Introductory Turbulence Modeling

Lectures Notes by
Ismail B. Celik

West Virginia University
Mechanical & Aerospace Engineering Dept.
P.O. Box 6106
Morgantown, WV 26506-6106

December 1999

TABLE OF CONTENTS

Page

NOMENCLATURE

English

B_k	production of turbulent kinetic energy by buoyancy
B_ε	production of turbulent dissipation by buoyancy
C_f	friction coefficient
D_k	destruction of turbulent kinetic energy
D_ε	destruction of turbulent dissipation
k	specific turbulent kinetic energy
l_{mfp}	mean-free path length
l_{mix}	mixing length
p	pressure
p^*	modified pressure
P_k	production of turbulent kinetic energy
P_ε	production of turbulent dissipation
q_i	Reynolds flux
S	source term
S_{ij}	mean strain rate tensor
t	time
u	velocity
u^+	dimensionless velocity (wall variables)
u_*	friction velocity
x	horizontal displacement (displacement in the streamwise direction)
y^+	dimensionless vertical distance (wall variables)

Greek Symbols

δ	boundary layer thickness
δ_v^*	displacement thickness
δ_{ij}	Kroenecker delta function
ε	dissipation rate of turbulent kinetic energy
ϕ	generic scalar variable
ϕ'	fluctuating component of time-averaged variable ϕ
Φ	mean component of time-averaged variable ϕ
$\bar{\phi}$	Reynolds time averaged variable ϕ
Γ	diffusion coefficient
Γ_t	turbulent diffusivity
κ	von-Karman constant
μ	molecular viscosity

ν_t	eddy viscosity
Π_{ij}	pressure-strain correlation tensor
ρ	density
σ	turbulent Prandtl-Schmidt number
σ_k	Prandtl-Schmidt number for k
τ_{ij}	Reynolds stress tensor
τ_w	wall shear stress
ω	dissipation per unit turbulent kinetic energy (specific dissipation)
Ω_{ij}	mean rotation tensor

1.0 INTRODUCTION

Theoretical analysis and prediction of turbulence has been, and to this date still is, the fundamental problem of fluid dynamics, particularly of computational fluid dynamics (CFD). The major difficulty arises from the random or chaotic nature of turbulence phenomena. Because of this unpredictability, it has been customary to work with the time averaged forms of the governing equations, which inevitably results in terms involving higher order correlations of fluctuating quantities of flow variables. The semi-empirical mathematical models introduced for calculation of these unknown correlations form the basis for turbulence modeling. It is the focus of the present study to investigate the main principles of turbulence modeling, including examination of the physics of turbulence, closure models, and application to specific flow conditions. Since turbulent flow calculations usually involve CFD, special emphasis is given to this topic throughout this study.

There are three key elements involved in CFD:

(1) grid generation
(2) algorithm development
(3) turbulence modeling

While for the first two elements precise mathematical theories exist, the concept of turbulence modeling is far less precise due to the complex nature of turbulent flow. Turbulence is three-dimensional and time-dependent, and a great deal of information is required to describe all of the mechanics of the flow. Using the work of previous investigators (e.g. Prandtl, Taylor, and von Karman), an ideal turbulence model attempts to capture the essence of the relevant physics, while introducing as little complexity as possible. The description of a turbulent flow may require a wide range of information, from simple definitions of the skin friction or heat transfer coefficients, all the way up to more complex energy spectra and turbulence fluctuation magnitudes and scales, depending on the particular application. The complexity of the mathematical models increases with the amount of information required about the flowfield, and is reflected by the way in which the turbulence is modeled, from simple mixing-length models to the complete solution of the full Navier-Stokes equations.

The Physics of Turbulence

In 1937, Taylor and von Karman proposed the following definition of turbulence:

> "Turbulence is an irregular motion which in general makes its appearance in fluids, gaseous or liquid, when they flow past solid surfaces or even when neighboring streams of the same fluid flow past or over one another"

Some of the key elements of turbulence are that it occurs over a large range of length and time scales, at high Reynolds number, and is fully three-dimensional and time-dependent. Turbulent flows are much more irregular and intermittent in contrast with laminar flow, and turbulence typically develops as an instability of laminar flow. For a real (i.e. viscous) fluid, these instabilities result from the interactions of the non-linear inertial terms and the viscous terms contained in the Navier-Stokes equations, which are very complex due to the fact that turbulence is rotational, three-dimensional, and time-dependant.

The rotational and three-dimensional natures of turbulence are closely linked, as vortex stretching is required to maintain the constantly fluctuating vorticity. As vortex stretching is absent in two-dimensional flows, turbulence must be three-dimensional. This implies that there are no two-dimensional approximations, thus making the problem of resolving turbulent flows a difficult problem.

The time-dependant nature of turbulence, with a wide range of time scales (i.e. frequencies), means that statistical averaging techniques are required to approximate random fluctuations. Time averaging, however, leads to correlations in the equations of motion that are unknown a priori. This is the classic closure problem of turbulence, which requires modeled expressions to account for the additional unknowns, and is the primary focus of turbulence modeling.

Turbulence is a continuous phenomenon that exists on a large range of length and time scales, which are still larger than molecular scales. In order to visualize turbulent flows, one often refers to turbulent eddies, which can be thought of as a local swirling motion whose characteristic dimension is on the order of the local turbulence length scale. Turbulent eddies also overlap in space, where larger eddies carry smaller ones. As there exists a large range of different scales (or turbulent eddy sizes), an energy cascade exists by which energy is transferred from the larger scales to the smaller scales, and eventually to the smallest scales where the energy is dissipated into heat by molecular viscosity. Turbulent flows are thus always dissipative.

Turbulent flows also exhibit a largely enhanced diffusivity. This turbulent diffusion greatly enhances the transfer of mass, momentum, and energy. The apparent stresses, therefore, may be of several orders of magnitude greater than in the corresponding laminar case.

The fact that the Navier-Stokes equations are non-linear for turbulent flows leads to interactions between fluctuations of different wavelengths and directions. The wavelengths of the motion may be as large as a characteristic scale on the order of the width of the flow, all the way to the smallest scales, which are limited by the viscous dissipation of energy. The action of vortex stretching is mainly responsible for spreading the motion over a wide range of wavelengths. Wavelengths which are nearly comparable to the characteristic mean-flow scales interact most strongly with the mean flow. This implies that the larger-scale turbulent eddies are most responsible for the energy transfer and enhanced diffusivity. In turn, these large eddies cause random stretching of the vortex elements of the smaller eddies, and energy is cascades down from the largest to the smallest scales.

Future chapters will examine some of the above aspects of turbulence as they relate to case specific issues.

A Brief History of Turbulence Modeling

The origin of the time-averaged Navier-Stokes equations dates back to the late nineteenth century when Reynolds (1895) published results from his research on turbulence. The earliest attempts at developing a mathematical description of the turbulent stresses, which is the core of the closure problem, were performed by Boussinesq (1877) with the introduction of the eddy viscosity concept. Neither of these authors, however, attempted to solve the time-averaged Navier-Stokes equations in any kind of systematic manner.

More information regarding the physics of viscous flow was still required, until Prandtl's discovery of the boundary layer in 1904. Prandtl (1925) later introduced the concept of the mixing-length model, which prescribed an algebraic relation for the turbulent stresses. This early development was the cornerstone for nearly all turbulence modeling efforts for the next twenty years. The mixing length model is now known as an algebraic, or zero-equation model. To develop a more realistic mathematical model of the turbulent stresses, Prandtl (1945) introduced the first one-equation model by proposing that the eddy viscosity depends on the turbulent kinetic energy, k, solving a differential equation to approximate the exact equation for k. This one-equation model improved the turbulence predictions by taking into account the effects of flow history

The problem of specifying a turbulence length scale still remained. This information, which can be thought of as a characteristic scale of the turbulent eddies, changes for different flows, and thus is required for a more complete description of the turbulence. A more complete model would be one that can be applied to a given turbulent flow by prescribing boundary and/or initial conditions. Kolmogorov (1942) introduced the first complete turbulence model, by modeling the turbulent kinetic energy k, and introducing a second parameter ω that he referred to as the rate of dissipation of energy per unit volume and time. This two-equation model, termed the k-ω model, used the reciprocal of ω as the turbulence time scale, while the quantity $k^{1/2}/\omega$ served as a turbulence length scale, solving a differential equation for ω similar to the solution method for k. Because of the complexity of the mathematics, which required the solution of nonlinear differential equations, it went virtually without application for many years, before the availability of computers.

Rotta (1951) pioneered the use of the Boussinesq approximation in turbulence models to solve for the Reynolds stresses. This approach is called a second-order or second-moment closure. Such models naturally incorporate non-local and history effects, such as streamline curvature and body forces. The previous eddy viscosity models failed to account for such effects. For a three-dimensional flow, these second-order closure models introduce seven equations, one for a turbulence length scale, and six for the Reynolds stresses. As with Kolmogorov's k-ω model, the complex nature of this model awaited adequate computer resources.

Thus, by the early 1950's, four main categories of turbulence models had developed:

(1) Algebraic (Zero-Equation) Models
(2) One-Equation Models
(3) Two-Equation Models
(4) Second-Order Closure Models

With increased computer capabilities beginning in the 1960's, further development of all four of these classes of turbulence models has occurred. The most important modern developments are given below for each class.

Algebraic (Zero-Equation) Models

Van Driest (1956) devised a viscous damping correction for the mixing-length model. This correction is still in use in most modern turbulence models. Cebeci and Smith (1974) refined the eddy viscosity/mixing-length concept for better use with attached boundary layers. Baldwin and Lomax (1978) proposed an alternative algebraic model to eliminate some of the difficulty in defining a turbulence length scale from the shear-layer thickness.

One-Equation Models

While employing a much simpler approach than two-equation or second-order closure models, one-equation models have been somewhat unpopular and have not showed a great deal of success. One notable exception was the model formulated by Bradshaw, Ferris, and Atwell (1967), whose model was tested against the best experimental data of the day at the 1968 Stanford Conference on Computation and Turbulent Boundary Layers. There has been some renewed interest in the last several years due to the ease with which one-equation models can be solved numerically, relative to more complex two-equation or second-order closure models.

Two-Equation Models

While Kolmogorov's k-ω model was the first two-equation model, the most extensive work has been done by Daly and Harlow (1970) and Launder and Spalding (1972). Launder's k-ε model is the most widely used two-equation turbulence model; here ε is the dissipation rate of turbulent kinetic energy. Independently of Kolmogorov, Saffman (1970) developed a k-ω model that shows advantages to the more well known k-ε model, especially for integrating through the viscous sublayer and in flows with adverse pressure gradients.

Second-Order Closure Models

Due to the increased complexity of this class of turbulence models, second-order closure models do not share the same wide use as the more popular two-equation or algebraic models. The most noteworthy efforts in the development of this class of models was performed by Donaldson and Rosenbaum (1968), Daly and Harlow (1970), and Launder, Reece, and Rodi (1975). The latter has become the baseline second-order closure model, with more recent contributions made by Lumley (1978), Speziale (1985, 1987a), Reynolds (1987), and many other thereafter, who have added mathematical rigor to the model formulation.

While the present study is not intended to be a complete catalogue of all turbulence models, more detailed description is given for some of the above models in later chapters. The concept of the closure problem will also be investigated, along with a discussion of case specific issues as they relate to different types of flows. We should note here that unfortunately there are not many text books in the literature which can be used for teaching turbulence modeling, in contrast to the existence of hundreds of thousands of journal and conference papers in the literature about this subject. We would mention the three books that present authors consider as the best for teaching purposes. These are due to Launder and Spalding (1972), Rodi (1980), and Wilcox (1993).

2.0 REYNOLDS TIME AVERAGING

A turbulence model is defined as a set of equations (algebraic or differential) which determine the turbulent transport terms in the mean flow equations and thus close the system of equations. Turbulence models are based on hypotheses about the turbulent processes and require empirical input in the form of model constants or functions; they do not simulate the details of the turbulent motion, but only the effect of turbulence on the mean flow behavior. The concept of Reynolds averaging and the averaged conservation equations are some of the main concepts that form the basis of turbulence modeling.

Since all turbulent flows are transient and three-dimensional, the engineer is generally forced to develop methods for averaged quantities to extract any useful information. The most popular method for dealing with turbulent flows is Reynolds averaging which provides information about the overall mean flow properties. The main idea behind Reynolds time-averaging is to express any variable, $\phi(x,t)$, which is a function of time and space, as the sum of a mean and a fluctuating component as given by

$$\phi(x,t) = \Phi(x) + \phi'(x,t) \tag{2.1}$$

Here we use the notation that the uppercase symbols denote the time average of that quantity. For stationary turbulence, this average is defined by

$$\Phi(x) = \overline{\phi(x,t)} = \lim_{\tau\to\infty} \frac{1}{\tau} \int_{t}^{t+\tau} \phi(x,t)dt \tag{2.2}$$

where, by definition, the average of the fluctuating component is zero. For engineering applications it is assumed that τ is much greater than the time scale of the turbulent fluctuations.

For some flows the average flow may vary slowly with time when compared to the time of the turbulent fluctuations. For these flows the definition given in Eq. (2.2) may be replaced by

$$\phi(x,t) = \Phi(x,t) + \phi'(x,t) \tag{2.3}$$

where

$$\Phi(x,t) = \overline{\phi(x,t)} = \frac{1}{\tau} \int_{t}^{t+\tau} \phi(x,t)dt \tag{2.4}$$

and it is assumed that the time scale of the turbulent fluctuations is much less than τ , and that τ is much less than the time scale relative to the mean flow (i.e. period of oscillations for an oscillating flow or wave). With the preceding definition in mind, the following rules apply to Reynolds time averaging.

1. The time average of any constant value (scalar or vector) is equal to the value of the constant given by

$$A = \bar{a} = a \tag{2.5}$$

2. The time average of a time-averaged quantity is the same as the time average itself

$$\bar{\bar{a}} = A \tag{2.6}$$

3. Because time averaging involves a definite integral, it is a linear operator in that the average of a sum equals the sum of the averages as in

$$\overline{a + b} = \bar{a} + \bar{b} = A + B \tag{2.7}$$

4. The time average of a mean quantity times a fluctuating quantity is zero since it is similar to a constant times the average of a fluctuating quantity.

$$\overline{\phi' \Psi} = \overline{\phi'}\, \Psi = 0 \tag{2.8}$$

5. The time average of the product of two variable quantities is given by

$$\overline{\phi\psi} = \overline{(\Phi + \phi')(\Psi + \psi')} \tag{2.9}$$

which can be written as

$$\overline{\phi\psi} = \overline{\Phi\Psi + \Phi\psi' + \Psi\phi' + \phi'\psi'}$$

Here we use the fact that the two uppercase quantities are averages, and use Eq. (2.6) and (2.8) to set the second and third terms to zero. Also, realizing that the average of the product of two fluctuating quantities is not necessarily zero gives

$$\overline{\phi\psi} = \Phi\Psi + \overline{\phi'\psi'} \tag{2.10}$$

6. The time average of a spatial derivative is given by

$$\overline{\frac{\partial \phi}{\partial x_i}} = \frac{\partial \overline{(\Phi + \phi')}}{\partial x_i} = \overline{\frac{\partial \Phi}{\partial x_i}} + \overline{\frac{\partial \phi'}{\partial x_i}} = \frac{\partial \left(\overline{\Phi}\right)}{\partial x_i} + \frac{\partial \left(\overline{\phi'}\right)}{\partial x_i} \tag{2.11}$$

Since the average of any fluctuating component is zero, the last term on the right is zero. This indicates that the average of the spatial derivative of a variable is equal to the derivative of the average of the variable, or

$$\overline{\frac{\partial \phi}{\partial x_i}} = \frac{\overline{\partial \Phi}}{\partial x_i} = \frac{\partial \Phi}{\partial x_i} \tag{2.12}$$

7. The Reynolds average of a derivative with respect to time is zero for stationary turbulence. For non-stationary turbulence, the term given by $\overline{\frac{\partial \phi(x,t)}{\partial t}}$ is the average of the time-derivative of a scalar quantity ϕ.

Applying Eq. (2.12) to the term given above yields

$$\overline{\frac{\partial \phi(x,t)}{\partial t}} = \frac{\partial \overline{\phi(x,t)}}{\partial t} \tag{2.13}$$

Applying Reynolds decomposition to the scalar in Eq. (2.13) yields

$$\overline{\phi(x,t)} = \overline{\Phi(x,t)} + \overline{\phi'(x,t)} = \Phi(x,t) \tag{2.14}$$

because the time-average of a mean quantity yields the mean quantity, while the time-average of a fluctuating component is zero. Substituting into Eq. (2.13) yields

$$\overline{\frac{\partial \phi(x,t)}{\partial t}} = \frac{\partial \Phi(x,t)}{\partial t} \tag{2.15}$$

Which shows that the average of a time derivative is equal to the time derivative of the average.

3.0 AVERAGED TRANSPORT EQUATIONS

With the concept of Reynolds time-averaging and the rules defining its application, we turn our attention to the general conservation equations governing fluid flow and transport phenomena. First, we consider incompressible fluids with constant properties. The equation of continuity is given by

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{3.1}$$

In general, the equation of continuity is

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0 \tag{3.2}$$

Here and after, the tensor (index) notation is used such that repeated indices indicates summation (e.g. $u_i u_i = {u_1}^2 + {u_2}^2 + {u_3}^2$ in three dimensions). Taking the Reynolds time-average of Eq. (3.1) gives

$$\frac{\partial(\overline{U_i + u_i'})}{\partial x_i} = \frac{\partial U_i}{\partial x_i} + \frac{\partial \overline{u_i'}}{\partial x_i} = 0 \tag{3.3}$$

or

$$\frac{\partial U_i}{\partial x_i} = 0 \tag{3.4}$$

For incompressible flows, it follows from Eq. (3.3) that the divergence of the fluctuating velocity components is also zero and is given by

$$\frac{\partial u_i'}{\partial x_i} = 0 \tag{3.5}$$

In addition to the continuity equation, the other governing equations for incompressible flow are the momentum equation given by

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_i u_j) = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}[2\mu s_{ji}] + \rho g_i \tag{3.6}$$

where

$$s_{ji} = \frac{1}{2}\left(\frac{\partial}{\partial x_j}(u_i) + \frac{\partial}{\partial x_i}(u_j)\right) \tag{3.7}$$

and the simplified thermal energy equation which is given by

$$\frac{\partial(T)}{\partial t} + \frac{\partial}{\partial x_i}(u_i T) = \frac{\partial}{\partial x_i}\left(\alpha \frac{\partial T}{\partial x_i}\right) + S \tag{3.8}$$

when ρc_p is constant.

A generic scalar transport equation can also be included in this set of equations and is given by

$$\frac{\partial(\rho\phi)}{\partial t}+\frac{\partial}{\partial x_i}(\rho u_i\phi)=\frac{\partial}{\partial x_i}\left[\Gamma\left(\frac{\partial\phi}{\partial x_i}\right)\right]+S_c+S_p\phi \tag{3.9}$$

Taking the Reynolds time average for Eqs. (3.6), (3.8), and (3.9) gives the Reynolds time averaged momentum equation as

$$\frac{\partial}{\partial t}(\rho U_i)+\frac{\partial}{\partial x_j}(\rho U_i U_j)=-\frac{\partial P}{\partial x_i}+\frac{\partial}{\partial x_j}(\mu\overline{S_{ji}})-\frac{\partial}{\partial x_j}(\overline{\rho u_i' u_j'}) \tag{3.10}$$

the Reynolds time averaged thermal energy equation as

$$\frac{\partial(\overline{T})}{\partial t}+\frac{\partial}{\partial x_i}(u_i\overline{T})=\frac{\partial}{\partial x_i}\left(\alpha\frac{\partial\overline{T}}{\partial x_i}-\overline{u_i'T'}\right)+\overline{S} \tag{3.11}$$

and the Reynolds averaged scalar transport equation as

$$\frac{\partial(\rho\Phi)}{\partial t}+\frac{\partial}{\partial x_i}(\rho U_i\Phi)=\frac{\partial}{\partial x_i}\left(\Gamma\frac{\partial\Phi}{\partial x_i}-\rho\overline{u_i'\phi'}\right)+\overline{S_c}+\overline{S_p}\Phi \tag{3.12}$$

The full derivations for Eqs. (3.10), (3.11), and (3.12) are given in Appendix C.

The Reynolds averaged continuity equation is basically the same as the unaveraged equation in that there are no new terms. However, additional flux terms arise in the momentum and scalar equations. The extra terms in the momentum equation are given by

$$\tau_{ij}' = -\rho\overline{u_i' u_j'} \tag{3.13}$$

which are known as the Reynolds stresses, and the extra terms in both the scalar transport and energy equation take the form

$$q_i = -\rho\overline{u_i'\phi'} \tag{3.14}$$

which are referred to as Reynolds (or turbulent) fluxes. These additional fluxes arise from the convective transport due to turbulent fluctuations. When an equation is time-averaged, the influence of the fluctuations over the averaging time period is included via these additional flux terms. In the course of Reynolds averaging of the conservation equations, these additional fluxes have been generated but no new equations were obtained to account for these new unknowns. Turbulence models provide closure to Eqs. (3.10), (3.11), and (3.12) by providing models for the fluxes given by Eqs. (3.13) and (3.14).

