level q^* rises. However, as more firms operate in this market, the increase in q^* becomes smaller since the derivative $\frac{\partial q^*}{\partial \beta}$ decreases in N, i.e.,

$$\frac{\partial \left(\frac{\partial q^*}{\partial \beta}\right)}{\partial N} = \frac{\theta N^2 c - \theta(N-1)2Nc}{(N^2c)^2} < 0.$$

Exercise #6 - Linear and Leontief Preferences

6. Consider an economy in which preferences are

$$\begin{aligned} &\text{Consumer 1:} \qquad U^1 = x_1^1 + x_2^1 \\ &\text{Consumer 2:} \qquad U^2 = \min\{x_1^2, x_2^2\} \end{aligned}$$

(a) Given the endowments $\omega^1 = (1, 2)$ and $\omega^2 = (3, 1)$, find the set of Pareto efficient allocations and the contract curve.

- For consumer 1, the indifference curves are found be solving for x_2^1, i.e., $x_2^1 = U^1 - x_1^1$, and thus are depicted as straight lines with a slope of -1. For consumer 2, the indifference curves are right angles with corners ("kinks") at consumption bundles with equal quantities of the two goods, $x_1^2 = x_2^2$. Figure 6.2 illustrates the Edgeworth box. Recall that there are 4 units of good 1, but only 3 units of good 2, explaining the rectangular shape of the Edgeworth box. The Pareto efficient allocations (PEAs) are at the corners of consumer 2's indifference curves.

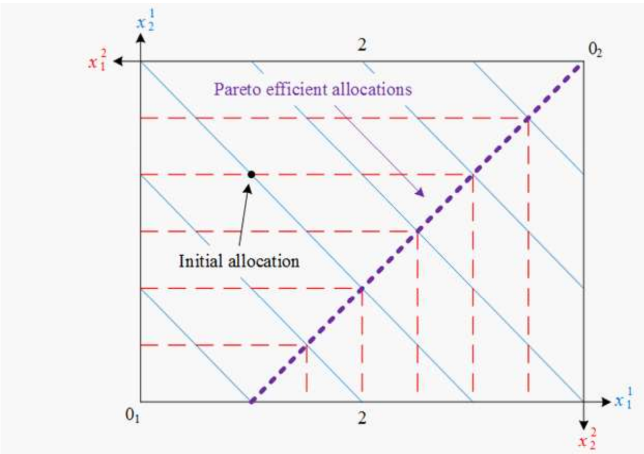

Figure 6.2. Edgeworth box and PEAs.

(b) Which allocations are competitive equilibria?

- The only equilibrium must be on the indifference curve of consumer 1 through the endowment point ω. This is shown by point e in figure 6.3, where consumer 1 is as well off as in his initial endowment. The budget line must therefore overlap his indifference curve. Any other price ratio will lead consumer 1 to choose a corner allocation (either spending all his income on good 1 alone if $p_1 < p_2$, or on good 2 alone if $p_2 < p_1$). In contrast, consumer 2 wish to consume at the corner of an indifference curve. Point e therefore must be the unique equilibrium (unique WEA) which is $((x_1^1, x_1^2), (x_2^1, x_2^2)) = ((2,1),(2,2))$.[25] As depicted in the figure WEA $\in$ PEA.

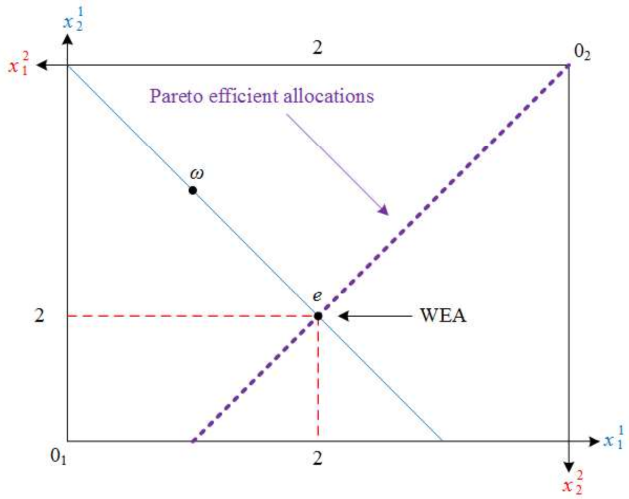

Figure 6.3. PEAs and WEA.

Exercise #7 - Finding Offer Curves for Different Preferences

7. Consider a two-good economy, where every person has the endowment $\omega = (0, 20)$. For each of the following preferences, solve the individual's UMP in order to find his demand curve. The use the endowment to identify his offer curve.

   (a) Cobb-Douglas type: $\alpha \log(x_1) + (1-\alpha)\log(x_2)$, where $\alpha \in (0,1)$.

   - Setting up the Lagrangian and normalizing the price of good 2, so $p_2 = 1$ and $p_1 = p$, we obtain

$$\mathcal{L} = \alpha \log(x_1) + (1-\alpha)\log(x_2) + \lambda\,[20 - px_1 - x_2]$$

[25] The WEA simultaneously satisfies $x_2^1 = 3 - x_1^1$ for consumer 1, $x_1^2 = x_2^2$ for consumer 2 (points at the kink of his indifference curve), and the feasibility conditions $x_1^1 + x_2^1 = 4$ and $x_1^2 + x_2^2 = 3$.

which yields first-order conditions

$$\begin{aligned}
\frac{\alpha}{x_1} - \lambda p &= 0 \\
\frac{1-\alpha}{x_2} - \lambda &= 0 \\
20 - px_1 - x_2 &= 0
\end{aligned}$$

Subtracting the first two equations from the third one, we find $\lambda = \frac{1}{20}$, and so the demands will be

$$x_1 = \frac{20\alpha}{p} \qquad \text{and} \qquad x_2 = 20(1-\alpha)$$

and the offer curve will simply be a horizontal straight line at $x_2^h = 20(1-\alpha)$. Since the offer curve depicts the relationship between the demand of good 2 and good 1, the offer curve in this case is just $x_2 = 20(1-\alpha)$, i.e., a horizontal straight line with height $20(1-\alpha)$ in the Edgeworth box.

(b) Perfect substitutes: $ax_1 + x_2$

- In this case, the consumer demands units of one of the good alone (when the slope of his indifference curve and budget line differs) or any bundle on his budget line (if their slopes coincide). In particular, since the $MRS_{1,2} = \frac{a}{1} = a$, and the price ration is $\frac{p_1}{p_2} = p$, the consumer only demands good 2 if $p > a$, i.e., $x = (0, 20)$; only good 1 if $p < a$, yielding a demand $x = \left(\frac{20}{a}, 0\right)$; and any point on the budget line $px_1 + x_2 = 20$ if $p = a$; as depicted in figures 6.4a and 6.4b.

