Structural reliability analysis based on the dynamic integrity of an attractor

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Motivation

- Stability, dynamic integrity, and reliability are essential properties of a dynamical system's attractor for safe design purposes.
- The ideas and procedures discussed herewith are applicable to any dynamical system, but will be illustrated within the context of structural stability.

A historical perspective on buckling strength

- Buckling load of “perfect” compressed rods: “Euler’s load”
- Buckling loads of “perfect” plates, shells and other slender structures: Timoshenko
- Imperfection sensitivity of buckling loads: von Kármán, Koiter, Donnel & Wan: “Koiter’s load”
- Statistical variation of buckling strength due to randomicity of imperfections: Thompson, Roorda
- A first reliability analysis of buckling: Perry & Chilver
- GIM and LIM dynamic integrity of a nonlinear system’s attractor: Soliman & Thompson
- IF dynamic integrity of a nonlinear attractor: Lenci & Rega
- Safe buckling strength: Rega: “Thompson’s load”

A historical perspective on buckling strength

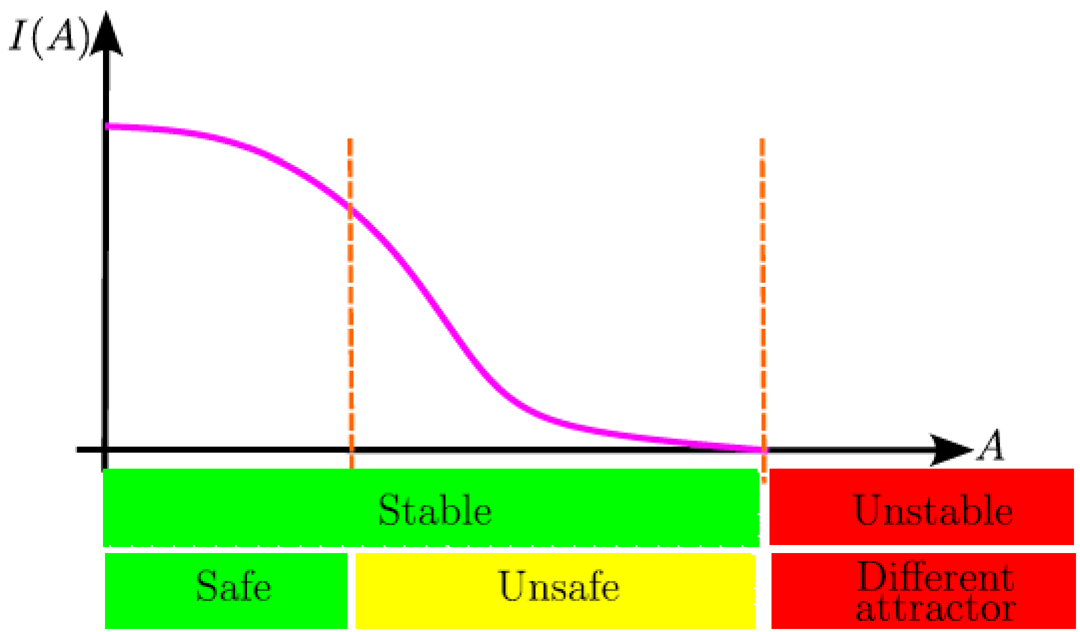
- However, a reliability measure for "Thompson's load" is still missing: the integrity measure may be affected by the statistical variation of "Koiter's load", considered as a random variable, depending, on its turn, on the imperfection randomicity.
- Hence, what is aimed is something like Perry&Chilver have done with respect to "Koiter's load", yet this time with respect to "Thompson's load".
- The probability that the integrity measure would be at least a safely chosen value depends on the load not overcoming an associated safely chosen "Thompson's load": the buckling strength considering structural stability, dynamic integrity and reliability.

A historical perspective on buckling strength

- To calculate the usual dynamic-integrity measures (GIM, LIM and IF) it is necessary to obtain the basin of attraction of the attractor of interest: this is still a computationally demanding task for systems with several degrees of freedom.
- A tool, such as the in-house code *Poli BoA*, is necessary for obtaining the basins of attraction and the associated dynamic-integrity measure chosen, generically referred to as I .
- This procedure is repeated for different values of a convenient system parameter *A* (e.g., applied load, imperfection, etc.), within a convenient range.

