Lista 2 - MAT2352

Cálculo para Funções de Várias Variáveis II

Problema 1.

(a) *Note que, como o cilindro é descrito por* $x^2 + y^2 = 1$*, a projeção no plano* xy *será um círculo de centro na origem e raio* 1.

Nosso intuito é descrever tal domínio utilizando coordenadas cilíndricas:

$$(x, y, z) \longmapsto (r\cos\theta, r\sin\theta, z)$$

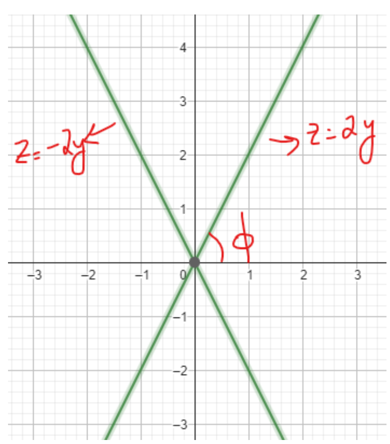
Assim, como a projeção em xy *é o círculo inteiro, sabemos que* $0 \leq \theta \leq 2\pi$*, e de mesma forma* $0 \leq r \leq 1$.

Basta verificarmos como z *varia nesse domínio. Observando as geratrizes do cone, que são dadas quando* $x = 0$

$$z^2 = 4y^2 \Rightarrow z = 2y \text{ ou } z = -2y$$

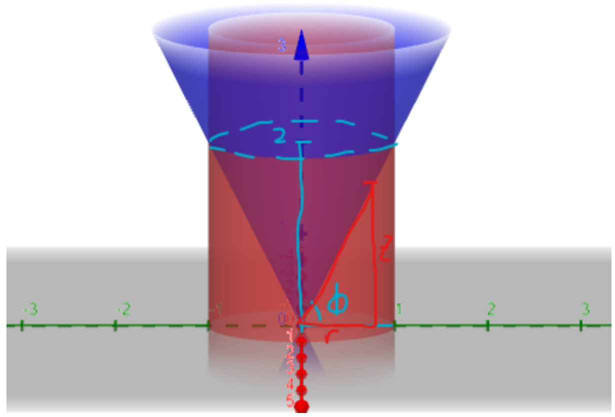
obtemos uma angulação de

$$\tan\phi = 2 \Rightarrow \phi = \arctan 2$$



Desta forma, z começa a ser viável a partir de $z=0$ e podemos subir com z até

$$\frac{z}{r} = \tan\phi = 2 \Rightarrow z = 2r$$



Finalmente o domínio em coordenadas cilíndricas é

$$D = \left\{(r,\theta,z) \in \mathbb{R}^3 : 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi, 0 \leq z \leq 2r\right\}$$

