Escola Politécnica da Universidade de São Paulo
Departamento de Engenharia Mecânica

PME-3211 – Mecânica dos Sólidos II

Aula #10

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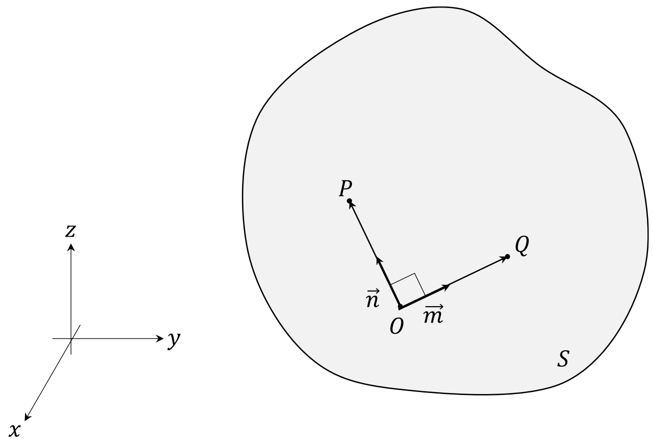
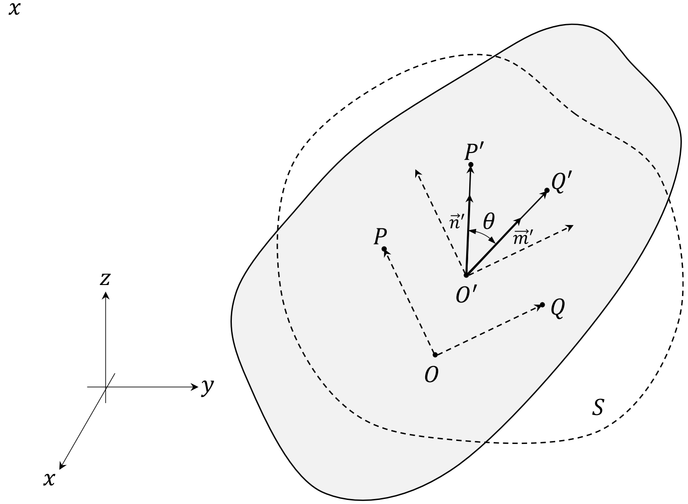
Distorção entre duas direções perpendiculares





$$d\vec{r}_p = (P - 0) = \begin{Bmatrix} dx_P \\ dy_p \\ dz_P \end{Bmatrix}$$

$$\vec{n} = \frac{(P - O)}{|P - O|} = \frac{d\vec{r}_P}{dr_p} = \frac{1}{dr_P}\begin{Bmatrix} dx_P \\ dy_p \\ dz_P \end{Bmatrix} = \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix}$$

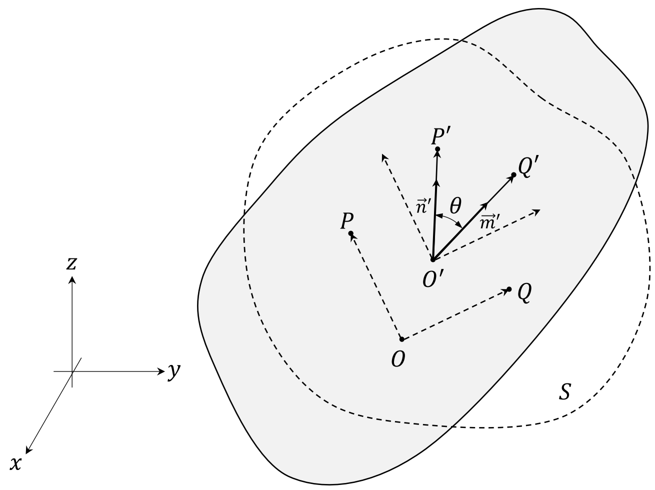
$$d\vec{r}_Q = (Q - O) = \begin{Bmatrix} dx_Q \\ dy_Q \\ dz_Q \end{Bmatrix}$$

