Solutions and Hints to Selected Problems

The Cartoon Guide to Calculus

Chapter 0.

1. ALL $t \neq \frac{1}{2}$.

2. ALL b EXCEPT $b = 4$ AND $b = -9$.

3. ALL $x \neq \pm 1$

4. THE INTERVAL $[-2, 2]$

5. ALL $\theta \neq \pm(\sqrt{\pi})/3$

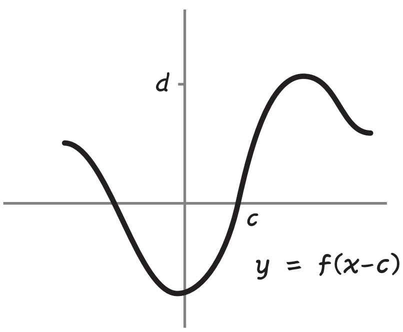
6. ALL $x \neq 0$

7. THE INTERVAL $(-\infty, 0)$

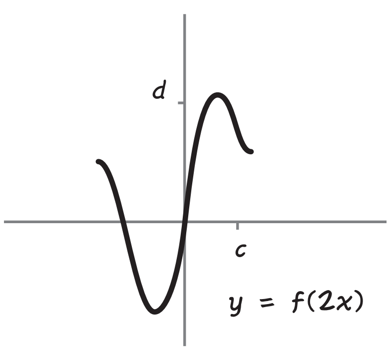
8. ALL REAL NUMBERS

9. THE INTERVAL $(1, \infty)$

10a.



10d.



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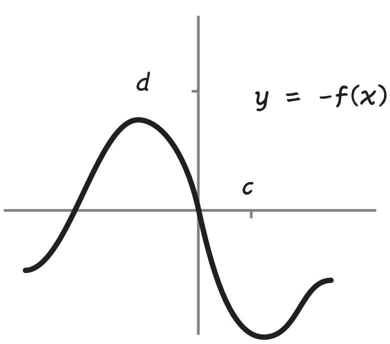


11b. DEEPEST INSIDE: $w(x) = x^2 - 1$;
MIDDLE: $v(w) = \ln w$
OUTSIDE: $u(v) = \sqrt{v}$
$h(x) = u(v(w(x)))$

11c. INSIDE: $g(x) = e^x$;
OUTSIDE: $f(x) = 4t^3 + t^2 + 6t - 99$
$h(x) = f(g(x))$

12. LET $y = x + c$. THEN

$$P(y) = b_0 + b_1(x + c) + b_2(x + c)^2 \ldots + b_n(x + c)^n$$

EXPANDING ALL THE BINOMIALS AND COLLECTING LIKE TERMS RESULTS IN A POLYNOMIAL

$$a_0 + a_1 x + a_2 x^2 \ldots + a_n x^n$$

THIS IS $P(x + c)$, SO

$$P(x) = a_0 + a_1(x - c) + a_2(x - c)^2 \ldots + a_n(x - c)^n$$

NOTE THAT $a_n = b_n$.

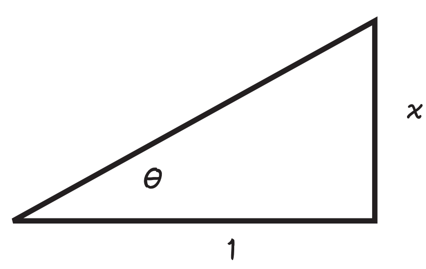
13. SORRY, BUT THIS QUASI-INCOHERENT PROBLEM SHOULD BE IGNORED. TRY IT WITH THIS FUNCTION INSTEAD, WHICH IS WHAT I INTENDED TO WRITE (SIGH...):

$$f(x) = x^2 \quad \text{FOR} \quad 0 < x \leq 1$$

$$f(x) = (x - 1)^2 - 1 \quad \text{FOR} \quad 1 < x < 2$$

THIS FUNCTION IS ONE-TO-ONE BUT NOT INCREASING ON ITS WHOLE DOMAIN.

14.



15. DOUBLING TIME IS $\frac{\ln 2}{r}$ YEARS.

Chapter 1.

1. LIMIT IS 6

2. LIMIT IS $6 + C$

3. LIMIT IS 1/4

4. LIMIT IS $-\infty$

5. LIMIT IS 6

6. LIMIT IS 0, AS THE DENOMINATOR GOES TO ∞.

7. SUBSTITUTING $y = 1/(x - 1)$ MAKES THE EXPRESSION (AFTER MUCH SIMPLIFICATION) EQUAL TO

$$3 + \frac{1}{y}$$

THE LIMIT AS $y \to \infty$ IS 3.

