PEF 5737 - Nonlinear dynamics and stability

Direct parametrization of invariant manifolds

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Objectives and references

- To present a big picture on the method of direct parametrization of invariant manifolds;
- Reference
  1. Alessandra Vizzaccaro, Andrea Opreni, Loïc Salles, Atilio Frangi, Cyril Touzé. High order direct parametrisation of invariant manifolds for model order reduction of finite element structures: application to large amplitude vibrations and uncovering of a folding point. Nonlinear Dynamics, 110(1), 521-575, 2022.

Full-order model

- Consider the full-order model (FOM) with N degrees of freedom

$$\boldsymbol{M}\ddot{\boldsymbol{U}} + \boldsymbol{C}\dot{\boldsymbol{U}} + \boldsymbol{K}\boldsymbol{U} + \boldsymbol{G}(\boldsymbol{U},\boldsymbol{U}) + \boldsymbol{H}(\boldsymbol{U},\boldsymbol{U},\boldsymbol{U}) = \boldsymbol{0} \tag{1}$$

  with linear mass, damping and stiffness matrices known. Tensors G_{rs} and H_{rst} are also known.

- Properties of the undamped linear system

$$(-\omega_j^2 \boldsymbol{M} + \boldsymbol{K})\boldsymbol{\Phi_j} = \boldsymbol{0} \tag{2}$$

$$\boldsymbol{\Phi_j}^T \boldsymbol{M} \boldsymbol{\Phi_j} = 1 \tag{3}$$

$$\boldsymbol{\Phi_j}^T \boldsymbol{K} \boldsymbol{\Phi_j} = \omega_j^2 \tag{4}$$

- Nodal displacements and velocities $\boldsymbol{U}, \boldsymbol{V} = \dot{\boldsymbol{U}} \Rightarrow$. Modal displacement and velocities $u_j = \boldsymbol{\Phi_j}^T \boldsymbol{M}\boldsymbol{U}$ and $v_j = \boldsymbol{\Phi_j}^T \boldsymbol{M}\boldsymbol{V}$
- Lightly damped systems, with $\boldsymbol{C} = a_1\boldsymbol{M} + a_2\boldsymbol{K} \Rightarrow$ Vector $\boldsymbol{\Phi_j}$ make $\boldsymbol{C}$ diagonal ($\boldsymbol{\Phi_j}^T \boldsymbol{C}\boldsymbol{\Phi_j} = 2\xi_j\omega_j$)

Full-order model

- Damped eigenvalue problem

$$(\Lambda_j^2 \boldsymbol{M} + \Lambda_j \boldsymbol{C} + \boldsymbol{K})\boldsymbol{\Phi_j} = \boldsymbol{0} \tag{5}$$

$$\Lambda_j = -\xi\omega_j + i\omega_j\sqrt{1-\xi_j^2} \tag{6}$$

$$\bar{\Lambda}_j = -\xi\omega_j - i\omega_j\sqrt{1-\xi_j^2} \tag{7}$$

- First-order FOM

$$\boldsymbol{M}\dot{\boldsymbol{V}} + \boldsymbol{C}\boldsymbol{V} + \boldsymbol{K}\boldsymbol{U} + \boldsymbol{G}(\boldsymbol{U},\boldsymbol{U}) + \boldsymbol{H}(\boldsymbol{U},\boldsymbol{U},\boldsymbol{U}) = \boldsymbol{0} \tag{8}$$

$$\boldsymbol{M}\dot{\boldsymbol{U}} = \boldsymbol{M}\boldsymbol{V} \tag{9}$$

- Since the first-order FOM has dimension $2N$, we sort the eigenvalues as $\Lambda_j = -\xi\omega_j + i\omega_j\sqrt{1-\xi_j^2}$ and $\Lambda_{j+N} = -\xi\omega_j - i\omega_j\sqrt{1-\xi_j^2}$
- Right eigenvectors of the first-order FOM

$$\left(\Lambda_s \begin{bmatrix} \boldsymbol{M} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{M} \end{bmatrix} + \begin{bmatrix} \boldsymbol{C} & \boldsymbol{K} \\ -\boldsymbol{M} & \boldsymbol{0} \end{bmatrix}\right) \mathbb{Y}_s = \boldsymbol{0} \tag{10}$$

$$\mathbb{Y}_s = \begin{Bmatrix} \Phi_s \Lambda_s \\ \Phi_s \end{Bmatrix} \in \mathbb{C} \tag{11}$$