$$\vec{m} = \frac{(Q - O)}{|Q - O|} = \frac{d\vec{r}_Q}{dr_Q} = \frac{1}{dr_Q}\begin{Bmatrix} dx_Q \\ dy_Q \\ dz_Q \end{Bmatrix} = \begin{Bmatrix} m_x \\ m_y \\ m_z \end{Bmatrix}$$

$$\vec{n}' = \frac{(P' - O')}{|P' - O'|}$$

$$\vec{m}' = \frac{(Q' - O')}{|Q' - O'|}$$

Distorção entre duas direções perpendiculares



$$\gamma = \frac{\pi}{2} - \theta$$

$$\theta = \frac{\pi}{2} - \gamma$$

$$\cos\theta = \cos\left(\frac{\pi}{2} - \gamma\right) = \operatorname{sen}\gamma$$

Para pequenas deformações

$$\gamma = \cos\theta$$

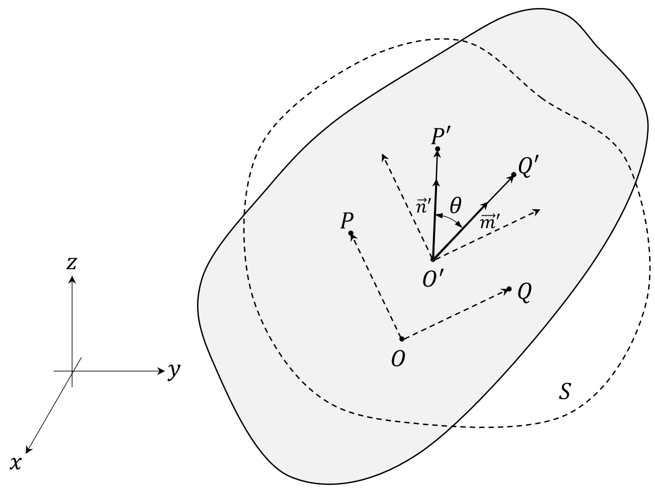
Mas

$$\vec{n}' \cdot \vec{m}' = \cos\theta$$

Então:

$$\gamma = \vec{n}' \cdot \vec{m}'$$

Distorção entre duas direções perpendiculares



$$|P' - O'| = (1 + \varepsilon_P)|P - O| = (1 + \varepsilon_P)dr_P$$

$$\vec{n}' = \frac{(P' - P) + (P - O) - (O' - O)}{(1 + \varepsilon_P)dr_P}$$

$$(P' - P) = \begin{Bmatrix} u(P) \\ v(P) \\ w(P) \end{Bmatrix}$$

$$(O' - O) = \begin{Bmatrix} u(O) \\ v(O) \\ w(O) \end{Bmatrix}$$

$$\vec{n}' = \frac{1}{dr_P(1 + \varepsilon_P)} \left[\begin{Bmatrix} dx_P \\ dy_P \\ dz_P \end{Bmatrix} + \begin{Bmatrix} u(P) - u(O) \\ v(P) - v(O) \\ w(P) - w(O) \end{Bmatrix} \right]$$

Distorção entre duas direções perpendiculares

$$\vec{n}' = \frac{1}{dr_P(1+\varepsilon_P)}\left[\begin{Bmatrix} dx_P \\ dy_P \\ dz_P \end{Bmatrix} + \begin{Bmatrix} u(P)-u(O) \\ v(P)-v(O) \\ w(P)-w(O) \end{Bmatrix}\right]$$

$$u(P)-u(O) = \frac{\partial u}{\partial x}dx_P + \frac{\partial u}{\partial y}dy_P + \frac{\partial u}{\partial z}dz_P$$

$$v(P)-v(O) = \frac{\partial v}{\partial x}dx_P + \frac{\partial v}{\partial y}dy_P + \frac{\partial v}{\partial z}dz_P$$

$$w(P)-w(O) = \frac{\partial w}{\partial x}dx_P + \frac{\partial w}{\partial y}dy_P + \frac{\partial w}{\partial z}dz_P$$

$$n_x' = \frac{dx_P + \frac{\partial u}{\partial x}dx_P + \frac{\partial u}{\partial y}dy_P + \frac{\partial u}{\partial z}dz_P}{dr_P(1+\varepsilon_P)}$$

$$n_x' = \frac{\left(1+\frac{\partial u}{\partial x}\right)n_x + \frac{\partial u}{\partial y}n_y + \frac{\partial u}{\partial z}n_z}{(1+\varepsilon_P)}$$

