Linear Stochastic Thermodynamics
Local Detailed Balance

$$\frac{W_{mn}(t)}{W_{nm}(t)} = e^{-\beta(E_m - E_n) + \beta\mu(n_m - n_n)}$$

Justification

Equilibrium Regime

$$W_{mn} P_n^e = W_{nm} P_m^e$$

$$\frac{W_{mn}}{W_{nm}} = \frac{P_m^e}{P_n^e} = \frac{e^{-\beta(E_m - \beta\mu n_m)}/Z}{e^{-\beta(E_n - \beta\mu n_n)}/Z}$$

$$= e^{-\beta(E_m - E_n) + \beta\mu(n_m - n_n)}$$

Close at the equilibrium regime, the transition rates is supposed to be similar

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$$W_{mn}(t) = C_{mn}(t)\, e^{\frac{\epsilon_n - \mu n_n}{T}}$$

Coupling between states $\underline{\underline{m}}$ and $\underline{\underline{n}}$

in such a way that

$$C_{mn}^{(t)} = C_{nm}^{(t)} \quad \text{for } m \neq n$$

$$C_{nn} = -\sum_{m \neq n} C_{mn}$$

Linear Description

$$\epsilon_n(t) = \epsilon_{0,n} + T_0 F_\epsilon g_\epsilon(t) \delta_{\epsilon,n}$$

$$\frac{1}{T(t)} = \frac{1}{T_0} + F_T g_T(t)$$

$$\frac{\mu(t)}{T_0} = \frac{\mu_0}{T_0} + F_\mu g_\mu(t)$$

$g_\epsilon(t)$, $g_T(t)$ $g_\mu(t)$ $\longrightarrow$ driving function (assumed here to be periodic)

F_ϵ, F_T and F_μ $\to$ strength of drivings that are supposed to be $<< 1$

$T_0 \to$ reference temperature

$\delta_{\epsilon,n} \to$ associated to the energy level

Let us assume that $W(t)$ can be expanded, up to first-order, in the thermodynamic forces.

$$W(t) = \left[C_{mn}^{(eq)} + \sum_\alpha F_\alpha C_{mn}^{\alpha} g_\alpha(t) \right] \left[\underbrace{\quad\quad}_{*} \right.$$

$\uparrow \alpha = \{\epsilon, T, \mu\}$

$$+ \frac{\epsilon_n(t)}{T(t)} = \left[\epsilon_{0,n} + T_0 F_\epsilon g_\epsilon(t) \delta_{\epsilon,n} \right] \left[\frac{1}{T_0} + F_T g_T(t) \right]$$

$$= \left[\frac{\epsilon_{0,n}}{T_0} + \epsilon_{0,n} F_T g_T(t) + F_\epsilon g_\epsilon(t) \delta_{\epsilon,n} \right]$$

$$- \frac{\mu(t) n_n}{T(t)} = - \left[\mu_0 + T_0 F_\mu g_\mu(t) \right] \left[\frac{1}{T_0} + F_T g_T(t) \right] n_n$$

$$= - \left[\frac{\mu_0}{T_0} + F_\mu g_\mu(t) + \mu_0 F_T g_T(t) \right] n_n$$

$$= C_{mn}(t)\, e^{\frac{\epsilon_{0,n} - \mu_0 n_n}{T_0}} \left[e^{(\epsilon_{0,n} - \mu_0 n_n) F_T g_T(t) + F_\epsilon g_\epsilon(t) \gamma_{\epsilon,n} - F_\mu g_\mu(t) n_n} \right]$$