4.0 DERIVATION OF THE REYNOLDS STRESS EQUATIONS

In the Reynolds averaged momentum equation, as was given in Eq. (3.10), the extra terms, which are commonly called the Reynolds stresses, can be expressed as a tensor

$$\tau_{ij} = -\overline{\rho u_i' u_j'} \tag{4.1}$$

where the first index indicates the plane along which the stress acts, and the second gives the coordinate direction. Here the primes indicate that this average stress is obtained from the turbulent fluctuation part of the instantaneous velocity, which is given by

$$u_i = U_i + u_i' \tag{4.2}$$

To find an equation for the Reynolds stresses, first consider the Navier-Stokes equations for incompressible fluids, given by

$$\frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} = \frac{-1}{\rho}\frac{\partial p}{\partial x_i} + \frac{1}{\rho}\frac{\partial}{\partial x_k}\left(\mu \frac{\partial u_i}{\partial x_k}\right) \tag{4.3}$$

or

$$\frac{\partial u_j}{\partial t} + u_k \frac{\partial u_j}{\partial x_k} = \frac{-1}{\rho}\frac{\partial p}{\partial x_j} + \frac{1}{\rho}\frac{\partial}{\partial x_k}\left(\mu \frac{\partial u_j}{\partial x_k}\right) \tag{4.4}$$

Note that these equations are written in a non-conservative form using the equation of continuity. With the eventual goal of finding the material, or substantial, time derivative of the Reynolds stress equations, realize that the material time derivative of the non-averaged terms can be written as

$$\frac{D(\rho u_i' u_j')}{Dt} = u_i' \frac{D(\rho u_j')}{Dt} + u_j' \frac{D(\rho u_i')}{Dt} \tag{4.5}$$

using the chain rule, and also that the Navier Stokes equations written in terms of u_i contain the fluctuating terms u_i'. With this in mind, intuitively multiply Eq. (4.3) by u_j' and multiply Eq. (4.4) by u_i' yielding

$$u_j' \frac{\partial u_i}{\partial t} + u_j' u_k \frac{\partial u_i}{\partial x_k} = u_j' \frac{-1}{\rho}\frac{\partial p}{\partial x_i} + u_j' \frac{1}{\rho}\frac{\partial}{\partial x_k}\left(\mu \frac{\partial u_i}{\partial x_k}\right) \tag{4.6}$$

and

$$u_i' \frac{\partial u_j}{\partial t} + u_i' u_k \frac{\partial u_j}{\partial x_k} = u_i' \frac{-1}{\rho}\frac{\partial p}{\partial x_j} + u_i' \frac{1}{\rho}\frac{\partial}{\partial x_k}\left(\mu \frac{\partial u_j}{\partial x_k}\right) \tag{4.7}$$

To obtain the derivative as given in Eq. (4.5), take the time average of Eqs. (4.6) and (4.7) and add them together to give

$$\underbrace{\overline{u_j'\frac{\partial u_i}{\partial t}+u_i'\frac{\partial u_j}{\partial t}}}_{\text{(I)}}+\underbrace{\overline{u_j'u_k\frac{\partial u_i}{\partial x_k}+u_i'u_k\frac{\partial u_j}{\partial x_k}}}_{\text{(II)}}=\underbrace{\overline{u_j'\frac{-1}{\rho}\frac{\partial p}{\partial x_i}+u_i'\frac{-1}{\rho}\frac{\partial p}{\partial x_j}}}_{\text{(III)}}$$
$$+\underbrace{\overline{u_j'\frac{1}{\rho}\frac{\partial}{\partial x_k}\left(\mu\frac{\partial u_i}{\partial x_k}\right)+u_i'\frac{1}{\rho}\frac{\partial}{\partial x_k}\left(\mu\frac{\partial u_j}{\partial x_k}\right)}}_{\text{(IV)}} \tag{4.8}$$

where the transient, convective, pressure, and viscous stress terms have been grouped together. To put Eq. (4.8) in the form of the Reynolds equation, each set of terms is considered separately.

First consider the unsteady term given by

$$\text{(I)} = \overline{u_j'\frac{\partial u_i}{\partial t}+u_i'\frac{\partial u_j}{\partial t}} \tag{4.9}$$

which can be re-written as

$$\text{(I)} = \overline{u_j'\frac{\partial(U_i+u_i')}{\partial t}+u_i'\frac{\partial(U_j+u_j')}{\partial t}}$$

or

$$\text{(I)} = \overline{u_j'\frac{\partial(U_i)}{\partial t}}+\overline{u_j'\frac{\partial(u_i')}{\partial t}}+\overline{u_i'\frac{\partial(U_j)}{\partial t}}+\overline{u_i'\frac{\partial(u_j')}{\partial t}}$$

Applying the rule that the average of a fluctuating quantity times a mean value is zero, the first and third terms disappear. Using the chain rule, and the fact that the average of a time derivative is equal to the time derivative of the average, gives the final form as

$$\text{(I)} = \overline{u_j'\frac{\partial u_i}{\partial t}+u_i'\frac{\partial u_j}{\partial t}}=\frac{\partial(\overline{u_j'u_i'})}{\partial t} \tag{4.10}$$

Second, consider the convective term in Eq. (4.8) given by

$$\text{(II)} = \overline{u_j'u_k\frac{\partial u_i}{\partial x_k}+u_i'u_k\frac{\partial u_j}{\partial x_k}} \tag{4.11}$$

Substituting for u_k, u_i, and u_j, the sum of their mean and fluctuating parts gives

$$(II) = \overline{u_j'(U_k + u'_k)\frac{\partial(U_i + u_i')}{\partial x_k}} + \overline{u_i'(U_k + u'_k)\frac{\partial(U_j + u_j')}{\partial x_k}}$$

which can be re-written as

$$(II) = \begin{aligned} &\overline{u_j' U_k \frac{\partial U_i}{\partial x_k}} + \overline{u_j' U_k \frac{\partial u_i'}{\partial x_k}} + \overline{u_j' u'_k \frac{\partial U_i}{\partial x_k}} + \overline{u_j' u'_k \frac{\partial u_i'}{\partial x_k}} + \overline{u_i' U_k \frac{\partial U_j}{\partial x_k}} \\ &+ \overline{u_i' U_k \frac{\partial u_j'}{\partial x_k}} + \overline{u_i' u'_k \frac{\partial U_j}{\partial x_k}} + \overline{u_i' u'_k \frac{\partial u_j'}{\partial x_k}} \end{aligned} \tag{4.12}$$

Using the rule for the mean of an already averaged quantity times a fluctuating variable (See Eq. (2.8)), the first and fifth terms in Eq. (4.12) are zero. The second and sixth term may be combined using the chain rule, and the fourth and eight terms may also be combined using the chain rule, resulting in

$$(II) = \overline{U_k \frac{\partial(u_i' u_j')}{\partial x_k}} + \overline{u_j' u'_k \frac{\partial U_i}{\partial x_k}} + \overline{u'_k \frac{\partial(u_i' u_j')}{\partial x_k}} + \overline{u_i' u'_k \frac{\partial U_j}{\partial x_k}} \tag{4.13}$$

Now for Eq. (4.13) consider the first, second and fourth terms. In the first, U_k may be treated as a constant and it may be removed from under the average sign; also the rule for a spatial derivative may be used. In the second and fourth terms the derivative of the mean velocities may be treated as a constant. Using these, Eq. (4.13) may be written as

$$(II) = U_k \frac{\partial(\overline{u_i' u_j'})}{\partial x_k} + \overline{u_j' u'_k} \frac{\partial U_i}{\partial x_k} + \overline{u'_k \frac{\partial(u_i' u_j')}{\partial x_k}} + \overline{u_i' u'_k} \frac{\partial U_j}{\partial x_k} \tag{4.14}$$

The third term in Eq. (4.14) can be modified by realizing that from the continuity equation, Eq. (3.5), it follows

$$\frac{\partial u_k'}{\partial x_k'} = \overline{u_i' u_j' \frac{\partial u_k'}{\partial x_k'}} = 0 \tag{4.15}$$

Using Eq. (4.15) and the chain rule, Eq (4.14) may be written in its final form as

$$(II) = U_k \frac{\partial(\overline{u_i' u_j'})}{\partial x_k} + \overline{u_j' u'_k} \frac{\partial U_i}{\partial x_k} + \frac{\partial(\overline{u_i' u_j' u'_k})}{\partial x_k} + \overline{u_i' u'_k} \frac{\partial U_j}{\partial x_k} \tag{4.16}$$

Third, consider the pressure term in Eq. (4.8) given as

$$(\text{III}) = \overline{u_j' \frac{-1}{\rho}\frac{\partial p}{\partial x_i} + u_i' \frac{-1}{\rho}\frac{\partial p}{\partial x_j}} \tag{4.17}$$

which can be re-written as

$$(\text{III}) = \overline{u_j' \frac{-1}{\rho}\frac{\partial P}{\partial x_i}} + \overline{u_j' \frac{-1}{\rho}\frac{\partial p'}{\partial x_i}} + \overline{u_i' \frac{-1}{\rho}\frac{\partial P}{\partial x_j}} + \overline{u_i' \frac{-1}{\rho}\frac{\partial p'}{\partial x_j}}$$

Here the first and third terms are zero. The second and fourth terms can be re-written using the chain rule as

$$-\overline{u_i' \frac{\partial p'}{\partial x_j}} = \overline{p' \frac{\partial u_i'}{\partial x_j}} - \overline{\frac{\partial (p' u_i')}{\partial x_j}} \tag{4.18}$$

and

$$-\overline{u_j' \frac{\partial p'}{\partial x_i}} = \overline{p' \frac{\partial u_j'}{\partial x_i}} - \overline{\frac{\partial (p' u_j')}{\partial x_i}} \tag{4.19}$$

Using the Kronecker delta function, δ_{ij} , the pressure expressions may be written in their final form as

$$(\text{III}) = \overline{\frac{1}{\rho} p' \left[\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i} \right]} - \frac{1}{\rho}\frac{\partial}{\partial x_k}\left[(\overline{p' u_i'})\delta_{jk} + (\overline{p' u_j'})\delta_{ik} \right] \tag{4.20}$$

Finally, consider the viscous stress term in Eq. (4.8) given by

$$(\text{IV}) = \overline{u_j' \frac{1}{\rho}\frac{\partial}{\partial x_k}\left(\mu \frac{\partial u_i}{\partial x_k} \right) + u_i' \frac{1}{\rho}\frac{\partial}{\partial x_k}\left(\mu \frac{\partial u_j}{\partial x_k} \right)} \tag{4.21}$$

Canceling the average U_i terms, as has been done in previous derivations, gives

$$(\text{IV}) = \frac{1}{\rho}\overline{u_j' \frac{\partial}{\partial x_k}\mu\left(\frac{\partial u_i'}{\partial x_k} \right)} + \frac{1}{\rho}\overline{u_i' \frac{\partial}{\partial x_k}\mu\left(\frac{\partial u_j'}{\partial x_k} \right)}$$

and applying the chain rule twice yields

$$(\text{IV}) = \frac{1}{\rho}\overline{\frac{\partial}{\partial x_k}\mu u_j'\left(\frac{\partial u_i'}{\partial x_k} \right)} + \frac{1}{\rho}\overline{\frac{\partial}{\partial x_k}\mu u_i'\left(\frac{\partial u_j'}{\partial x_k} \right)} - 2\frac{1}{\rho}\overline{\mu \frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}} \tag{4.22}$$

and

$$(IV) = \frac{1}{\rho}\frac{\partial}{\partial x_k}\mu\left(\frac{\partial\overline{u_i'u_j'}}{\partial x_k}\right) - 2\frac{1}{\rho}\mu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}} \tag{4.23}$$

which completes the manipulation of the viscous term.

Collecting the transient, convective, pressure, and viscous terms together gives

$$\begin{aligned}&\frac{\partial(\overline{u_j'u_i'})}{\partial t} + U_k\frac{\partial(\overline{u_i'u_j'})}{\partial x_k} + \overline{u_j'u_k'}\frac{\partial U_i}{\partial x_k} + \frac{\partial(\overline{u_i'u_j'u_k'})}{\partial x_k} + \overline{u_i'u_k'}\frac{\partial U_j}{\partial x_k} = \\ &\quad\frac{1}{\rho}\overline{p'\left[\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right]} - \frac{1}{\rho}\frac{\partial}{\partial x_k}\left[(\overline{p'u_j'})\delta_{ik} + (\overline{p'u_j'})\delta_{ik}\right] + \\ &\quad\frac{1}{\rho}\frac{\partial}{\partial x_k}\mu\left(\frac{\partial\overline{u_i'u_j'}}{\partial x_k}\right) - 2\frac{1}{\rho}\mu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}}\end{aligned} \tag{4.24}$$

which is equivalent to Eq. (4.8).

Multiplying Eq. (4.24) by -ρ and re-arranging terms gives

$$\begin{aligned}&-\frac{\partial(\rho\overline{u_j'u_i'})}{\partial t} - U_k\frac{\partial(\overline{\rho u_i'u_j'})}{\partial x_k} = \overline{\rho u_j'u_k'}\frac{\partial U_i}{\partial x_k} + \overline{\rho u_i'u_k'}\frac{\partial U_j}{\partial x_k} + \frac{\partial(\overline{\rho u_i'u_j'u_k'})}{\partial x_k} \\ &+2\mu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}} - \overline{p'\left[\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right]} + \frac{\partial}{\partial x_k}\left[(\overline{p'u_i'})\delta_{jk} + (\overline{p'u_j'})\delta_{ik}\right] - \frac{\partial}{\partial x_k}\mu\left(\frac{\partial\overline{u_i'u_j'}}{\partial x_k}\right)\end{aligned} \tag{4.25}$$

where the Reynolds Stress terms may be seen in the derivatives on the LHS.

Applying the definition given by Eq. (4.1) the Reynolds stress equation may now be written in its most recognizable form as

$$\frac{\partial\tau_{ij})}{\partial t} + U_k\frac{\partial(\tau_{ij})}{\partial x_k} = -\tau_{jk}\frac{\partial U_i}{\partial x_k} - \tau_{ik}\frac{\partial U_j}{\partial x_k} + \varepsilon_{ij} - \Pi_{ij} + \frac{\partial}{\partial x_k}\left(\upsilon\frac{\partial(\tau_{ij})}{\partial x_k} + C_{ijk}\right) \tag{4.26}$$

where

$$\varepsilon_{ij} = \overline{2\mu\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}}$$

$$\Pi_{ij} = \overline{p'\left[\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right]}$$

and

$$C_{ijk} = \left[(\overline{\rho u_i' u_j' u_k'}) + (\overline{p' u_i'})\delta_{jk} + (\overline{p' u_j'})\delta_{ik}\right]$$

As has been seen from the momentum equation, when turbulent flows are considered the averaged flux of momentum due to the turbulent fluctuations must be taken into account. Though the influence of these fluctuations is known, no direct means of calculating them exists. With the derivation of the Reynolds stress equation, the influences on the stress term can be identified, but with the derivation new terms (higher order correlations) are generated which of themselves are unknown. While the Reynolds equations provide insight into the nature of the turbulent stresses, the engineer must find some way to close the equations before they can be used. Finding closure equations for calculating these extra terms is the basis of turbulence modeling.

5.0 THE EDDY VISCOSITY/DIFFUSIVITY CONCEPT

The oldest proposal for modeling the turbulent or Reynolds stresses turns out to be a significant part of most turbulence models of practical use today. The main idea behind this model is Boussinesq's eddy-viscosity concept, which assumes that, in analogy to the viscous stresses in laminar flows, the turbulent stresses are proportional to the mean velocity gradient. This approach stems from treating turbulent eddies in a similar way that molecules are treated and analyzed in kinetic theory. Here eddies replace molecules as carriers of thermal energy and momentum. The eddy viscosity concept is best considered in light of molecular transport of momentum. For a purely shearing flow, the average stress due to molecular motion acting on a plane can be given by

$$\tau_{xy} = -\rho\overline{u''v''} \tag{5.1}$$

where u'' and v'' denote molecular velocities. A typical derivation of this shear stress can be accomplished by considering the shear flow illustrated in Figure 5.1 (Wilcox, 1993). Here the stress exerted in the horizontal direction by the fluid particles, or molecules, at point B on a plane at point A, whose normal is in the y-direction, can be given by

$$\tau_{xy} = \frac{\dot{m}(u_B - u_A)}{Area} \tag{5.2}$$

where the area in Eq. (5.2) is that of the vertical plane at A.

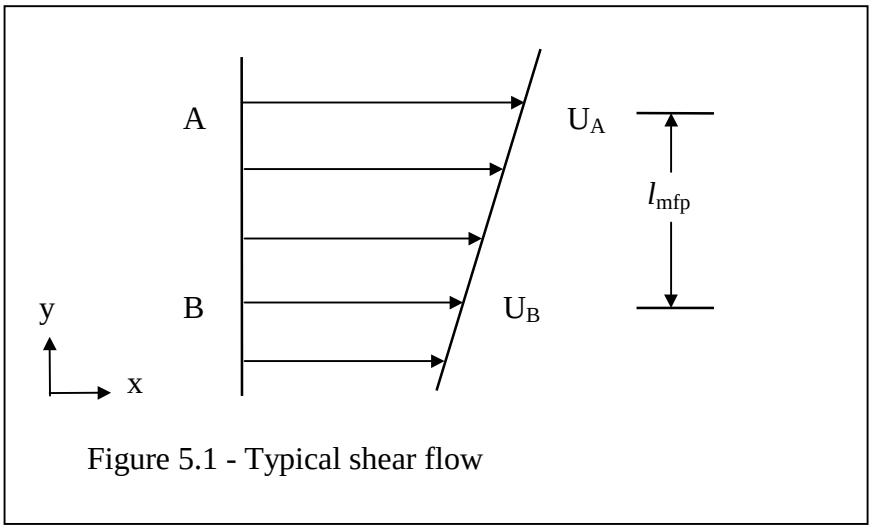


Figure 5.1 - Typical shear flow

For a perfect gas, the average vertical velocity can be taken to be the thermal velocity, v_{th}. A typical particle will move with this velocity along its mean free path, l_{mfp}, before colliding with another molecule and transferring its momentum.

If a molecule is considered to move along its mean free path along the vertical distance from B to A, then the shear stress on the lower side of plane A may be written as

$$\tau_{xy} = -C\rho v_{th} l_{mfp} \frac{dU}{dy} \tag{5.3}$$

where C is a proportionality constant. From kinetic theory, for a perfect gas this constant can be shown to be 0.5. This then allows the viscosity of a perfect gas to be defined by

$$\mu = \frac{1}{2}\rho v_{th} l_{mfp} \tag{5.4}$$

Now the shear stress given in Eq. (5.1) may be written as

$$\tau_{xy} = -\rho\overline{u'' v''} = \mu \frac{dU}{dy} \tag{5.5}$$

Realize that in Eq. (5.3) the Taylor series defining the velocity has been truncated after the first term. This approximation requires that (Wilcox, 1993)

$$l_{mfp} << \frac{dU}{dy}\left(\frac{d^2U}{dy^2}\right)^{-1} \tag{5.6}$$

This approximation also assumes that the horizontal velocity remains essentially constant at any plane. Since molecules will be transferring horizontal momentum to and from a plane, the velocity at any plane may be considered effectively constant only if the molecules experience many collisions on the time scale relative to the mean flow. This stipulation requires

$$l_{mfp} << v_{th}\left(\frac{dU}{dy}\right)^{-1} \tag{5.7}$$

Both of these stipulations are satisfied for virtually all flows of engineering interest, given that v_{th} is on the order of the speed of sound in the fluid and l_{mfp} is relatively small. They are mentioned here since anyone performing turbulence modeling and trying to mimic eddy transport, by analogy to molecular transport, must at least be aware of these stipulations.

Using a concept similar to the molecular viscosity for molecular stresses, the concept of the eddy viscosity may be used to model the Reynolds stresses. For general flow situations the eddy viscosity model may be written as

$$-\overline{u_i' u_j'} = \nu_t\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) - \frac{2}{3}k\delta_{ij} \tag{5.8}$$

where ν_t is the turbulent or eddy viscosity, and k is the turbulent kinetic energy. In contrast to the molecular viscosity, the turbulent viscosity is not a fluid property but depends strongly on the state of turbulence; ν_t may vary significantly from one point in the flow to another and also from flow to flow. The main problem in this concept is to determine the distribution of ν_t.

Inclusion of the second part of the eddy viscosity expression assures that the sum of the normal stresses is equal to 2k, which is required by definition of k. The normal stresses act like pressure forces, and thus the second part constitutes pressure. Eq. (5.8) is used to eliminate $\overline{u_i' u_j'}$ in the momentum equation. The second part can be absorbed into the pressure-gradient term so that, in effect, the static pressure is replaced as an unknown quantity by the modified pressure given by

$$p^* = p + \frac{2}{3}\bar{\rho}k \,. \tag{5.9}$$

Therefore, the appearance of k in Eq. (5.8) does not necessarily require the determination of k to make use of the eddy viscosity formulation; the main objective is then to determine the eddy viscosity.

In direct analogy to the turbulent momentum transport, the turbulent heat or mass transport is often assumed to be related to the gradient of the transported quantity, with eddies again replacing molecules as the carriers. With this concept, the Reynolds flux terms may be expressed using

$$-\overline{u_i' \phi'} = \Gamma_t \frac{\partial \bar{\phi}}{\partial x_i} \tag{5.10}$$

Here Γ_t is the turbulent diffusivity of heat or mass and has units equivalent to the thermal diffusivity of m^2/s. Like the eddy viscosity, Γ_t is not a fluid property but depends on the state of the turbulence. The eddy diffusivity is usually related to the turbulent eddy viscosity via

$$\Gamma_t = \frac{\nu_t}{\sigma} \tag{5.11}$$

where σ is the turbulent Prandtl or Schmidt number, which is a constant approximately equal to one.

As will be shown in later sections, the primary goal of many turbulence models is to find some prescription for the eddy viscosity to model the Reynolds stresses. These may range from the relatively simple algebraic models, to the more complex models such as the k-ε model, where two additional transport equations are solved in addition to the mean flow equations.

6.0 ALGEBRAIC TURULENCE MODELS: ZERO-EQUATION MODELS

The simplest turbulence models, also referred to as zero equation models, use a Boussinesq eddy viscosity approach to calculate the Reynolds stress. In direct analogy to the molecular transport of momentum, Prandtl's mixing length model assumes that turbulent eddies cling together and maintain their momentum for a distance, l_{mix}, and are propelled by some turbulent velocity, v_{mix}. With these assumptions, the Reynolds stress terms are modeled by

$$-\rho\overline{u'v'} = \rho v_{mix} l_{mix} \frac{dU}{dy} \tag{6.1}$$

for a two-dimensional shear flow as is shown in Figure 5.1. This model further postulates that the mixing velocity, v_{mix}, is of the same order of magnitude as the (horizontal) fluctuating velocities of the eddies, which can be supported through experimental results for a wide range of turbulent flows. With this assumption

$$v_{mix} \approx |u'| \approx |v'| \approx |w'| \approx l_{mix} \left| \frac{dU}{dy} \right| \tag{6.2}$$

or, in terms of the eddy, or turbulent, viscosity for a shear flow

$$\mu_t = \rho (l_{mix})^2 \left| \frac{dU}{dy} \right| \tag{6.3}$$

which can be implied from Eq. (6.1). This definition for the eddy viscosity can also be implied on dimensional grounds. With these definitions in mind, the objective of most algebraic models is to find some prescription for the turbulent mixing length, in order to provide closure to Eqs. (6.1) and (6.3).

The idea that a turbulent mixing length can be used in a similar way that the molecular mean free path is used to calculate the viscosity for a perfect gas provides a reasonable approach to calculating the eddy viscosity. However, this approach must be examined in light of the same assumptions made for the analysis of the molecular viscosity in Section 5.0. The appropriateness of this model can be questioned by considering the two requirements that were considered for the case of molecular mixing (Tennekes and Lumley, 1983), namely

$$l_{mix} << \frac{dU}{dy} \left(\frac{d^2U}{dy^2} \right)^{-1} \tag{6.4}$$

and

$$l_{mix} << v_{th} \left(\frac{dU}{dy} \right)^{-1} \tag{6.5}$$

It has been shown experimentally, that close to a wall l_{mix} is proportional to the normal distance from the wall. Also, near a solid surface the velocity gradient varies inversely with y, as deduced from the law of the wall for a turbulent velocity profile. Considering these facts in light of Eq. (6.4), the mixing length model does not give solid justification for the first order Taylor series truncation that is used in the molecular viscosity equation. The fact that the average time for collisions

$$\frac{v_{mix}}{l_{mix}} \approx \left|\frac{dU}{dy}\right| \tag{6.6}$$

is large, also guarantees that the momentum of an eddy will undergo changes due to other collisions before it travels the full distance of its mixing length (Tennekes and Lumley, 1983). This fact is not taken into consideration since the molecular length of travel is defined as the undisturbed distance that a molecule travels before collision. These comparisons show that the mixing length model does not have a strong theoretical background as it is usually perceived. Despite the shortcomings, however, in practice it can actually be calibrated to give good engineering and trend predictions.

