pm$ T or C side

$$\tau_{xy} = \frac{T r_{avg}}{J}, \quad J = 2I$$

$$\sigma_y = \frac{p r_1}{t}$$

$$\sigma_y = \frac{(10\ MPa)(38\ mm)}{2\ mm} = 190\ MPa$$

$$\sigma_x = 95\ MPa \pm \frac{(2 \times 10^6\ N\cdot mm)(39\ mm)}{\pi (39\ mm)^3 (2\ mm)} = \begin{matrix} 304.3, \\ -114.3 \end{matrix}\ MPa$$

$$\tau_{xy} = \frac{(3 \times 10^6)(39)}{2\pi (39)^3 (2)} = 157.0\ MPa$$

$\sigma_z = 0$ (outside), $\sigma_z = -p = -10\ MPa$ (inside)

Consider the four combinations of inside or outside wall on tension or compression side of bending.

(6.19, p.2)

$$\sigma_1, \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sigma_{\tau 3} \pm \tau_3 , \qquad \sigma_3 = \sigma_z$$

$$\tau_1, \tau_2, \tau_3 = \frac{|\sigma_2 - \sigma_3|}{2}, \frac{|\sigma_1 - \sigma_3|}{2}, \frac{|\sigma_1 - \sigma_2|}{2}$$

$$\sigma_{max} = MAX(\sigma_1, \sigma_2, \sigma_3), \qquad \tau_{max} = MAX(\tau_1, \tau_2, \tau_3)$$

Stresses in MPa

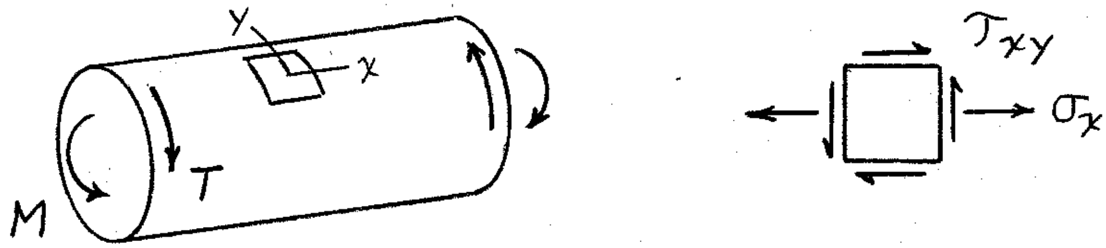
Location	σ_x	σ_y	σ_z	τ_{xy}	$\sigma_{\tau 3}$	τ_3
outside, T	304.3	190.0	0.0	157.0	247.1	167.0
inside, T	304.3	190.0	-10.0	157.0	247.1	167.0
outside, C	-114.3	190.0	0.0	157.0	37.9	218.6
inside, C	-114.3	190.0	-10.0	157.0	37.9	218.6

Location	σ_1	σ_2	σ_3	τ_1	τ_2	σ_{max}	τ_{max}
outside, T	414.2	80.1	0.0	40.1	207.1	414.2	207.1
inside, T	414.2	80.1	-10.0	45.1	212.1	414.2	212.1
outside, C	256.5	-180.7	0.0	90.4	128.2	256.5	218.6
inside, C	256.5	-180.7	-10.0	85.4	133.2	256.5	218.6

$\sigma_{max} = 414.2$ MPa on tension side of bending, same for inside or outside. ◀

$\tau_{max} = 218.6$ MPa on compression side, same inside or outside. ◀

6.20 Solid shaft, $d = 25$ mm, $M = 250$ N·m, $T = 600$ N·m. Find σ_{max}, τ_{max}.



$$\sigma_x = \frac{Mr}{I}, \quad I = \frac{\pi r^4}{4} \quad (T \text{ side, max. from NA})$$

$$\sigma_x = \frac{(250 \times 10^3\ \text{N·mm})(12.5\ \text{mm})}{\pi\,(12.5\ \text{mm})^4/4} = 163.0\ \text{MPa}$$

$$\tau_{xy} = \frac{Tr}{J}, \quad J = \frac{\pi r^4}{2} \quad (\text{outside surface})$$

$$\tau_{xy} = \frac{(600 \times 10^3)(12.5)}{\pi\,(12.5)^4/2} = 195.6\ \text{MPa}$$

$$\sigma_1, \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sigma_{\tau 3} \pm \tau_3, \qquad \sigma_3 = \sigma_z$$

$$\tau_1, \tau_2, \tau_3 = \frac{|\sigma_2 - \sigma_3|}{2}, \frac{|\sigma_1 - \sigma_3|}{2}, \frac{|\sigma_1 - \sigma_2|}{2}$$

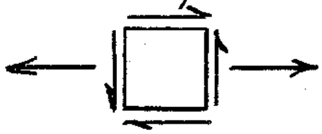
$$\sigma_{max} = \text{MAX}(\sigma_1, \sigma_2, \sigma_3), \qquad \tau_{max} = \text{MAX}(\tau_1, \tau_2, \tau_3)$$

Stresses in MPa

σ_x	σ_y	σ_z	τ_{xy}	$\sigma_{\tau 3}$	τ_3	σ_1	σ_2
163.0	0.0	0.0	195.6	81.5	211.9	293.4	-130.4

σ_3	τ_1	τ_2	σ_{max}	τ_{max}
0.0	65.2	146.7	293.4	211.9

6.21 Solid shaft, diameter d, moment M and torque T. (a) $\tau_{max} = f(d, M, T)$



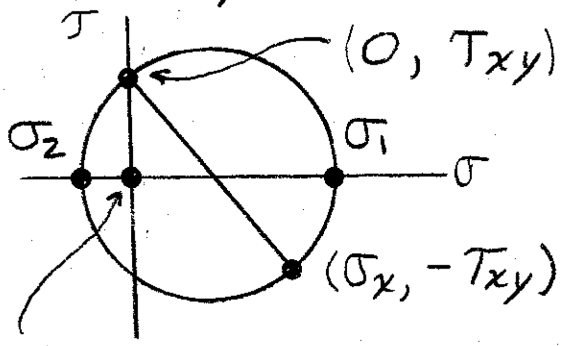
$\sigma_y = 0$

$$\sigma_x = \frac{Mr}{I} = \frac{M(d/2)}{\pi d^4/64} = \frac{32M}{\pi d^3}$$

$$\tau_{xy} = \frac{Tr}{J} = \frac{T(d/2)}{\pi d^4/32} = \frac{16T}{\pi d^3}$$

$$\sigma_1, \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}, \quad \sigma_3 = 0$$

Since $\sigma_y = 0$, $\sigma_1 > 0$, $\sigma_2 < 0$



Hence σ_1 and σ_2 determine τ_{max}, as $\sigma_3 = \sigma_z = 0$.

$$\tau_{max} = MAX\left[\frac{|\sigma_1 - \sigma_2|}{2}, \frac{|\sigma_2 - \sigma_3|}{2}, \frac{|\sigma_1 - \sigma_3|}{2}\right] = \frac{\sigma_1 - \sigma_2}{2}$$

$$\tau_{max} = \frac{1}{2}\left[2\sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2}\right] = \sqrt{\left(\frac{16M}{\pi d^3}\right)^2 + \left(\frac{16T}{\pi d^3}\right)^2}$$

$$\tau_{max} = \frac{16}{\pi d^3}\sqrt{M^2 + T^2}$$ ◀

(b) $d \geq ?$ for $\tau_{max} \leq 180$ MPa

$$d = \left[\frac{16}{\pi \tau_{max}}\sqrt{M^2 + T^2}\right]^{1/3}$$

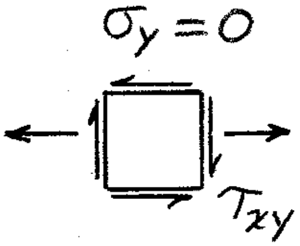
(6.21, p.2)

$M = 250,\ T = 600\ \text{N}\cdot\text{m}$

$$d = \left[\frac{16}{\pi(180\ \text{N/mm}^2)}\sqrt{(250{,}000\ \text{N}\cdot\text{mm})^2 + (600{,}000\ \text{N}\cdot\text{mm})^2}\right]^{\frac{1}{3}}$$

$d = 26.40\ \text{mm}$ ◀

6.22 Solid shaft, $d = 50$ mm, under axial load $P = 200$ kN and torque $T = 1500$ N·m. Find (a) τ_{max}, (b) σ_{max}.

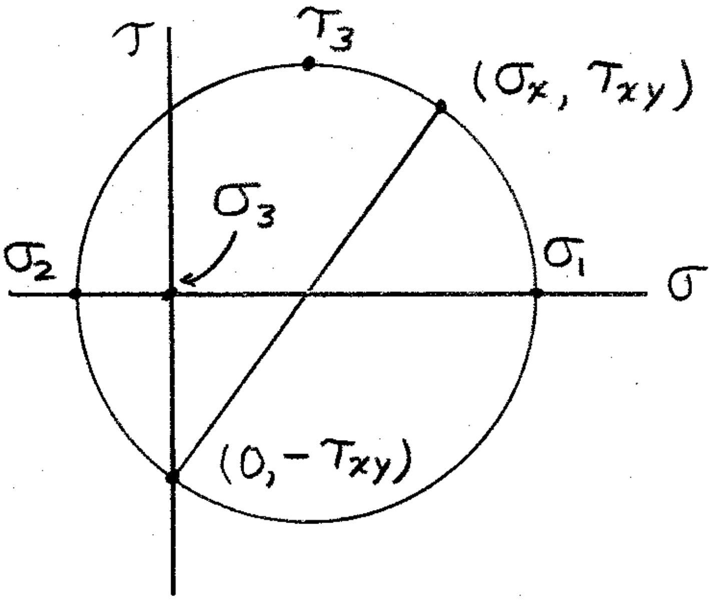


$$\sigma_x = \frac{P}{A} = \frac{4P}{\pi d^2} = \frac{4(200{,}000\text{ N})}{\pi(50\text{ mm})^2}$$

$$\sigma_x = 101.86\text{ MPa}$$

$$\tau_{xy} = \frac{Tc}{J} = \frac{T(d/2)}{\pi d^4/32} = \frac{16T}{\pi d^3}$$

$$\tau_{xy} = \frac{16(1.5\times10^6\text{ N}\cdot\text{mm})}{\pi(50\text{ mm})^3} = 61.12\text{ MPa}$$



$\sigma_z = \sigma_3 = 0$

$\sigma_{max} = \sigma_1$

$$\tau_{max} = \text{MAX}\left(\frac{|\sigma_1 - \sigma_2|}{2}, \frac{|\sigma_2 - \sigma_3|}{2}, \frac{|\sigma_1 - \sigma_3|}{2}\right)$$

$$\tau_{max} = \frac{|\sigma_1 - \sigma_2|}{2} = \tau_3$$

(6.22, p.2)

$$\sigma_1, \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

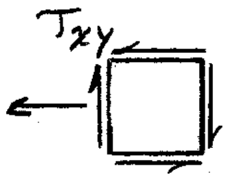
$$\sigma_1, \sigma_2 = 50.93 \pm 79.55 = 130.48, -28.62$$

$$\sigma_{max} = 130.5 \text{ MPa} \quad \blacktriangleleft$$

$$\tau_{max} = \frac{|\sigma_1 - \sigma_2|}{2} = 79.6 \text{ MPa} \quad \blacktriangleleft$$

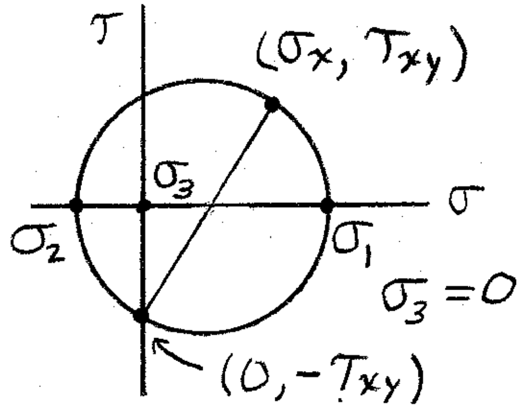
6.23 Solid shaft, diameter d, axial load P, torque T.

(a) $\tau_{max} = f(d, P, T)$



$\sigma_y = 0$

$$\sigma_x = \frac{P}{A} = \frac{4P}{\pi d^2}$$



$$\tau_{xy} = \frac{Tr}{J} = \frac{T(d/2)}{\pi d^4/32} = \frac{16T}{\pi d^3}$$

$$\tau_{max} = MAX\left(\frac{|\sigma_1 - \sigma_2|}{2}, \frac{\sigma_2 - \sigma_3}{2}, \frac{|\sigma_1 - \sigma_3|}{2}\right)$$

Noting Mohr's circle with $\sigma_1 > 0$, $\sigma_2 < 0$, $\sigma_3 = 0$,

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2}, \quad \sigma_1, \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tau_{max} = \sqrt{\frac{\sigma_x^2}{4} + \tau_{xy}^2} = \sqrt{\frac{1}{4}\left(\frac{4P}{\pi d^2}\right)^2 + \left(\frac{16T}{\pi d^3}\right)^2}$$

$$\tau_{max} = \frac{2}{\pi d^2}\sqrt{P^2 + \left(\frac{8T}{d}\right)^2}$$ ◀

(b) $P = 200{,}000\text{ N}$, $T = 1.5 \times 10^6\text{ N}\cdot\text{mm}$

What min. d so $\tau_{max} < 100\text{ MPa}$?

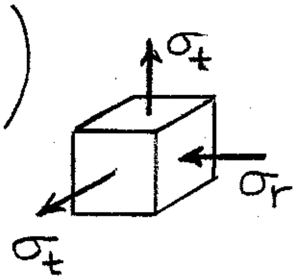
Cannot solve for d above closed form.

Substitute P, T, τ_{max} in units of N, mm, MPa

Solve iteratively for $d = 45.8\text{ mm}$. ◀

6.25 Thick-walled spherical pressure vessel.
(a) Derive $\tau_{max} = f(r_1, r_2, R, p)$; show that overall τ_{max} is at $R = r_1$.
(b) Determine $\sigma_1, \sigma_2, \sigma_3$, and τ_1, τ_2, τ_3, for $d_1 = 100$, $d_2 = 150$, $p = 300$ MPa, $R = r_1$.
(c) For (b), plot σ_r, σ_t, and τ_{max} vs. R.

