A general FEM formulation of nonlinear dynamics applied to accessing the statical loading effect upon the dynamic response of planar frames

Reyolando M L R F Brasil and Carlos E N Mazzilli

Department of Structural and Foundation Engineering, Escola Politécnica
Universidade de São Paulo, CP 61548, 05424-970 São Paulo, SP, Brazil

This paper initially discusses the dynamics of discrete structural systems of geometricaly nonlinear behaviour costituted by linear elastic materials. Two formulations are derived, namely *global* and *incremental*. They are both suitable to general FE modelling, as the matrix equations of motion are written in explicit form. Matrices and vectors involved are characterized in terms of constraint equations defined within the continuum discretization. In principle, such formulations are applicable to any structural theory, as the theories of beams, plates and shells. As an example, the Bernoulli-Euler beam element is studied herewith. Both global and incremental formulations capture the effect geometrical nonlinearities have upon inertial and elastic forces alike. The ANDROS FEM program, developed by the authors, which is based upon the global formulation, has been successfully used in several nonlinear analyses. From this general background, the paper proceeds to consider the effect statical loading may have upon the free undamped vibration frequencies of a structure. It is shown that the tangent stiffness matrix of the incremental formulation should be used in the resultant eingenvalue problem. In some cases, axial forces are seen to have a strong influence on the internal resonance tuning. It is shown, in a sample structure thus tuned and subjected to dynamical loading, that a nonlinear regime may appear in the response.

A GENERAL FEM FORMULATION

Discrete structural systems of geometrically nonlinear behaviour constituted by linear elastic material are initially considered. The equations of motion in explicit form are investigated, retaining nonlinearities both in the elastic and in the inertial forces. The foregoing formulation is based on Mazzilli (1988) and is already in adequate form for application to the finite-element method, thus allowing the analysis of large engineering systems. The secant matrices of mass, equivalent damping – there included the inertial damping – and stiffness, as well as the equivalent load vector, are characterized within the global equations of motion. Alternatively, the tangent matrices of mass, damping and stiffness, and the incremental equivalent load vector are deduced within the incremental equations of motion. These latter may be eventually used for consideration of elastoplastic behaviour, interpreting it as hypoelasticity in each time increment. The formulation is able to tackle conservative and non-conservative applied loads, as well as translation and rotation support excitations.

As the starting point, the generalized Lagrange's equations are recast. For a system with n degrees of freedom $Q_1, Q_2, \ldots, Q_n$, they are:

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{Q}_r}\right] - \left[\frac{\partial \mathcal{L}}{\partial Q_r}\right] = N^r, \quad r = 1\,to\,n$$

the Lagrangian function $\mathcal{L}$ being given by the difference between the kinetic energy T and the total potential energy V. The generalized forces N^r retain the effects of the nonconservative forces. One can also define $\mathcal{L}$ as the difference between the kinetic energy and the strain energy U, so that N^r would then also include the effects of applied conservative forces. Holonomic constraints are assumed, so that the position vector of a generic volume element $d\Omega$ of the deformable system is supposed to be a known function of the generalized coordinates and the time:

$$\vec{R} = \vec{R}(Q_1, Q_2, \ldots, Q_n, t)$$

If these functions are all linear in the generalized coordinates, geometrical nonlinearities will be lost. Therefore,

part of "MECHANICS PAN-AMERICA 1993" edited by MRM Crespo da Silva and CEN Mazzilli ASME Reprint No AMR134
Appl Mech Rev vol 46, no 11, part 2, November 1993

extreme caution should be taken when the finite-element technique is introduced into the formulation. The usual hypothesis that the displacement field within a finite element is given by a linear combination of nodal displacements may be inadequate for the nonlinear analysis. Hence, at the Solid Mechanics level it is necessary that the underlying structural theory be consistently nonlinear and further that at the structural discretization level, via the finite-element technique, nonlinearities are not subtly lost within the matrix equations of motion.

The kinetic energy is:

$$T = \frac{1}{2}\int_\Omega \dot{\vec{R}} \cdot \dot{\vec{R}} \rho d\Omega$$

where Ω is the region occupied by the undeformed system and ρ the specific mass. The velocity vector is:

$$\dot{\vec{R}} = \frac{\partial \vec{R}}{\partial Q_r}\dot{Q}_r + \frac{\partial \vec{R}}{\partial t}$$

and consequently:

$$T = \frac{1}{2}A^{rs}\dot{Q}_r\dot{Q}_s + B^r\dot{Q}_r + \frac{1}{2}C$$

with:

$$A^{rs} = \int_\Omega \frac{\partial \vec{R}}{\partial Q_r}\cdot\frac{\partial \vec{R}}{\partial Q_s}\rho d\Omega;$$
$$B^{r} = \int_\Omega \frac{\partial \vec{R}}{\partial Q_r}\cdot\frac{\partial \vec{R}}{\partial t}\rho d\Omega;$$
$$C = \int_\Omega \frac{\partial \vec{R}}{\partial t}\cdot\frac{\partial \vec{R}}{\partial t}\rho d\Omega$$

Recasting Lagrange's equations, one gets:

$$A^{rs}\ddot{Q}_s + \left[\frac{\partial A^{rs}}{\partial Q_t} - \frac{1}{2}\frac{\partial A^{st}}{\partial Q_r}\right]\dot{Q}_s\dot{Q}_t + \frac{\partial A^{rs}}{\partial t}\dot{Q}_s + \left[\frac{\partial B^r}{\partial Q_s} - \frac{\partial B^s}{\partial Q_r}\right]\dot{Q}_s \frac{\partial B^r}{\partial t} - \frac{1}{2}\frac{\partial C}{\partial Q_r} = -\frac{\partial U}{\partial Q_r} + N^r$$