Free shear flows

A flow is termed "free" if it can be considered to be unbounded by any solid surface. Since walls and boundary conditions at solid surfaces complicate turbulence models, two-dimensional, free shear flows form a good set of cases to study the applicability of a turbulence model. This stems from the fact that only one significant turbulent stress exists in two-dimensional flows. Three flows that can be considered are the far wake, the mixing layer, and the jet. A wake forms downstream of any object placed in the path of a flowing fluid, a mixing layer occurs between two parallel streams moving at different speeds, and a jet occurs when a fluid is injected into a second quiescent fluid. A far wake is shown in Figure 6.1.

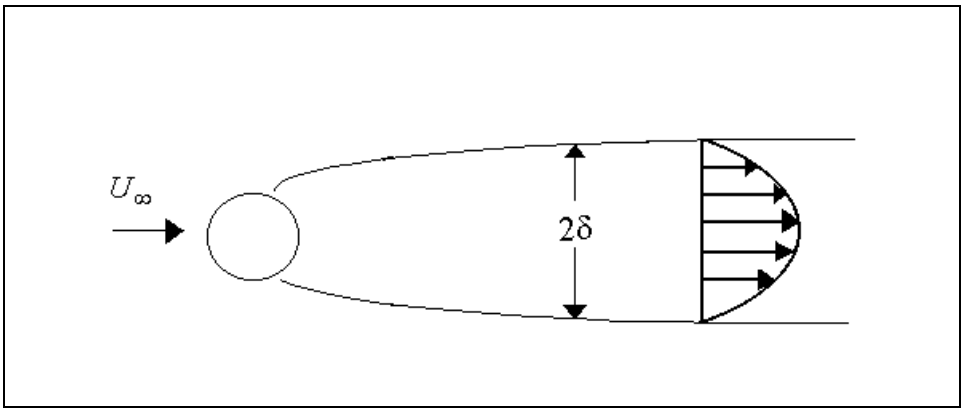


Figure 6.1 - Far Wake

For each of these cases the assumption can be made that the mixing length is some constant times the local layer width, δ, or

$$l_{mix} = \alpha\, \delta(x) \qquad (6.7)$$

This constant must be determined through some empirical input, as well as the governing equations. For all three flows the standard boundary layer (or thin shear layer) equations can be used. Two assumptions governing the eventual solution of the problem are that the pressure is constant and that molecular transport of momentum is negligible compared to the turbulent transport. From the similarity analysis and numerical solutions that are obtained for this class of flows, the values governing the mixing length are given in Table 6.1.

Table 6.1 - Mixing length constants for free shear flows (Wilcox, 1993)

Flow Type:	Far Wake	Plane Jet	Radial Jet	Plane Mixing Layer
$\frac{l_{mix}}{\delta}$	0.180	0.098	0.080	0.071

For each of the free shear cases, the analytical prediction of the velocity gives a sharp turbulent/non-turbulent interface. These interfaces, while existing in reality, are generally characterized by time fluctuations and have smooth properties when averaged, not sharp interfaces. This unphysical prediction of the mixing length model is characteristic of many turbulence models where a region has a turbulent/non-turbulent interface. The intermittent, transient nature of turbulence is therefore not accounted for at the interface in these solutions.

Wall bounded flows

In free shear flows the mixing length was shown to be constant across a layer and proportional to the width of the layer. For flows near a solid surface, a different prescription must be used, notably since the mixing length can not physically extend beyond the boundary established by the solid surface.

For flow near a flat wall or plate, using the experimental fact that momentum changes are negligible and the shear stress is approximately constant, it can be shown (Wilcox, 1993), using boundary layer theory, that

$$1.0 = (1.0 + \frac{\nu_t}{\nu})\frac{du^+}{dy^+} \tag{6.8}$$

where $u^+ = u/u_*$, $y^+ = yu_*/\nu$ ($u_* = \sqrt{\tau_w/\rho}$, τ_w = wall shear stress) are the familiar dimensionless velocity and vertical distance (also known as wall variables) as used in the law of the wall. In the viscous sublayer, where viscous forces dominate turbulent fluctuations, Eq. (6.8) becomes (see Appendix B)

$$u^+ = y^+ \tag{6.9}$$

For wall flows, based on experimental observations, it is taken that the mixing length is proportional to the distance from the surface, or in terms of the eddy viscosity as given by Eq. (6.3)

$$\nu_t = \kappa^2 y^2 \left|\frac{dU}{dy}\right| \tag{6.10}$$

Also, in the fully turbulent zone, effects of molecular viscosity are low compared with ν_t. This allows Eq. (6.8) to be written as

$$1.0 = \left(\kappa y^+ \frac{du^+}{dy^+}\right)^2 \tag{6.11}$$

which can be integrated (with the assumption that the turbulent shear stress is constant) to give

$$u^+ = \frac{1}{\kappa}\ln(y^+) + C \tag{6.12}$$

and shows that the mixing length concept is consistent with the experimental law of the wall if the mixing length is taken to vary in proportion to the distance normal from the surface. Since Eq. (6.12) is a good estimate only in the log layer, and not close to the wall in the viscous sublayer or in the outer layer, several modifications to the approximation for l_{mix} have been devised. Three of the better known modifications are the Van-Driest, Cebeci-Smith (1974), and Baldwin-Lomax (1978) models.

Van-Driest Model

To make l_{mix} approach zero more quickly in the viscous sublayer, this model specifies

$$l_{mix} = \kappa y\left[1 - \exp\left(y^+ / A_o^+\right)\right] \tag{6.13}$$

where $A_o^+ = 26$. This model is based primarily on experimental evidence, but it is also based on the idea that the Reynolds stress approaches zero near the wall in proportion to y^3. This has been shown to be the case in DNS simulations.

Cebeci-Smith Model (1974)

For wall boundary layers, this model provides a complete specification of the mixing length and eddy viscosity over the entire range of the viscous sublayer, log layer, and defect or outer layer. Here the eddy viscosity is given in the inner region by

$$\nu_t = (l_{mix})^2\left[\left(\frac{\partial U}{\partial y}\right)^2 + \left[\left(\frac{\partial V}{\partial x}\right)^2\right]\right]^{1/2} \qquad y \le y_m \tag{6.14}$$

with

$$l_{mix} = \kappa y\left[1 - \exp(-y^+ / A^+)\right] \tag{6.15}$$

$$A^+ = 26\left[1 + y\frac{dP/dx}{\rho u_\tau^2}\right]^{-1/2} \tag{6.16}$$

The outer layer viscosity is given by

$$\nu_t = \alpha U_e \delta_v^* F_{Kleb}(y;\delta) \qquad y > y_m \tag{6.17}$$

with

$$\alpha = 0.0168$$

and

$$F_{Kleb}(y;\delta) = \left[1 + 5.5\left(\frac{y}{\delta}\right)^6\right]^{-1} \tag{6.18}$$

Here y_m is the smallest value of y for which the inner and outer eddy viscosities are equal, δ is the boundary layer thickness, and δ_v^* is the velocity or displacement thickness for incompressible

flow. F_{Kleb} is Klebanoff's intermittency function, which was proposed to account for the fact that in approaching the free stream from within a boundary layer, the flow is sometimes laminar and sometimes turbulent (i.e. intermittent).

Some general improvements of this model are that it includes boundary layers with pressure gradients by modifying the value of A^+ in Van-Dreist's mixing length formula. This model is also very popular because of its ease of implementation in a computer program; here the main problem is the computation of δ and δ_v^*. It is also worth noting that, in general, the point where $y = y_m$ and the Reynolds stress is a maximum occurs well inside the log layer.

Baldwin-Lomax Model (1978)

The Baldwin-Lomax model (Baldwin and Lomax, 1978) was formulated to be used in applications where the boundary layer thickness, δ, and displacement thickness, δ_v^*, are not easily determined. As in the Cebeci-Smith model (Cebeci and Smith, 1974), this model also uses an inner and an outer layer eddy viscosity. The inner viscosity is given by

$$\nu_t = (l_{mix})^2 |\omega| \qquad y \le y_m \tag{6.19}$$

where the symbol, ω ,is the magnitude of the vorticity vector for three dimensional flows. Here the vorticity provides a more general parameter for determining the magnitude of the mixing velocity than the velocity gradient as is given in Eq. (6.2). The mixing length is calculated from the Van-Driest equation:

$$l_{mix} = \kappa y \left[1 - \exp(-y^+ / A_o^+)\right] \tag{6.20}$$

The outer viscosity is given by

$$\nu_t = \rho \alpha C_{cp} F_{Wake} F_{Kleb}(y, y_{max} / C_{Kleb}) \qquad y > y_m \tag{6.21}$$

where

$$F_{Wake} = \min\left[y_{max} F_{max}, C_{wk} y_{max} U^2_{dif} / F_{max}\right] \tag{6.22}$$

$$F_{max} = \frac{1}{\kappa}\left[\max_y (l_{mix} |\omega|)\right] \tag{6.23}$$

This model avoids the need to locate the boundary layer edge by calculating the outer layer length scale in terms of the vorticity instead of the displacement or thickness. By replacing $U_e \delta^*_v$ in the Cebici-Smith model by $C_{cp} F_{Wake}$

$$\text{if} \quad F_{Wake} = y_{max} F_{max} \quad \text{then} \quad \delta_v^* = \frac{y_{mxz}^2 \omega}{U_e} \tag{6.24}$$

$$\text{if} \quad F_{Wake} = C_{wk} y_{max} U_{dif}^2 / F_{max} \quad \text{then} \quad \delta = \frac{U_{dif}}{|\omega|} \tag{6.25}$$

Here y_{max} is the value of y at which l_{mix} of ω achieves its maximum value, and U_{dif} is the maximum value of the velocity for boundary layers. The constants in this model are given by

$\kappa = 0.40$	$\alpha = 0.0168$	$A^+_o = 26$
$C_{cp} = 1.6$	$C_{Kleb} = 0.3$	$C_{wk} = 1$

Both the Cebici-Smith (1974) and the Baldwin-Lomax (1978) models yield reasonable results for such applications as fully developed pipe or channel flow and boundary layer flow. One interesting observation for pipe and channel flow is that subtle differences in a model's prediction for Reynolds stress can lead to much larger differences in velocity profile predictions. This is a common accuracy dilemma with many turbulence models. Both models have been fine tuned for boundary layer flow and therefore provide good agreement with experimental data for reasonable pressure gradients and mild adverse pressure gradients.

For separated flows, algebraic models generally perform poorly due to their inability to account for flow history effects. The effects of flow history account for the fact that the turbulent eddies in a zone of separation occur on a time scale independent of the mean strain rate. Several modifications to the algebraic models have been proposed to improve their prediction of separated flows, most notably a model proposed by Johnson and King (1985). This model solves an extra ordinary differential equation, in addition to the Reynolds equations, to satisfy a "non-equilibrium" parameter that determines the maximum Reynolds stress.

In general, zero-equation algebraic models perform reasonably well for free shear flows; however, the mixing length specification for these flows is highly problem dependent. For wall bounded flows and boundary layer flows the Cebeci-Smith (1974) and Baldwin-Lomax (1978) models give good engineering predictions when compared to experimental values of the friction coefficients and velocity profiles. This is partially due to the modifications these models have received to match experimental data, especially for boundary layer flows. Neither model is reliable for predicting extraordinarily complex flows or separated flows, however they have historically provided sound engineering solutions for problems within their range of applicability. The Johnson-King (1985) model mentioned above gives better prediction for separated flows by solving an extra differential equation. This model is referred to as a "half-equation" model due to the fact that the additional equation solved is an ordinary differential equation. As will be shown in the next section on one and two equation models, a major classification of turbulence modeling is based on the solutions of additional partial differential equations that determine characteristic turbulent velocity and length scales.

7.0 ONE-EQUATION TURBULENCE MODELS

As an alternative to the algebraic or mixing length model, one-equation models have been developed in an attempt to improve turbulent flow predictions by solving one additional transport equation. While several different turbulent scales have been used as the variable in the extra transport equation, the most popular method is to solve for the characteristic turbulent velocity scale proportional to the square root of the specific kinetic energy of the turbulent fluctuations. This quantity is usually just referred to as the turbulence kinetic energy and is denoted by *k*. The Reynolds stresses are then related to this scale in a similar manner in which τ_{ij} was related to v_{mix} and l_{mix} in algebraic models. Since the modeled *k* equation is generally the basis for all one and two equation models, the exact differential equation for the turbulence kinetic energy, and the physical interpretation of the terms in the turbulence kinetic energy equation, will be considered first. Then, with some understanding of the exact equation, the methods used to model the *k* equation will be considered.

Exact Turbulent Kinetic Energy Equation

As has been mentioned, an obvious choice for the characteristic velocity scale in a turbulent flow is the square root of the kinetic energy of the turbulent fluctuations. Using the primed quantities to denote the velocity fluctuations, the Reynolds averaged kinetic energy of the turbulent eddies can be written on a per unit mass basis as

$$k = \frac{1}{2}\left(\overline{u'u'} + \overline{v'v'} + \overline{w'w'}\right) \tag{7.1}$$

An equation for *k* may be obtained by realizing that Eq. (7.1) is just 1/2 times the sum of the normal Reynolds stresses. By setting j = i and k = j in Eq. (4.26), which is equivalent to taking the trace of the Reynolds stress tensor, the *k* equation for an incompressible fluid can be deduced as

$$\rho\frac{\partial k}{\partial t} + \rho\frac{\partial}{\partial x_j}\left(U_j k\right) = \tau_{ij}\frac{\partial U_i}{\partial x_j} - \mu\overline{\frac{\partial u'_i}{\partial x_k}\frac{\partial u'_i}{\partial x_k}} + \frac{\partial}{\partial x_j}\left(\mu\frac{\partial k}{\partial x_j} - \frac{1}{2}\rho\overline{u_i' u_i' u_j'} - \overline{p' u_j'}\right) \tag{7.2}$$

I II III IV

The full derivation of Eq. (7.2) is given in Appendix C. As was seen in the derivation for the Reynolds stress equation, the *k* equation involves several higher order correlations of fluctuating velocity components, which cannot be determined. Therefore, before attempting to model Eq. (7.2), it is beneficial to consider the physical processes represented by each term in the *k* equation.

The first and second term on the left-hand side of Eq. (7.2) represent the rate of change of the turbulent kinetic energy in an Eulerian frame of reference (time rate of change plus advection) and are familiar from any generic scalar transport equation.

The first term on the right-hand side of Eq. (7.2) is generally known as the production term, and represents the specific kinetic energy per unit volume that an eddy will gain per unit time due to the mean (flow) strain rate. If the equation for the kinetic energy of the mean flow is considered, it can be shown that this term actually appears as a sink in that equation. This should be expected and further verifies that the production of turbulence kinetic energy is indeed a result of the mean flow losing kinetic energy.

The second term on the left-hand side of Eq. (7.2) is referred to as dissipation, and represents the mean rate at which the kinetic energy of the smallest turbulent eddies is transferred to thermal energy at the molecular level. The dissipation is denoted as ε and is given by

$$\text{(II)} = \varepsilon = \nu \overline{\frac{\partial u'_i}{\partial x_k} \frac{\partial u'_i}{\partial x_k}} \tag{7.3}$$

The dissipation is essentially the mean rate at which work is done by the fluctuating strain rate against fluctuating viscous stresses. It can be seen that because the fluctuating strain rate is multiplied by itself, the term given by Eq. (7.3) is always positive. Hence, the role of the dissipation will be to always act as a sink (or destruction) term in the turbulent kinetic energy equation. It should also be noted that the fluctuating strain rate is generally much larger than the mean rate of strain when the Reynolds number of a given flow is large.

The third term in Eq (7.2) represents the diffusion, not destruction, of turbulent energy by the molecular motion that is equally responsibly for diffusing the mean flow momentum. This term is given by

$$\text{(III)} = \frac{\partial}{\partial x_j}\left(\mu \frac{\partial k}{\partial x_j}\right) \tag{7.4}$$

and can be seen to have the same general form of any generic scalar diffusion term.

The last term in Eq. (7.2), containing the triple fluctuating velocity correlation and the pressure fluctuations, is given by

$$\text{(IV)} = \frac{\partial}{\partial x_j}\left(-\rho \overline{u_i' u_i' u_j'} - \overline{p' u_j'}\right) \tag{7.5}$$

The first part of this term represents the rate at which turbulent energy is transported through the flow via turbulent fluctuations. The second part of this term is equivalent to the flow work done on a differential control volume due to the pressure fluctuations. Essentially this amounts to a transport (or redistribution) by pressure fluctuations. Each of the terms included in Eq. (7.4) and

(7.5) have the tendency to redistribute the specific kinetic energy of the flow. This is directly analogous to the way in which scalar gradients and turbulent fluctuations transport any generic scalar. It is important, then, to realize that the transport by gradient diffusion, turbulent fluctuations, and pressure fluctuations can only redistribute the turbulent kinetic energy in a given flow. However, the first two terms on the RHS of Eq. (7.2) actually represent a source and a sink by which turbulent kinetic energy may be produced or destroyed.

Modeled Turbulent Kinetic Energy Equation

If the *k* equation is going to be used for any purpose other than lending some physical insight into the behavior of the energy contained by the turbulent fluctuations, some way of obtaining its solution must be sought. As with all Reynolds averaged, turbulence equations, the higher order correlations of fluctuating quantities produce more unknowns than available equations, resulting in the familiar closure problem. Therefore, if the *k* equation is to be solved, some correlation for the Reynolds stresses, dissipation, turbulent diffusion, and pressure diffusion must be specified in light of physical reasoning and experimental evidence. The specification of these terms in the specific turbulent kinetic energy equation is generally the starting point in all one and two equation turbulence models.

The first assumption made in modeling the *k* equation is that the Boussinesq eddy viscosity approximation is valid and that the Reynolds stresses can be modeled by

$$\tau_{ij} = \mu_t \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) - \frac{2}{3} \rho k \delta_{ij} \tag{7.6}$$

for an incompressible fluid. The eddy viscosity in Eq. (7.6) is based on a similar concept that was used for algebraic models and is generally given by

$$\mu_t \approx \rho k^{1/2} l \tag{7.7}$$

where *l* is some turbulence length scale. This length scale, *l*, may be similar to the mixing length, but is written without the subscript "mix" to avoid confusion with mixing length models. The characteristic velocity scale in Eq. (7.7) is taken to be the square root of *k*, since the turbulent fluctuations at a point in the flowfield are representative of the turbulent transport of momentum. The prescription for μ_t given by Eq. (7.7) is very similar to that given by

$$\mu_t \approx \rho \nu_{mix} l_{mix} \tag{7.8}$$

for the algebraic or mixing length model, where ν_{mix} was generally taken to be related to a single velocity gradient in the flow. Implicit in either prescription for the eddy viscosity is the assumption that the turbulence behaves on a time scale proportional to that of the mean flow. This is not a trivial assumption and will be shown to be responsible for many of the inaccuracies

that can result using an eddy viscosity model for certain classes of flows. It should be noted that Eq. (7.7) is an isotropic relation, which assumes that momentum transport in all directions is the same at a given point in space. However, if used appropriately, this concept yields predictions with acceptable accuracy for a wide range of engineering applications.

The physical reasoning used in modeling the dissipation term is not distinctive; however, the same general conclusion of how epsilon should scale can be inferred by several considerations. A first approximation for ε may be obtained by considering a steady, homogenous, thin shear layer where it can be shown that production and dissipation of turbulent kinetic energy balance (i.e. flow is in local equilibrium). Mathematically this may be written as

$$-\overline{u_i' u_j'}\frac{\partial U_i}{\partial x_j} = \nu\overline{\frac{\partial u'_i}{\partial x_k}\frac{\partial u'_i}{\partial x_k}} = \varepsilon \tag{7.9}$$

from Eq. (7.2). Since homogeneous, shear generated turbulence can be characterized by one length scale, *l*, and one velocity scale, u, the left hand side of Eq. (7.9) may be taken to scale approximately as u^3/l. Hence, for turbulent flows where production balances dissipation, if the turbulence can be characterized by one length and one velocity scale, the dissipation may be taken to scale according to

$$\varepsilon \approx \frac{u^3}{l} \tag{7.10}$$

where the length scale *l*, and the velocity scale u, are characteristic of the turbulent flow.

Another interpretation of the dissipation term may be to consider that the rate of energy dissipation is controlled by inviscid mechanisms, the interaction of the largest eddies. The large eddies cascade energy to the smaller scales, which adjust to accommodate the energy dissipation. Therefore the dissipation should scale in proportion to the scales determined by the largest eddies (see, e.g. Tennekes and Lumley, 1983). Using dimensional analysis, in terms of a single turbulent velocity scale, u, and a single turbulence length scale, *l*, leads to Eq. (7.10).

The estimate obtained in this fashion rests on one of the primary assumptions of turbulence theory; it claims that the large eddies lose most of their energy in one turnover time. It does not claim that the large eddy dissipates the energy directly, but that it is dissipative because it creates smaller eddies. This interpretation is independent of the assumption that production balances dissipation and may be taken as a good approximation for many types of flows. The absence of this assumption may be implied from the fact that it is already accounted for since at the small scales, turbulence production effectively balances dissipation. In other words, at high Reynolds numbers, the local equilibrium assumption is usually valid.

White (1991) comes to the same conclusion based on a somewhat more physically based argument. If an eddy of size *l* is moving with speed u, its energy dissipated per unit mass should be approximated by

$$\varepsilon \approx \frac{(drag)(velocity)}{\text{mass}} \approx \frac{(\rho u^2 l^2)u}{\rho l^3} \approx \frac{u^3}{l} \tag{7.11}$$

Given these arguments for the scales of the dissipation, and taking the square root of the turbulent kinetic energy as the characteristic velocity scale, the dissipation is generally modeled by

$$\varepsilon \approx \frac{k^{3/2}}{l} \tag{7.12}$$

The turbulent transport of *k*, and the pressure diffusion terms, are generally modeled by assuming a gradient diffusion transport mechanism. This method is similar to that used for generic scalar transport in a turbulent flow as was given in Section 5.0. While this method is normally applied to the terms involving the turbulent fluctuations, it is common practice to group the pressure fluctuation term, which is generally small for incompressible flows, with the gradient diffusion term. With these assumptions the turbulent transport of *k* and the pressure fluctuation term are given by

$$\frac{\mu_t}{\sigma_k}\frac{\partial k}{\partial x_j} = \left(-\frac{1}{2}\rho\overline{u_i' u_i' u_j'} - \overline{p' u_j'} \right) \tag{7.13}$$

where σ_k is a closure coefficient known as the Prandtl-Schmidt number for k.

