NOTE ON THE TRANSFINITE CASE OF HALL'S THEOREM ON REPRESENTATIVES

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1. P. Hall [1] proved the following theorem. Let N be a finite set and let, for $\nu \in N$, A_ν be an arbitrary set. Suppose that (*Hall's condition*), for every subset M of N,

$$|\cup (\nu \in M) A_\nu| \geqslant |M|,$$

where $|A|$ denotes the cardinal number of a set A. Then it is possible to find, for every $\nu \in N$, an element x_ν of A_ν such that $x_\mu \neq x_\nu$ for $\mu \neq \nu$. We express the conclusion by saying that $(x_\nu : \nu \in N)$ is a *representation* of the system $(A_\nu : \nu \in N)$. A system of sets is *representable* or *not representable* according to whether it does or does not have such a representation. Several proofs have been given of Hall's theorem. It is well-known that the proposition no longer holds if both N and at least one A_ν are allowed to be infinite. Hall's theorem has, however, been extended [2]† to the case when N is arbitrary but every A_ν is finite. In Hall's condition the set M is always taken to be finite. The known proofs of this extension use the case of finite N and, in addition, Tychonoff's theorem or arguments amounting in effect to a proof of the relevant case of Tychonoff's theorem.

In this note we give (i) a simpler proof of Hall's theorem for general N (Theorem 1) which does not assume the case of finite N (ii) an even more straightforward proof for the case of at most denumerable N, (iii) a further extension (Theorem 2) which covers the case of an arbitrary N when at most one A_ν is infinite, (iv) a description (Theorem 3) of a class on non-representable systems of sets which makes it unlikely that (iii) can be extended trivially.

The final version of (iii) is due to H. A. Jung and replaces an earlier, more complicated, version. It is published here by kind permission of Dr. Jung.

2. Capital letters denote sets. Throughout this note we consider a fixed system

$$(A_\nu : \nu \in N) \tag{1}$$

of sets and put, for $M \subseteq N$,

$$S(M) = \cup (\nu \in M) A_\nu.$$

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† Alternative proofs in [3] and [4].

The symbol $(A_\nu : \nu\in N)+(B)$ denotes the system of sets consisting of the sets A_ν, for $\nu\in N$, and the set B. More generally, the obvious meaning attaches to the symbol

$$(A_\nu : \nu\in N)+(B_\lambda : \lambda\in L).$$

Trivially, every representable system satisfies Hall's condition.

LEMMA. *Let the system $(A_\nu : \nu\in M)+(B)$ satisfy Hall's condition.*

(i) *If $|B|\geqslant 2$, then there is $x\in B$ such that $(A_\nu : \nu\in M)+(B-\{x\})$ satisfies Hall's condition.*

(ii) *If $|B|<\aleph_0$, then there is $\xi\in B$ such that $(A_\nu : \nu\in M)+(\{\xi\})$ satisfies Hall's condition.*

Proof. To prove (i), choose distinct elements x_0, x_1 of B. Assume that neither x_0 nor x_1 can be taken as x. Then, for $\lambda\in\{0, 1\}$, there is $M_\lambda\subseteq M$ such that $|P_\lambda|\leqslant|M_\lambda|<\aleph_0$, where $P_\lambda=S(M_\lambda)\cup(B-\{x_\lambda\})$. Then

$$\begin{aligned}|M_0|+|M_1|\geqslant|P_0|+|P_1| &= |P_0\cup P_1|+|P_0\cap P_1|\\ &\geqslant|S(M_0\cup M_1)\cup B|+|S(M_0\cap M_1)|\\ &\geqslant|M_0\cup M_1|+1+|M_0\cap M_1|\\ &=|M_0|+|M_1|+1\end{aligned}$$

which is a contradiction. Hence either x_0 or x_1 can be taken as x, and (i) is established. Now (ii) follows by applying (i) $|B|-1$ times. We shall only use (ii).