8. LIMIT IS 2

9. LIMIT IS ∞

10. $$x \sin\frac{1}{x} \le |x|$$

$$x \sin\frac{1}{x} \ge -|x|$$

LIMIT AS $x \to 0$ MUST BE 0.

12. TAKE **I** TO BE ANY INTERVAL AROUND L OF RADIUS $< |L|/2$. BY DEFINITION OF THE LIMIT, THERE MUST BE AN INTERVAL **J** AROUND a ON WHICH $|f(x) - L| < |L|/2$. BUT

$$|L| - |f(x)| \le |f(x) - L| \text{ SO}$$

$$|L| - |f(x)| < |L|/2 \text{ FROM WHICH}$$

$$|f(x)| > |L| - |L|/2 = |L|/2$$

13. CHOOSE THE INTERVAL **J** SO THAT IF x IS IN **J**, THEN

$$|f(x)| > |L|/2 \text{ AND } |f(x) - L| < \frac{\varepsilon L^2}{2}$$

Chapter 2.

1. $f'(x) = 3x^2 + 5$

3. $$P'(x) = \frac{1}{\sqrt{x}}(1 + \tfrac{1}{2}\ln x)$$

5. $$h'(x) = -\sin x + \frac{5}{3}x^{-4/3}$$

6. $$R'(x) = \frac{-2}{(x-1)^2}$$

8. $v'(t) = \tan t \sec t$

9. $$F'(x) = \frac{-1}{x(\ln x)^2}$$

11. $$Q'(x) = -529\,\frac{2x^3 - x^2 + 1}{(x^3 - x^2 - x - 1)^2}$$

13a. VELOCITY AT TIME t IS $A'(t) = -9.8t + 30$. PLUG IN $t = 3$ TO GET $A'(3) = (-9.8)(3) + 30 = 0.6$ M/SEC.

13b. HERE $A'(t) = -9.8t + 45$. THE HINT SUGGESTS THAT AT THE TOP OF ITS FLIGHT, THE BALL'S VELOCITY IS ZERO. SET $A'(t) = 0$ AND SOLVE FOR t. YOU SHOULD FIND THAT THE BALL REACHES ITS HIGHEST POINT AT AROUND $t = 4.6$ SECONDS. PLUG THAT INTO $A(t)$ TO FIND THE MAXIMUM HEIGHT. THE TOTAL TIME OF FLIGHT IS 9.2 SECONDS: 4.6 SECONDS GOING UP, AND 4.6 SECONDS COMING DOWN.

14a. **VERY** SORRY! SOLVING THIS DEPENDS ON MATERIAL IN THE NEXT CHAPTER! THE DERIVATIVE OF T IS

$$T'(t) = (250)(0.46)e^{-0.46t} = 115e^{-0.46t}$$

PLUG IN VALUES OF t TO FIND THE RATE OF HEATING. FOR INSTANCE,

$$T'(100) = 115e^{-46}, \text{ A VERY SMALL NUMBER!}$$

14b. THIS CAN BE SOLVED WITHOUT DIFFERENTIATING. THE ANSWER IS

$$t = \frac{\ln 250}{0.46} \approx 12 \text{ MINUTES}$$

15a. **DOUBLE**-SORRY! TAKING THIS DERIVATIVE ALSO REQUIRES THE CHAIN RULE, COVERED IN CHAPTER 3. THE TRAIL'S SLOPE, $A'(x)$, IS

$$A'(x) = 0.3 + (0.3)\sin\left(\frac{x}{20}\right) + \left(\frac{0.3}{20}\right)\cos\left(\frac{x}{20}\right)$$

17b. ASSUME f IS EVEN. THEN

$$f'(-x) = \lim_{h\to 0}\frac{f(-x+h) - f(-x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x-h) - f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h) - f(x)}{-h} = -f'(x)$$

Chapter 3.

1. $h(u) = f(g(u)) = \cos^2 u$, $v(x) = g(f(x)) = \cos(x^2)$

$h'(u) = -2 \sin u \cos u$, $v'(x) = -2x \sin(x^2)$

3a. $f'(t) = \frac{1}{2}(2t + 1)(1 + t + t^2)^{-1/2}$

3b. $g'(x) = -100 \sin x \cos x (\cos^2 x - \sin^2 x)^{24}$

3d. $P'(r) = 20r(r^2 + 7)^9$

3h. $F'(x) = \frac{1}{2} e^{\frac{x-a}{2}}$

3i. ERROR! SHOULD BE $u(t)$!

