Exercício 5. *Considere em* $\mathbb{C}^2$ *o operador hermitiano*

$$X = \begin{bmatrix} 1 & 1+i \\ 1-i & 1 \end{bmatrix}.$$

1. *Encontre os autovalores e autovetores desse operador.*
2. *Encontre a probabilidade de obter cada uma das respostas possíveis quando uma medição de* X *é aplicada a um sistema no estado*
$$|\psi\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$
3. *Encontre a probabilidade de obter cada uma das respostas possíveis quando uma medição de* X *é aplicada a um sistema no estado*
$$|\psi\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}.$$

1.

$$X = \begin{bmatrix} 1 & 1+i \\ 1-i & 1 \end{bmatrix}$$

$$X - \lambda I = \begin{bmatrix} 1-\lambda & 1+i \\ 1-i & 1-\lambda \end{bmatrix}$$

$$\det(X - \lambda I) = (1-\lambda)^2 - (1+i)(1-i)$$

$$= 1 - 2\lambda + \lambda^2 - (1 - (-1))$$

$$= 1 - 2\lambda + \lambda^2 - 1 - 1$$

$$= \lambda^2 - 2\lambda - 1$$

$$\Delta = 4 + 4 = 8$$

$$\lambda = \frac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}$$

possíveis valores que obtemos ao medir X.

$$\lambda_1 = 1 + \sqrt{2}$$

$$X - \lambda I = \begin{bmatrix} 1 - (1+\sqrt{2}) & 1+i \\ 1-i & 1-(1+\sqrt{2}) \end{bmatrix}$$

$$= \begin{bmatrix} -\sqrt{2} & 1+i \\ 1-i & -\sqrt{2} \end{bmatrix}$$

$$\begin{bmatrix} -\sqrt{2} & 1+i \\ 1-i & -\sqrt{2} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} -\sqrt{2}u + (1+i)v &= 0 \\ (1-i)u - \sqrt{2}\,v &= 0 \end{aligned}\right\}$$

$$-\sqrt{2}\,u = -(1+i)\,v$$

$$u = \frac{(1+i)\,v}{\sqrt{2}}$$

$$|v_1\rangle = \begin{bmatrix} (1+i)\,v/\sqrt{2} \\ v \end{bmatrix}$$

$$\|v_1\|^2 = \frac{|1+i|^2}{2} v^2 + v^2$$

exigência de normalização

$$= \frac{(1+1)}{2} v^2 + v^2 = 2v^2 = 1$$

$$v^2 = \frac{1}{2}$$

$$\Rightarrow v = 1/\sqrt{2}$$

$$|v_1\rangle = \begin{bmatrix} (1+i)/2 \\ \\ 1/\sqrt{2} \end{bmatrix}$$

autovetor de X com autovalor $\lambda_1 = 1+\sqrt{2}$

$$\lambda_2 = 1 - \sqrt{2}$$

$$X - \lambda I = \begin{bmatrix} 1-(1-\sqrt{2}) & 1+i \\ 1-i & 1-(1-\sqrt{2}) \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{2} & 1+i \\ 1-i & \sqrt{2} \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{2} & 1+i \\ 1-i & \sqrt{2} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\sqrt{2}\,u + (1+i)\,v = 0$$

$$u = -\frac{(1+i)\,v}{\sqrt{2}}$$

$$|v_2\rangle = \begin{bmatrix} -(1+i)/\sqrt{2}\; v \\ \\ v \end{bmatrix}$$

$$\|v_2\|^2 = \frac{\overbrace{|-1-i|^2}^{=2}}{2}\,v^2 + v^2 = 2v^2 = 1$$

$$\Rightarrow \quad v = \frac{1}{\sqrt{2}}$$

$$|v_2\rangle = \begin{bmatrix} -\frac{(1+i)}{2} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

autovetor com autovalor $\lambda_2 = 1 - \sqrt{2}$

2) Quero encontrar a probabilidade de cada uma das respostas possíveis ao realizar uma medição de X.

