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Evaluation of non-linear normal modes for finite-element models

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Abstract

In this paper, an alternative technique for the evaluation of non-linear normal modes is presented and applied to finite-element models. It is based upon the method of multiple scales, so that non-linear normal modes are evaluated as asymptotic expansions starting with the linear damped vibration modes. Non-linear normal modes are characterized both by the time response of all generalised coordinates and by explicit non-linear relationships between them and the modal variables. For the sake of an example, a non-conservative model of a cantilever beam, discretised with Bernoulli–Euler finite elements, is analysed. © 2002 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Vibration modes play an important role in linear dynamics. Hence, it comes as no surprise the desire to extend to non-linear systems all the benefits of modal analysis.

Vakakis and his co-workers [1] point out that Lyapunov in 1907 had already tried to obtain modal solutions for non-linear systems. But it was Rosenberg [2] who first proposed a definition for non-linear normal modes. According to Rosenberg, non-linear normal modes are synchronous motions for which there is a fixed relationship among the generalised coordinates, that is, all generalised coordinates execute equiperiodic motions, pass through the equilibrium position at the same time and reach their maximum displacements simultaneously. Besides, if a value is attributed to a generalised coordinate, all the other ones will be uniquely defined. Yet, such a definition was not general enough to include even the case of linear systems with non-proportional damping, for which the normal modes behave as non-standing waves, that is to say, they are non-synchronous.

In the 1990s Shaw and Pierre [3] proposed a redefiniton of non-linear normal modes in which not only the generalised displacements but also the velocities should be taken into account. According to them, a non-linear normal mode is a free-vibration motion which takes place in a two-dimensional invariant manifold embedded in the phase space of the system. This manifold must contain the equilibrium point and be tangent to the corresponding eigenspace of the linearised system. The advantage of such a definition is that it incorporates the previous one as a particular case and is suitable to both conservative and non-conservative systems. They also proposed a procedure to evaluate non-linear normal modes, which is here referred to as the invariant manifold technique.

Nayfeh and co-workers [4,5] also contributed to the subject. They studied the non-linear normal mode of undamped single degree of freedom systems using the

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method of multiple scales and showed the equivalence between their results and those of the invariant manifold technique for this simple case of a conservative system.

It has been Soares [6] who first succeeded in evaluating non-linear normal modes of full finite-element models. In his work, Soares applied the invariant manifold technique to plane frame structures, discretised with Bernoulli–Euler elements [7].

In the present paper, a new approach to the problem of evaluating non-linear normal modes of discrete systems, including finite-element models, is proposed. It adopts Shaw and Pierre's definition of non-linear normal modes, but relies upon the method of multiple scales, rather than the invariant manifold technique, to calculate both the time response and the non-linear relationships among the generalised coordinates of multi-degree of freedom non-conservative systems.

By the way, the very application of the method of multiple scales to full finite-element models seems to be a pioneering accomplishment.

2. Non-linear normal modes: temporal description of the generalised coordinates

Non-linear free-vibration equations of a fairly general class of elastic finite-element models can be written as:

$$M_{rs}\ddot{p}_s + D_{rs}\dot{p}_s + U_{,r} = 0, \tag{1}$$

where M_{rs}, D_{rs} and $U_{,r}$ stand for the generic elements of the mass matrix, equivalent damping matrix and elastic force vector, respectively; r and s vary from 1 to n (number of degrees of freedom of the model); summation over the repeated index s from 1 to n is implied. The notation $U_{,r}$ is used to denote the partial derivative of the strain energy U with respect to the generalised coordinate p_r, evaluated at the equilibrium position. It is further assumed that M_{rs}, D_{rs} and $U_{,r}$ depend on the generalised coordinates p_i, $i = 1, 2, \ldots, n$, and D_{rs} may also depend on the velocities $\dot{p}_i$, $i = 1, 2, \ldots, n$, as indicated:

$$M_{rs} = {}^0M_{rs} + {}^1M^i_{rs}p_i + {}^2M^{ij}_{rs}p_ip_j, \tag{2}$$

$$D_{rs} = {}^0D_{rs} + {}^1D^i_{rs}\dot{p}_i + {}^2D^{ij}_{rs}\dot{p}_ip_j, \tag{3}$$

$$U_{,r} = {}^0K_{rs}p_s + {}^1K^i_{rs}p_ip_s + {}^2K^{ij}_{rs}p_ip_jp_s. \tag{4}$$

In (2)–(4) summation over the repeated indices i and j, from 1 to n, is also implied; ${}^0M_{rs}$, ${}^0D_{rs}$ and ${}^0K_{rs}$ are the linear-theory generic elements of the mass, equivalent damping and stiffness matrices, respectively; ${}^1M_{rs}$, ${}^1D_{rs}$ and ${}^1K_{rs}$ stand respectively for the mass, equivalent damping and stiffness coefficients introducing quadratic non-linearities into (1); ${}^2M_{rs}$, ${}^2D_{rs}$ and ${}^2K_{rs}$ stand respectively for the mass, equivalent damping and stiffness coefficients introducing cubic non-linearities into (1). Systems ruled by a linear-elastic constitutive law and subjected to finite displacements (non-linear kinematics), though small strains, fit into the class of problems modelled by (1) and (2)–(4). That is the case of planar reticulated structures, as those studied in [7].

In order to obtain the temporal description of all generalised coordinates when the system is vibrating in a particular non-linear normal mode, one can use the method of multiple scales. Following the usual procedure [8], the generalised coordinates and the time derivatives are written as asymptotic expansions of a small dimensionless perturbation parameter ε, that is,

$$p_i(t) = \varepsilon p_{i1}(T_0, T_1, T_2, \ldots) + \varepsilon^2 p_{i2}(T_0, T_1, T_2, \ldots) + \varepsilon^3 p_{i3}(T_0, T_1, T_2, \ldots) + \cdots, \tag{5}$$

$$T_i = \varepsilon^i t, \tag{6}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \cdots, \tag{7}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} = D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2\left(2D_0D_2 + D_1^2\right) + \cdots, \tag{8}$$

$$D_i^n = \frac{\mathrm{d}^n}{\mathrm{d}T_i^n}. \tag{9}$$

Taking (2)–(9) into (1) and collecting terms of the same order of ε, it is possible to arrive at differential equations whose solutions and solvability conditions allow for the characterization of the non-linear normal modes.