3. THEOREM 1 (M. Hall). *Let $|A_\nu|<\aleph_0$ for $\nu\in N$, and suppose that the system (1) satisfies Hall's condition. Then (1) is representable.*

Proof. Let Ω be the set of all systems $(B_\nu : \nu\in N)$ such that $B_\nu\subseteq A_\nu$ for $\nu\in N$, and $(B_\nu : \nu\in N)$ satisfies Hall's condition. The assertion means that there is $(B_\nu : \nu\in N)\in\Omega$ such that $|B_\nu|=1$ for $\nu\in N$. Define a partial order on Ω by putting

$$(B_\nu' : \nu\in N)\leqslant(B_\nu'' : \nu\in N)$$

if and only if $B_\nu'\subseteq B_\nu''$ for $\nu\in N$. The finiteness of A_ν and of the sets M in Hall's condition implies that Zorn's lemma applies to $(\Omega, \leqslant)$ and yields a minimal element $(B_\nu : \nu\in N)$ of Ω. Let $\nu_0\in N$. Then, by our lemma, there is $\xi\in B_{\nu_0}$ such that the system $(B_\nu : \nu\in N-\{\nu_0\})+(\{\xi\})$ satisfies Hall's condition. Put $B'_{\nu_0}=\{\xi\}$ and $B_\nu'=B_\nu$ for $\nu\in N-\{\nu_0\}$. Then $(B_\nu' : \nu\in N)\in\Omega$ and, by the minimality of $(B_\nu : \nu\in N)$, we conclude that $|B_{\nu_0}|=1$. Hence $|B_\nu|=1$ for $\nu\in N$, and Theorem 1 follows.

4. If $|N|<\aleph_0$ then the assertion of Theorem 1 follows directly, without Zorn's lemma, after $|N|$ applications of our lemma. We should note here that if (1) satisfies Hall's condition then, on choosing $A_\nu'\subseteq A_\nu$ such that

$|A_\nu'| = \min(|N|, |A_\nu|)$, we conclude that $(A_\nu' : \nu \in N)$ satisfied Hall's condition. Hence we may assume, without loss of generality, that all $|A_\nu| \leqslant |N|$, and then (ii) of our lemma applies if $|N| < \aleph_0$.

Now consider the case $|N| = \aleph_0$, say $N = \{0, 1, \ldots\}$. We assume the hypothesis of Theorem 1, and we shall deduce the conclusion by repeated applications of our lemma. Let $\nu_0 \in N$. We consider the operation T_{ν_0} which turns the system (1) into the system $T_{\nu_0}(A_\nu : \nu \in N) = (A_\nu' : \nu \in N)$ and which is defined as follows. By our lemma there is $\xi \in A_{\nu_0}$ such that the system $(A_\nu : \nu \neq \nu_0) + (\{\xi\})$ satisfies Hall's condition. Fix ξ in some way and put $A'_{\nu_0} = \{\xi\}$; $A_\nu' = A_\nu$ for $\nu \neq \nu_0$. Now let

$$(B_{0\nu} : \nu \in N) = T_0(A_\nu : \nu \in N),$$

$$(B_{\lambda+1, \nu} : \nu \in N) = T_{\lambda+1}(B_{\lambda\nu} : \nu \in N) \text{ for } \lambda \in N.$$

Then $|B_{\nu\nu}| = 1$ for all ν, and $(B_{\nu\nu} : \nu \in N)$ satisfies Hall's condition. This proves Theorem 1 if $|N| = \aleph_0$.

5. A subset M of N is called a *critical index set* if $|S(M)| = |M| < \aleph_0$. Let N^* be the union of all critical index sets.

THEOREM 2. *Let* (1) *satisfy Hall's condition, and let* $|A_\nu| < \aleph_0$ *for* $\nu \in N$. *Then the system*

$$(A_\nu : \nu \in N) + (B) \tag{2}$$

is representable if and only if $B \nsubseteq S(N^*)$.

