B.6

Martin's Axiom

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Martin's axiom, known as MA, can be stated in a number of different ways. The topological form of MA is easy to remember. The Boolean algebra form is sometimes reassuringly familiar. But the partial order form is the useful one. It is an unfamiliar hurdle for nonlogicians. But it is more or less necessary for efficient use of MA; it is a first step in forcing techniques; and it is really not difficult.

If $\langle P, \geq \rangle$ is a partially ordered set, then $D \subset P$ is *dense* provided, for each $p \in P$, there is a $d \in D$ with $p \geq d$. A subset Q of P is *compatible* provided, for each finite subset F of Q, there is a $q \in P$ such that $p \geq q$ for all $p \in F$. ccc is read *countable chain condition* and $\langle P, \geq \rangle$ is ccc provided every pairwise incompatible subset is countable.

MARTIN'S AXIOM: *Suppose that $\langle P, \geq \rangle$ is a* ccc *partially ordered set and $\mathcal{D}$ is a family of less than 2^ω dense subsets of P. Then there is a compatible subset Q of P which meets every member of $\mathcal{D}$.*

In Section 6 of Chapter B.4 Burgess proves:

1. THEOREM. *If* ZFC *is consistent, then* ZFC + MA + $\neg$ CH *is consistent.*

MA is also known to be independent of $\neg$CH.

We begin by proving that the above partial order form of MA implies the topological form; actually they are equivalent.

A topological space is ccc if every family of disjoint open sets is countable.

2. THEOREM (MA)*. *If X is a* ccc *compact Hausdorff space, then X is not the union of less than 2^ω nowhere dense sets.*

PROOF. Suppose that $\omega \leq \lambda < 2^\omega$, that X is a ccc compact Hausdorff space, and that $\{X_\alpha\}_{\alpha\in\lambda}$ is a family of nowhere dense subsets of X.

Let P be the set of all nonempty open sets in X and partially order P by $p \leq q$ if $p \subset q$. Since X is ccc so is P. For $\alpha \in \lambda$, define $D_\alpha = \{p \in P \mid \bar{p} \cap X_\alpha = \emptyset\}$; each D_α is dense in P. Hence there is a compatible $Q \subset P$ which intersects every D_α. Q is a basis for a filter and X is compact, so there is an $x \in \bigcap\{\bar{q} \mid q \in Q\}$. For each $\alpha \in \lambda$ there is a $q \in Q$ with $\bar{q} \cap X_\alpha = \emptyset$, so $x \notin X_\alpha$. Hence $X \neq \bigcup_{\alpha\in\lambda} X_\alpha$. $\square$

If CH, then Theorem 2 is just the Baire category theorem, so as no surprise:

* We write "Theorem (MA)" to indicate that we use Martin's Axiom to prove the theorem.

3. THEOREM. CH *implies* MA.

PROOF. Suppose that $\{D_n\}_{n\in\omega}$ is a family of dense subsets of a ccc partially ordered set $\langle P, \geq\rangle$. By induction choose $d_n \in D_n$ such that $d_n > d_{n+1}$ for all n. Then $\{d_n\}_{n\in\omega}$ is a compatible subset of P which meets every D_n. □

A basic fact about partial orders is:

4. THEOREM (MA + $\neg$CH). *If* $\langle P, \geq\rangle$ *is a* ccc *partial order and* $R \subset P$ *is uncountable, then there is an uncountable compatible* $Q \subset R$.

PROOF. Without loss of generality $|R| = \omega_1$. Let

$$G = \{p \in P \mid |\{q \in R \mid p \text{ and } q \text{ are compatible}\}| = \omega\}.$$

Let G^* be a maximal pairwise incompatible subset of G. Let $P^* = \{p \in P \mid p$ is *not* compatible with any $q \in G^*\}$ and let $R^* = R \cap P^*$.

Since $\langle P, \geq\rangle$ is ccc, G^* is countable and $R - R^*$ is countable. Thus there is a one-to-one indexing $\{q_\alpha\}_{\alpha\in\omega_1}$ of R^*.

