Example 5.4

The continuity equation can be used to change the form of an expression. Write the expression $\rho\, D\tilde{u}/Dt + p\, \nabla \cdot \mathbf{V}$, which appears in the differential energy equation, in terms of enthalpy h rather than internal energy $\tilde{u}$. Recall that $h = \tilde{u} + p/\rho$ (see Eq. 1.7.11).

Solution

Using the definition of enthalpy, we can write

$$\frac{D\tilde{u}}{Dt} = \frac{Dh}{Dt} - \frac{1}{\rho}\frac{Dp}{Dt} + \frac{p}{\rho^2}\frac{D\rho}{Dt}$$

where we used

$$\frac{D}{Dt}\left(\frac{p}{\rho}\right) = \frac{1}{\rho}\frac{Dp}{Dt} - \frac{p}{\rho^2}\frac{D\rho}{Dt}$$

The desired expression is then

$$\rho\frac{D\tilde{u}}{Dt} + p\, \nabla \cdot \mathbf{V} = \rho\frac{Dh}{Dt} - \frac{Dp}{Dt} + \frac{p}{\rho}\frac{D\rho}{Dt} + p\, \nabla \cdot \mathbf{V}$$

The continuity equation (5.2.7) is introduced resulting in

$$\begin{aligned}\rho\frac{D\tilde{u}}{Dt} + p\, \nabla \cdot \mathbf{V} &= \rho\frac{Dh}{Dt} - \frac{Dp}{Dt} + \frac{p}{\rho}\frac{Dp}{Dt} + p\left(-\frac{1}{\rho}\frac{D\rho}{Dt}\right) \\ &= \rho\frac{Dh}{Dt} - \frac{Dp}{Dt}\end{aligned}$$

and enthalpy has been introduced.

5.3 DIFFERENTIAL MOMENTUM EQUATION

5.3.1 General Formulation

Suppose that we do not know the velocity field or the pressure field in an incompressible[3] flow of interest and we wish to solve differential equations to provide us with that information. The differential continuity equation is one differential equation to help us toward this end; however, it has three unknowns, the three velocity components. The differential momentum equation is a vector equation and thus provides us with three scalar equations. These component equations will aid us in our attempt to determine the velocity and pressure fields. There is a difficulty in deriving these equations, however, since we must use the stress components to determine the forces required in the momentum equation. Let us identify these stress components.

There are nine stress components that act at a particular point in a fluid flow. They are the nine components of the stress tensor τ_{ij}. We will not study the properties of a stress tensor in detail in this study of fluid mechanics since we do not have to maximize or minimize the stress (as would be required in a solid

[3]An incompressible flow, when referred to in a general discussion such as in this section, will generally refer to a constant-density flow. This is true in most fluid mechanics literature, including textbooks on the subject.

mechanics course); we must, however, use the nine stress components in our derivations, then relate the stress components to the velocity and pressure fields with the appropriate equations. The stress components that act at a point are displayed on two- and three-dimensional rectangular elements in Fig. 5.2. These elements are considered to be an exaggerated point, a cubical point; the stress components act in the positive direction on a positive face (a normal vector points in the positive coordinate direction) and in the negative direction on a negative face (a normal vector points in the negative coordinate direction). The first subscript on a stress component denotes the face upon which the component acts, and the second subscript denotes the direction in which it acts; the component τ_{xy} acts in the positive y-direction on a positive x-face and in the negative y-direction on a negative x-face, as displayed in Fig. 5.2a. A stress component that acts perpendicular to a face is referred to as a **normal stress**; the components σ_{xx}, σ_{yy}, and σ_{zz} are normal stresses. A stress component that acts tangential to a face is called a **shear stress**; the components τ_{xy}, τ_{yx}, τ_{xz}, τ_{zx}, τ_{yz}, and τ_{zy} are the shear stress components.

Normal stress: *A stress component that acts perpendicular to an area.*

Shear stress: *A stress component that acts tangential to an area.*

There are nine stress components that act at a particular point in a fluid. To derive the differential momentum equation, consider the forces acting on the infinitesimal fluid particle shown in Fig. 5.3. Only the forces acting on positive faces are shown. The stress components are assumed to be function of x, y, z, and t, and hence the values of the stress components change from face to face since the location of each face is slightly different. The body force is assumed to act in an arbitrary direction.

