2.1 INTRODUCTION

Fluid statics: *The study of fluids in which there is no relative motion between fluid particles.*

KEY CONCEPT *The only stress that exists where there is no motion is a normal stress, the pressure.*

Fluid statics is the study of fluids in which there is no relative motion between fluid particles. If there is no relative motion, no shearing stresses exist, since velocity gradients, such as du/dy, are required for shearing stresses to be present. The only stress that exists is a normal stress, the pressure, so it is the pressure that is of primary interest in fluid statics.

Three situations, depicted in Fig. 2.1, involving fluid statics will be investigated. These include fluids at rest, such as water pushing against a dam, fluids contained in devices that undergo linear acceleration, and fluids contained in rotating cylinders. In each of these three situations the fluid is in static equilibrium with respect to a reference frame attached to the boundary surrounding the fluid. In addition to the examples shown for fluids at rest, we consider instruments called manometers and investigate the forces of buoyancy. Finally, the stability of floating bodies such as ships will also be presented.

2.2 PRESSURE AT A POINT

We have defined pressure as being the infinitesimal normal compressive force divided by the infinitesimal area over which it acts. This defines the pressure at a point. One might question whether the pressure at a given point varies as the normal to the area changes direction. To show that this is not the case, even for fluids in motion with no shear, consider the wedge-shaped element of unit depth (in the z-direction) shown in Fig. 2.2. Assume that a pressure p acts on the hypotenuse and that a different pressure acts on each of the other areas, as shown. Since the forces on the two end faces are in the z-direction, we have not included them on the element. Now, let us apply Newton's second law to the element, for both the x- and y-directions:

$$\begin{aligned} \Sigma F_x = ma_x: &\qquad p_x \Delta y - p\Delta s \sin\theta = \rho \frac{\Delta x \Delta y}{2} a_x \\ \Sigma F_y = ma_y: &\qquad p_y\, \Delta x - \rho g \frac{\Delta x \Delta y}{2} - p\Delta s \cos\theta = \rho \frac{\Delta x \Delta y}{2} a_y \end{aligned} \tag{2.2.1}$$

where we have used $\Delta V = \Delta x\ \Delta y/2$ (we could include Δz in each term to account

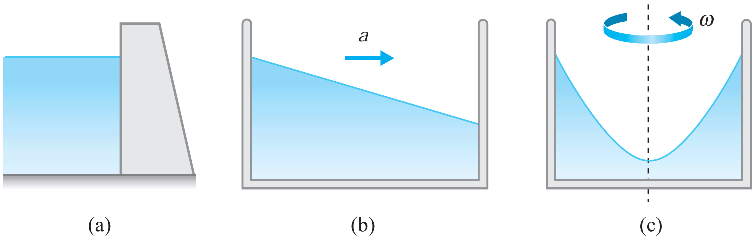

Fig. 2.1 Examples included in fluid statics: (a) liquids at rest; (b) linear acceleration; (c) angular rotation.

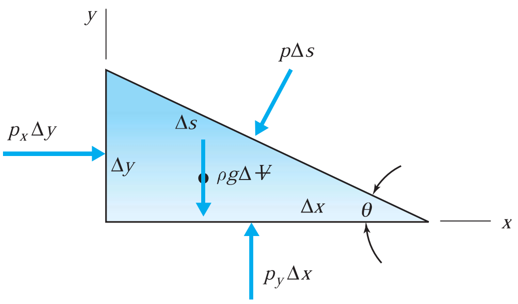

Fig. 2.2 Pressure at a point in a fluid.

for the depth). The pressures shown are due to the surrounding fluid and are the average pressure on the areas. Substituting

$$\Delta s \sin \theta = \Delta y \qquad \Delta s \cos \theta = \Delta x \tag{2.2.2}$$

we see that Eqs. 2.2.1 take the forms

$$\begin{aligned} p_x - p &= \frac{\rho a_x}{2} \Delta x \\ p_y - p &= \frac{\rho(a_y + g)}{2} \Delta y \end{aligned} \tag{2.2.3}$$

Note that in the limit as the element shrinks to a point, $\Delta x \to 0$ and $\Delta y \to 0$. Hence the right-hand sides in the equations above go to zero, even for fluids in motion, providing us with the result that, at a point,

$$p_x = p_y = p \tag{2.2.4}$$

Since θ is arbitrary, this relationship holds for all angles at a point. We could have analyzed an element in the xz-plane and concluded that $p_x = p_z = p$. Thus we conclude that the pressure in a fluid is constant at a point; that is, pressure is a scalar function. It acts equally in all directions at a given point for both a static fluid and a fluid that is in motion in the absence of shear stress.

KEY CONCEPT *Pressure in a fluid acts equally in all directions at a given point.*

2.3 PRESSURE VARIATION

A general equation is derived to predict the pressure variation of fluids at rest or fluids undergoing an acceleration while the relative position of fluid elements to one another remains the same (this eliminates shear stress). To determine the pressure variation in such fluids, consider the infinitesimal element displayed in Fig. 2.3, where the z-axis is in the vertical direction. The pressure variation from one point to another will be determined by applying Newton's second law; that is, the sum of the forces acting on the fluid element is equal to the mass times the acceleration of the element.

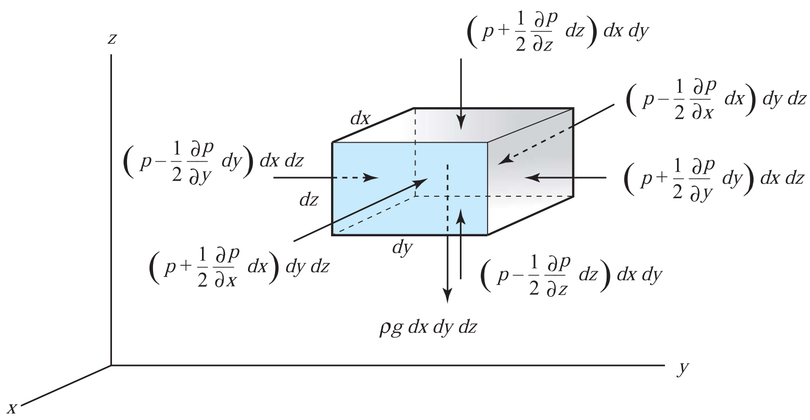

Fig. 2.3 Forces acting on an infinitesimal element that is at rest in the *xyz*-reference frame. The reference frame may be accelerating or rotating.

If we assume that a pressure p exists at the center of this element, the pressures at each of the sides can be expressed by using the chain rule from calculus with $p(x, y, z)$:

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz \tag{2.3.1}$$

If we move from the center to a face a distance $(dx/2)$ away, we see that the pressure is

$$p\left(x + \frac{dx}{2}, y, z\right) = p(x, y, z) + \frac{\partial p}{\partial x}\frac{dx}{2} \tag{2.3.2}$$

The pressures at all faces are expressed in this manner, as shown in Fig. 2.3. Newton's second law is written in vector form for a constant-mass system as

$$\Sigma\, \mathbf{F} = m\mathbf{a} \tag{2.3.3}$$

This results in the three component equations, assuming z to be vertical and using the mass as $\rho\, dx\, dy\, dz$,

$$\begin{aligned}
-\frac{\partial p}{\partial x}dx\,dy\,dz &= \rho a_x\,dx\,dy\,dz\\
-\frac{\partial p}{\partial y}dx\,dy\,dz &= \rho a_y\,dx\,dy\,dz\\
-\frac{\partial p}{\partial z}dx\,dy\,dz &= \rho(a_z + g)\,dx\,dy\,dz
\end{aligned} \tag{2.3.4}$$

where a_x, a_y, and a_z are the components of the acceleration of the element. Division by the element's volume $dx\,dy\,dz$ yields

$$\frac{\partial p}{\partial x} = -\rho a_x$$
$$\frac{\partial p}{\partial y} = -\rho a_y \tag{2.3.5}$$
$$\frac{\partial p}{\partial z} = -\rho(a_z + g)$$

The pressure differential in any direction can now be determined from Eq. 2.3.1 as

$$dp = -\rho a_x dx - \rho a_y dy - \rho(a_z + g)dz \tag{2.3.6}$$

where z is always vertical. Pressure differences between specified points can be found by integrating Eq. 2.3.6. This equation is useful in a variety of problems, as will be demonstrated in the remaining sections of this chapter.

2.4 FLUIDS AT REST

A fluid at rest does not undergo any acceleration. Therefore, set $a_x = a_y = a_z = 0$ and Eq. 2.3.6 reduces to

$$dp = -\rho g\,dz \tag{2.4.1}$$

or

$$\frac{dp}{dz} = -\gamma \tag{2.4.2}$$

This equation implies that there is no pressure variation in the x- and y-directions, that is, in the horizontal plane. The pressure varies in the z-direction, the vertical direction, only. Also note that dp is negative if dz is positive; that is, the pressure decreases as we move up and increases as we move down.

2.4.1 Pressures in Liquids at Rest

If the density can be assumed constant, Eq. 2.4.2 is integrated to yield

$$\Delta p = -\gamma \Delta z \quad \text{or} \quad p + \gamma z = \text{constant} \quad \text{or} \quad \frac{p}{\gamma} + z = \text{constant} \tag{2.4.3}$$

so that pressure increases with depth. Note that z is positive in the upward direction. The quantity $(p/\gamma + z)$ is often referred to as the *piezometric head*. If the point of interest were a distance h below a **free surface** (a surface separating a gas

Free surface: *A surface separating a gas from a liquid.*

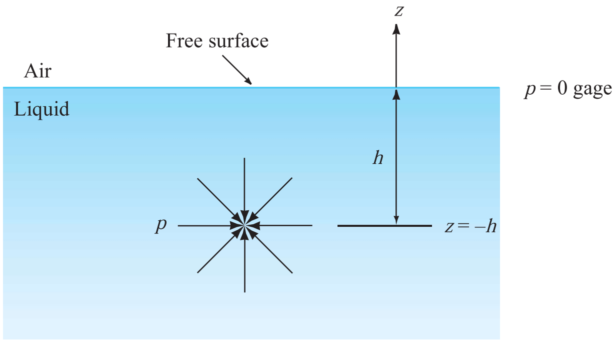

Fig. 2.4 Pressure below a free surface.

from a liquid), as shown in Fig. 2.4, Eq. 2.4.3 would result in

$$p = \gamma h \tag{2.4.4}$$

KEY CONCEPT *The equation $p = \gamma h$ is used to convert pressure to a height of liquid.*

where $p = 0$ at $h = 0$. This equation will be quite useful in converting pressure to an equivalent height of liquid. For example, atmospheric pressure is often expressed as millimeters of mercury; that is, the atmospheric pressure is equal to the pressure at a certain depth in a mercury column, and by knowing the specific weight of mercury, we can then determine that depth using Eq. 2.4.4.

