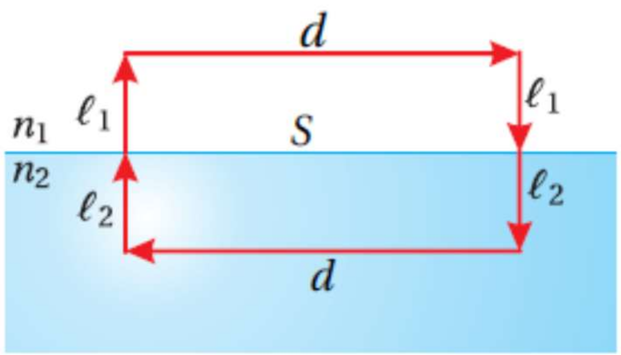
Condições de contorno

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad \text{(Gauss' Law)}$$

$$\nabla \cdot \mathbf{B} = 0 \quad \text{(Gauss' Law for magnetism)}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{(Faraday's Law)}$$

$$\nabla \times \frac{\mathbf{B}}{\mu_0} = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J} \quad \text{(Ampere's Law revised by Maxwell)}$$



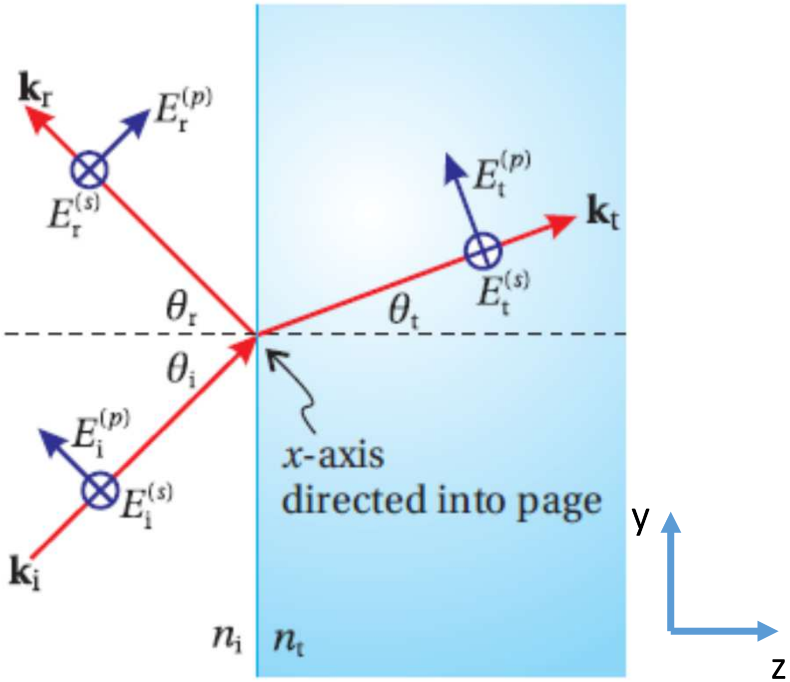
$$\oint_C \mathbf{E} \cdot d\ell = -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot \hat{\mathbf{n}}\, da \rightarrow \oint \mathbf{E} \cdot d\ell = E_{1||} d - E_{1\perp} \ell_1 - E_{2\perp} \ell_2 - E_{2||} d + E_{2\perp} \ell_2 + E_{1\perp} \ell_1 = (E_{1||} - E_{2||}) d$$

$$\int_S \mathbf{B} \cdot \hat{\mathbf{n}}\, da \rightarrow 0 \rightarrow E_{1||} = E_{2||} \quad \textbf{(1)}$$

$$\oint_C \mathbf{B} \cdot d\ell = \mu_0 \int_S \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \cdot \hat{\mathbf{n}}\, da \rightarrow \oint \mathbf{B} \cdot d\ell = B_{1||} d - B_{1\perp} \ell_1 - B_{2\perp} \ell_2 - B_{2||} d + B_{2\perp} \ell_2 + B_{1\perp} \ell_1 = (B_{1||} - B_{2||}) d$$

$$\int_S \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \cdot \hat{\mathbf{n}}\, da \rightarrow 0 \rightarrow B_{1||} = B_{2||} \quad \textbf{(2)}$$

Refração e reflexão na interface



$$\mathbf{k}_i = k_i\left(\hat{\mathbf{y}}\sin\theta_i + \hat{\mathbf{z}}\cos\theta_i\right)$$

$$\mathbf{k}_r = k_r\left(\hat{\mathbf{y}}\sin\theta_r - \hat{\mathbf{z}}\cos\theta_r\right)$$

$$\mathbf{k}_t = k_t\left(\hat{\mathbf{y}}\sin\theta_t + \hat{\mathbf{z}}\cos\theta_t\right)$$

$$\mathbf{E}_i = \left[E_i^{(p)}\left(\hat{\mathbf{y}}\cos\theta_i - \hat{\mathbf{z}}\sin\theta_i\right) + \hat{\mathbf{x}}E_i^{(s)}\right] e^{i\left[k_i\left(y\sin\theta_i + z\cos\theta_i\right) - \omega_i t\right]}$$

$$\mathbf{E}_r = \left[E_r^{(p)}\left(\hat{\mathbf{y}}\cos\theta_r + \hat{\mathbf{z}}\sin\theta_r\right) + \hat{\mathbf{x}}E_r^{(s)}\right] e^{i\left[k_r\left(y\sin\theta_r - z\cos\theta_r\right) - \omega_r t\right]}$$

$$\mathbf{E}_t = \left[E_t^{(p)}\left(\hat{\mathbf{y}}\cos\theta_t - \hat{\mathbf{z}}\sin\theta_t\right) + \hat{\mathbf{x}}E_t^{(s)}\right] e^{i\left[k_t\left(y\sin\theta_t + z\cos\theta_t\right) - \omega_t t\right]}$$

