Quando uma variável resposta Y é função de duas ou mais variáveis regressoras $(X_1, X_2, \ldots, X_k)$ temos uma REGRESSÃO LINEAR MÚLTIPLA.

O modelo estatístico é dado por:

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \ldots + \beta_k X_{ik} + \varepsilon_i$$

com: $i = 1, \ldots, n$ e $j = 1, \ldots, k$.

Considerando cada variável X_{ij} centrada em sua média $\bar{X}_j$, temos: $x_{ij} = X_{ij} - \bar{X}_j \Rightarrow X_{ij} = x_{ij} + \bar{X}_j$

Desta forma,

$$Y_i = \beta_0 + \beta_1(x_{i1} + \bar{X}_1) + \beta_2(x_{i2} + \bar{X}_2) + \ldots + \beta_k(x_{ik} + \bar{X}_k) + \varepsilon_i$$

$$Y_i = \underbrace{(\beta_0 + \beta_1 \bar{X}_1 + \ldots + \beta_k \bar{X}_k)}_{\alpha} + \beta_1 x_{i1} + \beta_2 x_{i2} + \ldots + \beta_k x_{ik} + \varepsilon_i$$

Matricialmente, o modelo RLM é dado por:

$$\underset{\sim}{Y} = \mathbb{X}\underset{\sim}{\theta} + \underset{\sim}{\varepsilon}$$

$\underset{\sim}{Y}$ é um vetor $n \times 1$ da variável aleatória Y.

$$\underset{\sim}{Y} = {}_n\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix}_1$$

$\mathbb{X}$ é a matriz $n \times p$ de delineamento, $p = k+1$

$$\mathbb{X} = {}_n\begin{bmatrix} 1 & X_{11} & X_{12} & \cdots & X_{1k} \\ 1 & X_{21} & X_{22} & \cdots & X_{2k} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & X_{n1} & X_{n2} & \cdots & X_{nk} \end{bmatrix}_{p=k+1}$$

$\underset{\sim}{\theta}$ é o vetor de parâmetros $p \times 1$, $p = k+1$

$$\underset{\sim}{\theta} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_k \end{bmatrix}_{p \times 1}$$

$\underset{\sim}{\varepsilon}$ é o vetor $n \times 1$ de erros

$$\underset{\sim}{\varepsilon} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}_{n \times 1}$$

Suposições:

i) A variável y é uma função LINEAR das k variáveis regressoras;

ii) As variáveis x_j são de natureza fixa;

iii) $E(\varepsilon_i) = 0 \Rightarrow E(\underset{\sim}{\varepsilon}) = \phi = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}_{n \times 1}$

iv) Erros são homocedásticos e independentes

$$\left.\begin{array}{l} Var(\varepsilon_i) = \sigma^2 \\ Cov(\varepsilon_i, \varepsilon_{i'}) = 0 \end{array}\right\} Var(\underset{\sim}{\varepsilon}) = \begin{bmatrix} \sigma^2 & 0 & \cdots & 0 \\ 0 & \sigma^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma^2 \end{bmatrix}_{n \times n} = I_n \sigma^2 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{n \times n} \sigma^2$$

v) Erros têm distribuição NORMAL

$$\varepsilon_i \sim N(0, \sigma^2) \Rightarrow \underset{\sim}{\varepsilon} \sim N(\phi, I\sigma^2)$$

$$y_i \sim N(\beta_0 + \beta_1 x_i, \sigma^2) \Rightarrow \underset{\sim}{Y} \sim N(X\underset{\sim}{\theta}, I\sigma^2)$$

ESTIMAÇÃO DOS PARÂMETROS – MMQ $(n > p)$

Considerando o modelo: $\underset{\sim}{Y} = X\underset{\sim}{\theta} + \underset{\sim}{\varepsilon}$, temos:

$$\underset{\sim}{\varepsilon} = \underset{\sim}{Y} - X\underset{\sim}{\theta} \Rightarrow \text{minimizar } \sum_{i=1}^{n} \varepsilon_i = \underset{\sim}{\varepsilon}^T \underset{\sim}{\varepsilon} = [\varepsilon_1, \dots \varepsilon_n]\begin{bmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_n \end{bmatrix}$$

$$\underset{\sim}{\varepsilon}^T \underset{\sim}{\varepsilon} = [\underset{\sim}{Y} - X\underset{\sim}{\theta}]^T [\underset{\sim}{Y} - X\underset{\sim}{\theta}]$$

OBS: $\begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix}\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} = a_{11}.a_{11} + a_{12}.a_{21} + a_{13}.a_{31}$

$$Z = [\underset{\sim}{Y}^T - \underset{\sim}{\theta}^T X^T][\underset{\sim}{Y} - X\underset{\sim}{\theta}]$$

$$Z = \underset{\sim}{Y}^T \underset{\sim}{Y} - \underbrace{\underset{\sim}{Y}^T X\underset{\sim}{\theta}}_{A^T} - \underset{\sim}{\theta}^T \underbrace{X^T \underset{\sim}{Y}}_{A} + \underset{\sim}{\theta}^T \underbrace{X^T X}_{B} \underset{\sim}{\theta}$$

$(A+B)^T = A^T + B^T$

$(AB)^T = B^T A^T$

$$\frac{\partial Z}{\partial \underset{\sim}{\theta}} = 0$$

$\frac{\partial A^T \underset{\sim}{\theta}}{\partial \underset{\sim}{\theta}} = \frac{\partial \underset{\sim}{\theta}^T A}{\partial \underset{\sim}{\theta}} = A$

$\frac{\partial \underset{\sim}{\theta}^T B \underset{\sim}{\theta}}{\partial \underset{\sim}{\theta}} = 2B\underset{\sim}{\theta}$

$$\hookrightarrow -A - A + 2B\underset{\sim}{\theta} = 0 \quad \begin{cases} A = X^T \underset{\sim}{Y} \\ B = X^T X \end{cases}$$

$$-X^T \underset{\sim}{Y} - X^T \underset{\sim}{Y} + 2X^T X \hat{\underset{\sim}{\theta}} = 0$$

$$-2X^T \underset{\sim}{Y} + 2X^T X \hat{\underset{\sim}{\theta}} = 0$$

$$X^T X \hat{\underset{\sim}{\theta}} = X^T \underset{\sim}{Y} \quad \text{(SEN)}$$

$$\underbrace{(X^T X)^{-1} X^T X}_{I} \hat{\underset{\sim}{\theta}} = (X^T X)^{-1} X^T \underset{\sim}{Y}$$

$$\boxed{\hat{\underset{\sim}{\theta}} = (X^T X)^{-1} X^T \underset{\sim}{Y}}$$

QUEM SÃO AS MATRIZES ?