Full-order model

- We sort the right eigenvectors as

$$\mathbb{Y}_j = \begin{Bmatrix} \Phi_j \Lambda_j \\ \Phi_j \end{Bmatrix}, \mathbb{Y}_{j+N} = \begin{Bmatrix} \Phi_j \bar{\Lambda}_j \\ \Phi_j \end{Bmatrix} = \bar{\mathbb{Y}}_j \tag{12}$$

- Left eigenvectors of the first-order FOM

$$\mathbb{X}_s^T \left(\Lambda_s \begin{bmatrix} \boldsymbol{M} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{M} \end{bmatrix} + \begin{bmatrix} \boldsymbol{C} & \boldsymbol{K} \\ -\boldsymbol{M} & \boldsymbol{0} \end{bmatrix} \right) = \boldsymbol{0} \tag{13}$$

$$\tag{14}$$

- The left eigenvectors are sorted as

$$\mathbb{X}_j = \frac{1}{\Lambda_j - \bar{\Lambda}_j} \begin{Bmatrix} \Phi_j \\ -\bar{\Lambda}_j \Phi_j \end{Bmatrix}, \mathbb{X}_{j+N} = \bar{\mathbb{X}}_j \tag{15}$$

- Properties

$$\mathbb{X}_r^T \begin{bmatrix} \boldsymbol{M} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{M} \end{bmatrix} \mathbb{Y}_s = \delta_{rs} \qquad \mathbb{X}_r^T \begin{bmatrix} \boldsymbol{C} & \boldsymbol{K} \\ -\boldsymbol{M} & \boldsymbol{0} \end{bmatrix} \mathbb{Y}_s = -\Lambda_r \delta_{rs} \tag{16}$$

Preparing the reduced-order model (ROM)

- $\mathcal{M} = \{m_1\ m_2\ \ldots\ m_n\}$ is a set of master modes. $n \ll N$ is the number of master modes and corresponds to the dimension of the ROM
- We define the following matrices

$$\boldsymbol{X} = [\mathbb{X}_{m_1}\ \mathbb{X}_{m_2} \ldots \mathbb{X}_{m_n}\ \bar{\mathbb{X}}_{m_1}\ \bar{\mathbb{X}}_{m_2} \ldots \bar{\mathbb{X}}_{m_n}] = [\boldsymbol{X_1}\ \boldsymbol{X_2}\ \ldots \boldsymbol{X_n}\ \boldsymbol{\bar{X}_1}\ \boldsymbol{\bar{X}_2}\ \ldots \boldsymbol{\bar{X}_n}] \quad (17)$$

$$\boldsymbol{Y} = [\mathbb{Y}_{m_1}\ \mathbb{Y}_{m_2} \ldots \mathbb{Y}_{m_n}\ \bar{\mathbb{Y}}_{m_1}\ \bar{\mathbb{Y}}_{m_2} \ldots \bar{\mathbb{Y}}_{m_n}] = [\boldsymbol{Y_1}\ \boldsymbol{Y_2}\ \ldots \boldsymbol{Y_n}\ \boldsymbol{\bar{Y}_1}\ \boldsymbol{\bar{Y}_2}\ \ldots \boldsymbol{\bar{Y}_n}] \quad (18)$$

$$\boldsymbol{\lambda} = diag(\Lambda_{m_1}\ \Lambda_{m_2}\ \ldots\ \Lambda_{m_n}\ \bar{\Lambda}_{m_1}\ \bar{\Lambda}_{m_2}\ \ldots\ \bar{\Lambda}_{m_n}) =$$

$$= diag(\lambda_1\ \lambda_2\ \ldots\ \lambda_n\ \bar{\lambda}_1\ \bar{\lambda}_2\ \ldots\ \bar{\lambda}_n) \quad (19)$$

$$\boldsymbol{\phi} = [\Phi_{m_1}\ \Phi_{m_2}\ \ldots\ \Phi_{m_n}\ \Phi_{m_1}\ \Phi_{m_2}\ \ldots\ \Phi_{m_n}] = [\Phi_1\ \Phi_2\ \ldots\ \Phi_n\ \Phi_1\ \Phi_2\ \ldots\ \Phi_n] \quad (20)$$