(a) $\sigma_t = C\left(\dfrac{r_2^3}{2R^3} + 1\right)$, $\sigma_r = -C\left(\dfrac{r_2^3}{R^3} - 1\right)$



where $C = \dfrac{p r_1^3}{r_2^3 - r_1^3}$ (Fig. A.6(b))

$\sigma_1, \sigma_2, \sigma_3 = \sigma_t, \sigma_t, \sigma_r$ (all τ's zero)

$$\tau_{max} = MAX\left(\frac{|\sigma_2 - \sigma_3|}{2}, \frac{|\sigma_1 - \sigma_3|}{2}, \frac{|\sigma_1 - \sigma_2|}{2}\right) = \frac{\sigma_t - \sigma_r}{2}$$

$$\tau_{max} = \frac{C}{2}\left[\frac{r_2^3}{2R^2} + 1 + \frac{r_2^3}{R^3} - 1\right] = \frac{C}{2}\left[\frac{3r_2^3}{2R^3}\right]$$

$$\tau_{max} = \frac{3 p r_1^3 r_2^3}{4R^3(r_2^3 - r_1^3)}$$

As $R\uparrow$, $\tau_{max}\downarrow$, so overall τ_{max} is at $R = r_1$. ◀

(b) $C = \dfrac{(300\text{ MPa})(50\text{ mm})^3}{(75^3 - 50^3)\text{ mm}^3} = 126.3$ MPa

$$\sigma_t = (126.3\text{ MPa})\left(\frac{(75\text{ mm})^3}{2(50\text{ mm})^3} + 1\right) = 339.5\text{ MPa}$$

$$\sigma_r = -126.3\left(\frac{75^3}{50^3} - 1\right) = -300\text{ MPa} \quad (-p)$$

$\sigma_1, \sigma_2, \sigma_3 = 339.5, 339.5, -300$ MPa ◀

(6.25, p. 2)

$$\tau_1, \tau_2, \tau_3 = \frac{|\sigma_2-\sigma_3|}{2}, \frac{|\sigma_1-\sigma_3|}{2}, \frac{|\sigma_1-\sigma_2|}{2}$$

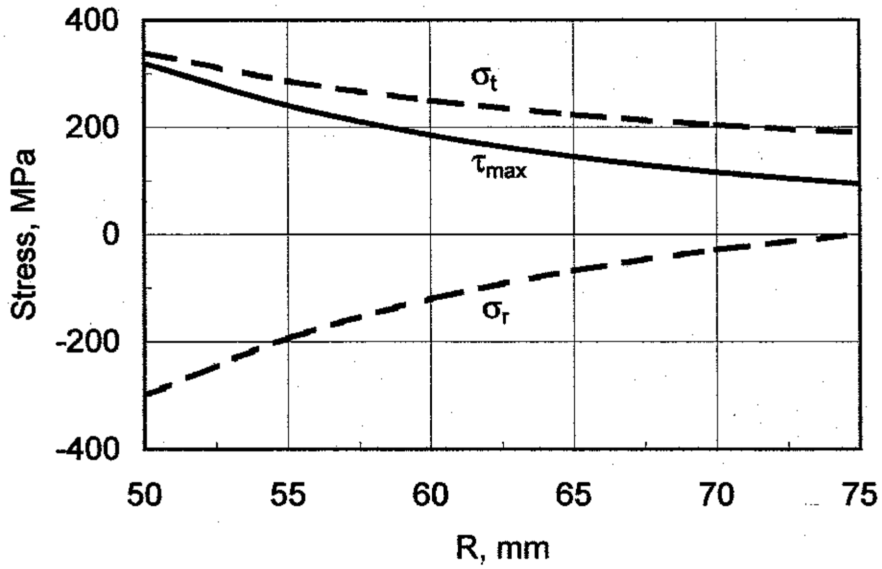
$\tau_1, \tau_2, \tau_3 = 319.7,\ 319.7,\ 0$ MPa ◀

(c) Calculate σ_r, σ_t, and τ_{max} for various R, where $r_1 = 50$, $r_2 = 75$ mm, $p = 300$ MPa.

Stresses in MPa

R, mm	σ_t	σ_r	τ_{max}
50.0	339.5	-300.0	319.7
55.0	286.5	-194.0	240.2
60.0	249.7	-120.4	185.0
65.0	223.3	-67.7	145.5
70.0	204.0	-29.0	116.5
75.0	189.5	0.0	94.7

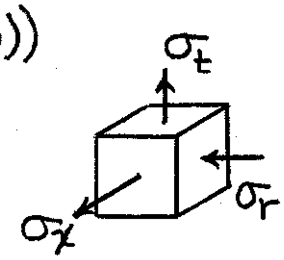
These values are plotted below.



◀

6.26 Thick-walled tube with closed ends.
(a) Derive $\tau_{max} = f(r_1, r_2, R, p)$; show that overall τ_{max} is at $R = r_1$.
(b) Determine $\sigma_1, \sigma_2, \sigma_3$, and τ_1, τ_2, τ_3, for $d_1 = 80$, $d_2 = 100$ mm, $p = 100$ MPa, $R = r_1$.
(c) For (b), plot $\sigma_r, \sigma_t, \sigma_x$, and τ_{max} vs. R.

(a) $$\sigma_x = \frac{p r_1^2}{r_2^2 - r_1^2} = C \qquad \text{(Fig. A.6 (b))}$$



$$\sigma_t = C\left(\frac{r_2^2}{R^2} + 1\right), \quad \sigma_r = -C\left(\frac{r_2^2}{R^2} - 1\right)$$

$$\sigma_1, \sigma_2, \sigma_3 = \sigma_x, \sigma_t, \sigma_r \quad \text{(all } \tau\text{'s zero)}$$

$$\tau_{max} = \text{MAX}(\tau_1, \tau_2, \tau_3)$$

$$\tau_1 = \frac{|\sigma_2 - \sigma_3|}{2} = \frac{C}{2}\left[\frac{r_2^2}{R^2} + 1 + \frac{r_2^2}{R^2} - 1\right] = \frac{C r_2^2}{R^2}$$

$$\tau_2 = \frac{|\sigma_1 - \sigma_3|}{2} = \frac{C}{2}\left[1 + \frac{r_2^2}{R^2} - 1\right] = \frac{C r_2^2}{2R^2}$$

$$\tau_3 = \frac{|\sigma_1 - \sigma_2|}{2} = \frac{C}{2}\left[\left|1 - \frac{r_2^2}{R^2} - 1\right|\right] = \frac{C r_2^2}{2R^2}$$

$$\tau_{max} = \frac{C r_2^2}{R^2} = \frac{p r_1^2 r_2^2}{R^2 (r_2^2 - r_1^2)}$$ As $R\uparrow$, $\tau_{max}\downarrow$, so overall τ_{max} at r_1. ◀

(b) $$C = \frac{(100\text{ MPa})(40\text{ mm})^2}{(50^2 - 40^2)\text{ mm}^2} = 177.8\text{ MPa} = \sigma_x$$

$$\sigma_t = (177.8\text{ MPa})\left(\frac{(50\text{ mm})^2}{(40\text{ mm})^2} + 1\right) = 455.6\text{ MPa}$$

(6.26, p. 2)

$$\sigma_r = -177.8\left(\frac{50^2}{40^2} - 1\right) = -100 \text{ MPa} \quad (-p)$$

$\sigma_1, \sigma_2, \sigma_3 = 177.8, 455.6, -100$ MPa

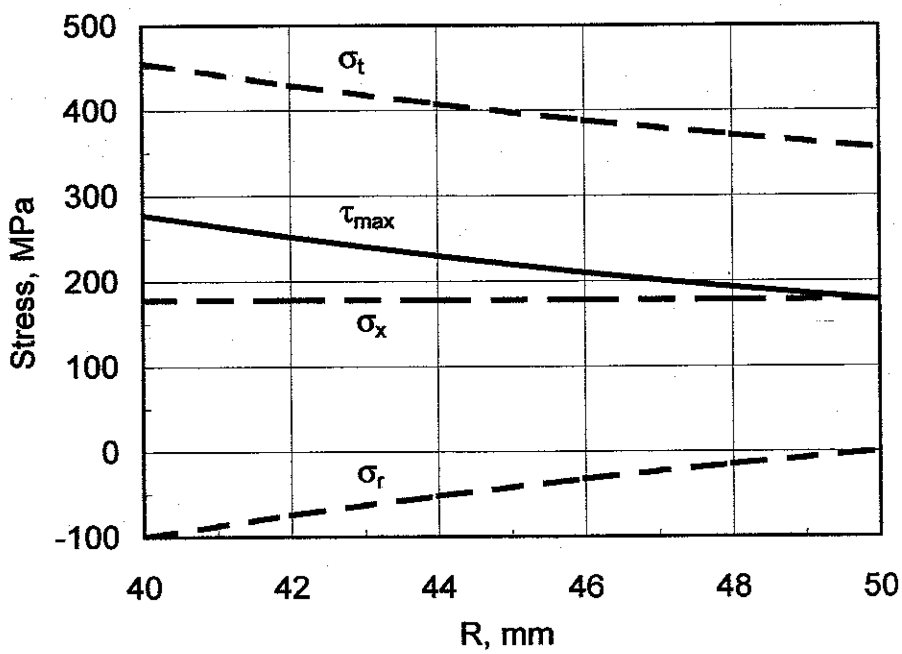
$\tau_1, \tau_2, \tau_3 = 277.8, 138.9, 138.9$ MPa

(c) Calculate $\sigma_x, \sigma_t, \sigma_r$ and τ_{max} for various R, where $r_1 = 40$, $r_2 = 50$ mm, $p = 100$ MPa.

Stresses in MPa

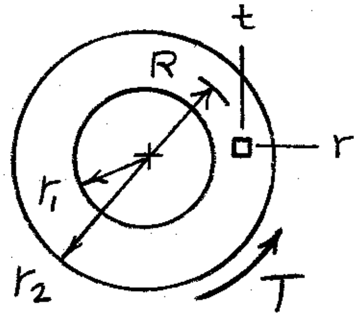
R, mm	σ_x	σ_t	σ_r	τ_{max}
40.0	177.8	455.6	-100.0	277.8
42.0	177.8	429.7	-74.2	252.0
44.0	177.8	407.3	-51.8	229.6
46.0	177.8	387.8	-32.3	210.0
48.0	177.8	370.7	-15.1	192.9
50.0	177.8	355.6	0.0	177.8

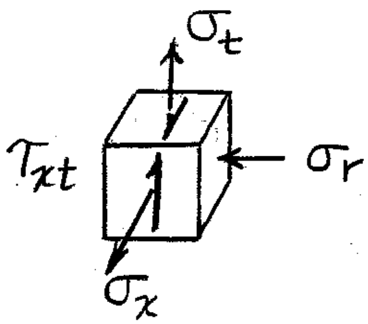
These values are plolled below.



6.27 Thick-walled tube, closed ends.

$p = 100\ \text{MPa}$, $T = 15\ \text{kN}\cdot\text{m}$, $r_1 = 30\ \text{mm}$, $r_2 = 40\ \text{mm}$, $\tau_{max} = ?$





$$\tau_{xt} = \frac{TR}{J}, \qquad J = \frac{\pi}{2}\left(r_2^4 - r_1^4\right)$$

$$\sigma_x = \frac{p r_1^2}{r_2^2 - r_1^2}$$

$$\sigma_t = \sigma_x\left(\frac{r_2^2}{R^2} + 1\right), \qquad \sigma_r = -\sigma_x\left(\frac{r_2^2}{R^2} - 1\right)$$

$$\sigma_r = \sigma_3$$

$$\sigma_1, \sigma_2 = \frac{\sigma_x + \sigma_t}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_t}{2}\right)^2 + \tau_{xt}^2}$$

$$\tau_{max} = \text{MAX}(\tau_1, \tau_2, \tau_3) = \text{MAX}\left(\frac{|\sigma_2 - \sigma_3|}{2}, \frac{|\sigma_1 - \sigma_3|}{2}, \frac{|\sigma_1 - \sigma_2|}{2}\right)$$

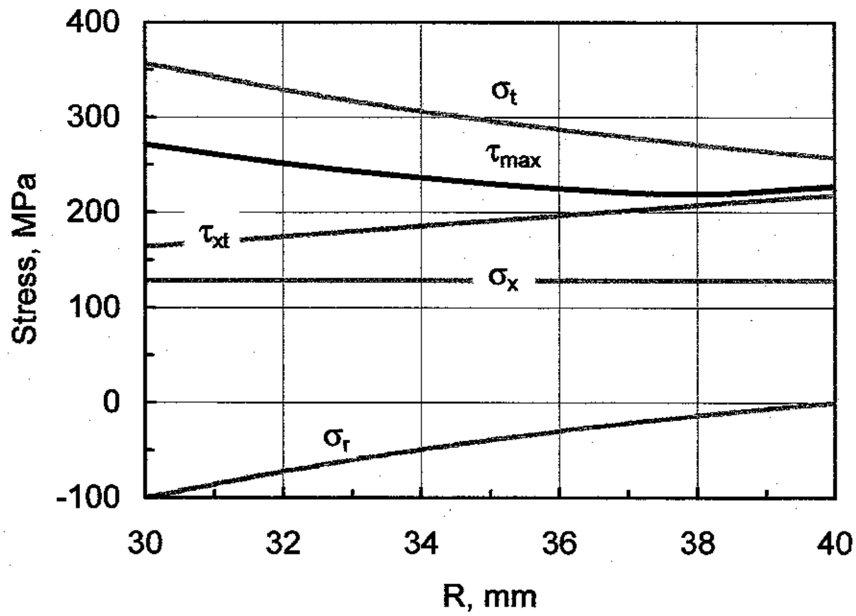
Calculate stresses σ_x, σ_t, σ_r, τ_{xt}, and from these σ_1, σ_2, σ_3, and τ_1, τ_2, τ_3, and finally τ_{max}. Do so for each of several values of R as in the table following.

(6.27, p.2)

Stresses in MPa

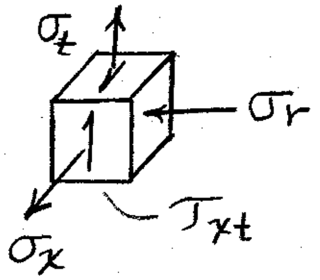
R, mm	σ_x	σ_t	$\sigma_r = \sigma_3$	τ_{xt}	σ_1	σ_2	τ_{max}
30.0	128.6	357.1	-100.0	163.7	442.5	43.2	271.3
32.0	128.6	329.5	-72.3	174.6	430.5	27.6	251.4
34.0	128.6	306.5	-49.4	185.5	423.3	11.8	236.3
36.0	128.6	287.3	-30.2	196.4	419.8	-3.9	225.0
38.0	128.6	271.0	-13.9	207.4	419.1	-19.4	219.2
40.0	128.6	257.1	0.0	218.3	420.4	-34.7	227.5

τ_{max} decreases with R, has a minimum around R = 38 mm, and increases beyond the. The overall τ_{max} = 271.3 MPa at r_1.



6.28 Thick-walled tube, closed ends

$p = 50$ MPa, $T = 12.5$ kN·m $= 12.5 \times 10^6$ N·mm,
$r_1 = 25$, $r_2 = 45$ mm (a) $\tau_{max} = ?$



$$\tau_{xt} = \frac{TR}{J}, \quad J = \frac{\pi}{2}(r_2^4 - r_1^4)$$

$$J = 5.828 \times 10^6 \text{ mm}^4$$

$$\sigma_x = \frac{p r_1^2}{r_2^2 - r_1^2}, \quad R \text{ varies}$$

$$\sigma_t = \sigma_x \left(\frac{r_2^2}{R^2} + 1\right), \quad \sigma_r = -\sigma_x \left(\frac{r_2^2}{R^2} - 1\right)$$

$$\sigma_r = \sigma_3$$

$$\sigma_1, \sigma_2 = \frac{\sigma_x + \sigma_t}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_t}{2}\right)^2 + \tau_{xt}^2}$$

$$\tau_{max} = MAX\left(\frac{|\sigma_2 - \sigma_3|}{2}, \frac{|\sigma_1 - \sigma_3|}{2}, \frac{|\sigma_1 - \sigma_2|}{2}\right)$$

For several R from r_1 to r_2, calculate σ_x, σ_t, σ_r, and τ_{xt}, then σ_1, σ_2, σ_3 and finally τ_{max}.