2.4.2 Pressures in the Atmosphere

Standard atmosphere: *Position at 40° latitude where calculations are standardized.*

For the atmosphere where the density depends on height [i.e., $\rho = \rho(z)$], we must integrate Eq. 2.4.1 along a vertical path. The atmosphere is divided into four layers: the *troposphere* (nearest Earth), the *stratosphere*, the *mesosphere*, and the *ionosphere*.[1] Because conditions change with time and latitude in the atmosphere with the layers being thicker at the equator and thinner at the poles, we base calculations on the **standard atmosphere**, which is at 40° latitude. In the standard atmosphere the temperature in the troposphere varies linearly with elevation, $T(z) = T_0 - \alpha z$, where the *lapse rate* $\alpha = 0.0065$ K/m (0.00357°R/ft) and T_0 is 288 K (518°R). In the part of the stratosphere between 11 and 20 km the temperature is constant at −56.5°C. (Commercial aircraft usually fly in the lower part of this constant-temperature region.) The temperature then increases again and reaches a maximum near 50 km; it then decreases to the edge of the ionosphere. The standard atmosphere is sketched in Fig. 2.5. Because the density of the air in the ionosphere is so low, it is possible for satellites to orbit the earth in this layer.

Figure 2.6 shows how atmospheric pressure varies with altitude on three mountains. A column of air from the outer atmosphere to a given point on Earth contains gases that exert a force equal to 14.7 lb on each square inch. This pressure is 1 atm or 760 mm Hg. At a higher altitude the pressure is less because the mass of the column of air from the outer atmosphere to that point is less. Examples of pressure on the three mountains are given on the right of Fig. 2.6.

[1] The ionosphere is composed of the thermosphere and exosphere.

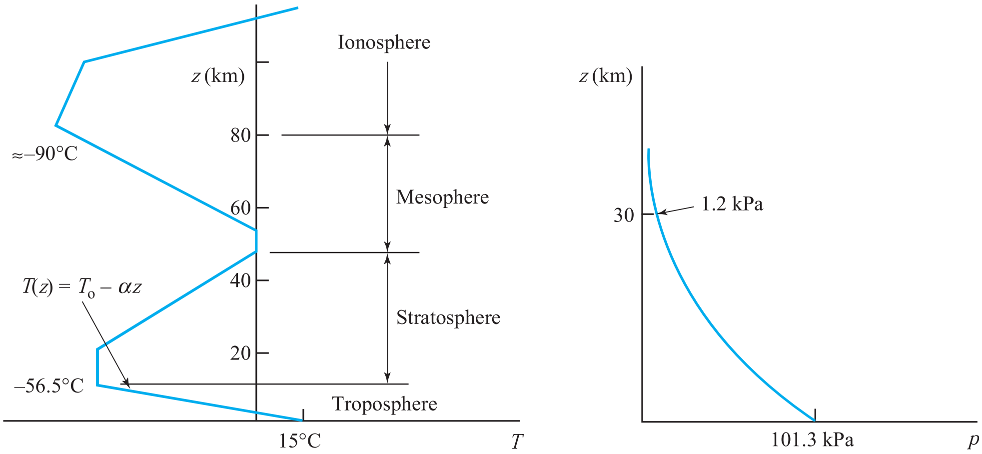

Fig. 2.5 Standard atmosphere.

To determine the pressure variation of the troposphere, we can use the ideal-gas law $p = \rho RT$ and Eq. 2.4.1; there results

$$dp = -\frac{pg}{RT}dz$$

or, collecting the pressure p on the left-hand side,

$$\frac{dp}{p} = -\frac{g}{RT}dz \qquad (2.4.5)$$

This can be integrated, between sea level and an elevation z in the troposphere:

$$\int_{p_{atm}}^{p}\frac{dp}{p} = -\frac{g}{R}\int_{0}^{z}\frac{dz}{T_0 - \alpha z} \qquad (2.4.6)$$

Upon integration this gives

$$\ln\frac{p}{p_{atm}} = \frac{g}{\alpha R}\ln\frac{T_0 - \alpha z}{T_0} \qquad (2.4.7)$$

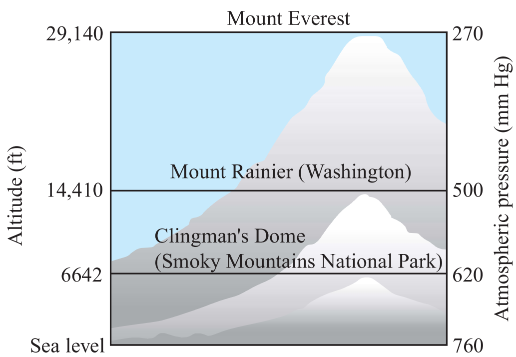

Fig. 2.6 Atmospheric pressure and altitude on three mountains.

which can be put in the form

$$p = p_{atm}\left(\frac{T_0 - \alpha z}{T_0}\right)^{g/\alpha R} \tag{2.4.8}$$

If we use standard atmospheric conditions in Eq. 2.4.8, we find that $p/p_{atm} = 0.999$ at $z = 10$ m. Consequently, we ignore changes in pressure in a gas such as air unless z is relatively large. At $z = 1000$ m, the pressure decreases by about 2%.

In the lower part of the stratosphere, where the temperature is constant, Eq. 2.4.5 is integrated again as follows:

$$\int_{p_s}^{p} \frac{dp}{p} = -\frac{g}{RT_s}\int_{z_s}^{z} dz \tag{2.4.9}$$

$$\ln\frac{p}{p_s} = -\frac{g}{RT_s}(z - z_s) \tag{2.4.10}$$

or

$$p = p_s \exp\left[\frac{g}{RT_s}(z_s - z)\right] \tag{2.4.11}$$

The subscript s denotes conditions at the troposphere–stratosphere interface. Properties of the standard atmosphere up to 80 km are listed in Appendix B.3.

Example 2.1

The atmospheric pressure is given as 680 mm Hg at a mountain location. Convert this to kilopascals and meters of water. Also, calculate the pressure decrease due to 500-m elevation increase, starting at 2000 m elevation, assuming constant density.

Solution

Use Eq. 2.4.4 and find, using $S_{Hg} = 13.6$ with Eq. 1.5.2,

$$\begin{aligned} p &= \gamma_{Hg}h \\ &= (9.81\ \text{kN/m}^3 \times 13.6) \times 0.680\ \text{m} = 90.7\ \text{kPa} \end{aligned}$$

To convert this to meters of water, we have

$$\begin{aligned} h &= \frac{p}{\gamma_{H_2O}} \\ &= \frac{90.7}{9.810} = 9.25\ \text{m of water} \end{aligned}$$

To find the pressure decrease, we use Eq. 2.4.3 and find the density in Table B.3:

$$\begin{aligned} \Delta p &= -\gamma\Delta z = -\rho g\Delta z \\ &= -1.007\ \text{kg/m}^3 \times 9.81\ \text{m/s}^2 \times 500\ \text{m} = -4940\ \text{Pa} \end{aligned}$$

where we used kg = N·s^2/m.

Note: Since gravity is known to three significant digits, we express the answer to three significant digits.

Example 2.2

Assume an isothermal atmosphere and approximate the pressure at 10 000 m. Calculate the percent error when compared with the values using Eq. 2.4.8 and from Appendix B.3. Use a temperature of 256 K, the temperature at 5000 m.

Solution

Integrate Eq. 2.4.5 assuming that T is constant, as follows:

$$\int_{101}^{p} \frac{dp}{p} = -\frac{g}{RT}\int_{0}^{z} dz$$

$$\ln \frac{p}{101} = -\frac{gz}{RT} \qquad \text{or} \qquad p = 101e^{-gz/RT}$$

Substituting $z = 10\,000$ m and $T = 256$ K, there results

$$p = 101e^{-9.81\times 10\,000/(287\times 256)}$$

$$= 26.57 \text{ kPa}$$

Using Eq. 2.4.8 we have

$$p = p_{\text{atm}}\left(\frac{T_0 - \alpha z}{T_0}\right)^{g/\alpha R}$$

$$= 101\left(\frac{288 - 0.0065 \times 10\,000}{288}\right)^{9.81/0.0065\times 287} = 26.3 \text{ kPa}$$

The actual pressure at 10 000 m is found from Table B.3 to be 26.50 kPa. Hence the percent errors are

$$\%\ \text{error} = \left(\frac{26.57 - 26.3}{26.3}\right) \times 100 = 1.03\%$$

$$\%\ \text{error} = \left(\frac{26.57 - 26.50}{26.50}\right) \times 100 = 0.26\%$$

Because the error is so small, we often assume the atmosphere to be isothermal. *Note:* When evaluating gz/RT we use $R = 287$ J/kg·K, not 0.287 kJ/kg·K. To observe that gz/R is dimensionless, which it must be since it is an exponent, use $\text{N} = \text{kg}\cdot\text{m/s}^2$ so that

$$\left[\frac{gz}{RT}\right] = \frac{(\text{m/s}^2)\text{m}}{(\text{J/kg}\cdot\text{K})\text{K}} = \frac{\text{m}^2/\text{s}^2}{\text{N}\cdot\text{m/kg}} = \frac{\text{m}^2/\text{s}^2}{(\text{kg}\cdot\text{m}^2/\text{s}^2)/\text{kg}} = \frac{\text{m}^2/\text{s}}{\text{m}^2/\text{s}}$$