A special though still rather general class of holonomic constraints is now considered. It is supposed that there exists a reference frame $Yy^1y^2y^3$, called "relative", with respect to which the system constraints are scleronomic. Support excitations are defined by rigid-body motion of the relative frame $Yy^1y^2y^3$ with respect to an inertial frame $Xx^1x^2x^3$. Calling $\vec{R}$ the position vector in the inertial frame, $\vec{r}$ the position vector in the relative frame and $\vec{S}$ the vector (Y-X), one writes:

$$\vec{R} = \vec{r} + \vec{S}; \quad \vec{S} = S^i\vec{e}_i; \quad \vec{r} = a^i_j y^j \vec{e}_i$$

where the matrix $[a^i_j]$ defines the relative frame rotation with respect to the inertial frame. It can be explicitly given in terms of the Euler angles (Mazzilli, 1988). For a planar problem, the matrix $[a^i_j]$ can be written in terms of the support rotation $\phi(t)$:

$$[a^i_j] = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

It was already observed that it is a feature of nonlinear structural theories to render nonlinear functions for the coordinates y^j in the variables $Q_1, Q_2, ..., Q_n$. Additionally, support excitations are defined by rotations through the matrix $[a^i_j(t)]$ or by translations through $S^i(t)$. Hence:

$$\vec{R} = [a^i_j(t)y^j(Q_1, Q_2, ..., Q_n) + S^i(t)]\vec{e}_i$$

Back into expressions for A^{rs}, B^r e C, and taking these into Lagrange's equations, one arrives at:

$$\begin{aligned} F^{rsjm}\delta_{jm}\ddot{Q}_s &\quad + \\ \left(F^{rsjm}_{,t} - \frac{1}{2}F^{stjm}_{,r}\right)\delta_{jm}\dot{Q}_s\dot{Q}_t &\quad + \\ (F^{rsjm} - F^{srjm})a^i_j\dot{a}^k_m\delta_{ik}\dot{Q}_s &\quad + \\ H^{rj}a^i_j\ddot{S}^k\delta_{ik} &\quad + \\ G^{rjm}a^i_j\ddot{a}^k_m\delta_{ik} &= -U_{,r} + N^r \end{aligned}$$

Notation $(.)_{,r}$ indicates partial differentiation with respect to Q_r, δ_{jm} being the Kronecker's symbol and:

$$F^{rsjm} = \int_\Omega \frac{\partial y^j}{\partial Q_r}\frac{\partial y^m}{\partial Q_s}\rho d\Omega;$$
$$G^{rjm} = \int_\Omega \frac{\partial y^j}{\partial Q_r}y^m\rho d\Omega;$$
$$H^{rj} = \int_\Omega \frac{\partial y^j}{\partial Q_r}\rho d\Omega$$

All terms dependent of F^{rsjm}, G^{rjm} e H^{rj} in the above equations of motion are inertial forces, there included the Coriolis forces:

$$(F^{rsjm} - F^{srjm})a^i_j\dot{a}^k_m\delta_{ik}\dot{Q}_s$$

and the centrifugal forces:

$$G^{rjm}a^i_j\ddot{a}^k_m\delta_{ik}$$

Since F^{rsjm}, G^{rjm} e H^{rj} may be functions of the generalized coordinates, it is clear that geometrical nonlinearities can in fact generate nonlinear inertial forces. The elastic force vector $U_{,r}$, on its turn, may also include nonlinear terms caused by geometrical nonlinearities, as it is well known even in Statics. It is supposed that the generalized force vector always include viscous damping:

$$N^r = P^r - \mu^{rs}\dot{Q}_s$$

The **global nonlinear equations of motion** can now be written in the compact form:

$$M^{rs}\ddot{Q}_s + D^{rs}\dot{Q}_s + U_{,r} = \mathcal{F}^r$$

where:

$$M^{rs} = F^{rsjm}\delta_{jm}$$

$$\begin{aligned} D^{rs} &= \mu^{rs} + (F^{rsjm} - F^{srjm})a^i_j \dot{a}^k_m \delta_{ik} \\ &+ \left(F^{rsjm}_{,t} - \frac{1}{2}F^{stjm}_{,r}\right)\delta_{jm}\dot{Q}_t \end{aligned}$$

$$\mathcal{F}^r = P^r - G^{rjm}a^i_j \ddot{a}^k_m \delta_{ik} - H^{rj}a^i_j \ddot{S}^k \delta_{ik}$$

It should be observed that the introduction of the stiffness matrix in the global equation of motion is, in some ways, arbitrary. In fact, what naturally appears there is the elastic force vector $U_{,r}$. Obviously, one could think of the secant stiffness matrix K^{rs}_S, as:

$$U_{,r} = K^{rs}_S Q_s$$

Nevertheless, several other matrices could be thought of, as the linear theory stiffness matrix, for example. Once a particular stiffness matrix K^{rs} has been chosen, one can add to both sides of the global equation of motion the term $K^{rs}Q_s$. Introducing now the definition:

$$\Delta\mathcal{F}^r = K^{rs}Q_s - U_{,r}$$

one finally gets for the global equations of motion:

$$M^{rs}\ddot{Q}_s + D^{rs}\dot{Q}_s + K^{rs}Q_s = \mathcal{F}^r + \Delta\mathcal{F}^r$$

In other words, the arbitrary choice of the stiffness matrix implies in adding to the load vector $\mathcal{F}^r$ a correcting term. It should be observed that, even in Statics, the global equilibrium equation:

$$U_{,r} = P^r$$

can well be written in the form:

$$K^{rs}Q_s = P^r + \Delta\mathcal{F}^r$$

and, in numerical nonlinear analysis, this latter suggests an iterative procedure equivalent to that of the modified Newton-Raphson method. In fact, choosing a particular matrix K^{rs} and supposing in the first iteration that $\Delta\mathcal{F}^r = 0$, one can obtain Q_s from the equilibrium equation. The new value for $\Delta\mathcal{F}^r$ would then come after its definition and so forth.

