PEF 5737 - Nonlinear dynamics and stability

Averaging method

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Programa de Pós-Graduação em Engenharia Civil
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Outline

1. Objective

2. Background on nonlinear differential equations (practical aspects)

3. Averaging method

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Objective

- To discuss the averaging method, commonly adopted in the analysis of nonlinear equations;

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2 Background on nonlinear differential equations (practical aspects)

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Bernoulli's equation

- Bernoulli's equation is a first-order differential equation with the form:

$$\dot{u} + p(t)u = q(t)u^n \tag{1}$$

with $n > 1$. If $n = 0$ or $n = 1$, Eq. 1 is linear and the solution is known.

- Multiplying both sides of Eq. 1 by $u^{-n}(1-n)$:

$$(1-n)u^{-n}\dot{u} + (1-n)p(t)u^{1-n} = (1-n)q(t) \tag{2}$$

- Change of variables: $z = u^{1-n}$. Hence, we use the chain rule to obtain $\dot{z} = (1-n)u^{-n}\dot{u}$.
- Using the above results on Eq. 1, we obtain the following linear equation:

$$\dot{z} + f(t)z = g(t) \tag{3}$$

where $f(t) = (1-n)p(t)$ and $g(t) = (1-n)q(t)$

- We solve Eq. 3 using an integrating factor μ. For this, we define

$$\mu = \mu(t) = exp\left\{\int f(t)dt\right\} \tag{4}$$

- Notice that

$$\frac{d}{dt}(z\mu) = \dot{z}\mu + z\dot{\mu} = \dot{z}\mu + z\mu f(t) \tag{5}$$

Bernoulli's equation

- Multiplying both sides of Eq. 3 by μ leads to:

$$\mu\dot{z} + f(t)z\mu = \mu g(t) \leftrightarrow \frac{d}{dt}(z\mu) = \mu g(t) \tag{6}$$

- The integration of Eq. 6 is easily made:

$$z\mu = \int \mu g(t)dt + c \leftrightarrow z = \frac{1}{\mu}\int \mu g(t)dt + \frac{c}{\mu} \tag{7}$$

- Once z is obtained, $u(t)$ can be obtained recalling that $z = u^{1-n}$.

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2 Background on nonlinear differential equations (practical aspects)

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General aspects

- We investigate analytical approximated solutions for the nonlinear equation with the form

$$\ddot{u} + u = \epsilon F(u, \dot{u}, t) \tag{8}$$

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- If $0 < \epsilon \ll 1$, the following *ansatz* are proposed:

$$u = a\cos(t+\psi) \tag{9}$$

$$\dot{u} = -a\sin(t+\psi) \tag{10}$$

with $a = a(t)$ and $\psi = \psi(t)$. Since $0 < \epsilon \ll 1$ (weak nonlinearities), $a(t)$ and $\psi(t)$ slowly vary in time.

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- Taking the derivative of Eq. 9 and substituting into Eq. 10

$$\dot{a}\cos(t+\psi) - a\sin(t+\psi)(1+\dot{\psi}) = -a\sin(t+\psi) \rightarrow \dot{a}\cos(t+\psi) - a\dot{\psi}\sin(t+\psi) = 0 \tag{11}$$

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- The derivative of Eq. 10 is:

$$\ddot{u} = -\dot{a}\sin(t+\psi) - a(1+\dot{\psi})\cos(t+\psi) \tag{12}$$

The averaging method and the slow-flow equations

- Using Eqs. 9 and 12 into the original equation (Eq. 8), one obtains:

$$\dot{a}\sin(t+\psi) + a\dot{\psi}\cos(t+\psi) = -F(a\cos(t+\psi), -a\sin(t+\psi), t) \tag{13}$$

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- Eqs. 12 and 13 define a system of ODEs in the variables $\dot{a}$ and $\dot{\psi}$. This system can be solved for $\dot{a}$ and $\dot{\psi}$. The result is given in Eqs. 14 and 15.

$$\dot{a} = -F(a\cos(t+\psi), -a\sin(t+\psi), t)\sin(t+\psi) \tag{14}$$

$$a\dot{\psi} = -F(a\cos(t+\psi), -a\sin(t+\psi), t)\cos(t+\psi) \tag{15}$$

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- Since a and ψ slowly vary (when compared to the "fast" scale given by the natural period of the linear system $T = 2\pi$), the RHS of Eqs. 14 and 15 can be replaced by their averaged values calculated considering one period of the "fast scale". This approach leads to the slow-flow equations, given by Eqs. 16 and 17.

$$\dot{a} = -\frac{1}{2\pi}\int_0^{2\pi} F(a\cos\phi, -a\sin\phi, t)\sin\phi d\phi \tag{16}$$

$$a\dot{\psi} = -\frac{1}{2\pi}\int_0^{2\pi} F(a\cos\phi, -a\sin\phi, t)\cos\phi d\phi \tag{17}$$

where $\phi = t + \psi$.

van der Pol equation

- Consider the van der Pol equation $\ddot{u} + \epsilon(u^2 - 1)\dot{u} + u = 0$. Putting it on the nomenclature herein adopted, we can notice that $F(u, \dot{u}, t) = \epsilon(1 - u^2)\dot{u}$.