2.4.3 Manometers

Hydrostatic Pressure, 517

Manometers are instruments that use columns of liquids to measure pressures. Three such instruments, shown in Fig. 2.7, are discussed to illustrate their use. Part (a) displays a U-tube manometer, used to measure relatively small pressures. In this case the pressure in the pipe can be determined by defining a point 1 at the center of the pipe and a point 2 at the surface of the right column. Then, using Eq. 2.4.3,

$$p_1 + \gamma z_1 = p_2 + \gamma z_2$$

where the datum from which z_1 and z_2 are measured is located at any desired position, such as through point 1. Since $p_2 = 0$ (gage pressure is selected; if absolute pressure is desired, we would select $p_2 = p_{atm}$) and $z_2 - z_1 = h$,

$$p_1 = \gamma h \tag{2.4.12}$$

Figure 2.7b shows a manometer used to measure relatively large pressures since we can select γ_2 to be quite large; for example, we could select γ_2 to be that of mercury so that $\gamma_2 = 13.6\ \gamma_{water}$. The pressure can be determined by

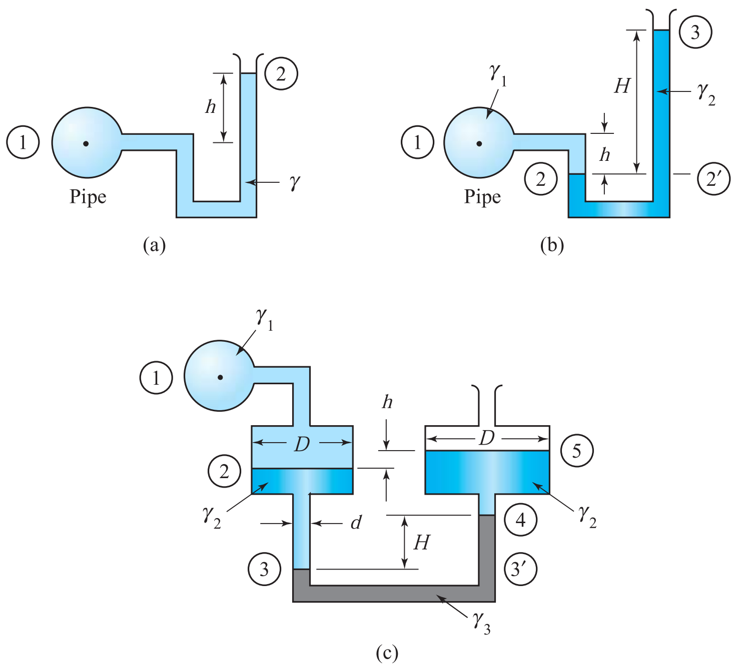

Fig. 2.7 Manometers: (a) U-tube manometer (small pressures); (b) U-tube manometer (large pressures); (c) micromanometer (very small pressure changes).

introducing the points indicated. This is necessary because Eq. 2.4.3 applies throughout one fluid; γ must be constant. The value of γ changes abruptly at point 2. The pressure at point 2 and at point 2′ is the same since the points are at the same elevation in the same fluid. Hence

$$\begin{aligned} p_2 &= p_2{}' \\ p_1 + \gamma_1 h &= p_3 + \gamma_2 H \end{aligned} \tag{2.4.13}$$

Setting $p_3 = 0$ (gage pressure is used) results in

$$p_1 = -\gamma_1 h + \gamma_2 H \tag{2.4.14}$$

Figure 2.6c shows a micromanometer that is used to measure very small pressure changes. Introducing the points indicated, requiring that $p_3 = p_3{}'$, we can write

$$p_1 + \gamma_1(z_1 - z_2) + \gamma_2(z_2 - z_3) = p_5 + \gamma_2(z_5 - z_4) + \gamma_3(z_4 - z_3) \tag{2.4.15}$$

Observe that $z_2 - z_3 + h = H + z_5 - z_4$ and set $p_5 = 0$; then

$$\begin{aligned} p_1 &= \gamma_1(z_2 - z_1) + \gamma_2(h - H) + \gamma_3 H \\ &= \gamma_1(z_2 - z_1) + \gamma_2 h + (\gamma_3 - \gamma_2)H \end{aligned} \tag{2.4.16}$$

Note that in all of the equations above for all three manometers, we have identified all interfaces with a point. This is always necessary when analyzing a manometer.

The micromanometer is capable of measuring small pressure changes because a small pressure change in p_1 results in a relatively large deflection H. The change in H due to a change in p_1 can be determined using Eq. 2.4.16. Suppose that p_1 increases by Δp_1 and, as a result, z_2 decreases by Δz; then h and H also change. Using the fact that a decrease in z_2 is accompanied by an increase in z_5 leads to an increase in h of $2\Delta z$ and, similarly, assuming that the volumes are conserved, it can be shown that H increases by $2\Delta z D^2/d^2$. Hence a pressure change Δp_1 can be evaluated from changes in deflections as follows:

$$\Delta p_1 = \gamma_1(-\Delta z) + \gamma_2(2\Delta z) + \frac{(\gamma_3 - \gamma_2)2\Delta z D^2}{d^2} \tag{2.4.17}$$

The rate of change in H with p_1 is

$$\frac{\Delta H}{\Delta p_1} = \frac{2\Delta z D^2/d^2}{\Delta p_1} \tag{2.4.18}$$

Using Eq. 2.4.17 we have

$$\frac{\Delta H}{\Delta p_1} = \frac{2D^2/d^2}{-\gamma_1 + 2\gamma_2 + 2(\gamma_3 - \gamma_2)D^2/d^2} \tag{2.4.19}$$

An example of this type of manometer is given in Example 2.4.

Example 2.3

Water and oil flow in horizontal pipelines. A double U-tube manometer is connected between the pipelines, as shown in Fig. E2.3. Calculate the pressure difference between the water pipe and the oil pipe.

Solution

We first identify the relevant points as shown in the figure. Begin at point ① and add pressure when the elevation decreases and subtract pressure when the elevation increases until point ⑤ is reached:

$$p_1 + \gamma(z_1 - z_2) - \gamma S_1(z_3 - z_2) - \gamma S_{air}(z_4 - z_3) + \gamma S_2(z_4 - z_5) = p_5$$

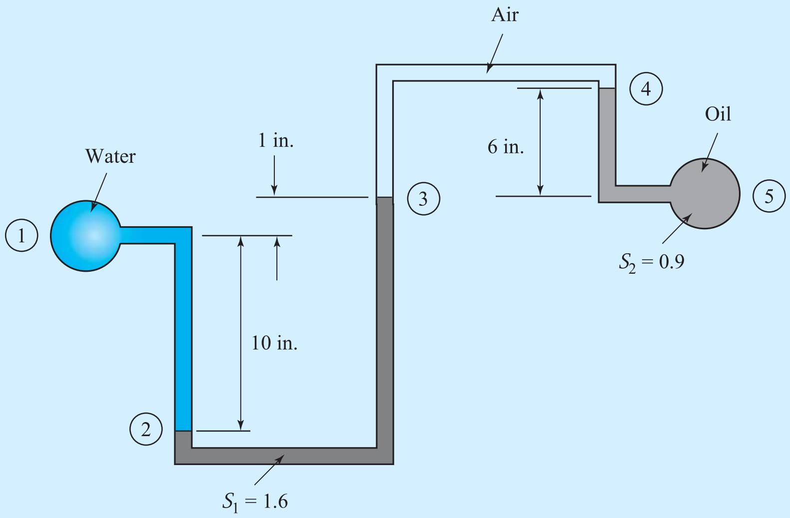

Fig. E2.3

where $\gamma = 62.4$ lb/ft^3, $S_1 = 1.6$, $S_2 = 0.9$, and $S_{air} \approx 0$. Thus

$$p_1 - p_5 = 62.4\left(-\frac{10}{12} + 1.6 \times \frac{11}{12} + 0 \times \frac{6}{12} - 0.9 \times \frac{6}{12}\right)$$
$$= 11.44 \text{ lb/ft}^2 \quad \text{or} \quad 0.0794 \text{ psi}$$

Note that by neglecting the weight of the air, the pressure at point 3 is equal to the pressure at point 4.

Example 2.4

For a given condition the liquid levels in Fig. 2.7c are $z_1 = 0.95$ m, $z_2 = 0.70$ m, $z_3 = 0.52$ m, $z_4 = 0.65$ m, and $z_5 = 0.72$ m. Further, $\gamma_1 = 9810\ \text{N/m}^3$, $\gamma_2 = 11\,500\ \text{N/m}^3$, and $\gamma_3 = 14\,000\ \text{N/m}^3$. The diameters are $D = 0.2$ m and $d = 0.01$ m. (a) Calculate the pressure p_1 in the pipe, (b) calculate the change in H if p_1 increases by 100 Pa, and (c) calculate the change in h of the manometer of Fig. 2.7a if $h = 0.5$ m of water and $\Delta p_1 = 100$ Pa.

Solution

(a) Referring to Fig. 2.7c, we have

$$h = 0.72 - 0.70 = 0.02\ \text{m}$$

$$H = 0.65 - 0.52 = 0.13\ \text{m}$$

Substituting the given values into Eq. (2.4.16) leads to

$$\begin{aligned} p_1 &= \gamma_1(z_2 - z_1) + \gamma_2 h + (\gamma_3 - \gamma_2)H \\ &= 9810(0.70 - 0.95) + 11\,500(0.02) + (14\,000 - 11\,500)(0.13) \\ &= -1898\ \text{Pa} \end{aligned}$$

(b) If the pressure p_1 is increased by 100 Pa to $p_1 = -1798$ Pa, the change in H is, using Eq. 2.4.19,

$$\Delta H = \Delta p_1 \frac{2D^2/d^2}{-\gamma_1 + 2\gamma_2 + 2(\gamma_3 - \gamma_2)D^2/d^2}$$

$$\Delta H = 100 \frac{2(20^2)}{-9810 + 2(11\,500) + 2(14\,000 - 11\,500) \times 20^2} = 0.0397\ \text{m}$$

Thus H increases by 3.97 cm as a result of increasing the pressure by 100 Pa.