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$$= e^{\frac{\epsilon_{0,n} - \mu_0 n_n}{T_0}} \left[C_{mn}^{(eq)} + \sum_\alpha F_\alpha C_{mn}^\alpha g_\alpha(t) \right] \left[1 + \sum_\alpha F_\alpha g_\alpha(t) \gamma_n^\alpha \right]$$

$$\gamma_n^\alpha = \begin{cases} \epsilon_{0,n} - \mu_0 n_n & \text{if } \alpha = T \\ + \gamma_{\epsilon,n} & \text{if } \alpha = \epsilon \\ - n_n & \text{if } \alpha = \mu \end{cases}$$

$$= \underbrace{e^{\frac{\epsilon_{0,n} - \mu_0 n_n}{T_0}}}_{\tilde{W}_n^{eq}} \left\{ C_{mn}^{(eq)} + \sum_\alpha F_\alpha g_\alpha(t) \left[C_{mn}^\alpha + C_{mn}^{(eq)} \gamma_n^\alpha \right] \right\}$$

$$= W_{mn}^{(eq)} + \sum_\alpha F_\alpha \underbrace{g_\alpha(t) \left[W_{mn}^{(eq)} \gamma_n^\alpha + \tilde{W}_n^{eq} C_{mn}^\alpha \right]}_{W_{mn}^\alpha(t)}$$

Let us assume that the probability distribution can be expanded up to linear order in terms of thermodynamic forces

$$p(t) = p^{eq} + \sum_\alpha F_\alpha p^\alpha(t)$$

$$\dot{p}(t) = \sum_\alpha F_\alpha \dot{p}^\alpha(t)$$

Important: We are considering the case of periodic drivings, so that $\frac{1}{T} \int_0^T g_\alpha(t) = 0$ and $g_\alpha(t) = g_\alpha(t+T)$

Since $\dot{P}_m = \sum_n W_{mn}(t) P_n(t)$, we compare the terms: ⑤

$$\sum_\alpha F_\alpha \dot{P}^\alpha(t) = \sum_n \Big[W_{mn}^{(eq)} + \sum_\alpha F_\alpha g_\alpha(t) W_{mn}^\alpha(t) \Big] P_n(t)$$

$$= \sum_n \Big[W_{mn}^{(eq)} + \sum_\alpha F_\alpha g_\alpha(t) W_{mn}^\alpha(t) \Big] \Big[P_n^{eq} + \sum_\alpha F_\alpha P_n^\alpha(t) \Big]$$

(steady state solution)

$$\sum_\alpha F_\alpha \dot{P}_n^\alpha(t) = \cancel{\sum_n W_{mn}^{(eq)} P_n^{(eq)}} + \sum_\alpha F_\alpha \left(W_{mn}^{(eq)} P_n^\alpha(t) + g_\alpha(t) W_{mn}^\alpha(t) \cdot P_n^{eq} \right)$$

$$\boxed{\dot{P}^\alpha(t) = W^{(eq)} P^\alpha(t) + g_\alpha(t) W^\alpha(t) P^{eq}}$$

$$\dot{P}^\alpha(t) - W^{(eq)} P^\alpha(t) = g_\alpha(t) W^\alpha(t) P^{eq}$$

$$P^\alpha(t) = e^{W^{(eq)}(t-\tau)} \int_{-\infty}^{t} e^{-W^{(eq)}(t'-\tau)} W^\alpha(t') P^{eq} dt'$$

$$= \int_{-\infty}^{t} e^{W^{(eq)}(t-t')} W^\alpha(t') P^{eq} g_\alpha(t') dt'$$

By introducing the new variables

$$\tau = t - t'$$

$$d\tau = -dt'$$

$$P^\alpha(t) = -\int_\infty^0 e^{W^{(eq)}\tau} W^\alpha(t-\tau) P^{eq} d\tau \quad (*P)$$

$$= \int_0^\infty e^{W^{(eq)}\tau} W^\alpha(t-\tau) g_\alpha(t-\tau) P^{eq} d\tau .$$

Analogously

$$\dot{p}^{\alpha}(t) = \int_0^{\infty} e^{W^{(eq)}\tau}\, \dot{w}^{\alpha}(t-\tau)\, p^{eq}\, d\tau$$

(From the property discussed in page 8-5), one can write $p^{\alpha}(t)$ as

$$p^{\alpha}_{m}(t) = \sum_{n,k} \int_0^{\infty} \left(e^{W^{(eq)}\tau}\right)_{mn} \left(W^{(eq)}\right)_{nk} \gamma^{\alpha}_{k} \, p^{eq}_{K}\, g_{\alpha}(t-\tau)\, d\tau$$

Hence $\Rightarrow$ the first-order probability distribution contribution can be obtained exactly in terms of equilibrium properties, the driving functions and the γ_{α}'s.

