**Solution** To prove uniform asymptotic stability in the large, we need to prove the following:

1. The origin is uniformly stable.
2. Every solution is uniformly bounded.
3. Every solution converges to the origin when $t \to \infty$ uniformly in $t_0$ and $\|\mathbf{x}_0\| \le \delta$, where $\delta$ is fixed but arbitrarily large. That is, given two real numbers $\delta > 0$ and $\mu > 0$, there is a real number $T(\mu, \delta)$ such that

   $$\|\mathbf{x}_0\| \le \delta$$

   implies

   $$\|\boldsymbol{\phi}(t; \mathbf{x}_0, t_0)\| \le \mu, \qquad \text{for all } t \ge t_0 + T(\mu, \delta)$$

   where $\boldsymbol{\phi}(t; \mathbf{x}_0, t_0)$ is the solution to the given differential equation

Since $\beta$ is continuous and $\beta(0) = 0$, we can take a $\delta(\epsilon) > 0$ such that $\beta(\delta) < \alpha(\epsilon)$ for any $\epsilon > 0$. Figure 5–10 shows the curves $\alpha(\|\mathbf{x}\|)$, $\beta(\|\mathbf{x}\|)$, and $V(\mathbf{x}, t)$. Noting that

$$V(\boldsymbol{\phi}(t; \mathbf{x}_0, t_0), t) - V(\mathbf{x}_0, t_0) = \int_{t_0}^{t} \dot{V}(\boldsymbol{\phi}(\tau; \mathbf{x}_0, t_0), \tau)\, d\tau < 0, \qquad t > t_0$$

if $\|\mathbf{x}_0\| \le \delta$, $t_0$ being arbitrary, we have

$$\alpha(\epsilon) > \beta(\delta) \ge V(\mathbf{x}_0, t_0) \ge V(\boldsymbol{\phi}(t; \mathbf{x}_0, t_0), t) \ge \alpha(\|\boldsymbol{\phi}(t; \mathbf{x}_0, t_0)\|)$$

for all $t \ge t_0$. Since $\alpha$ is nondecreasing and positive, this implies that

$$\|\boldsymbol{\phi}(t; \mathbf{x}_0, t_0)\| < \epsilon, \qquad \text{for } t \ge t_0, \|\mathbf{x}_0\| \le \delta$$

Hence, we have shown that for each real number $\epsilon > 0$ there is a real number $\delta > 0$ such that $\|\mathbf{x}_0\| \le \delta$ implies $\|\boldsymbol{\phi}(t; \mathbf{x}_0, t_0)\| \le \epsilon$ for all $t \ge t_0$. Thus, we have proved uniform stability.

Next, we shall prove that $\|\boldsymbol{\phi}(t; \mathbf{x}_0, t_0)\| \to 0$ when $t \to \infty$ uniformly in $t_0$ and $\|\mathbf{x}_0\| \le \delta$. Let us take any $0 < \mu < \|\mathbf{x}_0\|$ and find a $\nu(\mu) > 0$ such that $\beta(\nu) < \alpha(\mu)$. Let

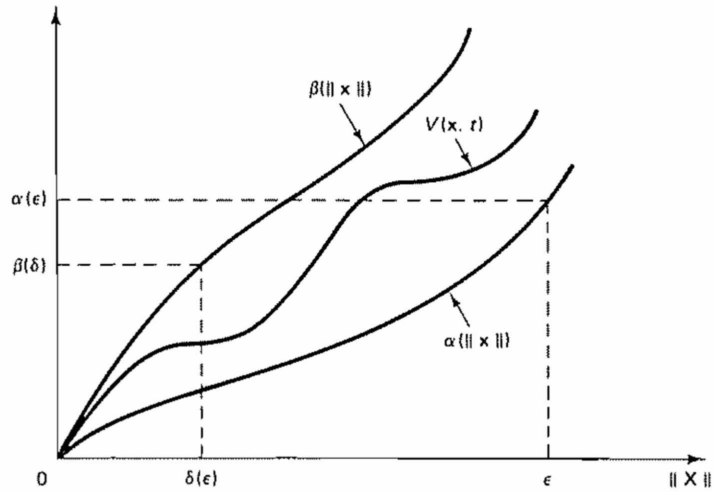


**Figure 5–10** Curves $\alpha(\|\mathbf{x}\|)$, $\beta(\|\mathbf{x}\|)$, and $V(\mathbf{x}, t)$