A historical perspective on buckling strength

- Typically, the graph *I(A)* will indicate a decrease in the dynamic-integrity measure *I* as the parameter *A* increases, even with a steep gradient, thus justifying the nomenclature of an erosion profile: Thompson's 'Dover Cliff'.



Probability density functions of input and output

- In usual applications the randomicity of parameter A comes from multiple and independent factors: it is a reasonable assumption that its probability density function is a normal distribution (central limit theorem), as recalled by Roorda.

- Hence, for the input distribution one writes $f(A) = \frac{1}{\sigma_A\sqrt{2\pi}} e^{-\frac{(A-\bar{A})^2}{2\sigma_A{}^2}}$

- It is also assumed that the output distribution is normal too: $f(I) = \frac{1}{\sigma_I\sqrt{2\pi}} e^{-\frac{(I-\bar{I})^2}{2\sigma_I{}^2}}$

Simplified procedure to evaluate the output distribution's statistical parameters

- Suppose the erosion profile $I(A)$ is known and $tan\alpha = \left|\frac{dI}{dA}\right|$ is the absolute value of its local slope.
- It is postulated that the output integrity measure standard deviation is given by $\sigma_I = \sigma_A\ tan\alpha$ about the expected value $\bar{I}$.
- Hence, for every point $(\bar{A}, \bar{I})$ of the erosion profile, it can be defined the cut-off region for which the integrity measure complies with $I \geq I_{ref}$, provided $A \leq A_{ref}$, leading to the probability assigned to safety as a direct reliability measure.

Formal procedure to evaluate the output distribution's statistical parameters

- Auxiliary variables: $\beta = A - \bar{A}$ and $\rho(\beta) = I(A) - I(\bar{A}) = I(\bar{A} + \beta) - I(\bar{A})$

- For $\rho(\beta)$ a power series expansion can be written : $\rho(\beta) = \sum_{n=1}^{\infty} \frac{1}{n!} a_n \beta^n$

- Using the notation $E[x]$ for the expected value of a generic variable x, and t for an auxiliary dummy variable, the moment-generating function can be written as:

$$M_\rho(t) = E[e^{\rho t}] = E\left[e^{t \sum_{n=1}^{\infty} \frac{1}{n!} a_n \beta^n}\right] = E\left[\prod_{n=1}^{\infty} e^{\frac{1}{n!} a_n \beta^n t}\right]$$

$$\cong 1 + t \sum_{n=2}^{\infty} \frac{1}{n!} a_n \mu_n + \frac{t^2}{2} \left(\sum_{n=1}^{\infty} \frac{1}{n!^2} {a_n}^2 \mu_{2n} + 2 \sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} \frac{1}{n!\, m!} a_n a_m \mu_{n+m} \right)$$

- $\mu_n = E[\beta^n]$ are the central moments of the input distribution. Notice that $\mu_1 = 0$

Formal procedure to evaluate the output distribution's statistical parameters

- The input moment-generating function can be used to obtain the expected value $\bar{\rho}$ (first central moment η_1) and the other central moments η_r of the output distribution, as follows:

$$\bar{\rho} = \bar{I} - I(\bar{A}) = \left[\frac{\partial M_\rho}{\partial t}\right]_{t=0} = \sum_{n=2}^{\infty} \frac{1}{n!} a_n \mu_n$$

$$\eta_r = \left[\frac{\partial^r M_{\rho-\bar{\rho}}}{\partial t^r}\right]_{t=0}, r = 2,3, \ldots$$