Let P' be the set of all finite, compatible in $\langle P, \geq\rangle$, subsets of P^*. Observe that if $F \in P'$ and $p \geq q$ for all $p \in F$, then $q \in P^*$. Thus P', partially ordered by reverse inclusion is ccc.

For each $\alpha \in \omega_1$, define $D_\alpha = \{F \in P' \mid F \cap \{q_\beta\}_{\beta>\alpha} \neq \emptyset\}$.

For each $F \in P'$ there is a $q \in P^*$ with $p \geq q$ for all $p \in F$. Since q is compatible with uncountably many q_β, each D_α is dense in P'. So there is a compatible $Q' \subset P'$ which meets every D_α. Thus $Q = \bigcup Q'$ is an uncountable compatible subset of P. □

Recall (see Chapter B.3) that a Souslin tree is an uncountable tree with no uncountable chains or antichains. In a tree a compatible set in reverse order is a chain; so Theorem 4 yields:

5. THEOREM (MA + $\neg$CH). *There is no Souslin tree.*

A Souslin tree can be used to build a Souslin line — a connected, linearly ordered, ccc space which is not separable. It is well known that the product of two Souslin lines is *not* ccc. However:

6. THEOREM (MA + $\neg$CH). *The product of any family of* ccc *spaces is* ccc.

PROOF. By a simple Δ-system argument (not using MA, see Theorem 5.8 of

Chapter B.3), if a product is not ccc, then some subproduct of finitely many factors is not ccc. So to prove Theorem 6 we need only show that the product of any two ccc spaces is ccc.

Suppose that X and Y are ccc and that $\{U_\alpha \times V_\alpha\}_{\alpha\in\omega_1}$ are disjoint, nonempty basic open sets in $X \times Y$. Let P be the set of all nonempty open sets in X partially ordered by inclusion. By Theorem 4 there is an uncountable compatible subset Q of $\{U_\alpha\}_{\alpha\in\omega_1}$. But if U_α and U_β belong to Q, $U_\alpha \cap U_\beta \neq \emptyset$ so $V_\alpha \cap V_\beta = \emptyset$. Thus $\{V_\alpha \mid U_\alpha \in Q\}$ is a family of disjoint open sets which contradicts the fact that Y is ccc. $\square$

MA was first discovered by MARTIN and SOLOVAY [1970], who observed that MA was inherent in a number of previous proofs of the consistency that there be no Souslin trees. Their original paper is a beautiful explanation of MA which I recommend. They stress the usefulness of MA as a viable alternative to CH. They point out that many of the traditional problems solved using CH can be solved using MA alone. Frequently the part of CH that is used is only MA. But almost equally often, a statement *true* under CH can be proved *false* under MA + $\neg$CH. Since MA + $\neg$CH is consistent with ZFC, any such statement is itself independent of ZFC.

MA is severely limited by the fact that it only says something interesting about cardinals λ where $\omega < \lambda < 2^\omega$. However these are precisely the cardinals which most often cause grief for nonlogicians. Certainly general topologists have found MA applicable to a rich variety of their problems. Those discussed by Juhász in Chapter B.7 give some feeling for its use and we will not attempt to give references for the multitude of other topological uses of MA.

An analyst who works with Banach spaces recently asked me two questions. Must a compact Hausdorff space of cardinality $\leq 2^\omega$ be sequentially compact? Must a ccc compact Hausdorff space with a point countable separating family of open F_σ's be metrizable? Neither question can be answered in ZFC. The analyst already knew that MA + $\neg$CH implies that the answer to both questions is *yes*. Analysts too are beginning to use and recognize MA.

An important recent result using MA is in algebra. Shelah has recently proved that Whitehead's problem is undecidable in ZFC: if V = L, then every W-group is free; but if MA + $\neg$CH, there is a W-group which is not free. There is an excellent description of Whitehead's problem and Shelah's solution written for the Monthly by EKLOF [1977] so we will avoid the necessary definitions and heartily recommend the reading of Eklof's article.

Another recommended Monthly article, written by SHOENFIELD [1975], gives an elegant, elementary discussion of MA and proves many of the same theorems given here.