Pequenas deformações:

$$n_x' = \left(1+\frac{\partial u}{\partial x}-\varepsilon_P\right)n_x + \frac{\partial u}{\partial y}n_y + \frac{\partial u}{\partial z}n_z$$

Analogamente:

$$n_y' = \frac{\partial v}{\partial x}n_x + \left(1+\frac{\partial v}{\partial y}-\varepsilon_P\right)n_y + \frac{\partial v}{\partial z}n_z$$

$$n_z' = \frac{\partial w}{\partial x}n_x + \frac{\partial w}{\partial y}n_y + \left(1+\frac{\partial w}{\partial z}-\varepsilon_P\right)n_z$$

$$m_x' = \left(1+\frac{\partial u}{\partial x}-\varepsilon_Q\right)m_x + \frac{\partial u}{\partial y}m_y + \frac{\partial u}{\partial z}m_z$$

$$m_y' = \frac{\partial v}{\partial x}m_x + \left(1+\frac{\partial v}{\partial y}-\varepsilon_Q\right)m_y + \frac{\partial v}{\partial z}m_z$$

$$m_z' = \frac{\partial w}{\partial x}m_x + \frac{\partial w}{\partial y}m_y + \left(1+\frac{\partial w}{\partial z}-\varepsilon_q\right)m_z$$

Distorção entre duas direções perpendiculares

$$\gamma(\vec{n},\vec{m}) = n'_x m'_x + n'_y m'_y + n'_z m'_z$$

Considerando pequenas deformações:

$$\begin{aligned}\gamma(\vec{n},\vec{m}) &\\ = (1-\varepsilon_P-\varepsilon_Q)&(n_x m_x + n_y m_y + n_z m_z)\\ + 2\left(\frac{\partial u}{\partial x}n_x m_x + \frac{\partial v}{\partial y}n_y m_y + \frac{\partial w}{\partial z}n_z m_z\right)&\\ + \left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)n_y m_x + \left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)n_z m_x&\\ + \left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)n_x m_y + \left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)n_x m_z&\\ + \left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)n_y m_z + \left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)n_z m_y&\end{aligned}$$

Como:

$$n_x m_x + n_y m_y + n_z m_z = 0$$

Então:

$$\begin{aligned}\gamma(\vec{n},\vec{m}) &\\ = 2\left(\frac{\partial u}{\partial x}n_x m_x + \frac{\partial v}{\partial y}n_y m_y + \frac{\partial w}{\partial z}n_z m_z\right)&\\ + \left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)n_y m_x + \left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)n_z m_x&\\ + \left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)n_x m_y + \left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)n_x m_z&\\ + \left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)n_y m_z + \left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)n_z m_y&\end{aligned}$$

Portanto:

$$\begin{aligned}\gamma(\vec{n},\vec{m}) &\\ = 2(\varepsilon_x n_x m_x + \varepsilon_y n_y m_y + \varepsilon_z n_z m_z)&\\ + \gamma_{xy}(n_y m_x + n_x m_y)&\\ + \gamma_{xz}(n_z m_x + n_x m_z)&\\ + \gamma_{yz}(n_y m_z + n_z m_y)&\end{aligned}$$

Distorção entre duas direções perpendiculares

$$\gamma(\vec{n},\vec{m}) = 2\left(\varepsilon_x n_x m_x + \varepsilon_y n_y m_y + \varepsilon_z n_z m_z\right) + \gamma_{xy}\left(n_y m_x + n_x m_y\right) + \gamma_{xz}\left(n_z m_x + n_x m_z\right) + \gamma_{yz}\left(n_y m_z + n_z m_y\right)$$

$$\Rightarrow \gamma(\vec{n},\vec{m}) = 2\begin{Bmatrix} m_x \\ m_y \\ m_z \end{Bmatrix}^t \begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & \varepsilon_z \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix}$$

$$=$$

$$\gamma(\vec{n},\vec{m}) = 2\begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix}^t \begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & \varepsilon_z \end{bmatrix} \begin{Bmatrix} m_x \\ m_y \\ m_z \end{Bmatrix}$$