Considerando n observações e duas variáveis X_1 e X_2, temos:

$$\underset{\sim}{Y} = {}_{n}\begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix}_{1} \qquad X = {}_{n}\begin{bmatrix} 1 & X_{11} & X_{12} \\ \vdots & \vdots & \vdots \\ 1 & X_{n1} & X_{n2} \end{bmatrix}_{p=3} \qquad X^T = {}_{p}\begin{bmatrix} 1 & \dots & 1 \\ X_{11} & \dots & X_{n1} \\ X_{12} & \dots & X_{n2} \end{bmatrix}_{n}$$

$$X^T X = {}_{p}\begin{bmatrix} 1 & \dots & 1 \\ X_{11} & \dots & X_{n1} \\ X_{12} & \dots & X_{n2} \end{bmatrix}_{n} {}_{n}\begin{bmatrix} 1 & X_{11} & X_{12} \\ \vdots & \vdots & \vdots \\ 1 & X_{n1} & X_{n2} \end{bmatrix}_{p} = {}_{p}\begin{bmatrix} n & \sum X_{i1} & \sum X_{i2} \\ & \sum X_{i1}^2 & \sum X_{i1}X_{i2} \\ \text{SIM} & & \sum X_{i2}^2 \end{bmatrix}_{p}$$

$\hookrightarrow$ simétrica

$$\underset{p}{\mathbf{X}^T}\underset{\sim}{y} = \begin{bmatrix} 1 & \cdots & 1 \\ x_{11} & \cdots & x_{n1} \\ x_{12} & \cdots & x_{n2} \end{bmatrix}_{n} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}_{1} = \begin{bmatrix} \sum y_i \\ \sum x_{i1} y_i \\ \sum x_{i2} y_i \end{bmatrix}_{1}$$

Considerando o estimador $\underset{\sim}{\hat{\theta}} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\underset{\sim}{y}$, precisamos da inversa da matriz $\mathbf{X}^T\mathbf{X}$.

Como calcular as inversas?

- Matriz 2×2

$$Q = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad Q^{-1} = \frac{1}{\det(Q)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$ em que $\det(Q) = ad - bc$

- Matriz 3×3

$$Q = \begin{bmatrix} a & b & c \\ b & e & f \\ c & f & h \end{bmatrix} \qquad Q^{-1} = \begin{bmatrix} A & B & C \\ B & E & F \\ C & F & H \end{bmatrix}$$

em que:

$$A = (eh - f^2)/z \qquad E = (ah - c^2)/z$$

$$B = -(bh - cf)/z \qquad F = -(af - bc)/z$$

$$C = (bf - ce)/z \qquad H = (ae - b^2)/z$$

$$z = aeh + 2bcf - af^2 - b^2h - c^2e \Rightarrow \det(Q)$$

Vamos verificar as matrizes e cálculos com o RLS!

Considerando o RLS

$$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i \qquad i = 1, \dots, n$$

$$\underset{\sim}{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \quad \underset{\sim}{\mathbb{X}} = \begin{bmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \quad \underset{\sim}{\theta} = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} \quad \underset{\sim}{\varepsilon} = \begin{bmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_n \end{bmatrix}$$

$$\mathbb{X}^T\mathbb{X} = \begin{bmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} = \begin{bmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix} \qquad \mathbb{X}^T y = \begin{bmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum x_i y_i \end{bmatrix}$$

$$(\mathbb{X}^T\mathbb{X})^{-1} = \frac{1}{n\sum x_i^2 - (\sum x_i)^2} \begin{bmatrix} \sum x_i^2 & -\sum x_i \\ -\sum x_i & n \end{bmatrix}$$

$$\underset{\sim}{\hat{\theta}} = (\mathbb{X}^T\mathbb{X})^{-1}\mathbb{X}^T \underset{\sim}{y} = \frac{1}{n\sum x_i^2 - (\sum x_i)^2} \begin{bmatrix} \sum x_i^2 & -\sum x_i \\ -\sum x_i & n \end{bmatrix} \begin{bmatrix} \sum y_i \\ \sum x_i y_i \end{bmatrix}$$

$$\underset{\sim}{\hat{\theta}} = \frac{1}{n\sum x_i^2 - (\sum x_i)^2} \begin{bmatrix} \sum x_i^2 . \sum y_i - \sum x_i \sum x_i y_i \\ -\sum x_i \sum y_i + n \sum x_i y_i \end{bmatrix}$$

$$\underset{\sim}{\hat{\theta}} = \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix} = \begin{bmatrix} \dfrac{\sum x_i^2 \sum y_i - \sum x_i \sum x_i y_i}{n\sum x_i^2 - (\sum x_i)^2} \\[2ex] \dfrac{n\sum x_i y_i - \sum x_i \sum y_i}{n\sum x_i^2 - (\sum x_i)^2} \end{bmatrix}$$

→ equivalente $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$

→ equivalente $\hat{\beta}_1 = \dfrac{SPXY}{SQX}$

Propriedades:

i) Os elementos de $\hat{\underset{\sim}{\theta}}$ são combinações lineares dos y_i

$$\hat{\underset{\sim}{\theta}} = \begin{bmatrix} \hat{\beta}_0 \\ \vdots \\ \hat{\beta}_K \end{bmatrix} \Rightarrow \hat{\beta}_j = \sum_{i=1}^{n} c_{ij} y_i$$

$$\hat{\underset{\sim}{\theta}} = \underbrace{(\mathbb{X}^T\mathbb{X})^{-1}\mathbb{X}^T}_{{}_p\mathbb{C}_n} \underset{\sim}{y}$$