Newton's second law applied to a fluid particle, for the x-component direction, is $\Sigma F_x = ma_x$. For the particle shown in Fig. 5.3, this takes the form

$$\left(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x}\frac{dx}{2}\right) dy\, dz + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\frac{dy}{2}\right) dx\, dz + \left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\frac{dz}{2}\right) dx\, dy$$
$$-\left(\sigma_{xx} - \frac{\partial \sigma_{xx}}{\partial x}\frac{dx}{2}\right) dy\, dz - \left(\tau_{yx} - \frac{\partial \tau_{yx}}{\partial y}\frac{dy}{2}\right) dx\, dz$$
$$-\left(\tau_{zx} - \frac{\partial \tau_{zx}}{\partial z}\frac{dz}{2}\right) dx\, dy + \rho g_x\, dx\, dy\, dz = \rho\, dx\, dy\, dz \frac{Du}{Dt} \tag{5.3.1}$$

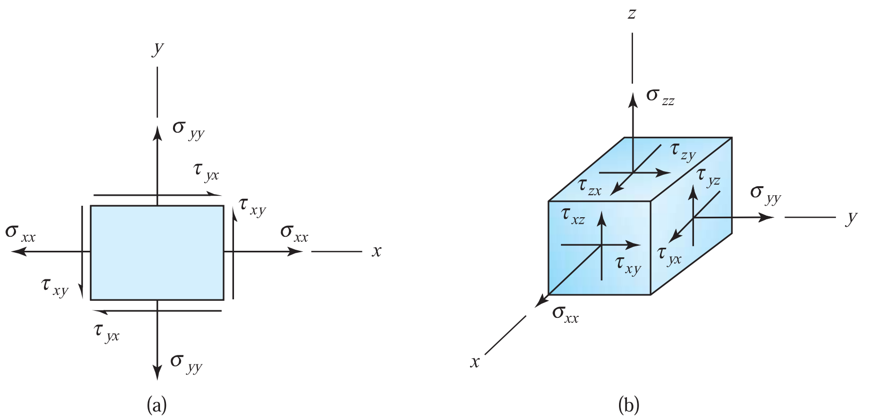

Fig. 5.2 Stress components in Cartesian coordinates: (a) two-dimensional stress components; (b) three-dimensional stress components.

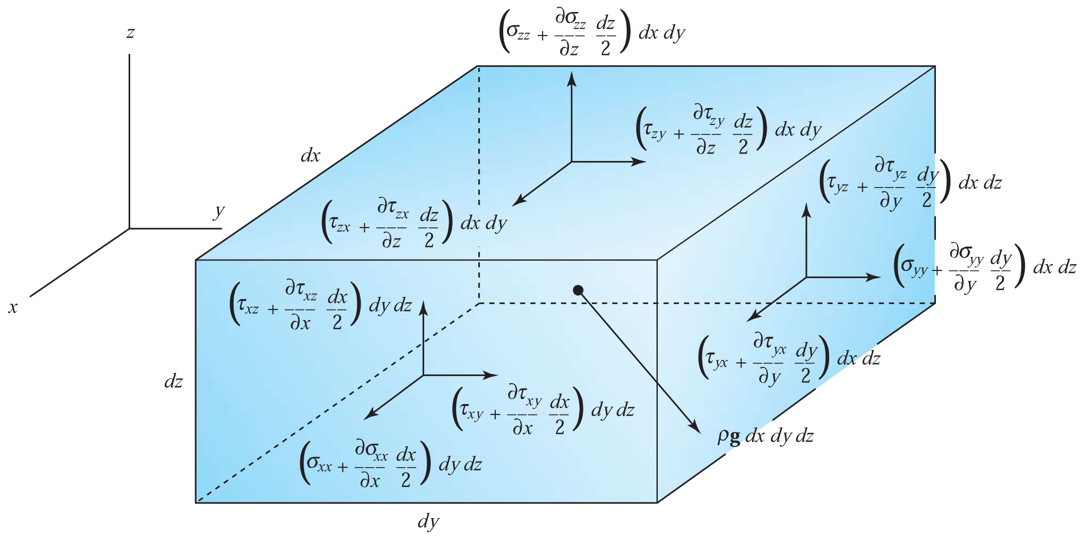

Fig. 5.3 Forces acting on an infinitesimal fluid particle.