$u'(t) = 6t^3(t^4 + 7)^{1/2}$

3k. $R'(x) = -\dfrac{10(t + 1)^4}{(t - 1)^6}$

5a. $\ln f(x) = 5 \ln x + x - \frac{1}{3} \ln(1 + x)$

$$\frac{f'(x)}{f(x)} = \frac{5}{x} + 1 - \frac{1}{1 + x}$$

$$f'(x) = \left(\frac{5}{x} + 1 - \frac{1}{1 + x}\right) x^5 e^x (1 + x)^{-1/3}$$

$$= \frac{x^2 + 13x + 15}{3x(x + 1)} x^5 e^x (1 + x)^{-1/3}$$

5b. $g'(x) = \dfrac{1}{\sqrt{x}} (\frac{1}{2} \ln x + 1) x^{\sqrt{x}}$

6a. $f^{-1}(y) = \arcsin(y - 2)$

6c. $f^{-1}(y) = 1 \pm \sqrt{y}$

7. $g(f(x)) = 1/f(x)$

9a. FLEA b. MOUSE, NOT FLEA c. NEITHER
d. MOUSE, NOT FLEA e. MOUSE, NOT FLEA
f. NEITHER g. MOUSE, NOT FLEA h. FLEA

Chapter 4.

1. $h' = \dfrac{\pi(R - h)^2 - V'}{\pi R^2}$

3. START WITH THE BASIC CIRCLE RELATIONS:

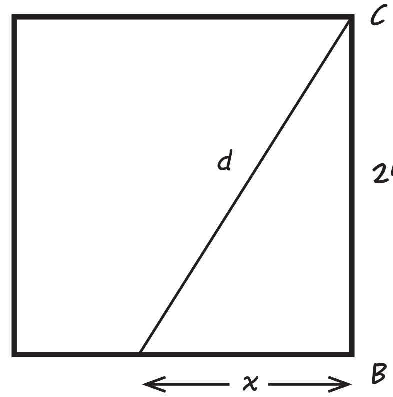
$C = 2\pi r$, $A = \pi r^2$

FROM THESE,

$C^2 = 4\pi A$, SO $A' = CC'/2\pi$

4. WHEN $y = 12$, $y' = 3/4$ METERS PER SECOND.

5.



LET x BE THE SNAIL'S DISTANCE FROM CORNER B. THE PROBLEM STATES THAT $x'(t) = -1$ (NEGATIVE BECAUSE THE DISTANCE IS GETTING SMALLER). ALSO:

$d^2 = x^2 + 25^2$ SO

$dd' = xx'$

WHEN $x = 15$, THEN,

$d = \sqrt{225 + 625} = \sqrt{950}$, AND

$d' = -\dfrac{15}{\sqrt{950}} \approx 0.49$ CM/SEC.

Chapter 5.

1a. $x = -\frac{1}{2}$ is a minimum.

1c. $h'(t) = 6t^2 - 6t - 36$, which can be factored:

$$h'(t) = 6(t - 3)(t + 2)$$

$t = 3$ is a minimum, $t = -2$ a maximum.

1e. $F'(\theta) = \cos\theta - \sin\theta$. This is zero when $\cos\theta = \sin\theta$, i.e., when the angle θ is either $\pi/4$, $(\pi/4) + 2\pi$, $(\pi/4) + 4\pi$, etc., where $\sin\theta = \cos\theta = \frac{1}{2}\sqrt{2}$, or $5\pi/4$, $(5\pi/4) + 2\pi$, etc., where $\sin\theta = \cos\theta = -\frac{1}{2}\sqrt{2}$.

$$F'' = -F, \text{ so}$$

$$F''(\pi/4) = -\sqrt{2}, \quad F''(5\pi/4) = \sqrt{2}$$

So the maxima are the points

$$\theta = (\pi/4) \pm 2\pi n, \quad n = 0, 1, 2, \ldots$$

And the minima are the points

$$\theta = (5\pi/4) \pm 2\pi n, \quad n = 0, 1, 2, \ldots$$

1f. Try implicit differentiation for this one!

$$A^2 = 4 - x^2, \text{ so } AA' = -x, \quad A' = -\frac{x}{A}$$

This can be zero only when $x = 0$, and it is fairly obvious that this must be a maximum.

1h. Same maxima and minima as problem 1e.

