$$= [1 \quad 0]\begin{bmatrix} z-1 & -T \\ 0 & z-1 \end{bmatrix}^{-1}\begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix} + 0$$

$$= \frac{T^2(z+1)}{2(z-1)^2} = \frac{T^2 z^{-1}(1+z^{-1})}{2(1-z^{-1})^2}$$

Problem A–5–17

Show that the following quadratic form is positive definite:

$$V(\mathbf{x}) = 10x_1^2 + 4x_2^2 + x_3^2 + 2x_1x_2 - 2x_2x_3 - 4x_1x_3$$

Solution The quadratic form $V(\mathbf{x})$ can be written as follows:

$$V(\mathbf{x}) = \mathbf{x}^T\mathbf{P}\mathbf{x} = [x_1 \quad x_2 \quad x_3]\begin{bmatrix} 10 & 1 & -2 \\ 1 & 4 & -1 \\ -2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Applying Sylvester's criterion, we obtain

$$10 > 0, \qquad \begin{vmatrix} 10 & 1 \\ 1 & 4 \end{vmatrix} > 0, \qquad \begin{vmatrix} 10 & 1 & -2 \\ 1 & 4 & -1 \\ -2 & -1 & 1 \end{vmatrix} > 0$$

Since all the successive principal minors of the matrix $\mathbf{P}$ are positive, $V(\mathbf{x})$ is positive definite.

Problem A–5–18

Consider the system defined by

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$$

Suppose that

$$\mathbf{f}(\mathbf{0}, t) = \mathbf{0}, \qquad \text{for all } t$$

Suppose also that there exists a scalar function $V(\mathbf{x}, t)$ that has continuous first partial derivatives If $V(\mathbf{x}, t)$ satisfies the conditions

1. $V(\mathbf{x}, t)$ is positive definite. That is, $V(\mathbf{0}, t) = 0$ and $V(\mathbf{x}, t) \geq \alpha(\|\mathbf{x}\|) > 0$ for all $\mathbf{x} \neq \mathbf{0}$ and all t, where α is a continuous nondecreasing scalar function such that $\alpha(0) = 0$.
2. The total derivative $\dot{V}(\mathbf{x}, t)$ is negative for all $\mathbf{x} \neq \mathbf{0}$ and all t, or $\dot{V}(\mathbf{x}, t) \leq -\gamma(\|\mathbf{x}\|) < 0$ for all $\mathbf{x} \neq \mathbf{0}$ and all t, where γ is a continuous nondecreasing scalar function such that $\gamma(0) = 0$.
3. There exists a continuous nondecreasing scalar function β such that $\beta(0) = 0$ and, for all t, $V(\mathbf{x}, t) \leq \beta(\|\mathbf{x}\|)$
4. $\alpha(\|\mathbf{x}\|)$ approaches infinity as $\|\mathbf{x}\|$ increases indefinitely, or

$$\alpha(\|\mathbf{x}\|) \to \infty, \qquad \text{as } \|\mathbf{x}\| \to \infty$$

then the origin of the system, $\mathbf{x} = \mathbf{0}$, is uniformly asymptotically stable in the large (This is Liapunov's main stability theorem)

Prove this theorem.