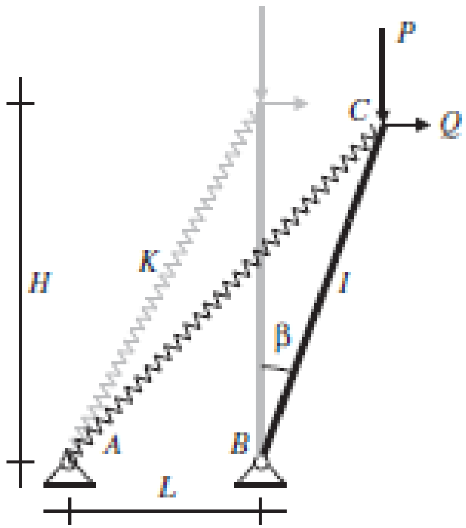
- From which the expected value and variance of the dynamic integrity measure are:

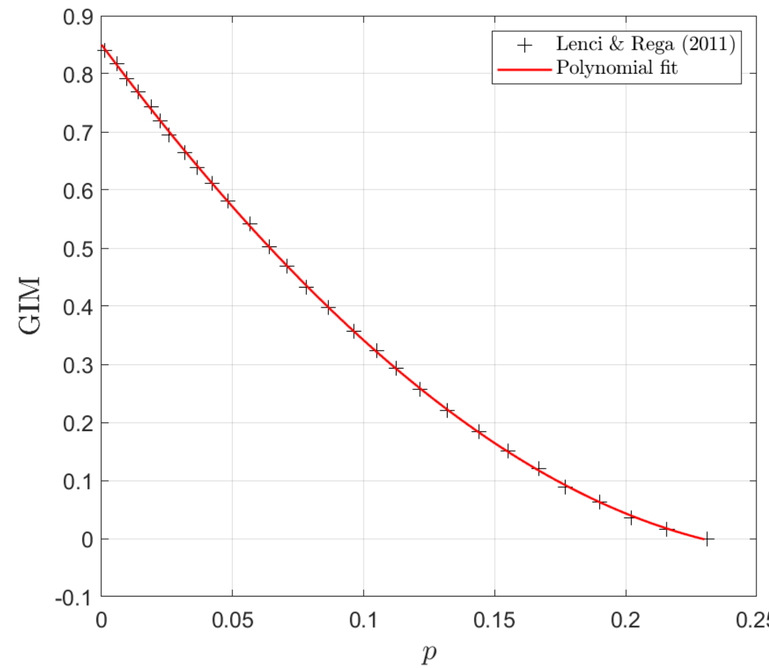
$$\bar{I} = I(\bar{A}) + \bar{\rho} = I(\bar{A}) + \sum_{n=2}^{\infty} \frac{1}{n!} a_n \mu_n$$

$$\eta_2 = \sum_{n=1}^{\infty} \frac{1}{n!^2} {a_n}^2 \mu_{2n} + 2 \sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} \frac{1}{n!\,m!} a_n a_m \mu_{n+m} - \left(\sum_{n=2}^{\infty} \frac{1}{n!} a_n \mu_n\right)^2$$

Application to an archetypal model

- Lenci and Rega's model and the erosion profile found for the integrity measure $I = GIM$ in function of the axial load $A = p$ for $q = 0.03$





- Suggestion of Thompson's load to be $\bar{A} = p_T \cong 0.175$ and the corresponding $I(\bar{A}) = GIM_T \cong 0.100$ (this will be referred to as Scenario 2).

Application to an archetypal model: simplified procedure

- A cubic polynomial fitting for the curve for $I(A) = b_0 + b_1 A + b_2 A^2 + b_3 A^3$, with $b_0 = 0.8500$, $b_1 = -6.0556$, $b_2 = 9.2479$ and $b_3 = 4.2930$.
- The local erosion profile slope in Scenario 2 is $tan\alpha = \left|\frac{dI}{dA}(0.175)\right| \cong 2.4244$.
- Assuming, for the sake of an example, a standard deviation $\sigma_A = 0.020$, the estimated output standard deviation would be $\sigma_I = 0.049$.
- Probability of 31.7% for GIM to be at least $GIM_T + \sigma_I = 0.149$ provided p is not larger than $p_T + \sigma_A = 0.195$.
- Probability of 50% for GIM to be at least $GIM_T = 0.100$ provided p is not larger than $p_T = 0.175$.
- Probability of 68.3% for GIM to be at least $GIM_T - \sigma_I = 0.051$ provided p is not larger than $p_T - \sigma_A = 0.155$.