Since the driving is periodic, from (*P) we see that it is also periodic.

Stochastic Thermodynamics

$$\dot{W}(t) = \sum_m \dot{\epsilon}_m P_m(t)$$

$$\dot{W}_{cHem}(t) = \sum_m \mu\, n_m \dot{P}_m(t)$$

$$\dot{Q}(t) = \sum_m (\epsilon_m - \mu n_m)\, \dot{P}_m(t)$$

First law of Thermodynamics

$$\dot{u}(t) = \dot{W}(t) + \dot{W}_{CHem}(t) + \dot{Q}(t).$$

$$\int_0^T \dot{u}(t)\, dt = u(T) - u(0) = 0$$

(periodic driving)

$$\int_0^T \dot{W}(t)\, dt + \int_0^T \dot{W}_{CHem}(t)\, dt + \int_0^T \dot{Q}(t)\, dt = 0$$

$$\dot{W}(t) = \sum_m \dot{\epsilon}_m P_m(t) = \sum_m T_0 F_{\epsilon} \dot{g}_{\epsilon}(t)\, \gamma_{\epsilon, m} P_m(t)$$

$$\bar{\dot{W}} = \frac{T_0 F_{\epsilon}}{T} \int_0^T dt\, \dot{g}_{\epsilon}(t) \sum_m \gamma_{\epsilon m} P_m(t)$$

By performing a partial integration, we have that

probability is (periodic)

$$\overline{\dot{W}} = \cancel{\frac{T_0 F_\epsilon}{T}\, g_\epsilon(t) \Big|_0^T \sum_m \gamma_{\epsilon,m} P_m(t)} \to 0$$

$$- \frac{T_0 F_\epsilon}{T} \underbrace{\int_0^T dt\, g_\epsilon(t) \sum_m \gamma_{\epsilon,m} \dot{P}_m(t)}_{-J_\epsilon} \qquad (\propto W)$$

$$\boxed{\overline{\dot{W}} = T_0 F_\epsilon J_\epsilon} \quad //$$

Chemical Work

$$\overline{\dot{W}}_{CHEM} = \frac{1}{T}\int_0^T \sum_m \left[\mu_0 + T_0 F_\mu g_\mu(t)\right] n_m \dot{P}_m(t)\, dt$$

probability is periodic

$$= \cancel{\frac{\mu_0}{T} \sum_m \int_0^T n_m \dot{P}_m(t)\, dt} \to 0 + T_0 F_\mu \frac{1}{T} \int_0^T \sum_m \overbrace{n_m}^{-\gamma_\mu^\alpha} \dot{P}_m(t)\, dt,$$

$$(\propto W_c).$$

$$= T_0 F_\mu J_\mu$$

Heat

$$\dot{Q} = -\dot{W} - \dot{W}_{CHEM} \qquad (*Q)$$
$$= -T_0 \left(F_\mu J_\mu + F_\epsilon J_\epsilon \right)$$

Entropy Production ⇒ From the

Schnakenberg expression:

$$\overline{\Pi}_i = \frac{1}{T} \sum_{m,n} \int_0^T dt \, W_{mn} P_n \ln \frac{W_{mn} P_n}{W_{nm} P_m}$$

We see that $\overline{\Pi}$ can be splitted in two parts:

$$\overline{\Pi} = \frac{1}{T} \sum_{mn} \int_0^T dt \, W_{mn} P_n \ln \frac{W_{mn}}{W_{nm}} + \underbrace{\frac{1}{T} \int_0^T dt \sum_{mn} W_{mn} P_n \ln \frac{P_n}{P_m}}_{(*)}$$

$$(*) \; \frac{1}{T} \int_0^T dt \, \cancel{\sum_{mn} W_{mn} P_n \ln P_n} \to 0 + \frac{1}{T} \int_0^T dt \, \underbrace{\sum_{mn} W_{mn} P_n}_{\sum_m \sum_n W_{mn} P_n \to \dot{P}_m} \ln P_m$$

since $\sum_m W_{mn} = 0$

$$= \frac{1}{T} \int_0^T dt \, \cancel{\sum_m \dot{P}_m \ln P_m} \to 0$$

since the probability is periodic.