Como a Jacobiana da transformação cilíndrica é dada por

$$J_{cil} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

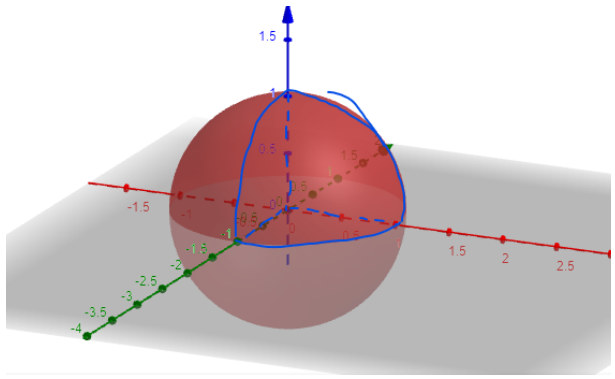
donde

$$det J_{cil} = r$$

e temos a integral

$$\begin{aligned}
\int\int\int_D x^2\ dxdydz &= \int_0^{2\pi}\int_0^1\int_0^{2r} r^2\cos^2\theta\cdot r\ dzdrd\theta \\
&= \int_0^{2\pi}\int_0^1 r^3\cos^2\theta\cdot z\Big|_0^{2r}\ drd\theta \\
&= \int_0^{2\pi}\int_0^1 2r^4\cos^2\theta\ drd\theta \\
&= \int_0^{2\pi}\frac{2}{5}r^5\cos^2\theta\Big|_0^1 \\
&= \int_0^{2\pi}\frac{2}{5}\cos^2\theta\ d\theta \\
&= \frac{2}{5}\int_0^{2\pi}\cos^2\theta\ d\theta \\
&= \frac{2}{5}\Big(\frac{1}{2}\theta+\frac{1}{4}\sin(2\theta)\Big)\Big|_0^{2\pi} \\
&= \frac{2}{5}\pi
\end{aligned}$$

(b) *Temos como domínio o oitavo da bola*



Descrevendo tal domínio em coordenadas esféricas

$$(x, y, z) \longmapsto (\rho \cos\theta \sin\phi, \rho \sin\theta \sin\phi, \rho \cos\phi)$$

temos que

$$0 \leq \theta \leq \frac{\pi}{2} \text{ (condição do primeiro ocatante - } x \geq 0, y \geq 0\text{)}$$

$$0 \leq \phi \leq \frac{\pi}{2} \text{ ,pois } z \geq 0$$
$$0 \leq \rho \leq 1$$

Ademais, a Jacobiana da transformação esférica é dada por

$$J_{esf} = \begin{bmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial \phi}{z} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{bmatrix} = \begin{bmatrix} \cos\theta\sin\phi & -\rho\sin\theta\sin\phi & \rho\cos\theta\cos\phi \\ \sin\theta\sin\phi & \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \\ \cos\phi & 0 & -\rho\sin\phi \end{bmatrix}$$

donde

$$detJ_{esf} = \rho^2 \sin\phi$$

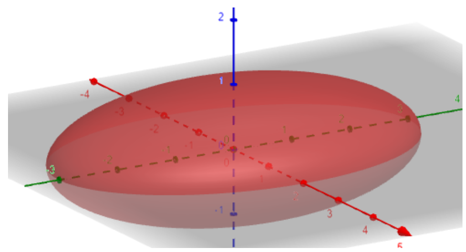
Finalmente

$$
\begin{aligned}
\int\int\int_D y^2 \ dxdydz &= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \rho^2\sin^2\theta\sin^2\phi\cdot\rho^2\sin\phi \ d\rho d\phi d\theta \\
&= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \rho^4\sin^2\theta\sin^3\phi \ d\rho d\phi d\theta \\
&= \int_0^{\pi/2}\int_0^{\pi/2} \frac{1}{5}\rho^5\sin^2\theta\sin^3\phi\Big|_0^1 d\phi d\theta \\
&= \int_0^{\pi/2}\int_0^{\pi/2} \frac{1}{5}\sin^2\theta\sin^3\phi \ d\phi d\theta \\
&= \int_0^{\pi/2} \frac{1}{5}\sin^2\theta\Big(\int_0^{\pi/2}\sin^3\phi \ d\phi\Big)d\theta \\
&= \int_0^{\pi/2} \frac{1}{5}\sin^2\theta\Big(-\cos\phi+\frac{1}{3}\cos^3\phi\Big)\Big|_0^{\pi/2} \ d\theta \\
&= \frac{1}{5}\int_0^{\pi/2}\sin^2\theta\frac{2}{3} \ d\theta \\
&= \frac{2}{15}\int_0^{\pi/2}\sin^2\theta \ d\theta \\
&= \frac{2}{15}\Big(\frac{1}{2}\theta-\frac{1}{4}\sin(2\theta)\Big)\Big|_0^{\pi/2} \\
&= \frac{\pi}{30}
\end{aligned}
$$

