Exercícios:

I. Curvas no plano e no espaço:

① Desenhe as imagens das seguintes curvas, indicando o sentido do percurso.

a) $\gamma(t) = (\cos^2 t, \operatorname{sen} t)$; $0 \leq t \leq 2\pi$.

$$\gamma(t) = (\underbrace{1 - \operatorname{sen}^2 t}_{x(t)}, \underbrace{\operatorname{sen} t}_{y(t)})$$

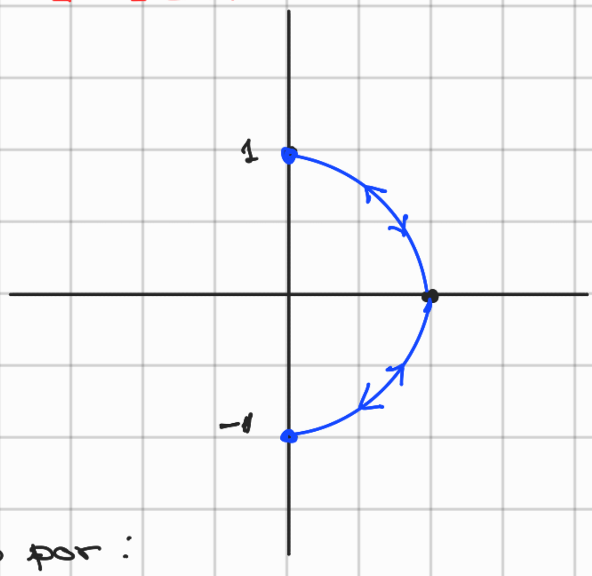
Observamos que:

$$x = x(t) = 1 - \operatorname{sen}^2 t$$

$$y = y(t) = \operatorname{sen} t$$

Então $x = 1 - y^2$

$$x - 1 = -y^2$$



O sentido do percurso é dado por:

$\gamma(0) = (1,0)$; $\gamma(\pi/2) = (0,1)$; $\gamma(\pi) = (1,0)$; $\gamma(3\pi/2) = (0,-1)$

$\gamma(2\pi) = (1,0)$

b) $\gamma(t) = (\sin t, \sin^2 t)$; $t \in \mathbb{R}$.

$$x = x(t) = \operatorname{sen} t$$

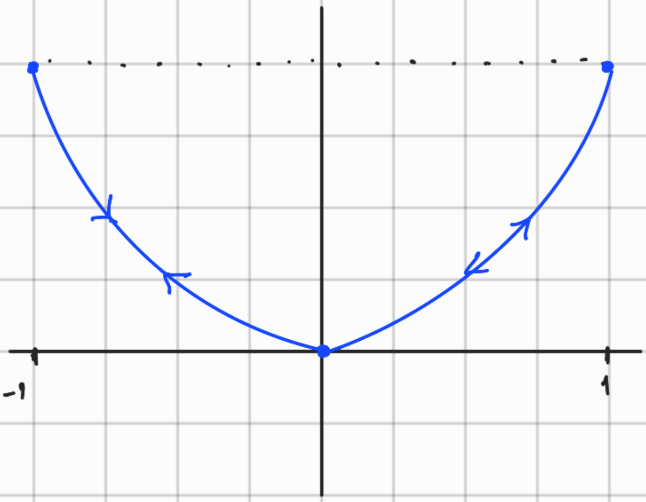
$$y = y(t) = \operatorname{sen}^2 t = x^2$$

O sentido do percurso é:

$\gamma(0) = (0,0)$; $\gamma(\pi/2) = (1,1)$

$\gamma(\pi) = (0,0)$; $\gamma(3\pi/2) = (-1,1)$

$\gamma(2\pi) = (0,0)$.



c) $\gamma(t) = (2 + \cos t, 3 + 4 \operatorname{sen} t)$; $t \in [-\pi, \pi]$.

$x = x(t) = 2 + \cos t \Rightarrow x - 2 = \cos t \Rightarrow x - 2 = \cos t$

$y = y(t) = 3 + 4 \operatorname{sen} t \Rightarrow y - 3 = 4 \operatorname{sen} t \Rightarrow \dfrac{y-3}{4} = \operatorname{sen} t$.