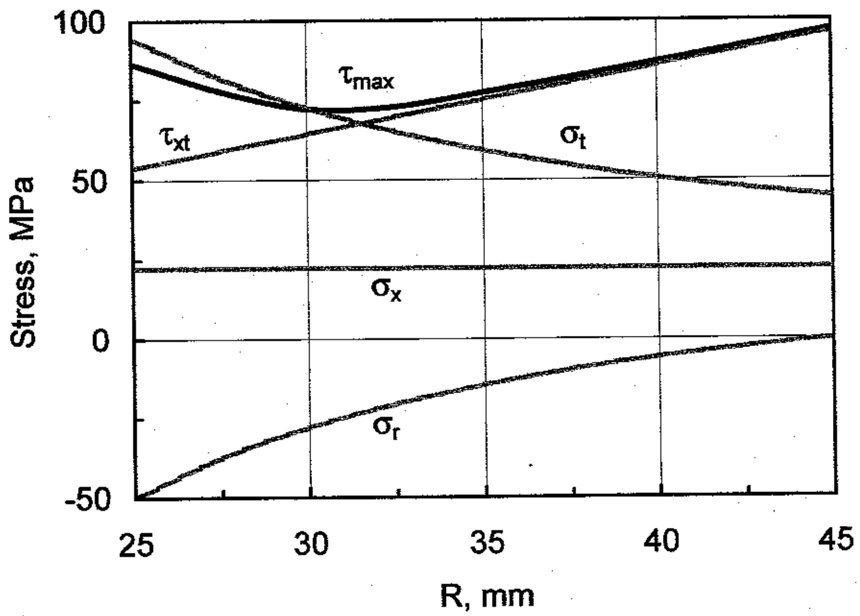
Stresses in MPa

R, mm	σ_x	σ_t	$\sigma_r = \sigma_3$	τ_{xt}	σ_1	σ_2	τ_{max}
25.0	22.3	94.6	-50.0	53.6	123.2	-6.2	86.6
27.5	22.3	82.1	-37.4	59.0	118.3	-13.9	77.9
30.0	22.3	72.5	-27.9	64.3	116.5	-21.6	72.2
32.5	22.3	65.1	-20.5	69.7	116.6	-29.2	72.9
35.0	22.3	59.2	-14.6	75.1	118.1	-36.5	77.3
37.5	22.3	54.5	-9.8	80.4	120.4	-43.6	82.0
40.0	22.3	50.6	-5.9	85.8	123.4	-50.5	87.0
42.5	22.3	47.3	-2.7	91.2	126.8	-57.2	92.0
45.0	22.3	44.6	0.0	96.5	130.6	-63.7	97.2

(6.28, p.2)

τ_{max} first decreases with R, then reaches a minimum, and increases to a maximum value of 97.2 MPa at $R = r_2$.

(b) Plot σ_x, σ_t, σ_r, τ_{xt}, and τ_{max}, all versus R.



6.29 Rotating annular disc, Fig. A.9, $r_1 = 90$, $r_2 = 300$, $t = 50$ mm, $f = 120$ rev/s, of alloy steel.
(a) Calculate and plot σ_r and σ_t versus R.
(b) Find and locate σ_{max}, τ_{max}.
(c) Show that σ_{max}, τ_{max} always at $R = r_1$.

(a) For various $z = R/r_2$, calculate σ_r, σ_t. Equations and details for $\rho\omega^2 r_2^2$ are below, and results and plot on the next pg.

$$\alpha = r_1/r_2, \qquad z = R/r_2$$

$$\sigma_r = \rho\omega^2 r_2^2 \left(\frac{3+\nu}{8}\right)\left[1+\alpha^2 - z^2 - \frac{\alpha^2}{z^2}\right]$$

$$\sigma_t = \rho\omega^2 r_2^2 \left(\frac{3+\nu}{8}\right)\left[1+\alpha^2 - \frac{1+3\nu}{3+\nu} z^2 + \frac{\alpha^2}{z^2}\right]$$

$$\sigma_x = 0 \qquad \text{(plane stress)}$$

$$\rho\omega^2 r_2^2 = \left(7.87\frac{\text{Mg}}{\text{m}^3} \times \frac{1000\text{ kg}}{\text{Mg}} \times \frac{\text{N}}{\text{kg}\cdot\text{m/s}^2}\right)\left(120\frac{\text{rev}}{\text{s}} \times 2\pi\frac{\text{rad}}{\text{rev}}\right)^2 (0.300\text{ m})^2$$

$$\rho\omega^2 r_2^2 = 402.7 \times 10^6 \frac{\text{N}}{\text{m}^2} = 402.7\text{ MPa}$$

(b) Since τ's are zero, $\sigma_r, \sigma_t, \sigma_x = \sigma_1, \sigma_2, \sigma_3$

$\sigma_{max} = 337.9$ MPa at inner wall. ◀

$$\tau_{max} = \text{MAX}\left(\frac{|\sigma_2 - \sigma_3|}{2}, \frac{|\sigma_1 - \sigma_3|}{2}, \frac{|\sigma_1 - \sigma_2|}{2}\right) = \frac{\sigma_t}{2}$$

$\tau_{max} = 168.9$ MPa at inner wall ◀

(6.29, p.2)

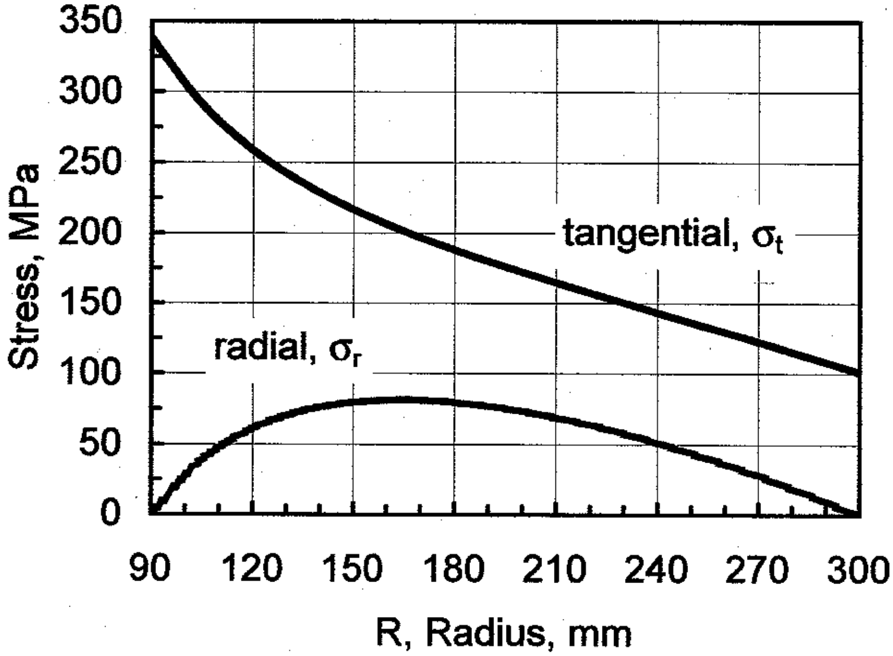
r_1 mm	r_2 mm	f cyc/s
90	300	120

ρ g/cm^3	ν	α r_1/r_2
7.87	0.293	0.30

ω rad/s	$\sqrt{r_1 r_2}$ mm	$\rho\, r_2^2 \omega^2$ MPa
754.0	164.3	402.7

ρ, ν from Tables 3.1, 5.2.

R mm	z R/r_2	σ_r MPa	σ_t MPa
90.0	0.300	0.0	337.9
104.0	0.347	36.6	293.4
118.0	0.393	58.6	262.4
132.0	0.440	71.5	239.4
146.0	0.487	78.4	221.2
160.0	0.533	81.1	206.2
174.0	0.580	80.6	193.2
188.0	0.627	77.6	181.5
202.0	0.673	72.6	170.7
216.0	0.720	66.0	160.4
230.0	0.767	57.9	150.5
244.0	0.813	48.5	140.6
258.0	0.860	37.9	130.9
272.0	0.907	26.3	121.1
286.0	0.953	13.6	111.1
300.0	1.000	0.0	101.0



(6.29, p.3)

(c) From equation for σ_t, only $-z^2$ and α^2/z^2 vary. Both decrease with R, so that σ_t also decreases. Thus, σ_t is maximum at $R = r_1$. Also $(\sigma_t - \sigma_r)$ is always positive (see below), so that $\sigma_t > \sigma_r$ for all R.

$$\sigma_t - \sigma_r = \rho \omega^2 r_2^2 \left(\frac{3+\nu}{8}\right)\left[-z^2\left(\frac{1+3\nu}{3+\nu} - 1\right) + 2\frac{\alpha^2}{z^2}\right]$$

$$\sigma_t - \sigma_r = \rho \omega^2 r_2^2 \left(\frac{3+\nu}{8}\right)\left[z^2 \frac{2(1-\nu)}{3+\nu} + 2\frac{\alpha^2}{z^2}\right]$$

which is always positive as $1 > \nu$.

Hence, σ_{max} is always σ_t at $R = r_1$.

Note that σ_r is never negative. To show this, substitute for α and z in σ_r.

$$\sigma_r = \rho \omega^2 r_2^2 \left(\frac{3+\nu}{8}\right)\left[1 + \frac{r_1^2}{r_2^2} - \frac{R^2}{r_2^2} - \frac{r_1^2}{r_2^2}\frac{r_2^2}{R^2}\right]$$

$$\sigma_r = \rho \omega^2 r_2^2 \left(\frac{3+\nu}{8}\right)\left[\frac{(R^2 - r_1^2)(r_2^2 - R^2)}{r_2^2 R^2}\right]$$

$R \geq r_1$ and $r_2 \geq R$, so $\sigma_r \geq 0$.

Since $\sigma_t > \sigma_r$ also, $\sigma_t > 0$, and

$$\tau_{max} = MAX\left(\frac{|\sigma_t - 0|}{2}, \frac{|\sigma_r - 0|}{2}, \frac{|\sigma_r - \sigma_t|}{2}\right) = \frac{\sigma_t}{2}$$

Hence, τ_{max} occurs at $R = r_1$ where σ_t is maximum.

(The above logic applies for plane stress, $\sigma_x = 0$. Extension to plane strain is needed.)

6.30 Rework Prob. 6.4 using cubic eqn., find direction cosines, and show equivalence with axis rotations. The equations needed are below, and results on pages following.

Obtain cubic equation.

$$I_1 = \sigma_x + \sigma_y + \sigma_z$$

$$I_2 = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2$$

$$I_3 = \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{zx}^2 - \sigma_z\tau_{xy}^2$$

$$\sigma^3 - \sigma^2 I_1 + \sigma I_2 - I_3 = 0$$

Solve for the three roots, σ_1, σ_2, and σ_3, and then apply:

$$\tau_1, \tau_2, \tau_3 = \frac{|\sigma_2 - \sigma_3|}{2}, \frac{|\sigma_1 - \sigma_3|}{2}, \frac{|\sigma_1 - \sigma_2|}{2}$$

$$\sigma_{max} = \mathrm{MAX}(\sigma_1, \sigma_2, \sigma_3), \qquad \tau_{max} = \mathrm{MAX}(\tau_1, \tau_2, \tau_3)$$

For each σ_i, solve for direction cosines l_i, m_i, and n_i.

$$(\sigma_x - \sigma_i)l_i + \tau_{xy}m_i + \tau_{zx}n_i = 0$$

$$\tau_{xy}l_i + (\sigma_y - \sigma_i)m_i + \tau_{yz}n_i = 0$$

$$\tau_{zx}l_i + \tau_{yz}m_i + (\sigma_z - \sigma_i)n_i = 0$$

$$l_i^2 + m_i^2 + n_i^2 = 1$$

(6.30, p.2)
Plot cubic to find approximate roots, then solve for each iteratively by invoking $f(\sigma) = 0$.

Stresses in MPa

σ_x	σ_y	σ_z	τ_{xy}	τ_{yz}	τ_{zx}
120	40	0	-30	0	0

Invariants

I_1	I_2	I_3
160	3,900	0

$$f(\sigma) = \sigma^3 - \sigma^2 I_1 + \sigma I_2 - I_3$$

Principal stresses

i	σ_i	$f(\sigma_i)$
1	**130.000**	0.00000
2	**30.000**	0.00000
3	**0.000**	0.00000

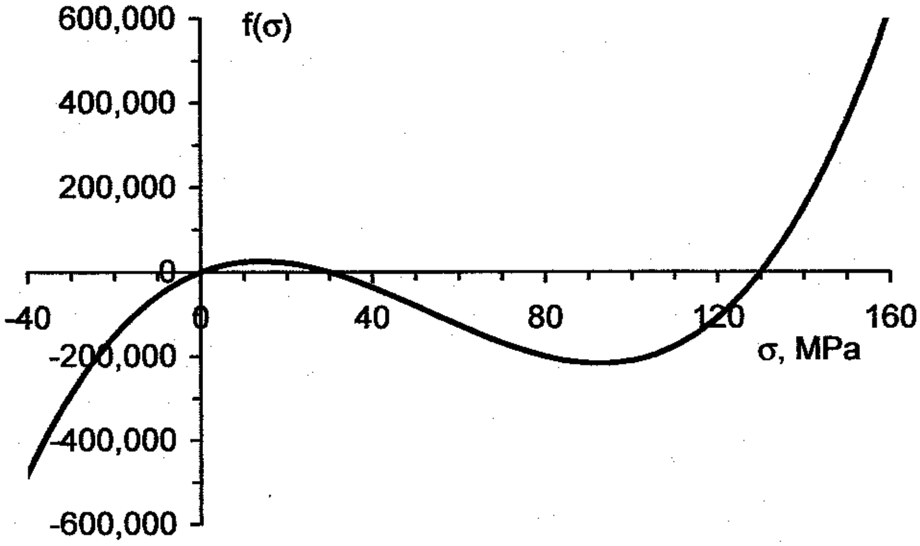
σ	f(σ)
-55	-864875
-50	-720000
-45	-590625
-40	-476000
-35	-375375
-30	-288000
-25	-213125
-20	-150000
-15	-97875
-10	-56000
-5	-23625
0	0
5	15625
10	24000
15	25875
20	22000
25	13125
30	0

σ	f(σ)
30	0
35	-16625
40	-36000
45	-57375
50	-80000
55	-103125
60	-126000
65	-147875
70	-168000
75	-185625
80	-200000
85	-210375
90	-216000
95	-216125
100	-210000
105	-196875
110	-176000
115	-146625

σ	f(σ)
115	-146625
120	-108000
125	-59375
130	0
135	70875
140	154000
145	250125
150	360000
155	484375
160	624000
165	779625
170	952000

σ_1	σ_2	σ_3	τ_1	τ_2	τ_3	σ_{max}	τ_{max}
130.00	30.00	0.00	15.00	65.00	50.00	130.00	65.00

(6.30, p. 3)



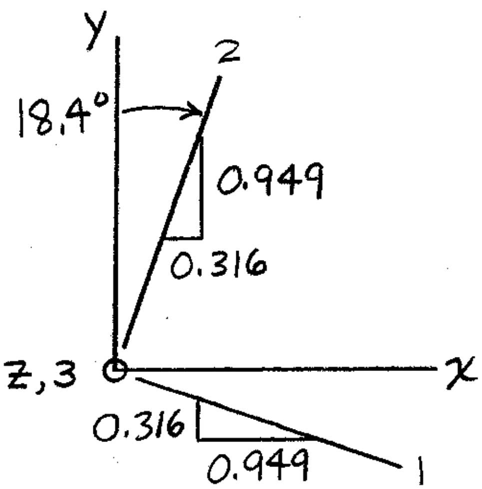
In this case, the cubic can also be solved without iteration.

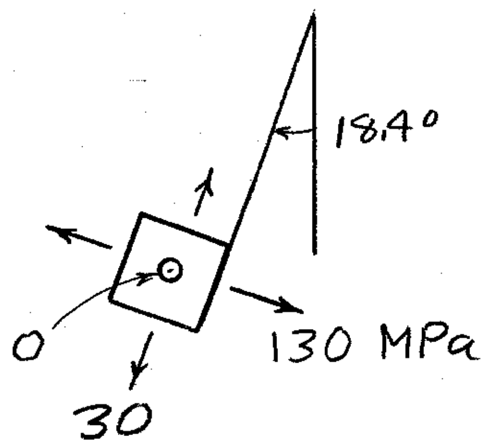
$$\sigma^3 - 160\sigma^2 + 3900\sigma = 0$$

$$(\sigma - 130)(\sigma - 30)(\sigma - 0) = 0$$

$\sigma_1 = 130,\ \sigma_2 = 30,\ \sigma_3 = 0$ MPa ◁

Direction cosines are calculated as on the next page. These agree with $\theta_n = 18.4°$ CW.