(c) Com o domínio

$$D = \left\{(x, y, z) \in \mathbb{R}^3 : \frac{x^2}{4} + \frac{y^2}{9} + z^2 \leq 1, x \geq 0\right\}$$

temos o elipsóide



Escrevendo em coordenadas esféricas (com $a = 2, b = 3, c = 1$)

$$(x, y, z) \longmapsto (2\rho\cos\theta\sin\phi, 3\rho\sin\theta\sin\phi, \rho\cos\phi)$$

temos, como a única restrição é $x \geq 0$

$$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

$$0 \leq \phi \leq \pi$$

$$0 \leq \rho \leq 1$$

Utilizando o Jacobiano da transformação

$$J_{esf} = \begin{bmatrix} 2\cos\theta\sin\phi & -2\rho\sin\theta\sin\phi & 2\rho\cos\theta\cos\phi \\ 3\sin\theta\sin\phi & 3\rho\cos\theta\sin\phi & 3\rho\sin\theta\cos\phi \\ \cos\phi & 0 & -\rho\sin\phi \end{bmatrix}$$

temos

$$detJ_{esf} = 2 \cdot 3 \cdot 1 \cdot \rho^2 \sin\phi = 6\rho^2 \sin\phi$$

e finalmente

$$
\begin{aligned}
\int\int\int_D x\ dxdydz &= \int_{-\pi/2}^{\pi/2}\int_0^{\pi}\int_0^1 2\rho\cos\theta\sin\phi\cdot 6\rho^2\sin\phi\ d\rho d\phi d\theta \\
&= 12\int_{-\pi/2}^{\pi/2}\int_0^{\pi}\int_0^1 \rho^3\cos\theta\sin^2\phi\ d\rho d\phi d\theta \\
&= 12\int_{-\pi/2}^{\pi/2}\int_0^{\pi} \frac{1}{4}\rho^4\cos\theta\sin^2\phi\Big|_0^1 d\phi d\theta \\
&= 3\int_{-\pi/2}^{\pi/2}\int_0^{\pi}\cos\theta\sin^2\phi\ d\phi d\theta \\
&= 3\Big(\int_{-\pi/2}^{\pi/2}\cos\theta\ d\theta\Big)\Big(\int_0^{\pi}\sin^2\phi\ d\phi\Big) \\
&= 3\Big[\sin\theta\Big]\Big|_{-\pi/2}^{\pi/2}\Big[\frac{1}{2}\phi-\frac{1}{4}\sin(2\phi)\Big]\Big|_0^{\pi} \\
&= 3\cdot 2\cdot\frac{\pi}{2} = 3\pi
\end{aligned}
$$

Problema 2. *Com o elipsóide*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

e transformando em coordenadas esféricas

$$(x, y, z) \longmapsto (a\rho\cos\theta\sin\phi, b\rho\sin\theta\sin\phi, c\rho\cos\phi)$$

temos

$$D = \{(\rho, \theta, \phi) \in \mathbb{R}^3 : 0 \leq \rho \leq 1, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi\}$$

Considerando o Jacobiano da transformação

$$J_{esf} = \begin{bmatrix} a\cos\theta\sin\phi & -a\rho\sin\theta\sin\phi & a\rho\cos\theta\cos\phi \\ b\sin\theta\sin\phi & b\rho\cos\theta\sin\phi & b\rho\sin\theta\cos\phi \\ c\cos\phi & 0 & -c\rho\sin\phi \end{bmatrix}$$