Sabemos que $\operatorname{sen}^2 t + \cos^2 t = 1 \Rightarrow (x-2)^2 + \left(\dfrac{y-3}{4}\right)^2 = 1$

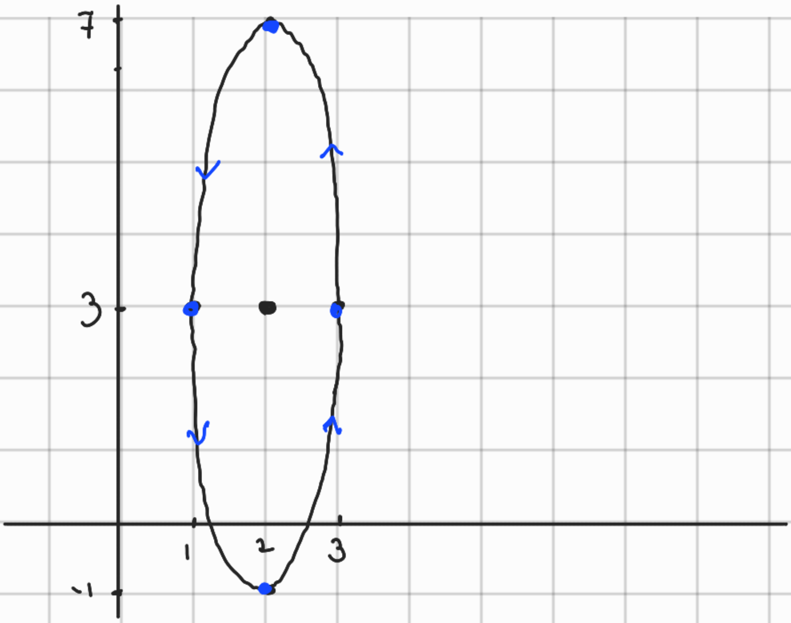
$$\Rightarrow \frac{(x-2)^2}{1} + \frac{(y-3)^2}{4^2} = 1$$

$\gamma(-\pi) = (1,3)$

$\gamma(-\pi/2) = (2,-1)$

$\gamma(0) = (3,3)$

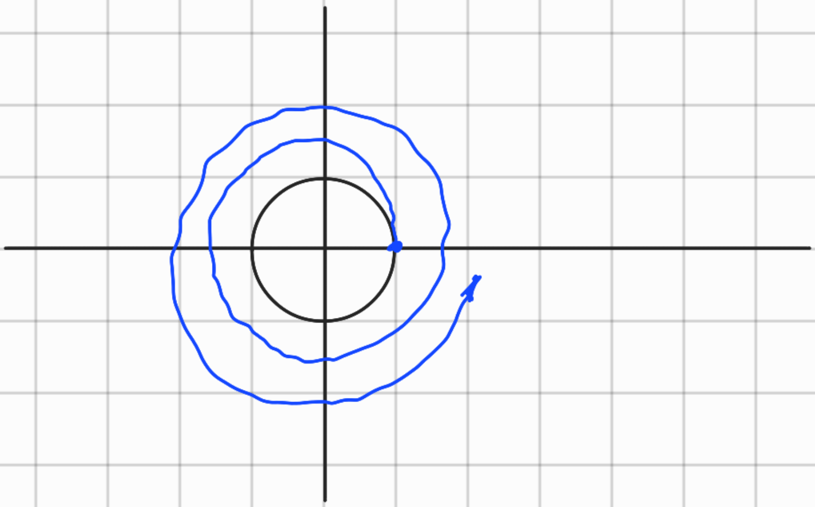
$\gamma(\pi/2) = (2,7)$



d) $\gamma(t) = (e^t \cos t, e^t \operatorname{sen} t)$, $t \geq 0$.