Shelah's solution of Whitehead's problem illustrates the relationship between MA + $\neg$CH and V = L often seen in topology. Roughly speaking, MA + $\neg$CH completely unravels the area between $\omega = 2^\omega$ while V = L holds it completely rigid. Anyway these two axioms are strongly contrasting and often yield contradictory theorems.

Let us turn now to some of the combinatorial consequences of MA:

7. THEOREM (MA). *Suppose that $\mathscr{A}$ and $\mathscr{B}$ are families of cardinality less than 2^ω of subsets of ω such that, for all finite subsets $\mathscr{C}$ of $\mathscr{A}$ and elements B of $\mathscr{B}$, $B - \bigcup \mathscr{C}$ is infinite. Then there is an $M \subset \omega$ such that $B - M$ is infinite for all $B \in \mathscr{B}$ but $A - M$ is finite for all $A \in \mathscr{A}$.*

PROOF. Index $\mathscr{A} = \{A_\alpha\}_{\alpha\in\lambda}$ and $\mathscr{B} = \{B_\alpha\}_{\alpha\in\lambda}$ for some $\lambda < 2^\omega$. Let $P = \{\langle H, K\rangle \mid H$ is a finite subset of λ and K is a finite subset of $\omega\}$. Define $\langle H, K\rangle \geq \langle H', K'\rangle$ in P provided $H \subset H'$, $K \subset K'$, and $(K' - K) \cap \bigcup_{\alpha\in H} A_\alpha = \emptyset$.

Any uncountable subset of P has two members with the same second element, say $\langle H, K\rangle$ and $\langle H', K\rangle$. Since $\langle H, K\rangle \geq \langle (H \cup H'), K\rangle$ and $\langle H', K\rangle \geq \langle (H \cup H'), K\rangle$, P is ccc.

For each $\alpha \in \lambda$, define $D_\alpha = \{\langle H, K\rangle \in P \mid \alpha \in H\}$. For $\alpha \in \lambda$ and $n \in \omega$, define $E_{\alpha,n} = \{\langle H, K\rangle \in P \mid |K \cap B_\alpha| > n\}$. It is easy to check that each member of $\mathscr{D} = \{D_\alpha \mid \alpha \in \lambda\} \cup \{E_{\alpha,n} \mid \alpha \in \lambda$ and $n \in \omega\}$ is dense in P. Since $|\mathscr{D}| = \lambda < 2^\omega$, MA implies that there is a compatible subset Q of P which meets every member of $\mathscr{D}$.

Define $M = \omega - \bigcup \{K \mid \langle H, K\rangle \in Q\}$. If $\alpha \in \lambda$ there is $\langle H, K\rangle \in Q \cap D_\alpha$. If $\langle H', K'\rangle$ is any other member of Q, since Q is compatible, there is $\langle H'', K''\rangle \in Q$ such that $\langle H'', K''\rangle \leq \langle H, K\rangle$ and $\langle H'', K''\rangle \leq \langle H', K'\rangle$. Since $K' \subset K''$ and $(K'' - K') \subset A_\alpha$, $(K' - A_\alpha) \subset K$. Thus $A_\alpha - M \subset K$ so $A_\alpha - M$ is finite.

Since for each $n \in \omega$ there is an $\langle H, K\rangle \in Q$ with $|K \cap B_\alpha| > n$, $B_\alpha - M$ is infinite for all $\alpha \in \lambda$. $\square$

The proof of Theorem 7 is typical of MA proofs. With $\lambda = \omega$, this theorem is a frequently used fact which does not need MA in its proof. But MA allows us to extend the theorem to all cardinals less than 2^ω although such an extension would be false without some set theoretic assumption beyond ZFC.