In a great number of cases it may be more convenient to work with the **incremental equations of motion**:

$$M^{rs}_T\delta\ddot{Q}_s + D^{rs}_T\delta\dot{Q}_s + K^{rs}_T\delta Q_s = \delta\mathcal{P}^r$$

where:

$$M^{rs}_T = M^{rs}$$

$$\begin{aligned} D^{rs}_T &= \mu^{rs} + (F^{rsjm} - F^{srjm})a^i_j \dot{a}^k_m \delta_{ik} \\ &+ (F^{rsjm}_{,t} + F^{rtjm}_{,s} - F^{stjm}_{,r})\delta_{jm}\dot{Q}_t \end{aligned}$$

$$\begin{aligned} K^{rs}_T &= U_{,rs} + G^{rjm}_{,s}a^i_j \ddot{a}^k_m \delta_{ik} + H^{rj}_{,s}a^i_j \ddot{S}^k \delta_{ik} \\ &+ F^{rtjm}_{,s}\delta_{jm}\ddot{Q}_t + \left(F^{rujm}_{,st} - \frac{1}{2}F^{utjm}_{,rs}\right)\delta_{jm}\dot{Q}_u\dot{Q}_t \\ &+ (F^{rtjm}_{,s} - F^{trjm}_{,s})a^i_j \dot{a}^k_m \delta_{ik}\dot{Q}_t + \mu^{rt}_{,s}\dot{Q}_t \end{aligned}$$

$$\begin{aligned} \delta\mathcal{P}^r &= \delta P^r - G^{rjm}(\delta a^i_j \ddot{a}^k_m + a^i_j \delta\ddot{a}^k_m)\delta_{ik} \\ &- H^{rj}(\delta a^i_j \ddot{S}^k + a^i_j \delta\ddot{S}^k)\delta_{ik} \\ &- (F^{rsjm} - F^{srjm})(\delta a^i_j \dot{a}^k_m + a^i_j \delta\dot{a}^k_m)\delta_{ik} \end{aligned}$$

A PLANAR STRUT FINITE ELEMENT

As an application example of the previous section general formulation, we consider now the 2-D strut finite element according to the Bernoulli-Euler theory (Mazzilli, 1990). From Figs 1a and 1b the following relations can be derived for the displacements u (local system $\bar{y}^1$ direction) and v (local system $\bar{y}^2$ direction) of the elemental mass dm situated in a cross section defined by the strut axis coordinate x and at a distance y from the cross section centroid:

$$\begin{aligned} u &= \bar{u} - y\sin\alpha \\ v &= \bar{v} + y(\cos\alpha - 1) \end{aligned}$$

where $\bar{u}$ and $\bar{v}$ are the u and v displacements of a material point on the strut axis and α is the cross-section rotation, for which rigorously:

$$\begin{aligned} \sin\alpha &= \frac{\bar{v}'}{\bar{\lambda}} \\ \cos\alpha &= \frac{1+\bar{u}'}{\bar{\lambda}} \\ \tan\alpha &= \frac{\bar{v}'}{1+\bar{u}'} \end{aligned}$$

Notation $(.)'$ denotes differentiation with respect to **x**. The axis stretching is:

$$\bar{\lambda} = \sqrt{(1+\bar{u}')^2 + (\bar{v}')^2}$$

Observe that the "local system" $Y\bar{y}^1\bar{y}^2$ is defined by a rotation θ, independent of time, with respect to the relative frame Yy^1y^2. In what follows here, the elemental