- Dimensions: $\boldsymbol{X}_{2N\times 2n}$, $\boldsymbol{Y}_{2N\times 2n}$, $\boldsymbol{\lambda}_{2n\times 2n}$, $\boldsymbol{\phi}_{N\times 2n}$

Definitions

- The dynamics will be described in a $2n$-dimensional manifold. The coefficient 2 is due to the pair displacement-velocity
- We introduce $\boldsymbol{z}_{2n\times 1}$ normal coordinates for describing the dynamics on the mentioned manifold $\Rightarrow \dot{\boldsymbol{z}} = \boldsymbol{f}(\boldsymbol{z})$
- Nonlinear mappings: $\boldsymbol{U} = \boldsymbol{\Psi}(\boldsymbol{z})$ and $\boldsymbol{V} = \boldsymbol{\Upsilon}(\boldsymbol{z})$
- Idea: We will expand the nonlinear mappings $\boldsymbol{\Psi}, \boldsymbol{\Upsilon}$ and the reduced dynamics $\boldsymbol{f}$ as polynomial functions of $\boldsymbol{z}$. The goal is to identify the coefficients of the mentioned expansion
- Notice that
  1. ROMs from the normal forms: Expand $\Rightarrow$ Reduce
  2. ROMs from DPIM: Reduce $\Rightarrow$ Expand

- Chain rule:

$$\dot{\boldsymbol{U}} = \boldsymbol{\nabla}_{\boldsymbol{z}}\boldsymbol{\Psi}(\boldsymbol{z})\dot{\boldsymbol{z}} = \boldsymbol{\nabla}_{\boldsymbol{z}}\boldsymbol{\Psi}(\boldsymbol{z})\boldsymbol{f}(\boldsymbol{z}) = \sum_{s=1}^{2n} \frac{\partial \boldsymbol{\Psi}(\boldsymbol{z})}{\partial z_s} f_s(\boldsymbol{z}) \tag{21}$$

$$\dot{\boldsymbol{V}} = \boldsymbol{\nabla}_{\boldsymbol{z}}\boldsymbol{\Upsilon}(\boldsymbol{z})\dot{\boldsymbol{z}} = \boldsymbol{\nabla}_{\boldsymbol{z}}\boldsymbol{\Upsilon}(\boldsymbol{z})\boldsymbol{f}(\boldsymbol{z}) = \sum_{s=1}^{2n} \frac{\partial \boldsymbol{\Upsilon}(\boldsymbol{z})}{\partial z_s} f_s(\boldsymbol{z}) \tag{22}$$

- Substituting the above quantities into the first-order equation of motion of the FOM

$$\begin{aligned} &\boldsymbol{M}\boldsymbol{\nabla}_{\boldsymbol{z}}\boldsymbol{\Upsilon}(\boldsymbol{z})\boldsymbol{f}(\boldsymbol{z}) + \boldsymbol{C}\boldsymbol{\Upsilon}(\boldsymbol{z}) + \boldsymbol{K}\boldsymbol{\Psi}(\boldsymbol{z}) + \boldsymbol{G}(\boldsymbol{\Psi}(\boldsymbol{z}), \boldsymbol{\Psi}(\boldsymbol{z})) + \\ &+ \boldsymbol{H}(\boldsymbol{\Psi}(\boldsymbol{z}), \boldsymbol{\Psi}(\boldsymbol{z}), \boldsymbol{\Psi}(\boldsymbol{z})) = \boldsymbol{0} \end{aligned} \tag{23}$$

$$\boldsymbol{M}\boldsymbol{\nabla}_{\boldsymbol{z}}\boldsymbol{\Psi}(\boldsymbol{z})\boldsymbol{f}(\boldsymbol{z}) = \boldsymbol{M}\boldsymbol{\Upsilon}(\boldsymbol{z}) \tag{24}$$