Two direct consequences of Theorem 7 which are sometimes more immediately applicable are:

8. Corollary (MA). *If $\mathscr{B}$ is a family of cardinality less than 2^ω of subsets of ω with each finite subset of $\mathscr{B}$ having infinite intersection, then there is an infinite subset L of ω such that $L - B$ is finite for all $B \in \mathscr{B}$.*

Proof. Define $\mathscr{A}^* = \{(\omega - B) \mid B \in \mathscr{B}\}$ and $B^* = \{\omega\}$. By Theorem 7 applied to $\mathscr{A}^*$, $\mathscr{B}^*$, there is an M such that $A - M$ is finite for all $A \in \mathscr{A}^*$ but $\omega - M$ is infinite. Define $L = \omega - M$. $\square$

9. Corollary (MA). *Suppose that $\mathscr{A}$ and $\mathscr{B}$ are families of cardinality less than 2^ω of subsets of ω such that $B - \bigcup \mathscr{C}$ is infinite for all $B \in \mathscr{B}$ and finite $\mathscr{C} \subset \mathscr{B}$. Then, if $|\mathscr{A} \cup \mathscr{B}| = \lambda$, there is an infinite subset L of ω such that $A \cap L$ is finite for all $A \in \mathscr{A}$ and $B \cap L$ is infinite for all $B \in \mathscr{B}$.*

Proof. By Theorem 7 there is an $M \subset \omega$ such that $A - M$ is finite for all $A \in \mathscr{A}$ and $B - M$ is infinite for all $B \in \mathscr{B}$. Define $L = \omega - M$. Then, for $B \in \mathscr{B}$, $B - M = B - (\omega - L) = L - (\omega - B) = L \cap B$ is infinite. Similarly $L \cap A$ is finite for all $A \in \mathscr{A}$. $\square$

A consequence of Corollary 8 is:

10. Corollary (MA). *If $\lambda < 2^\omega$, then 2^λ is sequentially compact.*

Proof. Let $\{f_n\}_{n\in\omega}$ be an infinite subset of 2^λ. We show that, if f is an arbitrary limit point of $\{f_n\}_{n\in\omega}$, then there is an $L \subset \omega$ such that $\{f_n\}_{n\in L}$ converges to f.

Let G be the set of all functions into 2 which f extends whose domain is a finite subset of λ. For $g \in G$ let $B_g = \{n \in \omega \mid f_n \text{ extends } g\}$. Since $|G| = \lambda$, the hypotheses of Corollary 8 are satisfied and there is an infinite $L \subset \omega$ such that $L - B_g$ is finite for all $g \in G$. L clearly has the desired property. $\square$

Actually, in Corollary 10, 2^λ may be replaced by any compact Hausdorff space of cardinality less than 2^{2^ω} (see Corollary 1 to Theorem 1.7 in Chapter B.7).

For an application of Corollary 9, define $F = \{f : \omega \to \omega\}$ with $f < g$ in F provided there is an $n \in \omega$ with $f(k) < g(k)$ for all $k > n$. A subset G of F is *dominating* provided, for every $f \in F$ there is a $g \in G$ with $g > f$.

$\{f_\alpha\}_{\alpha<\lambda} \subset F$ is called a *scale* if it is dominating and $\alpha < \beta < \lambda$ implies that $f_\alpha < f_\beta$.

11. COROLLARY (MA). *Every dominating family has cardinality* 2^ω, *and there is a scale.*

PROOF. Suppose that $\lambda < 2^\omega$ and that $\{f_\alpha\}_{\alpha\in\lambda} \subset F$. For $\alpha \in \lambda$, define $A_\alpha = \{\langle i,j\rangle \in \omega^2 \mid j \leq f(i)\}$ and for $i \in \omega$ define $B_i = \{i\} \times \omega$. By Corollary 9 there is an $L \subset \omega^2$ such that all $A_\alpha \cap L$ are finite and $B_i \cap L$ are infinite. Choose $f \in F$ with $(i, f(i)) \in L$ for all $i \in \omega$. Then $f_\alpha < f$ for all $\alpha \in \lambda$. $\square$

A trivial consequence of Corollary 11 is that 2^ω is regular. A consequence of Corollary 9 (which was first used to show the consistency of a counterexample to the normal Moore space conjecture) is the following.