De (1), as componentes x e y da incidência e reflexão (paralelas à interface) combinadas devem igualar a componente de transmissão na interface z = 0

$$\left[E_i^{(p)}\hat{\mathbf{y}}\cos\theta_i + \hat{\mathbf{x}}E_i^{(s)}\right] e^{i\left(k_i y\sin\theta_i - \omega_i t\right)} + \left[E_r^{(p)}\hat{\mathbf{y}}\cos\theta_r + \hat{\mathbf{x}}E_r^{(s)}\right] e^{i\left(k_r y\sin\theta_r - \omega_r t\right)}$$

$$= \left[E_t^{(p)}\hat{\mathbf{y}}\cos\theta_t + \hat{\mathbf{x}}E_t^{(s)}\right] e^{i\left(k_t y\sin\theta_t - \omega_t t\right)}$$

Refração e reflexão na interface

Isso deve valer para quaisquer valores de t e y. Logo, as fases das exponenciais devem ser iguais. Portanto:

$$\omega_i = \omega_r = \omega_t \equiv \omega$$

Da mesma forma, os termos espaciais dos expoentes devem se igualar. Logo:

$$k_i \sin\theta_i = k_r \sin\theta_r = k_t \sin\theta_t$$

$k_i = k_r = n_i\omega/c$ and $k_t = n_t\omega/c$ **Daí tiramos:**

$\theta_r = \theta_i$ **Lei da reflexão** **e** $n_i \sin\theta_i = n_t \sin\theta_t$ **Lei de Snell !!**

Refração e reflexão na interface

Coeficientes de Fresnel

Com a igualdade dos expoentes, pode-se escrever duas equações, para as componentes x e y

$$E_i^{(s)} + E_r^{(s)} = E_t^{(s)}$$

$$\left(E_i^{(p)} + E_r^{(p)}\right)\cos\theta_i = E_t^{(p)}\cos\theta_t$$

Quatro incógnitas. Precisamos mais equações !

Aí entra o campo magnético para nos ajudar

$$\mathbf{B} = \frac{\mathbf{k}\times\mathbf{E}}{\omega} = \frac{n}{c}\hat{\mathbf{u}}\times\mathbf{E}$$

$$\vec{u} = \frac{\vec{k}}{k}$$

$$\mathbf{B}_i = \frac{n_i}{c}\left[-\hat{\mathbf{x}}E_i^{(p)} + E_i^{(s)}\left(-\hat{\mathbf{z}}\sin\theta_i + \hat{\mathbf{y}}\cos\theta_i\right)\right]e^{i[k_i(y\sin\theta_i + z\cos\theta_i) - \omega_i t]}$$

$$\mathbf{B}_r = \frac{n_r}{c}\left[\hat{\mathbf{x}}E_r^{(p)} + E_r^{(s)}\left(-\hat{\mathbf{z}}\sin\theta_r - \hat{\mathbf{y}}\cos\theta_r\right)\right]e^{i[k_r(y\sin\theta_r - z\cos\theta_r) - \omega_r t]}$$

$$\mathbf{B}_t = \frac{n_t}{c}\left[-\hat{\mathbf{x}}E_t^{(p)} + E_t^{(s)}\left(-\hat{\mathbf{z}}\sin\theta_t + \hat{\mathbf{y}}\cos\theta_t\right)\right]e^{i[k_t(y\sin\theta_t + z\cos\theta_t) - \omega_t t]}$$

Aplicando a condição de contorno (2) para z = 0:

$$\frac{n_i}{c}\left[-\hat{\mathbf{x}}E_i^{(p)} + E_i^{(s)}\hat{\mathbf{y}}\cos\theta_i\right] + \frac{n_i}{c}\left[\hat{\mathbf{x}}E_r^{(p)} - E_r^{(s)}\hat{\mathbf{y}}\cos\theta_i\right] = \frac{n_t}{c}\left[-\hat{\mathbf{x}}E_t^{(p)} + E_t^{(s)}\hat{\mathbf{y}}\cos\theta_t\right]$$

Logo:

$$n_i\left(E_i^{(p)} - E_r^{(p)}\right) = n_t E_t^{(p)}$$

$$n_i\left(E_i^{(s)} - E_r^{(s)}\right)\cos\theta_i = n_t E_t^{(s)}\cos\theta_t$$