3. If one side of the rectangle is x, and the perimeter is P, then the adjacent side is $(P/2) - x$, and the area, as a function of x, is

$$A(x) = x\left(\frac{P}{2} - x\right)$$

This has a maximum when $x = P/4$.

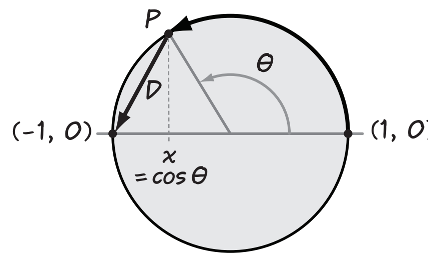
4a. $T = \dfrac{v_0 \sin\theta}{9.8}$

4b. $D'(\theta) = \dfrac{v_0^{\,2}}{4.9}(\cos^2\theta - \sin^2\theta)$

This is zero when $\cos\theta = \pm\sin\theta$, i.e., when

$\theta = \pi/4$, $3\pi/4$, etc., in other words, when the catapult is aimed upward at half a right angle. Note that this does not depend on v_0!!

5. You were asked for the lowest-cost route across a pond from (1, 0) to (−1, 0), when construction on land is cheaper than building across water.



The problem is something of a trick question, as we'll see. There are two basic ways to solve it, and the difference between them illustrates an important point about how to approach math problems.

First, let's take the direct approach, using the formulas and techniques laid out in recent chapters. Warning: hairy algebra ahead!

The cost per unit length over land was given as \$4, and over water as \$5. To make the algebra a little simpler, let r be the ratio of land cost per unit to water cost per unit: in this case $r = 4/5$. We can always assume $r < 1$. Otherwise, just build straight across the water!

The problem can be set up with either x or θ as the variable. Let's use the angle, as was suggested in the book.

We can write the cost as

$$C(\theta) = r\theta + D$$

because D is the length of road over water, and θ is the length of the arc. (Actually, C is only 1/5 of the cost, but minimizing this will minimize the cost also, won't it?)

By the Pythagorean theorem,

$$(1) \quad D^2 = 2 + 2\cos\theta$$

We seek critical points of the cost, i.e., values of θ such that $C'(\theta) = 0$. Now

$$(2) \quad C'(\theta) = r + D'(\theta)$$

Find D' by implicit differentiation of equation (1).

$$(3) \quad D' = \frac{-\sin\theta}{D}$$

AT A CRITICAL POINT, THEN, WHERE $C' = 0$,

$$(4)\quad D'(\theta) = -r \quad \text{OR} \quad \frac{-\sin\theta}{D} = -r \quad \text{OR}$$

$$(5)\quad \sin\theta = rD$$

NOW WE SOLVE FOR θ. SQUARING (5) GIVES

$$\sin^2\theta = r^2D^2$$

NOW SUBSTITUTE FOR D^2 FROM (1):

$$\sin^2\theta = r^2(2 + 2\cos\theta) \quad \text{OR}$$

$$1 - \cos^2\theta = r^2(2 + 2\cos\theta) \quad \text{OR}$$

$$\cos^2\theta + 2r^2\cos\theta + (2r^2 - 1) = 0$$

THIS IS A QUADRATIC EQUATION IN $\cos\theta$. APPLYING THE QUADRATIC FORMULA GIVES

$$\cos\theta = \tfrac{1}{2}(-2r^2 \pm \sqrt{4r^4 - 4(2r^2 - 1)}\,),$$

WHICH, TO MY IMMENSE RELIEF, SIMPLIFIES TO

$$= -r^2 \pm \sqrt{(r^2 - 1)^2}$$

$$= -r^2 \pm (r^2 - 1)$$

THE PLUS SIGN GIVES THE BORING SOLUTION $\cos\theta = -1$, $D = 0$, WHICH CORRESPONDS TO GOING ALL THE WAY AROUND BY LAND. (IN FACT, D' ISN'T DEFINED WHEN $D = 0$. SEE WHY?) LET'S LOOK AT THE SOLUTION WITH THE MINUS SIGN. IN THAT CASE, AT THE CRITICAL POINT,

$$(6)\quad \cos\theta = 1 - 2r^2$$

IS IT A MAXIMUM OR A MINIMUM? LET'S TRY THE SECOND DERIVATIVE TEST. r IS A CONSTANT, SO THE SECOND DERIVATIVE IS (APPARENTLY) SIMPLE. FROM (2),

$$C''(\theta) = D''(\theta)$$

AND D'' COMES FROM (3):

$$D'' = \frac{-D\cos\theta + D'\sin\theta}{D^2} \quad \text{OR}$$

$$(8)\quad D^2D'' = -D\cos\theta + D'\sin\theta$$

(D^2 IS POSITIVE, SO THIS HAS THE SAME SIGN AS D''.)