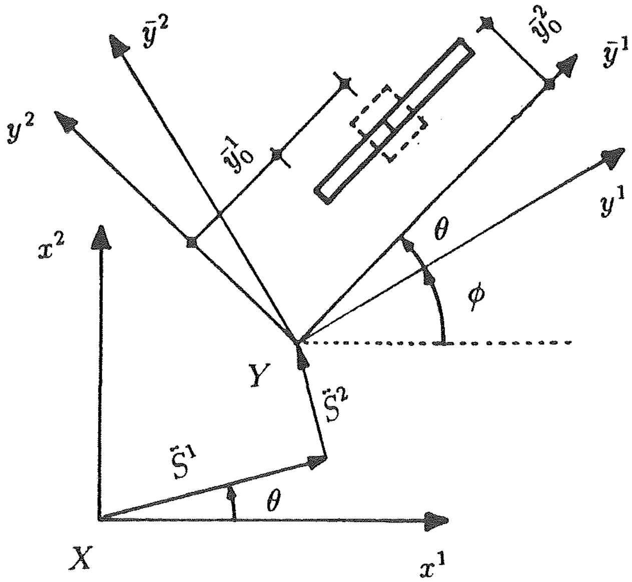

Fig 1a

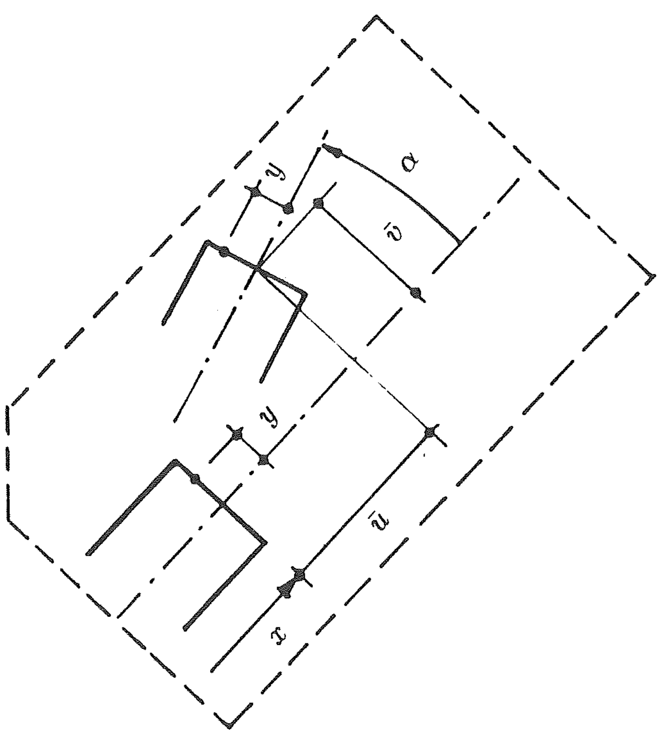

Fig 1b

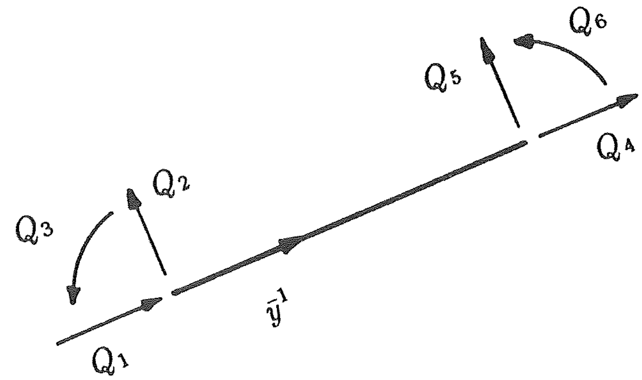

Fig 2

matrices and vectors will be defined for the local system. The constraint equations will then be:

$$\bar{y}^1 = \bar{y}_0^1 + x + \bar{u} - y\sin\alpha$$
$$\bar{y}^2 = \bar{y}_0^2 + \bar{v} + y\cos\alpha$$

The following approximations will be assumed from now on:

$$\sin\alpha \approx \alpha \approx \tan\alpha \approx \bar{v}'$$

$$\cos\alpha \approx 1 - \frac{1}{2}(\bar{v}')^2$$

The Bernoulli-Euler strut does not allow for shear strain and the longitudinal strain ϵ can be defined as:

$$\epsilon = \bar{\epsilon} - y\alpha' \approx \bar{\epsilon} - y\bar{v}''$$

$$\bar{\epsilon} = \bar{\lambda} - 1 \approx \bar{u}' + \frac{1}{2}(\bar{v}')^2$$

To formulate the finite element it is necessary to introduce the discretization, that is, the displacement field within the element should be defined in terms of the nodal displacements, interpreted as generalized coordinates (Q_1 to Q_6) and indicated in Fig 2.

As in the standard applications of the finite-element method, one can think of stating that the transversal displacements $\bar{v}$ result from a **linear combination** of the nodal displacements:

$$\bar{v} = Q_i\Psi_i \text{ , } sum\, from\, 1\, to\, 6,\, where:$$

$$\Psi_1(x) = \Psi_4 = 0$$

$$\Psi_2(x) = 1 - 3\frac{x^2}{\ell^2} + 2\frac{x^3}{\ell^3}$$

$$\Psi_3(x) = x - 2\frac{x^2}{\ell} + \frac{x^3}{\ell^2}$$

$$\Psi_5(x) = 3\frac{x^2}{\ell^2} - 2\frac{x^3}{\ell^3}$$

$$\Psi_6(x) = -\frac{x^2}{\ell} + \frac{x^3}{\ell^2}$$

Nevertheless, if the longitudinal displacements $\bar{u}$ are also supposed to result from a **linear combination** of the nodal displacements, important geometrical nonlinearities will be lost. Instead, we shall follow here the simple hypothesis proposed by Souza Lima and Venancio Filho (1982) for Statics, that is, the **constancy** of normal forces **N** inside the element. It is interesting to remark that an equivalent hypothesis was recently proposed by Salami and Morley (1992), still in Statics.

$$\begin{aligned} N &= \frac{EA}{\ell}\int_0^\ell \bar{\epsilon}dx \\ &= \frac{EA}{\ell}\left[\bar{u}(\ell)-\bar{u}(0)+\frac{1}{2}\int_0^\ell(\bar{v}')^2dx\right] \\ &= \frac{EA}{\ell}\left[Q_4-Q_1+\frac{1}{2}\alpha_{ij}(\ell)Q_iQ_j\right] \end{aligned}$$

where EA is the axial stiffness and:

$$\alpha_{ij}(x)=\int_0^x \Psi_i'(z)\Psi_j'(z)dz$$

Therefore:

$$\bar{u}'=\frac{1}{\ell}(Q_4-Q_1)+\frac{1}{2\ell}Q_iQ_j\alpha_{ij}(\ell)-\frac{1}{2}Q_iQ_j\Psi_i'\Psi_j'$$

and, after integration in **x**:

$$\bar{u}=Q_1+\left[\frac{1}{\ell}(Q_4-Q_1)+\frac{1}{2\ell}Q_iQ_j\alpha_{ij}(\ell)\right]x-\frac{1}{2}Q_iQ_j\alpha_{ij}(x)$$

or:

$$\bar{u}=Q_1\left(1-\frac{x}{\ell}\right)+Q_4\left(\frac{x}{\ell}\right)+\frac{1}{2}Q_iQ_j\left[\frac{x}{\ell}\alpha_{ij}(\ell)-\alpha_{ij}(x)\right]$$

It should be noted that the field of longitudinal displacements $\bar{u}$ has not resulted a linear function of the generalized coordinates!