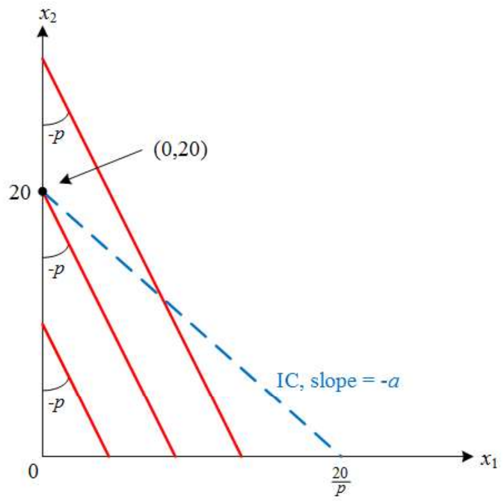

Figure 6.4a. Demand when $p > a$.

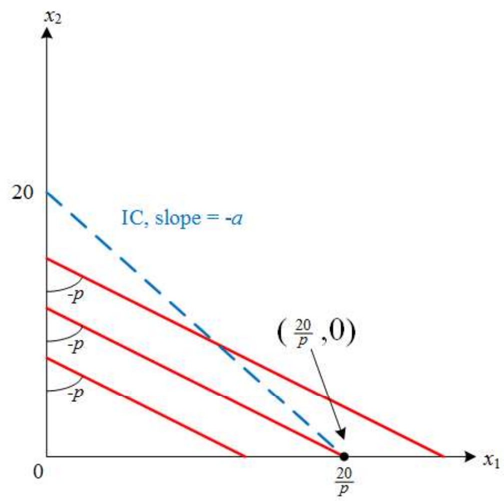

Figure 6.4b. Demand when $p < a$.

(c) Perfect complements: $\min\{ax_1, x_2\}$.

- Demand will be at the kink of the indifference curve, i.e., $ax_1 = x_2$, which together with the budget constraint $px_1 + x_2 = 20$ yields $px_1 + ax_1 = 20$, or $x_1 = \frac{20}{p+a}$. Hence, the demand for good 2 is $x_2 = ax_1 = a\frac{20}{p+a}$. That is, the offer curve satisfies $x_2 = ax_1$, thus being a straight line from the origin $(0,0)$ and with a positive slope $a > 0$.

(d) Consider now an economy where all individuals have the Cobb-Douglas preferences of part (a). There are two individuals: consumer A with $\alpha = \frac{1}{2}$ and endowment $\omega = (10, 0)$, and consumer B with $\alpha = \frac{3}{4}$ and $\omega = (0, 20)$. Find the WEA.

- If a person with preferences of $\alpha \log(x_1) + (1 - \alpha) \log(x_2)$ had an income of 10 units of commodity 1 (as opposed to 20 in part (a)) then, by analogy with part (a), demand would be

$$x^1 = \begin{bmatrix} 10\alpha \\ 10p(1-\alpha) \end{bmatrix}$$

and the offer curve will simply be a vertical straight line at $x_1^h = 10\alpha$. From our demands in part (a) and the equation above, we have $x_1^1 = 10(\frac{1}{2}) = 5$, $x_2^2 = 20(1 - \frac{3}{4}) = 5$. Given that there are 10 units in total of commodity 1 and 20 units in total of commodity 2 the materials balance condition then means that the equilibrium allocation must be

$$x^1 = \begin{bmatrix} 5 \\ 15 \end{bmatrix} \quad \text{and} \quad x^2 = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

Solving for p from our equilibrium we find that the equilibrium price ratio must be 3.

Exercise #8 - Barter Economies

8. Consider the following indirect utility functions for consumers A and B

$$\begin{aligned} v^A(\mathbf{p}, m) &= \ln m - \frac{1}{2}\ln p_1 - \frac{1}{2}\ln p_2 \\ v^B(\mathbf{p}, m) &= \left(\frac{1}{p_1} + \frac{1}{p_2}\right) m \end{aligned}$$

Exercise #9 - Pure Exchange Economy

9. Consider a pure-exchange economy with two individuals, A and B, each with utility function $u^i(x^i, y^i)$ where $i = \{A, B\}$, whose initial endowments are $e^A = (10, 0)$ and $e^B = (0, 10)$, that is, individual A (B) owns all units of good x (y, respectively).

   (a) Assuming that utility functions are $u^i(x^i, y^i) = \min\{x^i, y^i\}$ for all individuals $i = \{A, B\}$, find the set of PEAs and the set of WEAs.

   - *PEAs.* Since the utility functions are not differentiable we cannot follow the property of $MRS^A_{x,y} = MRS^B_{x,y}$ across consumers. Figure 6.5 helps us identify the set of PEAs. Points away from the 45^o-line, satisfying $y^A = x^A$, such as N, cannot be pareto efficient since we can still find other points, such as M, where consumer 2 is make better off while consumer 1 reaches the same utility level as under N. Once we are at points on the 45^o-line, such as M, we cannot find other points making at least once consumer better off (and keep the other consumer at least as well off). Hence, the set of PEAs is

$$\{(x^A, y^A), (x^B, y^B) : y^A = x^A \text{ and } y^B = x^B\}$$

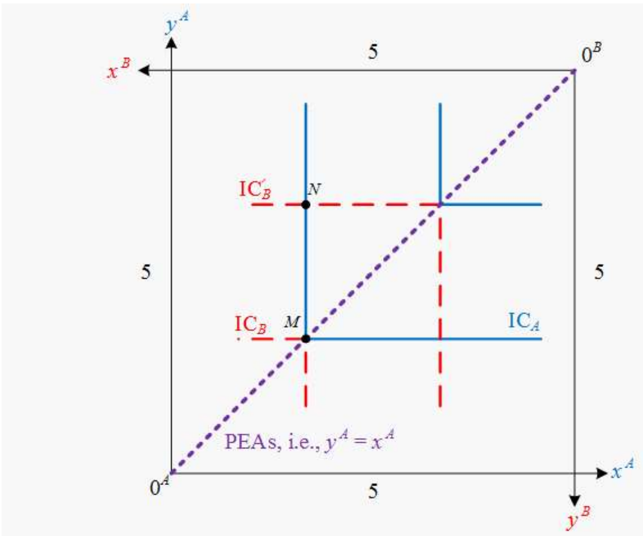

Figure 6.5. Edgeworth box and PEAs.

   - *WEAs.* Using good 2 as the numeraire, i.e., $p_2 = 1$, the price ratio becomes $\frac{p_1}{p_2} = p_1$. The budget line of both consumers therefore has a slope $-p_1$ and crosses the point representing the initial endowment e in figure 6.6 (where e lies at the lower right-hand corner). Therefore, the WEA is given by the vector $\{(5,5),(5,5)\}$, where every consumer enjoys 5 units of every good.