$\gamma(t) = e^t(\cos t, \operatorname{sen} t)$; $t \geq 0$

$e^t \to \infty$ quando $t \to \infty$

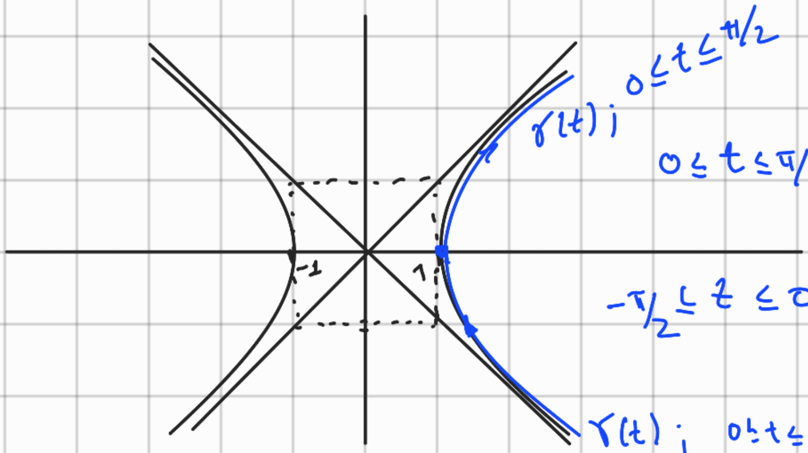


e) $\gamma(t) = (\sec t, \tan t)$; $t \in (-\pi/2, \pi/2)$

$\gamma(t) = (x(t), y(t))$

$x(t) = \sec t$
$y(t) = \tan t$ $\Rightarrow$ $\tan^2 t + 1 = \sec^2 t$

$y^2 + 1 = x^2 \rightarrow x^2 - y^2 = 1$ hipérbole



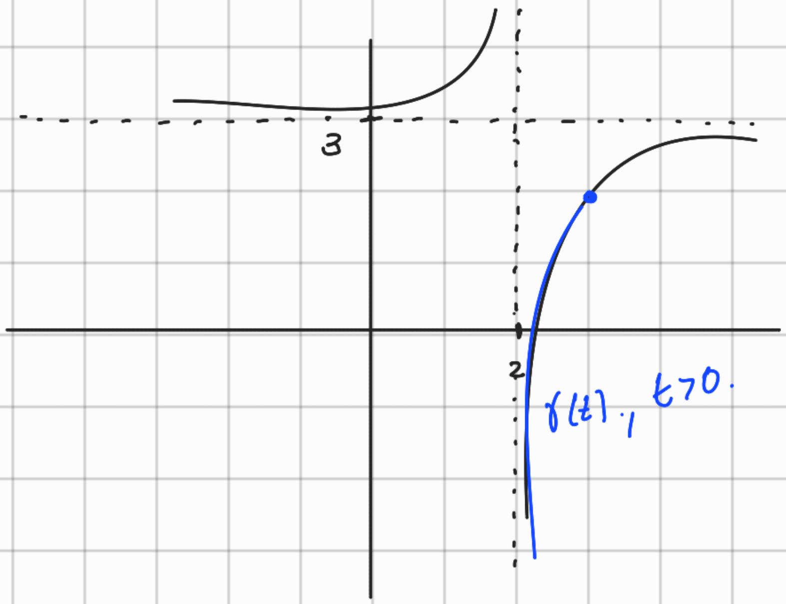
$\gamma(0) = (\sec 0, \tan 0) = (1,0)$

$0 \leq t \leq \pi/2$; $\gamma(t) = (\sec t, \tan t)$
aumenta aumenta

$-\pi/2 \leq t \leq 0$; $\gamma(t) = (\sec t, \tan t)$
posit negativo

f) $\gamma(t) = (2+e^{-t}, 3-e^{t}), \ t \geq 0$

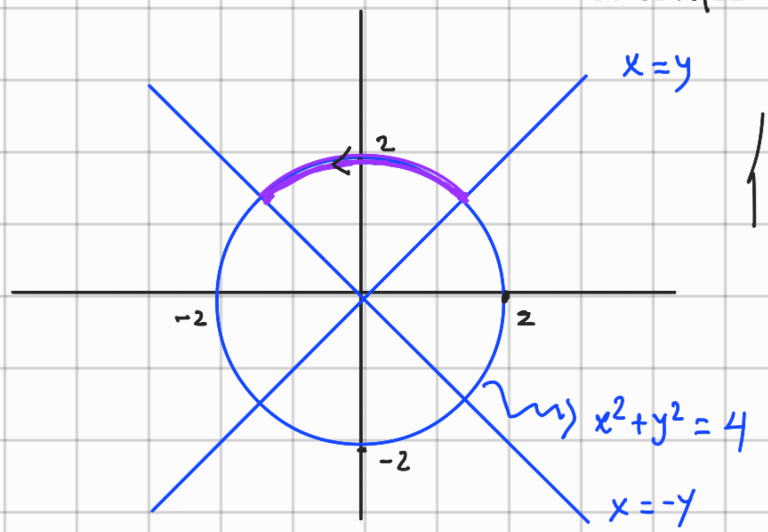
$$\begin{cases} x(t) = 2 + e^{-t} \Rightarrow x-2 = e^{-t} \Rightarrow \frac{1}{x-2} = e^{t} \\ y(t) = 3 - e^{t} \Rightarrow y = 3 - \frac{1}{x-2} \end{cases}$$



$\gamma(0) = (3,2)$