2.1. Equations of order ε

The equations of order ε correspond to the linear damped system. They read

$${}^0M_{rs}D_0^2p_{s1} + {}^0D_{rs}D_0p_{s1} + {}^0K_{rs}p_{s1} = 0. \tag{10}$$

The solution of (10) is sought in the form

$$p_{s1} = A_{s1}\mathrm{e}^{\Psi T_0}, \tag{11}$$

where $A_{s1} = A_{s1}(T_1, T_2, \ldots)$ and Ψ may be complex numbers. It follows from (11) and (10):

$$\left(\Psi^2\,{}^0M_{rs} + \Psi\,{}^0D_{rs} + {}^0K_{rs}\right)A_{s1} = 0 \qquad \forall r, s = 1\text{–}n, \tag{12}$$

which characterizes a damped eigenvalue problem. The solvability condition, in this case, coincides with the characteristic equation of the problem:

$$\det\left({}^0\mathbf{S}\right) = 0, \tag{13}$$

where ${}^0\mathbf{S} = \Psi^2\,{}^0\mathbf{M} + \Psi\,{}^0\mathbf{D} + {}^0\mathbf{K}$.

The roots of (13) are written in the form

$$\Psi_u = \alpha_u + \mathrm{i}\beta_u; \quad u = 1\text{–}n, \tag{14}$$

where α_u is minus the product of the damping ratio by the undamped linear frequency of mode u, and $|\beta_u|$ is the damped linear frequency of mode u.

Choosing a non-zero A^u_{v1} (for a certain degree of freedom v and mode u)—see Eq. (11)—as modal coordinate, it is possible to normalise the amplitudes of the other degrees of freedom with respect to this one:

$$\frac{A^u_{s1}}{A^u_{v1}} = \phi^u_s \Rightarrow A^u_{s1} = \phi^u_s A^u_{v1} = (\gamma^u_s + \mathrm{i}\delta^u_s)A^u_{v1}; \qquad s = 1\text{–}n, \text{ no sum in } u. \tag{15}$$

Of course, $\phi^u_v = 1$. Note that ϕ^u_s represents a term of the complex eigenvector.

It is proposed from this moment on a simplification in the notation with the elimination of the indexes u and v, which stand for the selected mode and the degree of freedom related to the modal variable, respectively. This simplification implies no restriction to the procedure, because it can be applied to any mode, provided any non-zero generalised coordinate is chosen as the modal variable. Let's write then

$$\Psi_u = \Psi = \alpha + \mathrm{i}\beta, \tag{16}$$

$$A^u_{v1} = A_1, \tag{17}$$

$$A^u_{s1} = A_{s1} = \phi_s A_1 = (\gamma_s + \mathrm{i}\delta_s)A_1. \tag{18}$$

Since the solution of (10) must be real:

$$p_{s1} = A_{s1}\mathrm{e}^{\Psi T_0} + \bar{A}_{s1}\mathrm{e}^{\overline{\Psi} T_0}, \tag{19}$$

where $\bar{A}_{s1}$ and $\overline{\Psi}$ represent the complex conjugates of A_{s1} and Ψ, respectively.

2.2. Equations of order ε^2

After collection of terms of order ε^2 in (1), one gets:

$$\begin{aligned} &{}^0M_{rs}D_0^2 p_{s2} + {}^0D_{rs}D_0 p_{s2} + {}^0K_{rs}p_{s2} \\ &\quad = -2\,{}^0M_{rs}D_0D_1p_{s1} - {}^0D_{rs}D_1p_{s1} - {}^1M^i_{rs}p_{i1}D_0^2p_{s1} \\ &\qquad - {}^1D^i_{rs}(D_0p_{i1}D_0p_{s1}) - {}^1K^i_{rs}p_{i1}p_{s1}. \end{aligned} \tag{20}$$

Terms on the right-hand side of (20) depend upon the solution of order ε. The solution of (20) is sought in the form

$$p_{s2} = A_{s2}\mathrm{e}^{\Psi T_0} + B_{s2}\mathrm{e}^{2\Psi T_0} + E_{s2}\mathrm{e}^{2\alpha T_0} + \text{c.c.}, \tag{21}$$

where c.c. stands for complex conjugate.

Taking (19) and (21) in (20) and collecting terms in $\mathrm{e}^{\Psi T_0}$:

$$\begin{aligned} &(\Psi^2\,{}^0M_{rs} + \Psi\,{}^0D_{rs} + {}^0K_{rs})A_{s2} \\ &\quad = -(2\Psi\,{}^0M_{rs} + {}^0D_{rs})D_1A_{s1}. \end{aligned} \tag{22}$$

One must bear in mind that in (13) it was imposed that the matrix ${}^0\mathbf{S}$ should be singular. So, for the system (22) to have solution, it is required that the determinant of the matrix ${}^1\mathbf{S}$—which results from the substitution of one of the columns of ${}^0\mathbf{S}$ by the vector of independent terms of (22)—be null. This is the solvability condition to be imposed on terms of order ε^2:

$$\det({}^1\mathbf{S}) = 0. \tag{23}$$

One arrives at a differential equation of the type:

$$(f + \mathrm{i}g)D_1A_1 = 0, \tag{24}$$

where f and g are real numbers. Eq. (24) is satisfied if $D_1A_1 = 0$, or $A_1 = A_1(T_2)$.

Note that the term in $\mathrm{e}^{\Psi T_0}$ in (21) does not add any new feature to the solution, since there is already a term of this type in p_{s1} (see (19)). It will therefore be disregarded from this point on.