where the component of the gravity vector **g** acting in the x-direction is g_x, and Du/Dt is the x-component acceleration of the fluid particle (see Eq. 3.2.9). After we divide by the volume $dx\,dy\,dz$, the equation above can be simplified to

$$\rho \frac{Du}{Dt} = \frac{\sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho g_x \tag{5.3.2}$$

Similarly, for the y- and z-directions we would have

$$\begin{aligned} \rho \frac{Dv}{Dt} &= \frac{\partial \tau_{xy}}{\partial x} + \frac{\sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho g_y \\ \rho \frac{Dw}{Dt} &= \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\sigma_{zz}}{\partial z} + \rho g_z \end{aligned} \tag{5.3.3}$$

We can, by taking moments about axes passing through the center of the infinitesimal element, show that

$$\tau_{yx} = \tau_{xy} \qquad \tau_{yz} = \tau_{zy} \qquad \tau_{xz} = \tau_{zx} \tag{5.3.4}$$

That is, the stress tensor is symmetric; so there are actually six independent stress components.

The stress tensor may be displayed in the usual way as

$$\tau_{ij} = \begin{pmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{pmatrix} \tag{5.3.5}$$

The subscripts i and j take on numerical values 1, 2, or 3. Then τ_{12} represents the element τ_{xy} in the first row, second column.

5.3.2 Euler's Equations

Good approximations to the components of the stress tensor for many flows, especially for flow away from a boundary (flow around an airfoil) or in regions of sudden change (flow through a contraction) are displayed by the array

$$\tau_{ij} = \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix} \tag{5.3.6}$$

For such flows, we have assumed the shear stress components that result from viscous effects to be negligibly small and the normal stress components to be equal to the negative of the pressure; this is precisely what we did in Fig. 3.16 when deriving Bernoulli's equation. If these stress components are introduced back into Eqs. 5.3.2 and 5.3.3 there results, for this frictionless flow,

KEY CONCEPT *We often assume shear stress components to be negligibly small.*

$$\begin{aligned} \rho \frac{Du}{Dt} &= -\frac{\partial p}{\partial x} + \rho g_x \\ \rho \frac{Dv}{Dt} &= -\frac{\partial p}{\partial y} + \rho g_y \\ \rho \frac{Dw}{Dt} &= -\frac{\partial p}{\partial z} + \rho g_z \end{aligned} \tag{5.3.7}$$

The scalar equations above can then be written as the vector equation

$$\rho \frac{D}{Dt}(u\hat{\mathbf{i}} + v\hat{\mathbf{j}} + w\hat{\mathbf{k}}) = -\left(\frac{\partial p}{\partial x}\hat{\mathbf{i}} + \frac{\partial p}{\partial y}\hat{\mathbf{j}} + \frac{\partial p}{\partial z}\hat{\mathbf{k}}\right) - \rho \mathbf{g} \tag{5.3.8}$$

In vector form, we have the well-known **Euler's equation**

Euler's equation: *The three differential equations that result from applying Newton's second law and neglecting viscous effects.*

$$\rho \frac{D\mathbf{V}}{Dt} = -\boldsymbol{\nabla} p - \rho \mathbf{g} \tag{5.3.9}$$

If we assume a constant-density, steady flow, Eq. 5.3.9 can be integrated along a streamline to yield Bernoulli's equation, a result that does not surprise us since the same assumptions were imposed when deriving Bernoulli's equation in Chapter 3; this will be illustrated in Example 5.6.

With the differential momentum equations in the form of Eqs. 5.3.7, we have added three additional equations to the continuity equation to give four equations and four unknowns, u, v, w, and p. With the appropriate boundary and initial conditions, a solution, yielding the velocity and pressure fields for this inviscid, incompressible flow, would be possible.