12. THEOREM (MA). *If X is a subset of the real numbers of cardinality less than* 2^ω, *then every subset of X is a relative G_δ set.*

PROOF. Suppose that $Y \subset X$ and that $\{U_n\}_{n\in\omega}$ is a countable open basis for the real numbers such that no pair of distinct numbers is in the intersection of infinitely many U_n's.

Index $Y = \{y_\alpha\}_{\alpha\in\lambda}$ and $X - Y = \{x_\alpha\}_{\alpha\in\lambda}$: we assume without loss of generality that neither is empty and repetitions do not matter. For $\alpha \in \lambda$, let $B_\alpha = \{n \in \omega \mid y_\alpha \in U_n\}$ and let $A_\alpha = \{n \in \omega \mid x_\alpha \in U_n\}$. Define $\mathcal{A} = \{A_\alpha\}_{\alpha\in\lambda}$ and $\mathcal{B} = \{B_\alpha\}_{\alpha\in\lambda}$. By Corollary 10 there is an $L \subset \omega$ such that $L \cap B_\alpha$ is infinite and $L \cap A_\alpha$ is finite for all $\alpha \in \lambda$. For $n \in \omega$ define $L_n = \bigcup_{m\in M} U_m$. Since each $B_\alpha \cap L$ is infinite, $y_\alpha \in \bigcap_{n\in\omega} L_n$. But each $A_\alpha \cap L$ is finite so $x_\alpha \notin \bigcap_{n\in\omega} L_n$. Thus Y is a relative G_δ. $\square$

If X is infinite, the cardinality of the set of all G_δ sets in X is 2^ω, and the cardinality of all subsets of X is 2^λ. Thus Theorem 12 yields:

13. COROLLARY (MA). *If* $\omega \leq \lambda < 2^\omega$, *then* $2^\omega = 2^\lambda$.

Another Baire category type theorem is:

14. THEOREM (MA). *Suppose that* $0 < \lambda < 2^\omega$, *that X is a space with a countable basis, and that $\{X_\alpha\}_{\alpha\in\lambda}$ is a family of nowhere dense subsets of X. Then $\bigcup_{\alpha\in\lambda} X_\alpha$ is the union of countably many nowhere dense sets.*

PROOF. Let $\{U_n\}_{n\in\omega}$ be a basis for the topology of X: make sure that each basis element is indexed with infinitely many different n. Then define $B_n = \{m \in \omega \mid U_m \subset U_n\}$. For each $\alpha \in \lambda$ define $A_\alpha = \{n \in \omega \mid U_n \cap X_\alpha \neq \emptyset\}$. Let $\mathcal{B} = \{B_n \mid n \in \omega\}$ and $\mathcal{A} = \{A_\alpha \mid \alpha \in \lambda\}$. By Corollary 9 there is an $L \subset \omega$ such that $L \cap B_n$ is infinite for all $n \in \omega$ but $L \cap A_\alpha$ is finite for all $\alpha \in \lambda$. Thus if $Y_n = X - \bigcup\{m \in L \mid m > n\}$, Y_n is nowhere dense and, for each $\alpha \in \lambda$, there is an n with $X_\alpha \subset Y_n$. Therefore $\bigcup_{\alpha\in\lambda} X_\alpha$ is the union of countably many nowhere dense sets. □

REMARK. All the consequences 7–14 of MA were in fact proved from 7. Actually, van Douwen has shown that the simpler 8 implies 7 (and hence 7–14). To see this, let $\mathcal{A}$, $\mathcal{B}$ be as in 7, and let

$$\mathcal{D} = \{[\omega - n]^{<\omega} \mid n \in \omega\} \cup \{\{s \in [\omega]^{<\omega} \mid s \cap B \neq \emptyset\} \mid B \in \mathcal{B}\} \\ \cup \{[\omega - A]^{<\omega} \mid A \in \mathcal{A}\}$$

(where $[I]^{<\omega}$ is the set of all finite subsets of I). Apply 8 to $\mathcal{D}$ to get an $L \subset [\omega]^{<\omega}$ with $L - X$ finite for all $X \in \mathcal{D}$, and let $M = \omega - \bigcup L$. It is known that 4–6 and 15–17 do not follow from 8.