(c) For the manometer in Fig. 2.7a, the pressure p_1 is given by $p = \gamma h$. Assume that initially $h = 0.50$ m. Thus the pressure initially is

$$p_1 = 9810 \times 0.50 = 4905\ \text{Pa}$$

Now if p_1 is increased by 100 Pa, h can be found:

$$p_1 = \gamma h$$

$$h = \frac{p_1}{\gamma} = \frac{5005}{9810} = 0.510\ \text{m}. \quad \therefore \Delta h = 0.510 - 0.5 = 0.01\ \text{m}$$

Thus an increase of 100 Pa increases h by 1 cm in the manometer shown in part (a), 25% of the change in the micromanometer.

2.4.4 Forces on Plane Areas

In the design of devices and objects that are submerged, such as dams, flow obstructions, surfaces on ships, and holding tanks, it is necessary to calculate the magnitudes and locations of forces that act on both plane and curved surfaces. In this section we consider only plane surfaces, such as the plane surface of general shape shown in Fig. 2.8. Note that a side view is given as well as a view showing

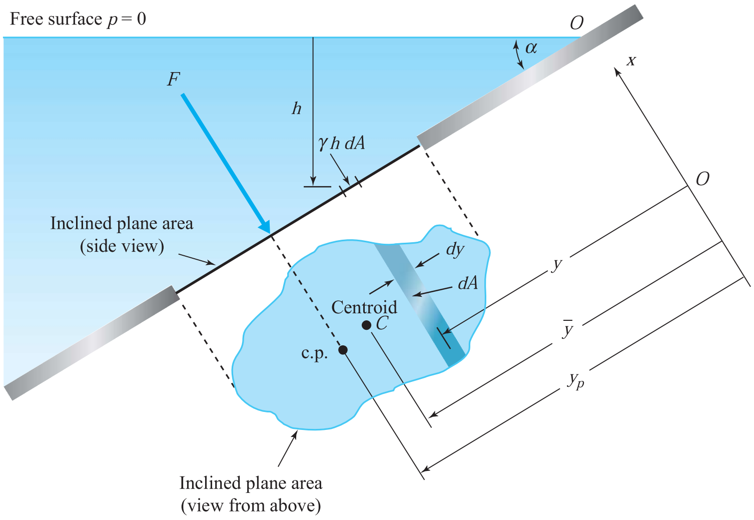

Fig. 2.8 Force on an inclined plane area.

the shape of the plane. The total force of the liquid on the plane surface is found by integrating the pressure over the area, that is,

$$F = \int_A p\, dA \tag{2.4.20}$$

where we usually use gage pressure. (Atmospheric pressure cancels out since it acts on both sides of the area.) The x and y coordinates are in the plane of the plane surface, as shown. Assuming that $p = 0$ at $h = 0$, we know that

$$\begin{aligned} p &= \gamma h \\ &= \gamma y \sin \alpha \end{aligned} \tag{2.4.21}$$

where h is measured vertically down from the free surface to the elemental area dA and y is measured from point O on the free surface. The force may then be expressed as

$$\begin{aligned} F &= \int_A \gamma h\, dA \\ &= \gamma \sin \alpha \int_A y\, dA \end{aligned} \tag{2.4.22}$$

The distance to a centroid is defined as

$$\bar{y} = \frac{1}{A} \int_A y\, dA \tag{2.4.23}$$

The expression for the force then becomes

$$\begin{aligned} F &= \gamma \bar{y} A \sin \alpha \\ &= \gamma \bar{h} A = p_C A \end{aligned} \tag{2.4.24}$$

where $\bar{h}$ is the vertical distance from the free surface to the centroid of the area and p_C is the pressure at the centroid. Thus we see that the magnitude of the force on a plane surface is the pressure at the centroid multiplied by the area. The force does not, in general, act at the centroid.

KEY CONCEPT *The force on a plane surface is the pressure at the centroid multiplied by the area.*

To find the location of the resultant force F, we note that the sum of the moments of all the infinitesimal pressure forces acting on the area A must equal the moment of the resultant force. Let the force F act at the point (x_p, y_p), the **center of pressure** (c.p.). The value of y_p can be obtained by equating moments about the x-axis:

Center of pressure: *The point where the resultant force acts.*

$$y_pF = \int_A yp\,dA$$

$$= \gamma \sin\alpha \int_A y^2\,dA = \gamma I_x \sin\alpha \qquad (2.4.25)$$

where the second moment of the area about the x-axis is

$$I_x = \int_A y^2\,dA \qquad (2.4.26)$$

The second moment of an area is related to the second moment of an area $\bar{I}$ about the centroidal axis by the parallel-axis-transfer theorem,

$$I_x = \bar{I} + A\bar{y}^2 \qquad (2.4.27)$$

Substitute Eqs. 2.4.24 and 2.4.27 into Eq. 2.4.25, and obtain

$$y_p = \frac{\gamma(\bar{I} + A\bar{y}^2)\sin\alpha}{\gamma\bar{y}A\sin\alpha} = \bar{y} + \frac{\bar{I}}{A\bar{y}} \qquad (2.4.28)$$

where $\bar{y}$ is measured parallel to the plane area to the free surface.

Centroids and moments for several areas are presented in Appendix C. Using the expression above, we can show that the force on a rectangular gate, with the top edge even with the liquid surface, as shown in Fig. 2.9, acts two-thirds of the way down. This is also obvious considering the triangular pressure distribution acting on the gate. Note that Eq. 2.4.28 shows that y_p is always greater

KEY CONCEPT *The force on a rectangular gate, with the top edge even with the liquid surface, acts two-thirds of the way down.*

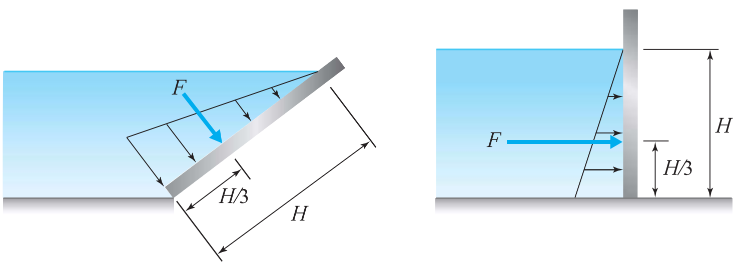

Fig. 2.9 Force on a plane area with top edge in a free surface.

than $\bar{y}$; that is, the resultant force of the liquid on a plane surface always acts below the centroid of the area, except on a horizontal area for which $\bar{y} = \infty$; then the center of pressure and the centroid coincide.

Similarly, to locate the x-coordinate x_p of the c.p., we write

$$x_p F = \int_A xp\, dA$$
$$= \gamma \sin\alpha \int_A xy\, dA = \gamma I_{xy} \sin\alpha \tag{2.4.29}$$

where the product of inertia of the area A is

$$I_{xy} = \int_A xy\, dA \tag{2.4.30}$$

Using the transfer theorem for the product of inertia,

$$I_{xy} = \bar{I}_{xy} + A\bar{x}\bar{y} \tag{2.4.31}$$

Equation 2.4.29 becomes

$$x_p = \bar{x} + \frac{\bar{I}_{xy}}{A\bar{y}} \tag{2.4.32}$$

We now have expressions for the coordinates locating the center of pressure.

Finally, we should note that the force F in Fig. 2.8 is the result of a *pressure prism* acting on the area. For the rectangular area shown in Fig. 2.10, the pressure increases, as shown by the pressure distribution in Fig. 2.10b. If we form the integral $\int p\, dA$, we obtain the volume of the pressure prism, which equals the force F acting on the area, shown in Fig. 2.10c. The force acts through the centroid of the volume. For the rectangular area shown in Fig. 2.10a, the volume could be divided into two volumes: a rectangular volume with centroid at its center, and a triangular volume with centroid one-third the distance from the appropriate base. The location of the force is then found by locating the centroid of the composite volume.

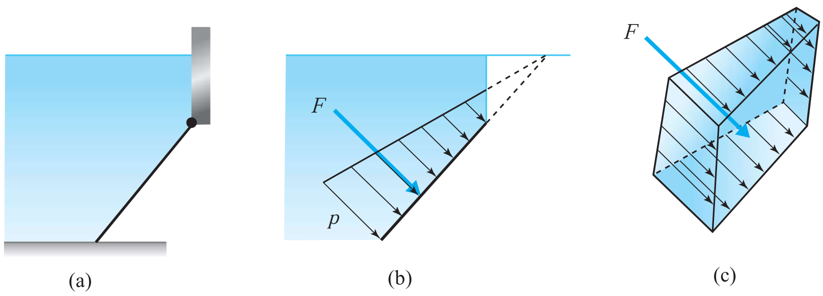

Fig. 2.10 Pressure prism: (a) rectangular area; (b) pressure distribution on the area; (c) pressure prism.

Example 2.5

A plane area of 80 cm × 80 cm acts as an escape hatch on a submersible in the Great Lakes. If it is on a 45° angle with the horizontal, what force applied normal to the hatch at the bottom edge is needed to just open the hatch, if it is hinged at the top edge when the top edge is 10 m below the surface? The pressure inside the submersible is assumed to be atmospheric.