LUCKILY, BY NOW WE CAN FIND ALL THOSE NUMBERS. AT OUR CRITICAL POINT θ, FROM (4) AND (6),

$$(9)\quad D' = -r \quad \text{AND} \quad \cos\theta = 1 - 2r^2$$

WE FIND D BY SUBSTITUTING $1 - 2r^2$ FOR $\cos\theta$ IN (1):

$$D^2 = 2 + 2(1 - 2r^2)$$

AND $\sin\theta$ FROM THE USUAL TRIG IDENTITY:

$$\sin^2\theta = 1 - (1 - 2r^2)^2$$

WORKING THESE OUT, YOU SHOULD FIND

$$(10)\quad D = 2(1 - r^2)^{\frac{1}{2}} \quad \sin\theta = 2r(1 - r^2)^{\frac{1}{2}}$$

AT THE CRITICAL POINT, THEN, WE CAN PLUG THESE VALUES INTO (8), AND AFTER AN ANNOYING AMOUNT OF ALGEBRA IN WHICH WE MUST BE VERY CAREFUL TO KEEP TRACK OF OUR MINUS SIGNS, WE FIND

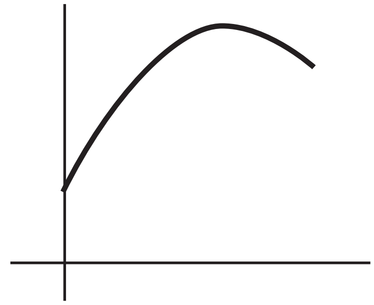
$$(11)\quad D^2D'' = -2(1 - r^2)^{3/2}$$

NOTE A COUPLE OF THINGS HERE: FIRST, IT'S O.K. TO TAKE THE SQUARE ROOT, BECAUSE $r < 1$. SECOND, $(1 - r^2)^{3/2}$ IS POSITIVE. THAT IS, EQUATION 11 SAYS THAT THE SECOND DERIVATIVE IS **NEGATIVE.** THIS CRITICAL POINT IS NOT A MINIMUM AT ALL: IT'S THE POINT WHERE THE COST OF BUILDING THE ROAD IS A LOCAL **MAXIMUM!!!**

THE OPTIMAL ROAD WILL **ALWAYS** BE EITHER STRAIGHT ACROSS OR THE LONG WAY AROUND, WHICHEVER IS CHEAPER. WHEN $r = 4/5$, AS GIVEN HERE, IT'S CHEAPEST TO GO STRAIGHT ACROSS, AT A COST OF $(5)(2) = 10$, RATHER THAN THE LONG WAY, WHICH WOULD COST $4\pi \approx 12.57$.

SO... WAS ALL THIS MATH A WASTE OF TIME? MAYBE, MAYBE NOT! WE CERTAINLY GOT A CALCULUS WORKOUT IN THE PROCESS, AND LEARNED SOMETHING ABOUT CHORDS IN A CIRCLE...

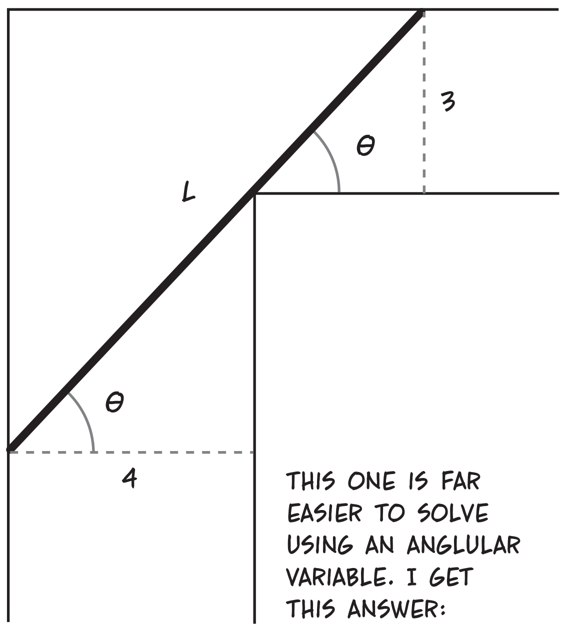
ON THE OTHER HAND, WE COULD HAVE STARTED BY MAKING A CRUDE GRAPH (OR USING A GRAPHING CALCULATOR). THEN WE'D HAVE SEEN THAT $C(\theta) = (4/5)\theta + (2 + 2\cos\theta)^{\frac{1}{2}}$ HAS ROUGHLY THIS SHAPE:



AND WE'D HAVE KNOWN AHEAD OF TIME THAT THERE WAS NO POINT IN LOOKING FOR A CHEAPER PATH GOING PARTWAY AROUND THE CIRCLE. THE LESSON IS: GET TO KNOW YOUR FUNCTION BEFORE YOU ATTACK IT!

Chapter 5 (cont'd).

6.



THIS ONE IS FAR EASIER TO SOLVE USING AN ANGLULAR VARIABLE. I GET THIS ANSWER:

$$\theta = \arctan \sqrt[3]{(3/4)} \approx 0.7375 \text{ RADIANS}$$

$$L = (4/\cos\theta) + (3/\sin\theta) \approx 9.86 \text{ METERS}$$

Chapter 6.

2. $\sqrt{67} \approx \sqrt{64} + \dfrac{1}{2\sqrt{64}}(67 - 64) = 8\frac{3}{16}$

4. $\arctan(1.1) \approx \arctan 1 + \dfrac{1}{1 + 1^2}(0.1)$

$$= \frac{\pi}{4} + \frac{1}{20}$$

5. LIMIT IS $\lim_{x \to 0} \dfrac{2x}{-\sin x} \cos(x)^2 = -2.$

7. LIMIT IS 4.

9. LIMIT IS 0.

12. L'HÔPITAL'S RULE DOES NOT APPLY.

13c. $P(x) = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \dfrac{x^8}{8!}$

Chapter 7.

1. $\dfrac{f(2) - f(0)}{2 - 0} = \dfrac{15 - 3}{2} = 6$

$$f'(x) = 3x^2 + 2$$

SET $f'(x) = 6$ AND SOLVE FOR x. ANSWER $= 2/\sqrt{3}$

2. $c = \ln(e^{-3} - e) - \ln 4$

3. YOU SHOULD FIND THIS EQUATION TO SOLVE FOR x:

$$\frac{8}{4 - x^2} = \frac{3}{2} \quad \text{OR}$$

$$(4 - x^2) = \frac{16}{3}$$

THIS HAS TWO SOLUTIONS, BUT ONLY ONE OF THEM IS ON THE INTERVAL [0, 2]. THE ANSWER IS $c = 4 - (4/\sqrt{3})$.

5. NOTE THAT THE FUNCTION IS EVEN.

6. $c = \arccos(\pm\sqrt{(a/\tan a)})$

7. THE FUNCTION IS INCREASING, AND THEREFORE CAN CROSS THE x-AXIS AT MOST ONCE. IT DOES, IN FACT, CROSS ONCE, AS YOU CAN SEE BY CONSIDERING THE FUNCTION'S VALUES WHEN x IS VERY SMALL AND VERY LARGE.

8a. THE DERIVATIVE $P'(x)$ IS ZERO AT ONLY ONE POINT, SO THERE CAN BE AT MOST TWO POINTS a AND b WITH $P(a) = P(b) = 0$.

8b AND 8c FOLLOW BY BOOTSTRAPPING ONE DEGREE AT A TIME.

10. $f(b) \leq 7(b - a) + 2$

11. THE FUNCTION IS NOT CONTINUOUS AT $x = 2$.

12. APPLY THE MEAN VALUE THEOREM TO THE FUNCTION $f - g$.

Chapter 8.

1. $E_{LOW} = 0 + 3(1)^2 + 3(2)^2 + 3(3)^2 = 42$

2. $E_{HIGH} = 3(1)^2 + 3(2)^2 + 3(3)^2 + 3(4)^2 = 90$

3. $\frac{1}{2}(E_{HIGH} + E_{LOW}) = 66$

4. $3((\frac{1}{2})^2 + (\frac{3}{2})^2 + (\frac{5}{2})^2 + (\frac{7}{2})^2) = 63$

5. $s(t) = t^3$, AND $s(4) - s(0) = 64$

6. USING HEIGHTS AT THE MIDPOINTS OF THE INTERVALS:

$$2\left(\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \frac{e^2 - 7}{e^2 + 7}\right) \approx 1.964$$

Chapter 9.