$$|\psi\rangle = \alpha\,|v_1\rangle + \beta\,|v_2\rangle$$

$$p(\lambda_1) = |\alpha|^2$$

$$p(\lambda_2) = |\beta|^2$$

$$\langle v_1|\psi\rangle = \alpha \underbrace{\langle v_1|v_1\rangle}_{=1} + \beta \underbrace{\langle v_1|v_2\rangle}_{=0}$$

$$\alpha = \langle v_1|\psi\rangle$$

$$\langle v_2|\psi\rangle = \alpha \underbrace{\langle v_2|v_1\rangle}_{=0} + \beta \underbrace{\langle v_2|v_2\rangle}_{=1}$$

$$\beta = \langle v_2 | \psi \rangle$$

$$|v_1\rangle = \begin{bmatrix} (1+i)/2 \\ 1/\sqrt{2} \end{bmatrix} \qquad |v_2\rangle = \begin{bmatrix} -(1+i)/2 \\ 1/\sqrt{2} \end{bmatrix}$$

$$|\psi\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\langle v_1 | v_2 \rangle = \left(\frac{1+i}{2}\right)^* \times \left(-\frac{(1+i)}{2}\right) + \frac{1}{\sqrt{2}}^* \times \frac{1}{\sqrt{2}}$$

conjugado

$$= \frac{1}{4}(1-i)(-1-i) + \frac{1}{2}$$

$$= \frac{1}{4}(-1-1) + \frac{1}{2} = -\frac{1}{2} + \frac{1}{2} = 0$$

$$|v_1\rangle = \begin{bmatrix} (1+i)/2 \\ 1/\sqrt{2} \end{bmatrix} \qquad |v_2\rangle = \begin{bmatrix} -(1+i)/2 \\ 1/\sqrt{2} \end{bmatrix}$$

$$|\psi\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\alpha = \langle v_1 | \psi \rangle$$

$$= \frac{(1+i)^*}{2} \times 1 + \frac{1}{\sqrt{2}}^* \times 0 = \frac{1-i}{2}$$

$$\beta = \langle v_2 | \psi \rangle$$

$$= -\frac{(1+i)^*}{2} \times 1 + \frac{1}{\sqrt{2}}^* \times 0 = -\frac{(1-i)}{2}$$

$$|\psi\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \alpha |v_1\rangle + \beta |v_2\rangle$$

$$= \frac{(1-i)}{2} |v_1\rangle - \frac{(1-i)}{2} |v_2\rangle$$

probabilidade de obter resposta λ_1 ao medir X

$$p(\lambda_1) = \left| \frac{(1-i)}{2} \right|^2 = \frac{2}{4} = \frac{1}{2}$$

$$p(\lambda_2) = \left| -\frac{(1-i)}{2} \right|^2 = \frac{2}{4} = \frac{1}{2}$$

probabilidade de obter λ_2

$$p(\lambda_1) = 1/2$$
$$p(\lambda_2) = 1/2$$

3. $|\psi\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$|\psi\rangle = \alpha |v_1\rangle + \beta |v_2\rangle$$

$\alpha = \langle v_1 | \psi \rangle$ $\quad\quad$ $\beta = \langle v_2 | \psi \rangle$

$p(\lambda_1) = |\alpha|^2$ $\quad\quad$ $p(\lambda_2) = |\beta|^2$

Exercício 20. *O Hamiltoniano de um sistema é representado pela matriz*

$$H = \hbar\omega \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Dois outros observáveis são representados pelas matrizes

$$A = \gamma \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

$$B = \mu \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

onde ω, γ, μ são constantes reais.

1. *Encontre os autovalores e autovetores de H, A e B.*
2. *Escreva a evolução temporal de cada um desses vetores.*
3. *Escreva a evolução temporal de um estado arbitrário desse sistema.*

1- Operador H

Autovetor $|u_1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ com autovalor $E_1 = \hbar\omega$

Autovetor $|u_2\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ com autovalor $E_2 = 2\hbar\omega$

Autovetor $|u_3\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ com autovalor $E_3 = -\hbar\omega$

Operador A

$$A = \gamma \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} -\lambda & \gamma & 0 \\ \gamma & -\lambda & 0 \\ 0 & 0 & 2\gamma - \lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (-\lambda)\times(-\lambda)\times(2\gamma-\lambda) - \gamma \times \gamma \times (2\gamma - \lambda)$$

$$= (\lambda^2 - \gamma^2)(2\gamma - \lambda)$$

$$\lambda = \pm\gamma \qquad \lambda = 2\gamma$$

$\lambda_1 = \gamma$

$$\begin{bmatrix} -\gamma & \gamma & 0 \\ \gamma & -\gamma & 0 \\ 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-\gamma u + \gamma v = 0 \Rightarrow u = v$$
$$\gamma w = 0 \Rightarrow w = 0$$

$$|v_1\rangle = \begin{bmatrix} u \\ u \\ 0 \end{bmatrix}$$

$$\|v_1\|^2 = u^2 + u^2 = 2u^2 = 1$$

condição de normalização

$$\Rightarrow \quad u = \frac{1}{\sqrt{2}}$$

$$|v_1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$ autovetor com autovalor $\lambda_1 = \gamma$