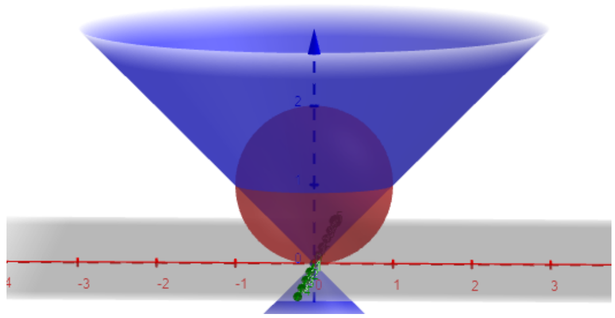
obtemos

$$det J_{esf} = abc\rho^2\sin\phi$$

Temos que o volume do elipsóide será

$$\begin{aligned} V = \int\int\int_D 1\ dxdydz &= \int_0^{2\pi}\int_0^{\pi}\int_0^1 abc\rho^2\sin\phi\ d\rho d\phi d\theta \\ &= abc\int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho^2\sin\phi\ d\rho d\phi d\theta \\ &= abc\int_0^{2\pi}\int_0^{\pi} \frac{1}{3}\rho^3\sin\phi\Big|_0^1\ d\phi d\theta \\ &= \frac{1}{3}abc\int_0^{2\pi}\int_0^{\pi}\sin\phi\ d\phi d\theta \\ &= \frac{1}{3}abc\int_0^{2\pi} -\cos\phi\Big|_0^{\pi}\ d\theta \\ &= \frac{1}{3}abc\int_0^{2\pi} 2\ d\theta \\ &= \frac{2}{3}abc\int_0^{2\pi}\ d\theta = \frac{4}{3}\pi abc \end{aligned}$$

Problema 3. *Temos como domínio*

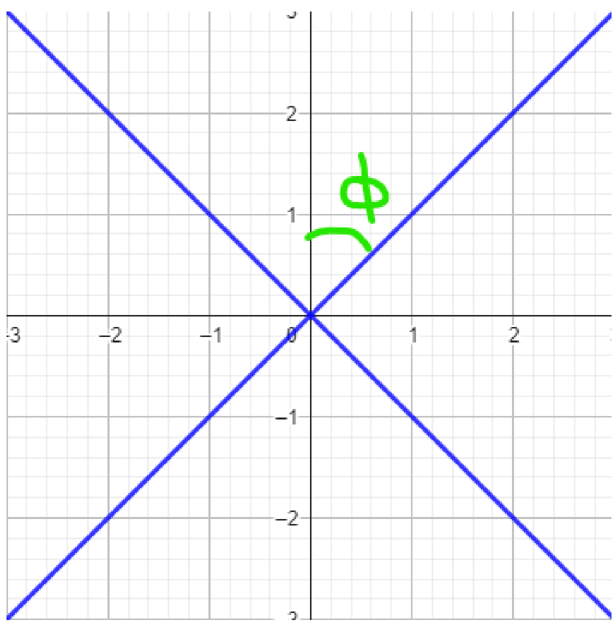


O cone tem como geratrizes as retas (tomando $x = 0$ na equação do cone)

$$z = y \text{ e } z = -y$$

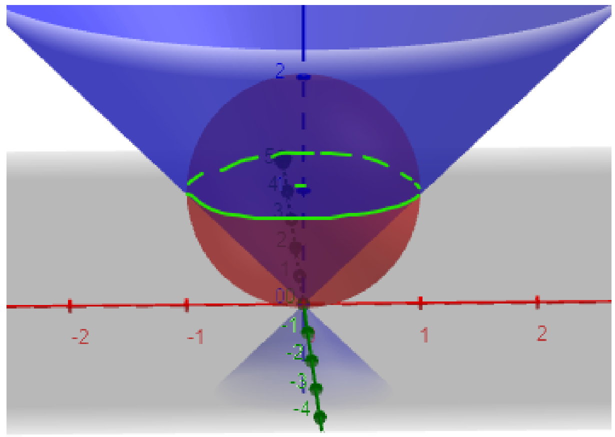
donde a inclinação determinada é de

$$\tan\phi = \frac{z}{y} \Rightarrow \tan\phi = 1 \Rightarrow \phi = \frac{\pi}{4}$$