$$\doteq \frac{1}{T}\int_0^T dt \sum_{m,n} W_{mn} P_n \ln \frac{W_{mn}}{W_{nm}}$$

$$\ln \frac{W_{mn}}{W_{nm}} = + \left[\frac{(\epsilon_n - \mu n_n)}{T} - \frac{(\epsilon_m - \mu n_m)}{T} \right]$$

$$= \frac{1}{T}\int_0^T \sum_{mn} W_{mn} P_n \left[\underbrace{\cancel{\frac{(\epsilon_n - \mu n_n)}{T}}}_{0 \text{ since } \sum_m W_{mn} = 0} - \frac{(\epsilon_m - \mu n_m)}{T} \right]$$

$$= -\frac{1}{T}\int_0^T \sum_m \dot{P}_m \frac{(\epsilon_m - \mu n_m)}{T} = \underbrace{-\frac{1}{T}\int_0^T dt \frac{\dot{Q}(t)}{T(t)}}$$

classical expression for the entropy production. $\left(\Delta S = \frac{\Delta Q}{T} \right)$

By using the relation:

$$\frac{1}{T(t)} = \frac{1}{T_0} + f_T g_T(t)$$

above equation can be splitted in two parts:

$$= -\frac{1}{T}\int_0^T \sum_m \dot{P}_m \frac{(\epsilon_m - \mu n_m)}{T_0}\, dt +$$

$$-\frac{1}{T}\int_0^T \sum_m F_T\, g_T(t)\, \dot{P}_m\, (\epsilon_m - \mu n_m)\, dt$$

The first term is just $F_\epsilon J_\epsilon + F_\mu J_\mu$ (see eq. (*Q)), whereas for the second term one can replace $\epsilon_m - \mu n_m \simeq \epsilon_{0,m} - \mu_0 n_m$ since the linear contribution has already been considered in $F_T\, g_T(t)$.

So that: $-\frac{F_T}{T}\int_0^T g_T(t)\, dt \sum_m (\epsilon_{0,m} - \mu_0 n_m)\, \dot{P}_m$

By comparing with eqs. (*W) and (*W$_c$), we see that above equation has the same structure, from which one write it as

$F_T \, J_T$ where

$$J_T = -\frac{1}{T}\int_0^T g_T(t)\,dt \sum_m \left(\epsilon_{0,m} - \mu_0 \, n_m \right) P_m^{\circ}(t)$$

and therefore

$$\overline{\Pi} = \underbrace{J_T F_T + J_\epsilon F_\epsilon + J_\mu F_\mu}_{\text{"tri-linear" form}} = \sum_\alpha F_\alpha J_\alpha \,,$$

where

$$J_T = -\frac{1}{T}\int_0^T g_T(t)\,dt \sum_m \left(\epsilon_{0m} - \mu_0 \, n_m \right) P_m^{\circ}(t)$$

$$J_\epsilon = -\frac{1}{T}\int_0^T g_\epsilon(t)\,dt \sum_m \gamma_{\epsilon,m} \, P_m^{\circ}(t)$$

$$J_\mu = \frac{1}{T}\int_0^T g_\mu(t)\,dt \sum_m n_m \, P_m^{\circ}(t) .$$

or generically

$$J_\alpha = -\frac{1}{T}\int_0^T g_\alpha(t)\,dt \sum_m \gamma_{\alpha,m} \, P_m^{\circ}(t)$$

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With
$$\gamma_{\alpha,m} = \begin{cases} f_{0,m} - \mu_0 n_m & \text{if } \alpha = T \\ -n_m & \text{if } \alpha = \mu \\ \gamma_{\epsilon,m} & \text{if } \alpha = \epsilon \end{cases}$$

$$\overline{\Pi} = \sum_{\alpha} J_{\alpha} F_{\alpha}$$

This expression for the mean entropy production has the same form than those discussed by mean of entropic arguments (a la Callen).