Collecting now terms in $\mathrm{e}^{2\alpha T_0}$, one obtains:

$$\begin{aligned} &(4\alpha^2\,{}^0M_{rs} + 2\alpha\,{}^0D_{rs} + {}^0K_{rs})E_{s2} \\ &\quad = -(\Psi^2\,{}^1M^i_{rs} + \Psi\overline{\Psi}\,{}^1D^i_{rs} + {}^1K^i_{rs})\bar{A}_{i1}A_{s1} \\ &\quad = -(\Psi^2\,{}^1M^i_{rs} + \Psi\overline{\Psi}\,{}^1D^i_{rs} + {}^1K^i_{rs})\bar{\phi}_i\phi_s\bar{A}_1A_1. \end{aligned} \tag{25}$$

So,

$$E_{s2} = \eta_s\bar{A}_1A_1 = (\sigma_s + \mathrm{i}\tau_s)\bar{A}_1A_1, \tag{26}$$

where σ_s and τ_s are real numbers, provided the matrix of coefficients is non-singular. Collecting finally terms in $\mathrm{e}^{2\Psi T_0}$, it is seen that

$$\begin{aligned} &(4\Psi^2\,{}^0M_{rs} + 2\Psi\,{}^0D_{rs} + {}^0K_{rs})B_{s2} \\ &\quad = -(\Psi^2\,{}^1M^i_{rs} + \Psi^2\,{}^1D^i_{rs} + {}^1K^i_{rs})A_{i1}A_{s1} \\ &\quad = -(\Psi^2\,{}^1M^i_{rs} + \Psi^2\,{}^1D^i_{rs} + {}^1K^i_{rs})\phi_i\phi_s(A_1)^2. \end{aligned} \tag{27}$$

So,

$$B_{s2} = \rho_s(A_1)^2 = (\epsilon_s + \mathrm{i}\xi_s)(A_1)^2, \tag{28}$$

where ϵ_s and ξ_s are real numbers, provided the matrix of coefficients is non-singular. In the case of a system with one eigenvalue exactly equal to twice another one, such as the case of internal resonance 2:1 in undamped models, the solution of (27) cannot be written as in (28).

2.3. Equations of order ε^3

After collection of terms of order ε^3 in (1), one gets:

$$\begin{aligned}&{}^0M_{rs}D_0^2p_{s3} + {}^0D_{rs}D_0p_{s3} + {}^0K_{rs}p_{s3}\\ &\quad = -{}^0M_{rs}\left[2D_0D_1p_{s2} + \left(2D_0D_2 + D_1^2\right)p_{s1}\right]\\ &\qquad - {}^0D_{rs}[D_1p_{s2} + D_2p_{s1}] - {}^1M_{rs}^i\{p_{i1}[D_0p_{s2}\\ &\qquad + 2D_0D_1p_{s1}] + p_{i2}D_0^2p_{s1}\} - {}^1D_{rs}^i\{D_0p_{i1}[D_0p_{s2}\\ &\qquad + D_1p_{s1}] + D_1p_{i1}D_0p_{s1} + D_0p_{i2}D_0p_{s1}\}\\ &\qquad - {}^1K_{rs}^i[p_{i1}p_{s2} + p_{i2}p_{s1}] - {}^2M_{rs}^{ij}\left[p_{i1}p_{j1}D_0^2p_{s1}\right]\\ &\qquad - {}^2D_{rs}^{ij}\left[D_0p_{i1}p_{j1}D_0p_{s1}\right] - {}^2K_{rs}^{ij}\left[p_{i1}p_{j1}p_{s1}\right].\end{aligned} \tag{29}$$

Having in mind that $D_1A_1 = 0$ and, therefore, that $D_1p_{s1} = 0$ and $D_1p_{s2} = 0$, for $i = 1$–n, Eq. (29) reads

$$\begin{aligned}&{}^0M_{rs}D_0^2p_{s3} + {}^0D_{rs}D_0p_{s3} + {}^0K_{rs}p_{s3}\\ &\quad = -{}^0M_{rs}[2D_0D_2p_{s1}] - {}^0D_{rs}[D_2p_{s1}] - {}^1M_{rs}^i\left[p_{i1}D_0p_{s2}\right.\\ &\qquad \left.+ p_{i2}D_0^2p_{s1}\right] - {}^1D_{rs}^i[D_0p_{i1}D_0p_{s2} + D_0p_{i2}D_0p_{s1}]\\ &\qquad - {}^1K_{rs}^i[p_{i1}p_{s2} + p_{i2}p_{s1}] - {}^2M_{rs}^{ij}\left[p_{i1}p_{j1}D_0^2p_{s1}\right]\\ &\qquad - {}^2D_{rs}^{ij}\left[D_0p_{i1}p_{j1}D_0p_{s1}\right] - {}^2K_{rs}^{ij}\left[p_{i1}p_{j1}p_{s1}\right].\end{aligned} \tag{30}$$

Therefore, the solution of (30) should be sought in the form

$$p_{s3} = A_{s3}e^{\Psi T_0} + C_{s3}e^{3\Psi T_0} + \text{c.c.} \tag{31}$$

After substitution of (19), (26), (28) and (31) in (29) and collection of terms in $e^{\Psi T_0}$, it comes out