$$\lambda_2 = -\gamma$$

$$A - \lambda_2 I = \begin{bmatrix} \gamma & \gamma & 0 \\ \gamma & \gamma & 0 \\ 0 & 0 & 3\gamma \end{bmatrix}$$

$$\begin{bmatrix} \gamma & \gamma & 0 \\ \gamma & \gamma & 0 \\ 0 & 0 & 3\gamma \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\gamma u + \gamma v = 0 \quad \Rightarrow \quad u = -v$$

$$3\gamma w = 0 \quad \Rightarrow \quad w = 0$$

$$|v_2\rangle = \begin{bmatrix} u \\ -u \\ 0 \end{bmatrix}$$

$$\|v_2\|^2 = u^2 + u^2 = 2u^2 = 1 \quad \Rightarrow \quad u = \frac{1}{\sqrt{2}}$$

$$|v_2\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

autovetor com autovalor $\lambda_2 = -\gamma$

$\lambda_3 = 2\gamma$

$$\begin{bmatrix} -2\gamma & \gamma & 0 \\ \gamma & -2\gamma & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2\gamma u + \gamma v = 0 \qquad (1)$$
$$\gamma u - 2\gamma v = 0 \qquad (2)$$

$(1) + 2 \times (2)$:

$$\gamma v - 4\gamma v = 0 \qquad -3\gamma v = 0$$
$$\Rightarrow v = 0$$

$$\Rightarrow u = 0$$

$$|v_3\rangle = \begin{bmatrix} 0 \\ 0 \\ w \end{bmatrix}$$

$$|v_3\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

autovetor de A com autovalor $\lambda_3 = 2\gamma$

2- Evolução temporal

Autovetor $|u_1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ com autovalor $E_1 = \hbar\omega$

$$|u_1(t)\rangle = e^{-iE_1 t/\hbar} |u_1\rangle$$

$$= e^{-i\omega t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Autovetor $|u_2\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ com autovalor $E_2 = 2\hbar\omega$

$$|u_2(t)\rangle = e^{-iE_2 t/\hbar} |u_2\rangle$$

$$= e^{-2i\omega t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Autovetor $|u_3\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ com autovalor $E_3 = -\hbar\omega$

$$|u_3(t)\rangle = e^{-iE_3 t/\hbar} |u_3\rangle$$

$$= e^{i\omega t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Evolução temporal de um estado arbitrário

$$\left\{ \begin{aligned} |\psi\rangle &= \alpha_1 |u_1\rangle + \alpha_2 |u_2\rangle + \alpha_3 |u_3\rangle \leftarrow \\ |\psi(t)\rangle &= \alpha_1 |u_1(t)\rangle + \alpha_2 |u_2(t)\rangle + \alpha_3 |u_3(t)\rangle \\ &= \alpha_1 \underline{e^{-i\omega t} |u_1\rangle} + \alpha_2 \underline{e^{-2i\omega t} |u_2\rangle} + \alpha_3 \underline{e^{\omega t} |u_3\rangle} \end{aligned} \right.$$

Evolução temporal dos autoestados de A

$$|v_1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} |u_1\rangle + \frac{1}{\sqrt{2}} |u_2\rangle$$

$$|v_1(t)\rangle = \frac{1}{\sqrt{2}} e^{-i\omega t} |u_1\rangle + \frac{1}{\sqrt{2}} e^{-2i\omega t} |u_2\rangle$$

$$|v_2\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} |u_1\rangle - \frac{1}{\sqrt{2}} |u_2\rangle$$

$$|v_2(t)\rangle = \frac{1}{\sqrt{2}} e^{-i\omega t} |u_1\rangle - \frac{1}{\sqrt{2}} e^{-2i\omega t} |u_2\rangle$$

$$|v_3\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$|v_3\rangle = |u_3\rangle$$

$$|v_3(t)\rangle = e^{i\omega t} |u_3\rangle$$

3-

Evolução temporal de um estado arbitrário

$$|\psi\rangle = \alpha_1|u_1\rangle + \alpha_2|u_2\rangle + \alpha_3|u_3\rangle \leftarrow$$

$$|\psi(t)\rangle = \alpha_1|u_1(t)\rangle + \alpha_2|u_2(t)\rangle + \alpha_3|u_3(t)\rangle$$

$$= \alpha_1 \underline{e^{-i\omega t}|u_1\rangle} + \alpha_2 \underline{e^{-2i\omega t}|u_2\rangle} + \alpha_3 \underline{e^{\omega t}|u_3\rangle}$$

solução da equação de Schrödinger

$$i\hbar \frac{d\,|\Psi(t)\rangle}{dt} = H\,|\Psi(t)\rangle$$

$$|\Psi(t)\rangle = f(t)\,|v\rangle$$

$$\frac{d}{dt}\,|\Psi(t)\rangle = f'(t)\,|v\rangle$$

$$i\hbar\, f'(t)\,|v\rangle = H f(t)\,|v\rangle$$

$$i\hbar f'(t)\,|v\rangle = f(t)\,H\,|v\rangle$$

$$H\,|v\rangle = \underbrace{\left[\frac{i\hbar\, f'(t)}{f(t)}\right]}_{\text{autovalor de } H}\,|v\rangle$$