Baire category theorems, Borel hierarchy, and measure theory type theorems go together. One consequence of the following theorem is that the union of any family of less than 2^ω measurable sets of real numbers is measurable. We use R for the real line and $m(X)$ for the Lebesgue measure of X.

15. THEOREM (MA). *If* $0 \leq \lambda < 2^\omega$ *and for each* $\alpha \in \lambda$, $X_\alpha \subset R$ *and* $m(X_\alpha) = 0$, *then* $m(\bigcup_{\alpha\in\lambda} X_\alpha) = 0$.

PROOF. Suppose that $\varepsilon < 0$. We prove Theorem 15 by showing that there is an open subset Y of R such that $m(Y) \leq \varepsilon$ and $X_\alpha \subset Y$ for all $\alpha \in \lambda$.

Define $P = \{U \subset R \mid U$ is open and $m(U) < \varepsilon\}$ and partially order P by reverse inclusion.

Define $\mathcal{B}$ to be a countable family of open intervals which form a basis for R. Let $\mathcal{B}^*$ be the set of all finite unions of members of $\mathcal{B}$.

If S is an uncountable subset of P, there is an uncountable subset S' of S and a $0 < n \in \omega$ such that $U \in S'$ implies that $m(S) + 1/n < \varepsilon$. For each $U \in S'$ choose $U^* \subset U$ such that $U^* \in \mathcal{B}^*$ and $m(U - U^*) < 1/n$. Since S' is uncountable and $\mathcal{B}^*$ is countable, there must be different terms U and V

of S' with $U^* = V^*$. Then U and V are compatible since $m(U \cup V) < \varepsilon$; thus $\langle P, \subset \rangle$ is ccc.

For each $\alpha \in \lambda$ define $D_\alpha = \{U \in P \mid X_\alpha \subset U\}$. Since D_α is dense there is a compatible $Q \subset P$ which meets every D_α. Let $Y = \bigcup Q$. For all $\alpha \in \lambda$, $X_\alpha \subset Q$. Since R is hereditarily Lindelöf, countably many members of Q cover Y so, if $m(Y) > \varepsilon$, there is some finite subset Q' of Q with $m(\bigcup Q') > \varepsilon$. However, since Q is compatible, there is a $Y' \in P$ with $\bigcup Q' \subset Y'$. Since $m(Y') < \varepsilon$ we have a contradiction. Thus $m(Y) \leq \varepsilon$ and Y has all of the desired properties. □

An Aronszajn tree is a tree $\langle T, \leq \rangle$ of cardinality ω_1 having no uncountable level or chain. A Souslin tree is an Aronszajn tree in which every antichain is countable (see Chapter B.3). MA + $\neg$CH denies the existence of a Souslin tree in an especially strong way:

16. THEOREM (Baumgartner; MA + $\neg$CH). *Every Aronszajn tree is the union of countably many antichains.*

(Such an Aronszajn tree is called *special.*)

PROOF. Let $\langle T, \leq \rangle$ be an Aronszajn tree. Our aim is to define a function $q : T \to \mathbb{Q}$ (where $\mathbb{Q}$ is the set of all rational numbers) such that $s < t$ in T implies that $q(s) < q(t)$. Since $q^{-1}(r)$ for each $r \in \mathbb{Q}$ will then be an antichain, Theorem 16 will then be proved.

Let $P = \{f : S \to \mathbb{Q} \mid S$ is a finite subset of T and $s < t$ in S implies that $f(s) < f(t)\}$. Define $f \geq g$ in P provided g extends f. For each $t \in T$ define $D_t = \{f \in P \mid t \in \text{domain of } f\}$. Each D_t is dense in P and $|T| = \omega_1$ so, if P is ccc, there is a compatible $P' \subset P$ which meets every D_t. Thus there is a $q : T \to \mathbb{Q}$ which extends every $f \in P'$, and this q has the desired properties.