$$\begin{aligned}&\left(\Psi^2\,{}^0M_{rs} + \Psi\,{}^0D_{rs} + {}^0K_{rs}\right)A_{s3}\\ &\quad = -2\Psi\phi_s\,{}^0M_{rs}D_2A_1 - \phi_s\,{}^0D_{rs}D_2A_1\\ &\qquad - {}^1M_{rs}^i\left[2\Psi\bar{\phi}_i\rho_s\bar{A}_1(A_1)^2 + 2\alpha\phi_i\eta_s\bar{A}_1(A_1)^2\right.\\ &\qquad \left.+ 2\alpha\phi_i\bar{\eta}_s\bar{A}_1(A_1)^2\right]e^{2\alpha T_0} - {}^1M_{rs}^i\left[\overline{\Psi}^2\bar{\phi}_s\rho_i\bar{A}_1(A_1)^2\right.\\ &\qquad \left.+ \Psi^2\phi_s\eta_i\bar{A}_1(A_1)^2 + \Psi^2\phi_s\bar{\eta}_i\bar{A}_1(A_1)^2\right]e^{2\alpha T_0}\\ &\qquad - {}^1D_{rs}^i\left[2\Psi\overline{\Psi}\bar{\phi}_i\rho_s\bar{A}_1(A_1)^2 + 2\alpha\Psi\phi_i\eta_s\bar{A}_1(A_1)^2\right.\\ &\qquad \left.+ 2\alpha\Psi\phi_i\bar{\eta}_s\bar{A}_1(A_1)^2\right]e^{2\alpha T_0} - {}^1D_{rs}^i\left[2\Psi\overline{\Psi}\bar{\phi}_s\rho_i\bar{A}_1(A_1)^2\right.\\ &\qquad \left.+ 2\alpha\Psi\phi_s\eta_i\bar{A}_1(A_1)^2 + 2\alpha\Psi\phi_s\bar{\eta}_i\bar{A}_1(A_1)^2\right]e^{2\alpha T_0}\\ &\qquad - {}^1K_{rs}^i\left[\bar{\phi}_i\rho_s\bar{A}_1(A_1)^2 + \phi_i\eta_s\bar{A}_1(A_1)^2\right.\\ &\qquad \left.+ \phi_i\bar{\eta}_s\bar{A}_1(A_1)^2\right]e^{2\alpha T_0} - {}^1K_{rs}^i\left[\bar{\phi}_s\rho_i\bar{A}_1(A_1)^2\right.\\ &\qquad \left.+ \phi_s\eta_i\bar{A}_1(A_1)^2 + \phi_s\bar{\eta}_i\bar{A}_1(A_1)^2\right]e^{2\alpha T_0}\\ &\qquad - {}^2M_{rs}^{ij}\left[\Psi^2\bar{\phi}_i\phi_j\phi_s\bar{A}_1(A_1)^2 + \Psi^2\phi_i\bar{\phi}_j\phi_s\bar{A}_1(A_1)^2\right.\\ &\qquad \left.+ \overline{\Psi}^2\phi_i\phi_j\bar{\phi}_s\bar{A}_1(A_1)^2\right]e^{2\alpha T_0} - {}^2D_{rs}^{ij}\left[\Psi\overline{\Psi}\bar{\phi}_i\phi_j\phi_s\bar{A}_1(A_1)^2\right.\\ &\qquad \left.+ \Psi^2\phi_i\bar{\phi}_j\phi_s\bar{A}_1(A_1)^2 + \Psi\overline{\Psi}\phi_i\phi_j\bar{\phi}_s\bar{A}_1(A_1)^2\right]e^{2\alpha T_0}\\ &\qquad - {}^2K_{rs}^{ij}\left[\phi_i\phi_j\bar{\phi}_s\bar{A}_1(A_1)^2 + \phi_i\bar{\phi}_j\phi_s\bar{A}_1(A_1)^2\right.\\ &\qquad \left.+ \bar{\phi}_i\phi_j\phi_s\bar{A}_1(A_1)^2\right]e^{2\alpha T_0}.\end{aligned} \tag{32}$$

Eq. (32) may be written as:

$$\begin{aligned}&\left(\Psi^2\,{}^0M_{rs} + \Psi\,{}^0D_{rs} + {}^0K_{rs}\right)A_{s3}\\ &\quad = (x_r + iy_r)D_2A_1 + (w_r + iz_r)\bar{A}_1(A_1)^2e^{2\alpha T_0},\end{aligned} \tag{33}$$

where x_r, y_r, w_r and z_r are real numbers.

Applying the solvability condition, which is analogous to that discussed for the terms of order ε^2, one arrives at the following differential equation:

$$(\hat{p} + i\hat{q})D_2A_1 + (\hat{r} + i\hat{s})\bar{A}_1(A_1)^2\exp\left(\frac{2\alpha}{\varepsilon^2}T_2\right) = 0, \tag{34}$$

where $\hat{p}$, $\hat{q}$, $\hat{r}$ and $\hat{s}$ are real numbers.

Expressing the amplitude A_1 in polar coordinates, one obtains:

$$A_1 = \frac{1}{2}ae^{i\theta}, \tag{35}$$

with $a = a(T_2)$ and $\theta = \theta(T_2)$ real functions.

Substituting (35) in (34) and solving the corresponding differential equations, one arrives at

$$\begin{aligned}&a(t) = \frac{a_0}{\sqrt{1 + \frac{\vartheta}{\alpha}(\varepsilon a_0)^2 - \frac{\vartheta}{\alpha}(\varepsilon a_0)^2e^{2\alpha t}}},\\ &\theta(t) = \theta_0 - \frac{\kappa}{2\vartheta}\ln\left|1 + \frac{\vartheta}{\alpha}(\varepsilon a_0)^2 - \frac{\vartheta}{\alpha}(\varepsilon a_0)^2e^{2\alpha t}\right|,\end{aligned} \tag{36}$$

where

$$\begin{aligned}&\vartheta = -\left\{\frac{\hat{r}\hat{p} + \hat{s}\hat{q}}{4\left[\left(\hat{p}\right)^2 + \left(\hat{q}\right)^2\right]}\right\},\\ &a_0 = a(0),\\ &\kappa = \left\{\frac{\hat{r}\hat{q} - \hat{s}\hat{p}}{4\left[\left(\hat{p}\right)^2 + \left(\hat{q}\right)^2\right]}\right\},\\ &\theta_0 = \theta(0).\end{aligned} \tag{37}$$

Again, as it happened to the solution of order ε^2, the term in $e^{\Psi T_0}$ in (31) does not add any new feature to the solution, since there is already a term of this type in p_{s1} (see (19)). It will therefore be disregarded from this point on.