02. Esboce e parametrize cada conjunto C como uma curva

a) $C = \{(x,y) \in \mathbb{R}^2 : \underbrace{x^2+y^2=4}_{\text{circunferência}} ; \ y \geq -x \ e \ y \geq x \}$.



Uma parametrização:

$$\begin{cases} x = 2\cos t \\ y = 2\,\text{sen}\, t \end{cases} ; \ \frac{\pi}{4} \leq t \leq \frac{3\pi}{4}$$

04. Esboce uma família de curvas de nível de:

b $f(x,y) = x - \sqrt{1-y^2}$.

Domínio da função f

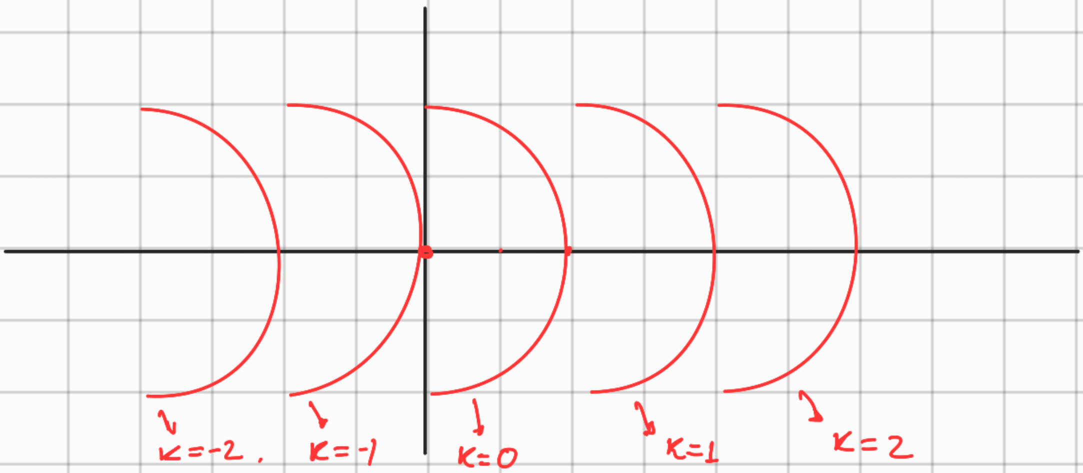
$D_f = \{(x,y) \in \mathbb{R}^2 : 1-y^2 \geq 0 \ e \ x \in \mathbb{R}\}$

$D_f = \{(x,y) \in \mathbb{R}^2 : x \in \mathbb{R}, \ y \in [-1,1]\}$

Curvas de nível: Dados $(x,y) \in D_f$ e $k \in \mathbb{R}$ temos

$$f(x,y) = k \Rightarrow x - \sqrt{1-y^2} = k$$
$$\Rightarrow x - k = \sqrt{1-y^2} \geq 0 \quad (x-k \geq 0 \Rightarrow x \geq k)$$
$$\Rightarrow (x-k)^2 + y^2 = 1 \rightsquigarrow \text{circunferencia raio 1 e centro } (k,0)$$