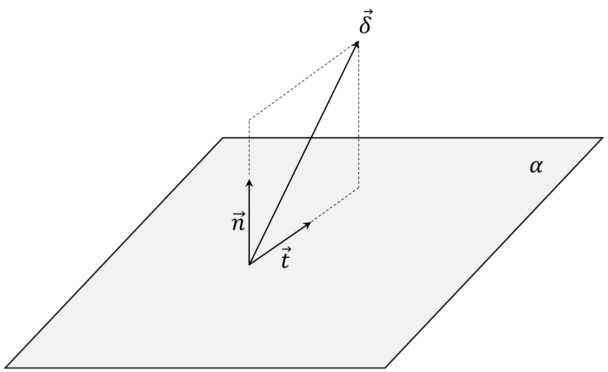
Na forma tensorial:

$$\gamma(\vec{n},\vec{m}) = 2(E[\vec{n}]\cdot\vec{m}) = 2(E[\vec{m}]\cdot\vec{n})$$

Vetor deformação

- Definição

$$\vec{\delta}(\vec{n}) = E[\vec{n}]$$



$\vec{n}$ e $\vec{t}$ são versores

- Componentes

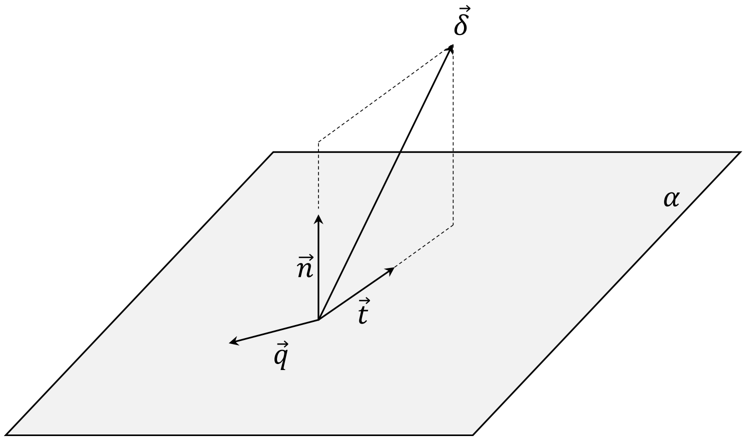
$$\vec{\delta}(\vec{n}) = a\,\vec{n} + b\,\vec{t}$$

$$a = \vec{\delta}(\vec{n}) \cdot \vec{n} = E[\vec{n}] \cdot \vec{n} = \varepsilon(\vec{n})$$

$$b = \vec{\delta}(\vec{n}) \cdot \vec{t} = E[\vec{n}] \cdot \vec{t} = \frac{1}{2}\gamma(\vec{n}, \vec{t})$$

$$\Rightarrow \vec{\delta}(\vec{n}) = \varepsilon(\vec{n})\,\vec{n} + \frac{1}{2}\gamma(\vec{n}, \vec{t})\,\vec{t}$$

Distorção máxima



- Seja o versor $\vec{q} \parallel \alpha$
- Então:

$$|\gamma(\vec{n},\vec{q})| \leq |\gamma(\vec{n},\vec{t})|$$

- Prova:

$$\gamma(\vec{n},\vec{q}) = 2E[\vec{n}]\cdot\vec{q} = 2\vec{\delta}(\vec{n})\cdot\vec{q}$$

$$= 2\left(\varepsilon(\vec{n})\vec{n} + \frac{1}{2}\gamma(\vec{n},\vec{t})\vec{t}\right)\cdot\vec{q}$$

$$= \gamma(\vec{n},\vec{t})(\vec{t}\cdot\vec{q})$$

mas

$$|\vec{t}\cdot\vec{q}| \leq 1$$

então:

$$|\gamma(\vec{n},\vec{q})| \leq |\gamma(\vec{n},\vec{t})|$$

- Notação:

$$\bar{\gamma}(\vec{n}) = \gamma(\vec{n},\vec{t})$$

Referência

Martins, C.A. *Introdução ao Estudo das Deformações.* Disponível no Moodle