1d)

$$\begin{cases} u u_x + x u_y = y \\ u(0,y) = -y \end{cases}$$

$$\begin{cases} \dfrac{dx(s,t)}{ds} = v, & x(0,t) = 0 \\ \dfrac{dy}{ds}(s,t) = x, & y(0,t) = t \\ \dfrac{dv}{ds}(s,t) = y, & v(0,t) = -t \end{cases} \iff \begin{cases} \dfrac{dZ}{ds}(s,t) = A \cdot Z(s,t) \\ Z(0,t) = \begin{pmatrix} 0 \\ t \\ -t \end{pmatrix} \end{cases}$$

onde $Z(s,t) = \begin{pmatrix} x(s,t) \\ y(s,t) \\ v(s,t) \end{pmatrix}$, $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

Autovalores de A: $\lambda_1 = 1$; $\lambda_2 = \frac{1}{2}(-1+i\sqrt{3})$; $\lambda_3 = \frac{1}{2}(-1-i\sqrt{3})$

Autovetores correspondentes:

$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} \frac{1}{2}(-1-i\sqrt{3}) \\ \frac{1}{2}(-1+i\sqrt{3}) \\ 1 \end{pmatrix}$, $v_3 = \begin{pmatrix} \frac{1}{2}(-1+i\sqrt{3}) \\ \frac{1}{2}(-1-i\sqrt{3}) \\ 1 \end{pmatrix}$

$B_1 = Re(v_2) = \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix}$, $B_2 = Im(v_2) = \begin{pmatrix} -\sqrt{3}/2 \\ \sqrt{3}/2 \\ 0 \end{pmatrix}$

$\cancel{Z(s,t) =}$ $z_1(s) = e^{s}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$z_2(s) = \left(B_1 \cos\left(\frac{\sqrt{3}}{2}s\right) - B_2 \operatorname{sen}\left(\frac{\sqrt{3}}{2}s\right)\right) e^{-\frac{1}{2}s} = \begin{pmatrix} -\frac{1}{2}\cos\frac{\sqrt{3}}{2}s + \frac{\sqrt{3}}{2}\operatorname{sen}\frac{\sqrt{3}}{2}s \\ -\frac{1}{2}\cos\frac{\sqrt{3}}{2}s - \frac{\sqrt{3}}{2}\operatorname{sen}\frac{\sqrt{3}}{2}s \\ \cos\frac{\sqrt{3}}{2}s \end{pmatrix} e^{-s/2}$$

$$z_3(s) = \left(B_2 \cos\left(\frac{\sqrt{3}}{2}s\right) + B_1 \operatorname{sen}\left(\frac{\sqrt{3}}{2}s\right)\right) e^{-\frac{1}{2}s} = \begin{pmatrix} -\frac{\sqrt{3}}{2}\cos\frac{\sqrt{3}}{2}s - \frac{1}{2}\operatorname{sen}\frac{\sqrt{3}}{2}s \\ \frac{\sqrt{3}}{2}\cos\frac{\sqrt{3}s}{2} - \frac{1}{2}\operatorname{sen}\frac{\sqrt{3}}{2}s \\ \operatorname{sen}\frac{\sqrt{3}}{2}s \end{pmatrix} e$$

$$z(s,t) = c_1 z_1(s) + c_2 z_2(s) + c_3 z_3(s)$$

$$\begin{pmatrix} 0 \\ t \\ -t \end{pmatrix} = z(0,t) = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} -\sqrt{3}/2 \\ \sqrt{3}/2 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} c_1 - \frac{1}{2}c_2 - \frac{\sqrt{3}}{2}c_3 = 0 \\ c_1 - \frac{1}{2}c_2 + \frac{\sqrt{3}}{2}c_3 = t \\ c_1 + c_2 = -t \end{cases}$$

$$\Rightarrow c_1 = 0 \quad c_2 = -t, \quad c_3 = \frac{t}{\sqrt{3}}$$

$$\Rightarrow z(s,t) = -t z_2(s) + \frac{t}{\sqrt{3}} z_3(s) =$$

$$= \begin{pmatrix} -2\sqrt{3}/3 \,\text{sen}\left(\frac{\sqrt{3}}{2}s\right) e^{-s/2} t \\ \left(\cos\frac{\sqrt{3}s}{2} + \frac{\sqrt{3}}{3}\text{sen}\frac{\sqrt{3}s}{2}\right) e^{-s/2} t \\ \left(-\cos\frac{\sqrt{3}s}{2} + \frac{\sqrt{3}}{3}\text{sen}\frac{\sqrt{3}s}{2}\right) t e^{-s/2} \end{pmatrix}$$

$$\Rightarrow \begin{cases} x(s,t) = -\frac{2\sqrt{3}}{3}\,\text{sen}\left(\frac{\sqrt{3}}{2}s\right) e^{-s/2} t & (1) \\ y(s,t) = \cos\left(\frac{\sqrt{3}}{2}s\right) e^{-s/2} t + \frac{\sqrt{3}}{3}\text{sen}\left(\frac{\sqrt{3}}{2}s\right) e^{-s/2} t & (2) \\ v(s,t) = -\cos\left(\frac{\sqrt{3}}{2}s\right) e^{-s/2} t + \frac{\sqrt{3}}{3}\text{sen}\left(\frac{\sqrt{3}}{2}s\right) e^{-s/2} t & (3) \end{cases}$$

De (1) e (2), $\frac{x}{2} + y = \cos\left(\frac{\sqrt{3}}{2}s\right) e^{-s/2} t$ (4)

~~Bes~~ Por (3) e (1) $v = -\cos\left(\frac{\sqrt{3}}{2}s\right) e^{-s/2} t - \frac{x}{2}$ (5)

Fazendo (4)+(5), temos

$$v + \frac{x}{2} + y = -\frac{x}{2} \Rightarrow \boxed{v = -x - y}$$

$$\boxed{u(x,y) = v(s,t) = -x - y.}$$