Example 5.5

A velocity field is proposed to be

$$u = \frac{10y}{x^2 + y^2} \qquad v = -\frac{10x}{x^2 + y^2} \qquad w = 0$$

(a) Is this a possible incompressible flow? (b) If so, find the pressure gradient ∇p assuming a frictionless air flow with the z-axis vertical. Use $\rho = 1.23$ kg/m^3.

Solution

(a) The continuity equation (5.2.9) is used to determine if the velocity field is possible. For this incompressible flow we have

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \cancelto{0}{\frac{\partial w}{\partial z}} = 0$$

Substituting in the velocity components, we have

$$\frac{\partial}{\partial x}\left(\frac{10y}{x^2 + y^2}\right) + \frac{\partial}{\partial y}\left(-\frac{10x}{x^2 + y^2}\right) = \frac{-10y(2x)}{(x^2 + y^2)^2} - \frac{-10x(2y)}{(x^2 + y^2)^2} = \frac{1}{(x^2 + y^2)^2}[-20xy + 20xy] = 0$$

The quantity in brackets is obviously zero; hence the velocity field given is a possible incompressible flow.

(b) The pressure gradient is found using Euler's equation. In component form we have the following:

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \cancelto{0}{\rho g_x}$$

$$\therefore \frac{\partial p}{\partial x} = -\rho\left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\cancelto{0}{\frac{\partial u}{\partial z}} + \cancelto{0}{\frac{\partial u}{\partial t}}\right]$$

$$= -1.23\left[\frac{10y}{x^2 + y^2}\,\frac{-20xy}{(x^2 + y^2)^2} + \frac{-10x}{x^2 + y^2}\,\frac{(x^2 + y^2)10 - 10y(2y)}{(x^2 + y^2)^2}\right]$$

$$= \frac{123x}{(x^2 + y^2)^2}$$

$$\rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \cancelto{0}{\rho g_y}$$

$$\therefore \frac{\partial p}{\partial y} = -\rho\left[u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\cancelto{0}{\frac{\partial v}{\partial z}} + \cancelto{0}{\frac{\partial v}{\partial t}}\right]$$

$$= -1.23\left[\frac{10y}{x^2 + y^2}\,\frac{(x^2 + y^2)(-10) + 10x(2x)}{(x^2 + y^2)^2} + \frac{-10x}{x^2 + y^2}\,\frac{20xy}{(x^2 + y^2)^2}\right]$$

$$= \frac{123y}{(x^2 + y^2)^2}$$

$$\rho \cancelto{0}{\frac{Dw}{Dt}} = -\frac{\partial p}{\partial z} + \rho g_z$$

$$\therefore \frac{\partial p}{\partial z} = \rho g_z = 1.23 \text{ kg/m}^3 \times (-9.81) \text{ m/s}^2 = -12.07 \text{ N/m}^3$$

Thus $\quad \nabla p = \dfrac{\partial p}{\partial x}\hat{\mathbf{i}} + \dfrac{\partial p}{\partial y}\hat{\mathbf{j}} + \dfrac{\partial p}{\partial z}\hat{\mathbf{k}} = \dfrac{123}{(x^2 + y^2)^2}(x\hat{\mathbf{i}} + y\hat{\mathbf{j}}) - 12.07\hat{\mathbf{k}} \text{ N/m}^3$

Example 5.6

Assume a steady, constant-density flow and integrate Euler's equation along a streamline in a plane flow.