Mas como $x \geq k$ então as curvas de nível são semi circunferencias, isto é:



c) $$f(x,y) = \frac{x^2}{x^2-y^2}$$

Domínio de f: $D_f = \{(x,y) \in \mathbb{R}^2 : x^2 - y^2 \neq 0\}$
$= \{(x,y) \in \mathbb{R}^2 : x \neq y \text{ e } x \neq -y\}$

Curvas de nível:

$$f(x,y) = k \iff \frac{x^2}{x^2-y^2} = k$$

Se $k = 0 \Rightarrow x^2 = 0 \Rightarrow x = 0$. (eixo y)

Se $k \neq 0 \Rightarrow x^2 = kx^2 - ky^2 \Rightarrow \boxed{y = \pm\sqrt{\frac{k-1}{k}}\, x}$ retas

Se $k = 1 \Rightarrow y = 0$; $x \neq 0$

Se $k = 2 \Rightarrow y = \pm\sqrt{\frac{1}{2}} \cdot x$

5. Encontre uma parametrização.

b. $f(x,y) = x - \sqrt{1-2y^2}$. , $k = 5$

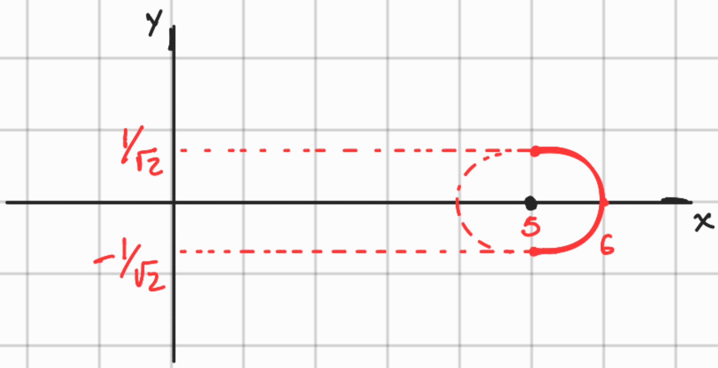
Domínio de f :

$$D_f = \{(x,y) \in \mathbb{R}^2 : 1-2y^2 \geq 0 , x \in \mathbb{R}\}$$
$$= \{(x,y) \in \mathbb{R}^2 : y \in [-1/\sqrt{2}, 1/\sqrt{2}] , x \in \mathbb{R}\}$$

Curva de nível $k=5$.

$$f(x,y) = 5 \Rightarrow x - \sqrt{1-2y^2} = 5$$
$$\Rightarrow x-5 = \sqrt{1-2y^2} \geq 0 \quad (x-5 \geq 0 \Rightarrow x \geq 5)$$
$$\Rightarrow (x-5)^2 + \left(\frac{y}{\frac{1}{\sqrt{2}}}\right)^2 = 1 \rightsquigarrow \text{elipse}$$

Como $x \geq 5$ então as curvas de nível são a metade da elipse.



6. Esboce os gráficos de.

c. $f(x,y) = \sqrt{x^2+9y^2}$.

- $D_f = \mathbb{R}^2$
- Curvas de nível :