Onsager coefficients

Given the current (flux) $J_{\alpha} = -\frac{1}{\tau}\int_0^{\tau} g_{\alpha}(t)\,dt \sum_m \gamma_{\alpha,m} \dot{P}_m(t)$

and $\dot{P}_m(t) = \sum_n W_{mn} P_n(t)$, we intend to show that J_α has the form that depends on thermodynamic forces. For instance

By replacing expressions for W_{mn} and $P_n(t)$ we have $\dot{P}_m(t)$ reads:

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$$\dot{P}_m(t) = \sum_n \left[W^{(eq)}_{mn} + \sum_{\alpha'} F_{\alpha'} g_{\alpha'}(t) W^{\alpha'}_{mn}(t) \right] \left[P^{eq}_n + \sum_\beta F_\beta P^\beta_n \right]$$

By considering only the linear terms

$$= \cancel{\sum_n W^{(eq)}_{mn} P^{eq}_n}^{\,0} + \underbrace{\sum_n \sum_{\alpha'} F_{\alpha'} g_{\alpha'}(t) W^{\alpha'}_{mn}(t) P^{eq}_n}_{1} +$$

$$\underbrace{\sum_\beta F_\beta \sum_n W^{(eq)}_{mn}}_{2} P^\beta_n(t)$$

$$\left(\text{where } P^\beta_n(t) = \int_0^\infty e^{W^{(eq)}\tau} W^\beta(t-\tau) P^{eq}_n \, d\tau \right)$$

$$\text{or } \sum_{n,k} \int_0^\infty [e^{W^{(eq)}\tau}]_{mn} [W^{eq}_{nk} \gamma^\beta_k] P^{eq}_k g_\beta(t-\tau) \, d\tau$$

Hence $J_\alpha = J^{(1)}_\alpha + J^{(2)}_\alpha$ where

$$J^{(1)}_\alpha = \sum_{\alpha'} F_{\alpha'} \left[-\frac{1}{T} \int_0^T g_\alpha(t) g_{\alpha'}(t) \sum_{n} \sum_{m} \gamma_{\alpha,m} W^{\alpha'}_{mn}(t) P^{eq}_n \right.$$

above expression can be simplified by noting that:

$$\sum_m \sum_n W^{\alpha'}_{mn}(t)\, p^{eq}_n = \sum_m \sum_n \left[W^{(eq)}_{mn}\, \gamma^{\alpha'}_n + \tilde{W}^{eq}_n\, C^{\alpha'}_{mn} \right] p^{eq}_n$$

But $\sum_n \tilde{W}^{eq}_n\, C^{\alpha'}_{mn}\, p^{eq}_n =$

$$= \sum_{n \neq m} C^{\alpha'}_{mn}\, \tilde{W}^{eq}_n\, p^{eq}_n + C^{\alpha'}_{mm}\, \tilde{W}^{eq}_m\, p^{eq}_m$$

By using the property $\sum_m C^{\alpha'}_{mn}\, \tilde{W}^{eq}_n = 0.$

$$C^{\alpha'}_{mm}\, \tilde{W}^{eq}_m = -\sum_{n;\, n \neq m} C^{\alpha'}_{nm}\, \tilde{W}^{eq}_m .$$

So that

$$\sum_n C^{\alpha'}_{mn}\, \tilde{W}^{eq}_n\, p^{eq}_n = \sum_{n \neq m} \left[C^{\alpha'}_{mn}\, \tilde{W}^{eq}_n\, p^{eq}_n - C^{\alpha'}_{\cancel{mn}\, nm}\, \tilde{W}^{eq}_{\cancel{n}\, m}\, p^{eq}_m \right]$$

0

due to the local detailed balance.