Collecting, finally, terms in $e^{3\Psi T_0}$, one obtains:

$$\begin{aligned}&\left(9\Psi^{2\,0}M_{rs}+3\Psi^{0}D_{rs}+{}^{0}K_{rs}\right)C_{s3}\\&\quad=-{}^{1}M^{i}_{rs}\left(2\Psi\phi_i\rho_s\right)(A_1)^3-{}^{1}M^{i}_{rs}\left(\Psi^2\phi_s\rho_i\right)(A_1)^3\\&\qquad-{}^{1}D^{i}_{rs}\left(2\Psi^2\phi_i\rho_s\right)(A_1)^3-{}^{1}D^{i}_{rs}\left(2\Psi^2\phi_s\rho_i\right)(A_1)^3\\&\qquad-{}^{1}K^{i}_{rs}(\phi_i\rho_s)(A_1)^3-{}^{1}K^{i}_{rs}\left(\phi_s\rho_i\right)(A_1)^3\\&\qquad-{}^{2}M^{ij}_{rs}\left(\Psi^2\phi_i\phi_j\phi_s\right)(A_1)^3-{}^{2}D^{ij}_{rs}\left(\Psi^2\phi_i\phi_j\phi_s\right)(A_1)^3\\&\qquad-{}^{2}K^{ij}_{rs}\left(\phi_i\phi_j\phi_s\right)(A_1)^3,\end{aligned}\tag{38}$$

and the solution is of the type

$$C_{s3}=\lambda_s(A_1)^3=(\mu_s+\mathrm{i}v_s)(A_1)^3,\tag{39}$$

where μ_s and v_s are real constants, provided the matrix of coefficients is non-singular. In the case of a system with one eigenvalue exactly equal to three times another one, such as the case of internal resonance 3:1 in undamped models, the solution of (38) cannot be written as in (39).

At this point, it is possible to write the time response for the generalised coordinates when the system is vibrating according to a certain non-linear normal mode. In complex notation

$$\begin{aligned}p_s&=\varepsilon A_{s1}e^{\Psi t}+\varepsilon^2B_{s2}e^{2\Psi t}+\varepsilon^2E_{s2}e^{2\alpha t}+\varepsilon^3C_{s3}e^{3\Psi t}\\&=\varepsilon\left(A_{s1}e^{\Psi t}+\bar{A}_{s1}e^{\overline{\Psi}t}\right)+\varepsilon^2\left(B_{s2}e^{2\Psi t}+\overline{B}_{s2}e^{2\overline{\Psi}t}\right)\\&\quad+\varepsilon^2\left(E_{s2}+\overline{E}_{s2}\right)e^{2\alpha t}+\varepsilon^3\left(C_{s3}e^{3\Psi t}+\overline{C}_{s3}e^{3\overline{\Psi}t}\right)\end{aligned}\tag{40}$$

or, in real notation

$$\begin{aligned}p_s&=\hat{a}e^{\alpha t}[\gamma_s\cos(\beta t+\theta)-\delta_s\,\mathrm{sen}(\beta t+\theta)]\\&\quad+\frac{1}{2}(\hat{a})^2e^{2\alpha t}[\epsilon_s\cos2(\beta t+\theta)-\xi_s\,\mathrm{sen}\,2(\beta t+\theta)+\sigma_s]\\&\quad+\frac{1}{4}(\hat{a})^3e^{3\alpha t}[\mu_s\cos3(\beta t+\theta)-v_s\,\mathrm{sen}\,3(\beta t+\theta)],\end{aligned}\tag{41}$$

where $\hat{a}=\varepsilon a$ and the other symbols are defined in (15), (26), (28), (36) and (39).

3. Regeneration of the generalised coordinates as functions of the modal variable

The aim here is to characterize the non-linear normal mode by writing the generalised coordinates in the form:

$$\begin{aligned}p_s&=F_{s1}U+F_{s2}V+F_{s3}U^2+F_{s4}UV+F_{s5}V^2\\&\quad+F_{s6}U^3+F_{s7}U^2V+F_{s8}UV^2+F_{s9}V^3,\end{aligned}\tag{42}$$

$$\begin{aligned}\dot{p}_s&=G_{s1}U+G_{s2}V+G_{s3}U^2+G_{s4}UV+G_{s5}V^2\\&\quad+G_{s6}U^3+G_{s7}U^2V+G_{s8}UV^2+G_{s9}V^3,\end{aligned}\tag{43}$$

with $s=1$–n and U and V are, respectively, a chosen non-zero generalised coordinate p_v and its corresponding velocity $\dot{p}_v$, that is:

$$U=p_v,\tag{44}$$

$$V=\dot{p}_v.\tag{45}$$

From (41) one obtains $\dot{p}_s$ by derivation with respect to time. If, now, p_s and $\dot{p}_s$, as determined from (41), are introduced on the left-hand side of (42) and (43), and $U=p_v$ and $V=\dot{p}_v$, as determined from (41), are introduced on the right-hand side of (42) and (43), one ends up with expressions in terms of the several coefficients F_{si}, G_{si}. After equating terms of the same power in ε, one obtains equations in which only linearly independent functions are present. The evaluation of F_{si} and G_{si} comes from the imposition that the coefficients of such functions on both sides of the equations should be the same. By doing so, it is possible to arrive at linear systems which, once solved, allow for the determination of all coefficients in (42) and (43),

The evaluation of the coefficients F_{si} is considered first.

- order ε

$$\begin{bmatrix}1&\alpha\\0&\omega\end{bmatrix}\begin{Bmatrix}F_{s1}\\F_{s2}\end{Bmatrix}=\begin{Bmatrix}\gamma_s\\\delta_s\end{Bmatrix},\tag{46}$$

with α, γ_s and δ_s defined in (18) and $\omega=(\mathrm{d}/\mathrm{d}t)(\beta t+\theta)=\beta+\dot{\theta}\cong\beta$.

- order ε^2

$$\begin{aligned}&\begin{bmatrix}1&\alpha&\alpha^2\\-1&-\alpha&(\omega^2-\alpha^2)\\0&-\omega&-2\omega\alpha\end{bmatrix}\begin{Bmatrix}F_{s3}\\F_{s4}\\F_{s5}\end{Bmatrix}\\&\quad=\begin{Bmatrix}\frac{\sigma_s+\epsilon_s}{2}-\left(\frac{\sigma_u+\epsilon_u}{2}\right)F_{s1}-(\sigma_u\alpha+\epsilon_u\alpha-\xi_u\omega)F_{s2}\\-\epsilon_s+\epsilon_uF_{s1}+2(\epsilon_u\alpha-\xi_u\omega)F_{s2}\\-\xi_s+\xi_uF_{s1}+2(\epsilon_u\omega+\xi_u\alpha)F_{s2},\end{Bmatrix}\end{aligned}\tag{47}$$

with ϵ_s, ξ_s and σ_s defined in (26) and (28).