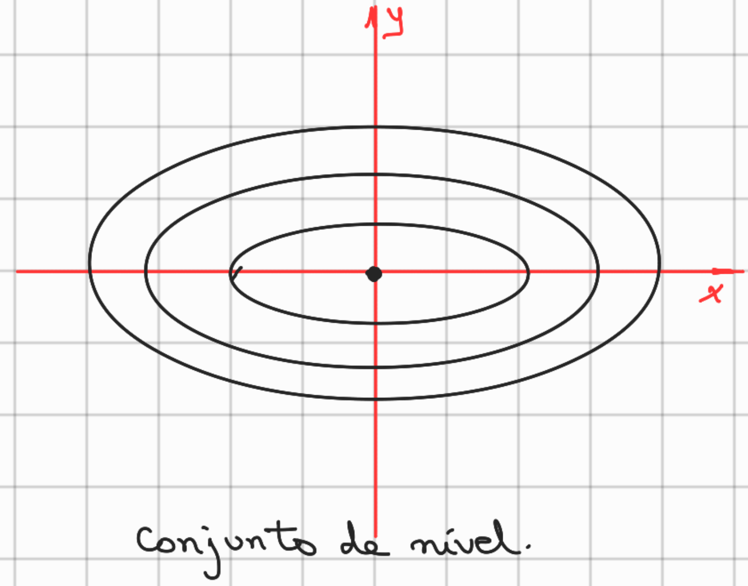
$$f(x,y) = k \Leftrightarrow \sqrt{x^2+9y^2} = k$$
$$\Leftrightarrow x^2 + 9y^2 = k^2$$
$$\Leftrightarrow \left(\frac{x}{k}\right)^2 + \left(\frac{y}{\frac{k}{3}}\right)^2 = 1 \quad \text{(elipses.)}$$

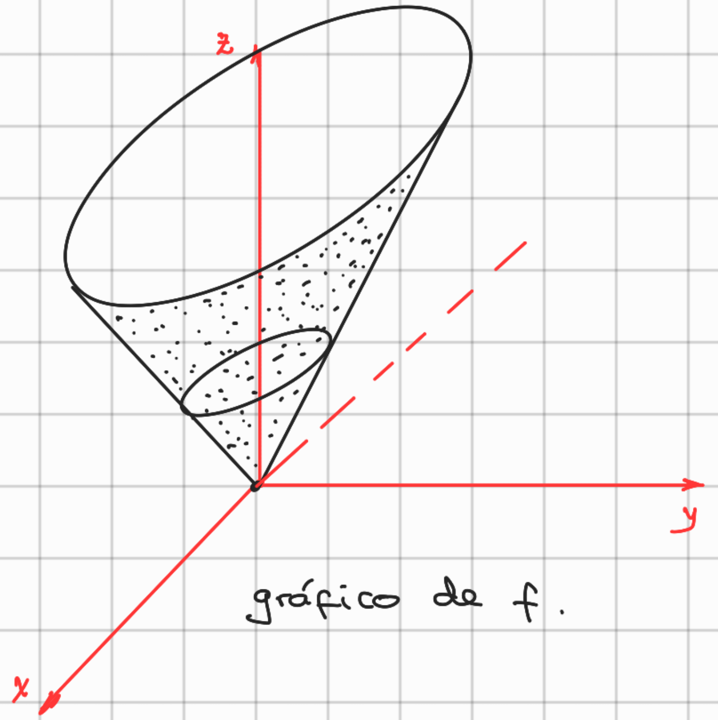
Se $k = 0 \Leftrightarrow x = y = 0$. então o conjunto de nível $N_0 = \{(0,0)\}$

Se $k < 0$ então $N_k = \phi$.

Se $k > 0$ então N_k é elipse.