Hence (9)

$$\sum_m \sum_n W^{\alpha'}_{mn}(t)\, P^{eq}_n = \sum_m \sum_n W^{(eq)}_{mn}\, \gamma^{\alpha'}_n\, P^{eq}_n \quad \text{and}$$

$$J^{(1)}_\alpha = \sum_{\alpha'} F_{\alpha'} \underbrace{\left[-\frac{1}{T}\int_0^T g_\alpha(t)\, g_{\alpha'}(t)\, dt \sum_n \sum_m \gamma_{\alpha,m} W^{(eq)}_{mn}\, \gamma^{\alpha'}_n P^{eq}_n \right]}_{\mathcal{L}_{\alpha,\alpha'}}$$

By defining above $\rightarrow$ $\mathcal{L}_{\alpha,\alpha'}$ $\hookrightarrow$ Onsager coefficient.
as $\mathcal{L}_{\alpha,\alpha'}$ (Onsager coefficient), we have that

$$J^{(1)}_\alpha = \sum_{\alpha'} \mathcal{L}_{\alpha,\alpha'} F_{\alpha'}$$

Now let us work with $J^{(2)}_\alpha$.

$$J^{(2)}_\alpha = -\frac{1}{T}\int_0^T g_\alpha(t)\, dt \sum_m \gamma_{\alpha,m} \sum_\beta F_\beta \sum_n W^{(eq)}_{mn} \int_0^\infty e^{W^{(eq)} z}\, \dot{w}(t-z)\, P^{eq}_n\, dz$$

$$= -\frac{1}{T}\sum_\beta \int_0^T dt\, g_\alpha(t) \int_0^\infty g_\beta(t-z)\, dz \sum_{K,\ell,m,n} \gamma_{\alpha,m} W^{eq}_{mn} \left(e^{W^{eq} z}\right)_{nK} W^{eq}_{K\ell}\, \gamma^\beta_\ell\, P^{eq}_\ell \Big] \times F_\beta \quad (*6)$$

Where we used again the expression

$$P_m^{\beta}(t) = \sum_{n,k} \int_0^{\infty} \left(e^{W^{eq}\tau}\right)_{mn} \left(W^{eq}\right)_{nk} \gamma_k^{\beta} P_k^{eq} g_{\beta}(t-\tau) d\tau$$

$$= \sum_{\beta} L_{\alpha,\beta} F_{\beta}, \text{ where}$$

$$L_{\alpha,\beta} = -\frac{1}{T}\left[\int_0^T dt \int_0^{\infty} d\tau \, g_{\alpha}(t) g_{\beta}(t-\tau) \sum_{k,l,m,n} \gamma_{\alpha,m} \times \right.$$

$$\left. \times W_{mn}^{eq} \left(e^{W^{eq}\tau}\right)_{nk} W_{kl}^{eq} \gamma_l^{\beta} P_l^{eq}\right]$$

The total matrix is the sum of two matrices

$$L_{\alpha,\beta} = L_{\alpha,\beta}^{(1)} + L_{\alpha,\beta}^{(2)}.$$

Next, we shall d we the reciprocal relations for $L_{\alpha,\beta}^{(1)}$ and $L_{\alpha,\beta}^{(2)}$

Reciprocal Relations

("adiabatic contribution" to ("local equilibrium state").

(1)

$$\mathcal{L}_{\alpha,\alpha'} = -\frac{1}{T}\int_0^T g_\alpha(t)\, g_{\alpha'}(t)\, dt \sum_n \sum_m \gamma_{\alpha,m} W^{(eq)}_{mn} \gamma_{\alpha',n} P^{eq}_n$$

$$t \to -t$$

$$= -\frac{1}{T}\int_{+0}^{-T} g_\alpha(-t)\, g_{\alpha'}(-t)\,(-dt) \sum_n \sum_m \gamma_{\alpha,m} W^{(eq)}_{mn} \gamma_{\alpha',n} P^{eq}_n$$

$$= -\frac{1}{T}\int_0^T \tilde{g}_\alpha(t)\, \tilde{g}_{\alpha'}(t)\, dt \sum_n \sum_m \gamma_{\alpha,m} W^{eq}_{mn} \gamma_{\alpha',n} P^{eq}_n$$