- order ε^3

$$\begin{bmatrix}1&\alpha&(\alpha^2-\omega^2)&(\alpha^3-3\alpha\omega^2)\\0&0&\omega^2&3\alpha\omega^2\\0&-\omega&-2\omega\alpha&-3\omega\alpha^2\\0&\omega&2\omega\alpha&(3\omega\alpha^2-\omega^3)\end{bmatrix}\begin{Bmatrix}F_{s6}\\F_{s7}\\F_{s8}\\F_{s9}\end{Bmatrix}$$

$$
= \left\{ \begin{array}{l}
\mu_s - \mu_u F_{s1} - 3(\mu_u \alpha - \nu_u \omega) F_{s2} - 2\epsilon_u F_{s3} \\
\quad -3(\epsilon_u \alpha - \xi_u \omega) F_{s4} + 4\left[\epsilon_u\left(\omega^2 - \alpha^2\right) + 2\xi_u \alpha\omega\right] F_{s5} \\[4pt]
-\frac{3}{4}\mu_s + \frac{3}{4}\mu_u F_{s1} + \frac{9}{4}(\mu_u \alpha - \nu_u \omega) F_{s2} - (\sigma_u - \epsilon_u) F_{s3} \\
\quad + \left[\frac{3}{2}\epsilon_u \alpha - 2\xi_u \omega - \sigma_u\left(\alpha + \frac{1}{2}\omega\right)\right] F_{s4} \\
\quad - \left[2\epsilon_u(2\omega^2 - \alpha^2)\right] F_{s5} - [6\xi_u \alpha\omega + 2\sigma_u \alpha^2] F_{s5} \\[4pt]
-\frac{3}{4}\nu_s + \frac{3}{4}\nu_u F_{s1} + \frac{9}{4}(\mu_u \omega + \nu_u \alpha) F_{s2} + 2\xi_u F_{s3} \\
\quad + \left[\frac{5}{2}\epsilon_u \omega + 3\xi_u \alpha + \frac{1}{2}\sigma_u \omega\right] F_{s4} \\
\quad + [6\epsilon_u \alpha\omega + 2\xi_u(2\alpha^2 - \omega^2) + 2\sigma_u \alpha\omega] F_{s5} \\[4pt]
\nu_s - \nu_u F_{s1} - 3(\mu_u \omega + \nu_u \alpha) F_{s2} - 2\xi_u F_{s3} \\
\quad -3(\epsilon_u \omega + \xi_u \alpha) F_{s4} - 4[2\epsilon_u \alpha\omega + \xi_u(\alpha^2 - \omega^2)] F_{s5}
\end{array} \right\}. \tag{48}
$$

with μ_s and ν_s defined in (39).

The coefficients G_{si} are evaluated from systems of linear algebraic equations which are very similar to the former ones (46)–(48), the differences being limited to the vectors of independent terms.

- order ε

$$
\begin{bmatrix} 1 & \alpha \\ 0 & \omega \end{bmatrix} \begin{Bmatrix} G_{s1} \\ G_{s2} \end{Bmatrix} = \begin{Bmatrix} \gamma_s \alpha - \delta_s \omega \\ \gamma_s \omega + \delta_s \alpha \end{Bmatrix}. \tag{49}
$$

with α, γ_s and δ_s defined in (18) and $\omega = (\mathrm{d}/\mathrm{d}t)(\beta t + \theta) = \beta + \dot{\theta} \cong \beta$.

- order ε^2

$$
\begin{bmatrix} 1 & \alpha & \alpha^2 \\ -1 & -\alpha & \left(\omega^2 - \alpha^2\right) \\ 0 & -\omega & -2\omega\alpha \end{bmatrix} \begin{Bmatrix} G_{s3} \\ G_{s4} \\ G_{s5} \end{Bmatrix}
= \left\{ \begin{array}{l}
\alpha(\sigma_s + \epsilon_s) - \omega\xi_s - \left(\frac{\sigma_u + \epsilon_u}{2}\right) G_{s1} \\
\quad -(\sigma_u \alpha + \epsilon_u \alpha - \xi_u \omega) G_{s2} \\
2(\epsilon_s \alpha - \xi_s \omega) + \epsilon_u G_{s1} + 2(\epsilon_u \alpha - \xi_u \omega) G_{s2} \\
-2(\epsilon_s \omega + \xi_s \alpha) + \xi_u G_{s1} + 2(\epsilon_u \omega + \xi_u \alpha) G_{s2}
\end{array} \right\}, \tag{50}
$$

with ϵ_s, ξ_s and σ_s defined in (26) and (28).

- order ε^3

$$
\begin{bmatrix} 1 & \alpha & (\alpha^2 - \omega^2) & (\alpha^3 - 3\alpha\omega^2) \\ 0 & 0 & \omega^2 & 3\alpha\omega^2 \\ 0 & -\omega & -2\omega\alpha & -3\omega\alpha^2 \\ 0 & \omega & 2\omega\alpha & (3\omega\alpha^2 - \omega^3) \end{bmatrix} \begin{Bmatrix} G_{s6} \\ G_{s7} \\ G_{s8} \\ G_{s9} \end{Bmatrix}
$$

