EXERCÍCIOS 3, 6, 9, 13 e 14 da Lista 6

9 (b)

$$T: M_2(\mathbb{R}) \longrightarrow M_2(\mathbb{R})$$

$$T(X) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} X$$

Se $X = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$

$$T(X) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} x+z & y+t \\ z & t \end{bmatrix}$$

Assim $\operatorname{Ker} T = \left\{ \begin{bmatrix} x & y \\ z & t \end{bmatrix} \mid T(X) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$

$= \left\{ \begin{bmatrix} x & y \\ z & t \end{bmatrix} \text{ com } \begin{bmatrix} x+z & y+t \\ z & t \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$

Temos então:

$$\left.\begin{array}{l} x+z = y+t = 0 \\ z = t = 0 \end{array}\right\} \Rightarrow x = 0 \text{ e } y = 0.$$

Logo, $\boxed{\operatorname{Ker} T = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}}$.

Com o Teo do Ker e da Im temos:

$\dim M_2(\mathbb{R}) = \dim \operatorname{Ker} T + \dim \operatorname{Im} T$

Como $\dim \operatorname{Ker} T = 0$, temos que $\dim \operatorname{Im} T = \dim M_2(\mathbb{R}) =$

Logo $\operatorname{Im} T$ é um subespaço de $M_2(\mathbb{R})$ de dimensão 4.

Assim, $\boxed{\operatorname{Im} T = M_2(\mathbb{R})}$. (Qualquer base de $M_2(\mathbb{R})$ é base de $\operatorname{Im} T$.)

9 (a) $T: \mathbb{R}^n \longrightarrow \mathbb{R}^{n-1}$

$T(x_1, \dots, x_n) = (x_1, \dots, x_{n-1})$

$\operatorname{Ker} T = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid (x_1, \dots, x_{n-1}) = (0, \dots, 0)\}$

Assim, $\boxed{\operatorname{Ker} T = \{(0, \dots, 0, x_n) \mid x_n \in \mathbb{R}\} = [\underbrace{(0, \dots, 0, 1)}_{e_n}]}$

Usando o Teorema do Núcleo ~~de~~ da Imagem,

$\underbrace{\dim \mathbb{R}^n}_{n} = \underbrace{\dim \operatorname{Ker} T}_{1} + \dim \operatorname{Im} T$

Logo $\dim \operatorname{Im} T = n - 1$. $\boxed{\text{Logo } \operatorname{Im} T = \mathbb{R}^{n-1}.}$

$(\operatorname{Im} T = (x_1, \dots, x_{n-1}) \mid x_i \in \mathbb{R}, \; i = 1, \dots, n-1.)$

9 (c) $T: P_n(\mathbb{R}) \longrightarrow P_{n+1}(\mathbb{R})$

$T(p(t)) = t\,p(t)$

$p(t) = a_0 1 + a_1 t + a_2 t^2 + \dots + a_n t^n$

$T(p(t)) = a_0 t + a_1 t^2 + \dots + a_n t^{n+1}$

$T(p(t)) = 0 \implies a_0 t + a_1 t^2 + \dots + a_n t^n = 0$

$\implies a_0 = a_1 = \dots = a_n = 0$.

Logo $\operatorname{Ker} T = \{0\}$

$\operatorname{Im} T = \{t\,p(t) \mid p(t) \in P_n(\mathbb{R})\}$

$\operatorname{Im} T = \{T(1), T(t), \dots, T(t^n)\}$ já que $\{1, t, \dots, t^n\}$ é uma base de $P_n[\mathbb{R}]$

Assim

$$\operatorname{Im} T = [t, t^2, \dots, t^{n+1}] =$$
$$\{ g(t) = b_0 \cdot 1 + b_1 t + \dots + b_{n+1} t^{n+1} \mid b_0 = 0 \}$$

13. (a) $T(x,y,z) = (-3y+4z,\ 3x+5z,\ -4x-4y)$

$$\operatorname{Ker} T = \{ (x,y,z) \in \mathbb{R}^3 \mid (-3y+4z,\ 3x+5z,\ -4x-4y) = (0,0,0) \}$$

Temos então o sistema:

$$\begin{cases} -3x + 4y = 0 \\ 3x + 5z = 0 \\ -4x - 4y = 0 \end{cases}$$

Matriz do sistema: $\begin{bmatrix} -3 & 4 & 0 \\ 3 & 0 & 5 \\ -4 & -4 & 0 \end{bmatrix}$ $\xrightarrow[L3/-4]{L2 \leftrightarrow L1,\ L2 \leftrightarrow L1+L2}$

Escalonando temos: $\begin{bmatrix} 3 & 0 & 5 \\ 0 & 4 & 5 \\ 1 & 1 & 0 \end{bmatrix} \longrightarrow$

$\xrightarrow{L1 \leftrightarrow L3,\ L3 \leftrightarrow L3-3L1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 4 & 5 \\ 0 & -3 & 0 \end{bmatrix} \xrightarrow{\frac{L3}{-3}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 4 & 5 \\ 0 & 1 & 0 \end{bmatrix}$

$\xrightarrow{L2 \leftrightarrow L3,\ L3-4L2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