Local detailed Balance

$$W^{eq}_{mn} P^{eq}_n = W^{eq}_{nm} P^{eq}_m$$

$$\sum_n \sum_m \gamma_{\alpha,m} W^{eq}_{nm} P^{eq}_m \gamma_{\alpha',n}$$

$$\sum_n \sum_m \gamma_{\alpha',n} W^{eq}_{nm} \gamma_{\alpha,m} P^{eq}_m$$

$$= -\frac{1}{T}\int_0^T \tilde{g}_\alpha(t)\, \tilde{g}_{\alpha'}(t)\, dt \sum_n \sum_m \gamma_{\alpha',n} W^{eq}_{nm} \gamma_{\alpha,m} P^{eq}_m$$

$$= \tilde{\mathcal{L}}^{(1)}_{\alpha',\alpha}$$

$$\mathcal{L}^{(2)}_{\alpha,\beta} = -\frac{1}{T}\int_0^T g_\alpha(t)\,dt \int_0^{\infty} g_\beta(t-\tau)\,d\tau \sum_{k,l,m,n} \gamma_{\alpha,m} W^{(eq)}_{mn} \left(e^{W^{(eq)}\tau}\right)_{nk} \times$$

$$\times W^{(eq)}_{kl}\, \gamma^{\beta}_{l}\, P^{eq}_{l}$$

By taking the transformation $t' = -t+\tau$

$$-t = t' - \tau \qquad \longleftrightarrow \qquad -dt = dt'$$

$$= -\frac{1}{T}\int_{\tau}^{-T+\tau} -g_\alpha(-t'+\tau)\,dt' \int_0^{\infty} g_\beta(-t')\,d\tau \quad \times$$

$$\times \underbrace{\sum_{k,l,m,n} \gamma_{\alpha,m} W^{(eq)}_{mn} \left(e^{W^{(eq)}\tau}\right)_{nk} W^{(eq)}_{kl}\, \gamma^{\beta}_{l}\, P^{eq}_{l}}_{(*)}$$

$$= \frac{1}{T}\int_0^{\infty} d\tau \int_{\tau}^{-T+\tau} g_\alpha(-t'+\tau)\, g_\beta(-t')\,dt' \quad \underbrace{\qquad\qquad}_{(*)}$$

Due to the periodicity of drivings, the integral $\int_{\tau}^{-T+\tau}$ can be shifted by $T-\tau$ reading

$$\int_T^0 \longrightarrow -\int_0^T$$

2)

$$= -\frac{1}{T}\int_0^{\infty} d\tau \int_0^{T} \overbrace{g_{\alpha}(-t'+\tau)}^{\tilde{g}_{\alpha}(t'-\tau)}\ \overbrace{g_{\beta}(-t')}^{\tilde{g}_{\beta}(t')} dt' \ \underbrace{\qquad}_{[\alpha']}$$

By imposing once again the local detailed balance we have that

$$\gamma_{\alpha,m} W^{(eq)}_{mn} \left(e^{W^{(eq)}\tau}\right)_{nk} \underbrace{W^{(eq)}_{k\ell}\, \gamma^{\beta}_{\ell}\, p^{eq}_{\ell}}$$

$$\gamma_{\alpha,m} W^{(eq)}_{mn} \left(e^{W^{(eq)}\tau}\right)_{nk} \quad W^{(eq)}_{\ell k}\, p^{(eq)}_{k}\, \gamma^{\beta}_{\ell}$$

$$\left(e^{W^{(eq)}\tau}\right)_{nk} p^{(eq)}_{k} = \left(e^{W^{(eq)}\tau}\right)_{kn} p^{(eq)}_{n}$$

$$W^{(eq)}_{mn}\, p^{(eq)}_{n} = W^{(eq)}_{nm}\, p^{(eq)}_{m}$$

$$= \sum_{k,\ell,m,n} \gamma^{\beta}_{\ell}\, W^{(eq)}_{\ell k} \left(e^{W^{(eq)}\tau}\right)_{kn} W^{(eq)}_{nm}\, \gamma_{\alpha,m}\, p^{eq}_{m}$$

$$= -\frac{1}{T}\int_0^{\infty} d\tau \int_0^{T} g_{\alpha}(t'-\tau)\, \tilde{g}_{\beta}(t') \sum_{k,\ell,m,n} \gamma^{\beta}_{\ell}\, W^{(eq)}_{\ell k} \left(e^{W^{(eq)}\tau}\right)_{kn} W^{(eq)}_{nm} \times \gamma_{\alpha,m}\, p^{eq}_{m} = \mathcal{L}_{\beta,\alpha}$$