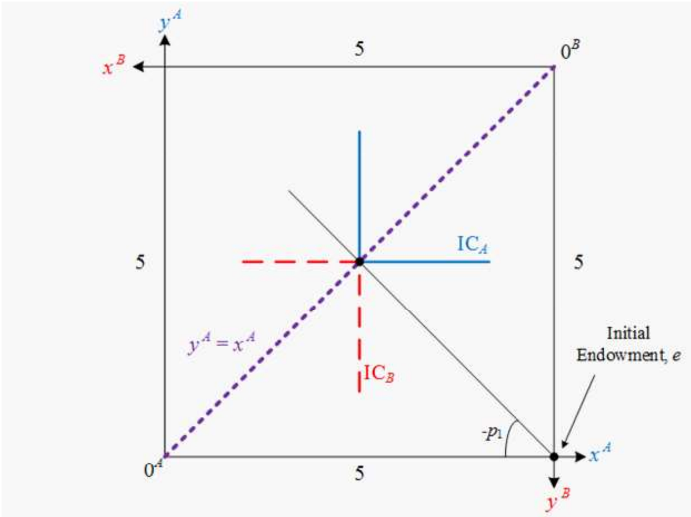

Figure 6.6. Edgeworth box and WEA.

(b) Assuming utility functions of $u^A(x^A, y^A) = x^A y^A$ and $u^B(x^B, y^B) = \min\{x^B, y^B\}$, find the set of PEAs and WEAs.

- *PEAs.* By the same argument as in question (a), the set of PEAs satisfies $y^A = x^A$, as depicted in figure 6.7. Point N cannot be efficient as we can still find other feasible points, such as M, where at least one consumer is made strictly better off (in this case consumer A). At points on the 45^o-line, however, we can no longer find alternatives that would constitute a Pareto improvement.

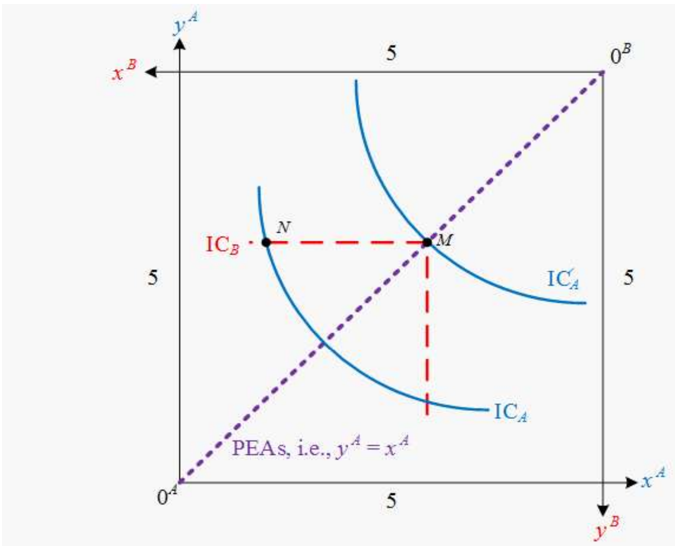

Figure 6.7. Edgeworth box and PEAs.

- *WEAs.* Using good y as the numeraire, $p_y = 1$, so that the price vector becomes $\mathbf{p} = (p_x, 1)$. Hence, *Consumer A's UMP is*

$$\max_{x^A, y^A} \quad x^A y^A$$
$$\text{subject to } p_x x^A + y^A = 10p_x$$

Taking first-order conditions

$$\begin{aligned} y^A - \lambda^A p_x &= 0 \\ x^A - \lambda^A &= 0 \\ p_x x^A + y^A &= 10p_x \end{aligned}$$

Combining the first two FOCs and rearranging, we have

$$p_x x^A = y^A$$

and substituting this equation into the third FOC yields

$$p_x x^A + p_x x^A = 10p_x \implies x^A = 5$$

and substituting this back into $p_x x^A = y^A$

$$y^A = 5p_x$$

Consumer B's UMP is not differentiable, but in equilibrium his Walrasian demands satisfy $x^B = y^B$. Substituting this into his budget constraint yields

$$p_x x^B + x^B = 10 \implies x^B = y^B = \frac{10}{p_x + 1}$$

Furthermore, the feasibility condition for good x entails

$$5 + \frac{10}{p_x + 1} = 10 + 0, \text{ or } p_x = 1$$

Therefore, the market of good x will clear at an equilibrium price of $p_x = 1$, i.e., $z_x(p_x, 1) = 0$ when $p_x = 1$. Since market y clears when market x does (by Walras' law), $z_y(p_x, 1)$ must also be zero when $p_x = 1$. Summarizing, the equilibrium price $p_x = 1$ yields a WEA

$$\{(5,5),(5,5)\}$$

Exercise #10 - Gross Substitutes

10. Consider an economy with two individuals, Amelia and Bernardo, with utility functions $u^A(x^A, y^A) = \min\{x^A, 2y^A\}$ for Amelia and $u^B(x^B, y^B) = \min\{2x^B, y^B\}$ for Bernardo, and initial endowments given by $\mathbf{e}^A = (1,0)$ and $\mathbf{e}^B = (0,1)$.

    (a) Find the Walrasian demands of each individual.

    - *Amelia.* The UMP of Amelia is

    $$\max_{x^A, y^A \geq 0} \quad \min\{x^A, 2y^A\}$$
    $$\text{subject to} \;\; p_x x^A + p_y y^A \leq p_x$$

    since she only owns one unit of good x, $\mathbf{e}^A = (1,0)$ the market value of her resources (as captured in the right-hand side of the budget constraint) is p_x. As she would consume (x^A, y^A) pairs at the kink of her L-shaped indifference curves, optimal consumption bundles satisfy $x^A = 2y^A$. Plugging $x^A = 2y^A$ into her budget line, $p_x x^A + p_y y^A = p_x$, yields

    $$p_x\left(2y^A\right) + p_y y^A = p_x$$

    and solving for y^A, we obtain Amelia's Walrasian demand of good y

    $$y^A = \frac{p_x}{2p_x + p_y}$$

    while her demand for good x is

    $$x^A = 2y^A = \frac{2p_x}{2p_x + p_y}$$

    - *Bernardo.* Similarly, Bernardo's utility maximizing bundles (x^B, y^B) satisfy $2x^B = y^B$ (bundles at the kink of his indifference curve) and $p_x x^B + p_y y^B = p_y$ (budget line since he only owns one unit of good y). Simultaneously solving price $p_1 = 1$ is absent. Therefore, the only equilibrium is stable.