(6.30, p.4)

For $\sigma_1 = 130.0$ MPa, solve for direction cosines l_1, m_1, and n_1.

$$(120-\sigma_1)l_1 - 30m_1 + 0n_1 = 0$$

$$-30l_1 + (40-\sigma_1)m_1 + 0n_1 = 0$$

$$0l_1 + 0m_1 + (0-\sigma_1)n_1 = 0$$

$$l_1^2 + m_1^2 + n_1^2 = 1$$

l_1	m_1	n_1
-0.9487	0.3162	0.0000

For $\sigma_2 = 30.0$ MPa, solve for direction cosines l_2, m_2, and n_2.

$$(120-\sigma_2)l_2 - 30m_2 + 0n_2 = 0$$

$$-30l_2 + (40-\sigma_2)m_2 + 0n_2 = 0$$

$$0l_2 + 0m_2 + (0-\sigma_2)n_2 = 0$$

$$l_2^2 + m_2^2 + n_2^2 = 1$$

l_2	m_2	n_2
0.3162	0.9487	0.0000

For $\sigma_3 = 0$ MPa, obtain direction cosines l_3, m_3, and n_3

from cross product: $(l_3, m_3, n_3) = (l_1, m_1, n_1)\mathbf{x}(l_2, m_2, n_2)$

l_3	m_3	n_3
0.0000	0.0000	1.0000

6.31 Rework Prob. 6.11 using cubic equation, find direction cosines, and show equilivalence with axis rotations. The equations needed are below, and results follow.

Obtain cubic equation.

$$I_1 = \sigma_x + \sigma_y + \sigma_z$$

$$I_2 = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2$$

$$I_3 = \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{zx}^2 - \sigma_z\tau_{xy}^2$$

$$\sigma^3 - \sigma^2 I_1 + \sigma I_2 - I_3 = 0$$

Solve for the three roots, σ_1, σ_2, and σ_3, and then apply:

$$\tau_1, \tau_2, \tau_3 = \frac{|\sigma_2 - \sigma_3|}{2}, \frac{|\sigma_1 - \sigma_3|}{2}, \frac{|\sigma_1 - \sigma_2|}{2}$$

$$\sigma_{max} = \mathrm{MAX}(\sigma_1, \sigma_2, \sigma_3), \qquad \tau_{max} = \mathrm{MAX}(\tau_1, \tau_2, \tau_3)$$

For each σ_i, solve for direction cosines l_i, m_i, and n_i.

$$(\sigma_x - \sigma_i)l_i + \tau_{xy}m_i + \tau_{zx}n_i = 0$$

$$\tau_{xy}l_i + (\sigma_y - \sigma_i)m_i + \tau_{yz}n_i = 0$$

$$\tau_{zx}l_i + \tau_{yz}m_i + (\sigma_z - \sigma_i)n_i = 0$$

$$l_i^2 + m_i^2 + n_i^2 = 1$$

(6.31, p.2)

Plot cubic to find approximate roots, then solve for each iteratively by invoking $f(\sigma) = 0$.

Stresses in MPa

σ_x	σ_y	σ_z	τ_{xy}	τ_{yz}	τ_{zx}
50	100	200	60	0	0

Invariants

I_1	I_2	I_3
350	31,400	280,000

$$f(\sigma) = \sigma^3 - \sigma^2 I_1 + \sigma I_2 - I_3$$

Principal stresses

i	σ_i	$f(\sigma_i)$
1	**140.000**	0.00000
2	**10.000**	0.00000
3	**200.000**	0.00000

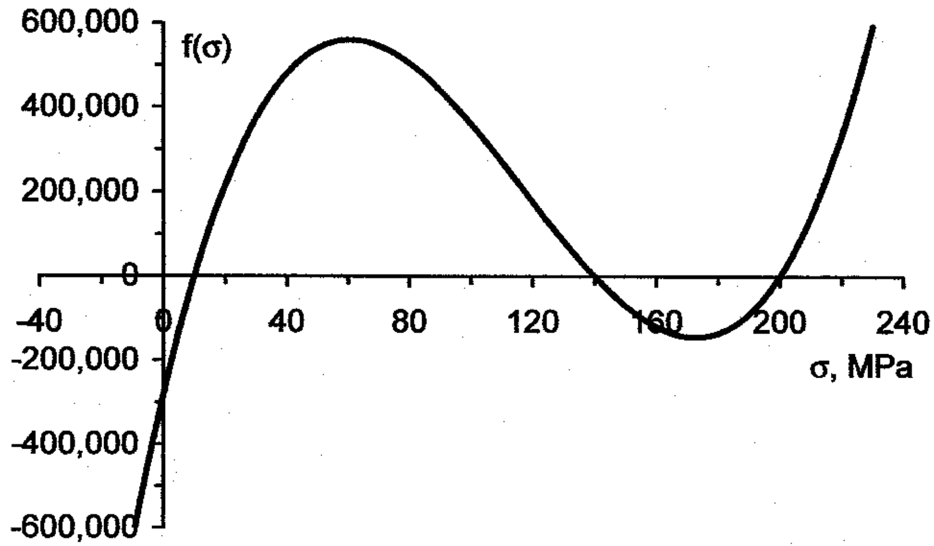
σ	$f(\sigma)$
-10	-630000
-5	-445875
0	-280000
5	-131625
10	0
15	115625
20	216000
25	301875
30	374000
35	433125
40	480000
45	515375
50	540000
55	554625
60	560000
65	556875
70	546000
75	528125

σ	$f(\sigma)$
75	528125
80	504000
85	474375
90	440000
95	401625
100	360000
105	315875
110	270000
115	223125
120	176000
125	129375
130	84000
135	40625
140	0
145	-37125
150	-70000
155	-97875
160	-120000

σ	$f(\sigma)$
160	-120000
165	-135625
170	-144000
175	-144375
180	-136000
185	-118125
190	-90000
195	-50875
200	0
205	63375
210	140000
215	230625
220	336000
225	456875
230	594000

σ_1	σ_2	σ_3	τ_1	τ_2	τ_3	σ_{max}	τ_{max}
140.00	10.00	200.00	95.00	30.00	65.00	200.00	95.00

(6.31, p. 3)



In this case, the cubic can also be solved without iteration. Substitute into Eq. 6.26.

$$\begin{vmatrix} (50-\sigma) & 60 & 0 \\ 60 & (100-\sigma) & 0 \\ 0 & 0 & (200-\sigma) \end{vmatrix} = 0$$

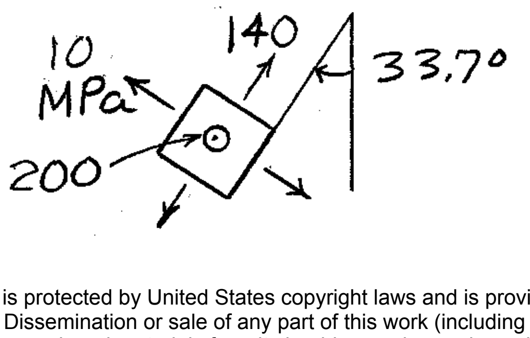
$$(200-\sigma)\left[(50-\sigma)(100-\sigma)-3600\right]=0$$

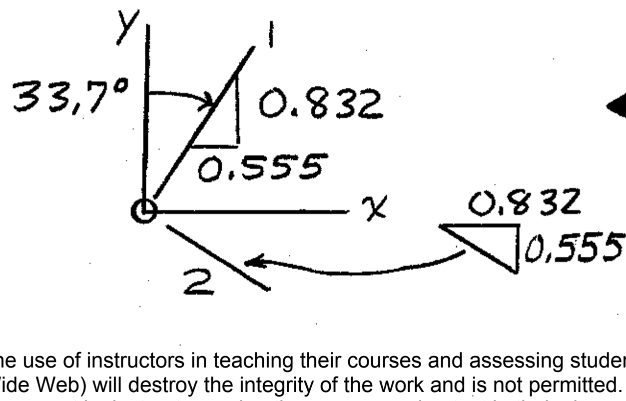
$$(200-\sigma)(\sigma^2-150\sigma+1400)=0$$

$$(200-\sigma)(\sigma-140)(\sigma-10)=0$$

$\sigma_1 = 140,\ \sigma_2 = 10,\ \sigma_3 = 200$ MPa ◁

Direction cosines from next page agree with $\theta_n = 33.7°$ CW.





◀

(6.31, p.4)

For $\sigma_1 = 140.0$ MPa, solve for direction cosines l_1, m_1, and n_1.

$$(50-\sigma_1)l_1 + 60m_1 + 0n_1 = 0$$

$$60l_1 + (100-\sigma_1)m_1 + 0n_1 = 0$$

$$0l_1 + 0m_1 + (200-\sigma_1)n_1 = 0$$

$$l_1^2 + m_1^2 + n_1^2 = 1$$

l_1	m_1	n_1
0.5547	0.8321	0.0000

For $\sigma_2 = 10.0$ MPa, solve for direction cosines l_2, m_2, and n_2.

$$(50-\sigma_2)l_2 + 60m_2 + 0n_2 = 0$$

$$60l_2 + (100-\sigma_2)m_2 + 0n_2 = 0$$

$$0l_2 + 0m_2 + (200-\sigma_2)n_2 = 0$$

$$l_2^2 + m_2^2 + n_2^2 = 1$$

l_2	m_2	n_2
-0.8321	0.5547	0.0000

For $\sigma_3 = 200$ MPa, obtain direction cosines l_3, m_3, and n_3

from cross product: $(l_3, m_3, n_3) = (l_1, m_1, n_1)\mathbf{x}(l_2, m_2, n_2)$

l_3	m_3	n_3
0.0000	0.0000	1.0000

6.32 $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}$ present. $\tau_{yz} = \tau_{zx} = 0$

Show that Eq. 6.27 reduces to Eq. 6.7 and $\sigma_z = \sigma_3$. Using the determinant:

$$\begin{vmatrix} (\sigma_x - \sigma) & \tau_{xy} & 0 \\ \tau_{xy} & (\sigma_y - \sigma) & 0 \\ 0 & 0 & (\sigma_z - \sigma) \end{vmatrix} = 0$$

$$(\sigma_z - \sigma)\left[(\sigma_x - \sigma)(\sigma_y - \sigma) - \tau_{xy}^2\right] = 0$$

$\sigma_z = \sigma = \sigma_3$ is one root

$$(\sigma_x - \sigma)(\sigma_y - \sigma) - \tau_{xy}^2 = 0$$

$$\sigma_x \sigma_y - \sigma\sigma_y - \sigma\sigma_x + \sigma^2 - \tau_{xy}^2 = 0$$

$$\sigma^2 - \sigma(\sigma_x + \sigma_y) + (\sigma_x \sigma_y - \tau_{xy}^2) = 0$$

Apply quadratic formula

$$\sigma = \frac{(\sigma_x + \sigma_y) \pm \sqrt{(\sigma_x + \sigma_y)^2 - 4(\sigma_x \sigma_y - \tau_{xy}^2)}}{2}$$

$$\sigma = \frac{\sigma_x + \sigma_y}{2} \pm \frac{1}{2}\sqrt{\sigma_x^2 + 2\sigma_x\sigma_y + \sigma_y^2 - 4\sigma_x\sigma_y + 4\tau_{xy}^2}$$

The remaing two roots are thus:

$$\sigma = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sigma_1, \sigma_2$$

6.33 to 6.39 For each given state of stress, find (a) σ_1, σ_2, σ_3, and τ_1, τ_2, τ_3, (b) σ_{max} and τ_{max}, and (c) 1, 2, 3 direction cosines. Employ Eqs. 6.18, 20, 25, and 28-31:

Obtain cubic equation.

$$I_1 = \sigma_x + \sigma_y + \sigma_z$$

$$I_2 = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2$$

$$I_3 = \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{zx}^2 - \sigma_z\tau_{xy}^2$$

$$\sigma^3 - \sigma^2 I_1 + \sigma I_2 - I_3 = 0$$

Solve for the three roots, σ_1, σ_2, and σ_3, and then apply :

$$\tau_1, \tau_2, \tau_3 = \frac{|\sigma_2 - \sigma_3|}{2}, \frac{|\sigma_1 - \sigma_3|}{2}, \frac{|\sigma_1 - \sigma_2|}{2}$$

$$\sigma_{max} = \mathrm{MAX}(\sigma_1, \sigma_2, \sigma_3), \qquad \tau_{max} = \mathrm{MAX}(\tau_1, \tau_2, \tau_3)$$

For each σ_i, solve for direction cosines l_i, m_i, and n_i.

$$(\sigma_x - \sigma_i)l_i + \tau_{xy}m_i + \tau_{zx}n_i = 0$$

$$\tau_{xy}l_i + (\sigma_y - \sigma_i)m_i + \tau_{yz}n_i = 0$$

$$\tau_{zx}l_i + \tau_{yz}m_i + (\sigma_z - \sigma_i)n_i = 0$$

$$l_i^2 + m_i^2 + n_i^2 = 1$$

6.33 For given state of stress, find the roots $\sigma_1, \sigma_2, \sigma_3$ of Eq. 6.28. Then Eqs. 6.18 and 6.20 give τ_1, τ_2, τ_3, and τ_{max}. Results follow:

Stresses in MPa

σ_x	σ_y	σ_z	τ_{xy}	τ_{yz}	τ_{zx}
-40	100	30	-50	12	0

Invariants

I_1	I_2	I_3
90	-4,844	-189,240

$$f(\sigma) = \sigma^3 - \sigma^2 I_1 + \sigma I_2 - I_3$$

Principal stresses

i	σ_i	$f(\sigma_i)$
1	**117.517**	0.00000
2	**28.664**	0.00000
3	**-56.180**	0.00000

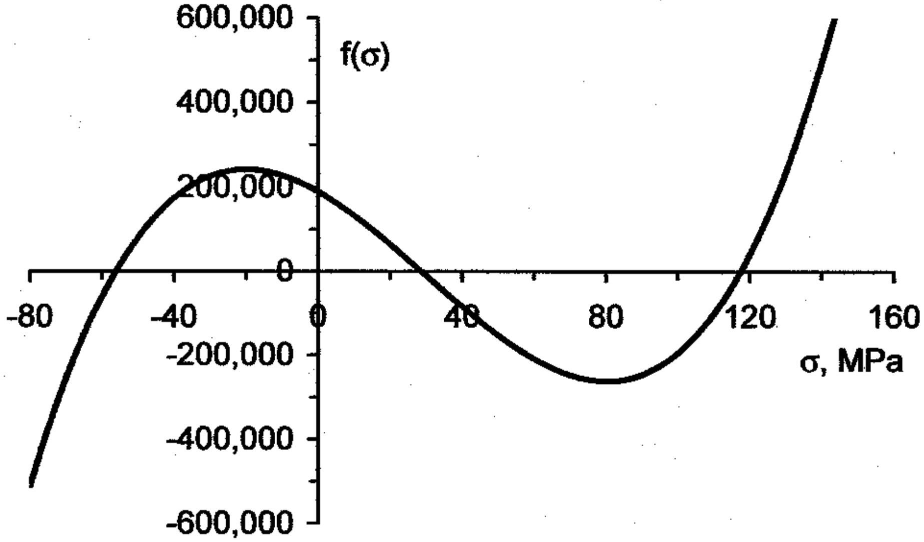
σ	$f(\sigma)$
-80	-511240
-75	-375585
-70	-255680
-65	-150775
-60	-60120
-55	17035
-50	81440
-45	133845
-40	175000
-35	205655
-30	226560
-25	238465
-20	242120
-15	238275
-10	227680
-5	211085
0	189240
5	162895

σ	$f(\sigma)$
5	162895
10	132800
15	99705
20	64360
25	27515
30	-10080
35	-47675
40	-84520
45	-119865
50	-152960
55	-183055
60	-209400
65	-231245
70	-247840
75	-258435
80	-262280
85	-258625
90	-246720

σ	$f(\sigma)$
90	-246720
95	-225815
100	-195160
105	-154005
110	-101600
115	-37195
120	39960
125	130615
130	235520
135	355425
140	491080
145	643235
150	812640
155	1000045
160	1206200

σ_1	σ_2	σ_3	τ_1	τ_2	τ_3	σ_{max}	τ_{max}
117.52	28.66	-56.18	42.42	86.85	44.43	117.52	86.85

(6.33, p.2)



Plot cubic to find approximate roots, then solve for each iteratively by invoking $f(\sigma)=0$.