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- Slow-flow equations (assuming that $a \neq 0$):

$$\dot{a} = -\frac{\epsilon}{2\pi}\int_0^{2\pi}(1 - a^2\cos^2\phi)\sin\phi(-a\sin\phi)d\phi \tag{18}$$

$$\dot{\psi} = -\frac{\epsilon}{a2\pi}\int_0^{2\pi}(1 - a^2\cos^2\phi)\cos\phi(-a\sin\phi)d\phi \tag{19}$$

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$$\dot{a} = \frac{\epsilon}{8}a(4 - a^2) \leftrightarrow \dot{a} - \frac{\epsilon}{2}a = -\frac{\epsilon}{8}a^3 \tag{20}$$

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- The solution of Eq. 20 can be obtained using the presented background. For this, we recognize the Bernoulli's equation with $p(t) = -\frac{\epsilon}{2}$, $q(t) = -\frac{\epsilon}{8}$ and $n = 3$. The change of variables is $z = a^{1-3} = a^{-2}$.

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- In the new variable, Eq. 20 is rewritten as:

$$\dot{z} + \epsilon z = \frac{\epsilon}{4} \tag{21}$$

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- Using the above result, $z(t)$ reads:

$$z = z(t) = \frac{1}{e^{\epsilon t}}\frac{1}{4}e^{\epsilon t} + \frac{c}{e^{\epsilon t}} = \frac{e^{\epsilon t} + 4c}{4e^{\epsilon t}} \tag{24}$$

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- Returning to the amplitude $a(t)$

$$z = a^{-2} \leftrightarrow a = z^{-1/2} \leftrightarrow a = \frac{1}{\sqrt{\frac{e^{\epsilon t}+4c}{4e^{\epsilon t}}}} = \frac{2e^{\epsilon\frac{t}{2}}}{\sqrt{e^{\epsilon t} + 4c}} \tag{25}$$

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- c can be determined using the initial condition $a(0)$. For this, consider $t = 0$ in Eq. 25 and the following equation holds:

$$a(0) = \frac{2}{\sqrt{1+4c}} \leftrightarrow c = \frac{1}{4}\left(\frac{4}{(a(0))^2} - 1\right) \tag{26}$$

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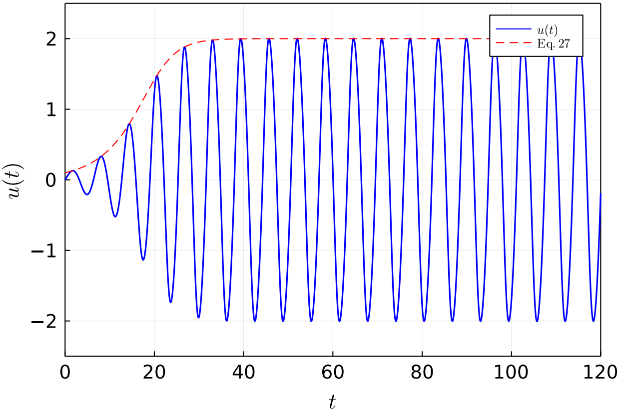
- By substituting Eq. 26 into Eq. 25 one obtains:

$$a(t) = \frac{2e^{\epsilon \frac{t}{2}}}{\sqrt{e^{\epsilon t} - 1 + \frac{4}{(a(0))^2}}} \tag{27}$$

van der Pol Equation

- The numerical solution of the van der Pol equation for $\epsilon = 0.3$, $u(0) = 0$ and $\dot{u}(0) = 0.1$ are shown in the figure below. The analytical solution for the instantaneous oscillation amplitude $a(t)$ given by Eq. 27 is also shown.



van der Pol equation

- Another type of study. The equilibrium point of Eq. 20 is $a_0 = 2$ (assuming $a > 0$). We study the stability of this equilibrium point considering $a = a_0 + \delta a$ (δa is a small disturbance superimposed to the equilibrium point).

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$$\dot{a} = \dot{a}_0 + \delta\dot{a} = \frac{\epsilon}{8}(a_0 + \delta a)(4 - (a_0 + \delta a)^2) = \frac{\epsilon}{8}(4a_0 - a_0^3 + (-3a_0^2 + 4)\delta a) + \mathcal{O}(\delta a)^2 \quad (28)$$

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- Since by definition $\dot{a}_0 = 0$, the linearized version of the above equation is $\delta\dot{a} = -\epsilon\delta a$. Therefore, δa (the disturbance) goes to zero provided $\epsilon > 0$. If we are not interested in the temporal evolution of the amplitude, this stability study suffices and the exact solution $a(t)$ does not need to be obtained.

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- Phase equation:

$$\dot{\psi} = -\frac{\epsilon}{a2\pi}\int_0^{2\pi}(1 - a^2\cos^2\phi)\cos\phi(-a\sin\phi)d\phi = 0 \quad (29)$$

Hence, the phase ψ does not vary in time. This means that the oscillation frequency is constant and equal to 1.

Bibliography

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- Rand, R.H. Lecture Notes on Nonlinear Vibrations, 2005.