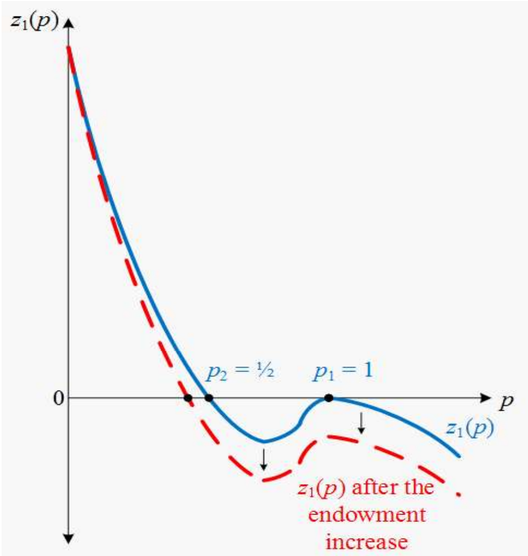

Figure 6.18. Excess demand and Stability.

Exercise #21 - Production Economy

21. Consider an economy with two consumers $i = \{A, B\}$, one firm and two goods $l = \{1, 2\}$. The individual endowments of individuals A and B are $\omega^A = \omega^B = (\frac{1}{2}, \frac{1}{2})$. The utility functions are

$$\begin{aligned} u^A(x_1^A, x_2^A) &= \ln(x_1^A) + \ln(x_2^A) \\ u^B(x_1^B, x_2^B) &= (x_1^B)^{\frac{1}{4}}(x_2^B)^{\frac{3}{4}} \end{aligned}$$

The firm produces good 2 using good 1 as input, the production function is $y_2 = \sqrt{y_1}$. The consumer B owns the firm (denote π the firm's profit). Good 2 is the numeraire good (i.e., $p_2 = 1$).

(a) Determine the demand for good 1 of the consumers and the firm.

- *Consumer A.* Starting with consumer A's utility maximization problem

$$\max_{x_1^A, x_2^A} \quad \ln(x_1^A) + \ln(x_2^A) + \lambda^A \left[p_1 \left(\frac{1}{2}\right) + p_2 \left(\frac{1}{2}\right) - p_1 x_1^A - p_2 x_2^A \right]$$

with first-order conditions

$$\begin{aligned}
\frac{1}{x_1^A} - \lambda^A p_1 &= 0 \\
\frac{1}{x_2^A} - \lambda^A p_2 &= 0 \\
p_1\left(\frac{1}{2}\right) + p_2\left(\frac{1}{2}\right) - p_1 x_1^A - p_2 x_2^A &= 0
\end{aligned}$$

Combining and rearranging the first two FOCs yields

$$p_1 x_1^A = p_2 x_2^A$$

Substituting this into the third FOC yields consumer A's demand for good 1

$$p_1\left(\frac{1}{2}\right) + p_2\left(\frac{1}{2}\right) - 2p_1 x_1^A = 0 \implies x_1^A = \frac{p_1 + p_2}{4p_1}$$

and plugging this value back into $p_1 x_1^A = p_2 x_2^A$ yields consumer A's demand for good 2

$$p_1\left(\frac{p_1 + p_2}{4p_1}\right) = p_2 x_2^A \implies x_2^A = \frac{p_1 + p_2}{4p_2}$$

- *Consumer B.* Next, we perform the same task for consumer B. His utility maximization problem is

$$\max_{x_1^B, x_2^B} \quad (x_1^B)^{\frac{1}{4}}(x_2^B)^{\frac{3}{4}} + \lambda^B\left[p_1\left(\frac{1}{2}\right) + p_2\left(\frac{1}{2}\right) + \pi - p_1 x_1^B - p_2 x_2^B\right]$$

with first-order conditions

$$\begin{aligned}
\frac{1}{4}(x_1^B)^{-\frac{3}{4}}(x_2^B)^{\frac{3}{4}} - \lambda^A p_1 &= 0 \\
\frac{3}{4}(x_1^B)^{\frac{1}{4}}(x_2^B)^{-\frac{1}{4}} - \lambda^A p_2 &= 0 \\
p_1\left(\frac{1}{2}\right) + p_2\left(\frac{1}{2}\right) + \pi - p_1 x_1^A - p_2 x_2^A &= 0
\end{aligned}$$

Combining and rearranging the first two FOCs yields

$$3p_1 x_1^B = p_2 x_2^B$$

Substituting this into the third FOC yields consumer B's demand for good 1

$$p_1\left(\frac{1}{2}\right)+p_2\left(\frac{1}{2}\right)+\pi-4p_1x_1^B=0 \implies x_1^B=\frac{p_1+p_2+2\pi}{8p_1}$$

and plugging this value back into $3p_1x_1^B=p_2x_2^B$ yields consumer B's demand for good 2

$$3p_1\left(\frac{p_1+p_2+2\pi}{8p_1}\right)=p_2x_2^B \implies x_2^B=\frac{3\left(p_1+p_2+2\pi\right)}{8p_2}$$

- *Firm.* Lastly, we find the firm's demand for good 1 by setting up its profit maximization problem

$$\max_{y_1,y_2} \quad p_2y_2-p_1y_1$$
$$\text{subject to } y_2=\sqrt{y_1}$$

substituting the constraint, we obtain an unconstrained maximization problem

$$\max_{y_1} \quad p_2\sqrt{y_1}-p_1y_1$$

and taking first-order conditions with respect to y_1 yields the firm's demand for good 1

$$\frac{p_2}{2\sqrt{y_1}}-p_1=0 \implies y_1=\frac{p_2^2}{4p_1^2}$$

Plugging this value back into $y_2=\sqrt{y_1}$ yields the firm's supply of good 2

$$y_2=\sqrt{\frac{p_2^2}{4p_1^2}}=\frac{p_2}{2p_1}$$

This also yields a value for the firm's profits, π of

$$\pi=p_2y_2-p_1y_1=p_2\frac{p_2}{2p_1}-p_1\frac{p_2^2}{4p_1^2}=\frac{p_2^2}{4p_1}$$

(b) Show that there is a unique equilibrium price p_1.

- The market clearing condition for good 1 is

$$x_1^A+x_1^B+y_1=1$$

substituting our demands from part (a) gives

$$\frac{p_1 + p_2}{4p_1} + \frac{p_1 + p_2 + 2\pi}{8p_1} + \frac{p_2^2}{4p_1^2} = 1$$

substituting our profit level, $\pi = \frac{p_2^2}{4p_1}$, our numeraire ($p_2 = 1$) and rearranging,

$$3p_1 + \frac{5}{2} = 5p_1^2$$

Solving for p_1 yields two solutions, only one of which is positive,

$$p_1 = \frac{3 + \sqrt{59}}{10} \approx 1.068$$

Hence, the equilibrium price ratio is $p_1 \simeq 1.068$.