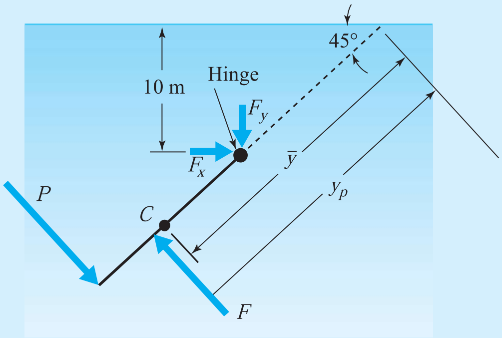

Fig. E2.5

Solution

First, a sketch of the hatch would be very helpful, as in Fig. E2.5. The force of the water acting on the hatch is

$$F = \gamma \bar{h} A$$
$$= 9810(10 + 0.4 \times \sin 45°)(0.8 \times 0.8) = 64\ 560 \text{ N}$$

The distance $\bar{y}$ is

$$\bar{y} = \frac{\bar{h}}{\sin 45°} = \frac{10 + 0.4 \times \sin 45°}{\sin 45°} = 14.542 \text{ m}$$

so that

$$y_p = \bar{y} + \frac{\bar{I}}{A\bar{y}}$$
$$= 14.542 + \frac{0.8 \times 0.8^3/12}{(0.8 \times 0.8) \times 14.542} = 14.546 \text{ m}$$

Taking moments about the hinge provides the needed force P to open the hatch:

$$0.8P = (y_p - \bar{y} + 0.4)F$$

$$\therefore \quad P = \frac{14.546 - 14.542 + 0.4}{0.8} 64\ 560 = 32\ 610 \text{ N}$$

Alternatively, we could have sketched the pressure prism, composed of a rectangular volume and a triangular volume. Moments about the top hinge would provide the desired force.

Example 2.6

Find the location of the resultant force F of the water on the triangular gate and the force P necessary to hold the gate in the position shown in Fig. E2.6a. Neglect the weight of the gate, as usual.

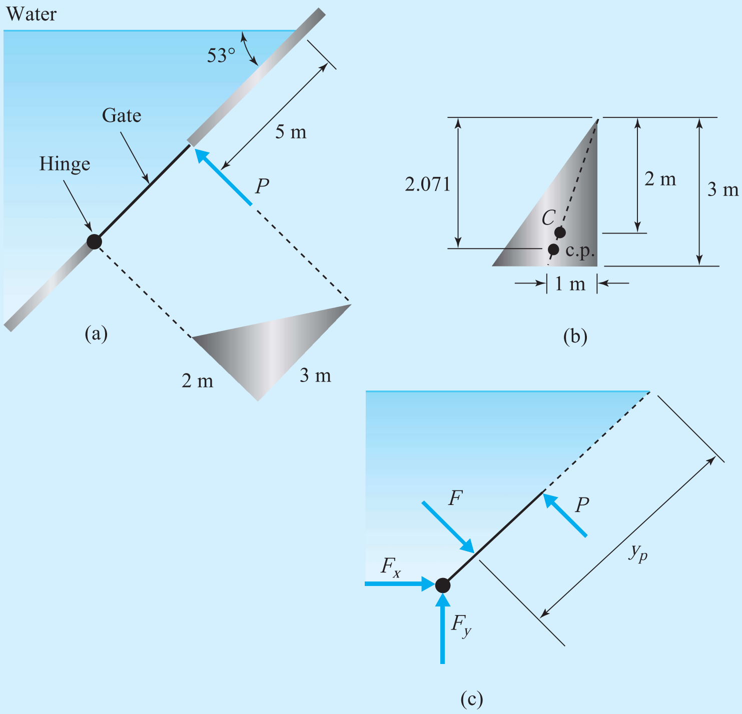

Fig. E2.6

Solution

First we draw a free-body diagram of the gate, including all the forces acting on the gate (Fig. E2.6c). The centroid of the gate is shown in Fig. E2.6b. The y-coordinate of the location of the resultant F can be found using Eq. 2.4.28 as follows:

$$\bar{y} = 2 + 5 = 7$$

$$y_p = \bar{y} + \frac{\bar{I}}{A\bar{y}}$$

$$= 7 + \frac{2 \times 3^3/36}{3 \times 7} = 7.071 \text{ m}$$

To find x_p we could use Eq. 2.4.32. Rather than that, we recognize that the resultant force must act on a line connecting the vertex and the midpoint of the opposite side since each infinitesimal force acts on this line (the moment of the resultant must equal the moment of its components). Thus, using similar triangles we have

$$\frac{x_p}{1} = \frac{2.071}{3}$$

$$\therefore \quad x_p = 0.690 \text{ m}$$

The coordinates x_p and y_p locate where the force due to the water acts on the gate.

If we take moments about the hinge, assumed to be frictionless, we can determine the force P necessary to hold the gate in the position shown:

$$\Sigma M_{\text{hinge}} = 0$$

$$\therefore \quad 3 \times P = (3 - 2.071)F$$

$$= 0.929 \times \gamma \bar{h} A$$

$$= 0.929 \times 9810 \times (7 \sin 53°) \times 3$$

where $\bar{h}$ is the vertical distance from the centroid to the free surface. Hence

$$P = 50\,900 \text{ N} \quad \text{or} \quad 50.9 \text{ kN}$$

2.4.5 Forces on Curved Surfaces

We do not use a direct method of integration to find the force due to the hydrostatic pressure on a curved surface. Rather, a free-body diagram that contains the curved surface and the liquids directly above or below the curved surface is identified. Such a free-body diagram contains only plane surfaces upon which unknown fluid forces act; these unknown forces can be found as in the preceding section.

As an example, let us determine the force of the curved gate on the stop, shown in Fig. 2.11a. The free-body diagram, which includes the water contained directly above the gate, is shown in Fig. 2.11b; F_1 and F_2 are due to the surrounding water and are the resultant forces of the pressure distributions shown; the body force F_W is due to the weight of the water shown. In Fig. 2.11c the gate is the free body; the forces F_x and F_y are the horizontal and vertical components, respectively, of the force acting on the hinge. By summing moments about an axis passing through the hinge, we can determine the force P acting on the stop.

If the curved surface is a quarter circle, the problem can be greatly simplified. This is observed by considering the free-body diagram of the gate only (Fig. 2.11c). The horizontal force F_H acting on the gate is equal to F_1 of Fig. 2.11b, and the component F_V is equal to the combined force $F_2 + F_W$ of Fig. 2.11b. Now, F_H and F_V are due to the differential pressure forces acting on the circular arc; each differential pressure force acts through the center of the circular arc. Hence the resultant force $\mathbf{F}_H + \mathbf{F}_V$ (this is a vector addition) must act through the center. Consequently, we can locate the components F_H and F_V at the center of the quarter circle, resulting in a much simpler problem. Example 2.7 will illustrate.

KEY CONCEPT *The resultant force $\mathbf{F}_H + \mathbf{F}_V$ must act through the center of the circular arc.*

If the pressure on the free surface is p_0, we can simply add a depth of liquid necessary to provide p_0 at the location of the free surface and then work the resulting problem with a fictitious free surface located the appropriate distance above the original free surface. Or, the pressure force $p_0 A$ is added to the force F_2 of Fig. 2.11b.

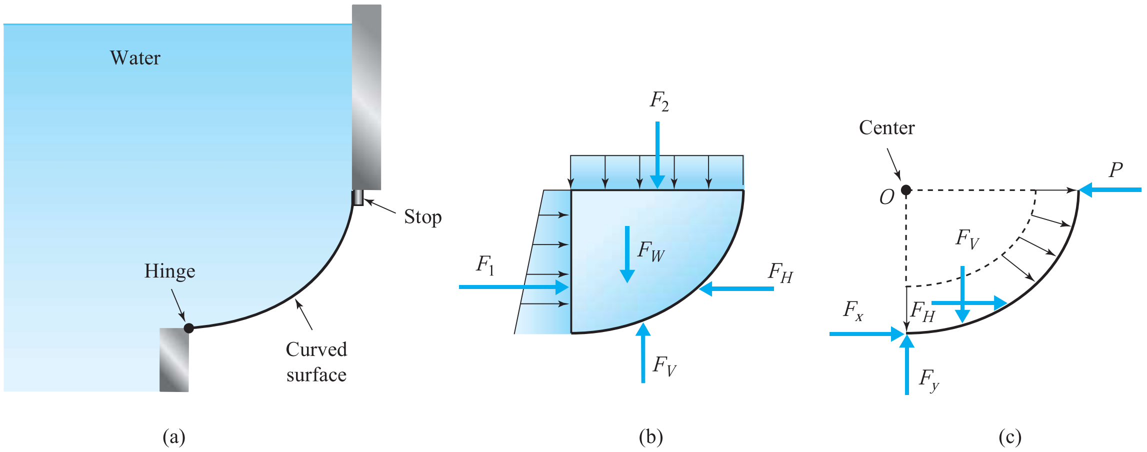

Fig. 2.11 Forces acting on a curved surface: (a) curved surface; (b) free-body diagram of water and gate; (c) free-body diagram of gate only.

Example 2.7

Calculate the force P necessary to hold the 4-m-wide gate in the position shown in Fig. E2.7a. Neglect the weight of the gate.