The constraint equations can now be written in compact form as:

$$\bar{y}^j=\gamma^j(x,Q_1,Q_2,...,Q_6)+y\delta^j(x,Q_1,Q_2,...,Q_6)$$

where:

$$\gamma^1=\bar{y}_0^1+x+Q_i\Phi_i(x)+\frac{1}{2}Q_iQ_j\beta_{ij}(x)$$

$$\delta^1=-Q_i\Psi_i'$$

$$\gamma^2=\bar{y}_0^2+Q_i\Psi_i$$

$$\delta^2=1-\frac{1}{2}Q_iQ_j\Psi_i'\Psi_j'$$

In the above equations, $\Phi_i(x)$, $i=1,2,...,6$, are the standard interpolation functions of the linear analysis:

$$\Phi_1(x)=1-\frac{x}{\ell}$$

$$\Phi_4(x)=\frac{x}{\ell}$$

$$\Phi_2(x)=\Phi_3(x)=\Phi_5(x)=\Phi_6(x)=0$$

and

$$\beta_{ij}(x)=\left[\frac{x}{\ell}\alpha_{ij}(\ell)-\alpha_{ij}(x)\right]$$

One is now ready to write in explicit form the elemental matrices and vectors used in the **global equations of motion**. In the local system, they are written as:

$$\bar{M}^{rs}=\rho A\int_0^\ell(\gamma^1_{,r}\gamma^1_{,s}+\gamma^2_{,r}\gamma^2_{,s})dx+\rho I\int_0^\ell(\delta^1_{,r}\delta^1_{,s}+\delta^2_{,r}\delta^2_{,s})dx$$

$$\begin{aligned} \bar{D}^{rs} &= \mu^{rs}-2\dot{\phi}\left[\rho A\int_0^\ell(\gamma^1_{,r}\gamma^2_{,s}-\gamma^1_{,s}\gamma^2_{,r})dx\right] \\ &+ \dot{Q}_i\rho A\int_0^\ell\left(\frac{1}{2}\gamma^1_{,ri}\gamma^1_{,s}+\gamma^1_{,r}\gamma^1_{,si}-\frac{1}{2}\gamma^1_{,sr}\gamma^1_{,i}\right)dx \\ &+ \dot{Q}_i\rho I\int_0^\ell\delta^2_{,si}\delta^2_{,r}dx \end{aligned}$$

$$\begin{aligned} \bar{U}_{,r} &= \frac{EA}{\ell}\left[\ell Q_i\Phi_i'+\frac{1}{2}\alpha_{ij}(\ell)Q_iQ_j\right]\left[\ell\Phi_r'+Q_k\alpha_{kr}(\ell)\right] \\ &+ EIQ_i\int_0^\ell\Psi_i''(x)\Psi_r''(x)dx \end{aligned}$$

$$\begin{aligned} \bar{\mathcal{F}}^r &= \bar{K}^{rs}Q_s-\bar{U}_{,r}+\bar{P}^r \\ &+ \dot{\phi}^2\rho A\int_0^\ell(\gamma^1_{,r}\gamma^1+\gamma^2_{,r}\gamma^2)dx \\ &+ \dot{\phi}^2\rho I\int_0^\ell(\delta^1_{,r}\delta^1+\delta^2_{,r}\delta^2)dx \\ &+ \ddot{\phi}\rho A\int_0^\ell(\gamma^1_{,r}\gamma^2-\gamma^2_{,r}\gamma^1)dx \\ &+ \ddot{\phi}\rho I\int_0^\ell(\delta^1_{,r}\delta^2-\delta^2_{,r}\delta^1)dx \\ &- \rho A\ddot{S}^1\left[\cos\phi\int_0^\ell\gamma^1_{,r}dx+\sin\phi\int_0^\ell\gamma^2_{,r}dx\right] \\ &- \rho A\ddot{S}^2\left[-\sin\phi\int_0^\ell\gamma^1_{,r}dx+\cos\phi\int_0^\ell\gamma^2_{,r}dx\right] \end{aligned}$$

In the above equations $\phi(t)$ stands for the support rotation characterized by the imposed angle between the relative Yy^1 axis and the inertial Xx^1 axis.

The matrices and vectors of the **incremental formulation** are now explicitly written for the local system:

$$\bar{M}^{rs}_T=\rho A\int_0^\ell(\gamma^1_{,r}\gamma^1_{,s}+\gamma^2_{,r}\gamma^2_{,s})dx+\rho I\int_0^\ell(\delta^1_{,r}\delta^1_{,s}+\delta^2_{,r}\delta^2_{,s})dx$$

$$\begin{aligned} \bar{D}^{rs}_T &= \mu^{rs}-2\dot{\phi}\left[\rho A\int_0^\ell(\gamma^1_{,r}\gamma^2_{,s}-\gamma^1_{,s}\gamma^2_{,r})dx\right] \\ &+ 2\left[\rho A\int_0^\ell\gamma^1_{,r}\gamma^1_{,si}dx+\rho I\int_0^\ell\delta^2_{,r}\delta^2_{,si}dx\right]\dot{Q}_i \end{aligned}$$