(c) Assume that the production function is now $y_2 = y_1$, and thus satisfies constant returns to scale. Determine the equilibrium price and allocation (i.e., the WEA).

- Recall that the isoprofit line $\pi = p_2 y_2 - p_1 y_1$ can be rewritten as $y_2 = \frac{\pi}{p_2} + \frac{p_1}{p_2} y_1$. Since the production function $y_2 = y_1$ satisfies constant returns to scale, output decisions are $y_2 = y_1 = +\infty$ if $p_2 > p_1$, $y_2 = y_1 = 0$ if $p_2 < p_1$, or any point in the line $y_2 = y_1$ if $p_2 = p_1$; as depicted in figures 6.19a and 6.19b. (See the chapter on production theory for more details on this type of technology).

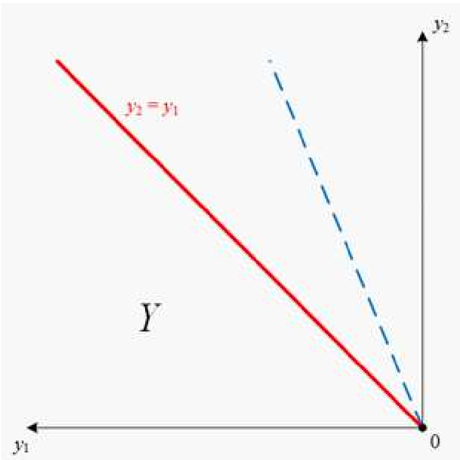

Figure 6.19a. Corner solution.

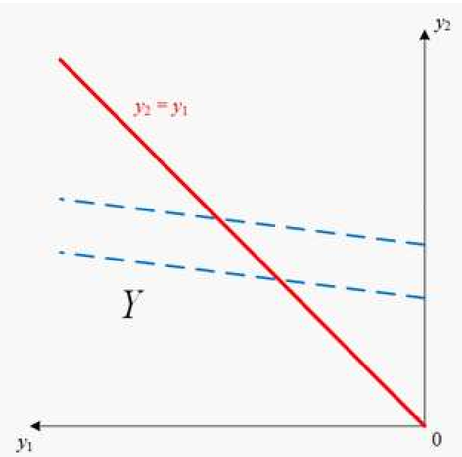

Figure 6.19b. Infinite production plan.

- Let us solve for the equilibrium.
  - In the case $p_2 > p_1$, $y_1 = +\infty$ so that the market for good 1 cannot clear,

i.e., $z_1(p) \neq 0$. There is no equilibrium in this case.

– In the case $p_2 < p_1$, $y_1 = 0$ (and $\pi = 0$). The market clearing condition for good 1 becomes $x_1^A + x_1^B = 1$. Substituting the demands from part (a), and the numeraire yields

$$\frac{p_1 + 1}{4p_1} + \frac{p_1 + 1}{8p_1} + 0 = 1$$

which, solving for p_1 gives $p_1 = \frac{3}{5}$. Howver, this result entails

$$p_1 = \frac{3}{5} < 1 = p_2$$

which contradicts our initial assumption of $p_2 < p_1$. Hence, there is no equilibrium in this case either.

– In this case $p_2 = p_1$, and the demands for good 1 of both consumers A and B reduce to $x_1^A = \frac{p_1+p_1}{4p_1} = \frac{2p_1}{4p_1} = \frac{1}{2}$ and $x_1^B = \frac{p_1+p_1}{8p_1} = \frac{2p_1}{8p_1} = \frac{1}{4}$ (given that $\pi = 0$, whatever y_1 is). The market clearing condition for good 1, $x_1^A + x_1^B + y_1 = 1$, thus becomes $\frac{1}{2} + \frac{1}{4} + y_1 = 1$. Solving for y_1, this implies that $y_1 = \frac{1}{4} \geq 0$. Hence, there is only one equilibrium. Using $p_1 = p_2$, i.e., $\frac{p_1}{p_2} = 1$, and $x_1^A = \frac{1}{2}$ and $x_1^B = \frac{1}{4}$, we can find the remaining elements of the WEA,

$$x_2^A = \frac{p_1 + p_1}{4p_1} = \frac{1}{2} \qquad \text{and} \qquad x_2^B = \frac{3(p_1 + p_1)}{8p_1} = \frac{3}{4}$$

and $y_2 = y_1 = \frac{1}{4}$.

(d) Consider the exchange economy consisting of consumers A and B (in other words, eliminate the firm). Determine the equilibrium (price and allocation).

- The Walrasian demands of A and B are already known from part (a).Since there is no firm $\pi = 0$. The market clearing condition for good 1 is $x_1^A + x_1^B = 1$. Substituting our Walrasian demands yields

$$\frac{p_1 + p_2}{4p_1} + \frac{p_1 + p_2}{8p_1} = 1$$

using our numeraire $p_2 = 1$ yields

$$\frac{p_1 + 1}{4p_1} + \frac{p_1 + 1}{8p_1} = 1$$

Solving for p_1 yields $p_1 = \frac{3}{5}$. Substituting this back into our demand functions yields our equilibrium quantities $x_1^A = \frac{2}{3}$, $x_2^A = \frac{2}{5}$, $x_1^B = \frac{1}{3}$, and $x_2^B = \frac{3}{5}$.

Exercise #22 - Production Economy with CRTS

22. Consider an economy with two consumers $i = \{A, B\}$, one firm (that produces good 2 using good 1 as input) and two goods $l = \{1, 2\}$. Consumer B owns the firm. Good 2 is the numeraire good (i.e., $p_2 = 1$). Consider that consumers' preferences are given by

$$u^A(x_1^A, x_2^A) = x_1^A + 4\sqrt{x_2^A} \quad \text{and} \quad u^B(x_1^B, x_2^B) = x_1^B + 2\sqrt{x_2^B}$$

while their endowments are

$$\omega^A = (4, 12) \quad \text{and} \quad \omega^B = (8, 8)$$

The production function is $y_2 = 3y_1$, and the firm operates in a perfectly competitive market facing prices $p_1 > 0$ and $p_2 > 0$. Compute the equilibrium price and allocation.