Combining the modeled terms for the Reynolds stress, dissipation, turbulent diffusion, and pressure diffusion with the transient, convective, and viscous terms gives the modeled *k* equation as

$$\rho\frac{\partial k}{\partial t} + \rho\frac{\partial}{\partial x_j}\left(U_j k\right) = \tau_{ij}\frac{\partial U_i}{\partial x_j} - \rho\varepsilon + \frac{\partial}{\partial x_j}\left((\mu + \frac{\mu_t}{\sigma_t}\frac{\partial k}{\partial x_j} \right) \tag{7.14}$$

with the dissipation and eddy viscosity given by

$$\varepsilon = \mathrm{C_D}\frac{k^{3/2}}{l} \tag{7.15}$$

$$\mu_t = C_\mu \rho k^{1/2} l \tag{7.16}$$

where C_D and C_μ are constants.

An obvious requirement is still the determination of the length scale, *l*, which for a one-equation model must still be specified algebraically in accordance to some mean flow parameters. Since the one-equation model is based primarily on several of the same assumptions as the mixing length model, for equilibrium flows it can be shown that the length scale *l* can be made

proportional to l_{mix}. For many one-equation models, the algebraic prescriptions used for the mixing length provide a logical starting place. One popular version of this model (Emmons, 1954; Glushko, 1965) involves setting $\sigma_k = 1.0$, with C_D varying between 0.07 and 0.09, along with length scale prescriptions similar to those for the mixing length model. This model is somewhat simple, involving less closure coefficients than several mixing length models; however, the necessary prescription of l by algebraic equations, which vary from problem to problem, is a definite limitation. Hence even with a characteristic velocity scale, one-equation models are still incomplete.

Typical applications for one-equation models involve primarily the same types of flows as did mixing length models. One-equation models have a somewhat better history for prediction of separated flows; however, they share most of the failures of the mixing length model. The specification of the mixing length by an algebraic formula is still almost entirely dependent on empirical data, and is usually incapable of including transport effects on the length scale. As will be seen in Section 8.0, the desire to include transport effects on the length scale will be the primary reason for introducing two-equation models such as the k-ε and k-ω models.

8.0 TWO-EQUATION TURBULENCE MODELS

Two-equation models have been the most popular models for a wide range of engineering analysis and research. These models provide independent transport equations for both the turbulence length scale, or some equivalent parameter, and the turbulent kinetic energy. With the specification of these two variables, two-equation models are complete; no additional information about the turbulence is necessary to use the model for a given flow scenario. While this is encouraging in that these models may appear to apply to a wide range of flows, it is instructive to understand the implicit assumptions made in formulating a two-equation model. While complete in that no new information is needed, the two-equation model is to some degree limited to flows in which its fundamental assumptions are not grossly violated. Specifically, most two-equation models make the same fundamental assumption of local equilibrium, where turbulent production and dissipation balance. This assumption further implies that the scales of the turbulence are locally proportional to the scales of the mean flow; therefore, most two-equation models will be in error when applied to non-equilibrium flows. Though somewhat restricted, two-equation models are still very popular and can be used to give results well within engineering accuracy when applied to appropriate cases.

This chapter will first focus on the second variable, which is solved in addition to the turbulence kinetic energy; specifically the two additional variables, ε and ω, will be considered. Generally, ε is defined as the dissipation, or rate of destruction of turbulence kinetic energy per unit time, and ω can be defined either as the rate at which turbulent kinetic energy is dissipated or as the inverse of the time scale of the dissipation. Both variables are related to each other and to the length scale, l, which has thus far been associated with zero and one-equation models. Here we introduce the mathematical expression for ω for further reference (Kolmogorov, 1942)

$$\omega = c\frac{k^{1/2}}{l} \tag{8.1}$$

where c is a constant.

General Two-Equation Model Assumptions

The first major assumption of most two-equation models is that the turbulent fluctuations, u', v', and w', are locally isotropic or equal. While this is true of the smaller eddies at high Reynolds numbers, the large eddies are in a state of steady anisotropy due to the strain rate of the mean flow, though u', v', and w' are almost always of the same magnitude. Implicit in this assumption is that the normal Reynolds stresses are equal at a point in the flowfield.

The second major assumption of most two-equation models is that the production and dissipation terms, given in the *k*-equation, are approximately equal locally. This is known as the local equilibrium assumption. This assumption follows from the fact that the Reynolds stresses must be estimated at every point in the flowfield. To allow the Reynolds stresses to be calculated

using local scales, most two-equation models assume that production equals dissipation in the *k*-equation or

$$\tau_{ij} S_{ij} = \rho\varepsilon \qquad \text{or} \qquad S_{ij} = \frac{\rho\varepsilon}{\tau_{ij}} \tag{8.2}$$

where τ_{ij} is the turbulent stress tensor. This implies that the turbulent and mean flow quantities are locally proportional at any point in the flow. Hence, with the ratio of turbulent and mean flow quantities equal to some local quantity, the eddy viscosity is defined to be the proportionality constant between the Reynolds stresses and the mean strain rate. This is the essence of the Boussinesq approximation, which was given in Chapter 5.0 by

$$-\overline{u_i' u_j'} = \nu_t \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) - \frac{2}{3} k \delta_{ij} \tag{8.3}$$

where the second quantity on the right involving *k* has been included to ensure the correct normal Reynolds stress in the absence of any strain rate. Since the turbulence and mean scales are proportional, the eddy viscosity can be estimated based on dimensional reasoning by using either the turbulent or mean scales. This implies that

for the k-ε model

$$\mu_t \propto \frac{\rho k^2}{\varepsilon} \tag{8.4}$$

for the k-ω model

$$\mu_t \propto \frac{\rho k}{\omega} \tag{8.5}$$

If production does not balance dissipation locally then the ratio of the Reynolds stresses to the mean strain rate is not a local constant and μ_t will be a function of both turbulent and mean scales.

Another way to understand this assumption of local equilibrium is to realize that transport effects, while included for the turbulent scales, are neglected for the turbulent Reynolds stresses. Local scales are then used in estimating the Reynolds stresses. Otherwise the Reynolds stresses would depend on the local conditions with some history effects.

In predominantly parabolic (uni-directional) flow, if the flow is slowly varying, then upstream scales are approximately the same as local scales of turbulence, and the equilibrium assumption will work. Also, if the turbulence is evolving at a sufficiently rapid rate, such that the effects of past events do not dominate the dynamics, the estimates based on local scales will work. While there are many cases where the local equilibrium assumption does exist for a wide range of shear flows, two-equation models should be carefully applied as the models can be expected to perform poorly for non-equilibrium flows. Typical flows where two-equation models have been shown to

fail are flows with sudden changes in mean strain rate, curved surfaces, secondary motions, rotating and stratified fluids, flows with separation, and three dimensional flows.

The k-ε Model

The first step in formulating the k-ε model is in the consideration of the second variable, ε. Since ε has already been defined by

$$\varepsilon = \nu \overline{\frac{\partial u'_i}{\partial x_k} \frac{\partial u'_i}{\partial x_k}} \tag{8.6}$$

the starting point for calculating the second variable should be the exact equation for epsilon. This can be done by performing the operation

$$\mathbf{L}(\,) = \nu \overline{\frac{\partial u'_i}{\partial x_j} \frac{\partial}{\partial x_j}(\,)} \tag{8.7}$$

to both sides of the instantaneous, incompressible, Navier-Stokes equations. The derivation of the ε-equation is given in Appendix C. After a tedious manipulation, the exact equation of ε can be written as

$$\begin{aligned} &\rho \frac{\partial \varepsilon}{\partial t} + \rho U_j \frac{\partial \varepsilon}{\partial x_j} = -2\mu \left[\overline{u'_{i,k} u'_{j,k}} + \overline{u'_{k,i} u'_{k,j}} \right] \frac{\partial U_i}{\partial x_j} - 2\mu \overline{u'_k u'_{i,j}} \frac{\partial^2 U_i}{\partial x_k \partial x_j} \\ &-2\mu \overline{u'_{i,k} u'_{i,m} u'_{k,m}} - 2\mu\nu \overline{u'_{i,km} u'_{i,km}} \\ &+ \frac{\partial}{\partial x_j} \left(\mu \frac{\partial \varepsilon}{\partial x_j} - \mu \overline{u'_j u'_{i,m} u'_{i,m}} - 2\nu \overline{p'_m u'_{j,m}} \right) \end{aligned} \tag{8.8}$$

The physical processes represented in the ε equation involve several double and triple correlations of fluctuating velocity components. While somewhat formidable, the following physical interpretations exist for the terms in the epsilon equation. The first two terms on the right-hand side of Eq. (8.8) represent the production of dissipation due to interactions between the mean flow and the products of the turbulent fluctuations. The next two terms on the right-hand side represent the destruction rate of dissipation due to the turbulent velocity fluctuations. Finally, the last three terms in Eq. (8.8) represent the transport or spatial re-distribution of dissipation due to viscous diffusion, turbulent fluctuations, and pressure-velocity fluctuations.

As already mentioned, the k-ε turbulence model is based on Eq. (8.8). It should be realized that what is actually needed in the model is a length or time scale relevant to the large, energy containing eddies, which are responsible for the majority of the turbulent stresses and fluxes. However, the mechanism of the smallest eddies, which physically accomplishes the dissipation,

are what is actually represented by Eq. (8.8). This leads to some questions as to how relevant the exact dissipation equation is when the desired quantity is a length scale, characteristic of the large eddies. A counter-argument in favor of the use of Eq. (8.8) stems from the same scale estimates for ε used in Chapter 7.0, which argue that ε is determined by the energy transfer from the large scales, and that Eq. (8.8) builds a link between the small-scale motion and the large eddies. Actually, since the closure approximations used in modeling Eq. (8.8) are based primarily on large-eddy scales, it is implied that the modeled equation is actually more of an empirical equation representing the transfer of energy from the large eddies to the small eddies.

Since the modeled *k*-equation was already investigated in Chapter 7.0, the method used to model the ε-equation will be considered here. The first term on the RHS side of Eq. (8.8) is given by

$$P_\varepsilon = -2\mu\left[\overline{u'_{i,k}\,u'_{j,k}} + \overline{u'_{k,i}\,u'_{k,j}}\right]\frac{\partial U_i}{\partial x_j} - 2\mu\,\overline{u'_k\,u'_{i,j}}\,\frac{\partial^2 U_i}{\partial x_k \partial x_j} \tag{8.9}$$

and is generally taken to give the production of dissipation as was previously mentioned. This interpretation follows from the presumption that the product of the turbulent fluctuations normally have negative correlations in regions of positive velocity gradients. As this term is regarded as a source, or production, of ε, the first step in modeling the term is to consider the assumption of local equilibrium, namely

$$P_k = \tau_{ij}\frac{\partial U_i}{\partial x_j} \approx \varepsilon \tag{8.10}$$

which implies

$$P_\varepsilon \propto \frac{P_k}{t_{ch}} = \left[\tau_{ij}\frac{\partial U_i}{\partial x_j}\right]\left(t_{ch}\right)^{-1} \tag{8.11}$$

where t_{ch} is a characteristic time scale for the production of ε. Since, for local equilibrium, the rate at which *k* is produced is equal to ε, then the only time scale that can define the rate of change of ε must scale with *k*. This implies that

$$t_{ch} = \frac{k}{\varepsilon} \tag{8.12}$$

which gives

$$P_\varepsilon = C_{\varepsilon 1}\frac{k}{\varepsilon}P_k \tag{8.13}$$

Hence, using the assumption of local isotropy and local equilibrium, Eq. (8.9) is replaced by Eq. (8.13), which closes the unknown velocity correlations.

In a similar manner, the quantity represented by the third and fourth terms on the RSH of Eq. (8.8), given by

$$D_\varepsilon = -2\mu\overline{u'_{i,k}\, u'_{i,m}\, u'_{k,m}} - 2\mu\nu\overline{u'_{i,km}\, u'_{i,km}} \tag{8.14}$$

can be taken to represent the time rate at which dissipation is destroyed or dissipated. Using dimensional analysis, this can be thought of as being given by

$$D_\varepsilon \propto \frac{\varepsilon}{t_{ch}} \tag{8.15}$$

which implies from Eq. (8.12) that

$$D_\varepsilon = C_{\varepsilon 2}\frac{\varepsilon^2}{k} \tag{8.16}$$

Finally, the diffusion due to molecular action and turbulent fluctuations is modeled using a standard gradient diffusion approach. Using a turbulent Prandtl number for ε, given by σ_ε, this implies

$$\frac{\partial}{\partial x_j}\left(\mu\frac{\partial\varepsilon}{\partial x_j} - \mu\overline{u'_j\, u'_{i,m}\, u'_{i,m}} - 2\nu\,\overline{p'_m\, u'_{j,m}}\right) = \frac{\partial}{\partial x_j}\left((\mu + \frac{\mu_t}{\sigma_\varepsilon}\frac{\partial\varepsilon}{\partial x_j}\right) \tag{8.17}$$

Collecting the modeled terms for the ε-equation along with the modeled *k*-equation, the standard k-ε turbulence model is given by

the *k*-equation

$$\rho\frac{\partial k}{\partial t} + \rho U_j\frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_j}\left((\mu + \frac{\mu_t}{\sigma_k})\frac{\partial k}{\partial x_j}\right) + \tau_{ij}\frac{\partial U_i}{\partial x_j} - \rho\varepsilon \tag{8.18}$$

the ε–equation

$$\rho\frac{\partial\varepsilon}{\partial t} + \rho U_j\frac{\partial\varepsilon}{\partial x_j} = \frac{\partial}{\partial x_j}\left((\mu + \frac{\mu_t}{\sigma_\varepsilon})\frac{\partial\varepsilon}{\partial x_j}\right) + C_{\varepsilon 1}\frac{\varepsilon}{k}\tau_{ij}\frac{\partial U_i}{\partial x_j} - C_{\varepsilon 2}\rho\frac{\varepsilon^2}{k} \tag{8.19}$$

and the eddy viscosity

$$\mu_t = \frac{\rho C_\mu k^2}{\varepsilon} \tag{8.20}$$

with the closure coefficients given by

$C_{\varepsilon 1} = 1.44$, $C_{\varepsilon 2} = 1.92$, $C_\mu = 0.09$, $\sigma_k = 1.0$, $\sigma_\varepsilon = 1.3$

As explained later, these coefficients are determined in a systematic manner applying a semi-empirical procedure and optimization (see, e.g. Rodi, 1980 for details).

The k-ω Model

Another popular two-equation model is the *k*-ω model, which will be presented here in the form given by Wilcox (1988a). In contrast to the *k*-ε model, which solves for the dissipation or rate of destruction of turbulent kinetic energy, the *k*-ω model solves for only the rate at which that dissipation occurs. Dimensionally, ω can be related to ε by $\omega \propto \varepsilon / k$. Another interpretation is the inverse of the time scale on which dissipation takes place, which is set by the largest eddies. The equation governing ω has traditionally been formulated based on physical reasoning in light of the processes normally governing the transport of a scalar in a fluid. Considering the processes of convection, diffusion, production, and destruction or dissipation, the model equation for ω is given by

$$\rho \frac{\partial \omega}{\partial t} + \rho U_j \frac{\partial \omega}{\partial x_j} = \frac{\partial}{\partial x_j}\left((\mu + \sigma \mu_t) \frac{\partial \omega}{\partial x_j} \right) + P_\omega - D_\omega \tag{8.21}$$

The production and dissipation of ω are modeled by analogy with the reasoning used to model the production and dissipation of ε in the epsilon equation. In light of the dimensions involved in *k*, ε, and ω, the production and dissipation terms for ω are given by

$$P_\omega = \alpha \frac{\omega}{k} \tau_{ij} \frac{\partial U_i}{\partial x_j} = \alpha \frac{\omega}{k} P_k \tag{8.22}$$

and

$$\mathrm{D}_\omega = \beta \rho \omega^2 \tag{8.23}$$

Combining these with the ω-equation and the modeled *k*-equation gives the *k*-ω model (Wilcox, 1988a) as

the *k*-equation:

$$\rho \frac{\partial k}{\partial t} + \rho U_j \frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_j}\left((\mu + \frac{\mu_t}{\sigma_k}) \frac{\partial k}{\partial x_j} \right) + \tau_{ij} \frac{\partial U_i}{\partial x_j} - \beta^* \rho k \omega \tag{8.24}$$

the ω-equation:

$$\rho \frac{\partial \omega}{\partial t} + \rho U_j \frac{\partial \omega}{\partial x_j} = \frac{\partial}{\partial x_j}\left((\mu + \frac{\mu_t}{\sigma_\omega}) \frac{\partial \omega}{\partial x_j} \right) + \alpha \frac{\omega}{k} \tau_{ij} \frac{\partial U_i}{\partial x_j} - \beta \rho \omega^2 \tag{8.25}$$

and the eddy viscosity:

$$\mu_t = \frac{\rho k}{\omega} \tag{8.26}$$

with the closure coefficients given by

$$\alpha = 5/9, \qquad \beta = 3/40, \qquad \beta^* = 9/100, \qquad \sigma_\omega = 1/2, \qquad \sigma_k = 1/2$$

Determination of Closure Coefficients

The determination of the closure coefficients used in turbulence models is not rigorously established, especially since the models involve many assumptions and arguments based on physical reasoning. Therefore, the popular approach for the determination of closure coefficients is to set the values in such a way that the model obtains reasonable agreement with experimentally observed properties of turbulence. This approach intrinsically adopts a high degree of presumption in that a constant determined from one application may not necessarily be the best one for a wide range of turbulent flows. The method presented by Wilcox for the *k*-ω model is given as an example.

Two turbulent flow cases are considered in establishing the closure coefficients for the *k*-ω model. These flows are the decaying (no production) of homogeneous, isotropic turbulence, and a boundary layer where the coefficients are chosen to ensure agreement with the law of the wall.

For decaying, homogeneous, isotropic turbulence, the asymptotic solution for *k* is given from Eq. (8.19) and (8.20) by

$$k \approx t^{-\beta^*/\beta} \tag{8.27}$$

From the boundary layer version of Eq. (8.19) and (8.20), a relationship between the von Karman constant, κ, from the law of the wall, and the coefficients in the *k*-ω model, may be deduced as

$$\alpha = \frac{\beta}{\beta^*} - \sigma_k \frac{\kappa^2}{\sqrt{\beta^*}} \tag{8.28}$$

and from the log-layer it can be shown that the Reynolds stress must satisfy

$$\tau_{xy} = \sqrt{\beta^*}\rho k \tag{8.29}$$

For decaying, homogeneous, isotropic turbulence, experimental data shows $\beta^*/\beta \approx 6/5$. Also, in the log-layer the ratio given by $\tau_{xy}/\rho k \approx 3/10$. Analysis of the defect layer and sublayer indicate that $\sigma_k = \sigma_\omega = 1/2$ should be chosen. Using these values, along with the standard value of the von Karman coefficient, $\kappa = 0.41$, yields the five closure coefficients given above when combined with Eq. (8.27), (8.28), and (8.29).

Applications of Two-Equation Models

In investigating the usefulness of two-equation models, results from the k-ε and k-ω model will be discussed in light of their ability to match experiments. This section will give some overview on how the models perform rather than give an in depth treatment on how to obtain solutions to the equations, as most solutions must be generated numerically anyway. Namely, free shear flows, boundary layers, pipe flows, and separated flows will be examined (Wilcox, 1993).

For the free shear flows given by jets, wakes, and mixing layers both models perform reasonably well. Spreading rates for these flows as predicted by the k-ε model are generally predicted to within 30% for the far wake and round jet, 15% for the mixing layer, and 5% for the plane jet. The k-ω model predictions are somewhat closer for each of the four cases; however, it is quite sensitive to free stream boundary conditions. This sensitivity does allow a degree of freedom in calibrating the model, and hence the increased accuracy obtained by using the k-ω model should be judged with caution.

For pipe flows at Re = 40,000, the k-ω model has been shown to give predictions within 6% of DNS predictions for such quantities as the mean velocity, Reynolds stress, and skin friction. The model is slightly inaccurate at predicting the value of k near the wall, but does accurately predict the value of k within 4% over 80% of the pipe diameter.

One comment on these two models is necessary before comparing their predictions for boundary layer flows. The k-ω model has been shown to reliably predict the law of the wall when the model is used to resolve the viscous sublayer, thereby eliminating the need to use a wall function, except for computational efficiency. The k-ε model, in its present form, has been shown to necessitate either a low Reynolds number modification, or the use of a wall function, when applied to a wall bounded flow. In light of these characteristics of the models, the following comparisons were made using a k-ω model that resolved the viscous sublayer, and a k-ε model that used a wall function. This may bias the k-ω model somewhat in the following evaluation given by Wilcox; at the Stanford Conference on Complex Turbulent Flows (1981). It was concluded that differential methods that integrate to the wall give better results than models using wall functions. A more objective comparison between the k-ω and k-ε models may be for versions of the models both integrating to the wall. Model characteristics for wall bounded flows will be examined in more detail in Chapter 9.0 on numerical considerations for wall bounded flows.

For a constant pressure incompressible boundary layer flow at a Reynolds number close to 10^6, both models perform very well, predicting values of the friction coefficient and mean velocity within 5%. However, for incompressible boundary layers with adverse pressure gradients, the k-ω model is claimed to be somewhat superior to the k-ε model. For one case simulated at a Reynolds number around 10^6, the friction coefficient for the k-ω model gave predictions within 5%, while the k-ε model predictions showed error as high as 20%.

An additional flow with adverse pressure gradient, the backward facing step, again shows the k-ω model to give superior performance to the k-ε model. Wilcox attributes the poor performance of the k-ε model for flows with adverse pressure gradients to the turbulence length scale predicted by the k-ε model in the near-wall region. Using a perturbation analysis of the defect layer, values of the turbulence length scale were predicted to be too large near the wall for the k-ε model. This can be rectified by using the correct length scale as prescribed by van Driest (see Eq. 6.13).