${}_p\mathbb{C}_n \Rightarrow$ matriz de coeficientes $\quad \mathbb{C} = {}_p\begin{bmatrix} c_{11} & c_{12} \cdots & c_{1n} \\ c_{21} & c_{22} \cdots & c_{2n} \\ \vdots & \vdots & \vdots \\ c_{p1} & c_{p2} & c_{pn} \end{bmatrix}_n$

$$\hat{\underset{\sim}{\theta}} = \mathbb{C}\underset{\sim}{y} = \begin{bmatrix} c_{11} & c_{12} \cdots & c_{1n} \\ c_{21} & c_{22} \cdots & c_{2n} \\ \vdots & & \\ c_{p1} & c_{p2} \cdots & c_{pn} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \sum c_{1i} y_i \\ \sum c_{2i} y_i \\ \vdots \\ \sum c_{pi} y_i \end{bmatrix}$$

ii) $\hat{\underset{\sim}{\theta}}$ é um estimador não viesado para $\underset{\sim}{\theta}$.

$E(A\underset{\sim}{y}) = A\,E(\underset{\sim}{y})$

$$E(\hat{\underset{\sim}{\theta}}) = E[\underbrace{(\mathbb{X}^T\mathbb{X})^{-1}\mathbb{X}^T}_{A}\underset{\sim}{y}] = (\mathbb{X}^T\mathbb{X})^{-1}\mathbb{X}^T E(\underset{\sim}{y}) = \underbrace{(\mathbb{X}^T\mathbb{X})^{-1}\mathbb{X}^T\mathbb{X}}_{I}\underset{\sim}{\theta} = \underset{\sim}{\theta}$$

iii) A matriz de covariâncias de $\hat{\underset{\sim}{\theta}}$ é dada por:

$Var(A\underset{\sim}{y}) = A\,Var(\underset{\sim}{y})\,A^T$

$$Var(\hat{\underset{\sim}{\theta}}) = var[\underbrace{(\mathbb{X}^T\mathbb{X})^{-1}\mathbb{X}^T}_{A}y] = (\mathbb{X}^T\mathbb{X})^{-1}\mathbb{X}^T\,Var(\underset{\sim}{y})\,\mathbb{X}(\mathbb{X}^T\mathbb{X})^{-1}$$

$$= (\mathbb{X}^T\mathbb{X})^{-1}\mathbb{X}^T\, I\sigma^2\, \mathbb{X}(\mathbb{X}^T\mathbb{X})^{-1}$$

$$= \underbrace{(\mathbb{X}^T\mathbb{X})^{-1}\mathbb{X}^T\mathbb{X}}_{I}(\mathbb{X}^T\mathbb{X})^{-1}\sigma^2$$

$$= (\mathbb{X}^T\mathbb{X})^{-1}\sigma^2$$

Para o modelo RLS, temos:

$$Var[\hat{\underset{\sim}{\theta}}] = \underbrace{\frac{1}{\underset{n\sum x_i^2}{n\sum X_i^2 - (\sum X_i)^2}} \begin{bmatrix} \sum X_i^2 & -\sum X_i \\ -\sum X_i & n \end{bmatrix}}_{(\mathbb{X}^T\mathbb{X})^{-1}} \cdot \sigma^2$$

$$= \frac{1}{n\sum x_i^2} \begin{bmatrix} \sum X_i^2 & -\sum X_i \\ -\sum X_i & n \end{bmatrix} \sigma^2$$

$$= \begin{bmatrix} \frac{\sum X_i^2}{n\sum x_i^2} {}^{*} & \frac{-\sum X_i}{n\sum x_i^2} \\ \frac{-\sum X_i}{n\sum x_i^2} & \frac{n}{n\sum x_i^2} \end{bmatrix} \sigma^2$$

$(*)$ $\sum X_i^2 = \sum X_i^2 - \frac{2(\sum X_i)^2}{n} + \frac{2(\sum X_i)^2}{n}$

$= \sum X_i^2 - 2\bar{X}(\sum X_i) + 2\bar{X}(\sum X_i) \cdot \frac{n}{n}$

$= \sum X_i^2 - 2\bar{X}(\sum X_i) + 2n\bar{X}^2$

$= \underbrace{\sum X_i^2 - 2\bar{X}(\sum X_i) + n\bar{X}} + n\bar{X}$

$= \sum(X_i - \bar{X})^2 + n\bar{X}$

$= \sum x_i^2 + n\bar{X}$

$$= \begin{bmatrix} \frac{\sum x_i^2 + n\bar{X}}{n\sum x_i^2} & -\frac{\sum X_i/n}{\sum x_i^2} \\ -\frac{\sum X_i/n}{\sum x_i^2} & \frac{1}{\sum x_i^2} \end{bmatrix} \sigma^2$$

$$= \begin{bmatrix} \frac{1}{n} + \frac{\bar{X}}{\sum x_i^2} & -\frac{\bar{X}}{\sum x_i^2} \\ -\frac{\bar{X}}{\sum x_i^2} & \frac{1}{\sum x_i^2} \end{bmatrix} \sigma^2$$

$Var(\hat{\beta}_0)$

$Var(\hat{\beta}_1)$

$Cov(\hat{\beta}_0, \hat{\beta}_1)$

Vimos que:

$Var(\hat{\beta}_0) = \left(\frac{1}{n} + \frac{\bar{X}}{\sum x_i^2}\right)\sigma^2$