O sistema tem apenas a solução trivial.

$$\operatorname{Ker} T = \{(0,0,0)\}$$
$$\dim \operatorname{Im} T = 3 \implies \operatorname{Im} T = \mathbb{R}^3.$$

13 (b)

$T(p(t)) = p(t+1)$

Se $p(t) = a_0 \cdot 1 + a_1 t + a_2 t^2 + a_3 t^3$, então

$$T(p(t)) = a_0 \cdot 1 + a_1 (t+1) + a_2 (t+1)^2 + a_3 (t+1)^3$$

$$= a_0 \cdot 1 + a_1 t + a_1 \cdot 1 + a_2 t^2 + 2a_2 t + a_2 \cdot 1$$
$$+ a_3 t^3 + 3a_3 t^2 + 3a_3 t + a_3 \cdot 1$$

$$= (a_0 + a_1 + a_2 + a_3) \cdot 1 + (a_1 + 2a_2 + 3a_3) t$$
$$+ (a_2 + 3a_3) t^2 + a_3 t^3$$

$\operatorname{Ker} T = \{ p(t) \mid T(p(t)) = 0 \}$

$$\Rightarrow \begin{cases} a_0 + a_1 + a_2 + a_3 = 0 \\ a_1 + 2a_2 + 3a_3 = 0 \\ a_2 + 3a_3 = 0 \\ a_3 = 0 \end{cases}$$

Logo $\operatorname{Ker} T = \{0\}$

e $\operatorname{Im} T = P_3(\mathbb{R})$, já que

$$\dim P_3(\mathbb{R}) = \underbrace{\dim \operatorname{Ker} T}_{= 0} + \dim \operatorname{Im} T.$$

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(c) $T : P_n(\mathbb{R}) \longrightarrow P_n(\mathbb{R})$

$T(p(t)) = p'(t)$

$$T(a_0 + a_1 t + \cdots + a_n t^n) = a_1 + 2a_2 t + \cdots + 3a_n t^{n-1}$$

Assim $\mathrm{Ker}\,T = \{ p(t) \mid a_1 = a_2 = \dots = a_n = 0 \}$

$\mathrm{Ker}\,T = \{$ polinômios constantes $\}$

$\mathrm{Ker}\,T = \{ p(t) = a_0 \}$
$a_0 \in \mathbb{R}$

$\{1, t, t^2, \dots, t^n\}$ é uma base de $P_n(\mathbb{R})$

$\{T(1), T(t), \dots, T(t^n)\}$ gera $\mathrm{Im}\,T$

$\mathrm{Im}\,T = [1, t, \dots, t^{n-1}] = P_{n-1}(\mathbb{R})$.

14: $T: P_2(\mathbb{R}) \longrightarrow M_2(\mathbb{R})$

$$T(at^2 + bt + c) = \begin{bmatrix} a-2b & b+c \\ c-3a & a+b+c \end{bmatrix}$$

$$\mathrm{Ker}\,T = \left\{ at^2 + bt + c \;\middle|\; \begin{bmatrix} a-2b & b+c \\ c-3a & a+b+c \end{bmatrix} = 0 \right\}$$

Temos o sistema

$$\begin{aligned} a - 2b &= 0 \\ b + c &= 0 \\ c - 3a &= 0 \\ a + b + c &= 0 \end{aligned}$$

Matriz do sistema

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 1 \\ -3 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \rightarrow$$

$$\rightarrow \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 1 \\ 0 & 6 & 3 \\ 0 & 3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Logo, $\operatorname{Ker} T = \{ 0 \}$

$$\operatorname{Im} T = \left\{ \begin{bmatrix} a-2b & b+c \\ c-3a & a+b+c \end{bmatrix} \mid a, b, c \right.$$

$$= \left\{ a \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} + b \begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix} + c \left[\right. \right.$$

$$\operatorname{Im} T = \left[\underbrace{\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}}_{M_1}, \underbrace{\begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix}}_{M_2}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right.$$

$$\dim P_2(\mathbb{R}) = \dim \operatorname{Ker} T + \dim \operatorname{Im} T$$

$$3 = 0 + \dim \operatorname{Im} T \Rightarrow$$

$$\dim \operatorname{Im} T = 3$$

Logo, como $\dim \operatorname{Im} T = 3$ e $\operatorname{Im} T$

$\{M_1, M_2, M_3\}$ é LI e é entã

base de $\operatorname{Im} T$.

⑥ $T : M_n(\mathbb{R}) \longrightarrow M_n(\mathbb{R})$

$T(X) = AX - XA$, onde A

matr

Mostrar que T é linear

(1) $T(X_1+X_2) = A(X_1+X_2) - (X_1+X$

$= AX_1 + AX_2 - X_1A - X_2A$

$= AX_1 - X_1A + AX_2 - X_2A$

$= T(X_1) + T(X_2)$

(2) $T(aX) = A(aX) - (aX)A$

$= a\,AX - a(XA)$

$= a(AX - XA) = a$

$$\mathrm{Ker}\,T = \{ X \in M_n(\mathbb{R}) \mid \underbrace{AX = XA}_{\text{são as matrizes}} \}$$

que comutam com a matriz A