Proof. (i) Let $(x_\nu : \nu \in N) + (y)$ be a representation of (2). Then, for every critical index set M,

$$|\{x_\nu : \nu \in M\}| \leqslant |S(M)| = |M| = |\{x_\nu : \nu \in M\}|.$$

Hence $S(M) = \{x_\nu : \nu \in M\}$ and therefore $S(N^*) = \{x_\nu : \nu \in N^*\}$,

$$y \in B - \{x_\nu : \nu \in N\} \subseteq B - \{x_\nu : \nu \in N^*\} = B - S(N^*),$$

so that $B \nsubseteq S(N^*)$.

(ii) Let $B \nsubseteq S(N^*)$. Choose $y \in B - S(N^*)$. By definition of N^*, the sytem

$$(A_\nu : \nu \in N^*) + (A_\nu - \{y\} : \nu \in N - N^*) \tag{3}$$

satisfies Hall's condition. Hence, by Theorem 1, there is a representation $(x_\nu : \nu \in N^*) + (x_\nu : \nu \in N - N^*)$ of (3). Then $y \in B - S(N^*) \subseteq B - \{x_\nu : \nu \in N^*\}$, and $(x_\nu : \nu \in N) + (y)$ is a representation of (2). This proves Theorem 2.

6. We now come to a negative result.

THEOREM 3. *Let* $(a_\nu : \nu \in N)$ *be a representation of* (1), *and let* L *be a set such that*

$$|L| > |\cup (\nu \in N)(A_\nu - \{a_\nu\})|.$$

Let $B_\lambda \subseteq \{a_\nu : \nu \in N\}$ for $\lambda \in L$. Then the system

$$(A_\nu : \nu \in N) + (B_\lambda : \lambda \in L) \tag{4}$$

is not representable.

Proof. Let $(x_\nu : \nu \in N) + (y_\lambda : \lambda \in L)$ be a representation of (4). Put

$$N_0 = \{\nu : x_\nu \neq a_\nu\}; \quad N_1 = \{\nu : x_\nu = a_\nu\}.$$

Then

$$\begin{aligned}\{x_\nu : \nu \in N_1\} \cup \{y_\lambda : \lambda \in L\} &= \{a_\nu : \nu \in N_1\} \cup \{y_\lambda : \lambda \in L\} \\ &\subseteq \{a_\nu : \nu \in N\},\end{aligned}$$

and $x_\nu \neq y_\lambda$ for $\nu \in N_1$ and $\lambda \in L$. Hence $\{y_\lambda : \lambda \in L\} \subseteq \{a_\nu : \nu \in N_0\}$,

$$|L| \leqslant |N_0| \leqslant |\cup(\nu \in N)(A_\nu - \{a_\nu\})| < |L|$$

which is the desired contradiction.

7. In conclusion we disprove what might appear to be a plausible extension of Theorem 2 to the case of more than one infinite set. Let $|A_\nu| < \aleph_0$ for $\nu \in N$, and let (1) satisfy Hall's condition. Suppose also that (4) is representable. Then it follows by an easy extension of part (i) of the proof of Theorem 2, that the system

$$(B_\lambda - S(N^*) : \lambda \in L) \tag{5}$$

is representable. By Theorem 2 it is, in fact, true that, vice versa, whenever (1) satisfies Hall's condition and (5) is representable then (4) is representable provided, however, that $|L| = 1$. We now show that this converse proposition is no longer valid if $|L| \geqslant 2$.

To this end put, in Theorem 3,

$$N = \{1, 2, \ldots\}; \quad A_\nu = \{0, \nu\} \text{ and } a_\nu = \nu \text{ for } \nu \in N.$$

Let $2 \leqslant |L| \leqslant \aleph_0$ and $B_\lambda = N$ for $\lambda \in L$. Then $N^* = \varnothing$, (1) satisfies Hall's condition, and (5) is representable. However, Theorem 3 applies and shows, as is easily verified directly, that (4) is not representable.

References

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