$Var(\hat{\beta}_1) = \frac{\sigma^2}{\sum x_i^2}$

$Cov(\hat{\beta}_0, \hat{\beta}_1) = -\frac{\bar{X}\sigma^2}{\sum x_i^2}$

iv) Como resultado dos itens i, ii e iii e que $\underset{\sim}{Y} \sim N(X\underset{\sim}{\theta}, I\sigma^2)$

temos que: $\hat{\underset{\sim}{\theta}} \sim N\left(\underset{\sim}{\theta}, (X^T X)^{-1}\sigma^2\right)$

v) Dada uma combinação linear dos parâmetros

$$\underset{\sim}{c}^T \underset{\sim}{\theta} = [c_0 \ c_1 \ \dots \ c_k]\begin{bmatrix}\theta_0 \\ \theta_1 \\ \vdots \\ \theta_k\end{bmatrix} = c_0\theta_0 + c_1\theta_1 + \dots + c_k\theta_k$$

Um estimador de mínimos quadrados, não viesado e de variância mínima é dado por: $\underset{\sim}{c}^T \hat{\underset{\sim}{\theta}}$

$$E(\underset{\sim}{c}^T\hat{\underset{\sim}{\theta}}) = \underset{\sim}{c}^T E(\hat{\underset{\sim}{\theta}}) = \underset{\sim}{c}^T \underset{\sim}{\theta}$$

$$Var(\underset{\sim}{c}^T\hat{\underset{\sim}{\theta}}) = \underset{\sim}{c}^T Var(\hat{\underset{\sim}{\theta}})\underset{\sim}{c} = \underset{\sim}{c}^T (X^T X)^{-1}\underset{\sim}{c} \cdot \sigma^2$$

Desta forma, $\underset{\sim}{c}^T\hat{\underset{\sim}{\theta}} \sim N\left(\underset{\sim}{c}^T\underset{\sim}{\theta}, \underset{\sim}{c}^T (X^T X)^{-1}\underset{\sim}{c}\,\sigma^2\right)$

vi) A aproximação de mínimos quadrados ($\hat{\underset{\sim}{Y}}$) para $\underset{\sim}{Y}$ é dada por:

$$\hat{\underset{\sim}{Y}} = X\hat{\underset{\sim}{\theta}} = \underbrace{X(X^T X)^{-1}X^T}_{\text{Hat matrix (matriz de projeção)}}\underset{\sim}{Y} = \overset{\text{H ou P}}{H}\underset{\sim}{Y}$$

$$E(\hat{\underset{\sim}{Y}}) = E(H\underset{\sim}{Y}) = H\,E(\underset{\sim}{Y}) = HX\underset{\sim}{\theta} = X\underbrace{(X^T X)^{-1}X^T X}_{I}\underset{\sim}{\theta} = X\underset{\sim}{\theta} = \underset{\sim}{Y}$$

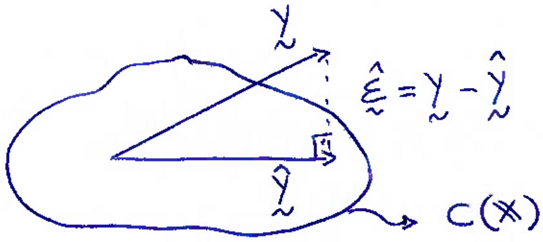
$$\begin{aligned}Var(\hat{\underset{\sim}{Y}}) &= Var(H\underset{\sim}{Y}) = H\,Var(\underset{\sim}{Y})\,H^T \\ &= X(X^T X)^{-1}X^T I\sigma^2 X(X^T X)^{-1}X^T \\ &= X\underbrace{(X^T X)^{-1}X^T X}_{I}(X^T X)^{-1}X^T\sigma^2 \\ &= \underbrace{X(X^T X)^{-1}X^T}_{H}\sigma^2 = H\sigma^2\end{aligned}$$

$$\hat{\underset{\sim}{Y}} \sim N(\underset{\sim}{Y}, H\sigma^2) \Rightarrow \hat{\underset{\sim}{Y}} \sim N\left(X\underset{\sim}{\theta}, X(X^T X)^{-1}X^T\sigma^2\right)$$

ANÁLISE DE VARIÂNCIA E TESTE F

Relembrando!

A matriz $H = \mathbb{X}(\mathbb{X}^T\mathbb{X})^{-1}\mathbb{X}^T$ é o projetor ortogonal de $\underset{\sim}{y}$ no espaço coluna de $\mathbb{X} \Rightarrow C(\mathbb{X})$.



Pelo teorema de Pitágoras, temos:

$$\|\underset{\sim}{y}\|^2 = \|\hat{\underset{\sim}{y}}\|^2 + \|\hat{\underset{\sim}{\varepsilon}}\|^2$$

* O vetor de observações $\underset{\sim}{y}$, pode ser decomposto na soma de dois vetores ortogonais
  - ↳ o vetor $\hat{\underset{\sim}{y}}$ do espaço de estimação;
  - ↳ o vetor $\hat{\underset{\sim}{\varepsilon}}$ do espaço resíduo.

Essa decomposição é o fundamento da análise de variância.

Considerando a decomposição,

$\|\underset{\sim}{y}\|^2 = \|\hat{\underset{\sim}{y}}\|^2 + \|\hat{\underset{\sim}{\varepsilon}}\|^2$, temos:

$$\|\underset{\sim}{y}\|^2 = \underset{\sim}{y}^T\underset{\sim}{y}$$

$$\|\hat{\underset{\sim}{y}}\|^2 = \|\mathbb{X}\hat{\underset{\sim}{\theta}}\|^2 = (\mathbb{X}\hat{\underset{\sim}{\theta}})^T(\mathbb{X}\hat{\underset{\sim}{\theta}})$$

$$\|\hat{\underset{\sim}{\varepsilon}}\|^2 = \|\underset{\sim}{y} - \mathbb{X}\hat{\underset{\sim}{\theta}}\|^2 = (\underset{\sim}{y} - \mathbb{X}\hat{\underset{\sim}{\theta}})^T(\underset{\sim}{y} - \mathbb{X}\hat{\underset{\sim}{\theta}})$$