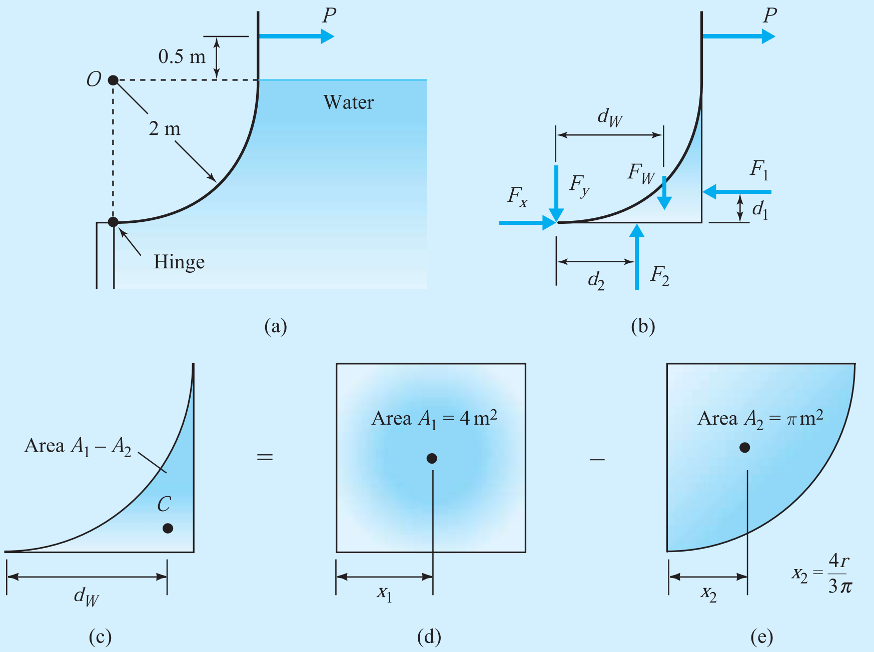

Fig. E2.7

Solution

The first step is to draw a free-body diagram. One choice is to select the gate and the water directly below the gate, as shown in Fig. E2.7b. To calculate P, we must determine F_1, F_2, F_W, d_1, d_2, and d_W; then moments about the hinge will allow us to find P. The force components are given by

$$F_1 = \gamma \bar{h}_1 A_1$$
$$= 9810 \times 1 \times (2 \times 4) = 78\,480 \text{ N}$$
$$F_2 = \gamma \bar{h}_2 A_2$$
$$= 9810 \times 2 \times (2 \times 4) = 156\,960 \text{ N}$$
$$F_W = \gamma V\!\!\!\!-_{\text{water}}$$
$$= 9810 \times 4\left(4 - \frac{\pi \times 2^2}{4}\right) = 33\,700 \text{ N}$$

The distance d_W is the distance to the centroid of the volume. It can be determined by considering the area as the difference of a square and a quarter circle as shown in Fig. E2.7c–e. Moments of areas yield

$$d_W(A_1 - A_2) = x_1 A_1 - x_2 A_2$$
$$d_W = \frac{x_1 A_1 - x_2 A_2}{A_1 - A_2}$$
$$= \frac{1 \times 4 - (4 \times 2/3\pi) \times \pi}{4 - \pi} = 1.553 \text{ m}$$

The distance $d_2 = 1$ m. Because F_1 is due to a triangular pressure distribution (see Fig. 2.9), d_1 is given by

$$d_1 = \frac{1}{3}(2) = 0.667 \text{ m}$$

Summing moments about the frictionless hinge gives

$$2.5P = d_1 F_1 + d_2 F_2 - d_W F_W$$
$$P = \frac{0.667 \times 78.5 + 1 \times 157.0 - 1.553 \times 33.7}{2.5} = 62.8 \text{ kN}$$

Rather than the somewhat tedious procedure above, we could observe that all the infinitesimal forces that make up the resultant force ($\mathbf{F}_H + \mathbf{F}_V$) acting on the circular arc pass through the center O, as noted in Fig. 2.11c. Since each infinitesimal force passes through the center, the resultant force must also pass through the center. Hence we could have located the resultant force ($\mathbf{F}_H + \mathbf{F}_V$) at point O. If F_V and F_H were located at O, F_V would pass through the hinge, producing no moment about the hinge. Then, realizing that $F_H = F_1$ and summing moments about the hinge gives

$$2.5P = 2F_H$$

Therefore,

$$P = 2 \times \frac{78.48}{2.5} = 62.8 \text{ kN}$$

This was obviously much simpler. All we needed to do was calculate F_H and then sum moments!

Example 2.8

Find the force P needed to hold the gate in the position shown in Fig. E2.8a if P acts 3 m from the y-axis. The parabolic gate is 150 cm wide.

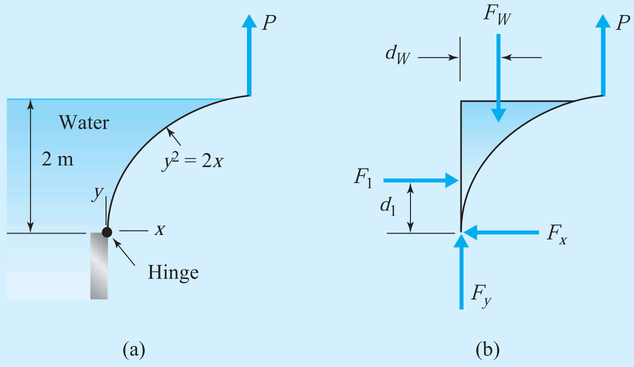

Fig. E2.8

Solution

A free-body diagram of the gate and the water directly above the gate is shown in Fig. E2.8b. The forces are found to be

$$F_1 = \gamma \bar{h} A$$
$$= 9810 \times 1 \times (2 \times 1.5) = 29\,430 \text{ N}$$
$$F_W = \gamma \forall$$
$$= 9810 \int_0^2 1.5x\, dy = 14\,715 \int_0^2 \frac{y^2}{2}\, dy = 14\,715 \frac{2^3}{6} = 19\,620 \text{ N}$$

The distance d_1 is $\frac{1}{3}(2) = 0.667$ m since the top edge is in the free surface. The distance d_W through the centroid is found using a horizontal strip:

$$d_W = \frac{\int_0^2 x(x/2)\, dy}{\int_0^2 x\, dy} = \frac{\frac{1}{8}\int_0^2 y^4\, dy}{\frac{1}{2}\int_0^2 y^2\, dy} = \frac{1}{4}\frac{2^5/5}{2^3/3} = 0.6 \text{ m}$$

Sum moments about the hinge and find P as follows:

$$3P = d_1 F_1 + d_W F_W$$
$$= 0.667 \times 29\,430 + 0.6 \times 19\,620 \qquad \therefore P = 10\,470 \text{ N}$$

2.4.6 Buoyancy

The law of buoyancy, known as Archimedes' principle, dates back some 2200 years to the Greek philosopher Archimedes. Legend has it that Hiero, king of Syracuse, suspected that his new gold crown may have been constructed of materials other than pure gold, so he asked Archimedes to test it. Archimedes probably made a lump of pure gold that weighed the same as the crown. The lump was discovered to weigh more in water than the crown weighed in water, thereby proving to Archimedes that the crown was not pure gold. The fake material possessed a larger volume to have the same weight as gold, hence it displaced more water. *Archimedes' principle* is: There is a buoyancy force on an object equal to the weight of displaced liquid.

KEY CONCEPT
Archimedes' principle states the buoyancy force on an object equals the weight of displaced liquid.

To prove the law of buoyancy, consider the submerged body shown in Fig. 2.12a. In part (b) a cylindrical free-body diagram is shown that includes the submerged body with weight W and liquid having a weight F_W; the cross-sectional area A is the maximum cross-sectional area of the body. From the diagram we see that the resultant vertical force acting on the free-body diagram due to the water only (do not include W) is equal to

$$\Sigma F = F_2 - F_1 - F_W \qquad (2.4.33)$$

This resultant force is by definition the *buoyant force* F_B. It can be expressed as

$$F_B = \gamma(h_2 A - h_1 A - \forall_W) \qquad (2.4.34)$$

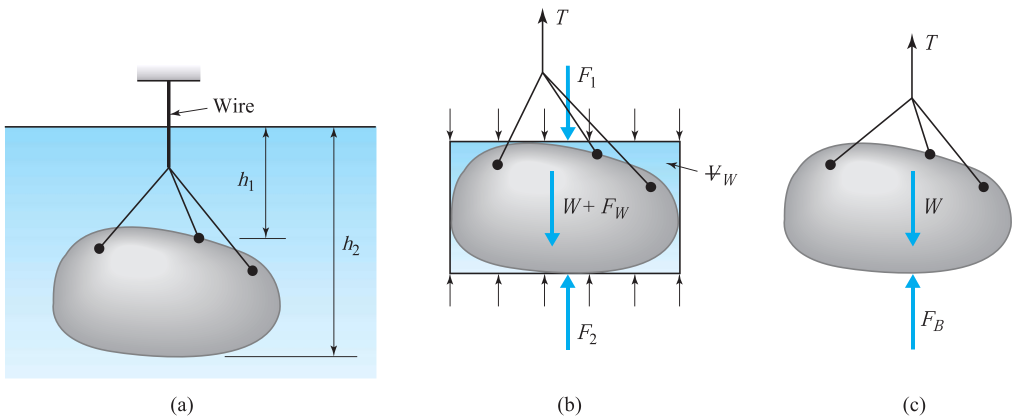

Fig. 2.12 Forces on a submerged body: (a) submerged body; (b) free-body diagram; (c) free body showing the buoyant force F_B.

where $\mathcal{V}_W$ is the liquid volume included in the free-body diagram. Recognizing that the volume of the submerged body is

$$\mathcal{V}_B = (h_2 - h_1)A - \mathcal{V}_W \tag{2.4.35}$$

we see from Eq. 2.4.34 that

$$F_B = \gamma \mathcal{V}_{\text{displaced liquid}} \tag{2.4.36}$$

thereby proving the law of buoyancy.

The force necessary to hold the submerged body in place (see Fig. 2.12c) is equal to

$$T = W - F_B \tag{2.4.37}$$

where W is the weight of the submerged body.

For a floating object, as in Fig. 2.13, the buoyant force is

$$F_B = \gamma \mathcal{V}_{\text{displaced liquid}} \tag{2.4.38}$$

Obviously, $T = 0$, so that Eq. 2.4.36 gives

$$F_B = W \tag{2.4.39}$$

where W is the weight of the floating object.

KEY CONCEPT *The buoyant force acts through the centroid of the displaced liquid volume.*

From the foregoing analysis it is apparent that the buoyant force F_B acts through the centroid of the displaced liquid volume. For the floating object, the weight of the object acts through its center of gravity, so the center of gravity of the object must lie on the same vertical line as the centroid of the liquid volume.

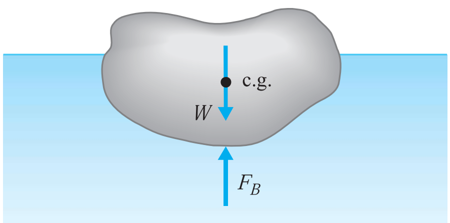

Fig. 2.13 Forces on a floating object.