It remains to prove that P is ccc. Assume that $\{f_\alpha\}_{\alpha \in \omega_1} \subset P$ and that the domain of f_α is S_α. We prove that P is ccc by showing that there are $\alpha < \beta$ in ω_1 and $f \in P$ such that f extends both f_α and f_β.

By a Δ-system argument (see Chapter B.3) we can choose an $n \in \omega$, a $k \in \omega$, and an infinite (uncountable) subset M of ω_1 such that:

(a) For all $\alpha \in M$, S_α has n terms $s_{0\alpha}, s_{1\alpha}, \ldots, s_{n-1,\alpha}$.

(b) $\alpha \neq \beta$ in M and $i \in n$ imply $f_\alpha(s_{i\alpha}) = f_\beta(s_{i\beta})$.

(c) $\alpha \neq \beta$ in M and $i \in k$ imply $s_{i\alpha} = s_{i\beta}$.

(d) $\alpha < \beta$ in M and i and j in $n - k$ imply that the level in T to which $s_{i\alpha}$ belongs precedes the level in T to which $s_{j\beta}$ belongs.

(e) $i \in n - k$ implies that $\{s_{i\alpha}\}_{\alpha \in \mathcal{M}}$ is an antichain.

To see that you can get (e), recall that if M' is any uncountable subset of ω_1, then $\{s_{i\alpha}\}_{\alpha \in M'}$ must contain an uncountable antichain since otherwise M' (with the induced order) would be a Souslin tree which is denied by $\mathrm{MA} + \neg \mathrm{CH}$.

Suppose that (i, j) is a pair of numbers in $n - k$. We now use Ramsey's theorem: $\omega \to (\omega)^2_2$ (see Chapter B.3). Let $A = \{$pairs (α, β) of terms of $M \mid \alpha < \beta$ and $s_{\alpha i} < s_{\beta j}\}$ and let $B = \{$pairs (α, β) of terms of $M \mid \alpha < \beta$ and $s_{\alpha i} \not< s_{\beta j}\}$. There can be no $\alpha < \beta < \gamma$ all of whose pairs are in A since then $s_{\alpha i} < s_{\gamma j}$ and $s_{\beta i} < s_{\gamma j}$ which by (d) would mean that $s_{\alpha i} < s_{\beta i}$ would contradict (e). So there is an infinite $M' \subset M$ all of whose pairs are in B.

For different i and j in $n - k$ choose successively smaller infinite M's until we have an infinite $M^* \subset M$ such that for all $\alpha < \beta$ in M^* and i and j in $n - k$, $s_{\alpha i} \not< s_{\beta j}$. Then, for $\alpha < \beta$ in M^*, $f : (S_\alpha \cup S_\beta) \to \mathbb{Q}$ defined by $f(s)$ is $f_\alpha(s)$ for $s \in S_\alpha$, and $f_\beta(s)$ for $s \in S_\beta$ is well defined and belongs to P and extends both f_α and f_β. $\square$

A similar proof (with a different proof that P is ccc) would show that under $\mathrm{MA} + \neg \mathrm{CH}$, every Aronszajn tree $\langle T, \leq \rangle$ is normal (under the topology induced by taking all sets of the form $\{x \in T \mid s < x \leq t\}$ where $s < t$ in T as a basis).

We close with a combinatorial theorem. A family $\mathcal{A}$ of sets is called *almost disjoint* if the intersection of each pair of distinct elements of $\mathcal{A}$ is finite.

17. THEOREM (Wage; MA). *Suppose that κ is regular and $\omega < \kappa \leq \lambda < 2^\omega$. If $\mathcal{A}$ is a family of λ almost disjoint countable subsets of κ, then there is a $B \subset \kappa$ of cardinality κ such that $\mathcal{A} \cup \{B\}$ is almost disjoint.*

PROOF. Let $\mathcal{A} = \{A_\alpha\}_{\alpha \in \lambda}$. Define $P = \{\langle H, K \rangle \mid H$ is a finite subset of $\mathcal{A}$ and K is a finite subset of $\kappa\}$. Partially order P by $\langle H, K \rangle \geq \langle H', K' \rangle$ in P provided $H \subset H'$, $K \subset K'$, and $(K' - K) \cap (\bigcup H) = \emptyset$.