2. $\frac{2}{15}x^5 + C$

4. $\frac{-1}{1-x} + C$

6. $\ln(9 + x^2) + C$

7. $\frac{1}{2}\arcsin(\frac{x}{2}) + C$

9. $\sin^2 x + C$

12. NOTE THAT $3x^2 = \frac{d}{dx}(x^3 + 1)$.
THE ANTIDERIVATIVE IS $\frac{1}{2}e^{(x^3+1)} + C$.

15. $\ln|x + 1| + C$

16. DO A PARTIAL FRACTION DECOMPOSITION TO GET

$\frac{1}{2}\ln|x - 1| - \frac{1}{2}\ln|x + 1| + C$

17. TRIVIAL

19. $\frac{1}{3}\sin^3\theta - \frac{1}{3}\cos^3\theta + C$

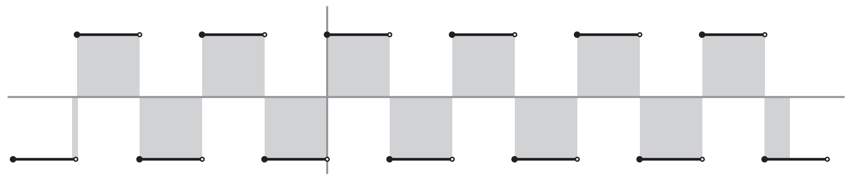
21. $\ln|t^3 - t^2 + 1| + C$

23. $-x^2 + C$ WHEN $x \le 0$
$x^2 + C$ WHEN $x \ge 0$

25. $\ln|f(x)| + C$

Chapter 10.

1. HERE IS THE GRAPH $y = g(x)$ WITH SHADED REGIONS SHOWING THE DEFINITE INTEGRAL. SQUARES ABOVE THE x-AXIS CANCEL THOSE BELOW THE x-AXIS. THEREFORE, THE INTEGRAL IS EQUAL TO THE AREA OF THE ONE EXCESS SQUARE ABOVE THE AXIS MINUS THE TWO SLIVERS AT EITHER END.



ANSWER IS $1 - 0.086 - 0.358 = 0.556$

2. THERE'S AN ERROR IN THE BOOK. IN THE SUMMATION, $1/n$ SHOULD BE T/n. THEN S_n BECOMES

$$\sum_{i=1}^{n}\left(\frac{iT}{n}\right)^2\left(\frac{T}{n}\right) = \frac{T^3}{n^3}\sum_{i=1}^{n} i^3 = T^3\left(\frac{2n^3 + \text{LOWER ORDER TERMS}}{6n^3}\right)$$

AND THE LIMIT AS $n \to \infty$ IS $\frac{1}{3}T^3$.

4. ON ANY SUBINTERVAL CONTAINING $x = 2$, THE FUNCTION IS UNBOUNDED, I.E., ITS VALUES $\to \infty$ AS $x \to 2$, SO THERE CAN BE NO MAXIMUM VALUE ON THAT INTERVAL.

Chapter 11.

1. $120 + 18 + 138$

3. $\frac{1}{51}(2^{51} - 1)$

5. $\frac{(-1)^n}{n+1}$

6. $\frac{1}{2}(\arcsin 1 - \arcsin(\frac{\sqrt{2}}{2}))$

$= \frac{1}{2}(\pi/2 - \pi/4) = \pi/8$

8. -1

10. $\frac{1}{2}e^9$

12. THIS DEPENDS ON THE FACT THAT $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$. SINCE $M(b - a)$ IS AN UPPER SUM OF $|f(x)|$ ON THE INTERVAL, IT FOLLOWS THAT

$$\int_a^b |f(x)|\,dx \le M(b - a)$$

AND THE OTHER INEQUALITY FOLLOWS.

14. $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots\ldots\ldots$

ISN'T THAT A BEAUTY? (NOT TO MENTION ALMOST UNBELIEVABLE.)

Chapter 12.

2. $\frac{1}{2}(1 + x^2)^{-1} + C$ BY SUBSTITUTION

4. $-\ln|\cos u| + C$ OR $\ln|\sec u| + C$

5. SUBSTITUTE $y = 3x - 1$

6. SUBSTITUTE $x = \sin\theta$. THEN $dx = \cos\theta\, d\theta$ AND

$$\int \sqrt{1 - x^2}\, dx = \int \cos^2\theta\, d\theta$$

WHICH WAS EVALUATED IN THE BOOK (OR RATHER, $\int \sin^2 x\, dx$ WAS, AND $\cos^2 x = 1 - \sin^2 x$).