Desta forma,

$$\begin{aligned}
\underset{\sim}{y}^T\underset{\sim}{y} &= (\mathbb{X}\hat{\underset{\sim}{\theta}})^T(\mathbb{X}\hat{\underset{\sim}{\theta}}) + (\underset{\sim}{y} - \mathbb{X}\hat{\underset{\sim}{\theta}})^T(\underset{\sim}{y} - \mathbb{X}\hat{\underset{\sim}{\theta}}) \\
&= (\hat{\underset{\sim}{\theta}}^T\mathbb{X}^T)(\mathbb{X}\hat{\underset{\sim}{\theta}}) + (\underset{\sim}{y}^T - \hat{\underset{\sim}{\theta}}^T\mathbb{X}^T)(\underset{\sim}{y} - \mathbb{X}\hat{\underset{\sim}{\theta}}) \\
&= \hat{\underset{\sim}{\theta}}^T\mathbb{X}^T\mathbb{X}\hat{\underset{\sim}{\theta}} + \underset{\sim}{y}^T\underset{\sim}{y} - \underset{\sim}{y}^T\mathbb{X}\hat{\underset{\sim}{\theta}} - \hat{\underset{\sim}{\theta}}^T\mathbb{X}^T\underset{\sim}{y} + \hat{\underset{\sim}{\theta}}^T\mathbb{X}^T\mathbb{X}\hat{\underset{\sim}{\theta}}
\end{aligned}$$

Usando o resultado do SEN $\mathbb{X}^T\mathbb{X}\hat{\underset{\sim}{\theta}} = \mathbb{X}^T\underset{\sim}{y}$, teremos:

$$\underset{\sim}{y}^T\underset{\sim}{y} = \hat{\underset{\sim}{\theta}}^T X^T \underset{\sim}{y} + \underset{\sim}{y}^T\underset{\sim}{y} - \underset{\sim}{y}^T X \hat{\underset{\sim}{\theta}} - \cancel{\hat{\underset{\sim}{\theta}}^T X^T \underset{\sim}{y}} + \cancel{\hat{\underset{\sim}{\theta}}^T X^T \underset{\sim}{y}}$$

$$\underbrace{\underset{\sim}{y}^T\underset{\sim}{y}}_{\text{SQTotal (não corrigida)}} = \underbrace{\hat{\underset{\sim}{\theta}}^T X^T \underset{\sim}{y}}_{\text{SQParâmetros}} + \big(\underbrace{\underset{\sim}{y}^T\underset{\sim}{y} - \underset{\sim}{y}^T X \hat{\underset{\sim}{\theta}}}_{\text{SQ Resíduos}}\big)$$

OBS: $\text{SQTotal} = \underbrace{\sum y_i^2}_{\text{SQTotal (NCorr)}} - \underbrace{\frac{(\sum y_i)^2}{n}}_{\text{correção}}$

↳ correção ↳ está embutida na SQParâmetros *

$$\text{SQParâmetros} = \text{SQRegressão} + \text{Correção}$$

veja a seguir

ou seja,

$$\text{SQTotal (NC)} = \text{SQParâmetros} + \text{SQResíduos}$$

ou

$$\text{SQTotal} = \text{SQRegressão} + \text{SQResíduo}$$

* Considerando a RLS com variável X centrada

$$y_i = \alpha + \beta_1 \underbrace{(x_i - \bar{x})}_{x_i} + \varepsilon_i$$

$$\text{SQParam} = \hat{\underset{\sim}{\theta}}^T X^T \underset{\sim}{y} = [\hat\alpha \ \ \hat\beta_1]\begin{bmatrix}\sum y_i \\ \sum x_i y_i\end{bmatrix}$$

$$\hat\alpha = \bar{y} \qquad \hat\beta_1 = \frac{\sum x_i y_i}{\sum x_i^2}$$

$$= \hat\alpha \sum y_i + \hat\beta_1 \sum x_i y_i$$

$$= \frac{\sum y_i}{n}\sum y_i + \frac{\sum x_i y_i}{\sum x_i^2}\sum x_i y_i$$

$$= \underbrace{\frac{(\sum y_i)^2}{n}}_{\text{correção}} + \underbrace{\frac{(\sum x_i y_i)^2}{\sum x_i^2}}_{\text{SQRegressão}}$$

$$\text{SQTotal (NC)} = \text{SQParam} + \text{SQResíduo}$$

$$\text{SQTotal (NC)} = \text{correção} + \text{SQReg} + \text{SQResíduo}$$

$$\text{SQTotal (NC)} - \text{correção} = \text{SQReg} + \text{SQResíduo}$$

$$\text{SQTotal} = \text{SQRegressão} + \text{SQResíduo}$$

NÚMERO DE GRAUS DE LIBERDADE

O número de graus de liberdade associados à uma SQ é dado pela característica da matriz idempotente de sua forma quadrática.

$SQ \Rightarrow \underset{\sim}{Y}^T A \underset{\sim}{Y}$ sendo A uma matriz idempotente ($A^2 = A$) então, o traço (A) = gl da SQ.

- O traço de uma matriz corresponde a soma dos elementos de sua diagonal principal.
- propriedades do traço:

$tr(A) = tr(A^T)$ $\quad tr(A+B) = tr(A) + tr(B)$

$tr(cA) = c\,tr(A)$ $\quad tr(A.B) = tr(BA)$

SQTotal (NC)

$$SQTotal\,(NC) = \underset{\sim}{Y}^T \underset{\sim}{Y} = \underset{\sim}{Y}^T I \underset{\sim}{Y}$$ ⇒ FORMA QUADRÁTICA

↳ É IDEMPOTENTE?

$I.I = I$ sim.