Homological equations

- Since the expansion is made around an equilibrium position, the zero-order term is not included in the expansion.
- Assuming that o is the maximum order employed in the expansion and that $[\ ]_p$ contains only monomials of order p in the components of $\boldsymbol{z}$, we have

$$\boldsymbol{\Psi}(\boldsymbol{z}) = \sum_{p=1}^{o} [\boldsymbol{\Psi}(\boldsymbol{z})]_p \tag{25}$$

$$\boldsymbol{\Upsilon}(\boldsymbol{z}) = \sum_{p=1}^{o} [\boldsymbol{\Upsilon}(\boldsymbol{z})]_p \tag{26}$$

$$\boldsymbol{f}(\boldsymbol{z}) = \sum_{p=1}^{o} [\boldsymbol{f}(\boldsymbol{z})]_p \tag{27}$$

- The order-p homological equations read

$$\begin{aligned} &\boldsymbol{M}[\boldsymbol{\nabla}_{\boldsymbol{z}}\boldsymbol{\Upsilon}(\boldsymbol{z})\boldsymbol{f}(\boldsymbol{z})]_p + \boldsymbol{C}[\boldsymbol{\Upsilon}(\boldsymbol{z})]_p + \boldsymbol{K}[\boldsymbol{\Psi}(\boldsymbol{z})]_p + [\boldsymbol{G}(\boldsymbol{\Psi}(\boldsymbol{z}),\boldsymbol{\Psi}(\boldsymbol{z}))]_p + \\ &+ [\boldsymbol{H}(\boldsymbol{\Psi}(\boldsymbol{z}),\boldsymbol{\Psi}(\boldsymbol{z}),\boldsymbol{\Psi}(\boldsymbol{z}))]_p = \boldsymbol{0} \qquad (28) \\ &\boldsymbol{M}[\boldsymbol{\nabla}_{\boldsymbol{z}}\boldsymbol{\Psi}(\boldsymbol{z})\boldsymbol{f}(\boldsymbol{z})]_p = \boldsymbol{M}[\boldsymbol{\Upsilon}(\boldsymbol{z})]_p \qquad (29) \end{aligned}$$

- As in other methods, the solution is started with $p = 1$ and, then, the homological equations are solved in sequential order

Homological equation for $p = 1$

- For $p = 1$, the nonlinear terms associated with tensors G_{rs} and H_{rst} do not appear. Hence, we have:

$$\boldsymbol{M}[\boldsymbol{\nabla}_{\boldsymbol{z}}\boldsymbol{\Upsilon}(\boldsymbol{z})\boldsymbol{f}(\boldsymbol{z})]_1 + \boldsymbol{C}[\boldsymbol{\Upsilon}(\boldsymbol{z})]_1 + \boldsymbol{K}[\boldsymbol{\Psi}(\boldsymbol{z})]_1 = \boldsymbol{0} \tag{30}$$

$$\boldsymbol{M}[\boldsymbol{\nabla}_{\boldsymbol{z}}\boldsymbol{\Psi}(\boldsymbol{z})\boldsymbol{f}(\boldsymbol{z})]_1 = \boldsymbol{M}[\boldsymbol{\Upsilon}(\boldsymbol{z})]_1 \tag{31}$$

- Notice that $p = 1$ corresponds to the linear part of the expansion. The unknowns for this problem read:

$$[\boldsymbol{\Upsilon}(\boldsymbol{z})]_1 = \boldsymbol{\Upsilon}^{(1)}_{N\times 2n}\boldsymbol{z} \tag{32}$$

$$[\boldsymbol{\Psi}(\boldsymbol{z})]_1 = \boldsymbol{\Psi}^{(1)}_{N\times 2n}\boldsymbol{z} \tag{33}$$

$$[\boldsymbol{f}(\boldsymbol{z})]_1 = \boldsymbol{f}^{(\boldsymbol{1})}_{\boldsymbol{2n\times 2n}}\boldsymbol{z} \tag{34}$$