7. SUBSTITUTE $y = 2x + 5$.

8. INTEGRATE BY PARTS TWICE TO GET

$$\frac{e^x}{2}(\sin x - \cos x) + C$$

10. INTEGRATE BY PARTS TWICE TO GET

$$(x(\ln x)^2 - 2x\ln x + 2x)\Big|_1^5$$

$$= (5\ln 5)^2 - 10\ln 5 + 8$$

$$= 56.662880725164843...$$

12. $x\arctan x - \frac{1}{2}\ln(1 + x^2) + C$

Chapter 13.

1. NOTE: STRICTLY SPEAKING, THE VOLUME, BEING "SOUTH OF THE EQUATOR," SHOULD BE NEGATIVE... BUT WE DON'T LIKE THAT, SO WE COMPENSATE BY INTEGRATING IN THE NEGATIVE DIRECTION, FROM ZERO DOWNWARD, TO PRODUCE A POSITIVE RESULT.

BY THE PYTHAGOREAN THEOREM, THE RADIUS OF A SLICE AT HEIGHT y IS $\sqrt{R^2 - y^2}$, SO THE SLICE'S AREA IS πy^2 OR

$$\pi(R^2 - y^2)$$

WHICH PRODUCES THE INTEGRAL GIVEN IN THE PROBLEM. IT WORKS OUT TO

$$V = \pi R D^2 - \tfrac{1}{3}\pi D^3$$

THE VOLUME OF THE HEMISPHERE ABOVE THE WATER LEVEL.

YOU CAN ALSO WORK OUT THE VOLUME OF WATER DIRECTLY. PUT THE HEMISPHERE IN POSITIVE TERRITORY, WITH ITS BASE RESTING ON THE ORIGIN. AT HEIGHT y, A SLICE OF WATER HAS RADIUS $\sqrt{R^2 - (R - y)^2}$ SO THE WATER UP TO HEIGHT h HAS THIS VOLUME:

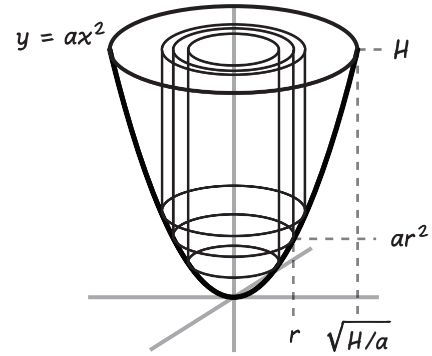
$$\pi \int_0^h R^2 - (R - y)^2\, dy$$

WHICH GIVES

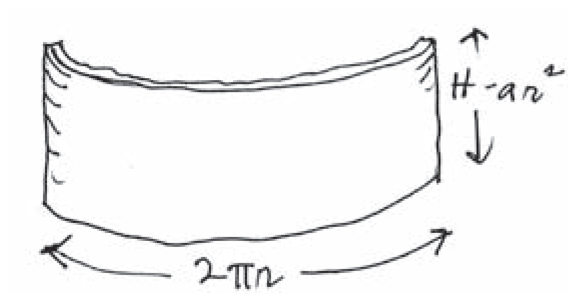
$$\pi(R^2 h - \tfrac{1}{3}(R^3 - (R - h)^3)))$$

2. THE INTEGRAL EQUALS -1.

3. HERE THE RADIUS OF A CYLINDER IS r, AND ITS HEIGHT IS $H - ar^2$. AS IN THE GLUE BLAST EXAMPLE, WE TREAT A THIN CYLINDRICAL SHELL AS A ROLLED-UP RECTANGLE, SO ITS VOLUME IS



$$2\pi r(H - ar^2)\, dr$$



INTEGRATING GIVES THE TOTAL VOLUME:

$$V = 2\pi \int_0^{\sqrt{H/a}} rH - ar^3\, dr = \pi\left(\frac{H^2}{a} - \frac{1}{2}\frac{H^2}{a}\right) = \frac{\pi H^2}{2a}$$

4. $\int_1^\infty \frac{\pi}{x^2}\, dx = \pi$

5. MEASURING FROM THE BOTTOM UP, A HORIZONTAL LINE ACROSS THE DAM AT HEIGHT t HAS LENGTH

$$L(t) = 200 + \frac{t}{2} \text{ METERS}$$

AND THE TOTAL FORCE IS GIVEN BY THE INTEGRAL

$$F = \int_0^{175} (9.8)(200t) + 4.9t^2\, dt$$