Então, $tr(I) = n$ é o grau de liberdade associado à SQTotal (NC)

Correção

$$C = \frac{(\Sigma Y_i)^2}{n}$$

$$C = [Y_1 \cdots Y_n] \underbrace{\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}}_{\underset{\sim}{1}} \underbrace{[1 \cdots 1]}_{\underset{\sim}{1}^T} \begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix} \cdot \frac{1}{n}$$

$$\underset{\sim}{J} = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix}_{n \times n}$$

$$C = \underset{\sim}{Y}^T \underbrace{\underset{\sim}{1}\,\underset{\sim}{1}^T}_{\underset{\sim}{J}} \underset{\sim}{Y} \cdot \frac{1}{n} = \underset{\sim}{Y}^T \underset{\sim}{J}\,\underset{\sim}{Y} \cdot \frac{1}{n}$$

Desta forma,

$$SQTotal = \underset{\sim}{y}^T \underset{\sim}{y} - \underset{\sim}{y}^T J \underset{\sim}{y} \frac{1}{n}$$

$$= \underset{\sim}{y}^T I \underset{\sim}{y} - \underset{\sim}{y}^T J \underset{\sim}{y} \cdot \frac{1}{n}$$

$$= \underset{\sim}{y}^T \underbrace{\left[I - \frac{1}{n} J\right]}_{\text{É IDEMPOTENTE?}} \underset{\sim}{y} \quad \Rightarrow \text{FORMA QUAD.}$$

$$\left[I - \frac{1}{n}J\right]\left[I - \frac{1}{n}J\right] = II - I\frac{1}{n}J - \frac{1}{n}JI + \frac{1}{n^2}JJ$$

$$= I - \frac{1}{n}J - \frac{1}{n}J + \frac{\cancel{n}J}{n^{\cancel{2}}}$$

$$= I - \frac{1}{n}J - \cancel{\frac{1}{n}J} + \cancel{\frac{J}{n}}$$

$$= I - \frac{1}{n}J \rightarrow \text{idempotente}$$

$$* \; JJ = \begin{array}{c} \\ n \end{array}\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix}_n \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix}_{\substack{n \quad\; n}} = \begin{bmatrix} n & \cdots & n \\ \vdots & & \vdots \\ n & \cdots & n \end{bmatrix}_{\substack{n \quad\; n}} = n\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix} = nJ$$

Então

$$tr\left(I - \frac{1}{n}J\right) = tr(I) - tr\left(\frac{1}{n}J\right)$$

$$= tr(I) - \frac{1}{n} tr(J)$$

$$= n - \frac{1}{n} \cdot n = n - 1$$

gl associado à SQTotal !

SQRegressão

$$SQReg = SQParam - C$$

$$= \hat{\underset{\sim}{\theta}}^T X^T \underset{\sim}{y} - \frac{1}{n} \underset{\sim}{y}^T J \underset{\sim}{y}$$

$$= \left[(X^TX)^{-1} X \underset{\sim}{y}\right]^T X^T \underset{\sim}{y} - \frac{1}{n} \underset{\sim}{y}^T J \underset{\sim}{y}$$

$$= \underset{\sim}{y}^T \underbrace{X^T (X^TX)^{-1} X^T}_{H} \underset{\sim}{y} - \frac{1}{n} \underset{\sim}{y}^T J \underset{\sim}{y}$$

$$= \underset{\sim}{y}^T H \underset{\sim}{y} - \frac{1}{n} \underset{\sim}{y}^T J \underset{\sim}{y}$$

$$= \underset{\sim}{y}^T \underbrace{\left[H - \frac{1}{n}J\right]}_{\text{É IDEMPOTENTE?}} \underset{\sim}{y}$$

$$\left[H - \frac{1}{n}J\right]\left[H - \frac{1}{n}J\right] =$$

$$= HH - H\frac{1}{n}J - \frac{1}{n}JH + \frac{1}{n^2}JJ$$

OBS:
$HJ = JH = J$
$H^2 = H$
$JJ = nJ$

$$= H - \frac{1}{n}J - \frac{1}{n}J + \frac{\cancel{n}J}{n^{\cancel{2}}}$$

$$= H - \frac{1}{n}J - \cancel{\frac{1}{n}J} + \cancel{\frac{1}{n}J}$$

$$= H - \frac{1}{n}J$$

↳ IDEMPOTENTE !

Então, $tr(H - \frac{1}{n}J) = tr(H) - \frac{1}{n} tr(J)$

$$= tr(\underbrace{X(X^TX)^{-1}}_{A}\underbrace{X^T}_{B}) - \frac{1}{n} tr(J)$$

$*$ $tr(AB) = tr(BA)$ ver propriedades

$$= tr(\underbrace{X^TX(X^TX)^{-1}}_{I_p}) - \frac{1}{n} tr(J)$$

$$= tr(I) - \frac{1}{n} tr(J)$$

$$= p - \frac{1}{n} n = \underline{p-1} \;//$$ gl associado à SQReg

SQResíduo

$$SQResíduo = SQTotal - SQReg$$

$$= SQTotal\,(NC) - SQParam$$

$$= \underset{\sim}{y}^T I \underset{\sim}{y} - \underset{\sim}{y}^T H \underset{\sim}{y}$$

$$= \underset{\sim}{y}^T [I - H] \underset{\sim}{y}$$

↳ É IDEMPOTENTE?

$$[I-H][I-H] = II - IH - HI + HH$$

$$= I - H - \cancel{H} + \cancel{H}$$

$$= I - H \rightarrow \text{IDEMPOTENTE!}$$

Então, $tr(I-H) = tr(I) - tr(H)$

$$= \underline{n - p} \;//$$ gl associado à SQResíduo

TABELA ANAVA e TESTE F

SQRegressão ⇒ o quanto reduz a SQResíduo quando incluímos um parâmetro (ou variável) no modelo.

modelo + simples $\xrightarrow{\text{(inclusão de variáveis)}}$ modelo + completo

SQRes (maior) $\xrightarrow{\text{essa redução}}$ SQRes (menor)

↓

SQReg (do modelo + completo)

por isso notação $R(\cdot) = SQ$

Vamos considerar 3 modelos

M1: $y_i = \beta_0 + \varepsilon_i$

M2: $y_i = \beta_0 + \beta_1 X_i + \varepsilon_i$

M3: $y_{ij} = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \varepsilon_i$

$R(\beta_0)$ = Soma Quadrado de parâmetros do M1

$R(\beta_0, \beta_1)$ = SQParâmetros do M2

$R(\beta_0, \beta_1, \beta_2)$ = SQParâmetros do M3

Vimos que:

SQTotal (NC) = SQParâmetros + SQResíduos

$\Sigma y_i^2 \rightarrow$ é sempre a mesma

quanto maior o número de parâmetros no modelo ↑SQParam e ↓SQRes

Logo, a SQReg (considerando M1 → M2) é dada por

$R(\beta_1|\beta_0) = R(\beta_0, \beta_1) - R(\beta_0)$

= SQParam (M2) − SQParam (M1)

= [SQTotal (NC) − SQRes (M2)] − [SQTotal (NC) − SQRes (M1)]

= $\cancel{\text{SQTotal}}$ (NC) − SQRes (M2) − $\cancel{\text{SQTotal}}$ (NC) + SQRes (M1)

= SQRes (M1) − SQRes (M2)

* A SQRegressão, ao considerar o M2, equivale à redução na SQResíduos devido a inclusão do parâmetro β_1 no modelo $y_i = \beta_0 + \varepsilon_i$ (com apenas β_0)

A SQReg (considerando M2 → M3) é dada por:

$R(\beta_2|\beta_0, \beta_1) = R(\beta_0, \beta_1, \beta_2) - R(\beta_0, \beta_1)$

= SQParam (M3) − SQParam (M2)

= [SQTotal (NC) − SQRes (M3)] − [SQTotal (NC) − SQRes (M2)]

= SQRes (M2) − SQRes (M3)

* A SQReg, ao considerar o M3, equivale à redução na SQResíduos devido a inclusão do parâmetro β_2 no modelo $y_{ij} = \beta_0 + \beta_1 X_i + \varepsilon_i$

Análise de Variância SEQUENCIAL

Nessa análise os parâmetros vão sendo adicionados sequencialmente.

Considerando uma variável resposta Y e duas variáveis regressoras (X_1 e X_2), temos o seguinte esquema de ANAVA.

FV	GL	SQ	QM	Fc
Parâmetros	$p=3$	$R(\beta_0, \beta_1, \beta_2)$	QMPar	$F_{par} = \frac{QMPar}{QMRes}$
β_0	1	$R(\beta_0) = C$	—	
$\beta_1 \mid \beta_0$	1	$R(\beta_0, \beta_1) - R(\beta_0)$	QM_1	$F_1 = QM_1/QMRes$
$\beta_2 \mid \beta_0, \beta_1$	1	$R(\beta_0, \beta_1, \beta_2) - R(\beta_0, \beta_1)$	QM_2	$F_2 = QM_2/QMRes$
Resíduo	$n-p=n-3$	por diferença	QMRes	
Total (NC)	n	$SQTotal(NC) = \underline{Y}^T\underline{Y}$		

Em Fpar, temos:

$H_0: \beta_0 = \beta_1 = \beta_2 = 0$ vs $H_a: \beta_j \neq 0$, para pelo menos 1 j.

Em F_1, temos

$H_0: \beta_1 = 0$ vs $H_a: \beta_1 \neq 0$

(ignorando a presença da X_2)

Em F_2, temos:

$H_0: \beta_2 = 0$ vs $H_a: \beta_2 \neq 0$

(ajustado para X_1)

Muito importante $\Rightarrow$ depende da ordem da variável no modelo

Considerando as Somas de Quadrados "Corrigidas"

FV	GL	SQ	QM	Fc
Regressão	$p-1=2$	$R(\beta_0,\beta_1,\beta_2)-R(\beta_0)$	$QMReg$	F_{reg}
↳ Par - C				
$\beta_1 \mid \beta_0$	1	$R(\beta_0,\beta_1)-R(\beta_0)$	QM_1	F_1
$\beta_2 \mid \beta_0,\beta_1$	1	$R(\beta_0,\beta_1,\beta_2)-R(\beta_0,\beta_1)$	QM_2	F_2
Resíduo	$n-p=n-3$	por diferença	$QMRes$	—
Total	$n-1$	$\underset{\sim}{Y}^T\underset{\sim}{Y} - C$ ↳ $R(\beta_0)$	—	—

Em F_{reg}, temos:

$H_0: \beta_1 = \beta_2 = 0$ vs $H_a: \beta_1$ ou $\beta_2 \neq 0$

Em F_1 e F_2 não há alterações (Tabela anterior)

Análise de Variância PARCIAL

Nessa análise verifica-se a contribuição de cada variável, dado que todas as outras já estão no modelo.

$$R(\beta_K \mid \beta_0,\ldots,\beta_{K-1},\beta_{K+1},\ldots,\beta_m) = \frac{\hat{\beta}_K^2}{(X_{KK}^T X)^{-1}}$$

TESTE DE HIPÓTESES

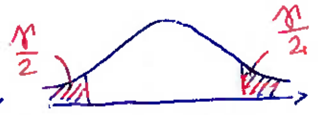
Seja, $\underset{\sim}{c}^T\underset{\sim}{\theta}$ uma combinação linear dos parametros e seu estimador $\underset{\sim}{c}^T\hat{\underset{\sim}{\theta}}$. Então,

$$\underset{\sim}{c}^T\hat{\underset{\sim}{\theta}} \sim N\left(\underset{\sim}{c}^T\underset{\sim}{\theta};\ \underset{\sim}{c}^T(X^TX)^{-1}\underset{\sim}{c}\,\sigma^2\right)$$

Hipóteses como:

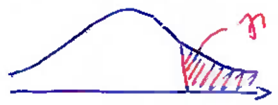
$H_0: \underset{\sim}{c}^T\underset{\sim}{\theta} = \underset{\sim}{c}^T\underset{\sim}{\theta}_0$ vs

$H_a: \underset{\sim}{c}^T\underset{\sim}{\theta} \neq \underset{\sim}{c}^T\underset{\sim}{\theta}_0 \rightarrow$ BILATERAL



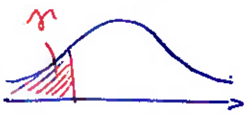
ou

$H_a: \underset{\sim}{c}^T\underset{\sim}{\theta} > \underset{\sim}{c}^T\underset{\sim}{\theta}_0 \rightarrow$ UNI. DIREITA



ou

$H_a: \underset{\sim}{c}^T\underset{\sim}{\theta} < \underset{\sim}{c}^T\underset{\sim}{\theta}_0 \rightarrow$ UNI. ESQUERDA



