us denote by $\epsilon'(\mu, \delta) > 0$ the minimum of the continuous nondecreasing function $\gamma(\|\mathbf{x}\|)$ on the compact set $\nu(\mu) \le \|\mathbf{x}\| \le \epsilon(\delta)$. Let us define

$$T(\mu, \delta) = \frac{\beta(\delta)}{\epsilon'(\mu, \delta)} > 0$$

Suppose that $\|\boldsymbol{\phi}(t; \mathbf{x}_0, t_0)\| > \nu$ over the time interval $t_0 \le t \le t_1 = t_0 + T$. Then we have

$$0 < \alpha(\nu) \le V(\boldsymbol{\phi}(t_1; \mathbf{x}_0, t_0), t_1) \le V(\mathbf{x}_0, t_0) - (t_1 - t_0)\epsilon' \le \beta(\delta) - T\epsilon' = 0$$

which is a contradiction. Hence, for some t in the interval $t_0 \le t \le t_1$, such as an arbitrary t_2, we have

$$\|\mathbf{x}_2\| = \|\boldsymbol{\phi}(t_2; \mathbf{x}_0, t_0)\| = \nu$$

Therefore,

$$\alpha(\|\boldsymbol{\phi}(t; \mathbf{x}_2, t_2)\|) < V(\boldsymbol{\phi}(t; \mathbf{x}_2, t_2), t) \le V(\mathbf{x}_2, t_2) \le \beta(\nu) < \alpha(\mu)$$

for all $t \ge t_2$. Hence,

$$\|\boldsymbol{\phi}(t; \mathbf{x}_0, t_0)\| < \mu$$

for all $t \ge t_0 + T(\mu, \delta) \ge t_2$, which proves uniform asymptotic stability. Since $\alpha(\|\mathbf{x}\|) \to \infty$ as $\|\mathbf{x}\| \to \infty$, there exists for arbitrarily large δ a constant $\epsilon(\delta)$ such that $\beta(\delta) < \alpha(\epsilon)$. Moreover, since $\epsilon(\delta)$ does not depend on t_0, the solution $\boldsymbol{\phi}(t; \mathbf{x}_0, t_0)$ is uniformly bounded. We thus have proved uniform asymptotic stability in the large.

Problem A–5–19

In z plane analysis, an $n \times n$ matrix $\mathbf{G}$ whose n eigenvalues are less than unity in magnitude is called a stable matrix. Consider an $n \times n$ Hermitian (or real symmetric) matrix $\mathbf{P}$ that satisfies the following matrix equation:

$$\mathbf{G}^*\mathbf{P}\mathbf{G} - \mathbf{P} = -\mathbf{Q} \tag{5–137}$$

where $\mathbf{Q}$ is a positive definite $n \times n$ Hermitian (or real symmetric) matrix. Prove that if matrix $\mathbf{G}$ is a stable matrix then a matrix $\mathbf{P}$ that satisfies Equation (5–137) is unique and is positive definite. Prove that matrix $\mathbf{P}$ can be given by

$$\mathbf{P} = \sum_{k=0}^{\infty} (\mathbf{G}^*)^k \mathbf{Q}\mathbf{G}^k$$

Prove also that although the right-hand side of this last equation is an infinite series the matrix is finite. Finally, prove that if Equation (5–137) is satisfied by positive definite matrices $\mathbf{P}$ and $\mathbf{Q}$, then matrix $\mathbf{G}$ is a stable matrix. Assume that all eigenvalues of $\mathbf{G}$ are distinct and all eigenvectors of $\mathbf{G}$ are linearly independent.

Solution Let us assume that there exist two matrices $\mathbf{P}_1$ and $\mathbf{P}_2$ that satisfy Equation (5–137). Then

$$\mathbf{G}^*\mathbf{P}_1\mathbf{G} - \mathbf{P}_1 = -\mathbf{Q} \tag{5–138}$$

and

$$\mathbf{G}^*\mathbf{P}_2\mathbf{G} - \mathbf{P}_2 = -\mathbf{Q} \tag{5–139}$$

By subtracting Equation (5–139) from Equation (5–138), we obtain

$$\mathbf{G}^*\hat{\mathbf{P}}\mathbf{G} - \hat{\mathbf{P}} = \mathbf{0} \tag{5–140}$$