$$
= \left\{ \begin{array}{l}
3(\mu_s \alpha - \nu_s \omega) - \mu_u G_{s1} - 3(\mu_u \alpha - \nu_u \omega) G_{s2} \\
\quad - 2\epsilon_u G_{s3} - 3(\epsilon_u \alpha - \xi_u \omega) G_{s4} \\
\quad + 4\left[\epsilon_u\left(\omega^2 - \alpha^2\right) + 2\xi_u \alpha\omega\right] G_{s5} \\[4pt]
-\frac{9}{4}\left(\mu_s \alpha - \nu_s \omega\right) + \frac{3}{4}\mu_u G_{s1} + \frac{9}{4}(\mu_u \alpha - \nu_u \omega) G_{s2} \\
\quad - (\sigma_u - \epsilon_u) G_{s3} + \left[\frac{3}{2}\epsilon_u \alpha - 2\xi_u \omega \right. \\
\quad \left. - \sigma_u\left(\alpha + \frac{1}{2}\omega\right)\right] G_{s4} - \left[2\epsilon_u(2\omega^2 - \alpha^2) \right. \\
\quad \left. + 6\xi_u \alpha\omega + 2\sigma_u \alpha^2\right] G_{s5} \\[4pt]
-\frac{9}{4}(\mu_s \omega + \nu_s \alpha) + \frac{3}{4}\nu_u G_{s1} + \frac{9}{4}(\mu_u \omega + \nu_u \alpha) G_{s2} \\
\quad + 2\xi_u G_{s3} + \left[\frac{5}{2}\epsilon_u \omega + 3\xi_u \alpha + \frac{1}{2}\sigma_u \omega\right] G_{s4} \\
\quad + [6\epsilon_u \alpha\omega + 2\xi_u(2\alpha^2 - \omega^2) + 2\sigma_u \alpha\omega] G_{s5} \\[4pt]
3(\mu_s \omega + \nu_s \alpha) - \nu_u G_{s1} - 3(\mu_u \omega + \nu_u \alpha) G_{s2} \\
\quad - 2\xi_u G_{s3} - 3(\epsilon_u \omega + \xi_u \alpha) G_{s4} \\
\quad - 4[2\epsilon_u \alpha\omega + \xi_u(\alpha^2 - \omega^2)] G_{s5}
\end{array} \right\}, \tag{51}
$$

with μ_s and ν_s defined in (39).

4. Numerical example

For the sake of an example, the elastic cantilever beam model of Fig. 1, with non-proportional damping, was analysed and its second non-linear normal mode evaluated. Large-amplitude free-vibration motion was considered, so that geometric non-linearity should be taken into account. The beam was discretised with 10 Bernoulli–Euler finite elements (30 active degrees of freedom). The equations of motion can be written in the form (1). Details on the beam-element formulation are presented in [7].

Fig. 2 shows a plot of the time evolution of the vertical displacement of the beam tip (the 29th degree of

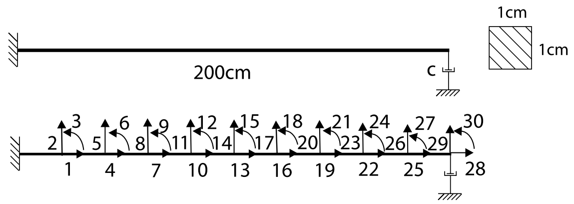

Fig. 1. Cantilever beam and finite-element model. Mechanical constants: $\rho = 7800$ kg/m^3, $E = 2.1 \times 10^{11}$ N/m^2, $c = 0.7$ N s/m.

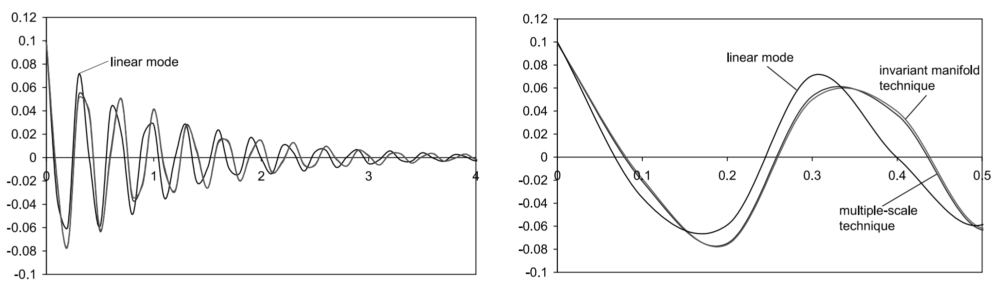

Fig. 2. Time evolution of the tip of the beam vertical displacement.

Table 1
Coefficients of the manifold associated with the second non-linear normal mode of the beam for p_{30} and $\dot{p}_{30}$

Coefficient	Multiple scales		Invariant manifold	
	p_{30}	$\dot{p}_{30}$	p_{30}	$\dot{p}_{30}$
U	2.3896	4.0551	2.3896	4.0551
V	−0.0005959	2.3907	−0.0005959	2.3907
U^2	0.0	0.0	0.0	0.0
UV	0.0	0.0	0.0	0.0
V^2	0.0	0.0	0.0	0.0
U^3	−1.0372	−46.0378	0.2045	−42.533
U^2V	0.0001147	−9.2005	−0.00287	−7.9776
UV^2	0.0004572	0.01787	0.0006399	0.01703
V^3	0.0	0.0004522	-4.155×10^{-7}	0.0006394

freedom, taken together with its corresponding velocity as modal variables) for certain initial conditions, just for the sake of an example. Three curves are presented in the plot, namely, the linear solution, the one obtained by the technique here proposed and the one obtained by the invariant manifold technique [6]. The last two almost coincide, which is seen as a validation test for the technique proposed here.

Re-generation—as proposed in Section 3—of the generalised coordinate associated with the rotation of the tip of the beam (the 30th degree of freedom) in terms of the modal variables, which are here chosen to be the vertical displacement of the tip of the beam and its velocity, leads to the coefficients presented in Table 1.

In spite of apparent numerical discrepancies in the coefficients of cubic terms according to the multiple-scale approach and the invariant manifold technique, they have an overall effect which is amazingly equivalent. In fact, Figs. 3 and 4 show that the temporal responses are in excellent agreement. It should be noted that this agreement is not because the non-linear coefficients are unimportant, since it can also be observed in these figures that the non-linear responses are quite different from the linear one.

Another interesting feature of this problem, explored by Soares in his doctoral thesis [9], is that even the linear solution does not characterize a standing wave for the normal modes, that is, the nodal points are not fixed during the structure vibration. As mentioned before, this is typical of systems with non-proportional damping and

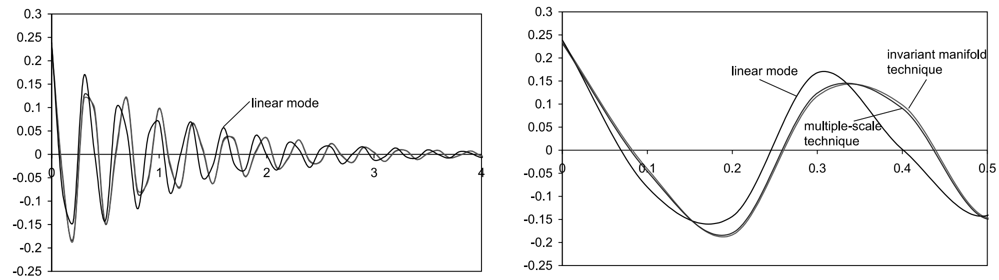

Fig. 3. Time evolution of the tip of the beam rotation.