Hydrometer: *An instrument used to measure the specific gravity of liquids.*

A **hydrometer**, an instrument used to measure the specific gravity of liquids, operates on the principle of buoyancy. A sketch is shown in Fig. 2.14. The upper part, the stem, has a constant diameter. When placed in pure water the specific gravity is marked to read 1.0. The force balance is

$$W = \gamma_{\text{water}} \forall \tag{2.4.40}$$

where W is the weight of the hydrometer and $\forall$ is the submerged volume below the $S = 1.0$ line. In an unknown liquid of specific weight γ_x, a force balance would be

$$W = \gamma_x(\forall - A\Delta h) \tag{2.4.41}$$

where A is the cross-sectional area of the stem. Equating these two expressions gives

$$\Delta h = \frac{\forall}{A}\left(1 - \frac{1}{S_x}\right) \tag{2.4.42}$$

where $S_x = \gamma_x/\gamma_{\text{water}}$. For a given hydrometer, $\forall$ and A are fixed so that the quantity Δh is dependent only on the specific gravity S_x. Thus the stem can be calibrated to read S_x directly. Hydrometers are used to measure the amount of antifreeze in the radiator of an automobile, or the charge in a battery since the density of the fluid changes as H_2SO_4 is consumed or produced.

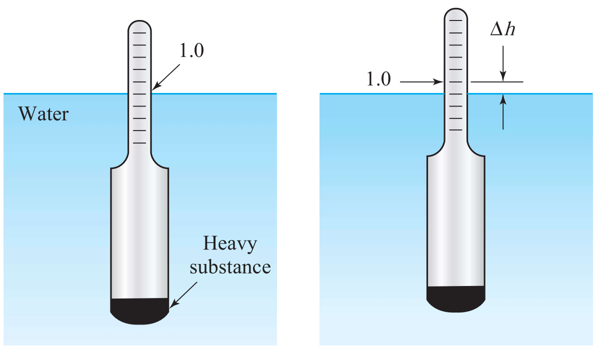

Fig. 2.14 Hydrometer: (a) in water; (b) in an unknown liquid.

Example 2.9

The specific weight and the specific gravity of a body of unknown composition are desired. Its weight in air is found to be 200 lb, and in water it weighs 150 lb.

Solution

The volume is found from a force balance when submerged as follows (see Fig. 2.12c):

$$T = W - F_B$$

$$150 = 200 - 62.4\forall \qquad \therefore \forall = 0.801 \text{ ft}^3$$

The specific weight is then

$$\gamma = \frac{W}{\forall} = \frac{200}{0.801} = 250 \text{ lb/ft}^3$$

The specific gravity is found to be

$$S = \frac{\gamma}{\gamma_{\text{water}}} = \frac{250}{62.4} = 4.01$$

2.4.7 Stability

KEY CONCEPT *A floating object has vertical stability.*

The notion of stability can be demonstrated by considering the vertical stability of a floating object. If the object is raised a small distance, the buoyant force decreases and the object's weight returns the object to its original position. Conversely, if a floating object is lowered slightly, the buoyant force increases and the larger buoyant force returns the object to its original position. Thus a floating object has vertical stability, since a small departure from equilibrium results in a restoring force.

Center of buoyancy: *Centroid of a floating body.*

Consider now the rotational stability of a submerged body, shown in Fig. 2.15. In part (a) the center of gravity G of the body is above the centroid C (also referred to as the **center of buoyancy**) of the displaced volume, and a small angular rotation results in a moment that will continue to increase the rotation; hence the body is unstable and overturning would result. If the center of gravity is below the centroid, as in part (c), a small angular rotation provides a restoring moment and the body is stable. Part (b) shows neutral stability for a body in which the center of gravity and the centroid coincide, a situation that is encountered whenever the density is constant throughout the submerged body.

Next, consider the rotational stability of a floating body. If the center of gravity is below the centroid, the body is always stable, as with the submerged body of Fig. 2.15c. The body may be stable, though, even if the center of gravity is above the centroid, as sketched in Fig. 2.16a. When the body rotates the centroid of the volume of displaced liquid moves to the new location C', shown in part (b). If the centroid C' moves sufficiently far, a restoring moment develops and the body is stable, as shown. This is determined by the *metacentric height* $\overline{GM}$ defined as the distance from G to the point of intersection of the buoyant force before rotation

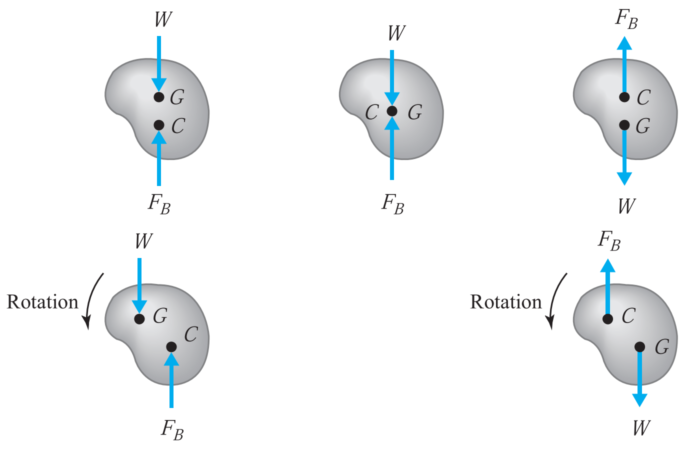

Fig. 2.15 Stability of a submerged body: (a) unstable; (b) neutral; (c) stable.

with the buoyant force after rotation. If $\overline{GM}$ is positive, as shown, the body is stable; if $\overline{GM}$ is negative (M lies below G), the body is unstable.

KEY CONCEPT *If GM is positive, the body is stable.*

To determine a quantitative relationship for the distance $\overline{GM}$ refer to the sketch of Fig. 2.17, which shows a uniform cross section. Let us find an expression for $\bar{x}$, the x-coordinate of the centroid of the displaced liquid volume. It can be found by considering the volume to be the original volume plus the added wedge with cross-sectional area DOE minus the subtracted wedge with cross-sectional area AOB; to locate the centroid of a composite volume, we take moments as follows:

$$\bar{x}\mathcal{V} = \bar{x}_0\mathcal{V}_0 + \bar{x}_1\mathcal{V}_1 - \bar{x}_2\mathcal{V}_2 \tag{2.4.43}$$

where $\mathcal{V}_0$ is the original volume below the water line, $\mathcal{V}_1$ is the area DOE times the length $\bar{x}_1$, and $\mathcal{V}_2$ is the area AOB times the length $\bar{x}_2$; the cross section is assumed to be uniform so that the length l is constant for the body. The quantity $\bar{x}_0$, the x-coordinate of point C, is zero. The remaining two terms can best be represented by integrals so that

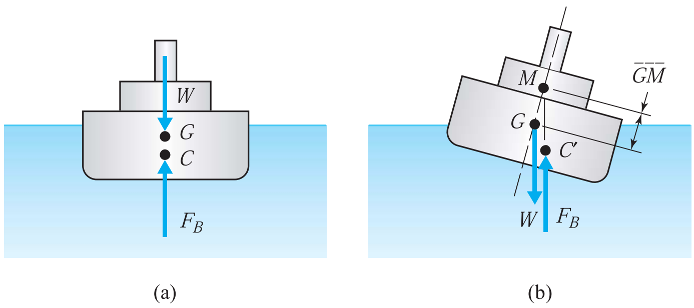

Fig. 2.16 Stability of a floating body: (a) equilibrium position; (b) rotated position.

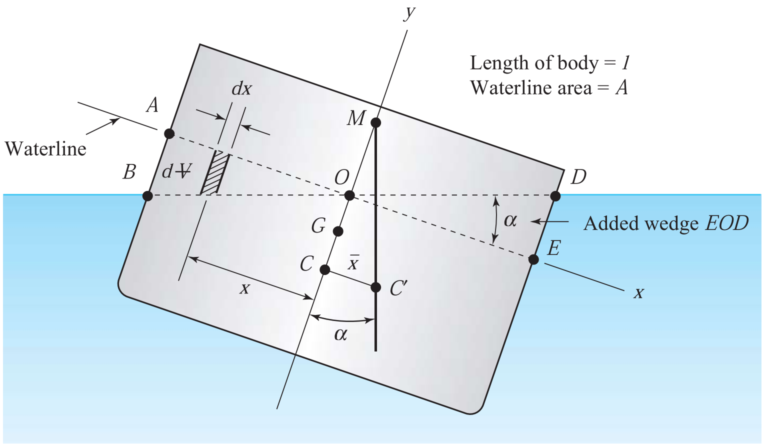

Fig. 2.17 Uniform cross section of a floating body.

$$\bar{x}\,\forall = \int_{\forall_1} x\, d\forall - \int_{\forall_2} x\, d\forall \tag{2.4.44}$$

Then $d\forall = x \tan\alpha\, dA$ in volume 1 and $d\forall = -x \tan\alpha\, dA$ in volume 2, where $dA = l\, dx$, l being the constant length of the body. The equation above becomes

$$\begin{aligned}\bar{x}\,\forall &= \tan\alpha \int_{A_1} x^2\, dA + \tan\alpha \int_{A_2} x^2\, dA \\ &= \tan\alpha \int_{A} x^2\, dA \\ &= I_O \tan\alpha \end{aligned}\tag{2.4.45}$$

where I_O is the second moment (moment of inertia) of the waterline area about an axis passing through the origin O. The waterline area would be the length $\overline{AE}$ times the length l of the body if l were of constant length. Using $\bar{x} = \overline{CM}\tan\alpha$, we can write

$$\overline{CM}\,\forall = I_O \tag{2.4.46}$$

or, with $\overline{CG} + \overline{GM} = \overline{CM}$, we have

$$\overline{GM} = \frac{I_O}{\forall} - \overline{CG} \tag{2.4.47}$$

For a given body orientation, if $\overline{GM}$ is positive, the body is stable. Even though this relationship (2.4.47) is derived for a floating body with uniform cross section, it is applicable for floating bodies in general. We will apply it to a floating cylinder in the following example.