Como queremos que fique dentro da esfera, sabemos então que

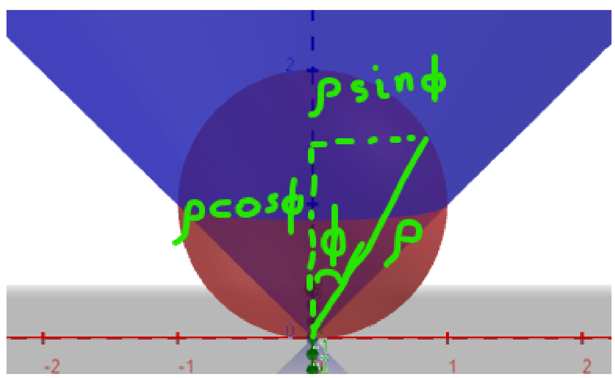
$$0 \le \phi \le \frac{\pi}{4}$$



A projeção no eixo xy será um círculo (desenho em verde) logo

$$0 \leq \theta \leq 2\pi$$

O raio, pode ser encontrado pela equação da esfera



$$x^2 + y^2 + z^2 = 2az$$

assim

$$\rho^2 \sin^2 \phi + \rho^2 \cos^2 \phi = 2a\rho \cos \phi$$

donde

$$\rho = 2a \cos \phi$$

Finalmente o domínio de integração será, em coordenadas esféricas

$$D = \{(\rho, \theta, \phi) \in \mathbb{R}^3 : 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \frac{\pi}{4}, 0 \leq \rho \leq 2a \cos \phi\}$$

donde

$$\begin{aligned}
\int\int\int_D \delta(x,y,z)\ dxdydz &= \int_0^{2\pi}\int_0^{\pi/4}\int_0^{2a\cos\phi}(\rho^2\cos^2\theta\sin^2\phi+\rho^2\sin^2\theta\cos^2\phi)\rho^2\sin\phi\ d\rho d\phi d\theta \\
&= \int_0^{2\pi}\int_0^{\pi/4}\int_0^{2a\cos\phi}\rho^4\sin^3\phi\ d\rho d\phi d\theta \\
&= \int_0^{2\pi}\int_0^{\pi/4}\frac{1}{5}\rho^5\sin^3\phi\Big|_0^{2a\cos\phi}\ d\phi d\theta \\
&= \int_0^{2\pi}\int_0^{\pi/4}\frac{32a^5}{5}\sin^3\phi\cos^5\phi\ d\phi d\theta \\
&= \frac{32a^5}{5}\int_0^{2\pi}\int_0^{\pi/4}-\frac{1}{6}(\cos^6\phi)'\sin^2\phi\ d\phi d\theta \\
&= -\frac{32a^5}{30}\int_0^{2\pi}\Big[\cos^6\phi\sin^2\phi\Big|_0^{\pi/4}-\int_0^{\pi/4}2\sin\phi\cos^7\phi\ d\phi\Big]d\theta \\
&= -\frac{32a^5}{30}\int_0^{2\pi}\frac{1}{16}+\frac{1}{4}\cos^8\phi\Big|_0^{\pi/4}\ d\theta \\
&= -\frac{32a^5}{30}\int_0^{2\pi}\frac{1}{16}-\frac{1}{4}+\frac{1}{64}\ d\theta \\
&= \frac{32a^5}{30}\int_0^{2\pi}\frac{11}{64}\ d\theta \\
&= \frac{32a^5}{30}\cdot\frac{11}{64}\cdot 2\pi = \frac{11}{30}a^5\pi
\end{aligned}$$

Problema 4.

(a) Paralelepípedo $[1,2]\times[0,3]\times[-3,1]$

A fronteira será as faces desse paralelepípedo, a saber

$$\begin{aligned}
\textit{Face } 1 &= \{x=1, y\in[0,3], z\in[-3,1]\}\\
\textit{Face } 2 &= \{x=2, y\in[0,3], z\in[-3,1]\}\\
\textit{Face } 3 &= \{y=0, x\in[1,2], z\in[-3,1]\}\\
\textit{Face } 4 &= \{y=1, x\in[1,2], z\in[-3,1]\}\\
\textit{Face } 5 &= \{z=-3, x\in[1,2], y\in[0,3]\}\\
\textit{Face } 6 &= \{z=1, x\in[1,2], y\in[0,3]\}
\end{aligned}$$

(b) Mesma resposta do item (a).

(c) A fronteira do semiespaço aberto é o plano $x=0$.

(d) Mesma resposta do item (b).

(e) A fronteira da bola aberta de centro em P *e raio* r *é a casca da bola*

$$S(P,r)=\{Q\in\mathbb{R}^3 : d(P,Q)=r\}$$

(f) Mesma resposta do item (e).

(g) A fronteira do $\mathbb{R}^3$ *é vazia pela própria definição de fronteira.*