- With the above definitions, the homological equations for $p = 1$ become:

$$\boldsymbol{M}\boldsymbol{\Upsilon}^{(1)}\boldsymbol{f}^{(1)}\boldsymbol{z} + \boldsymbol{C}\boldsymbol{\Upsilon}^{(1)}\boldsymbol{z} + \boldsymbol{K}\boldsymbol{\Psi}^{(1)}\boldsymbol{z} = \boldsymbol{0} \tag{35}$$

$$\boldsymbol{M}\boldsymbol{\Psi}^{(1)}\boldsymbol{f}^{(1)}\boldsymbol{z} = \boldsymbol{M}\boldsymbol{\Upsilon}^{(1)}\boldsymbol{z} \tag{36}$$

Homological equation for $p = 1$

- The last set of equation must be valid for any $\boldsymbol{z}$. Hence, we obtain:

$$\begin{bmatrix} \boldsymbol{M} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{M} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\Upsilon}^{(1)} \\ \boldsymbol{\Psi}^{(1)} \end{Bmatrix} \boldsymbol{f}^{(1)} + \begin{bmatrix} \boldsymbol{C} & \boldsymbol{K} \\ -\boldsymbol{M} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\Upsilon}^{(1)} \\ \boldsymbol{\Psi}^{(1)} \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \end{Bmatrix} \tag{37}$$

- The above equation has the same form of the right eigenvalue problem and, then,

$$\begin{Bmatrix} \boldsymbol{\Upsilon}^{(1)} \\ \boldsymbol{\Psi}^{(1)} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{\phi\lambda} \\ \boldsymbol{\phi} \end{Bmatrix} = \boldsymbol{Y} \tag{38}$$

$$\boldsymbol{f}^{(1)} = \boldsymbol{\lambda} \tag{39}$$

- Conclusion (as expected): The linear part of the expansion matches the solution of the linearized problem. Higher-order terms ($p > 1$) correspond to corrections applied to the linearized problem

An introduction to the case with $p > 1$

- With the results for $p = 1$ at hand, the expansions read:

$$\boldsymbol{\Psi}(\boldsymbol{z}) = \boldsymbol{\phi z} + \sum_{p=2}^{o} [\boldsymbol{\Psi}(\boldsymbol{z})]_p \quad (40)$$

$$\boldsymbol{\Upsilon}(\boldsymbol{z}) = \boldsymbol{\phi \lambda z} + \sum_{p=2}^{o} [\boldsymbol{\Upsilon}(\boldsymbol{z})]_p \quad (41)$$

- Explicitly (similar expansion also valid for $p > 1$

$$[\boldsymbol{\Psi}(z)]_p = \sum_{i_1=1}^{2n} \sum_{i_2=1}^{2n} \dots \sum_{i_p=1}^{2n} \boldsymbol{\Psi}^{(p)}_{\{i_1 i_2 \dots i_p\}} z_{i_1} z_{i_2} \dots z_{i_p} \quad (42)$$

- For a more compact nomenclature, it is introduced set $\mathcal{I} = \{i_1 i_2 \dots i_p\}$ with p elements (cardinality). The monomial associated with a certain set $\mathcal{I}^*$ is $\pi^*_{\mathcal{I}} = z_{i_1} z_{i_2} \dots$ and $\pi^{(p)}_{\mathcal{I}}$ represents all the monomials of order p for gathering all the monomials of order p.
- For example, the set $\mathcal{I} = \{1134\}$ is associated with the monomial $z_1^2 z_3 z_4$.
- With the introduced nomenclature, the expansion for $[\boldsymbol{\Psi}]_p$ is $\sum_{\mathcal{I}} \boldsymbol{\Psi}^{(p)}_{\mathcal{I}} \pi^{(p)}_{\mathcal{I}}$. Similar expression is valid for $[\boldsymbol{\Upsilon}(\boldsymbol{z})]_p$