Portanto $N_k \neq \phi \Leftrightarrow k \geq 0$, logo $Im(f) = [0, \infty)$





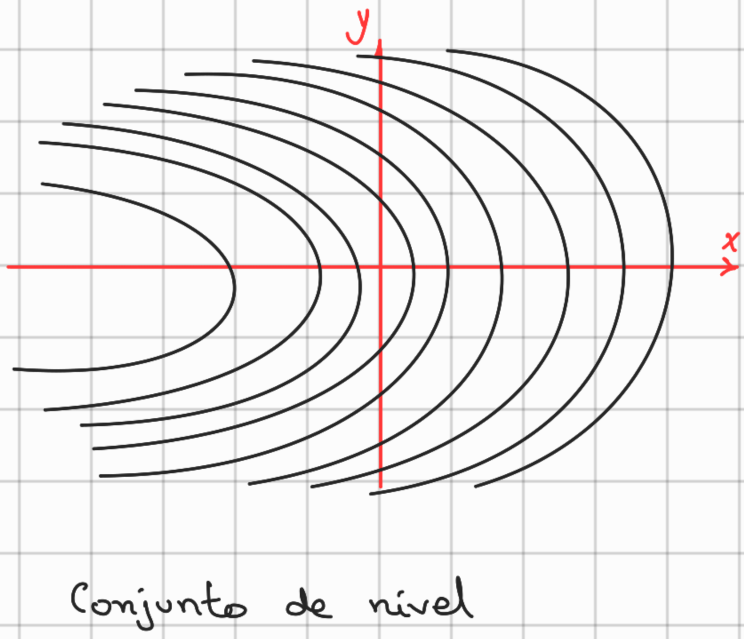
g. $f(x,y) = y^2 + x$

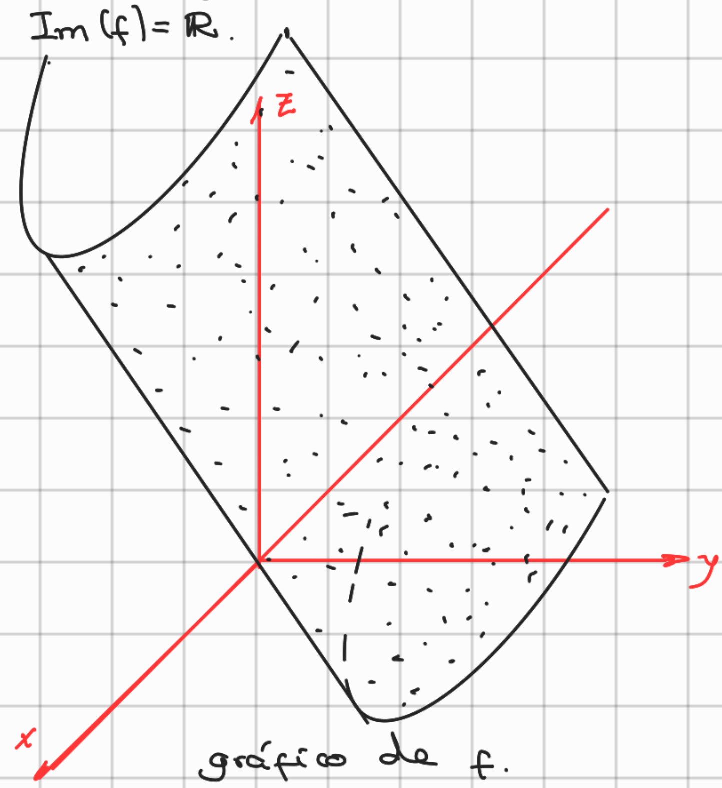
Domínio de f : $\mathbb{R}^2$

Conjunto de nível:

$f(x,y) = k \Leftrightarrow y^2 + x = k \Leftrightarrow y^2 = -(x-k)$. parábola.

Para todo $k \in \mathbb{R}$ temos que o conjunto de nível N_k é parábola. Portanto $Im(f) = \mathbb{R}$.





9. Encontre uma parametrização para C.

a. $C = \{(x,y,z) \in \mathbb{R}^3 : \underbrace{x^2+y^2+z^2=1}_{(I)} \text{ e } \underbrace{z=x+1}_{(II)}\}$ e $P = \left(-\frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{1}{2}\right)$

Sustituindo (II) em (I) : $x^2+y^2+(x+1)^2=1$

$$2x^2+y^2+2x=0$$

$$\left(x+\tfrac{1}{2}\right)^2 + \frac{y^2}{2} = \frac{1}{4}$$

$$\left(\frac{x+1/2}{1/2}\right)^2 + \left(\frac{y}{1/\sqrt{2}}\right)^2 = 1 \quad (\text{elipse})$$

A parametrização é:

$$\begin{cases} \dfrac{x+1/2}{1/2} = \cos t \Rightarrow x(t) = \dfrac{1}{2}\cos t - \dfrac{1}{2} \\ \dfrac{y}{1/\sqrt{2}} = \operatorname{sen} t \Rightarrow y(t) = \dfrac{1}{\sqrt{2}}\operatorname{sen} t = \dfrac{\sqrt{2}}{2}\operatorname{sen} t \\ z = x+1 \Rightarrow z(t) = \dfrac{1}{2}\cos t + \dfrac{1}{2} \end{cases}$$