$|v\rangle$ é autovetor com autovalor

$$\frac{i\hbar\, f'(t)}{f(t)} = E$$

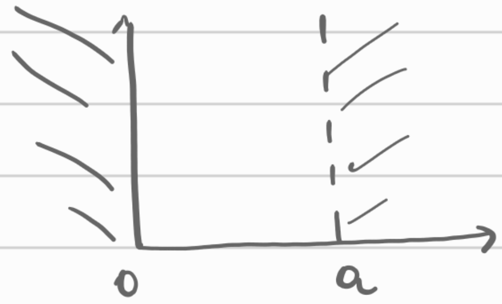
$$f'(t) = \frac{E}{i\hbar}\, f(t)$$

$$f(t) = e^{Et/i\hbar}$$

$$= e^{-iEt/\hbar}$$

Exemplo da aula passada

1)



$$U(x) = \begin{cases} 0, & 0 < x < L \\ \infty, & \text{cc.} \end{cases}$$

$$\psi_n(x) = \sqrt{\frac{2}{a}}\, \text{sen}\left(\frac{n\pi x}{a}\right), \qquad E_n = \frac{\hbar^2 \pi^2 n^2}{2ma^2}$$

Estados estacionários (soluções separáveis):

$$\Psi_n(x,t) = T(t)\, \psi_n(x)$$

$$= e^{-iE_n t/\hbar} \sqrt{\frac{2}{a}}\, \text{sen}\left(\frac{n\pi x}{a}\right)$$

$$\Psi_n(x,t) = \sqrt{\frac{2}{a}}\, e^{-iE_n t/\hbar}\, \text{sen}\left(\frac{n\pi x}{a}\right)$$

Em geral:

$$\Psi(x,t) = \sum_{n=1}^{\infty} c_n \Psi_n(x,t)$$

$$= \sum_{n=1}^{\infty} c_n \sqrt{\frac{2}{a}}\, e^{-iE_n t/\hbar}\, \text{sen}\left(\frac{n\pi x}{a}\right)$$

condições iniciais

3)

$$\psi_1(r,\theta,\phi) = \frac{1}{\sqrt{\pi}\, a_0^{3/2}} e^{-r/a_0}$$

$$\Psi_1(t,r,\theta,\phi) = \frac{1}{\sqrt{\pi}\, a_0^{3/2}} e^{-iE_1 t/\hbar} e^{-r/a_0}$$

$$E_n = -\frac{13{,}6}{n^2} \quad eV$$

$$E_1 = -13{,}6 \quad eV$$

Átomo de manganês, 25 elétrons

$n, l, m_l, m_s = \pm \frac{1}{2}$

$E_n = \frac{-13{,}6}{n^2}$ eV

s:	$l = 0$
p:	$l = 1$
d:	$l = 2$
f:	$l = 3$

$|L| = \hbar \sqrt{l(l+1)}$

$L_z = \hbar m_l$

$l = 0, 1, 2, \dots, n-1$

$m_l = -l, -l+1, \dots, 0, \dots, l-1, l$

$n = 1$	$l = 0$	$m_l = 0$	$m_s = \pm 1/2$	2 elet.
$n = 2$	$l = 0$	$m_l = 0$	$m_s = \pm 1/2$	2 eletr.
$n = 2$	$l = 1$	$m_l = -1, 0, 1$	$m_s = \pm 1/2$	6 eletr
$n = 3$	$l = 0$	$m_l = 0$	$m_s = \pm 1/2$	2 eletr.
$n = 3$	$l = 1$	$m_l = -1, 0, 1$	$m_s = \pm 1/2$	6 eletr.
$n = 4$	$l = 0$	$m_l = 0$	$m_s = \pm 1/2$	2 eletr.
$n = 3$	$l = 2$	$m_l = 0, \pm 1, \pm 2$		5 eletr.

↑	↑	↑	↑	↑
m_l: −2	−1	0	1	2

↑ ↓ ↑ ↓ ↑