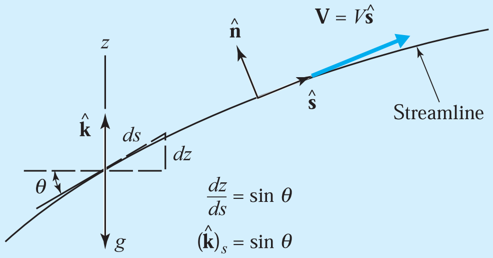

Fig. E5.6

Solution

First, let us express the substantial derivative in streamline coordinates. Since the velocity vector is tangent to the streamline, we can write

$$\mathbf{V} = V\hat{\mathbf{s}}$$

where $\hat{\mathbf{s}}$ is the unit vector tangent to the streamline and V is the magnitude of the velocity, as shown in Fig. E5.6. The substantial derivative is then, for this plane flow,

$$\frac{D\mathbf{V}}{Dt} = \frac{\partial \mathbf{V}}{\partial t} + V\frac{\partial (V\hat{\mathbf{s}})}{\partial s} + \overset{0}{\cancel{(V)_n}}\frac{\partial \mathbf{V}}{\partial n} = \frac{\partial \mathbf{V}}{\partial t} + V\frac{\partial V}{\partial s}\hat{\mathbf{s}} + V^2\frac{\partial \hat{\mathbf{s}}}{\partial s}$$

The quantity $\partial\hat{\mathbf{s}}/\partial s$ results from the change of the unit vector $\hat{\mathbf{s}}$; the unit vector cannot change magnitude (it must always have a magnitude of 1), it can only change direction. Hence the derivative $\partial\hat{\mathbf{s}}/\partial s$ is in a direction normal to the streamline and does not enter the streamwise component equation. For a steady flow $\partial\mathbf{V}/\partial t = 0$. Consequently, in the streamwise direction, Euler's equation (5.3.9) takes the form

$$\rho V\frac{\partial V}{\partial s} = -\frac{\partial p}{\partial s} - \rho g\frac{\partial z}{\partial s}$$

recognizing that the component of $\hat{\mathbf{k}}$ along the streamline can be expressed as $(\hat{\mathbf{k}})_s = \partial z/\partial s$ (see the sketch above). Note that we use partial derivatives in this equation since velocity and pressure also vary with the normal coordinate.

The equation above can be written, assuming constant density so that $\partial\rho/\partial s = 0$, as

$$\frac{\partial}{\partial s}\left(\rho\frac{V^2}{2} + p + \rho g z\right) = 0$$

Integrating along the streamline results in

$$\rho\frac{V^2}{2} + p + \rho g z = \text{const.}$$

or

$$\frac{V^2}{2} + \frac{p}{\rho} + gz = \text{const.}$$

This is, of course, Bernoulli's equation. We have integrated along a streamline assuming constant density, steady flow, negligible viscous effects, and an inertial reference frame, so it is to be expected that Bernoulli's equation will emerge.

5.3.3 Navier–Stokes Equations

Newtonian fluids: *Fluids that possess a linear relationship between stress and the velocity gradients.*

Isotropic fluid: *A fluid whose properties are independent of direction at a given position.*

Many fluids exhibit a linear relationship between the stress components and the velocity gradients. Such fluids are called **Newtonian fluids** and include common fluids such as water, oil, and air. If in addition to linearity, we require that the fluid be **isotropic**,[4] it is possible to relate the stress components and the velocity gradients using only two fluid properties, the *viscosity* μ and the *second coefficient of viscosity* λ. The stress–velocity gradient relations, often referred to as the *constitutive equations*,[5] are stated as follows:

$$\begin{aligned} \sigma_{xx} &= -p + 2\mu\frac{\partial u}{\partial x} + \lambda\,\boldsymbol{\nabla}\cdot\mathbf{V} & \tau_{xy} &= \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ \sigma_{yy} &= -p + 2\mu\frac{\partial v}{\partial y} + \lambda\,\boldsymbol{\nabla}\cdot\mathbf{V} & \tau_{xz} &= \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \sigma_{zz} &= -p + 2\mu\frac{\partial w}{\partial z} + \lambda\,\boldsymbol{\nabla}\cdot\mathbf{V} & \tau_{yz} &= \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \end{aligned} \tag{5.3.10}$$

For most gases, and for monatomic gases exactly, the second coefficient of viscosity is related to the viscosity by

$$\lambda = -\frac{2}{3}\mu \tag{5.3.11}$$

a condition that is known as *Stokes's hypothesis*. With this relationship the negative average of the three normal stresses is equal to the pressure; that is,

$$-\frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) = p \tag{5.3.12}$$

Using Eqs. 5.3.10, this can be shown to always be true for a liquid in which $\boldsymbol{\nabla}\cdot\mathbf{V} = 0$, and with Stokes's hypothesis it is also true for a gas.