In conclusion, two-equation models have proven that they perform well for a wide range of flows of engineering interest. Their application is limited, however, to flows that closely follow the implicit assumptions upon which most two-equation models are based. The most important assumptions for these models are that the flow does not depart far from local equilibrium, and that the Reynolds number is high enough that local isotropy is approximately satisfied. The k-ω model is supposedly better than the k-ε model for flows with adverse pressure gradient, though many variants of the k-ε model with correction for other factors such as streamline curvature, buoyancy, swirl, etc. for such cases exist. Generally, neither model is capable of giving quantitatively good results for more complicated flows such as flows with sudden changes in mean strain rate, curved surfaces, secondary motions, and separation. While two-equation models may be able to give qualitative results for such flows, generally a further level of complexity is needed in the model to obtain close agreement with experiments. As will be shown in Chapter 11.0, a class of models, known as algebraic stress models, attempt to end the need to assume local isotropy, while second order closure models attempt to account for history effects on the stresses and remove the need for assuming local equilibrium. Another method called Large Eddy Simulation (LES), where the most important eddies responsible for the turbulent fluctuations are resolved explicitly, will also be presented in Chapter 11.0.

9.0 WALL BOUNDED FLOWS

Most flows of engineering interest involve situations where the flow is constrained, at least to some degree, by a solid boundary. Since wall bounded flows are rather common, appropriate methods for applying two-equation models to such flows are important and will be dealt with in some detail in this chapter. Initially, some fundamental properties of turbulent flows over solid boundaries will be considered, including properties of the turbulence in the viscous sublayer and log layer for flows with zero or small pressure gradients. The behavior of k, ε, and ω will be considered in light of near wall turbulence phenomena, and the ability of the modeled equations to capture this phenomena will be investigated. The use and implementation of wall functions, as well as methods for resolving the viscous sublayer, will be discussed for both the k-ω and k-ε models. This chapter will also include a discussion of the numerical implementation of wall functions.

Review of Turbulent Flow Near a Wall

In a flow bounded by a wall, different scales and physical processes are dominant in the inner portion near the wall, and the outer portion approaching the free stream. These layers are typically known as the inner and outer layers, though traditionally the flow near a wall has been analyzed in terms of three layers:

1. The inner layer, or sublayer, where viscous shear dominates
2. The outer layer, or defect layer, where large scale turbulent eddy shear dominates
3. The overlap layer, or log layer, where velocity profiles exhibit a logarithmic variation

The matching of the inner and outer regions has been useful for establishing flow properties in this layer separating the areas of viscous and inertial domination. In the inner layer, convection and pressure gradients are assumed negligible. This is in line with the original reasoning of Prandtl and von Karman. These early researchers reasoned that in the inner layer the important variables determining the mean velocity are the viscosity, density, wall shear stress, and normal distance from the wall, y. The combination of these variables implies a form for the mean velocity, which can be written as

$$\mathrm{U} = f(\tau_w, \rho, \mu, \mathrm{y}) \tag{9.1}$$

For the outer layer, viscous effects are assumed negligible but the pressure gradient and convection must be included. Then the relevant variables that will influence the mean velocity are the wall shear stress, density, boundary layer thickness, pressure gradient, and the distance from the wall. Combination of these variables gives

$$\mathrm{U} = f\left(\tau_w, \rho, \delta, \mathrm{y}, \frac{\partial \mathrm{P}}{\partial \mathrm{x}}\right) \tag{9.2}$$

The log layer, where the inner and outer layers merge, can be determined from functional analysis (see e.g. Schlichting, 1979) by requiring the same asymptotic solution for the first derivative of the velocity with respect to the normal distance from the wall. This analysis leads to the logarithmic velocity profile law of the wall, given by

$$U^+ = \frac{1}{\kappa}\ln(y^+) + B \tag{9.3}$$

where $\kappa \cong 0.42$, $B \approx 5.0$,

$$u_\tau = \left(\frac{\tau_w}{\rho}\right)^{1/2} \quad \text{and} \quad y^+ = \frac{y u_\tau}{\nu} \quad \text{and} \quad U^+ = \frac{U}{u_\tau} \tag{9.4a, b, c}$$

It should be emphasized that Eq. (9.3), referred to as the log law, has been experimentally confirmed by many investigators. While this is a somewhat vague review of the physics underlying most wall bounded flows, it serves to set the stage for investigating how boundary conditions may be applied at solid surfaces for two-equation turbulence models.

Two-Equation Model Behavior Near a Solid Surface

For investigating the behavior of two-equation models near a solid boundary, Wilcox (1993) employs perturbation analysis to determine limiting forms of the equations very near the wall in the viscous sublayer, and in the inertial part of the sublayer where the law of the wall is valid. These investigations are informative in that they give insight into how such quantities as k, ε, and ω behave near solid surfaces, and what kind of numerical treatment is necessary for avoiding significant errors in such cases.

Establishing flow properties in the log layer, although it is not formally a separate layer, permits independent analysis of the sublayer and defect layer. Knowledge of flow properties in the log layer is also important in that it forms the basis for boundary conditions for many two-equation models. In the log-layer region, where effects of the pressure gradient, molecular diffusion, and convection are small compared to other quantities, the momentum equation reduces to

$$0 = \frac{\partial}{\partial y}\left(\nu_t \frac{\partial U}{\partial y}\right) \tag{9.5}$$

Reducing the *k*, ε, and ω equations, and assuming that the eddy viscosity varies linearly from the wall as was done for the mixing length model, the equations for the *k*-ω model in the log layer are given by

$$U = \frac{u_\tau}{\kappa}\ln(y^+) + \text{const} \tag{9.6}$$

$$k = \frac{(u_\tau)^2}{\sqrt{\beta^*}} \qquad \text{and} \qquad \omega = \frac{u_\tau}{\sqrt{\beta^*}\kappa y} \tag{9.7a, b}$$

and the log layer equations for the k-ε model are given by

$$U = \frac{u_\tau}{\kappa}\ln(y^+) + \text{const} \tag{9.8}$$

$$k = \frac{(u_\tau)^2}{\sqrt{C_\mu}} \qquad \text{and} \qquad \varepsilon = \frac{(u_\tau)^3}{\kappa y} \tag{9.9a, b}$$

These relations define a key set of equations that can be used to implement boundary conditions for solutions where it is undesirable to resolve the entire viscous sublayer.

Very close to the wall in the laminar sublayer, the effects of turbulence die out very quickly. The limiting value of k, ε, and ω approaching the wall from a point just outside the viscous sublayer can be inferred by considering the mean and instantaneous momentum and continuity equations and by expanding the instantaneous fluctuating velocity components in a Taylor Series (Wilcox, 1993). In this way, it can be shown that, near the wall

$$k \approx y^2 \tag{9.10}$$

$$\varepsilon \approx \frac{2\kappa\nu}{y^2} \tag{9.11}$$

$$\omega \approx \frac{2\nu}{\beta^* y^2} \tag{9.12}$$

Neither the k-ε nor the k-ω model reproduces these theoretical trends, though the k-ω model does come close enough to allow the viscous sublayer to be resolved without significant error. Resolving the viscous sublayer using the standard k-ε model cannot be done without corrections, which are usually added to the model as damping of the length scales as the wall is approached.

With some background about how the theoretical values behave near a solid boundary, the task of numerically accounting for solid boundaries will now be considered. The most popular choice, as has been previously mentioned, is to match the velocity profile to the law of the wall at the first grid node above the solid boundary. Effectively this approach uses the law of the wall as a constitutive equation for the wall shear stress in terms of the velocity at the first grid point. Using this approach, the value of the velocity may be used to determine τ_w, which may in turn be used to set the values of k and ε or ω using Eqs. (9.7a, b) and (9.9a, b). It should be noted that the application of wall functions may not yield good results when applied under conditions differing drastically from those under which the law of the wall is derived. Primarily, effects of

pressure gradients, buoyancy fluxes, and non-equilibrium type situations may alter the velocity profile near the wall in a manner that is not accounted for by the standard law of the wall.

Effects of Surface Roughness

When the boundary is smooth, the shear stress at the surface is transmitted to the flow via a viscous sublayer. The velocity in this sublayer varies linearly, as in laminar Couette flow, like $U = u_\tau^2\, y/\nu$, which implies that the thickness of the sublayer is on the order of $\delta_{lam} = (const)\nu/u_\tau^2$. Here the constant is the value of the velocity where it ceases to behave according to a linear relationship. As has been mentioned already, this implies a law given by

$$U^+ = \frac{1}{\kappa}\ln(y^+) + B \tag{9.13}$$

where the constant of integration B accounts for conditions at the boundary; the effect of B on the flow is to add a uniform velocity to the entire flow with no change in its internal structure.

For a rough surface, the roughness height may be larger than δ_{lam}. In that case, the stress is transmitted by pressure forces in the wakes of the roughness elements, rather than by viscosity. The form of the profile given in the log layer is then

$$U^+ = \frac{1}{\kappa}\ln(\frac{y}{k_s}) + 8.5 \tag{9.14}$$

where k_S is called Nikuradse's equivalent roughness height, and is related to the roughness geometry at the boundary.

Resolution of the Viscous Sublayer

The alternative to using wall functions is to solve the governing equations fully through the viscous sublayer up to the wall, where no-slip boundary conditions can be applied. As previously mentioned, this approach requires low Reynolds number modifications for the *k*-ε model, which include viscous damping functions similar to the Van Driest formulation given in Chapter 6.0. Several such modifications exist, the most popular being the Jones-Launder model, the Launder-Sharma model, the Lam-Bremhorst model, and the Chien model. All four of these models give the correct asymptotic behavior previously discussed. One advantage of the *k*-ω model is that it requires no modification and can be directly integrated to the wall with acceptable results. Some areas of concern when resolving the viscous sublayer include computational demands and possible convergence difficulties. Resolution of the viscous sublayer requires a major increase in the number of computational cells near the wall to accurately predict the velocity profile. In

addition, some of the viscous damping functions have been known to be quite stiff when the equations are being solved numerically; this can be attributed to the dissipation's time scale that is encountered approaching the wall. Equation (9.9b) shows that ε appears to approach infinity near the wall as y approaches zero.

Application of Wall Functions

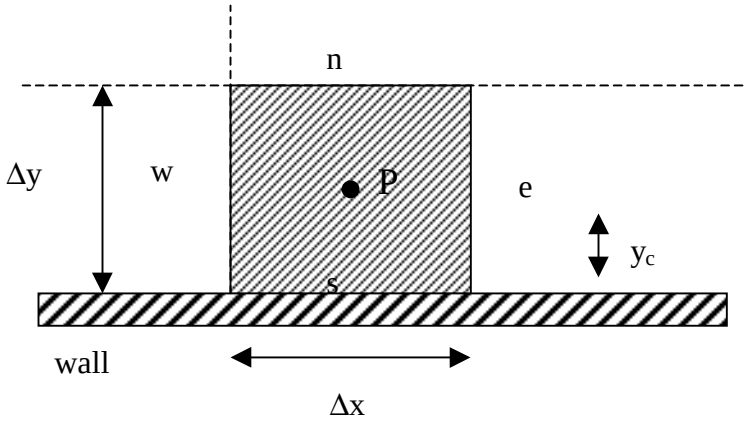
Numerically, a possible source of error may occur in the epsilon equation near the wall as it varies in inverse proportion to y as is indicated by Eq. (9.9b) for the case where wall functions are being used. Such a functional relationship may cause significant errors if the grid is not sufficiently fine near the first grid node next to the wall.

Estimates of y^+ can be made before a numerical simulation from known equations for flows such as zero pressure boundary layers, pipe flows, and channel flows is performed. Several useful equations and experimental correlations are hence included here as a reference for making such approximations.

Generalized Wall Functions:

To avoid computational cost, we can use wall functions to bridge the log-layer to the wall, i.e. skip the sub-layer and the buffer layer and enforce boundary conditions at $y=y_c$.

Standard approach (See Appendix B):



Enforce boundary conditions at $y = y_c$

** Couette flow analysis; equilibrium, i.e. $P_k=E$ and constant shear stress assumptions lead to (See work book for details)

$y_c^+ > 11.6$ (match point)

$$U_c^+ = \frac{1}{\kappa}\ln\left(\mathrm{E}y_c^+\right); \quad \mathrm{E} \cong 9.0, \quad \mathrm{K} = 0.41$$

$$k_c = U_*^2 \Big/ \sqrt{C_\mu}, \quad \mathrm{E}_c = U_*^3 \Big/ \kappa y_c, \quad \omega_c = \frac{k_2^{1/2}}{(\beta^*)^{1/4}\kappa y}$$

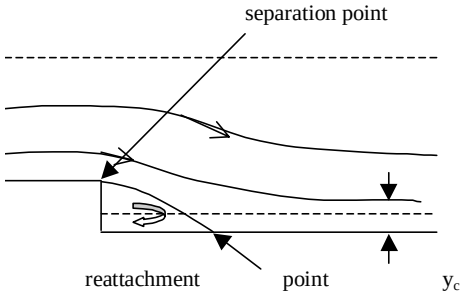
$$\mathrm{T}_c^+ = \frac{1}{\kappa}\ln\left(\mathbf{E}_h \mathrm{y}_c\right); \quad \mathbf{E}_h = \mathrm{func(Pr)}$$

Deficiencies of the standard approach:

τ_w=0 at separation points, hence k=0!

and heat flux $\dot{q}_h = 0$

According to experiments, k = k_{max}, $\dot{q}_u$ = maximum (heat transfer) at reattachment point and k ≠ 0 at separation points.



Remedy:

$$U_* \to k^{1/2} \quad \left(k = U_*^2 \Big/ \sqrt{C_\mu}, \text{ if } -\overline{u'v'} \Big/ k \cong 0.3\right)$$

let $S = U_*^2 \Big/ U_c^2$ (friction factor)

$$S^{1/2} = \frac{U_*}{U_c} = \frac{\kappa}{\ln\left(Ey\frac{U_c}{\nu}\cdot\frac{U_*}{U_c}\right)} = \frac{\kappa}{\ln\left(E\,\mathrm{Re}_c\, S^{1/2}\right)}$$

The above equation can be written, for computational purposes, in various ways.

For iterative solution: $S = \dfrac{\kappa S^{1/2}}{\ln\left[E\,\mathrm{Re}_c\, S^{1/2}\right]} = \dfrac{\kappa C_\mu^{1/4} k_c^{1/2}}{U_c \ln\left[E\,\mathrm{Re}_c\, C_\mu^{1/4} \dfrac{k^{1/2}}{U_c}\right]}$

For explicit non-iterative solution:

$$S = \frac{\mathrm{K} C_\mu^{1/4} k_c^{1/2}}{U_c \ln\left[E C_\mu^{1/4} \dfrac{y_c k_c^{1/2}}{\nu}\right]}$$

to calculate S and U_*, hence τ_w

$$\tau_w = \rho S U_c^2$$

$$P_k = -\overline{u'v'}\frac{\partial \bar{u}}{\partial y} = U_*^2 \frac{\partial \overline{U}}{\partial y} = S \cdot U_c^2 \left(\frac{U_c}{\alpha y_c}\right) : \alpha \text{ is a factor for approximating the derivative}$$

Also, the production and dissapation terms in the k-equation need to be calculated from the following relations:

$$(\mathrm{i})\ \varepsilon = P_k \Rightarrow C_\mu^{3/4} k^{3/2} \Big/ Ky$$

$$(ii) = U^{*2} \frac{U_c}{\alpha y_c} = \frac{U_*^3}{\alpha y_c S_c^{1/2}} = \frac{C_\mu^{3/4} k^{3/2}}{\alpha y_c \mathrm{K}} \ln\left[E y_c \frac{k^{1/2}}{\nu}\right]$$

- (See also two- and three layer models, Nallasamy (1987), as well as Patel et al. (1985).)
- TEACH Manual
- CAST Manual

Heat Transfer:

$$(H_c - H_w)\frac{\rho C_\mu^{1/4} k^{1/2}}{Q_w} = \frac{P_{r_t}}{\mathrm{K}} \ln\left[C_\mu^{1/4} \frac{y k^{1/2}}{\nu}\right] + C_Q(P_r)$$

$$C_Q = 12.5\,\mathrm{Pr}^{2/3} + 2.12 \ln \mathrm{Pr} - B^*;\quad B^* = \begin{cases} -5.3 & \mathrm{Pr} > 0.5 \\ -1.3 & \mathrm{Pr} \le 0.5 \end{cases}$$

H = Static (Stagnation) enthalpy

$$\frac{\partial H}{\partial y} = \frac{\mathrm{Pr}}{\mu} Q_w$$

Q_w = Wall Heat Flux

$$-\rho\overline{u'h'} = \frac{\mu_t}{\mathrm{Pr}_t}\frac{\partial H}{\partial x};\quad -\rho\overline{v'h'} = \frac{\mu_t}{\mathrm{Pr}_t}\frac{\partial H}{\partial y}$$

also note $$-\rho\overline{v'\vartheta'} = \frac{\mu_t}{\mathrm{Pr}_t}\frac{\partial T}{\partial y}$$

For $y_c^+ < 11.6$ use sub-layer equations

$$U_c^+ = y_c^+;\qquad \tau_w = \mu U_e \Big/ y_c$$

$$T_c^+ = \mathrm{Pr}\, y^+;\qquad T^+ = \frac{(\tau_w - T_c)}{\left(Q_w \Big/ \rho\varphi U_*\right)}$$

$k \sim y^2$, $\varepsilon \sim \dfrac{2\nu k}{y^2}$ or $\varepsilon_w = 2\nu\left(\dfrac{\partial k^{1/2}}{\partial y}\right)^2$

$$k = \frac{U_*^2}{C_\mu^{1/2}}\left(y \Big/ y_0\right)\qquad y_0 = 11.6$$

$$\varepsilon = \frac{2U_*^3}{C_\mu^{1/4} y_0^+}$$

Alternatively set k=0, $\dfrac{\partial \varepsilon}{\partial y} = 0\left(\text{or } \varepsilon = \nu\dfrac{\partial^2 k}{\partial y^2}\right)$

at the wall, y=0

Smooth Boundary Layers:

Given x (longitudinal distance along the boundary layer), some estimate for U_e, ρ, and μ

$$\mathrm{Re}_x = \frac{U_e x}{\nu}\qquad C_f = \frac{0.455}{\ln^2(0.06\,\mathrm{Re}_x)}\qquad u_\tau = U_e\left(\frac{C_f}{2}\right)^{1/2}$$

$$C_f = \frac{\tau_w}{1/2\,\rho U_e^2}\qquad u_\tau = \left(\frac{\tau_w}{\rho}\right)^{1/2}\qquad y^+ = \frac{y u_\tau}{\nu}$$

can be used to determine y values that give $30 < y^+ < 0.1\delta^+(x)$, which are usually accepted as valid values for y^+ in the log layer. The boundary layer thickness, if needed, may be estimated from

$$\frac{\delta}{x} \approx \frac{0.37}{Re_x^{1/5}}$$

Fully Rough Boundary Layers:

For boundary layers that are considered fully rough ($k^+ = ku_\tau/\nu > 15$)

$$C_f = \left[2.87 + 1.58\log\left(\frac{x}{k_s}\right)\right]^{-2.5}$$

may be used to obtain some estimate of the friction coefficient for zero pressure boundary layers if comparisons to model predictions are needed. Numerically, the wall node must be placed at some distance above the equivalent roughness height k_S. This follows from the physical argument that the flow cannot exist inside the wall, whose edges effectively protrude a distance k_s into the flow.

Fully developed smooth pipe flow:

For favorable pressure gradients, the law of the wall closely holds over the entire pipe. Using this assumption:

$$U_{avg} = \frac{Q}{A} \qquad \frac{U_{avg}}{u_\tau} = \frac{1}{\kappa}\ln\left[\frac{r_{pipe}u_\tau}{\nu}\right] + B - \frac{3}{2\kappa}$$

and

$$\frac{1}{\Lambda^{1/2}} = 0.8686\ln\left(Re_D\,\Lambda^{1/2}\right) - 0.8$$

$$\frac{\Lambda}{8} = \frac{\tau_w}{\rho U_{avg}^2} \qquad \tau_w = \rho U_{avg}^2\frac{\Lambda}{8} \qquad u_\tau = \left(\frac{\tau_w}{\rho}\right)^{1/2} \qquad y^+ = \frac{yu_\tau}{\nu}$$

10.0 EFFECTS OF BUOYANCY

While most turbulence is generated as a function of shear, in flows with appreciable density gradients the turbulent eddies may receive or lose energy due to the effects of buoyancy. By dividing the density into a fluctuating quantity and a mean quantity as was done for the other variables, it can be shown that the mean momentum equation is given by

$$\rho\frac{\partial}{\partial t}U_i + \rho U_j\frac{\partial U_i}{\partial x_j} = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\mu\overline{S_{ji}}\right) - \frac{\partial}{\partial x_j}\left(\overline{\rho u_i' u_j'}\right) + (\rho+\rho')g_i \tag{10.1}$$

where the density fluctuations are neglected in every term except in the gravity term, as in the Boussinesq approximation. The specific turbulent kinetic energy equation needs to be modified to include buoyancy effects as

$$\begin{aligned}\rho\frac{\partial k}{\partial t} + \rho\frac{\partial}{\partial x_j}\left(U_j k\right) &= \tau_{ij}\frac{\partial U_i}{\partial x_j} - \mu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_k}} + \overline{u_i'\rho'}g \\ &+ \frac{\partial}{\partial x_j}\left(\mu\frac{\partial k}{\partial x_j} - \frac{1}{2}\rho\overline{u_i' u_i' u_j'} - \overline{p' u_j'}\right)\end{aligned} \tag{10.2}$$

The extra term represented by

$$B_k = \overline{u_i'\rho'}g \tag{10.3}$$

represents the production of turbulent kinetic energy by buoyancy, and is the rate of work done against buoyancy forces by turbulent motion. This is essentially a transfer of either potential energy to turbulent kinetic energy, as would be the case in an unstably stratified fluid, or a transfer of turbulent kinetic energy to potential energy, as would be the case for the mixing of a heavy fluid with a lighter fluid against the action of gravity. A full derivation of the mean mass, momentum, and turbulent kinetic energy equations are given in Appendix D.

The buoyant production term is usually modeled as

$$\overline{u_i'\rho'}g = |g_i|\frac{\mu_t}{\rho\sigma_\rho}\frac{\partial\rho}{\partial x_i} \tag{10.4}$$

Here σ_ρ is a kind of turbulent Prandtl number. The density derivative is used since a negative density gradient usually corresponds to a stably stratified flow and the term acts as a sink on turbulent kinetic energy. For positive density gradients an unstable stratification usually exists and this term acts as a source in the equation.