$$
\begin{aligned}
\bar{U}_{,rs} &= \frac{EA}{\ell}[\ell\Phi'_s + \alpha_{is}(\ell)Q_i][\ell\Phi'_r + Q_j\alpha_{jr}(\ell)] \\
&+ \frac{EA}{\ell}\left[\ell Q_i\Phi'_i + \frac{1}{2}\alpha_{ij}Q_iQ_j\right]\alpha_{rs} \\
&+ EI\int_0^\ell \Psi''_r(x)\Psi''_s(x)dx
\end{aligned}
$$

$$
\begin{aligned}
\bar{K}_T^{rs} &= \bar{U}_{,rs} + \mu^{ri}_{,s}Q_i \\
&- \dot{\phi}^2\rho A\int_0^\ell(\gamma^1_{,rs}\gamma^1 + \gamma^1_{,r}\gamma^1_{,s} + \gamma^2_{,r}\gamma^2_{,s})dx \\
&+ \dot{\phi}^2\rho I\int_0^\ell(\delta^1_{,r}\delta^1_{,s} + \delta^2_{,rs}\delta^2 + \delta^2_{,r}\delta^2_{,s})dx \\
&- \ddot{\phi}\rho A\int_0^\ell(\gamma^1_{,rs}\gamma^2 + \gamma^1_{,r}\gamma^2_{,s} - \gamma^2_{,r}\gamma^1_{,s})dx \\
&+ \ddot{\phi}\rho I\int_0^\ell(\delta^1_{,r}\delta^2_{,s} - \delta^2_{,rs}\delta^1 - \delta^2_{,r}\delta^1_{,s})dx \\
&+ \rho A\ddot{S}^1\left[\cos\phi\int_0^\ell\gamma^1_{,rs}dx + \sin\phi\int_0^\ell\gamma^2_{,rs}dx\right] \\
&+ \rho A\ddot{S}^2\left[-\sin\phi\int_0^\ell\gamma^1_{,rs}dx + \cos\phi\int_0^\ell\gamma^2_{,rs}dx\right] \\
&+ \ddot{Q}_i\rho A\int_0^\ell(\gamma^1_{,rs}\gamma^1_{,i} + \gamma^1_{,r}\gamma^1_{,si})dx \\
&+ \ddot{Q}_i\rho I\int_0^\ell(\delta^2_{,rs}\delta^2_{,i} + \delta^2_{,r}\delta^2_{,si})dx \\
&+ \left[\rho A\int_0^\ell\gamma^1_{,rs}\gamma^1_{,ij}dx + \rho I\int_0^\ell\delta^2_{,rs}\delta^2_{,ij}dx\right]\dot{Q}_i\dot{Q}_j \\
&- 2\dot{\phi}\left[\rho A\int_0^\ell(\gamma^1_{,rs}\gamma^2_{,i} - \gamma^1_{,is}\gamma^2_{,r})dx\right]\dot{Q}_i
\end{aligned}
$$

$$
\begin{aligned}
\delta\bar{\mathcal{P}}^r &= \delta\bar{P}^r \\
&+ 2\dot{\phi}\rho A\left[\int_0^\ell(\gamma^1_{,r}\gamma^1 + \gamma^2_{,r}\gamma^2)dx\right]\delta\dot{\phi} \\
&+ 2\dot{\phi}\rho I\left[\int_0^\ell(\delta^1_{,r}\delta^1 + \delta^2_{,r}\delta^2)dx\right]\delta\dot{\phi} \\
&+ 2\rho A\left[\int_0^\ell(\gamma^1_{,r}\gamma^2_{,s} - \gamma^1_{,s}\gamma^2_{,r})dx\right]\delta\dot{\phi} \\
&+ 2\rho\left[I\int_0^\ell(\delta^1_{,r}\delta^2_{,s} - \delta^1_{,s}\delta^2_{,r})dx\right]\delta\dot{\phi} \\
&+ \rho A\left[\int_0^\ell(\gamma^1_{,r}\gamma^2 - \gamma^2_{,r}\gamma^1)dx\right]\delta\ddot{\phi} \\
&+ \rho I\left[\int_0^\ell(\delta^1_{,r}\delta^2 - \delta^2_{,r}\delta^1)dx\right]\delta\ddot{\phi} \\
&- \rho A\ddot{S}^1\left[-\sin\phi\int_0^\ell\gamma^1_{,r}dx + \cos\phi\int_0^\ell\gamma^2_{,r}dx\right]\delta\phi \\
&- \rho A\ddot{S}^2\left[-\cos\phi\int_0^\ell\gamma^1_{,r}dx - \sin\phi\int_0^\ell\gamma^2_{,r}dx\right]\delta\phi \\
&- \rho A\left[-\sin\phi\int_0^\ell\gamma^1_{,r}dx + \cos\phi\int_0^\ell\gamma^2_{,r}dx\right]\delta\ddot{S}^2 \\
&- \rho A\left[-\cos\phi\int_0^\ell\gamma^1_{,r}dx - \sin\phi\int_0^\ell\gamma^2_{,r}dx\right]\delta\ddot{S}^1
\end{aligned}
$$

Note that all matrices and vectors of both the global and the incremental equations of motion can be explicitly given in terms of the generalized coordinates, by considering the expressions for γ^1, γ^2, δ^1 and δ^2 previously written and the derivatives:

$$\gamma^1_{,r} = \Phi_r + Q_i\beta_{ir} \qquad \gamma^1_{,rs} = \beta_{rs}$$

$$\gamma^2_{,r} = \Psi_r \qquad \gamma_{,rs} = 0$$

$$\delta^1_{,r} = -\Psi'_r \qquad \delta^1_{,rs} = 0$$

$$\delta^2_{,r} = -Q_i\Psi'_i\Psi'_r \qquad \delta^2_{,rs} = -\Psi'_r\Psi'_s$$

The authors implemented at the Computational Mechanics Laboratory of Escola Politécnica da Universidade de São Paulo, the ANDROS system of FEM programs to perform nonlinear dynamical analysis of structures, based upon the global formulation (Mazzilli and Brasil, 1992). It has been successfully used in several nonlinear analyses reported in a number of papers (Brasil and Mazzilli, 1991), (Brasil and Mazzilli, 1992).