If we substitute the constitutive equations into the differential momentum equations (5.3.2) and (5.3.3), there results, using Stokes's hypothesis,

$$\begin{aligned} \rho\frac{Du}{Dt} &= -\frac{\partial p}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \frac{\mu}{3}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \\ \rho\frac{Dv}{Dt} &= -\frac{\partial p}{\partial y} + \rho g_y + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) + \frac{\mu}{3}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \\ \rho\frac{Dw}{Dt} &= -\frac{\partial p}{\partial z} + \rho g_z + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) + \frac{\mu}{3}\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \end{aligned} \tag{5.3.13}$$

Homogeneous fluid: *A fluid whose properties are independent of position.*

where we have assumed a **homogeneous fluid**, that is, fluid properties (e.g., the viscosity) are independent of position.

For an incompressible flow the continuity equation allows the equations above to be reduced to

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[4]The condition of isotropy exists if the fluid properties are independent of direction. Polymers are examples of anisotropic fluids.

[5]Details of the development of the constitutive equations can be found in any textbook on the subject of continuum mechanics.

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$
$$\rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \rho g_y + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \quad (5.3.14)$$
$$\rho \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \rho g_z + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

These are the **Navier–Stokes equations**, named after Louis M. H. Navier (1785–1836) and George Stokes (1819–1903); with these three differential equations and the differential continuity equation we have four equations and four unknowns, u, v, w, and p. The viscosity and density are fluid properties that are assumed to be known. With the appropriate boundary and initial conditions the equations can hopefully be solved. Several relatively simple geometries allow for analytical solutions; some of the solutions are presented in Chapter 7. Numerical solutions have also been determined for many flows of interest; computational methods are presented in Chapter 14. Because the equations are nonlinear partial differential equations (the acceleration terms cause the equations to be nonlinear as observed in Eqs. 3.2.9), we cannot be assured that the solution we find will actually be realized in the laboratory; that is, the solutions are not unique. For example, a laminar flow and a turbulent flow may have the identical initial and boundary conditions, yet the two flows (the two solutions) are very different.

Navier–Stokes equations: *The three differential equations that result from applying Newton's second law to an incompressible, isotropic, homogeneous fluid.*

The Navier–Stokes equations have not been solved for a turbulent flow. All turbulent flows are unsteady and three-dimensional and hence the time-derivative terms must be retained. This requires an initial condition on all dependent variables; i.e., u, v, w, and p must be known at all points in the flow field at $t = 0$. Such information would be extremely difficult, if not impossible, to obtain. To avoid this situation, time-averaged quantities are introduced for turbulent flows. This subject will be studied in a later chapter.

We can express the Navier–Stokes equations in vector form by multiplying Eqs. 5.3.14 by $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$, respectively, and adding. We recognize that

$$\frac{Du}{Dt}\hat{\mathbf{i}} + \frac{Dv}{Dt}\hat{\mathbf{j}} + \frac{Dw}{Dt}\hat{\mathbf{k}} = \frac{D\mathbf{V}}{Dt}$$
$$\frac{\partial p}{\partial x}\hat{\mathbf{i}} + \frac{\partial p}{\partial y}\hat{\mathbf{j}} + \frac{\partial p}{\partial z}\hat{\mathbf{k}} = \boldsymbol{\nabla} p \quad (5.3.15)$$
$$\nabla^2 u\hat{\mathbf{i}} + \nabla^2 v\hat{\mathbf{j}} + \nabla^2 w\hat{\mathbf{k}} = \nabla^2 \mathbf{V}$$

where we have introduced the Laplacian

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad (5.3.16)$$

Combining the above, the Navier–Stokes equations (5.3.14) take the vector form

$$\rho \frac{D\mathbf{V}}{Dt} = -\boldsymbol{\nabla} p + \rho \mathbf{g} + \mu \nabla^2 \mathbf{V} \quad (5.3.17)$$