A similar modification is made to the epsilon equation by adding the source term given by (Rodi, 1980)

$$B_\varepsilon = \frac{C_{\varepsilon 1}\varepsilon}{k} C_{\varepsilon 3} \max(B_k, 0) \tag{10.5}$$

where $C_{\varepsilon 3}$ is a constant. This term increases ε for unstable stratification, and gives no change in ε for stable stratification. With this modification, the epsilon equation is given by

$$\frac{\partial}{\partial t}(\rho\varepsilon) + \frac{\partial}{\partial x_j}(\rho u_j \varepsilon) = C_{\varepsilon 1}\frac{\varepsilon}{k}\tau_{ij}\frac{\partial u_i}{\partial x_j} + \frac{\varepsilon}{k}C_{\varepsilon 3}\max(G_k, 0) - C_{\varepsilon 2}\rho\frac{\varepsilon^2}{k} + \frac{\partial}{\partial x_j}\left[\left(\mu + \frac{\mu_t}{\sigma_k}\right)\frac{\partial \varepsilon}{\partial x_j}\right] \tag{10.6}$$

Streamline Curvature and Rotational Effects:

The effects of streamline curvature and rotation can be accounted for by analogy to the buoyancy effects (see Bradshaw, 1969; Rodi, 1979, Sloan et al., 1986). Here appropriate Richardson numbers must be defined. For a streamline with curvature R, and velocity U tangent to the streamline the appropriate Richardson number is

$$\mathrm{Ri_g} = \frac{2\mathrm{U}\left(\frac{\partial(\mathrm{UR})}{\partial \mathrm{n}}\right)}{\mathrm{R}^2\left(\frac{\partial \mathrm{U}}{\partial \mathrm{n}}\right)^2}$$

where the subscript ,n denotes differentiation w.r.t. to the normal direction to the streamline. For stable flow $\mathrm{Ri_g} > 0$.

For rotating (or swirling) flows Bradshaw (1969) suggest a gradient Richardson number defined by

$$\mathrm{Ri_g} = \frac{2\left(\mathrm{W}/\mathrm{r}\right)\left[\frac{\partial \mathrm{U}}{\partial \mathrm{r}} + 2\,\mathrm{W}/\mathrm{r}\right]}{\left(\frac{\partial \mathrm{U}}{\partial \mathrm{r}}\right)^2}$$

where $W = \Omega r$ is the swirl velocity in tangential direction, Ω is the angular velocity, U is the velocity in axial direction, and r is the distance in the radial direction. In the literature various Richardson numbers are used to account for the influence of rotation. For example Rodi (1979) suggests the use of a so called flux Richardson number defined by

$$Ri_f = \frac{2\left(W/r\right)\left[\frac{\partial U}{\partial r} - W/r\right]}{\left[\left(\frac{\partial U}{\partial r}\right)^2 + \left(\frac{\partial W}{\partial r} - W/r\right)^2\right]}$$

It is usually the source terms of the ε-equation that is modified by making the model constants a function of Richardson number to account for the above mentioned effects.
For rotating flows the constnat Cε2 is modified as

$$(C_{\varepsilon 2})_{mod} = C_{\varepsilon 2}\,(1 - C_g\, Ri_g)$$

For zero-equation models it is the mixing length that is modified (see Rodi, 1980)

11.0 ADVANCED MODELS

As was mentioned in Chapter 8.0, the eddy viscosity approximation for determining the Reynolds stresses is not a good model for flows with sudden changes in mean strain rate, curved surfaces, secondary motions, rotating and stratified flows, separated, and three- dimensional flows. The primary assumptions of local isotropy and local equilibrium are the main reasons that two-equation models typically fail to give anything more than qualitative results for such applications. Inherent in these assumptions is that the normal Reynolds stresses are equal and that flow history effects on the Reynolds stresses are negligible. Two types of models aimed at improving the predictive capabilities for flows where such assumptions are not valid are algebraic stress models and second moment closure models.

Algebraic Stress Models

The primary distinction of an algebraic stress model (ASM) is the assumption that the Reynolds stresses are given by a series expansion of functionals, of which the Boussinesq approximation only contains the first terms. While there have been numerous efforts in this area by a number of different investigators, and the forms of different algebraic stress models may vary, the stress-strain/vorticity constitutive relationship for models which carry the quadratic or cubic terms in the expansion can be written in the following canonical form for incompressible flows (Apsley *et al*, 1997):

$$\begin{aligned} a_{ij} = &-2C_\mu \frac{k}{\tilde{\varepsilon}} S_{ij} + C_1 \frac{k^2}{\tilde{\varepsilon}^2}\left(S_{ik}S_{ij} - \frac{1}{3}S_{kl}S_{kl}\delta_{ij} \right) + C_2 \frac{k^2}{\tilde{\varepsilon}^2}\left(S_{ik}\Omega_{jk} + S_{jk}\Omega_{ik} \right) \\ &+ C_3 \frac{k^2}{\tilde{\varepsilon}^2}\left(\Omega_{ik}\Omega_{jk} - \frac{1}{3}\Omega_{kl}\Omega_{kl}\delta_{ij} \right) + C_4 \frac{k^3}{\tilde{\varepsilon}^3}\left(S_{ik}\Omega_{jl} + S_{jk}\Omega_{il} \right)S_{kl} \\ &+ C_5 \frac{k^3}{\tilde{\varepsilon}^3}\left(\Omega_{ik}\Omega_{kl}S_{lj} + \Omega_{jk}\Omega_{kl}S_{li} - \frac{2}{3}\Omega_{kl}S_{lm}\Omega_{mk}\delta_{ij} \right) + C_6 \frac{k^3}{\tilde{\varepsilon}^3} S_{kl}S_{kl}S_{ij} \\ &+ C_7 \frac{k^3}{\tilde{\varepsilon}^3}\Omega_{kl}\Omega_{kl}S_{ij} \end{aligned} \tag{11.1}$$

where a_{ij} is the anisotropy tensor, given by

$$a_{ij} \equiv \frac{\overline{u_i u_j}}{k} - \frac{2}{3}k\delta_{ij} \tag{11.2}$$

Here, S_{ij} and Ω_{ij} are the mean strain-rate and mean vorticity tensors. (see Equation 11.8)

Note that in Eq. (11.1) allows for models (e.g. Craft *et al*, 1997) that make a distinction between the dissipation rate ε and its homogeneous portion $\tilde{\varepsilon} = \varepsilon - 2\nu\left(\frac{\partial k^{1/2}}{\partial y} \right)^2$.

There are many different variants of this model which adopt different values for the coefficients, e.g. Lien and Leschziner (1993); Craft *et al* (1997), and which may carry only the quadratic terms in the expansion or proceed through the cubic terms. Shih *et al.*'s (1993) model employs Eq. (11.1) only through the quadratic terms (up to and including the fourth term on the RHS), and utilizes the following values for the coefficients:

$$C_\mu = \frac{2/3}{1.25 + \bar{S} + 0.9\bar{\Omega}} \tag{11.3}$$

$$(C_1, C_2, C_3) = (3, 15, 19)\frac{1}{1000 + \bar{S}^3} \tag{11.4}$$

where $\bar{S}$ and $\bar{\Omega}$ are the strain and vorticity invariants, respectively, given by

$$\bar{S} = \frac{k}{\varepsilon}\sqrt{2S_{ij}S_{ij}} \qquad\qquad \bar{\Omega} = \frac{k}{\varepsilon}\sqrt{2\Omega_{ij}\Omega_{ij}} \tag{11.5}$$

The cubic extension of the model, including all of the terms on the RHS of Eq. (11.1), uses the following values for the remaining coefficients:

$$(C_4, C_5, C_6, C_7) = (80, 0, -16, 16)C_\mu{}^3 \tag{11.6}$$

Gatski and Speziale (1993) derive an explicit algebraic stress model, including the quadratic terms in the expansion, in terms of the turbulent kinetic energy $\kappa \equiv \frac{1}{2}\overline{u_i u_i}$. Here, the Reynolds stress tensor is given by

$$\tau_{ij} = \frac{2}{3}\kappa\delta_{ij} - 2C_\mu{}^* \frac{\kappa^2}{\varepsilon}S_{ij} - \beta_1 \frac{\kappa^3}{\varepsilon^2}\left(S_{ik}\Omega_{kj} + S_{jk}\Omega_{ki}\right) + \beta_2 \frac{\kappa^3}{\varepsilon^2}\left(S_{ik}S_{kj} - \frac{1}{3}S_{mn}S_{mn}\delta_{ij}\right) \tag{11.7}$$

where $\beta_1 = \frac{1}{2}g^2(2 - C_4)\left(\frac{4}{3} - C_2\right)$, $\beta_2 = g^2(2 - C_3)\left(\frac{4}{3} - C_2\right)$, and $C_\mu{}^* = \frac{1}{2}g\left(\frac{4}{3} - C_2\right)$, which has an approximate value of 0.09. Here,

$$S_{ij} = \frac{1}{2}\left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i}\right) \qquad\qquad \Omega_{ij} = \frac{1}{2}\left(\frac{\partial \bar{v}_i}{\partial x_j} - \frac{\partial \bar{v}_j}{\partial x_i}\right) \tag{11.8}$$

where the overbars denote ensemble averaging, and

$$g = \left(\frac{1}{2}C_1 + \frac{P_k}{\varepsilon} - 1\right)^{-1} \tag{11.9}$$

where $P_k = -\tau_{ij}\,\partial \bar{v}_i / \partial x_j$ represents the production of turbulent kinetic energy.
In the above expressions, C_1, C_2, C_3, and C_4 are the model constants that arrive in the pressure-strain correlation tensor, given by

$$\Pi_{ij} = -C_1 \varepsilon b_{ij} + C_2 \kappa S_{ij} + C_3 \kappa \left(b_{ik} S_{jk} + b_{jk} S_{ik} - \frac{2}{3} b_{mn} S_{mn} \delta_{ij} \right) + C_4 \kappa \left(b_{ik} W_{jk} + b_{jk} W_{ik} \right) \qquad (11.9)$$

where

$$b_{ij} = \frac{\tau_{ij} - \frac{2}{3}\kappa \delta_{ij}}{2\kappa} \qquad (11.10)$$

is the anisotropy tensor, and

$$W_{ij} = \Omega_{ij} + \eta_{mji}\omega_m \qquad (11.11)$$

is the absolute vorticity tensor, where ω_m and η_{mji} represent the angular velocity and permutation tensor, respectively. While the model constants given above have been examined by many investigators, the values determined in two noteworthy models are given below, as determined by Launder, Reece, and Rodi (1975), and Gibson and Launder (1978):

Launder, Reece, and Rodi Model

$C_1 = 3.0$ $C_2 = 0.8$ $C_3 = 1.75$ $C_4 = 1.31$ (11.12)

Gibson and Launder Model

$C_1 = 3.6$ $C_2 = 0.8$ $C_3 = 1.2$ $C_4 = 1.2$ (11.13)

Several formulations which are similar to, or variants of, Eq. (11.1) and Eq. (11.7) exist. The primary advantage of using such a non-linear constitutive equation is the ability to predict anisotropy in the normal Reynolds stresses. This is very useful for cases where secondary flows occur, such as when a pipe or channel is curved. The primary disadvantage of using a formulation similar to Eq. (11.1) or Eq. (11.7) is that the local equilibrium assumption is still present, leaving flow history effects unaccounted for.

Second Order Closure Models: RSM

With the price being a rather large degree of complexity, second order closure models seek to model the exact differential equation describing the Reynolds stress tensor, which was given in Chapter 2.0 by

$$\frac{\partial \tau_{ij})}{\partial t} + U_k \frac{\partial(\tau_{ij})}{\partial x_k} = -\tau_{jk}\frac{\partial U_i}{\partial x_k} - \tau_{ik}\frac{\partial U_j}{\partial x_k} + \varepsilon_{ij} - \Pi_{ij} + \frac{\partial}{\partial x_k}\left(\upsilon \frac{\partial(\tau_{ij})}{\partial x_k} + C_{ijk} \right) \qquad (11.14)$$

Inherent in such a model are the effects of flow history, as the Reynolds stress equation does include convection and diffusion terms for the stresses. Also, the Reynolds stress equations include production and body force terms that can respond to effects given by streamline curvature, rotation, and buoyancy. Another advantage is that there is no reason why the normal Reynolds stresses as predicted from the modeled equation should be equal. The major computational requirement for these models is that for a three-dimensional flow, six additional transport equations for the Reynolds stresses must be solved in addition to the typical Navier-Stokes and continuity equations. In most cases, the k and ε equations must also be solved. A major difficulty in working with such models is that the user must be skilled in tensor calculus to understand different models.

Large Eddy Simulation

At the highest level of complexity in numerical simulations of turbulence, Large Eddy Simulation (LES) ranks second only to direct numerical simulation (DNS) as being computationally intensive. The primary idea behind LES is to simulate only the larger scales of turbulence that are set by the geometry or specific flow conditions, and to account for the influence of the neglected smaller scales by use of a model. As such they need to be three-dimensional and transient calculations with relatively accurate numerical schemes. Since turbulence is known to be more isotropic at smaller scales, it is believed that the usual model assumptions involved in the eddy viscosity models should not matter as long as the grid is sufficiently small, since the neglected scales are roughly proportional to the mesh size.

The first step in LES is to filter the Navier-Stokes equations to determine which scales will be kept and which scales will be discarded. This may be interpreted as weakly analogous to Reynolds averaging where only the time averaged values are kept; however, LES differs in the fact that the filtering keeps the turbulence occurring on scales above a desired minimum filter width.

A generalized filter can be defined by

$$\langle u_i(x,t)\rangle = \iiint G(x-\xi:\Delta)u_i(\xi,t)d^3\xi \tag{11.15}$$

where the filter function is interpreted to keep values of u_i occurring on scales larger than the filter width Δ. The filter function G, is basically some function which is effectively zero for values of u_i occurring at the small scales. Here < > indicates a filtered variable.

Consider the Navier-Stokes equation for incompressible flow in terms of the mean flow quantities

$$\frac{\partial U_i}{\partial t} + \frac{\partial U_i U_j}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu\nabla^2 U_i \tag{11.16}$$

By Leibnitz' rule, the filtering operation commutes with partial differentiation with respect to time, i.e.,

$$\left\langle \frac{\partial U_i}{\partial t} \right\rangle = \frac{\partial \langle U_i \rangle}{\partial t} = \frac{\partial \overline{U_i}}{\partial t} \tag{11.17}$$

Note that $\langle \phi \rangle = \overline{\phi}$ both imply spatial filtering. Assuming that the filtering operator also commutes with spatial differentiation, then

$$\left\langle \frac{\partial U_i U_j}{\partial x_j} \right\rangle = \frac{\partial \langle U_i U_j \rangle}{\partial x_j} \tag{11.18}$$

With the assumption that the density does not vary (incompressible), then the pressure gradient can be filtered by

$$\left\langle -\frac{1}{\rho}\frac{\partial P}{\partial x_i} \right\rangle = -\frac{1}{\rho}\left\langle \frac{\partial P}{\partial x_i} \right\rangle = -\frac{1}{\rho}\frac{\partial \langle P \rangle}{\partial x_i} = -\frac{1}{\rho}\frac{\partial \overline{P}}{\partial x_i} \tag{11.19}$$

The diffusion terms due to molecular viscosity are filtered according to

$$\left\langle \nu \nabla^2 U_i \right\rangle = \nu \left\langle \nabla^2 U_i \right\rangle = \nu \nabla^2 \langle U_i \rangle = \nu \nabla^2 \overline{U_i} \tag{11.20}$$

Finally, the filtered Navier-Stokes equations are given by

$$\frac{\partial \overline{U_i}}{\partial t} + \frac{\partial \langle U_i U_j \rangle}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \overline{P}}{\partial x_i} + \nu \nabla^2 \overline{U_i} \tag{11.21}$$

Note that in Eq. (11.21), the overbar denotes a filtered variable rather than a time-averaged quantity. The difficulty in Eq. (11.21) comes from the nonlinear term, $\langle U_i U_j \rangle$. To handle this term, an analogy to Reynolds decomposition for time averaging is used, whereby the instantaneous velocity is broken into a large-scale component and a subgrid-scale component by

$$U_i = \overline{U_i} + u_i' \tag{11.22}$$

where $\overline{U_i}$ denotes the large-scale component of the velocity field that is resolved, and u_i' denotes the small-scale component which is filtered out. With this, the nonlinear advection term becomes

$$\left\langle U_i U_j \right\rangle = \left\langle \overline{U_i}\,\overline{U_j} \right\rangle + \left\langle \overline{U_i} u_j' \right\rangle + \left\langle u_i' \overline{U_j} \right\rangle + \left\langle u_i' u_j' \right\rangle \tag{11.23}$$

Substituting Eq. (11.23) into Eq. (11.21) yields

$$\frac{\partial \overline{U_i}}{\partial t} + \frac{\partial \overline{U_i}\,\overline{U_j}}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \overline{P}}{\partial x_i} - \frac{\partial \tau_{ij}}{\partial x_j} + \nu \nabla^2 \overline{U_i} \tag{11.24}$$

where

$$\tau_{ij} = \underbrace{\left\langle \overline{U_i} u_j' \right\rangle + \left\langle u_i' \overline{U_j} \right\rangle}_{C_{ij}} + \underbrace{\left\langle u_i' u_j' \right\rangle}_{R_{ij}} \tag{11.25}$$

The terms represented by C_{ij} are called "cross-terms" and physically represent random forcing produced by subgrid-scales on the large scales. The terms represented by R_{ij} are called "subgrid Reynolds stresses", analogous to the turbulent stresses that result from Reynolds averaging of the Navier-Stokes equations. It is the purpose of subgrid scale (SGS) models to resolve the influence of these additional unknown subgrid scale stresses that result from the filtering process.

Subgrid scale (SGS) models are similar to the turbulence model used in RANS (Reynolds Averaged Navier-Stokes) calculations, in that its primary purpose is to provide the influence of the small scales, usually based on some gradient diffusion hypothesis. Here, the rationale follows that the role of the small eddies is predominantly to accept energy from the larger scales and dissipate it, and that this transfer of energy is considered to be a one-way process. This influence acts primarily as a sink for the energy of the large scales to ensure that the statistics of the large scales are correct in that they continually dissipate their energy. Some popular subgrid scale models include the Smagorinsky (see e.g. Ciafalo, 1994; Piomelli, 1993a), *k*-ε, and dynamic Smagorinsky model (see e.g. Ghosal *et al*, 1995; Piomelli, 1993b).

12.0 CONCLUSIONS

With increased computer resources and speed now available at reasonable cost, many turbulence models are being developed and tested. With the advent of faster and more powerful computers, the limitations of some of the more complex modeling issues are being eliminated. The present study examined many of the numerical, theoretical, and practical issues involved in computational efforts at accurately resolving turbulent flows. Much of the focus of this work was devoted to examining the closure problem and the more popular algebraic and two-equation models, as well as investigating case specific issues as they relate to different types of flows.

The additional terms (i.e. the Reynolds stresses and turbulent fluxes) that arise as a result of time-averaging the equations of motion form the heart of the closure problem in turbulence modeling. These extra terms involve correlations between fluctuating velocity components, and are not known a priori. The solution of the conservation equations requires some empirical input to formulate mathematical models for these additional terms, and thus close the set of equations. This is the purpose of turbulence modeling.

The earliest proposal for modeling the Reynolds stresses is still in use in many turbulence models today. It involves the Boussinesq eddy viscosity/diffusivity concept, which assumes that the turbulent stresses are proportional to the mean velocity gradient. The primary goal of many turbulence models, then, is to provide some prescription for the eddy viscosity to model the Reynolds stresses. These may range from relatively simple algebraic models, to the more complex two-equation models, such as the k-ε model, where two additional transport equations are solved in addition to the mean flow equations.

There are four main categories of turbulence models:

(1) Algebraic (Zero-Equation) Models
(2) One-Equation Models
(3) Two-Equation Models
(4) Second-Order Closure Models

Each of these has received some attention from various researchers in the past, while algebraic models and two-equation models have received the most attention and met with the most success in practical application. It was not until recently that computational resources were sufficient to effectively apply second-order closure models, due to the large number of equations and complexity involved. Two-equation models, especially the k-ε model, have enjoyed the most success and popularity. While this study is not intended to be a complete catalogue of all turbulence models, there are several models that are examined in this work. The reader is referred to other sources given in the bibliography for more detailed information regarding specific models.

The influence of geometric effects and flow conditions are also examined (see Chapters 9.0 and 10.0). Specifically, the present study examines wall-bounded flows, including flow near a solid

surface, surface roughness, and resolution of the viscous sublayer, as well as the effects of buoyancy, which may effect the manner in which turbulent eddies receive or dissipate turbulent kinetic energy. More complex issues such as second order closure models, and large eddy simulation (LES) are briefly discussed in Chapter 11.0. Again, for more detailed information the reader is referred to other sources.

While not being a complete examination of all turbulence models* in use, this study provides a foundation for understanding the main concepts involved in turbulence modeling, as well as case specific issues for application to certain types of turbulent flows.

* A long list of commonly used turbulence models is presented in Appendix F for further reference.

ACKNOWLEDGEMENTS:

This work book has been prepared with the help of my former graduate students, Mr. Matthew R. Umbel and Wesley M. Wilson. Their contributions are greatly appreciated. Many thanks are due to Steven Rowan, and my student Andrew Burt for their help in typing many additional parts to this document.

BIBLIOGRAPHY

[1] Apsley, D. D., Chen, W. L., Leschziner, M. A., and Lien, F. S. (1997), "Non-Linear Eddy-Viscosity Modeling of Separated Flows," Report No. TFD/97/02, University of Manchester Institute of Science and Technology, Manchester, UK.

[2] Baldwin, B. S. and Lomax, H. (1978), "Thin-Layer Approximation and Algebraic Model for Separated Turbulent Flows," AIAA Paper 78-257, Huntsville, AL.

[3] Bardina, J. E., Huang, P. G., and Coakley, T. J., (1997) “Turbulence Modeling Validation, Testing, and Development,” NASA Technical Memorandum 110446, Ames Research Center.

[4] Boussinesq, J. (1877), "Theorie de l'Ecoulement Tourbillant," Mem. Presentes par Divers Savants Acad. Sci. Inst. Fr., Vol. 23, pp. 46-50.

[5] Bradshaw, P. (1969) "The Analogy Between Streamline Curvature and Buoyancy in Turbulent Shear Flow," J.Fluid Mechanics, vol 36, pp. 177-191.

[6] Bradshaw, P., Ferriss, D. H. and Atwell, N. P. (1967), "Calculation of Boundary Layer Development Using the Turbulent Energy Equations," Journal of Fluid Mechanics, Vol. 23, Pt. 3, pp. 593-616.

[7] Cebeci, T. and Bradshaw P., (1988) Physical and Computational Aspects of Convective Heat Transfer, Springer, New York.

[8] Chen, H.C. and Patel, V.C. (1987) “Practical Near-Wall turbulence Models for Complex Flows including Separation,” AIAA Paper 87-1300.

[9] Chen, C.-J. and Jaw, S.-Y., (1998) Fundamentals of Turbulence Modeling, Taylor & Francis, Washington, DC.

[10] Cebeci, T. and Smith, A. M. O. (1974), Analysis of Turbulent Boundary Layers, Ser. In Appl. Math. & Mech., Vol. XV, Academic Press.