Let $\{\langle H_\alpha, K_\alpha \rangle\}_{\alpha \in \omega_1}$ be an uncountable subset of P. By a Δ-system argument (see Chapter B.3), there is an uncountable $M \subset \omega_1$, $n \in \omega$, $H \subset \lambda$, and $K \subset \kappa$ such that $\alpha \neq \beta$ in M implies that $H_\alpha \cap H_\beta = H$ and $K_\alpha \cap K_\beta = K$, and $\alpha \in M$ implies that $K_\alpha - K$ has n terms. Observe that $\{K_\alpha - K\}_{\alpha \in M}$ are disjoint and $\{\bigcup (H_\alpha - H)\}_{\alpha \in M}$ are almost disjoint. Let C be any infinite countable subset of M. Since each $\bigcup_{\alpha \in C} H_\alpha$ is countable and $\{K_\alpha - K\}_{\alpha \in M}$ are disjoint, we can choose an $L \subset M$ such that $|L| > n$ and $(\bigcup_{\alpha \in C} \cup H_\alpha) \cap (\bigcup_{\beta \in L} K_\beta - K) = \emptyset$. Since $|L| > n$, $|K_\alpha - K| = n$ for all

$\alpha \in C$, and $\{\bigcup(H_\beta - H)\}_{\beta\in L}$ are almost disjoint, there is an $\alpha \in C$ and $\beta \in L$ such that $(K_\alpha - K) \cap (\bigcup(H_\beta - H)) = \emptyset$. Hence $\langle H_\alpha, K_\alpha\rangle \geq \langle (H_\alpha \cup H_\beta), (K_\alpha \cup K_\beta)\rangle$ and $\langle H_\beta, K_\beta\rangle \geq \langle (H_\alpha \cup H_\beta), (K_\alpha \cup K_\beta)\rangle$. Thus P is ccc.

For $\alpha \in \lambda$, let $X_\alpha = \{\langle H, K\rangle \in P \mid \alpha \in H\}$ and, for $\beta \in \kappa$ let $Y_\beta = \{\langle H, K\rangle \in P \mid (\kappa - B) \cap K \neq \emptyset\}$. Since both X_α and Y_β are dense in P for all $\alpha \in \lambda$ and $\beta \in \kappa$, there is a compatible Q which meets every X_α and Y_β. Let $B = \bigcup\{K \subset \kappa \mid \langle H, K\rangle \in Q\}$. Since $Q \cap Y_\beta \neq \emptyset$ for all $\beta \in \kappa$, $|B| = \kappa$. If $\alpha \in \lambda$, there is an $\langle H, K\rangle \in Q \cap X_\alpha$. If $\langle H', K'\rangle$ is any other member of Q, there is $\langle H'', K''\rangle \in P$ with $\langle H, K\rangle \geq \langle H'', K''\rangle$ and $\langle H', K'\rangle \geq \langle H'', K''\rangle$. Since $(K'' - K) \cap A_\alpha = \emptyset$ and $K' \subset K''$, $K' \cap A_\alpha \subset K$. Thus $B \cap A_\alpha \subset K$ so $B \cap A_\alpha$ is finite. □

From this mixed bag of theorems can we generalize about *when* we should expect MA to be useful? In topology MA + $\neg$CH can be used to construct a variety of normal but not collectionwise normal spaces. MA can be used to deny the existence of certain pathologies in ccc spaces. MA often has something to say about problems involving compact Hausdorff spaces. For instance many of the traditional theorems about $\beta\mathbb{N}$ (the Stone–Čech compactification of the integers) proved using CH have a more general version proved using only MA. Baire category and measure theory theorems proved traditionally for countable sets can often be extended to sets of cardinality less than 2^ω. And in any field of mathematics when one would like to prove a theorem for ω_1 sets and one knows an inductive proof of the theorem for countable sets, there is a natural setup for applying (MA + $\neg$CH). The difficulty is in proving that this natural partial order is ccc. It may not be ccc, in which case MA may not be applicable.

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