Podem ser testadas com a estatística:

$$t_c = \frac{\underset{\sim}{c}^T\hat{\underset{\sim}{\theta}} - \underset{\sim}{c}^T\underset{\sim}{\theta}_0}{\sqrt{\hat{var}(\underset{\sim}{c}^T\hat{\underset{\sim}{\theta}})}} = \frac{\underset{\sim}{c}^T\hat{\underset{\sim}{\theta}} - \underset{\sim}{c}^T\underset{\sim}{\theta}_0}{\sqrt{\underset{\sim}{c}^T(X^TX)^{-1}\underset{\sim}{c}\ QMRes}} \sim t_{\underbrace{n-p}_{gl\ resíduo}}$$

Exemplos:

1) $H_0: \beta_1 = 0$ ignorando X_2 $\Rightarrow$ $\underset{\sim}{C}^T = (0,1,0)$

$\underset{\sim}{\theta}^T = (\beta_0, \beta_1, \beta_2)$

$$\underset{\sim}{C}^T \underset{\sim}{\theta}_0 \Rightarrow (0,1,0)\begin{bmatrix}\beta_0\\ \beta_1\\ \beta_2\end{bmatrix} = 0 \Rightarrow \beta_1 = 0$$

2) $H_0: \beta_1 = \beta_2 \Rightarrow \beta_1 - \beta_2 = 0 \Rightarrow$ $\underset{\sim}{C}^T = (0, 1, -1)$

$\underset{\sim}{\theta}^T = (\beta_0, \beta_1, \beta_2)$

$$\underset{\sim}{C}^T \underset{\sim}{\theta}_0 \Rightarrow [0, 1, -1]\begin{bmatrix}\beta_0\\ \beta_1\\ \beta_2\end{bmatrix} = 0 \Rightarrow \beta_1 - \beta_2 = 0$$

PRINCÍPIO DO RESÍDUO CONDICIONAL (PRC)

Testes de hipóteses mais gerais podem ser feitos por meio da comparação de modelos e uso do PRC.

Modelo Completo versus Modelo Reduzido

$\Downarrow$

alterando a matriz X

EXEMPLO: Considerar n observações e K variáveis regressoras. O modelo completo é dado por:

$$Y_{ij} = \beta_0 + \beta_1 W_{i1} + \beta_2 W_{i2} + \dots + \beta_K W_{iK} + \varepsilon_{ij} \text{ , em que}$$

$\rightarrow$ W_{ij} são os próprios X_{ij} ou funções deles (Ex: X_{i1}^2 ou $X_{i1} X_{i2}$ ...)

$\rightarrow$ $\underset{\sim}{\varepsilon} \sim N(\phi, I\sigma^2)$

Matricialmente, $\underset{\substack{\sim\\ n\times 1}}{Y} = \underset{n\times p}{W_c}\, \underset{p\times 1}{\underset{\sim}{\theta}_c} + \underset{n\times 1}{\underset{\sim}{\varepsilon}_c}$ $\quad p = K+1$ parâmetros

Pelo método de mínimos quadrados, temos:

$\hat{\underset{\sim}{\theta}}_c = (W_c^T W_c)^{-1} W_c \underset{\sim}{Y}$

$\hat{Y}_c = W_c \hat{\underset{\sim}{\theta}}_c$

$SQTotal(NC) = \underset{\sim}{Y}^T \underset{\sim}{Y}$ (n gl)

$SQPar_c = \hat{\underset{\sim}{\theta}}_c^T W_c^T \underset{\sim}{Y}$ (p gl)

$SQRes_c = SQTotal(NC) - SQPar_c$ ($n-p$ gl)

Agora, considere o teste da seguinte hipótese

$$H_0: \beta_{m+1} = \beta_{m+2} = \dots = \beta_K = 0$$

Sob Ho, o modelo reduzido é dado por:

$$Y_i = \beta_0 + \beta_1 W_{i1} + \dots + \beta_m W_{im} + \varepsilon_i$$

Matricialmente, $\underset{n\times 1}{\underset{\sim}{Y}} = \underset{n\times q}{W_r}\,\underset{q\times 1}{\underset{\sim}{\hat{\theta}_r}} + \underset{n\times 1}{\varepsilon_r}$ $\quad q = m+1$ parâmetros $\quad \boxed{q < p}$

Pelo método dos mínimos quadrados, temos:

$$\underset{\sim}{\hat{\theta}_r} = (W_r^T W_r)^{-1} W_r^T \underset{\sim}{Y}$$

$$\hat{Y}_r = W_r \underset{\sim}{\hat{\theta}_r}$$

$$SQTotal(NC) = \underset{\sim}{Y}^T \underset{\sim}{Y} \quad (n \text{ gl})$$

$$SQPar_r = \underset{\sim}{\hat{\theta}_r^T} W_r^T \underset{\sim}{Y} \quad (q \text{ gl} \Rightarrow q = m+1)$$

$$SQRes_r = SQTotal(NC) - SQPar_r \quad (n-q)$$

Logo, quando passamos do modelo reduzido (q parâmetros) para o modelo completo (p parâmetros) há uma redução na Soma de Quadrados do Resíduo, dada por:

$$\begin{aligned} R(\beta_{m+1}, \beta_{m+2}, \dots, \beta_K \mid \beta_0, \beta_1, \dots, \beta_m) &= R(\beta_0, \dots, \beta_K) - R(\beta_0, \dots, \beta_m) \\ &= SQPar_c - SQPar_r \\ &= SQRes_r - SQRes_c \end{aligned} \Bigg\} (p-q \text{ gl})$$

Temos ainda que:

$$F = \frac{\dfrac{SQRes_r - SQRes_c}{p-q}}{QMRes_c} \sim F_{p-q,\, n-p;\, \gamma}$$

e rejeita-se a hipótese Ho com um nível $100\gamma\%$ de significância se

$$F > F_{p-q,\, n-p;\, \gamma}$$

VER OS DEMAIS EXEMPLOS NA APOSTILA.