[11] Celik, I. And Zhang, W-M. (1995) “Calculation of Numerical Uncertainty Using Richardson Extrapolation: Application to Some Simple Turbulent Flow Calculations,” Journal of Fluids Engineering, Vol. 117, pp. 439-445

[12] Ciafalo, M. (1994), "Large-eddy Simulation: Models and Applications," Adv. Heat Transfer, Vol. 25, pp. 322-408.

[13] Craft, T. J., Launder, B. E., and Suga, K. (1997), "Prediction of Turbulent Transitional Phenomena with a Nonlinear Eddy-Viscosity Model," International Journal of Heat and Fluid Flow, Vol. 18, p. 15.

[14] Daly, B. J. and Harlow, F. H. (1970), "Transport Equations in Turbulence," Physics of Fluids, Vol. 13, pp. 2634-2649.

[15] Donaldson, C. DuP. And Rosenbaum, H. (1968), "Calculation of the Turbulent Shear Flows Through Closure of the Reynolds Equations by Invariant Modeling," ARAP Report No. 127, Aeronautical Research Associates of Princeton, Princeton, NJ.

[16] Emmons, H. W. (1954), "Shear Flow Turbulence," Proceedings of the 2nd U.S. Congress of Applied Mechanics, ASME.

[17] Ferziger, J.H. (1989) “Estimation and reduction of Numerical Errors,”Forum on Methods of Estimating Uncertainty Limits in Fluid Flow Computations, ASME Winter Annual Meeting, San Francisco, CA

[18] Frisch, U., (1995) Turbulence: The Legacy of A. N. Kolmogorov, Cambridge University Press, New York, NY.

[19] Gatski, T. B., and Speziale, C. G. (1993), "On Explicit Algebraic Stress Models for Complex Turbulent Flows," Journal of Fluid Mechanics, Vol. 254, pp. 59-78.

[20] Ghosal, S., Lund. T., Moin, P. and Akselvoll, K. (1995), "A Dynamic Localization Model for Large-eddy Simulation of Turbulent Flows," Journal of Fluid Mechanics, Vol. 286, pp. 229-255.

[21] Gibson, M. M. and Launder, B. E. (1978), "Ground Effects on Pressure Fluctuations in the Atmospheric Boundary Layer," Journal of Fluid Mechanics, Vol. 86, pp. 491-511.

[22] Glushko, G. (1965), "Turbulent Boundary Layer on a Flat Plate in an Incompressible Fluid," Izvestia Academy Nauk. SSSR Mekh., No. 4, p. 13.

[23] Haase, W., Chaput, E., Elsholz, E., Leschziner, M. and Müller, U. Editors (1997) “ECARP-European computational Aerodynamics Research Project: Validation of CFD Codes and Assessment of Turbulence Models,” Notes on Numerical Fluid Mechanics, Vol. 58, pp. IX-X.

[24] Hanjalic, K. and Launder, B.E.(1972), “ A Reynolds Stress Model of Turbulence and its Application to Thin Shear Flows,” Journal of Fluid Mechanics, vol. 52, No. 4, pp. 609-638.

[25] Hogg, S. and Lerschziner, M.A. (1989) “Computation of Highly Swirling Confined Flow with a Reynolds Stress Turbulence Model,” AIAA Journal, vol. 27, No. 1, pp. 57-63.

[26] Johnson, D. A. and King, L. S. (1985), "A Mathematical Simple Turbulence Closure Model for Attached and Separated Turbulent Boundary Layers," AIAA Paper 84-0175.

[27] Kade and Yaglom (1972), “Heat and Mass Transfer Laws for Fully Turbulent Wall Flow,”International Journal of Heat & Mass Transfer, Vol. 15, pp. 2329-2351.

[28] Kolmogorov, A. N. (1942), "Equations of Turbulent Motion of an Incompressible Fluid," Izvestia Academy of Sciences, USSR; Physics, Vol. 6, Nos. 1 and 2, pp. 56-58.

[29] Launder, B. E. and Spalding, D. B. (1972*)*, Mathematical Models of Turbulence, Academic Press, London.

[30] Launder, B. E., Reece, G. J. and Rodi, W. (1975), "Progress in the Development of a Reynolds-Stress Turbulence Closure," Journal of Fluid Mechanics, Vol. 68, Pt. 3, pp. 537-566.

[31] Leschziner, M. A., (1998) “Advances in Modelling Physically Complex Flows with Anisotropy-Resolving Closures and Related Validation,” Proc. of FEDSM’98, 1998 ASME Fluids Engineering Division Summer Meeting, June 21-25, 1998, Washington, DC.

[32] Lien, F. S. and Leschziner, M. A. (1993), "A Pressure-Velocity Solution Strategy for Compressible Flow and its Application to Shock/Boundary-Layer Interaction Using Second-Moment Turbulence Closure," Journal of Fluids Engineering, Vol. 115, pp. 717-725.

[33] Lumley (1970), "Toward a Turbulent Constitutive Equation." Journal of Fluid Mechanics, Vol. 41, pp. 413-434.

[34] Lumley, J. L. (1978), "Computational Modeling of Turbulent Flows," Adv. Appl. Math., Vol. 18, pp. 123-176.

[35] Menter, F.R., (1994) “Two-Equation Eddy-Viscosity Turbulence Models for Engineering Applications,” AIAA Journal, Vol. 32, pp. 1598-1605.

[36] Menter, F.R., and Scheuerer, G., (1998) “Turbulence Modeling with Perspective To CFD,”Lecture Notes, AEA Technology Users Conference, Wilmington, DE.

[37] Nallasamy, M. (1987), “Turbulence Models and Their Applications to the Prediction of Internal Flows: A Review” Computers & Fluids, vol. 15, No. 2, pp. 151-194.

[38] Patel, V.C., Rodi, W., and Scheuerer, G. (1985), “Turbulence Models for Near-wall and Low Re number flows, A Review”, AIAA Journal, Vol 23, No.9, pp 1308-1319.

[39] Piomelli, U. (1993a), "Applications of Large Eddy Simulations in Engineering," In Large Eddy Simulation of Complex Engineering and Geophysical Flows, Eds. B. Galperin and S. A. Orszag, pages 119-141, Cambridge University Press, New York.

[40] Piomelli, U. (1993b), "High Reynolds Number Calculations Using the Dynamic Subgrid-Scale Stress Model," Physics of Fluids A, Vol. 5, pp. 1484-1490.

[41] Prandtl, L. (1925), "Uber die ausgebildete Turbulenz," *ZAMM*, Vol. 5, pp. 136-139.

[42] Prandtl, L. (1945), "Uber ein neues Formelsystem fur die ausgebildete Turbulenz," Nacr. Akad. Wiss. Gottingen, Math-Phys. Kl. 1945, pp6-19.

[43] Reynolds, O. (1895), "On the Dynamical Theory of Incompressible Viscous Fluids and the Determination of the Criterion," Philosophical Transactions of the Royal Society of London, Series A, Vol. 186, p. 123.

[44] Reynolds, W. C. (1987), "Fundamentals of Turbulence for Turbulence Modeling and Simulation," In Lecture Notes for von Karman Institute, AGARD Lecture Series No.86, pp. 1-66, New York: NATO.

[45] Rodi, W. (1979) "Influence of Buoyancy and Rotation on Equations for Turbulent Length Scale," , Proc. of the 2nd Int. Symp. on Turbulent Shear Flows, pp. 10.37-10.42, Imperial College, London.

[46] Rodi, W. (1980), "Turbulence Models and Their Application in Hydraulics - A State-of-the-Art Review", An IAHR (International Association of Hydraulic Research) Publication, Delft, The Netherlands.

[47] Rosten, H.I., and Worrell, J.K. (1988) "Generalized Wall functions for Turbulent Flows," Phoem'cs Journal, cham Ltd, Bakery House 40 High Street, Wimbledon, London SW196 AU

[48] Rotta, J. C. (1951), "Statistische Theorie nichthomogener Turbulenz," Zeitschrift fur Physik, Vol. 129, pp. 547-572.

[49] Saffman, P. G. (1970), "A Model for Inhomogeneous Turbulent Flow," *Proc. Roy. Soc. London.*, Vol. A317, pp. 417-433.

[50] Saffman, P. G. (1976), "Development of a Complete Model for the Calculation of Turbulent Shear Flows," April 1976 Symposium on Turbulence and Dynamical Systems, Duke Univ., Durham, NC.

[51] Schlichting, H. (1979), Boundary Layer Theory, Seventh Ed., McGraw-Hill, New York.

[52] Shih, T-H., Zhu, J., and Lumley, J. L. (1993), "A Realisable Reynolds Stress Algebraic Equation Model," NASA TM105993.

[53] Shih, T-H. (1996), “Developments in Computational Modeling of Turbulent Flows,” NASA Contract Report #198458, february, Lewis Research Center.

[54] Spalart, P.R. and Athmaras, S.R. (1994), La Recherche Aérospatiale, Vol. 1, pp. 5-21.

[55] Sloan, D.G., Smith, P.J., and Smoot, L.D. (1986) "Modeling of Swirl in Turbulent Flow Systems," Prog, Enenrgy Combust. Sci., Vol 12, pp. 163-250

[56] Speziale, C. G. (1985), "Modeling the Pressure-Gradient-Velocity Correlation for Turbulence," Physics of Fluids, Vol. 28, pp. 69-71.

[57] Speziale, C. G. (1987a), "Second-Order Closure Models for Rotating Turbulent Flows," Q. Appl. Math., Vol. 45, pp. 721-733.

[58] Speziale, C. G. (1987b), "On Nonlinear k-l and k-ε Models of Turbulence," Journal of Fluid Mechanics, Vol. 178, pp. 459-475.

[59] Speziale, C. G., (1998) “Turbulence Modeling for Time-Dependent RANS and VLES: A Review,” AIAA Journal, Vol. 36, No. 2, pp. 173-184.

[60] Tennekes, H. and Lumley, J. L. (1983), A First Course in Turbulence, MIT Press, Cambridge, MA.

[61] Van Driest, E. R. (1956), "On Turbulent Flow Near a Wall," Journal of the Aeronautical Sciences, Vol. 23, p. 1007.

[62] Vandromme, D., (1995) “Overview of Turbulence Models for CFD in Industry,” von Karman Institute for Fluid Dynamics, Lecture Series 1995-03, April 3-7, 1995.

[63] Van Dyke, M., (1982) An Album of Fluid Motion, The Parabolic Press, Stanford, CA.

[64] Von Karman, T. (1930), "Mechanische Ahnlichkeit und Turbulenz," Proc. Int. Congr. Appl. Mech., 3rd, Stockholm, Part 1, pp. 85-105.

[65] V. Yakhot and S.A. Orszag, (1986) Renormalization group analysis of turbulence: 1. Basic theory. Journal of Scientific Computing 1(1): 3-51.

[66] V. Yakhot, S.A. Orszag, S. Thangam, T.B. Gatski, and C.G. Speziale, (1992) Development of turbulence models for shear flows by expansion technique. Physics of Fluids A, 4(7):510-20.

[67] White, F. M. (1991), Viscous Fluid Flow, 2nd Edition, McGraw-Hill, Inc., New York, NY.

[68] Wilcox, D. C. (1988a), "Reassessment of the Scale Determining Equation for Advanced Turbulence Models," AIAA Journal, Vol. 26, No. 11, pp. 1299-1310.

[69] Wilcox, D. C. (1993), Turbulence Modeling for CFD, DCW Industries, Inc., La Canada, CA.

[70] Woods, W. A., and Clark, D. G., Ed., (1988) Visualized Flow: Fluid Motion in Basic and Engineering Situations Revealed by Flow Visualization, Permagon Press, Oxford, UK.

[71] Zeman, O. (1990) “Dilatation Dissipation: The Concept and Application in Modeling Compressible Mixing Layers,” Physics of Fluids A, Vl. 2, No. 2, pp. 178-188.

APPENDICES

Appendix A: Taylor Series Expansion

The Taylor Series expansions of the fluctuating velocity components is given below:

$$u'\big|_{y=0.0} = u'\big|_{y} - \frac{\partial u'}{\partial y} y + \vartheta\left(y^2\right) \tag{A.1}$$

$$v'\big|_{y=0.0} = v'\big|_{y} - \frac{\partial v'}{\partial y} y + \vartheta\left(y^2\right) \tag{A.2}$$

$$w'\big|_{y=0.0} = w'\big|_{y} - \frac{\partial w'}{\partial y} y + \vartheta\left(y^2\right) \tag{A.3}$$

and

$$u'\big|_{y} = A(x,z,t)y + \vartheta\left(y^2\right) \tag{A.4}$$

$$v'\big|_{y} = B(x,z,t)y^2 + \vartheta\left(y^2\right) \tag{A.5}$$

$$w'\big|_{y} = C(x,z,t)y + \vartheta\left(y^2\right) \tag{A.6}$$

then

$$k \sim \frac{1}{2}\left(\overline{A^2} + \overline{C^2}\right)y^2 + \vartheta\left(y^3\right) + \left(B^2 y^4\right) \tag{A.7}$$

where

$$\left(B^2 y^4\right) \approx 0$$

so,

$$k \sim \frac{1}{2}\left(\overline{A^2} + \overline{C^2}\right)y^2 + \vartheta\left(y^3\right) \tag{A.8}$$

and

$$\varepsilon \sim \nu\left(\overline{A^2} + \overline{C^2}\right) + \vartheta(y) \sim \nu\left[\left(\frac{\partial u'}{\partial y}\right)^2 + \left(\frac{\partial w'}{\partial y}\right)^2\right] \tag{A.9}$$

Assuming $\varepsilon = \beta^* w k$

$$k \sim y^2 \qquad \varepsilon \sim \frac{2k\nu}{y^2} \qquad w \sim \frac{2k\nu}{\beta^* y^2} \tag{A.10}$$

as $y \to 0.0$

Appendix B: Sublayer Analysis

(1) Symmetry:

$$\frac{\partial w'}{\partial z} = 0.0 \qquad \text{(B.1)}$$

(2) In the sublayer, convection and the pressure gradient in the x-direction (dominant flow direction parallel to the wall) are negligible. So, from the mean U-momentum

$$\frac{\partial^2 \bar{u}}{\partial y^2} = 0.0 \qquad \text{(B.2)}$$

and from the instantaneous U-momentum

$$\frac{\partial^2 u}{\partial y^2} = 0.0 \qquad \text{(B.3)}$$

Eqs. (B.2) and (B.3) imply that

$$\frac{\partial^2 u'}{\partial y^2} = 0.0 \qquad \text{(B.4)}$$

and applying the condition that $u'|_{y=0} = 0$ then

$$\frac{\partial u'}{\partial y} = A(x,z,t) \qquad \text{(B.5)}$$

or

$$u' = A(x,z,t)\cdot y \qquad \text{(B.6)}$$

(3) Instantaneous continuity equation:

$$\frac{\partial w'}{\partial z} + \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0 \qquad \text{(B.7)}$$

Substituting Eqs. (B.1) and (B.6) into Eq. (B.7) yields

$$y\frac{\partial}{\partial x}[A(x,z,t)] + \frac{\partial v'}{\partial y} = 0 \qquad \text{(B.8)}$$

or

$$\frac{\partial v'}{\partial y} = B(x,z,t)\cdot y \qquad \text{(B.9)}$$

(4) Using a similar argument to that used in (2):

(a) Convection is negligible; the pressure gradient in the z-direction is negligible

(b) The Reynolds stresses are negligible

(c) The z-derivative of any quantity is negligible (symmetry - Eq. (B.1))

(d)
$$\frac{\partial}{\partial y}\left(\nu\frac{\partial\overline{w}}{\partial y}\right) >> \frac{\partial}{\partial x}\left(\nu\frac{\partial\overline{w}}{\partial x}\right) \tag{B.10}$$

and

$$\frac{\partial}{\partial y}\left(\nu\frac{\partial w}{\partial y}\right) >> \frac{\partial}{\partial x}\left(\nu\frac{\partial W}{\partial y}\right) \tag{B.11}$$

With this then

$$\frac{\partial^2 w'}{\partial y^2} = 0 \tag{B.12}$$

or

$$\frac{\partial w'}{\partial y} = C(x,z,t) \tag{B.13}$$

Mean Flow in the Sublayer:

Integrating Eq. (B.2) yields

$$\frac{\partial\overline{u}}{\partial y} = C_1 = \frac{\tau_w}{\rho\nu} = \frac{u_*^{\,2}}{\nu} \tag{B.14}$$

Integrating once more gives

$$\overline{u} = \frac{u_*^{\,2}}{\nu}y + C_2 \tag{B.15}$$

But at y=0, $\overline{u}$ = 0, hence C_2 = 0. Then it follows that

$$\frac{\overline{u}}{u_*} = \frac{yu_*}{\nu} \tag{B.16}$$

or

$$u^+ = y^+$$

which is Eq. (6.9).

Near-Wall Values of k-ε, and wall-functions

Very near the wall we assume first the equations of motion can be simplified to one-dimensional Couette flow. It follows that shear stress is constant in this region $\tau = \tau_w$ = constant and in the log-layer laminar contribution to shear stress is negligible. All this amounts to assuming an equilibrium wall layer, i.e. $P_k = \varepsilon$. Hence, it follows that

$$P_k = -\overline{uv}\frac{\partial U}{\partial y} = \varepsilon \tag{B.17}$$

$$\tau = -\rho\,\overline{uv} = \rho\nu_t \frac{\partial U}{\partial y} = \tau_w \tag{B.18}$$

Note also

$$\nu_t = C_\mu \frac{k^2}{\varepsilon} \tag{B.19}$$

From B.17 and B.18

$$\tau = \rho \frac{\varepsilon}{\partial U/\partial y} \tag{B.20}$$

Substituting for ε from B.19 and using B.18 we obtain

$$\tau = \tau_w = \rho C_\mu^{1/2} k \tag{B.21a}$$

$$k = u_*^2 / c_\mu^{1/2} \tag{B.21b}$$

or

$$k = \frac{\tau_w/\rho}{C_\mu^{1/2}} = U_*^2 / C_\mu^{1/2}$$

From log law $U/U_* = u^+ = \frac{1}{\kappa}\ln y + B$, $k \cong 0.41$, B = 5.5 (B.22a)

$$\frac{\partial U}{\partial y} = \frac{U_*^\lambda}{\kappa y} \tag{B.22b}$$

Substituting (B.22b) into B.20 yields

$$\varepsilon = \frac{U_*^3}{\kappa y} \tag{B.23a}$$

(B.23a) can be written in terms of k by making use of B.21, which gives

$$U_* = C_\mu^{1/2} k^{1/2} \tag{B.23b}$$

$$\varepsilon = \frac{C_\mu^{3/4} k^{3/2}}{\kappa y} \tag{B.24}$$

As a side note, B.24 implies a dissipation length scale, given by

$$l_e = \frac{\kappa y}{C_\mu^{3/4}} \tag{B.25}$$

In what follows we present how wall-functions, e.g. logarithmic velocity profile (Eq. B.22b) can be employed (or imposed) in a typical CFD application.

There are two distinct approaches that use the same information in different ways. The first alternative is referred to as "standard wall functions" and the second as "generalized wall functions"

(i) Standard wall function approach

We let $s = U_*/U_2$ in Eq. (B.22a) and rearrange it to obtain

$$s = \frac{K}{\ln(Es\,\mathrm{Re}_2)} \tag{B.26}$$

Here $E \cong g$ (from $\frac{1}{\kappa}\ln E = B$), $\mathrm{Re}_2 = \frac{U_2 y_2}{\nu}$,

The subscript "2" denotes a value at the first grid node inside the calculation domain near the wall being considered. For curved walls U_2 is the velocity component tangent (or parallel) to the wall, and y_2 is the normal distance from the wall. After calculating s from (B.26) iteratively, u_* and hence τ_w are calculated from

$$u_* = u_2 s \tag{B.27}$$

$$\tau_w = \rho u_*^2 \tag{B.28}$$

From Eq. (B.18) it is seen that

$$\nu_t = \frac{\tau_w/\rho}{|\partial U/\partial y|} = \frac{u_*^2}{U_2/y_2} = \frac{u_*^2 y_2}{|U_2|} \tag{B.29}$$

Since $U_2 = (u_*/\kappa)\ln(Ey^+)$, (B.29) can also be written as

$$\nu_t = \frac{\kappa u_* y_2}{\ln(Ey^+)} \tag{B.30}$$

It follows from (B.29) and (B.30) that

$$\frac{\tau_w}{\rho} = \frac{\kappa u_* U_2}{\ln(Ey^+)} \tag{B.31}$$

Once u_* is calculated the near wall values of k and ε are prescribed from Eqs. (B.21b) and (B.23) respectively.

(ii) Generalized wall function approach

Experience show that approach (i) does not work well when there is flow separation and/or reattachment on the walls. In such situations u_* (hence τ_w) is calculated from Eq.(B.23b) and ε is calculated from (B.24).

Furthermore, the production of k, P_k, near the wall (i.e. at the first node) is modified to include τ_w directly where τ_w is determined from (B.31) hence imposing log-law. The turbulent eddy viscosity is still modified according to (B.29) and ε is calculated from (B.24).

In general from Eq.c.3.8 and c.3..9

$$P_h = -\overline{u_i u_j}\frac{\partial U_i}{\partial X_j} \tag{B.32}$$

$$-\overline{u_i u_j} = 2\nu_t S_{ij} - k\delta_{ij} \tag{B.33}$$

For compressible flow

$$P_k = \nu_t\left(\frac{\partial U_i}{\partial X_i} + \frac{\partial U_j}{\partial X_i}\right)\frac{\partial U_i}{\partial X_j} \tag{B.34}$$

Near the wall we separate the shear stress in Eq. (B.34) and replace it with τ_w.