Then the direction cosines for the 1, 2, 3 principal axes are obtained from Eqs. 6.25, 6.30, and 6.31. Details are on the next page.

(6.33, p. 3)

For $\sigma_1 = 117.52$ MPa, solve for direction cosines l_1, m_1, and n_1.

$$(-40-\sigma_1)l_1 - 50m_1 + 0n_1 = 0$$

$$-50l_1 + (100-\sigma_1)m_1 + 12n_1 = 0$$

$$0l_1 + 12m_1 + (30-\sigma_1)n_1 = 0$$

$$l_1^2 + m_1^2 + n_1^2 = 1$$

l_1	m_1	n_1
-0.3000	0.9451	0.1296

For $\sigma_2 = 28.66$ MPa, solve for direction cosines l_2, m_2, and n_2.

$$(-40-\sigma_2)l_2 - 50m_2 + 0n_2 = 0$$

$$-50l_2 + (100-\sigma_2)m_2 + 12n_2 = 0$$

$$0l_2 + 12m_2 + (30-\sigma_2)n_2 = 0$$

$$l_2^2 + m_2^2 + n_2^2 = 1$$

l_2	m_2	n_2
0.0803	-0.1103	0.9906

For $\sigma_3 = -56.18$ MPa, obtain direction cosines l_3, m_3, and n_3

from cross product: $(l_3, m_3, n_3) = (l_1, m_1, n_1)\mathbf{x}(l_2, m_2, n_2)$

l_3	m_3	n_3
0.9506	0.3076	-0.0428

6.34 For given state of stress, find the roots $\sigma_1, \sigma_2, \sigma_3$ of Eq. 6.28. Then Eqs. 6.18 and 6.20 give τ_1, τ_2, τ_3, and τ_{max}. Results follow:

Stresses in MPa

σ_x	σ_y	σ_z	τ_{xy}	τ_{yz}	τ_{zx}
0	0	0	0	100	100

Invariants

I_1	I_2	I_3
0	-20,000	0

$$f(\sigma) = \sigma^3 - \sigma^2 I_1 + \sigma I_2 - I_3$$

Principal stresses

i	σ_i	$f(\sigma_i)$
1	**141.421**	0.00000
2	**0.000**	0.00000
3	**-141.421**	0.00000

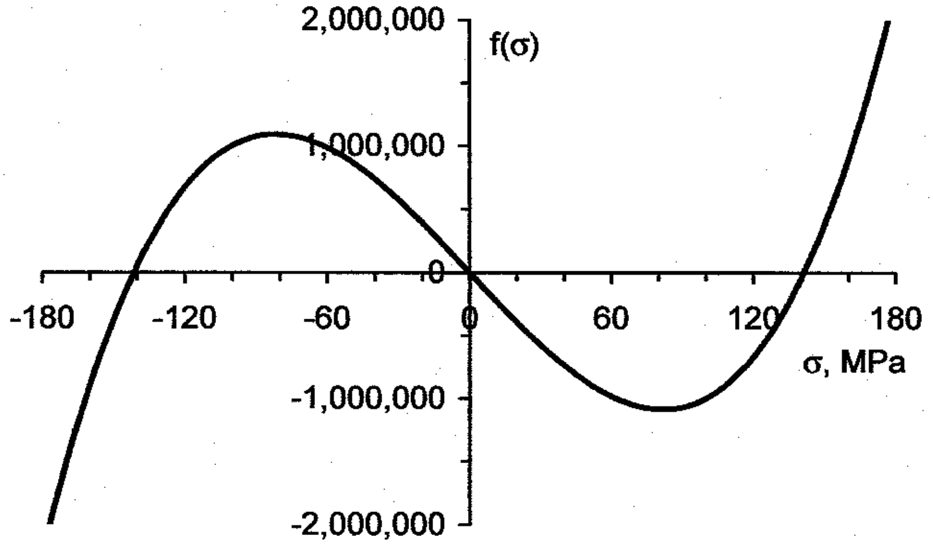
σ	$f(\sigma)$
-200	-4000000
-190	-3059000
-180	-2232000
-170	-1513000
-160	-896000
-150	-375000
-140	56000
-130	403000
-120	672000
-110	869000
-100	1000000
-90	1071000
-80	1088000
-70	1057000
-60	984000
-50	875000
-40	736000
-30	573000

σ	$f(\sigma)$
-30	573000
-20	392000
-10	199000
0	0
10	-199000
20	-392000
30	-573000
40	-736000
50	-875000
60	-984000
70	-1057000
80	-1088000
90	-1071000
100	-1000000
110	-869000
120	-672000
130	-403000
140	-56000

σ	$f(\sigma)$
140	-56000
150	375000
160	896000
170	1513000
180	2232000
190	3059000
200	4000000

σ_1	σ_2	σ_3	τ_1	τ_2	τ_3	σ_{max}	τ_{max}
141.42	0.00	-141.42	70.71	141.42	70.71	141.42	141.42

(6.34, p.2)



Plot cubic to find approximate roots, then solve for each iteratively by invoking $f(\sigma)=0$.

In this case the cubic can be solved without iteration.

$$\sigma^3 - 20{,}000\sigma = 0$$

$$(\sigma - 100\sqrt{2})(\sigma + 100\sqrt{2})(\sigma - 0) = 0$$

$$\sigma_1 = 141.42,\ \sigma_2 = 0,\ \sigma_3 = -141.42 \text{ MPa} \qquad \triangleleft$$

Then the direction cosines for the 1, 2, 3 principal axes are obtained from Eqs. 6.25, 6.30, and 6.31. Details are on the next page.

(6.34, p.3)

For $\sigma_1 = 141.42$ MPa, solve for direction cosines l_1, m_1, and n_1.

$$(0-\sigma_1)l_1 + 0m_1 + 100n_1 = 0$$

$$0l_1 + (0-\sigma_1)m_1 + 100n_1 = 0$$

$$100l_1 + 100m_1 + (0-\sigma_1)n_1 = 0$$

$$l_1^2 + m_1^2 + n_1^2 = 1$$

l_1	m_1	n_1
0.5000	0.5000	0.7071

For $\sigma_2 = 0$ MPa, solve for direction cosines l_2, m_2, and n_2.

$$(0-\sigma_2)l_2 + 0m_2 + 100n_2 = 0$$

$$0l_2 + (0-\sigma_2)m_2 + 100n_2 = 0$$

$$100l_2 + 100m_2 + (0-\sigma_2)n_2 = 0$$

$$l_2^2 + m_2^2 + n_2^2 = 1$$

l_2	m_2	n_2
0.7071	-0.7071	0.0000

For $\sigma_3 = -141.42$ MPa, obtain direction cosines l_3, m_3, and n_3

from cross product: $(l_3, m_3, n_3) = (l_1, m_1, n_1)\mathbf{x}(l_2, m_2, n_2)$

l_3	m_3	n_3
0.5000	0.5000	-0.7071

6.35 For given state of stress, find the roots $\sigma_1, \sigma_2, \sigma_3$ of Eq. 6.28. Then Eqs. 6.18 and 6.20 give τ_1, τ_2, τ_3, and τ_{max}. Results follow:

Stresses in MPa

σ_x	σ_y	σ_z	τ_{xy}	τ_{yz}	τ_{zx}
0	0	50	0	300	300

Invariants

I_1	I_2	I_3
50	-180,000	0

$$f(\sigma) = \sigma^3 - \sigma^2 I_1 + \sigma I_2 - I_3$$

Principal stresses

i	σ_i	$f(\sigma_i)$
1	**450.000**	0.00000
2	**0.000**	0.00000
3	**-400.000**	0.00000

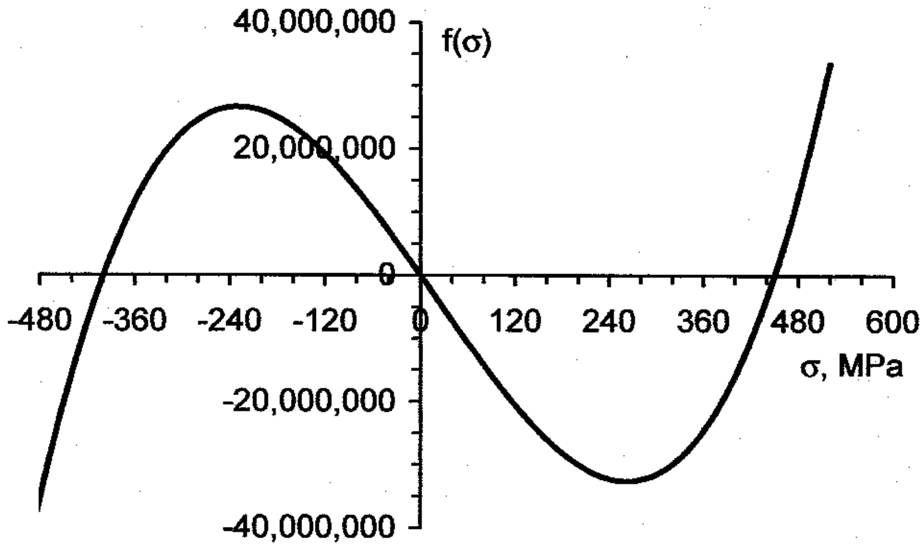
σ	$f(\sigma)$
-500	-47500000
-480	-35712000
-460	-25116000
-440	-15664000
-420	-7308000
-400	0
-380	6308000
-360	11664000
-340	16116000
-320	19712000
-300	22500000
-280	24528000
-260	25844000
-240	26496000
-220	26532000
-200	26000000
-180	24948000
-160	23424000

σ	$f(\sigma)$
-160	23424000
-140	21476000
-120	19152000
-100	16500000
-80	13568000
-60	10404000
-40	7056000
-20	3572000
0	0
20	-3612000
40	-7216000
60	-10764000
80	-14208000
100	-17500000
120	-20592000
140	-23436000
160	-25984000
180	-28188000

σ	$f(\sigma)$
180	-28188000
200	-30000000
220	-31372000
240	-32256000
260	-32604000
280	-32368000
300	-31500000
320	-29952000
340	-27676000
360	-24624000
380	-20748000
400	-16000000
420	-10332000
440	-3696000
460	3956000
480	12672000
500	22500000
520	33488000

σ_1	σ_2	σ_3	τ_1	τ_2	τ_3	σ_{max}	τ_{max}
450.00	0.00	-400.00	200.00	425.00	225.00	450.00	425.00 ◀

(6.35, p.2)



Plot cubic to find approximate roots, then solve for each iteratively by invoking $f(\sigma)=0$.

In this case the cubic can be solved without iteration.

$$\sigma_3 - 50\sigma^2 - 180{,}000\,\sigma = 0$$

$$(\sigma-450)(\sigma+400)(\sigma-0) = 0$$

$$\sigma_1 = 450,\ \sigma_2 = 0,\ \sigma_3 = -400 \text{ MPa} \quad \triangleleft$$

Then the direction cosines for the 1, 2, 3 principal axes are obtained from Eqs. 6.25, 6.30, and 6.31. Details are on the next. page

(6.35, p.3)

For $\sigma_1 = 450.00$ MPa, solve for direction cosines l_1, m_1, and n_1.

$$(0-\sigma_1)l_1 + 0m_1 + 300n_1 = 0$$

$$0l_1 + (0-\sigma_1)m_1 + 300n_1 = 0$$

$$300l_1 + 300m_1 + (50-\sigma_1)n_1 = 0$$

$$l_1^2 + m_1^2 + n_1^2 = 1$$

l_1	m_1	n_1
0.4851	0.4851	0.7276

For $\sigma_2 = 0$ MPa, solve for direction cosines l_2, m_2, and n_2.

$$(0-\sigma_2)l_2 + 0m_2 + 300n_2 = 0$$

$$0l_2 + (0-\sigma_2)m_2 + 300n_2 = 0$$

$$300l_2 + 300m_2 + (50-\sigma_2)n_2 = 0$$

$$l_2^2 + m_2^2 + n_2^2 = 1$$

l_2	m_2	n_2
0.7071	-0.7071	0.0000

For $\sigma_3 = -400.00$ MPa, obtain direction cosines l_3, m_3, and n_3

from cross product: $(l_3, m_3, n_3) = (l_1, m_1, n_1)\mathbf{x}(l_2, m_2, n_2)$

l_3	m_3	n_3
0.5145	0.5145	-0.6860

6.36 For given state of stress, find the roots $\sigma_1, \sigma_2, \sigma_3$ of Eq. 6.28. Then Eqs. 6.18 and 6.20 give τ_1, τ_2, τ_3, and τ_{max}. Results follow:

Stresses in MPa

σ_x	σ_y	σ_z	τ_{xy}	τ_{yz}	τ_{zx}
65	-120	-45	30	0	50

Invariants

I_1	I_2	I_3
-100	-8,725	691,500

$$f(\sigma) = \sigma^3 - \sigma^2 I_1 + \sigma I_2 - I_3$$

Principal stresses

i	σ_i	$f(\sigma_i)$
1	**88.107**	0.00000
2	**-62.469**	0.00000
3	**-125.638**	0.00000

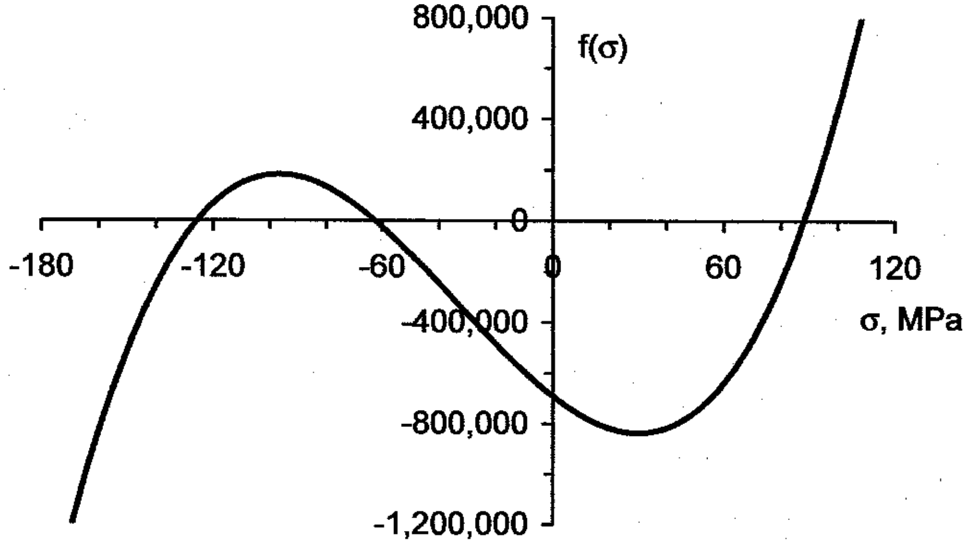
σ	$f(\sigma)$
-180	-1713000
-173	-1366892
-166	-1061846
-159	-795804
-152	-566708
-145	-372500
-138	-211122
-131	-80516
-124	21376
-117	96612
-110	147250
-103	175348
-96	182964
-89	172156
-82	144982
-75	103500
-68	49768
-61	-14156

σ	$f(\sigma)$
-61	-14156
-54	-86214
-47	-164348
-40	-246500
-33	-330612
-26	-414626
-19	-496484
-12	-574128
-5	-645500
2	-708542
9	-761196
16	-801404
23	-827108
30	-836250
37	-826772
44	-796616
51	-743724
58	-666038

σ	$f(\sigma)$
58	-666038
65	-561500
72	-428052
79	-263636
86	-66194
93	166332
100	436000
107	744868
114	1094994
121	1488436

σ_1	σ_2	σ_3	τ_1	τ_2	τ_3	σ_{max}	τ_{max}
88.11	-62.47	-125.64	31.58	106.87	75.29	88.11	106.87 ◀

(6.36, p.2)



Plot cubic to find approximate roots, then solve for each iteratively by invoking $f(\sigma) = 0$.