$\Rightarrow \gamma(t) = \frac{1}{2}\left(\cos t - 1, \sqrt{2}\operatorname{sen} t, \cos t + 1\right)$

Observamos

$\gamma(0) = (0,0,1)$

$\gamma(\pi/2) = \left(-\frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{1}{2}\right)$

$\gamma(\pi) = (-1,0,0)$

$\gamma(3\pi/2) = \left(-\frac{1}{2}, -\frac{\sqrt{2}}{2}, \frac{1}{2}\right)$

$\gamma(2\pi) = (0,0,1)$.

logo a parametrização é

$\gamma(t) = \frac{1}{2}\left(\cos t - 1, \sqrt{2}\operatorname{sen} t, \cos t + 1\right)$; $t \in [0, 2\pi]$.

sua derivada é:

$\gamma'(t) = \frac{1}{2}\left(-\operatorname{sen} t, \sqrt{2}\cos t, -\operatorname{sen} t\right)$

o vetor tangente no ponto $P = \left(-\frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{1}{2}\right) = \gamma(\pi/2)$ é

$\gamma'(\pi/2) = -\frac{1}{2}(1,0,1) = -\frac{1}{2}\vec{v}$; $\vec{v} = (1,0,1)$.

A equação à reta tangente no ponto $P = \gamma(\pi/2)$ é:

$$X = \gamma(\pi/2) + \lambda\vec{v} = \left(-\tfrac{1}{2}, \tfrac{\sqrt{2}}{2}, \tfrac{1}{2}\right) + \lambda(1,0,1) \; ; \; \lambda \in \mathbb{R}.$$

e. $C = \{(x,y,z) \in \mathbb{R}^3 : x = z \text{ e } x^2+y^2 = z\}$ e $P = (1/2, 1/2, 1/2)$.

substituindo : $x^2 - x + y^2 = 0$

$$\Rightarrow \left(x - 1/2\right)^2 + y^2 = 1/4.$$

$$\left(\frac{x - 1/2}{1/2}\right)^2 + \left(\frac{y}{1/2}\right)^2 = 1$$

A parametrização é:

$$\begin{cases} \dfrac{x - 1/2}{1/2} = \cos t \Rightarrow x(t) = \frac{1}{2}\cos t + \frac{1}{2} \\ 2y = \operatorname{sen} t \Rightarrow y(t) = \frac{1}{2}\operatorname{sen} t \\ z = x \Rightarrow z(t) = \frac{1}{2}\cos t + 1/2. \end{cases}$$

$\gamma(t) = \frac{1}{2}(\cos t + 1, \operatorname{sen} t, \cos t + 1)$

observamos

$\gamma(0) = (1,0,1)$ $\gamma(3\pi/2) = (1/2, -1/2, 1/2)$

$\gamma(\pi/2) = (1/2, 1/2, 1/2)$ $\gamma(2\pi) = (1, 0, 1)$

$\gamma(\pi) = (0,0,0)$

A parametrização é:

$\gamma(t) = \frac{1}{2}(\cos t + 1, \operatorname{sen} t, \cos t + 1)$; $t \in [0, 2\pi)$.

$\Rightarrow \gamma'(t) = \frac{1}{2}(-\operatorname{sen} t, \cos t, -\operatorname{sen} t)$

o vetor tangente no ponto $P = (1/2, 1/2, 1/2) = \gamma(\pi/2)$. é

$$\gamma'(\pi/2) = \frac{1}{2}(-1, 0, -1) = -\frac{1}{2}\underbrace{(1,0,1)}_{\vec{v}}$$

Logo a equação tangente no ponto $P = \gamma(\pi/2)$ é

$$X = \gamma(\pi/2) + \lambda\vec{v} = (1/2, 1/2, 1/2) + \lambda(1,0,1) \; ; \; \lambda \in \mathbb{R}.$$