- *Consumer A.* Setting up consumer A's utility maximization problem,

$$\max_{x_1^A, x_2^A} \; x_1^A + 4\sqrt{x_2^A} + \lambda^A \left[p_1(4) + p_2(12) - p_1 x_1^A - p_2 x_2^A\right]$$

with first-order conditions

$$\begin{aligned} 1 - \lambda^A p_1 &= 0 \\ \frac{2}{\sqrt{x_2^A}} - \lambda^A p_2 &= 0 \\ p_1(4) + p_2(12) - p_1 x_1^A - p_2 x_2^A &= 0 \end{aligned}$$

combining the first two first-order conditions and rearranging yields

$$\frac{1}{p_1} = \lambda^A = \frac{\frac{2}{\sqrt{x_2^A}}}{p_2} \quad \text{or} \quad x_2^A = \frac{4p_1^2}{p_2^2}$$

Substituting this into the third first-order condition yields

$$p_1(4) + p_2(12) - p_1 x_1^A - p_2\left(\frac{4p_1^2}{p_2^2}\right) = 0 \implies x_1^A = \frac{4p_1 + 12p_2}{p_1} - \frac{4p_1}{p_2}$$

Which is positive if and only if $p_2 > \frac{p_1}{6}(\sqrt{13} - 1)$. Otherwise, consumer A would be at a corner solution where $x_1^A = 0$ and $x_2^A > 0$. In particular, he would spend all his income on good 2, that is,

$$p_2 x_2^A = p_1(4) + p_2(12) \implies x_2^A = \frac{4p_1 + 12p_2}{p_2}$$

In summary, consumer A's demand is

$$(x_1^A, x_2^A) = \begin{cases} \left(\frac{4p_1+12p_2}{p_1} - \frac{4p_1}{p_2}, \frac{4p_1^2}{p_2^2}\right) & \text{if } p_2 > \frac{p_1}{6}(\sqrt{13}-1) \approx 0.434p_1 \\ \left(0, \frac{4p_1+12p_2}{p_2}\right) & \text{if } p_2 \leq \frac{p_1}{6}(\sqrt{13}-1) \end{cases}$$

- *Consumer B*. Setting up consumer B's utility maximization problem,

$$\max_{x_1^B, x_2^B} \quad x_1^B + 2\sqrt{x_2^B} + \lambda^B \left[p_1(8) + p_2(8) - p_1 x_1^B - p_2 x_2^B\right]$$

with first-order conditions

$$\begin{aligned} 1 - \lambda^B p_1 &= 0 \\ \frac{1}{\sqrt{x_2^B}} - \lambda^B p_2 &= 0 \\ p_1(8) + p_2(8) - p_1 x_1^B - p_2 x_2^B &= 0 \end{aligned}$$

combining the first two first-order conditions and rearranging yields

$$\frac{1}{p_1} = \lambda^B = \frac{\frac{1}{\sqrt{x_2^B}}}{p_2} \quad \text{or} \quad x_2^B = \frac{p_1^2}{p_2^2}$$

Substituting this into the third first-order condition yields

$$p_1(8) + p_2(8) - p_1 x_1^B - p_2\left(\frac{p_1^2}{p_2^2}\right) = 0 \implies x_1^B = \frac{8p_1 + 8p_2}{p_1} - \frac{p_1}{p_2}$$

Which is positive if and only if $p_2 > \frac{p_1}{4}(\sqrt{6}-2)$. Otherwise, consumer B would be at a corner solution where $x_1^B = 0$ and $x_2^B > 0$. In particular, he would spend all his income on good 2, that is,

$$p_2 x_2^B = p_1(8) + p_2(8) \implies x_2^B = \frac{8p_1 + 8p_2}{p_2}$$

In summary, consumer B's demand is

$$(x_1^B, x_2^B) = \begin{cases} \left(\frac{8p_1+8p_2}{p_1} - \frac{p_1}{p_2}, \frac{p_1^2}{p_2^2}\right) & \text{if } p_2 > \frac{p_1}{4}(\sqrt{6}-2) \approx 0.112p_1 \\ \left(0, \frac{8p_1+8p_2}{p_2}\right) & \text{if } p_2 \leq \frac{p_1}{4}(\sqrt{6}-2) \end{cases}$$

- *Equilibrium.* To solve for the equillibrium, we distinguish between 2 cases (active/inactive firm).
  - In the case in which the firm is active, we have $\frac{p_1}{p_2} = 3$ in equilibrium. Hence, $p_2 = \frac{p_1}{3}$. Howver, that entails $p_2 < \frac{p_1}{6}(\sqrt{13} - 1)$ and from our above analysis consumer A will be at a corner solution yielding demands

$$\begin{aligned} (x_1^A, x_2^A) &= (0, 24) \quad \text{for individual } A \text{, and} \\ (x_1^B, x_2^B) &= \left(\frac{23}{3}, 9\right) \quad \text{for individual } B \end{aligned}$$

    The market clearing condition is

$$x_1^A + x_1^B + y_1 = 12$$

    Substituting our demands, we can find the input level in equilibrium

$$0 + \frac{23}{3} + y_1 = 12 \implies y_1 = \frac{13}{3}$$

    Finally, since $y_2 = 3y_1$, equilibrium output is $y_2 = 3 \cdot \frac{13}{3} = 13$.
  - In the case in which the firm is inactive, we have that $\frac{p_1}{p_2} > 3$ in equilibrium. Recall that consumer A's consumption of good 1, when positive, is

$$x_1^A = \frac{4p_1 + 12p_2}{p_1} - \frac{4p_1}{p_2}$$

    which can be rewritten as

$$x_1^A = 4\frac{p_2}{p_1}\left(\frac{p_1}{p_2} + 3 - \left(\frac{p_1}{p_2}\right)^2\right)$$

    Let $X = \frac{p_1}{p_2} > 3$. Our condition becomes

$$\frac{4}{X}(-X^2 + X + 3)$$

    For $X > 3$, the quadratic term in the parenthesis is negative, as is its derivative as depicted in Figure 6.20. This implies that $x_1^A < 0$ for all $\frac{p_1}{p_2} > 3$. Hence, consumer A would not consume positive amounts of good 1, $x_1^A = 0$, but instead use all his resources on good 2, i.e., $x_2^A = \frac{4p_1+12p_2}{p_2}$. Since individual B

is the only consumer of good 1, $x_1^B = 12$.

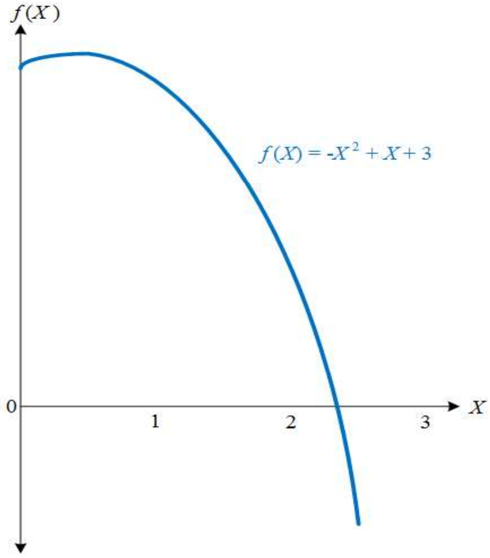

Figure 6.20. Equilibrium price ratio.