Then the direction cosines for the 1, 2, 3 principal axes are obtained from Eqs. 6.25, 6.30, and 6.31. Details are on the next page.

(6.36, p.3)

For $\sigma_1 = 88.11$ MPa, solve for direction cosines l_1, m_1, and n_1.

$$(65-\sigma_1)l_1 + 30m_1 + 50n_1 = 0$$

$$30l_1 + (-120-\sigma_1)m_1 + 0n_1 = 0$$

$$50l_1 + 0m_1 + (-45-\sigma_1)n_1 = 0$$

$$l_1^2 + m_1^2 + n_1^2 = 1$$

l_1	m_1	n_1
0.9277	0.1337	0.3485

For $\sigma_2 = -62.47$ MPa, solve for direction cosines l_2, m_2, and n_2.

$$(65-\sigma_2)l_2 + 30m_2 + 50n_2 = 0$$

$$30l_2 + (-120-\sigma_2)m_2 + 0n_2 = 0$$

$$50l_2 + 0m_2 + (-45-\sigma_2)n_2 = 0$$

$$l_2^2 + m_2^2 + n_2^2 = 1$$

l_2	m_2	n_2
0.3251	0.1695	-0.9304

For $\sigma_3 = -125.64$ MPa, obtain direction cosines l_3, m_3, and n_3

from cross product: $(l_3, m_3, n_3) = (l_1, m_1, n_1)\mathbf{x}(l_2, m_2, n_2)$

l_3	m_3	n_3
-0.1835	0.9764	0.1138

6.37 For given state of stress, find the roots $\sigma_1, \sigma_2, \sigma_3$ of Eq. 6.28. Then Eqs. 6.18 and 6.20 give τ_1, τ_2, τ_3, and τ_{max}. Results follow:

Stresses in MPa

σ_x	σ_y	σ_z	τ_{xy}	τ_{yz}	τ_{zx}
25	50	40	20	-30	0

Invariants

I_1	I_2	I_3
115	2,950	11,500

$$f(\sigma) = \sigma^3 - \sigma^2 I_1 + \sigma I_2 - I_3$$

Principal stresses

i	σ_i	$f(\sigma_i)$
1	**79.866**	0.00000
2	**30.397**	0.00000
3	**4.737**	0.00000

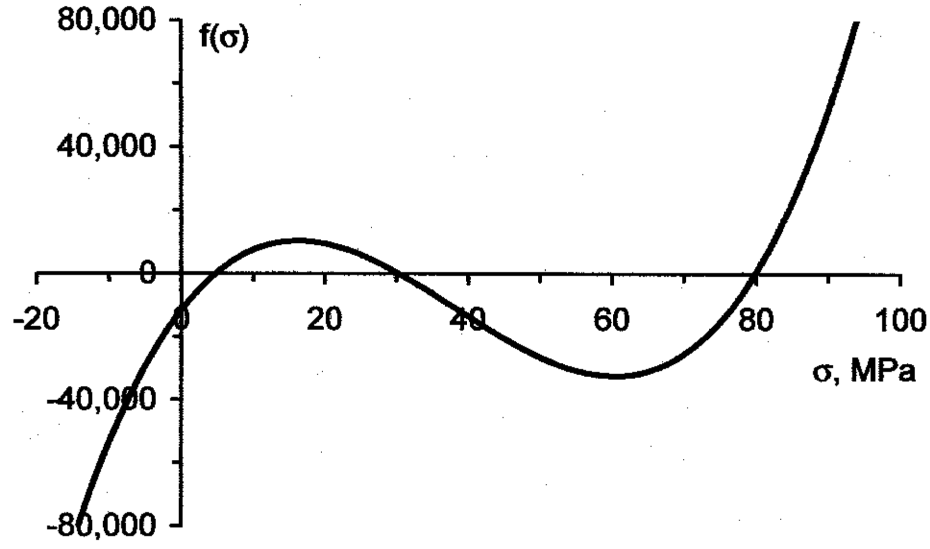
σ	$f(\sigma)$
-20	-124500
-17	-99798
-14	-78084
-11	-59196
-8	-42972
-5	-29250
-2	-17868
1	-8664
4	-1476
7	3858
10	7500
13	9612
16	10356
19	9894
22	8388
25	6000
28	2892
31	-774

σ	$f(\sigma)$
31	-774
34	-4836
37	-9132
40	-13500
43	-17778
46	-21804
49	-25416
52	-28452
55	-30750
58	-32148
61	-32484
64	-31596
67	-29322
70	-25500
73	-19968
76	-12564
79	-3126
82	8508

σ	$f(\sigma)$
82	8508
85	22500
88	39012
91	58206
94	80244
97	105288
100	133500

σ_1	σ_2	σ_3	τ_1	τ_2	τ_3	σ_{max}	τ_{max}
79.87	30.40	4.74	12.83	37.56	24.73	79.87	37.56

(6.37, p.2)



Plot cubic to find approximate roots, then solve for each iteratively by invoking $f(\sigma) = 0$.

Then the direction cosines for the 1, 2, 3 principal axes are obtained from Eqs. 6.25, 6.30, and 6.31. Details are on the next page.

(6.37, p. 3)

For $\sigma_1 = 79.87$ MPa, solve for direction cosines l_1, m_1, and n_1.

$$(25-\sigma_1)l_1 + 20m_1 + 0n_1 = 0$$

$$20l_1 + (50-\sigma_1)m_1 - 30n_1 = 0$$

$$0l_1 - 30m_1 + (40-\sigma_1)n_1 = 0$$

$$l_1^2 + m_1^2 + n_1^2 = 1$$

l_1	m_1	n_1
0.2796	0.7672	-0.5773

For $\sigma_2 = 30.40$ MPa, solve for direction cosines l_2, m_2, and n_2.

$$(25-\sigma_2)l_2 + 20m_2 + 0n_2 = 0$$

$$20l_2 + (50-\sigma_2)m_2 - 30n_2 = 0$$

$$0l_2 - 30m_2 + (40-\sigma_2)n_2 = 0$$

$$l_2^2 + m_2^2 + n_2^2 = 1$$

l_2	m_2	n_2
0.7488	0.2021	0.6312

For $\sigma_3 = 4.74$ MPa, obtain direction cosines l_3, m_3, and n_3

from cross product : $(l_3, m_3, n_3) = (l_1, m_1, n_1)\mathbf{x}(l_2, m_2, n_2)$

l_3	m_3	n_3
0.6009	-0.6088	-0.5179

6.38 For the given state of stress, find the roots $\sigma_1, \sigma_2, \sigma_3$ of Eq. 6.28. Then Eqs. 6.18 and 6.20 give τ_1, τ_2, τ_3, and τ_{max}. Results follow:

Stresses in MPa

σ_x	σ_y	σ_z	τ_{xy}	τ_{yz}	τ_{zx}
82	16	40	15	-5	22

Invariants

I_1	I_2	I_3
138	4,498	30,386

$$f(\sigma) = \sigma^3 - \sigma^2 I_1 + \sigma I_2 - I_3$$

Principal stresses

i	σ_i	$f(\sigma_i)$
1	**93.265**	0.00000
2	**35.577**	0.00000
3	**9.158**	0.00000

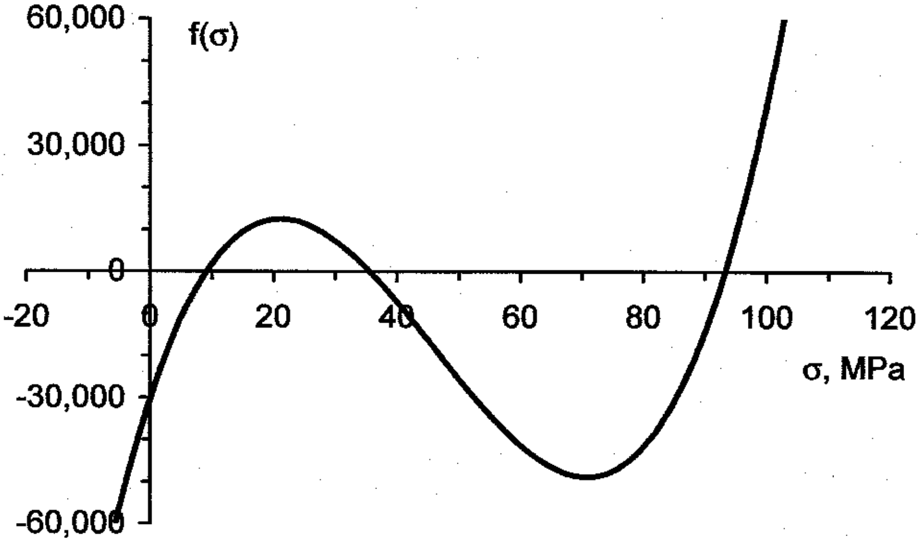
σ	$f(\sigma)$
-20	-183546
-17	-151647
-14	-123150
-11	-97893
-8	-75714
-5	-56451
-2	-39942
1	-26025
4	-14538
7	-5319
10	1794
13	6963
16	10350
19	12117
22	12426
25	11439
28	9318
31	6225

σ	$f(\sigma)$
31	6225
34	2322
37	-2229
40	-7266
43	-12627
46	-18150
49	-23673
52	-29034
55	-34071
58	-38622
61	-42525
64	-45618
67	-47739
70	-48726
73	-48417
76	-46650
79	-43263
82	-38094

σ	$f(\sigma)$
82	-38094
85	-30981
88	-21762
91	-10275
94	3642
97	20151
100	39414
103	61593
106	86850
109	115347
112	147246
115	182709
118	221898
121	264975
124	312102

σ_1	σ_2	σ_3	τ_1	τ_2	τ_3	σ_{max}	τ_{max}
93.27	35.58	9.16	13.21	42.05	28.84	93.27	42.05 ◀

(6.38, p.2)



Plot cubic to find approximate roots, then solve for each iteratively by invoking $f(\sigma) = 0$.

Then the direction cosines for the 1, 2, 3 principal axes are obtained from Eqs. 6.25, 6.30, and 6.31. Details are on the next page.

(6.38, p.3)

For $\sigma_1 = 93.27$ MPa, solve for direction cosines l_1, m_1, and n_1.

$$(82-\sigma_1)l_1 + 15m_1 + 22n_1 = 0$$

$$15l_1 + (16-\sigma_1)m_1 - 5n_1 = 0$$

$$22l_1 - 5m_1 + (40-\sigma_1)n_1 = 0$$

$$l_1^2 + m_1^2 + n_1^2 = 1$$

l_1	m_1	n_1
0.9182	0.1547	0.3647

For $\sigma_2 = 35.58$ MPa, solve for direction cosines l_2, m_2, and n_2.

$$(82-\sigma_2)l_2 + 15m_2 + 22n_2 = 0$$

$$15l_2 + (16-\sigma_2)m_2 - 5n_2 = 0$$

$$22l_2 - 5m_2 + (40-\sigma_2)n_2 = 0$$

$$l_2^2 + m_2^2 + n_2^2 = 1$$

l_2	m_2	n_2
0.2706	0.4276	-0.8625

For $\sigma_3 = 9.16$ MPa, obtain direction cosines l_3, m_3, and n_3

from cross product: $(l_3, m_3, n_3) = (l_1, m_1, n_1)\mathbf{x}(l_2, m_2, n_2)$

l_3	m_3	n_3
-0.2893	0.8906	0.3508

6.39 For given state of stress, find the roots $\sigma_1, \sigma_2, \sigma_3$ of Eq. 6.28. Then Eqs. 6.18 and 6.20 give τ_1, τ_2, τ_3, and τ_{max}. Results follow:

Stresses in MPa

σ_x	σ_y	σ_z	τ_{xy}	τ_{yz}	τ_{zx}
10	20	-10	-20	10	-30

Invariants

I_1	I_2	I_3
20	-1,500	-5,000

$$f(\sigma) = \sigma^3 - \sigma^2 I_1 + \sigma I_2 - I_3$$

Principal stresses

i	σ_i	$f(\sigma_i)$
1	**48.695**	0.00000
2	**3.218**	0.00000
3	**-31.913**	0.00000

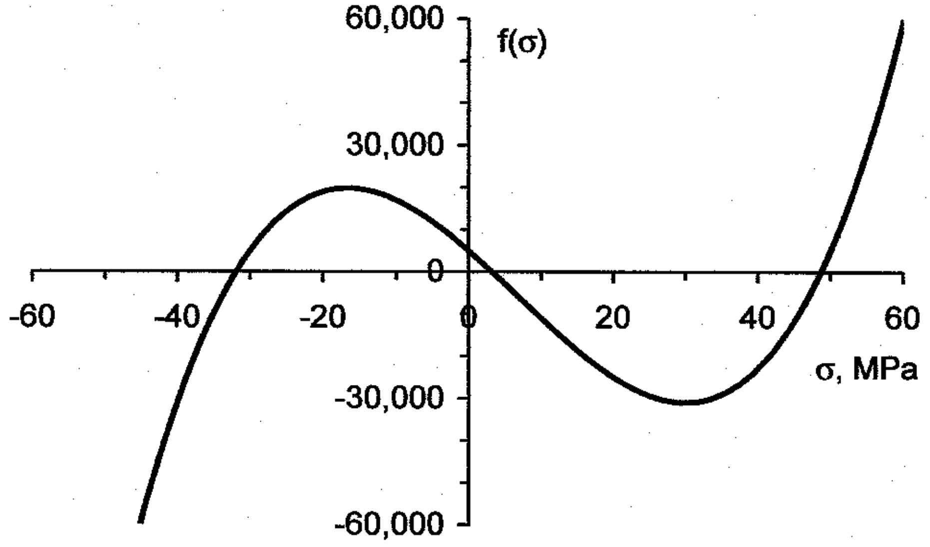
σ	$f(\sigma)$
-60	-193000
-57	-159673
-54	-129784
-51	-103171
-48	-79672
-45	-59125
-42	-41368
-39	-26239
-36	-13576
-33	-3217
-30	5000
-27	11237
-24	15656
-21	18419
-18	19688
-15	19625
-12	18392
-9	16151

σ	$f(\sigma)$
-9	16151
-6	13064
-3	9293
0	5000
3	347
6	-4504
9	-9391
12	-14152
15	-18625
18	-22648
21	-26059
24	-28696
27	-30397
30	-31000
33	-30343
36	-28264
39	-24601
42	-19192

σ	$f(\sigma)$
42	-19192
45	-11875
48	-2488
51	9131
54	23144
57	39713
60	59000
63	81167
66	106376
69	134789
72	166568
75	201875

σ_1	σ_2	σ_3	τ_1	τ_2	τ_3	σ_{max}	τ_{max}
48.70	3.22	-31.91	17.57	40.30	22.74	48.70	40.30

(6.39, p.2)



Plot cubic to find approximate roots, then solve for each iteratively by invoking $f(\sigma)=0$.

Then the direction cosines for the 1, 2, 3 principal axes are obtained from Eqs. 6.25, 6.30, and 6.31. Details are on the next page.