INFLUENCE OF AXIAL FORCES ON UNDAMPED VIBRATION FREQUENCIES OF PLANAR FRAMED STRUCTURES

If a considerable level of static load is applied to planar framed structures, resulting in high axial forces in some members, their natural frequencies of undamped free vibration may change considerably. The undamped equation of motion for those (small) vibrations about the deformed configuration is:

$$m_T^{rs}\,\delta\ddot{q}_s + k_T^{rs}\,\delta q_s = 0$$

where k_T^{rs} stands for the coefficients of the tangent stiffness matrix for that level of static loading. The coefficients of the tangent mass matriz m_T^{rs} are those of m^{rs} and can usually be made equal to those of the linear theory mass matrix. These are the two matrices one should use in the eigenvalue problem solution to find the natural frequencies of free undamped vibrations for the statically loaded deformed structure.

One of the programs included in the ANDROS system performs a static nonlinear analysis of planar framed structures, via the Modified Newton-Raphson algorithm, to obtain the tangent stiffness matrix at a certain level of loading. The system also features a standard eigenvalues routine, based on Holseholder-QL algorithm, to give the frequencies of undamped free vibrations about the statically loaded deformed configuration of the structure.

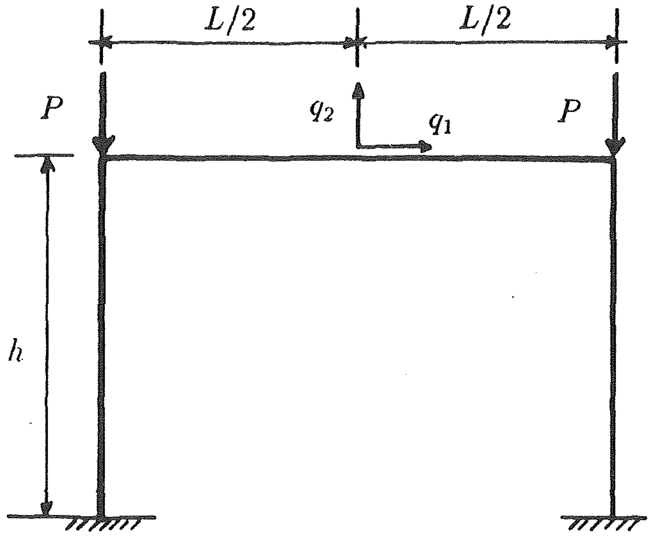

Fig 3

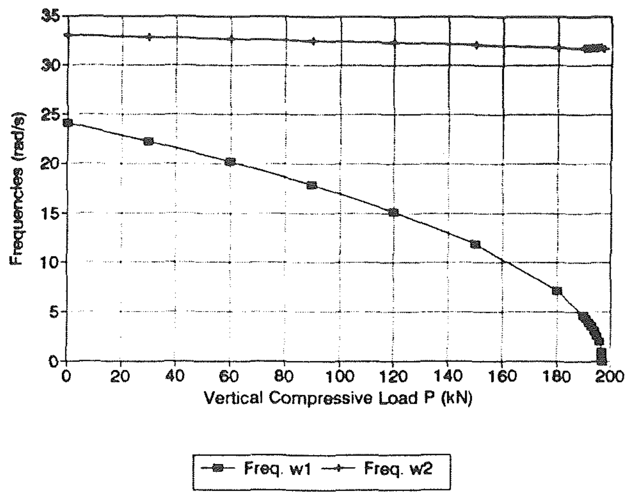

Fig 4

The simple portal frame of Fig 3 was analysed. It is constituted by a four-element horizontal beam and two three-element columns clamped in their bases. Two compressive vertical loads P are applied to the top of the columns. These forces could, as an example, be given by prestressing of the columns. The geometrical and material properties of the analysed model are:

$$A = 3.48 \times 10^{-3} m^2; \quad I = 1.55 \times 10^{-3} m^4;$$

$$L = 7.5m; \quad h = 3.0m;$$

$$E = 2.05 \times 10^{11} Pa; \quad \rho = 7.85 \times 10^3 kg/m^3$$

This structure was analysed for several levels of the P load. The values of its two lowest natural frequencies, namely: ω_1 (the sway mode frequency) and ω_2 (the first symmetrical mode frequency), are plotted in Fig 4 against the variation of P. One can see that ω_1 decreases according to a curve, reaching a zero value for the buckling load predicted by Timoshenko (1961), whereas ω_2 is nearly constant.

ANALYSIS OF A FRAME UNDER NEAR EXTERNAL AND INTERNAL RESONANCE CONDITIONS

As another example of the capabilities of the ANDROS FEM system, a full nonlinear dynamical analysis of a portal frame, with the same geometrical and material properties of the structure shown in Fig 3, is presented. When the load P is set to the value of 110 kN, one can see, in Fig 4, that the near internal resonance condition ($2\omega_1 \approx \omega_2$) is reached between the sway mode and the first symmetrical mode. It was considered a Rayleigh type viscous (linear) damping matrix to yield 0.5% damping rates for these first two modes.