Finally, substituting in consumer B's demand yields

$$x_1^B = \frac{8p_1 + 8p_2}{p_1} - \frac{p_1}{p_2} = 12$$

and solving for $\frac{p_1}{p_2}$ yields

$$\frac{p_1}{p_2} = 2\sqrt{3} - 2 \approx 1.464 < 3$$

which contradicts our original condition, i.e. $\frac{p_1}{p_2} > 3$. Hence, there is no equilibrium in this case.

Exercise #23 - Pareto and Efficient allocations in the Household

23. Consider an economy with two individuals, Ann and Bartholomew, each with utility funciton

$$u^A(x^A, l^A) = x^A l^A \quad \text{and} \quad u^B(x^B, l^B) = (x^B)^2 l^B$$

where x denotes a consumption good while l represents hours of leisure. In addition, Ann owns the only firm in this economy and has 20 hours to dedicate to either work (L^A) or leisure (l^A), $20 = L^A + l^A$; whereas Bartholomew does not own any assets in this economy (poor husband!), but has 30 hours to spend, $30 = L^B + l^B$. Ann's firm produces units of good x with labor hours using a Cobb-Douglas production technology $x = \sqrt{L}$, where $L \equiv L^A + L^B$.

(a) Find the set of PEAs.

- Allocations (x^A, l^A) and (x^B, l^B) for consumers and (x, L) for the firm are efficient if they solve

$$\max_{x^A, l^A, x^B, l^B, x, L} \quad x^A l^A$$
$$\text{subject to} \quad (x^B)^2 l^B \geq \bar{u}^B$$
$$\left.\begin{array}{r} x^A + x^B \leq x \\ l^A + l^B + L \leq 20 + 30 \end{array}\right\} \text{feasibility}$$
$$x \leq \sqrt{L} \quad \text{(technological feasibility)}$$

The Lagrangian of the above maximization problem is

$$\begin{aligned} \mathcal{L} &= x^A l^A + \lambda_1 \left[(x^B)^2 l^B - \bar{u}^B \right] - \lambda_2 \left[x^A + x^B - x \right] \\ &\quad - \lambda_3 \left[l^A + l^B + L - 50 \right] - \lambda_4 \left[x - \sqrt{L} \right] \end{aligned}$$

Before taking FOCs, note that both consumers' utility functions are strictly increasing and preferences are strictly convex, and so is the firm's technology, thus yielding interior solutions. Taking FOCs yields

$$\frac{\partial \mathcal{L}}{\partial x^A} = l^A - \lambda_2 = 0, \quad \frac{\partial \mathcal{L}}{\partial l^A} = x^A - \lambda_3 = 0 \tag{1}$$
$$\frac{\partial \mathcal{L}}{\partial x^B} = 2\lambda_1 x^B l^B - \lambda_2 = 0, \quad \frac{\partial \mathcal{L}}{\partial l^B} = \lambda_1 (x^B)^2 - \lambda_3 = 0 \tag{2}$$
$$\frac{\partial \mathcal{L}}{\partial x} = \lambda_2 - \lambda_4 = 0, \quad \frac{\partial \mathcal{L}}{\partial L} = -\lambda_3 + \frac{1}{2}\lambda_4 \frac{1}{\sqrt{L}} = 0 \tag{3}$$

and the four binding constraints. Dividing the two expressions in (1), we obtain

$$\frac{l^A}{x^A} = \frac{\lambda_2}{\lambda_3} \tag{4}$$

Operating similarly with the two expressions in (2) yields

$$\frac{2l^B}{x^B} = \frac{\lambda_2}{\lambda_3} \tag{5}$$

and solving for λ_4 in the two expressions in (3),

$$2\sqrt{L} = \frac{\lambda_2}{\lambda_3} \tag{6}$$

Combining (4)-(6) yields a double equality

$$\underbrace{\frac{l^A}{x^A}}_{MRS^A_{l,x}} = \underbrace{\frac{2l^B}{x^B}}_{MRS^B_{l,x}} = \underbrace{2\sqrt{L}}_{MRTS_{l,x}} \tag{7}$$

Hence, a PEA requires that the marginal rate of substitution between the consumption good and leisure coincides across consumers and, in turn, it must also coincide with the rate at which the firm can transform labor into the consumption good. Condition (7) together with the feasibility conditions in the four binding constraints, characterize the set of PEAs.

(b) Find the set of WEAs.

- A price vector for the single good in the economy, x, and the single input, labor, i.e., (p, w), and an allocation $\{(x^A, l^A), (x^B, l^B), (x, L)\}$ are a WEA if and only if

  1. The price vector (p, w) satisfies

     $$x^A(p, w) + x^B(p, w) = x(p, w)$$

     and

     $$l^A(p, w) + l^B(p, w) + L(p, w) = 50; \text{ and}$$

  2. (x^i, l^i) maximizes individual i's UMP, and (x, L) maximizes the firm's PMP. Let us seperately solve UMP_A, UMP_B, and PMP.

- UMP_A. Consumer A solves

$$\max_{x^A, l^A \geq 0} \quad x^A l^A$$
$$\text{subject to } \; px^A = \pi + w(20 - l^A)$$

which indicates that Ann can use the profits of the firm she owns, π, and the salary of working $L^A = 20 - l^A$ hours (i.e., the hours she does not dedicate to leisure) to purchase units of good x at a price p. The Lagrangian associated to this UMP$_A$ is

$$\mathcal{L} = x^A l^A - \lambda \left[p x^A - \pi - w(20 - l^A) \right]$$

Taking FOCs we obtain

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x^A} &= l^A - p\lambda = 0 \\ \frac{\partial \mathcal{L}}{\partial l^A} &= x^A - w\lambda = 0 \end{aligned}$$

which simplify to

$$\frac{l^A}{x^A} = \frac{p}{w}, \text{ i.e., } MRS^A_{x,l} = \text{ price ratio} \tag{8}$$

and consumer A's budget line

$$p x^A = \pi + w(20 - l^A)$$

Hence, A's demand for good x is

$$p x^A = \pi + w \left(20 - \underbrace{\frac{p}{w} x^A}_{l^A} \right)$$

which, solving for x^A yields

$$x^A = \frac{\pi + 20w}{2p}$$

Therefore, her labor supply is

$$l^A = \frac{p}{w} x^A = \frac{p}{w} \left(\frac{\pi + 20w}{2p} \right) = \frac{\pi + 20w}{2w}$$