(6.39, p.3)

For $\sigma_1 = 48.70$ MPa, solve for direction cosines l_1, m_1, and n_1.

$$(10-\sigma_1)l_1 - 20m_1 - 30n_1 = 0$$

$$-20l_1 + (20-\sigma_1)m_1 + 10n_1 = 0$$

$$-30l_1 + 10m_1 + (-10-\sigma_1)n_1 = 0$$

$$l_1^2 + m_1^2 + n_1^2 = 1$$

l_1	m_1	n_1
0.6574	-0.6116	-0.4402

For $\sigma_2 = 3.22$ MPa, solve for direction cosines l_2, m_2, and n_2.

$$(10-\sigma_2)l_2 - 20m_2 - 30n_2 = 0$$

$$-20l_2 + (20-\sigma_2)m_2 + 10n_2 = 0$$

$$-30l_2 + 10m_2 + (-10-\sigma_2)n_2 = 0$$

$$l_2^2 + m_2^2 + n_2^2 = 1$$

l_2	m_2	n_2
0.4488	0.7870	-0.4232

For $\sigma_3 = -31.91$ MPa, obtain direction cosines l_3, m_3, and n_3

from cross product: $(l_3, m_3, n_3) = (l_1, m_1, n_1)\mathbf{x}(l_2, m_2, n_2)$

l_3	m_3	n_3
0.6053	0.0807	0.7919

6.40 For $\sigma_x = 90$, $\sigma_y = 130$, $\sigma_z = -60$ MPa, with $\tau_{xy} = \tau_{yz} = \tau_{zx} = 0$: (a) Find $\sigma_1, \sigma_2, \sigma_3$, and τ_{max}. (b) Show for all $\tau_{xy} = \tau_{yz} = \tau_{zx}$ cases that $\sigma_1, \sigma_2, \sigma_3 = \sigma_x, \sigma_y, \sigma_z$, and $(1, 2, 3) = (x, y, z)$.

(a)
$$\begin{vmatrix} (\sigma_x - \sigma) & \tau_{xy} & \tau_{zx} \\ \tau_{xy} & (\sigma_y - \sigma) & \tau_{yz} \\ \tau_{zx} & \tau_{yz} & (\sigma_z - \sigma) \end{vmatrix} = 0$$

$$\begin{vmatrix} (90 - \sigma) & 0 & 0 \\ 0 & (130 - \sigma) & 0 \\ 0 & 0 & (-60 - \sigma) \end{vmatrix} = 0$$

$$(90 - \sigma)(130 - \sigma)(-60 - \sigma) = 0$$

$$\sigma_1 = 90,\ \sigma_2 = 130,\ \sigma_3 = -60 \text{ MPa}$$ ◀

$$\tau_{max} = \text{MAX}\left(\frac{|\sigma_2 - \sigma_3|}{2}, \frac{|\sigma_1 - \sigma_3|}{2}, \frac{|\sigma_1 - \sigma_2|}{2}\right)$$

$$\tau_{max} = \text{MAX}(95, 75, 20) = 95 \text{ MPa}$$ ◀

(b)
$$\begin{vmatrix} (\sigma_x - \sigma) & 0 & 0 \\ 0 & (\sigma_y - \sigma) & 0 \\ 0 & 0 & (\sigma_z - \sigma) \end{vmatrix} = 0$$

$$(\sigma_x - \sigma)(\sigma_y - \sigma)(\sigma_z - \sigma) = 0$$

$\sigma_1, \sigma_2, \sigma_3 = \sigma_x, \sigma_y, \sigma_z$ in all such cases. ◀

Also, $(l_1, m_1, n_1) = (1, 0, 0)$, $(l_2, m_2, n_2) = (0, 1, 0)$, and $(l_3, m_3, n_3) = (0, 0, 1)$. Hence, in all such cases, $(1, 2, 3) = (x, y, z)$. ◀

(6.40, p.2) Details showing $(1,2,3) = (x,y,z)$.

For $\sigma_1 = \sigma_x$, solve for direction cosines l_1, m_1, and n_1.

$$(\sigma_x - \sigma_1)l_1 + 0m_1 + 0n_1 = 0$$

$$0l_1 + (\sigma_y - \sigma_1)m_1 + 0n_1 = 0$$

$$0l_1 + 0m_1 + (\sigma_z - \sigma_1)n_1 = 0$$

$$l_1^2 + m_1^2 + n_1^2 = 1$$

l_1	m_1	n_1
1	0	0

For $\sigma_2 = \sigma_y$, solve for direction cosines l_2, m_2, and n_2.

$$(\sigma_x - \sigma_2)l_2 + 0m_2 + 0n_2 = 0$$

$$0l_2 + (\sigma_y - \sigma_2)m_2 + 0n_2 = 0$$

$$0l_2 + 0m_2 + (\sigma_z - \sigma_2)n_2 = 0$$

$$l_2^2 + m_2^2 + n_2^2 = 1$$

l_2	m_2	n_2
0	1	0

For $\sigma_3 = \sigma_z$, obtain direction cosines l_3, m_3, and n_3

from cross product: $\quad (l_3, m_3, n_3) = (l_1, m_1, n_1)\mathbf{x}(l_2, m_2, n_2)$

l_3	m_3	n_3
0	0	1

6.41 $\sigma_x = 50,\ \sigma_y = 10,\ \tau_{xy} = -15$ MPa (Prob. 6.1)

$$\sigma_h = \frac{\sigma_x + \sigma_y + \sigma_z}{3} = 20 \text{ MPa} \blacktriangleleft$$

$$\tau_h = \frac{1}{3}\sqrt{(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)}$$

$$\tau_h = \frac{1}{3}\sqrt{(\sigma_x - \sigma_y)^2 + \sigma_y^2 + \sigma_x^2 + 6\,\tau_{xy}^2} = 24.83 \text{ MPa} \blacktriangleleft$$

A less direct procedure is to take $\sigma_1, \sigma_2,$ and $\sigma_3 = 0$ from Prob. 6.1 solution, and substitute these into Eqs. 6.34 and 6.35, obtaining the same result as above

6.42 $\sigma_x = 50,\ \sigma_y = 10,\ \sigma_z = -20,\ \tau_{xy} = -15$ MPa

$\tau_{yz} = \tau_{zx} = 0$ (Prob. 6.10)

$$\sigma_h = \frac{\sigma_x + \sigma_y + \sigma_z}{3} = 13.33 \text{ MPa} \quad \blacktriangleleft$$

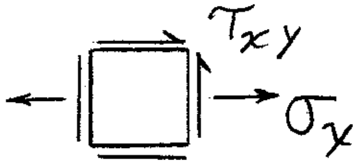
$$\tau_h = \frac{1}{3}\sqrt{(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)}$$

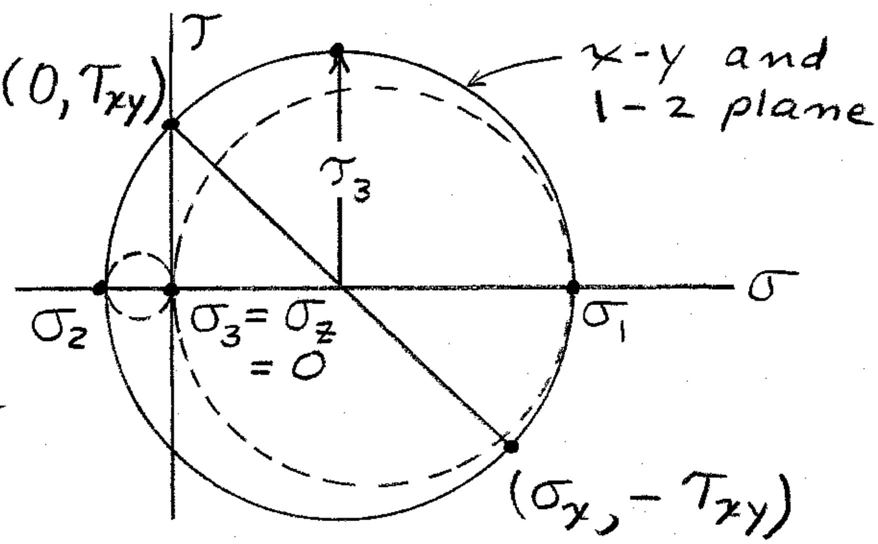
$$\tau_h = 31.18 \text{ MPa} \quad \blacktriangleleft$$

A less direct procedure is to take σ_1, σ_2, and σ_3 from the Prob. 6.10 solution, and substitute these into Eqs. 6.34 and 6.35, obtaining the same result as above.

6.43 σ_x, τ_{xy} nonzero

Determine $\sigma_{max}, \tau_{max}, \tau_h$





Mohr's circle for x-y plane must enclose origin.

$$\sigma_1 = \sigma_{max}, \quad \tau_3 = \frac{\sigma_1 - \sigma_2}{2} = \tau_{max}$$

$$\tau_3 = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \frac{1}{2}\sqrt{\sigma_x^2 + 4\tau_{xy}^2} = \tau_{max}$$ ◀

$$\sigma_1, \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \tau_3 = \frac{\sigma_x}{2} \pm \frac{1}{2}\sqrt{\sigma_x^2 + 4\tau_{xy}^2}$$

$$\sigma_{max} = \frac{1}{2}\left(\sigma_x + \sqrt{\sigma_x^2 + 4\tau_{xy}^2}\right)$$ ◀

$$\tau_h = \frac{1}{3}\sqrt{(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6\left(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2\right)}$$

$$\tau_h = \frac{\sqrt{2}}{3}\sqrt{\sigma_x^2 + 3\tau_{xy}^2}$$ ◀

6.44 σ_x, σ_y are nonzero

Determine $\sigma_{max}, \tau_{max}, \tau_h$

$$\sigma_1, \sigma_2, \sigma_3 = \sigma_x, \sigma_y, 0$$

$$\sigma_{max} = MAX(\sigma_x, \sigma_y) \blacktriangleleft$$

$$\tau_{max} = MAX\left(\frac{|\sigma_1 - \sigma_2|}{2}, \frac{|\sigma_2 - \sigma_3|}{2}, \frac{|\sigma_3 - \sigma_1|}{2}\right)$$

$$\tau_{max} = MAX\left(\frac{|\sigma_x - \sigma_y|}{2}, \frac{|\sigma_x|}{2}, \frac{|\sigma_y|}{2}\right) \blacktriangleleft$$

$$\tau_h = \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}$$

$$\tau_h = \frac{1}{3}\sqrt{(\sigma_x - \sigma_y)^2 + \sigma_x^2 + \sigma_y^2}$$

$$\tau_h = \frac{\sqrt{2}}{3}\sqrt{\sigma_x^2 - \sigma_x \sigma_y + \sigma_y^2} \blacktriangleleft$$

6.45 Express τ_h in terms of τ_1, τ_2, τ_3

$$\tau_1, \tau_2, \tau_3 = \frac{|\sigma_1 - \sigma_2|}{2}, \frac{|\sigma_2 - \sigma_3|}{2}, \frac{|\sigma_3 - \sigma_1|}{2}$$

$$\tau_h = \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}$$

$$\tau_h = \frac{1}{3}\sqrt{(2\tau_1)^2 + (2\tau_2)^2 + (2\tau_3)^2}$$

$$\tau_h = \frac{2}{3}\sqrt{\tau_1^2 + \tau_2^2 + \tau_3^2} \quad \blacktriangleleft$$

6.46 Thick-walled tube, closed ends.

$$\sigma_x = \frac{p r_1^2}{(r_2^2 - r_1^2)}, \quad \sigma_t = \sigma_x\left(\frac{r_2^2}{R^2} + 1\right), \quad \sigma_r = -\sigma_x\left(\frac{r_2^2}{R^2} - 1\right)$$

$$\sigma_x, \sigma_t, \sigma_r = \sigma_1, \sigma_2, \sigma_3 \quad (\text{all } \tau\text{'s} = 0)$$

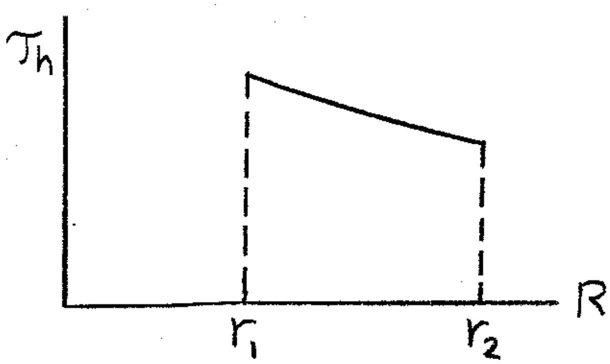
$$\tau_h = \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}$$

$$\tau_h = \frac{1}{3}\,\frac{p r_1^2}{r_2^2 - r_1^2}\sqrt{\left[1 - \left(\frac{r_2^2}{R^2} + 1\right)\right]^2 + \left[\left(\frac{r_2^2}{R^2} + 1\right) + \left(\frac{r_2^2}{R^2} - 1\right)\right]^2 + \left[-\left(\frac{r_2^2}{R^2} - 1\right) - 1\right]^2}$$

$$\tau_h = \frac{1}{3}\,\frac{p r_1^2}{r_2^2 - r_1^2}\,\frac{r_2^2}{R^2}\sqrt{1^2 + 2^2 + 1^2}$$

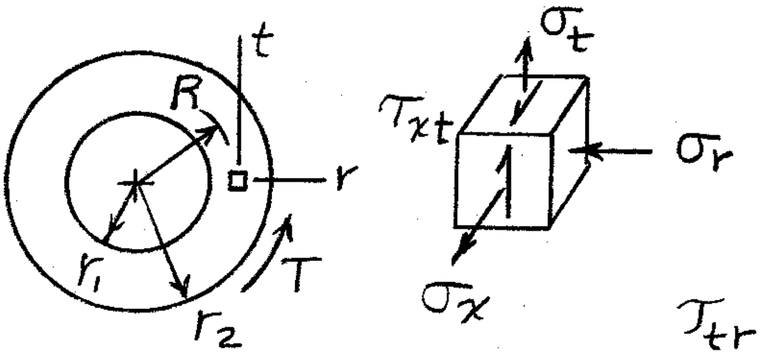
$$\tau_h = \sqrt{\frac{2}{3}}\;\frac{p r_1^2 r_2^2}{R^2(r_2^2 - r_1^2)}$$ ◀

As $R\uparrow$, $\tau_h\downarrow$, so τ_h is max, at $R = r_1$ ◀



6.47 Thick-walled tube, closed ends.

highest $\tau_h = ?$ (Eq. 6.38)



$p = 100$ MPa

$T = 15$ kN·m

$r_1 = 30$ mm

$r_2 = 40$ mm

$\tau_{tr} = \tau_{rx} = 0$

$$\tau_{xt} = \frac{TR}{J}, \qquad J = \frac{\pi}{2}\left(r_2^4 - r_1^4\right)$$

$$\sigma_x = \frac{p r_1^2}{r_2^2 - r_1^2}, \quad \sigma_t = \sigma_x\left(\frac{r_2^2}{R^2} + 1\right), \quad \sigma_r = -\sigma_x\left(\frac{r_2^2}{R^2} - 1\right)$$

$$\tau_h = \frac{1}{3}\sqrt{(\sigma_x - \sigma_t)^2 + (\sigma_t - \sigma_r)^2 + (\sigma_r - \sigma_x)^2 + 6(\tau_{xt}^2 + 0 + 0)}$$

Calculate τ_h for various R.