In a forced motion analysis at this level of load, harmonic vertical support excitation was applied, of the form

$$\ddot{S}^2 = \ddot{S}_o \sin \Omega t$$

at near resonance with the symmetrical mode ($\Omega \approx \omega_2$).

For a better understanding of the effects of the geometrical nonlinearities, one should analyse Fig 5 where q_1 stationary amplitude is plotted. The ground acceleration amplitude $\ddot{S}_o$ is kept constant (equal to 0.25 g) while its frequency Ω varies around ω_2. These conditions of near internal and external resonances lead to the "saturation" of the symmetrical mode, through which the energy is pumped into the system, with consequent rapid growing of the lateral vibrations (which otherwise would be near null), due to transference of the surplus energy. Although these postcritical vibrations are found to be theoretically stable, the considerable amplitude of the sway movement may lead to structural damage or even to failure.

CONCLUDING REMARKS

This paper fits into a research line which starts with consistent and general formulations of analytical dynamics, having in mind a robust modelling. The next step would be its application to the finite-element method which would put at reach the analysis of complex engineering problems. Stationary basic solutions could then be surveyed, in special basic statical equilibrium configurations. It would follow a modal analysis considering small perturbations around the chosen basic state, which would give evidences of the relevant modes to be kept in a low-dimension version of the large-size problem. Here, intuitive engineering reasoning - such as the consideration of the energy imparted to the selected modes and the possibility of internal and external resonances -, together with computational techniques for location of in-

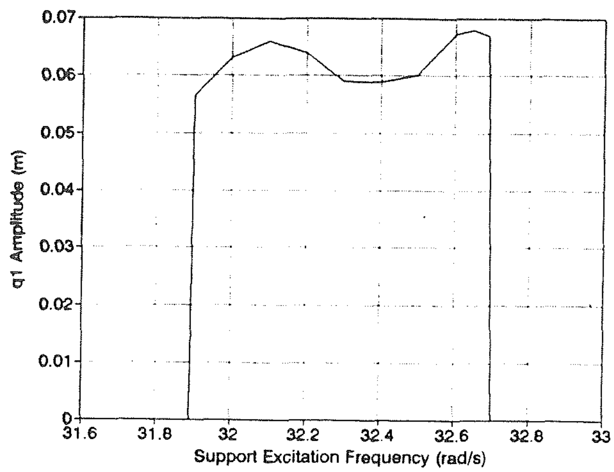

Fig 5

variant manifolds and their tangencies are required. Nonlinear parametric studies for the associated few-degree-of-freedom system would follow, supplying a valuable qualitative knowledge of alternative competing regimes. These studies would them be used to define the quantitative analysis to be performed in the original large-size system via the finite-element method. An effort in this way was made by Mazzilli and Brasil (1993). As a matter of fact, such a research line is already being pursued at LMC – Computational Mechanics Laboratory – of Escola Politécnica, University of São Paulo. At this moment, a general consistent formulation of analytical dynamics – which is part of the subject of this paper – is already available. Based upon it the ANDROS finite-element program was developed (Mazzilli and Brasil, 1992). ANDROS was capable of capturing in large systems the expected nonlinear phenomena after the study of associated simple systems.

Initial work is already under way on the condensation of multiple into few-degree-of-freedom systems, to which perturbation analyses can be applied in automatic fashion with the help of symbolic computation. So far this has been done following very much the intuitive engineering reasoning and considerable help is expected from the applied mathematicians in the more rigorous search of the invariant manifolds.

REFERENCES

Mazzilli C E N (1988), Dinâmica não-linear e estabilidade: uma formulação para sistemas submetidos a excitação de suporte ou a carregamentos não-conservativos, Associate Professor Thesis (in Portuguese) Escola Politécnica da Universidade de São Paulo, São Paulo, Brazil

Mazzilli C E N (1990), Nonlinear Finite Element Formulation in Dynamics, Tech Rep BT/PEF-9008, Escola Politécnica da Universidade de São Paulo

Souza Lima V M and Venâncio Filho F (1982), Considerações sobre a não-linearidade geométrica em estruturas reticuladas, in Proc of III Escola de Matemática Aplicada (in Portuguese), LNCC/CNPq, Rio de Janeiro, Brazil

Salami A T and Morley C T (1992), Finite element analysis of geometric non-linearity in plane frameworks, *The Structural Engineer*, **70** (15)

Mazzilli C E N and Brasil R M L R F (1992), ANDROS - A Finite-Element Program for Nonlinear Dynamics, Tech Report BT/PEF/9213, Escola Politécnica da USP, Dep. de Eng. de Estruturas e Fundações, São Paulo, Brazil

Brasil R M L F and Mazzilli C E N (1991), Dynamical Analysis of Framed Structures of Geometrical Nonlinear Behavior: a Finite Element Approach, in *Proc. of the II Pan American Congress of Applied Mechanics*, Valparaiso, Chile, 232-235

Brasil R M L F and Mazzilli C E N (1992), Nonlinear FE Analysis in Dynamics: Simplifications Aiming at Small Computers, in *Proc. of the IV International Conference on Education, Practice and Promotion of Computational Methods in Engineering using Small Computers*, Dalian, China, 531-536

Timoshenko S P (1961), *Theory of Elastic Stability*, McGraw-Hill, New York, NY

Mazzilli C E N and Brasil R M L R F (1993), Effect of Axial Forces on the Nonlinear Vibrations of a Three-Time Redundant Portal Frame: Analytical and Numerical Studies, submitted for publication to *Nonlinear Dynamics*