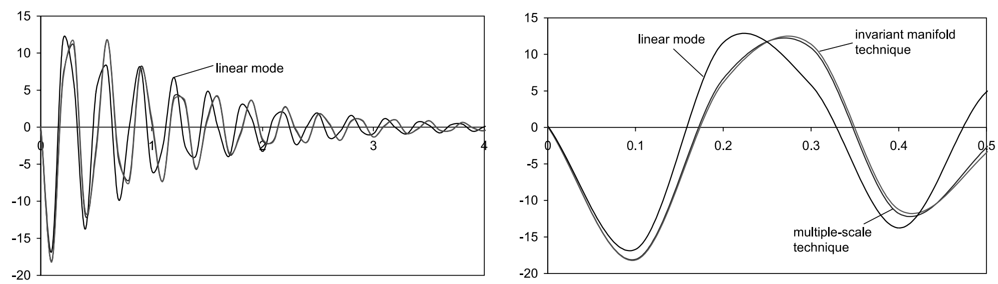

Fig. 4. Time evolution of the tip of the beam rotation velocity.

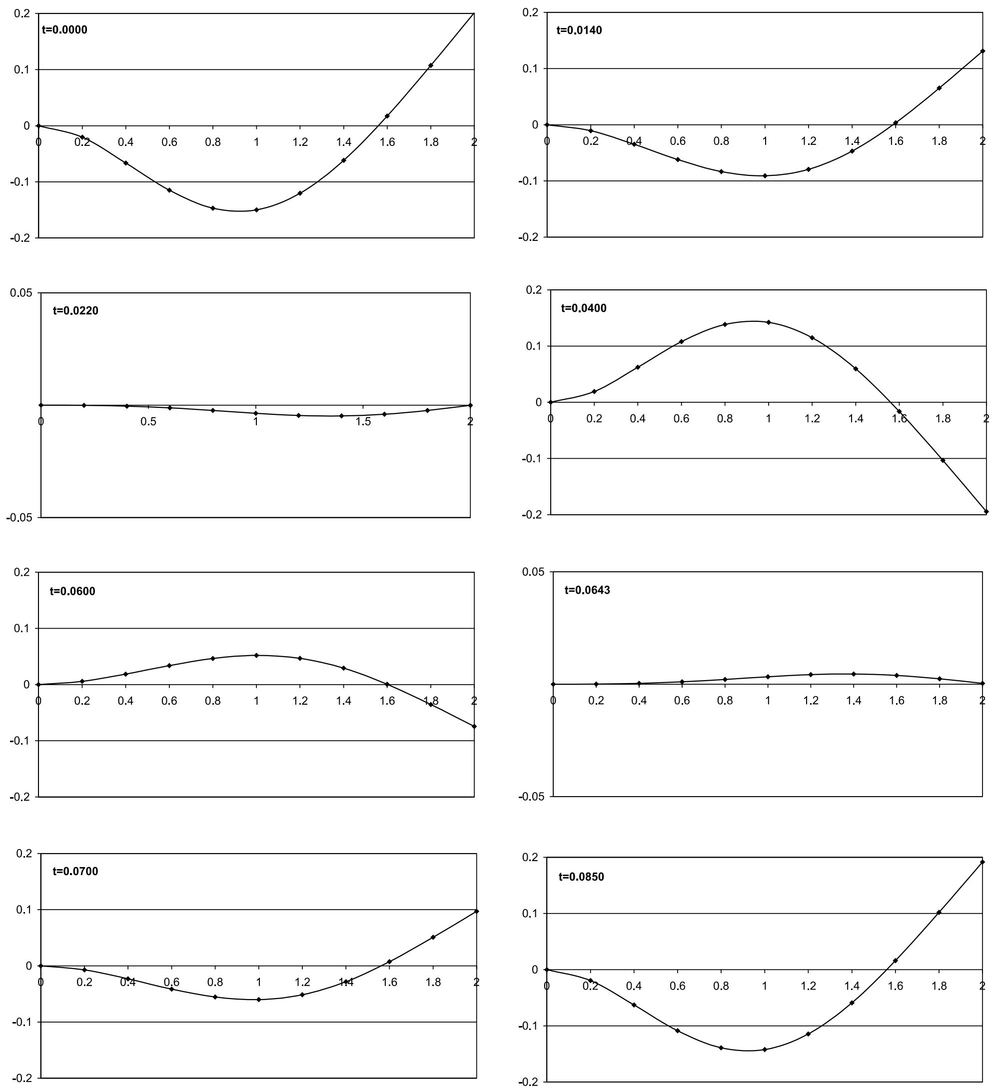

Fig. 5. Time evolution of the displacements of the cantilever beam with non-proportional damping. Inital conditions: $U(0) = 0.201$, $V(0) = 0.000$ (i.e. $a_0 = 2.15$, $\theta_0 = -0.052$).

the same feature has also been correctly captured by the technique here presented for the non-linear system (see Fig. 5). Note that the nodal point is moving along the beam axis as the motion takes place. It stays approximately at 0.5 m from the beam tip most of the time, but also reaches the beam tip at $t = 0.022$ and 0.0643 s.

5. Conclusions

In this paper it is presented an alternative method for the evaluation of non-linear normal modes of finite-element models.

The procedure relies upon the definition of non-linear normal modes of vibration proposed by Shaw and Pierre [3] and is based on the application of the method of multiple scales to the non-linear free-vibration equations of motion. Both the time evolution of the generalised coordinates and the topological characterization of the invariant manifold associated with the non-linear normal mode are pursued.

As an illustrative example, it is chosen a model of a cantilever beam with non-proportional damping. The results obtained are compared with those of the linear theory and the invariant manifold technique, with excellent agreement with the latter.

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