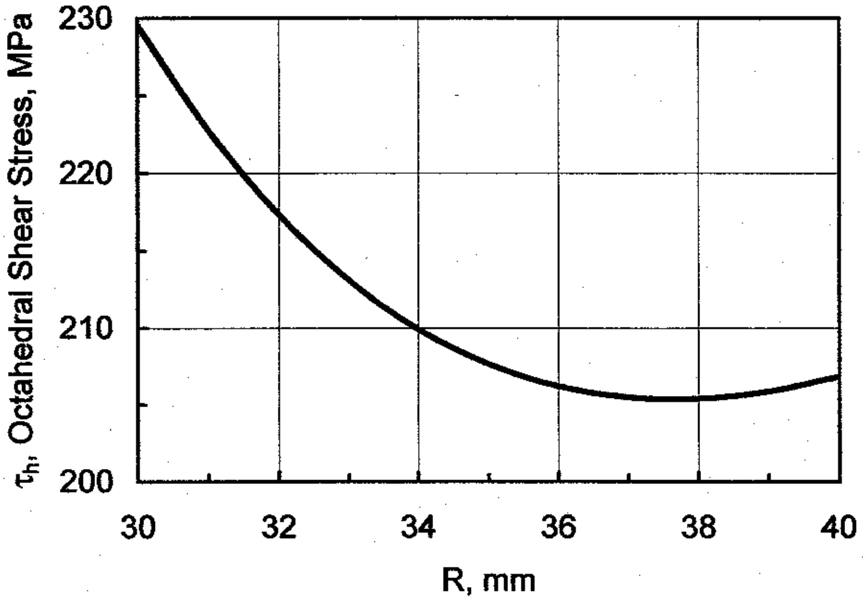
p MPa	T kN-m	r_1 mm	r_2 mm
100	15	30	40

R mm	Stresses in MPa σ_x	σ_t	σ_r	τ_{xt}	τ_h
30.00	128.6	357.1	-100.0	163.7	229.6
32.00	128.6	329.5	-72.3	174.6	217.3
34.00	128.6	306.5	-49.4	185.5	209.9
36.00	128.6	287.3	-30.2	196.4	206.2
38.00	128.6	271.0	-13.9	207.4	205.4
40.00	128.6	257.1	0.0	218.3	206.8

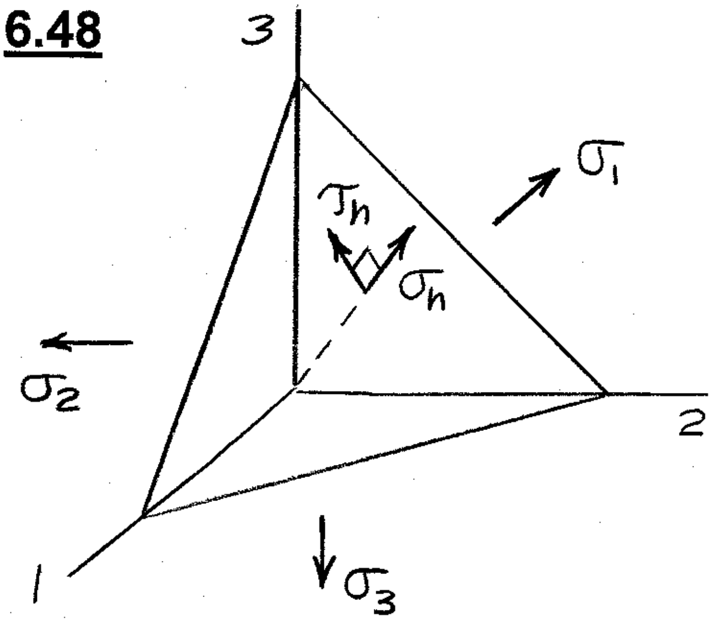
The highest τ_h occurs at the inner wall, where: $\tau_h = 229.6$ MPa ◀

(Plot is on next page.)

(6.47, p.2)


230
220
210
200
τ_h, Octahedral Shear Stress, MPa
30
32
34
36
38
40
R, mm

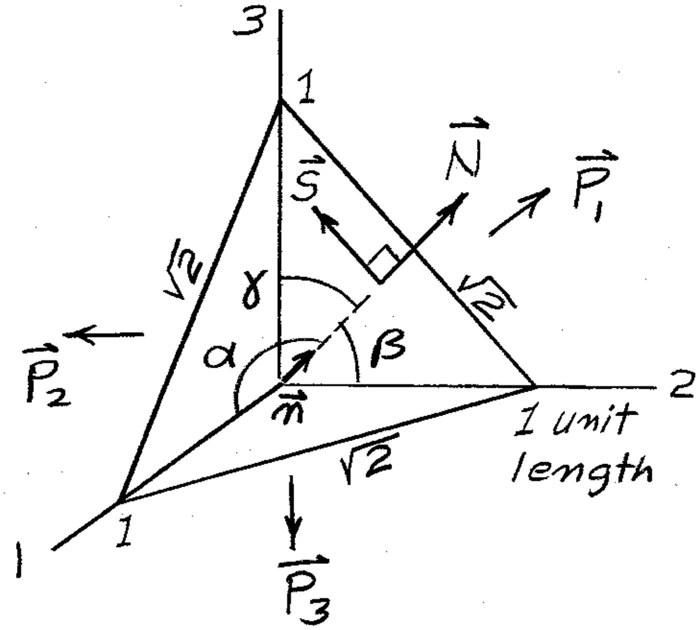
6.48



$$\sigma_h = f(\sigma_1, \sigma_2, \sigma_3)$$

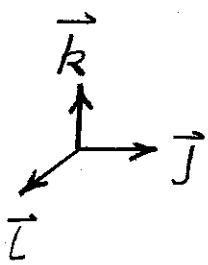
$$\tau_h = g(\sigma_1, \sigma_2, \sigma_3)$$

Multiply by relative areas to get forces, and then invoke equilibrium.



$$A_1 = A_2 = A_3 = \frac{1}{2}$$

$$A_h = \frac{\sqrt{3}}{2}$$



unit vectors

$$P_1 = \sigma_1 A_1 = \frac{\sigma_1}{2}, \quad P_2 = \sigma_2 A_2 = \frac{\sigma_2}{2}, \quad P_3 = \sigma_3 A_3 = \frac{\sigma_3}{2}$$

$$\vec{P} = -\vec{P}_1 - \vec{P}_2 - \vec{P}_3 = -\frac{\sigma_1}{2}\vec{i} - \frac{\sigma_2}{2}\vec{j} - \frac{\sigma_3}{2}\vec{k}$$

$\Sigma F_{\vec{n}} = 0$, $\Sigma \vec{F} = 0$ gives two equations

$$N = \sigma_h A_h = \frac{\sqrt{3}}{2}\sigma_h, \quad S = \tau_h A_h = \frac{\sqrt{3}}{2}\tau_h$$

(6.48, p.2)

$$\vec{n} = \cos\alpha\,\vec{i} + \cos\beta\,\vec{j} + \cos\gamma\,\vec{k} \quad \text{(unit vector)}$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1, \quad \alpha = \beta = \gamma$$

$$\cos\alpha = \cos\beta = \cos\gamma = 1/\sqrt{3}$$

$$\Sigma F_{\vec{n}} = \vec{P}\cdot\vec{n} + N = 0$$

$$\Sigma F_{\vec{n}} = -\tfrac{1}{2}(\sigma_1\vec{i} + \sigma_2\vec{j} + \sigma_3\vec{k})\cdot(\vec{i} + \vec{j} + \vec{k})\frac{1}{\sqrt{3}} + (\sqrt{3}/2)\sigma_h$$

$$\sigma_h = \frac{2}{\sqrt{3}}\,\frac{1}{2}\,\frac{1}{\sqrt{3}}(\sigma_1 + \sigma_2 + \sigma_3) = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} \quad \blacktriangleleft$$

$$\Sigma\vec{F} = \vec{P} + \vec{N} + \vec{S} = 0$$

$$\vec{N} = N\vec{n} = \frac{\sqrt{3}}{2}\sigma_h\,(\vec{i} + \vec{j} + \vec{k})\frac{1}{\sqrt{3}}$$

$$\vec{N} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{6}(\vec{i} + \vec{j} + \vec{k})$$

$$\vec{S} = -\vec{P} - \vec{N} = \tfrac{1}{2}(\sigma_1\vec{i} + \sigma_2\vec{j} + \sigma_3\vec{k}) - \vec{N}$$

$$\vec{S} = \frac{1}{6}\Big[(2\sigma_1 - \sigma_2 - \sigma_3)\vec{i} + (2\sigma_2 - \sigma_1 - \sigma_3)\vec{j} + (2\sigma_3 - \sigma_1 - \sigma_3)\vec{k}\Big] = S_x\vec{i} + S_y\vec{j} + S_z\vec{k}$$

$S = \sqrt{S_x^2 + S_y^2 + S_z^2}$, After manipulation:

$$\tau_h = \frac{2}{\sqrt{3}}S = \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} \quad \blacktriangleleft$$

6.49 Strains on mild steel surface from Prob. 5.17: $\varepsilon_x = 190 \times 10^{-6}$, $\varepsilon_y = -760 \times 10^{-6}$, and $\gamma_{xy} = 300 \times 10^{-6}$. Find ε_1, ε_2, ε_3, and $\gamma_1, \gamma_2, \gamma_3$.

$$\varepsilon_1, \varepsilon_2 = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$\varepsilon_1, \varepsilon_2 = \left[\frac{190 - 760}{2} \pm \sqrt{\left(\frac{190 + 760}{2}\right)^2 + \left(\frac{300}{2}\right)^2}\right] \times 10^{-6}$$

$$\varepsilon_1, \varepsilon_2 = [-285 \pm 498.1] \times 10^{-6} = 213.1 \times 10^{-6}, \quad -783.1 \times 10^{-6}$$ ◀

$$\varepsilon_z = \varepsilon_3 = \frac{-\nu}{1-\nu}(\varepsilon_x + \varepsilon_y), \quad \nu = 0.293 \text{ (Table 5.2)}$$

$$\varepsilon_3 = 236.2 \times 10^{-6}$$ ◀

$$\gamma_1, \gamma_2, \gamma_3 = |\varepsilon_2 - \varepsilon_3|, \ |\varepsilon_1 - \varepsilon_3|, \ |\varepsilon_1 - \varepsilon_2|$$

$$\gamma_1, \gamma_2, \gamma_3 = 1019 \times 10^{-6}; \ 23.1 \times 10^{-6}; \ 996.2 \times 10^{-6}$$ ◀

6.51 Strain gage rosette on Ti alloy surface.

$\varepsilon_x = 3800 \times 10^{-6}$, $\varepsilon_y = 160 \times 10^{-6}$, $\varepsilon_{45} = 2340 \times 10^{-6}$.

Find ε_1, ε_2, ε_3, and γ_1, γ_2, γ_3. No yielding.

From Ex. 6.9: $\gamma_{xy} = 2\varepsilon_{45} - \varepsilon_x - \varepsilon_y$

$$\gamma_{xy} = [2(2340) - 3800 - 160] \times 10^{-6} = 720 \times 10^{-6}$$

$$\varepsilon_1, \varepsilon_2 = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$\varepsilon_1, \varepsilon_2 = [1980 \pm 1855] \times 10^{-6} = 3835 \times 10^{-6}, 125 \times 10^{-6}$$ ◀

$$\varepsilon_z = \varepsilon_3 = \frac{-\nu}{1-\nu}(\varepsilon_x + \varepsilon_y), \quad \nu = 0.361 \text{ (Table 5.2)}$$

$$\varepsilon_3 = -2237 \times 10^{-6}$$ ◀

$$\gamma_1, \gamma_2, \gamma_3 = |\varepsilon_2 - \varepsilon_3|, |\varepsilon_1 - \varepsilon_3|, |\varepsilon_1 - \varepsilon_2|$$

$$\gamma_1, \gamma_2, \gamma_3 = 2362 \times 10^{-6}, 6072 \times 10^{-6}, 3711 \times 10^{-6}$$ ◀

6.52 Strain gage rosette on Al alloy free surface. $\varepsilon_x = 1200 \times 10^{-6}$, $\varepsilon_y = -650 \times 10^{-6}$, and $\varepsilon_{45} = 1900 \times 10^{-6}$. No yielding. Estimate $\sigma_x, \sigma_y, \tau_{xy}$ and σ_{max}, τ_{max}.

From Ex. 6.9: $\gamma_{xy} = 2\varepsilon_{45} - \varepsilon_x - \varepsilon_y$

$$\gamma_{xy} = [2(1900) - 1200 + 650] \times 10^{-6} = 3250 \times 10^{-6}$$

Hooke's Law for $\sigma_z = \tau_{yz} = \tau_{zx} = 0$ (surface)

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu\sigma_y], \quad \varepsilon_y = \frac{1}{E}[\sigma_y - \nu\sigma_x]$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G}, \quad G = \frac{E}{2(1+\nu)}$$

$E = 70{,}300$ MPa, $\nu = 0.345$ (Table 5.2)

$G = 26{,}134$ MPa, $\tau_{xy} = (32.5 \times 10^{-4})(26{,}134 \text{ MPa})$

$\tau_{xy} = 84.93$ MPa ◀

$$12 \times 10^{-4} = \frac{1}{70{,}300 \text{ MPa}}[\sigma_x - 0.345\sigma_y]$$

$$-6.5 \times 10^{-4} = \frac{1}{70{,}300 \text{ MPa}}[\sigma_y - 0.345\sigma_x]$$

Solve simultaneously

$\sigma_x = 77.86$, $\sigma_y = -18.83$ MPa ◀

$$\sigma_{max} = MAX(\sigma_1, \sigma_2, \sigma_3)$$

$$\tau_{max} = MAX(\tau_1, \tau_2, \tau_3)$$

(6.52, p.2)

$$\sigma_1, \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}, \quad \sigma_3 = 0$$

$$\sigma_1, \sigma_2 = 29.52 \pm 97.73 = 127.25, -68.21 \text{ MPa}$$

$$\sigma_{max} = 127.25 \text{ MPa} \quad \blacktriangleleft$$

$$\tau_{max} = MAX\left[\frac{|\sigma_1 - \sigma_2|}{2}, \frac{|\sigma_2 - \sigma_3|}{2}, \frac{|\sigma_3 - \sigma_1|}{2}\right]$$

$$\tau_{max} = 97.73 \text{ MPa} \quad \blacktriangleleft$$

6.53 Strain gage rosette as in Fig. 6.16(b). Develop equations to calculate ε_y and γ_{xy} from ε_x, ε_{60}, and ε_{120}.

From Ex. 6.9:

$$\varepsilon_\theta = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$\varepsilon_{60} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}(-0.5) + \frac{\gamma_{xy}}{2}\left(\frac{\sqrt{3}}{2}\right)$$

$$\varepsilon_{120} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}(-0.5) + \frac{\gamma_{xy}}{2}\left(-\frac{\sqrt{3}}{2}\right)$$

$$\varepsilon_{60} - \varepsilon_{120} = \frac{\sqrt{3}}{2}\gamma_{xy}, \quad \gamma_{xy} = \frac{2}{\sqrt{3}}(\varepsilon_{60} - \varepsilon_{120}) \quad \blacktriangleleft$$

$$\varepsilon_{60} + \varepsilon_{120} = (\varepsilon_x + \varepsilon_y) + (\varepsilon_x - \varepsilon_y)(-0.5)$$

$$\varepsilon_{60} + \varepsilon_{120} = \varepsilon_x/2 + 3\varepsilon_y/2$$

$$\varepsilon_y = \frac{1}{3}(2\varepsilon_{60} + 2\varepsilon_{120} - \varepsilon_x) \quad \blacktriangleleft$$

6.54 Develop equations for ε_h and γ_h.

$$\sigma_h = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}$$

$$\tau_h = \frac{1}{3}\sqrt{(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)}$$

$\sigma_x, \sigma_y, \sigma_z \longrightarrow \varepsilon_x, \varepsilon_y, \varepsilon_z$, hence $\sigma_h \rightarrow \varepsilon_h$

$\tau_{xy}, \tau_{yz}, \tau_{zx} \longrightarrow \frac{\gamma_{xy}}{2}, \frac{\gamma_{yz}}{2}, \frac{\gamma_{zx}}{2}$, hence $\tau_h \rightarrow \frac{\gamma_h}{2}$

$$\varepsilon_h = \frac{\varepsilon_1 + \varepsilon_2 + \varepsilon_3}{3} = \frac{\varepsilon_v}{3} = \frac{1}{3}\left(\begin{matrix}\text{volumetric}\\ \text{strain}\end{matrix}\right)$$ ◀

$$\gamma_h = \frac{2}{3}\sqrt{(\varepsilon_x - \varepsilon_y)^2 + (\varepsilon_y - \varepsilon_z)^2 + (\varepsilon_z - \varepsilon_x)^2 + \frac{3}{2}(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{zx}^2)}$$ ◀

The octahedral planes for stress and strain are the same only in cases where the principal axes for stress and strain coincide, hence for isotropic materials in general, and for the orthotropic case if the principal stresses are parallel to the planes of material symmetry.