Application to an archetypal model : simplified procedure

- Other simulations for $\sigma_A = 0.020$ with probabilities of lower bounds of GIM and upper bounds of p:

Prob. [%]	Scenario 1		Scenario 2		Scenario 3		Scenario 4		Scenario 5	
	$(\bar{p}; \overline{GIM})$ (0.215; 0.018)		$(\bar{p}; \overline{GIM})$ (0.175; 0.100)		$(\bar{p}; \overline{GIM})$ (0.135; 0.208)		$(\bar{p}; \overline{GIM})$ (0.095; 0.362)		$(\bar{p}; \overline{GIM})$ (0.055; 0.546)	
	$GIM \geq$	$p \leq$	$GIM \geq$	$p \leq$	$GIM \geq$	$p \leq$	$GIM \geq$	$p \leq$	$GIM \geq$	$p \leq$
31.73	0.092	0.215	0.149	0.195	0.208	0.175	0.279	0.155	0.357	0.135
50	0.053	0.195	0.100	0.175	0.150	0.155	0.212	0.135	0.282	0.115
68.27	0.004	0.175	0.051	0.155	0.092	0.135	0.145	0.115	0.207	0.095

Application to an archetypal model : simplified procedure

- Notice that for $p \leq 0.175$, there is a 31.73% probability that $GIM \geq 0.208$ (Scenario 3), 50% probability that $GIM \geq 0.100$ (Scenario 2), but a 68.27% probability that GIM would not be acceptable, since $GIM_T - \sigma_I$ is almost null (Scenario 1)!
- Alternatively, for $p \leq 0.155$, there is a 31.73% probability that $GIM \geq 0.279$ (Scenario 4), 50% probability that $GIM \geq 0.150$ (Scenario 3), and a 68.27% probability that $GIM \geq 0.051$ (Scenario 2), still positive.
- Finally, for $p \leq 0.135$, there is a 31.73% probability that GIM would be anti-economically large (Scenario 5), 50% probability that $GIM \geq 0.212$ (Scenario 4), and a 68.27% probability that $GIM \geq 0.092$ (Scenario 3).
- Would $p_T = 0.155$ or $p_T = 0.135$ be a better choice than $p_T = 0.175$ for a safe & economical engineering design?

Application to an archetypal model : formal procedure

- Only Scenario 2 is focused.
- Due to probability distribution symmetry, $\mu_3 = 0$.
- The only non-null central moment is $\mu_2 = \sigma_A^2 = 0.0004$.
- Cubic polynomial fitting is $\rho(\beta) = a_1\beta + a_2\beta^2 + a_3\beta^3$, where $a_1 = b_1 + 2b_2\bar{A} + 3b_3\bar{A}^2 = -2.4244$, $a_2 = b_2 + 3b_3\bar{A} = 11.5017$ and $a_3 = b_3 = 4.2930$.
- Hence $\bar{\rho} = \frac{1}{2}a_2\mu_2 \cong 0.002$ and $\eta_2 = a_1^2\mu_2 - \frac{1}{4}a_2^2\mu_2^2 = 0.002$.
- Finally, for the output variable $I = GIM$, the following statistical properties are found: $\bar{I} = I(\bar{A}) + \bar{\rho} \cong 0.102$, to be compared with $GIM_T \cong 0.100$, and $\sqrt{\eta_2} \cong 0.048$, to be compared with $\sigma_I = 0.049$, indicating a very good agreement with the simplified method.

Conclusion

- Both procedures were illustrated for an archetypal model in which a safe threshold is searched for the buckling load so as not to undergo a critical reduction or fractalization of the basin of attraction of the desired equilibrium configuration.
- A critical reasoning is raised about the choice of the safe, yet still economical, Thompson's load for the problem at hand.
- The easiness of application of both the simplified and the formal procedures, which closely agree with each other, gives hope to their adoption in the engineering design practice.
- One should not underestimate the determination of the erosion curve stage, which still poses some difficulties for systems with large number of degrees of freedom, thus emphasizing the importance of reduced-order models.

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