Refração e reflexão na interface

Coeficientes de Fresnel

Podemos escrever: $E_i^{(s)} + E_r^{(s)} = E_t^{(s)}$ **(a)** $\quad E_i^{(s)} - E_r^{(s)} = \dfrac{n_t \cos\theta_t}{n_i \cos\theta_i} E_t^{(s)}$ **(b)**

Somando (a) e (b) temos: $2E_i^{(s)} = \left[1 + \dfrac{n_t \cos\theta_t}{n_i \cos\theta_i}\right] E_t^{(s)}$ **(c)** → $\dfrac{E_t^{(s)}}{E_i^{(s)}} = \dfrac{2n_i \cos\theta_i}{n_i \cos\theta_i + n_t \cos\theta_t}$

Subtraindo (a) de (b) temos: $2E_r^{(s)} = \left[1 - \dfrac{n_t \cos\theta_t}{n_i \cos\theta_i}\right] E_t^{(s)}$

e dividindo por (c) → $\dfrac{E_r^{(s)}}{E_i^{(s)}} = \dfrac{n_i \cos\theta_i - n_t \cos\theta_t}{n_i \cos\theta_i + n_t \cos\theta_t}$

Refração e reflexão na interface

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Finalmente, e usando a Lei de Snell:

$$r_s \equiv \frac{E_r^{(s)}}{E_i^{(s)}} = \frac{n_i \cos\theta_i - n_t \cos\theta_t}{n_i \cos\theta_i + n_t \cos\theta_t} = \frac{\sin\theta_t \cos\theta_i - \sin\theta_i \cos\theta_t}{\sin\theta_t \cos\theta_i + \sin\theta_i \cos\theta_t} = \frac{\sin(\theta_t - \theta_i)}{\sin(\theta_t + \theta_i)}$$

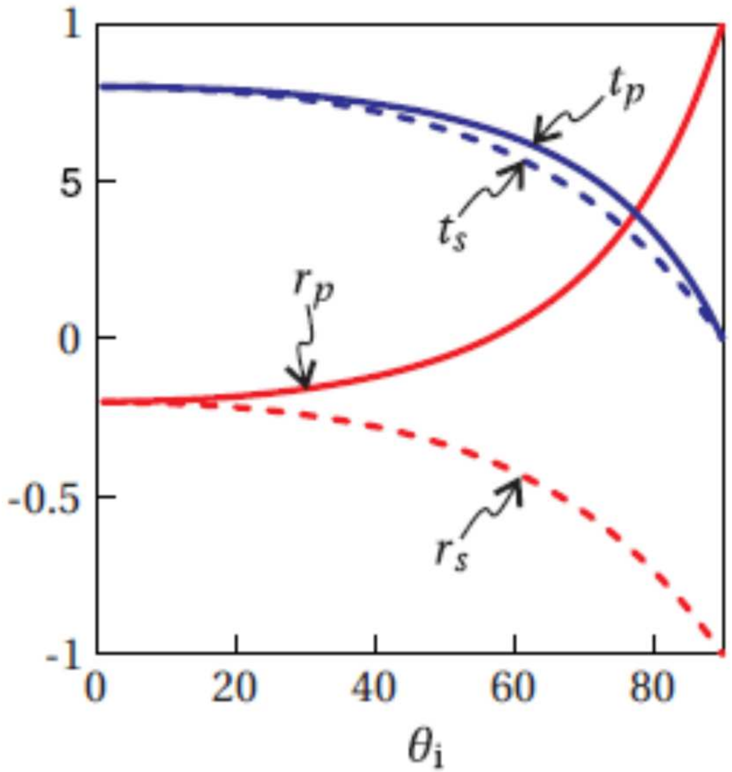
$$t_s \equiv \frac{E_t^{(s)}}{E_i^{(s)}} = \frac{2 n_i \cos\theta_i}{n_i \cos\theta_i + n_t \cos\theta_t} = \frac{2 \sin\theta_t \cos\theta_i}{\sin\theta_t \cos\theta_i + \sin\theta_i \cos\theta_t} = \frac{2 \sin\theta_t \cos\theta_i}{\sin(\theta_t + \theta_i)}$$

$$r_p \equiv \frac{E_r^{(p)}}{E_i^{(p)}} = \frac{n_i \cos\theta_t - n_t \cos\theta_i}{n_i \cos\theta_t + n_t \cos\theta_i} = \frac{\sin\theta_t \cos\theta_t - \sin\theta_i \cos\theta_i}{\sin\theta_t \cos\theta_t + \sin\theta_i \cos\theta_i} = \frac{\tan(\theta_t - \theta_i)}{\tan(\theta_t + \theta_i)}$$

$$t_p \equiv \frac{E_t^{(p)}}{E_i^{(p)}} = \frac{2 n_i \cos\theta_i}{n_i \cos\theta_t + n_t \cos\theta_i} = \frac{2 \sin\theta_t \cos\theta_i}{\sin\theta_t \cos\theta_t + \sin\theta_i \cos\theta_i} = \frac{2 \sin\theta_t \cos\theta_i}{\sin(\theta_t + \theta_i) \cos(\theta_t - \theta_i)}$$

Refração e reflexão na interface

Coeficientes de Fresnel



Coeficientes de Fresnel para interface ar-vidro