- *UMP$_B$*. Consumer B solves

$$\max_{x^B, l^B \geq 0} \quad (x^B)^2 l^B$$
$$\text{subject to } \; p x^B = w(30 - l^B)$$

where consumer B has 30 hours at his disposal, but does not enjoy firm profits. The Lagrangian associated to UMP$_B$ is

$$\mathcal{L} = (x^B)^2 l^B - \lambda \left[p x^B - w(30 - l^B) \right]$$

Taking FOCs we obtain

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x^B} &= 2x^B l^B - p\lambda = 0 \\ \frac{\partial \mathcal{L}}{\partial l^B} &= (x^B)^2 - w\lambda = 0 \end{aligned}$$

which simplify to

$$\frac{2l^B}{x^B} = \frac{p}{w}, \text{ i.e., } MRS^B_{x,l} = \text{ price ratio} \tag{9}$$

and his budget line

$$p x^B = w(30 - l^B)$$

Hence, consumer B's demand for good x is

$$p x^B = w \left(30 - \underbrace{\frac{p}{2w} x^B}_{l^B} \right)$$

which, solving for x^B yields

$$x^B = \frac{20w}{p}$$

Therefore, his labor supply is

$$l^B = \frac{p}{2w} x^B = \frac{p}{2w} \left(\frac{20w}{p} \right) = 10$$

- *PMP*. Finally, the firm chooses q and L to solve

$$\max_{q,L \geq 0} \; pq - wL$$
$$\text{subject to } \; q \leq \sqrt{L}$$

However, since the constraint will bind at the optimum, i.e., $q = \sqrt{L}$ we can rewrite the PMP as an unconstrained maximization problem with a single choice variable

$$\max_{L} \; p\sqrt{L} - wL$$

Taking FOCs with respect to L yields

$$p\frac{1}{2}\frac{1}{\sqrt{L}} - w = 0, \text{ or } 2\sqrt{L} = \frac{p}{w} \tag{10}$$

i.e., the marginal rate at which labor can get transformed into units of the consumption good must coincide with the price ratio. Therefore, for a WEA, we need conditions (8)-(10) to hold. In particular, conditions (8) and (9) imply

$$\frac{l^A}{x^A} = \frac{2l^B}{x^B} = \frac{p}{w}$$

i.e., $MRS^A_{x,l} = MRS^B_{x,l} = \frac{p}{w}$; and from condition (10)

$$\frac{l^A}{x^A} = \frac{2l^B}{x^B} = 2\sqrt{L} \tag{11}$$

i.e., $MRS^A_{x,l} = MRS^B_{x,l} = MRTS_{x,l}$.

- Putting everything together, we can start by combining our feasibility and technological feasibility conditions to obtain

$$x^A + x^B = \sqrt{L}$$

multiplying both sides by 2,

$$2(x^A + x^B) = 2\sqrt{L}$$

which allows us to make a substitution from equation (11). Plugging in the second equality from equation (11) yields

$$2(x^A + x^B) = \frac{2l^B}{x^B} = \frac{p}{w}$$

and plugging in the Walrasian demands for x^A and x^B gives

$$2\left(\frac{\pi + 20w}{2p} + \frac{20w}{p}\right) = \frac{p}{w}$$

Let the price of labor, w, serve as the numeraire, i.e., $w = 1$, and solving the above equation for p yields

$$p = \sqrt{\pi + 60}$$

Next, using our feasibility constraint for labor,

$$l^A + l^B + L = 50$$

and since $l^B = 10$,

$$L = 40 - l^A$$

we can use this equation, and our equation for prices to solve for the profit level, π

$$\pi = p\sqrt{L} - L = \sqrt{\pi + 60}\sqrt{40 - \underbrace{\frac{\pi + 20}{2}}_{l_A}} - \left(40 - \frac{\pi + 20}{2}\right)$$

Simplifying,

$$\pi + 60 = 2\sqrt{(\pi + 60)\left(30 - \frac{\pi}{2}\right)}$$

Squaring both sides,

$$(\pi + 60)^2 = 4(\pi + 60)\left(30 - \frac{\pi}{2}\right)$$

and solving for π yields a profit of $\pi = 20$. Plugging this back into our equation for prices gives

$$p = \sqrt{\pi + 60} = \sqrt{80} \approx 8.94$$

and substituting this value into the Walrasian demands gives the unique WEA $\{(x^A, l^A), (x^B, l^B), (x, L)\} = \{(2.23, 20), (2.23, 10), (4.47, 20)\}$ with price vector $(p, w) = (8.94, 1)$ and equilibrium profits of $\pi = 20$.

(c) Is the WEA you found in part (b) part of the set of PEAs?

- We only need to check that the WEA from part (b) (1) satisfies condition (7), i.e., $MRS^A_{x,l} = MRS^B_{x,l} = MRTS_{x,l}$, which does as described at the end of the firm's PMP; and (2) satisfies the four feasibility conditions, that is

$$\begin{aligned}
\frac{l^A}{x^A} &= \frac{2l^B}{x^B} = 2\sqrt{L} \\
(x^B)^2 l^B &\geq \bar{u}^B = 0 \\
x^A + x^B &= x \\
l^A + l^B + L &= 50 \\
q &= \sqrt{L}
\end{aligned}$$

For the first equation, plugging in our values yields $\frac{20}{2.23} = \frac{2 \cdot 10}{2.23} = 2\sqrt{20}$. For the second, consumer B reaches a utility level of $(2.23)^2 * 10 = 50 > 0$. For the third through fifth equations, we find that they hold by definition due to being used to find the WEA in part (b).

Exercise #24 - Equilibrium with Production

24. Consider an economy with two goods, 1 and 2, both of them being produced by using capital and labor. Firms are price takers, and output prices are determined in the international market. The output factors of goods 1 and 2 are

$$\begin{aligned}
q_1 &= (K_1)^{\frac{1}{4}} (L_1)^{\frac{3}{4}} \\
q_2 &= (K_2)^{\frac{3}{4}} (L_2)^{\frac{1}{4}}
\end{aligned}$$

(a) Find the marginal cost for each firm.

- *Firm 1.* Starting with firm 1's cost minimization problem,

$$\begin{aligned}
&\min_{K_1, L_1 \geq 0} \quad w_K K_1 + w_L L_1 \\
\text{subject to} \quad & q_1 = (K_1)^{\frac{1}{4}} (L_1)^{\frac{3}{4}}
\end{aligned}$$

We can express this minimization problem as a maximization problem of the negative of the objective function, or

$$\begin{aligned}
&\max_{K_1, L_1 \geq 0} \quad - w_K K_1 - w_L L_1 \\
\text{subject to} \quad & q_1 = (K_1)^{\frac{1}{4}} (L_1)^{\frac{3}{4}}
\end{aligned}$$