Example 2.10

A 0.25-m-diameter cylinder is 0.25 m long and composed of material with specific weight 8000 N/m^3. Will it float in water with the ends horizontal?

Solution

With the ends horizontal, I_O will be the second moment of the circular cross section,

$$I_O = \frac{\pi d^4}{64} = \frac{\pi \times 0.25^4}{64} = 0.000192 \text{ m}^4$$

The displaced volume will be

$$V\!\!\!/ = \frac{W}{\gamma_{\text{water}}} = \frac{8000 \times \pi/4 \times 0.25^2 \times 0.25}{9810} = 0.0100 \text{ m}^3$$

The depth the cylinder sinks in the water is

$$depth = \frac{V\!\!\!/}{A} = \frac{0.01}{\pi \times 0.25^2/4} = 0.204 \text{ m}$$

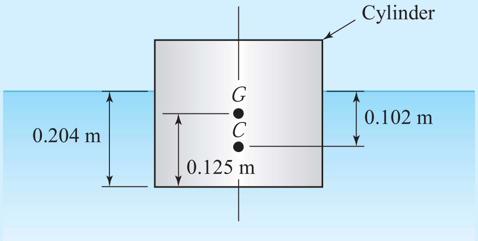

Fig. E2.10

Hence, the distance $\overline{CG}$, as shown in Fig. E2.10, is

$$\overline{CG} = 0.125 - \frac{0.204}{2} = 0.023 \text{ m}$$

Finally,

$$\overline{GM} = \frac{0.000192}{0.01} - 0.023 = -0.004 \text{ m}$$

This is a negative value showing that the cylinder will not float with ends horizontal. It would undoubtedly float on its side.

2.5 LINEARLY ACCELERATING CONTAINERS

In this section the fluid will be at rest relative to a reference frame that is linearly accelerating with a horizontal component a_x and a vertical component a_z. Then Eq. 2.3.6 simplifies to

$$dp = -\rho a_x \, dx - \rho(g + a_z) \, dz \qquad (2.5.1)$$

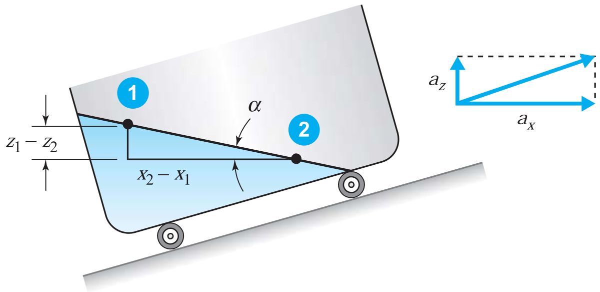

Fig. 2.18 Linearly accelerating tank.

Integrating between two arbitrary points 1 and 2 results in

$$p_2 - p_1 = -\rho a_x(x_2 - x_1) - \rho(g + a_z)(z_2 - z_1) \tag{2.5.2}$$

If points 1 and 2 lie on a constant-pressure line, such as the free surface in Fig. 2.18, then $p_2 - p_1 = 0$ and we have

$$\frac{z_1 - z_2}{x_2 - x_1} = \tan\alpha = \frac{a_x}{g + a_z} \tag{2.5.3}$$

where α is the angle that the constant-pressure line makes with the horizontal.

KEY CONCEPT *We often utilize the conservation of mass and equate the volumes before and after the acceleration is applied.*

In the solution of problems involving liquids, we must often utilize the conservation of mass and equate the volumes before and after the acceleration is applied. After the acceleration is initially applied, sloshing may occur. Our analysis will assume that sloshing is not present; either sufficient time passes to dampen out time-dependent motions, or the acceleration is applied in such a way that such motions are minimal.

Example 2.11

The tank shown in Fig. E2.11a is accelerated to the right. Calculate the acceleration a_x needed to cause the free surface, shown in Fig. E2.11b, to touch point A. Also, find p_B and the total force acting on the bottom of the tank if the tank width is 1 m.

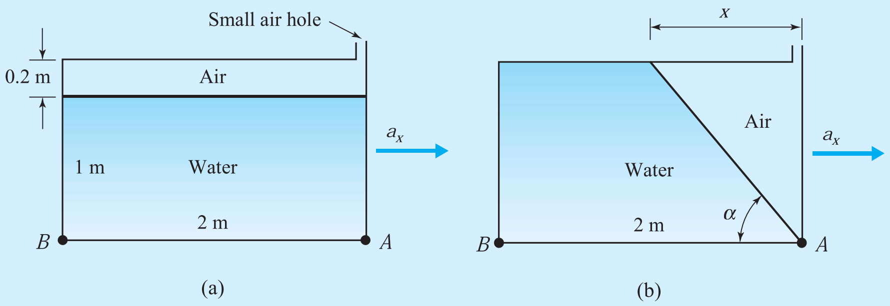

Fig. E2.11

Solution

The angle the free surface takes is found by equating the air volume (actually, areas since the width is constant) before and after since no water spills out:

$$0.2 \times 2 = \tfrac{1}{2}(1.2x)$$
$$x = 0.667 \text{ m}$$

The quantity $\tan \alpha$ is now known. It is

$$\tan \alpha = \frac{1.2}{0.667} = 1.8$$

Using Eq. 2.5.3, we find a_x to be, letting $a_z = 0$,

$$\begin{aligned} a_x &= g \tan \alpha \\ &= 9.81 \times 1.8 = 17.66 \text{ m/s}^2 \end{aligned}$$

We can find the pressure at B by noting the pressure dependence on x. At A, the pressure is zero. Hence, Eq. 2.5.2 yields

$$\begin{aligned} p_B - \overset{0}{\cancel{p_A}} &= -\rho a_x (x_B - x_A) \\ p_B &= -1000 \times 17.66(-2) \\ &= 35\,300 \text{ Pa} \quad \text{or} \quad 35.3 \text{ kPa} \end{aligned}$$

To find the total force acting on the bottom of the tank, we realize that the pressure distribution is decreasing linearly from $p = 35.3$ kPa at B to $p = 0$ kPa at A. Hence, we can use the average pressure over the bottom of the tank:

$$\begin{aligned} F &= \frac{p_B + p_A}{2} \times \text{area} \\ &= \frac{35\,300 + 0}{2} \times 2 \times 1 = 35\,300 \text{ N} \end{aligned}$$

2.6 ROTATING CONTAINERS

In this section we consider the situation of a liquid contained in a rotating container, such as that shown in Fig. 2.19. After a relatively short time the liquid reaches static equilibrium with respect to the container and the rotating rz-reference frame. The horizontal rotation will not alter the pressure distribution in the vertical direction. There will be no variation of pressure with respect to the θ-coordinate. Applying Newton's second law ($\Sigma F_r = ma_r$) in the r-direction to the element shown, using $\sin d\theta/2 \cong d\theta/2$, yields

KEY CONCEPT *Horizontal rotation will not alter the pressure distribution in the vertical direction.*

$$-\frac{\partial p}{\partial r} dr\, rd\theta\, dz - prd\theta\, dz - p\, dr\, d\theta\, dz - \frac{\partial p}{\partial r}(dr)^2 d\theta\, dz$$
$$+ 2p\frac{d\theta}{2} dr\, dz + prd\theta\, dz = -\rho\, rd\theta\, dr\, dz\, r\omega^2 \qquad (2.6.1)$$

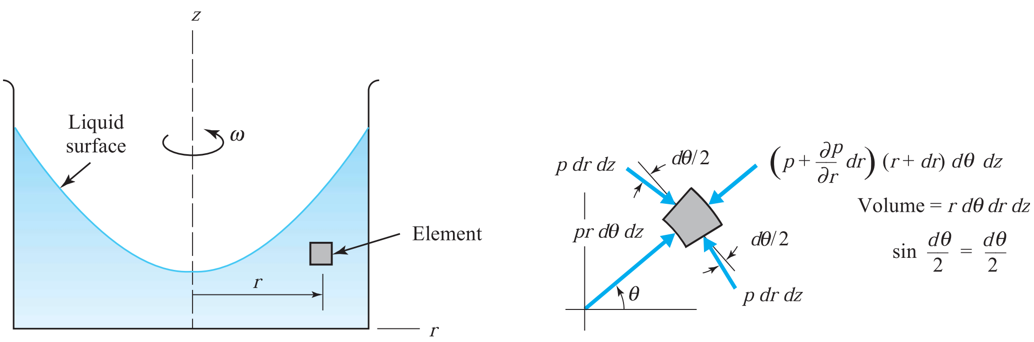

Fig. 2.19 Rotating container: (a) liquid cross section; (b) top view of element.

where the acceleration is $r\omega^2$ toward the center of rotation. Simplify and divide by the volume $rd\theta\, dr\, dz$; then

$$\frac{\partial p}{\partial r} = \rho r \omega^2 \tag{2.6.2}$$

where we have neglected the higher-order term that contains the differential dr. The pressure differential then becomes

$$\begin{aligned} dp &= \frac{\partial p}{\partial r} dr + \frac{\partial p}{\partial z} dz \\ &= \rho r \omega^2\, dr - \rho g\, dz \end{aligned} \tag{2.6.3}$$

where we have used the static pressure variation given by Eq. 2.3.5 with $a_z = 0$. We can now integrate between any two points (r_1, z_1) and (r_2, z_2) to obtain

$$p_2 - p_1 = \frac{\rho\omega^2}{2}(r_2^2 - r_1^2) - \rho g(z_2 - z_1) \tag{2.6.4}$$

If the two points are on a constant-pressure surface, such as the free surface, locating point 1 on the z-axis so that $r_1 = 0$, there results

$$\frac{\omega^2 r_2^2}{2} = g(z_2 - z_1) \tag{2.6.5}$$

KEY CONCEPT *The free surface is a paraboloid of revolution.*

which is the equation of a parabola. Hence the free surface is a paraboloid of revolution. The equations above can now, with the conservation of mass, be used to solve problems of interest.