A further modification may be necessary to avoid possible singularity in Eq.(B.26) which is to write

$$\ln(Ey^+) \cong \ln(1 + Ey^+) \tag{B.35}$$

Calculation of the Production Term

The production term in the turbulent kinetic energy equation is defined as

$$P_k = \tau_{ij}\frac{\partial U_i}{\partial x_j} \tag{B.36}$$

where the turbulent stresses are modeled by

$$\tau_{ij} = 2\nu_t S_{ij} - \frac{2}{3}k\delta_{ij}, \quad \text{i.e. } \tau_{12} = 2\nu_t S_{12} = \nu_t\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right) \tag{B.37}$$

and $$\frac{\partial U_i}{\partial x_j} = \frac{1}{2}\left[\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right] + \frac{1}{2}\left[\frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i}\right] = S_{ij} + \Omega_{ij} \tag{B.38}$$

where S_{ij} = The symmetric strain rate tensor

Ω_{ij} = Anti-symmetric mean vorticity tensor

thus $$S_{ij} = \frac{1}{2}\left[\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right], \tag{B.39}$$

and $\Omega_{ij} = \frac{1}{2}\left[\frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i}\right]$ (B.40)

Therefore the turbulent kinetic energy production is given by

$$P_k = \left(2\nu_t S_{ij} - \frac{2}{3}k\delta_{ij}\right)\left(S_{ij} + \Omega_{ij}\right) \tag{B.41}$$

Expanding Eq.(6) for P_k we get

$$P_k = \overbrace{2\nu_t S_{ij} S_{ij}}^{I} - \overbrace{\frac{2}{3}kS_{ii}}^{II} + \overbrace{2\nu_t S_{ij}\Omega_{ij}}^{III} - \overbrace{\frac{2}{3}k\Omega_{ii}}^{IV} \tag{B.42}$$

Term III is equal to zero because it involves multiplication of a symmetric tensor by an anti-symmetric tensor. Term IV is also zero because $\Omega_{ii} = 0$, this can be easily seen by inspecting equation (5). Thus the turbulent kinetic energy production expression reduces to

$$P_k = 2\nu_t S_{ij} S_{ij} - \frac{2}{3}kS_{ii} \tag{B.43}$$

where $S_{ij}S_{ij}$ is expanded as follows

$$\begin{bmatrix} S_{11}S_{11} + S_{21}S_{21} + S_{31}S_{31} \\ S_{12}S_{12} + S_{22}S_{22} + S_{32}S_{32} \\ S_{13}S_{13} + S_{23}S_{23} + S_{33}S_{33} \end{bmatrix} = 2(S_{12}S_{12} + S_{13}S_{13} + S_{23}S_{23}) + S_{11}^2 + S_{22}^2 + S_{33}^2 \tag{B.44}$$

The different components of the strain tensor are calculated as follows

$$S_{12} = \frac{1}{2}\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right),\ S_{13} = \frac{1}{2}\left(\frac{\partial U}{\partial z} + \frac{\partial W}{\partial x}\right),\ S_{23} = \frac{1}{2}\left(\frac{\partial V}{\partial z} + \frac{\partial W}{\partial y}\right) \tag{B.45a}$$

$$S_{11} = \frac{\partial U}{\partial x},\ S_{22} = \frac{\partial V}{\partial y},\ S_{33} = \frac{\partial W}{\partial z} \tag{B.45b}$$

$$2\nu_t \underbrace{(S_{11}^2 + S_{22}^2 + S_{33}^2)}_{Normal\ Stresses} + 4\nu_t(S_{12}^2 + S_{13}^2 + S_{23}^2) - \frac{2}{3}k\vec{\nabla}\cdot\vec{U} \tag{B.46}$$

but

$$\vec{\nabla}\cdot\vec{U} = \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = S_{11} + S_{22} + S_{33} \tag{B.47}$$

substituting Eq.(B.47) in Eq.(B.46) and using Eq.(B.37) for τ yields

$$P_k = 2\mu_t(S_{11}^2 + S_{22}^2 + S_{33}^2) - \frac{2}{3}k\rho(S_{11} + S_{22} + S_{33}) + 2(\tau_{12}S_{12} + \tau_{13}S_{13} + \tau_{23}S_{23}) \tag{B.48}$$

The shear parallel to xy-plane is given by

$$\tau_w = \tau_{31}\hat{i} + \tau_{32}\hat{j} = \tau_{13}\hat{i} + \tau_{23}\hat{j} \tag{B.49}$$

τ_w can be calculated from the log-law using u_{tan} which is defined as

$$U_{\tan} = \sqrt{U^2 + V^2}, \theta = \arcsin\left(\frac{V}{U_{\tan}}\right) \tag{B.50}$$

then

$$\begin{aligned} \tau_{13}^{bw} &= \tau_{31}^{bw} = \tau_{bw}\cos\theta \\ \tau_{23}^{bw} &= \tau_{32}^{bw} = \tau_{bw}\sin\theta \\ \tau_{12}^{bw} &= \tau_{21}^{bw} = 0; \quad \tau_{xy}^{bw} = \tau_{yx}^{bw} = 0 \end{aligned} \tag{B.51}$$

The shear stresses at the south wall are calculated from

$$U_{\tan} = \sqrt{U^2 + W^2}, \theta = \arcsin\left(\frac{W}{U_{\tan}}\right) \tag{B.52}$$

$$\begin{aligned} \tau_{21}^{sw} &= \tau_{12}^{sw} = \tau_{sw}\cos\theta = \tau_{xy} \\ \tau_{23}^{sw} &= \tau_{32}^{sw} = \tau_{sw}\sin\theta = \tau_{yz} \\ \tau_{13}^{sw} &= \tau_{31}^{sw} = 0; \quad \tau_{zx}^{sw} = \tau_{xz}^{sw} = 0 \end{aligned} \tag{B.53}$$

Appendix C: Conservation Equations

C.1 Conservation of Mass

The conservation of mass, or continuity, equation is given as

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0 \qquad \text{(C.1.1)}$$

Breaking the density and velocity into mean and fluctuating components yields

$$\rho = \overline{\rho} + \rho' \qquad u_i = \overline{u_i} + u_i' \qquad \text{(C.1.2a,b)}$$

Substituting Eqs. (C.1.2a) and (C.1.2b) into Eq. (C.1.1) yields

$$\frac{\partial}{\partial t}\left(\overline{\rho} + \rho'\right) + \frac{\partial}{\partial x_i}\left[\left(\overline{\rho} + \rho'\right)\left(\overline{u_i} + u_i'\right)\right] = 0 \qquad \text{(C.1.3)}$$

Neglecting ρ', and with $\frac{\partial}{\partial x_i}\left(\overline{\rho u_i'}\right)$ equal to zero when Eq. (C.1.3) is time-averaged, this yields the mean continuity equation.

$$\frac{\partial \overline{\rho}}{\partial t} + \frac{\partial}{\partial x_i}\left(\overline{\rho} \cdot \overline{u_i}\right) = 0 \qquad \text{(C.1.4)}$$

(*) - The divergence of the fluctuating velocity is also zero (incompressible only), by subtracting Eq. (C.1.4) from Eq. (C.1.1).

C.2 Conservation of Momentum

For the case of incompressible flow, the conservation of momentum equation is given by

$$\frac{\partial}{\partial t}(\rho u_i) + u_j \frac{\partial(\rho u_i)}{\partial x_j} = \rho g_i - \frac{\partial P_i}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\tau_{ji}\right) \qquad \text{(C.2.1)}$$

The stress tensor may be rewritten in terms of the strain rate tensor by

$$\tau_{ji} = 2\mu S_{ji} \qquad \text{(C.2.2)}$$

where

$$S_{ji} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right) \tag{C.2.3}$$

Substituting Eq. (C.2.2) into Eq. (C.2.1) yields

$$\frac{\partial}{\partial t}(\rho u_i) + u_j \frac{\partial(\rho u_i)}{\partial x_j} = \rho g_i - \frac{\partial P_i}{\partial x_i} + \frac{\partial}{\partial x_j}(2\mu S_{ji}) \tag{C.2.4}$$

By assuming that ρ is constant (incompressible), and applying the chain rule to the second term in Eq. (C.2.4) we find

$$\rho \frac{\partial u_i}{\partial t} + \rho \frac{\partial}{\partial x_j}(u_i u_j) - \rho u_i \frac{\partial u_j}{\partial x_j} = \rho g_i - \frac{\partial P_i}{\partial x_i} + \frac{\partial}{\partial x_j}(2\mu S_{ji}) \tag{C.2.5}$$

The third term in Eq. (C.2.5) is zero because of the incompressibility condition. Therefore,

$$\rho \frac{\partial u_i}{\partial t} + \rho \frac{\partial}{\partial x_j}(u_i u_j) = \rho g_i - \frac{\partial P_i}{\partial x_i} + \frac{\partial}{\partial x_j}(2\mu S_{ji}) \tag{C.2.6}$$

The instantaneous quantities can be broken up into mean and fluctuating components by

$$u_i = \overline{u_i} + u_i' \qquad u_j = \overline{u_j} + u_j' \qquad P_i = \overline{P_i} + P_i' \tag{C.2.7a,b,c}$$

$$S_{ji} = \overline{S_{ji}} + S_{ji}' \tag{C.2.7d}$$

Then, substituting Eqs. (C.2.7a) - (C.2.7d) into Eq. (C.2.6) yields

$$\rho \frac{\partial}{\partial t}\left(\overline{u_i} + u_i'\right) + \rho \frac{\partial}{\partial x_j}\left[\left(\overline{u_i} + u_i'\right)\left(\overline{u_j} + u_j'\right)\right] = \rho g_i - \frac{\partial}{\partial x_i}\left(\overline{P_i} + P_i'\right) + \frac{\partial}{\partial x_j}\left[2\mu\left(\overline{S_{ji}} + S_{ji}'\right)\right] \tag{C.2.8}$$

Expanding the second term on the LHS of Eq. (C.2.8) yields

$$\rho \frac{\partial}{\partial t}\left(\overline{u_i} + u_i'\right) + \rho \frac{\partial}{\partial x_j}\left[\overline{u_i} \cdot \overline{u_j} + u_i'\overline{u_j} + \overline{u_i}u_j' + u_i'u_j'\right] = \rho g_i - \frac{\partial}{\partial x_i}\left(\overline{P_i} + P_i'\right) + \frac{\partial}{\partial x_j}\left[2\mu\left(\overline{S_{ji}} + S_{ji}'\right)\right] \tag{C.2.9}$$

Taking the time-average, or Reynolds average, of Eq. (C.2.9), realizing that

$\frac{1}{T}\int \overline{u_i} dt = \overline{u_i}$ (by definition of the time-average of a constant)

$\frac{1}{T}\int \overline{u_i} u_j' dt = 0$ (the time-average of a constant multiplied by a fluctuating component is equal to zero)

$\frac{1}{T}\int u_i' u_j' dt \neq 0$ also $\frac{1}{T}\int u_j' u_i' dt \neq 0$

With the above definitions, then the time-average of Eq. (C.2.9) becomes

$$\rho\frac{\partial}{\partial t}\left(\overline{u_i}\right)+\rho\frac{\partial}{\partial x_j}\left[\overline{u_i}\cdot\overline{u_j}+\overline{u_i'u_j'}\right]=\rho g_i-\frac{\partial\overline{P_i}}{\partial x_i}+\frac{\partial}{\partial x_j}\left[2\mu\left(\overline{S_{ji}}\right)\right] \tag{C.2.10}$$

or

$$\rho\frac{\partial}{\partial t}\left(\overline{u_i}\right)+\rho\frac{\partial}{\partial x_j}\left[\overline{u_i}\cdot\overline{u_j}\right]=-\rho\frac{\partial}{\partial x_j}\left(\overline{u_i'u_j'}\right)+\rho g_i-\frac{\partial\overline{P_i}}{\partial x_i}+\frac{\partial}{\partial x_j}\left[2\mu\left(\overline{S_{ji}}\right)\right] \tag{C.2.11}$$

The second term in Eq. (C.2.11) can be rewritten as

$$\rho\frac{\partial}{\partial x_j}\left[\overline{u_i}\cdot\overline{u_j}\right]=\rho\overline{u_j}\frac{\partial\left(\overline{u_i}\right)}{\partial x_j}+\rho\overline{u_i}\frac{\partial\left(\overline{u_j}\right)}{\partial x_j} \tag{C.2.12}$$

But the last term in Eq. (C.2.12) is zero from the incompressibility condition. Then

$$\rho\frac{\partial}{\partial x_j}\left[\overline{u_i}\cdot\overline{u_j}\right]=\rho\overline{u_j}\frac{\partial\left(\overline{u_i}\right)}{\partial x_j} \tag{C.2.13}$$

Substituting Eq. (C.2.13) into Eq. (C.2.11) yields

$$\rho\frac{\partial}{\partial t}\left(\overline{u_i}\right)+\rho\overline{u_j}\frac{\partial}{\partial x_j}\left(\overline{u_i}\right)=\rho g_i-\frac{\partial\overline{P_i}}{\partial x_i}+\frac{\partial}{\partial x_j}\left[2\mu\left(\overline{S_{ji}}\right)-\rho\overline{u_i'u_j'}\right] \tag{C.2.14}$$

where the last term, $-\rho\overline{u_i'u_j'}$, is the Reynolds stress tensor.

C.3 Derivation of the k-Equation from the Reynolds Stresses

The specific turbulent kinetic energy k is defined according to

$$k=\frac{1}{2}\left(\overline{u'^2}+\overline{v'^2}+\overline{w'^2}\right) \tag{C.3.1a}$$

If the turbulence is isotropic (i.e. $\overline{u'^2}=\overline{v'^2}=\overline{w'^2}$) then

$$k=\frac{3}{2}\overline{u_i'^2} \tag{C.3.1b}$$

The equation of continuity for incompressible flow is given by

$$\frac{\partial \rho}{\partial t} + u_i' \frac{\partial \rho}{\partial x_i} = 0 \tag{C.3.2}$$

or equivalently by $\partial u_i'/\partial x_i = 0$. Using continuity, the Reynolds stresses may be written as

$$\begin{aligned} -\frac{\partial}{\partial t}\left(\overline{u_i'u_j'}\right) - u_k \frac{\partial}{\partial x_k}\left(\overline{u_i'u_j'}\right) &= \overline{u_j'u_k'}\frac{\partial u_i}{\partial x_k} + \overline{u_i'u_k'}\frac{\partial u_j}{\partial x_k} + \frac{\partial}{\partial x_k}\left(\overline{u_i'u_j'u_k'}\right) + 2\nu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}} \\ &- \overline{\frac{\rho'}{\rho}\left[\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right]} + \left(\frac{1}{\rho}\right)\frac{\partial}{\partial x_k}\left[\left(\overline{\rho' u_i'}\right)\delta_{jk} + \left(\overline{\rho' u_j'}\right)\delta_{ik}\right] - \frac{1}{\rho}\frac{\partial}{\partial x_k}\left[\mu\frac{\partial}{\partial x_k}\left(\overline{u_i'u_j'}\right)\right] \end{aligned} \tag{C.3.3}$$

If we now set j = i and k = j, as would be done to derive the equations for τ_{xx}, τ_{yy}, and τ_{zz}, then Eq. (C.3.3) becomes

$$\begin{aligned} -\frac{\partial}{\partial t}\left(\overline{u_i'^2}\right) - u_j \frac{\partial}{\partial x_j}\left(\overline{u_i'^2}\right) &= 2\overline{u_i'u_j'}\frac{\partial u_i}{\partial x_j} + \frac{\partial}{\partial x_j}\left(\overline{u_i'u_i'u_j'}\right) + 2\nu\overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} \\ &- \overline{\frac{\rho'}{\rho}\left[\frac{\partial u_i'}{\partial x_i} + \frac{\partial u_i'}{\partial x_i}\right]} + \left(\frac{1}{\rho}\right)\frac{\partial}{\partial x_j}\left[2\left(\overline{\rho' u_i'}\right)\delta_{ij}\right] - \frac{1}{\rho}\frac{\partial}{\partial x_j}\left[\mu\frac{\partial}{\partial x_j}\left(\overline{u_i'u_i'}\right)\right] \end{aligned} \tag{C.3.4}$$

The fourth term on the RHS of Eq. (C.3.4) is zero from the incompressibility condition. Also, since the Kronecker delta function is only non-zero when the subscripts are equal (i.e. i = j), then the fifth term on the RHS can be rewritten as

$$\left(\frac{1}{\rho}\right)\frac{\partial}{\partial x_j}\left[2\left(\overline{\rho' u_i'}\right)\delta_{ij}\right] = \left(\frac{1}{\rho}\right)\frac{\partial}{\partial x_j}\left[2\left(\overline{\rho' u_j'}\right)\right] \tag{C.3.5}$$

Substituting for Eq. (C.3.5), and multiplying both sides of Eq. (C.3.4) by -1/2, then

$$\begin{aligned} \frac{1}{2}\frac{\partial}{\partial t}\left(\overline{u_i'u_i'}\right) + u_j\frac{1}{2}\frac{\partial}{\partial x_j}\left(\overline{u_i'u_i'}\right) &= -\overline{u_i'u_j'}\frac{\partial u_i}{\partial x_j} - \nu\overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} - \frac{\partial}{\partial x_j}\left[\left(\overline{u_i'u_i'u_j'}\right)\frac{1}{2} + \frac{1}{\rho}\left(\overline{\rho' u_j'}\right)\right] \\ &+ \frac{1}{2\rho}\frac{\partial}{\partial x_j}\left[\mu\frac{\partial}{\partial x_j}\left(\overline{u_i'u_i'}\right)\right] \end{aligned} \tag{C.3.6}$$

where summation is implied by repeated indices. By definition, the dissipation of turbulent kinetic energy, ε, is given by

$$\varepsilon = \nu\overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} \tag{C.3.7}$$

Using this, along with the definition of the turbulent kinetic energy k given by Eq. (C.3.1), then Eq. (C.3.6) becomes

$$\frac{\partial k}{\partial t}+u_j\frac{\partial k}{\partial x_j}=-\overline{u_i'u_j'}\frac{\partial u_i}{\partial x_j}-\varepsilon-\frac{\partial}{\partial x_j}\left[\frac{1}{2}\left(\overline{u_i'u_i'u_j'}\right)+\frac{1}{\rho}\left(\overline{p'u_j'}\right)\right]+\frac{1}{\rho}\frac{\partial}{\partial x_j}\left[\mu\frac{\partial k}{\partial x_j}\right] \tag{C.3.8}$$

or

$$\rho\frac{\partial k}{\partial t}+\rho u_j\frac{\partial k}{\partial x_j}=\tau_{ij}\frac{\partial u_i}{\partial x_j}-\rho\varepsilon+\frac{\partial}{\partial x_j}\left[\mu\frac{\partial k}{\partial x_j}\right]-\frac{\partial}{\partial x_j}\left[\frac{1}{2}\left(\overline{\rho u_i'u_i'u_j'}\right)+\left(\overline{p'u_j'}\right)\right] \tag{C.3.9}$$

C.4 Derivation of the Dissipation Rate Equation

The Navier- Stokes equation can be written as:

$$\frac{\partial U_i}{\partial t}+U_j\frac{\partial U_i}{\partial x_j}=-\frac{1}{\rho}\frac{\partial P}{\partial x_i}+\frac{1}{\rho}\frac{\partial}{\partial x_j}\left(\mu\frac{\partial U_i}{\partial x_j}\right) \tag{C.4.1}$$

Substituting the mean and the fluctuating components in to (C.4.1) yields

$$\begin{aligned}&\frac{\partial\overline{U}_i}{\partial t}+\frac{\partial u_i'}{\partial t}+\overline{U}_j\frac{\partial}{\partial x_j}\left(\overline{U}_i+u_i'\right)+u_j'\frac{\partial}{\partial x_j}\left(\overline{U}_i+u_i'\right)=\\&-\frac{1}{\rho}\frac{\partial\overline{P}}{\partial x_i}-\frac{1}{\rho}\frac{\partial p'}{\partial x_i}+\frac{1}{\rho}\frac{\partial}{\partial x_j}\left(\mu\frac{\partial}{\partial x_j}\left(\overline{U}_i+u_i'\right)\right)\end{aligned} \tag{C.4.2}$$

After time averaging we get

$$\frac{\partial\overline{U}_i}{\partial t}+\overline{U}_j\frac{\partial\overline{U}_i}{\partial x_j}+\overline{u_j'\frac{\partial u_i'}{\partial x_j}}=-\frac{1}{\rho}\frac{\partial\overline{P}}{\partial x_i}+\frac{1}{\rho}\frac{\partial}{\partial x_j}\left(\mu\frac{\partial\overline{U}_i}{\partial x_j}\right) \tag{C.4.3}$$

Subtracting Eq. (C.4.3) from Eq. (C.4.2) yields

$$\frac{\partial u_i'}{\partial t}+\overline{U}_j\frac{\partial u_i'}{\partial x_j}+u_j'\frac{\partial}{\partial x_j}\left(\overline{U}_i+u_i'\right)-\overline{u_j'\frac{\partial u_i'}{\partial x_j}}=-\frac{1}{\rho}\frac{\partial p'}{\partial x_i}+\frac{1}{\rho}\frac{\partial}{\partial x_j}\left(\mu\frac{\partial u_i'}{\partial x_j}\right)$$

$$\overline{\frac{\partial}{\partial x_j}\left(u_i'u_j'\right)}=\overline{u_j'\frac{\partial u_i'}{\partial x_j}}+\underbrace{\overline{u_i'\frac{\partial u_j'}{\partial x_j}}}_{=0\,by\,continuity}$$

This becomes

$$\frac{\partial u_i'}{\partial t}+u_j'\frac{\partial}{\partial x_j}\left(\overline{U}_i+u_i'\right)+\overline{U}_j\frac{\partial u_i'}{\partial x_j}=-\frac{1}{\rho}\frac{\partial P'}{\partial x_i}+\frac{1}{\rho}\frac{\partial}{\partial x_j}\left(\mu\frac{\partial u_i'}{\partial x_j}\right)+\frac{\partial}{\partial x_j}\overline{u_i'u_j'} \qquad \text{(C.4.4)}$$

Differentiation of this equation with respect to x_l and multiplication by $\frac{\partial u_i}{\partial x_l}$ and averaging of that equation yields

$$\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial}{\partial x_k}\left(\frac{\partial u_i'}{\partial t}\right)}=\frac{\partial}{\partial t}\left\{\overline{\frac{1}{2}\left(\frac{\partial u_i'}{\partial x_k}\right)^2}\right\}$$

$$\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial}{\partial x_k}\left(u_j'\frac{\partial \overline{U_i}}{\partial x_j}\right)}=\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}\frac{\partial \overline{U_i}}{\partial x_j}}+\frac{\partial^2\overline{U_i}}{\partial x_k\partial x_j}\overline{u_j'\frac{\partial u_i'}{\partial x_k}}$$

$$\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial}{\partial x_k}\left(u_j'\frac{\partial u_i'}{\partial x_j}\right)}=\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}\frac{\partial u_i'}{\partial x_j}}+\overline{u_j'\frac{\partial}{\partial x_j}\left\{\frac{1}{2}\left(\frac{\partial u_i'}{\partial x_k}\right)^2\right\}}$$

$$\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial}{\partial x_k}\left(\overline{U_j}\frac{\partial u_i'}{\partial x_j}\right)}=\frac{\partial \overline{U_j}}{\partial x_k}\overline{\frac{\partial u_j'}{\partial x_k}\frac{\partial u_i'}{\partial x_j}}+\overline{U_j}\frac{\partial}{\partial x_j}\left\{\overline{\frac{1}{2}\left(\frac{\partial u_i'}{\partial x_k}\right)^2}\right\}$$

$$\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial}{\partial x_k}\left(-\frac{1}{\rho}\frac{\partial p'}{\partial x_i}\right)}=-\overline{\frac{1}{\rho}\frac{\partial u_i'}{\partial x_k}\frac{\partial^2 p'}{\partial x_k\partial x_i}}$$

$$\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial}{\partial x_k}\left(\frac{\partial}{\partial x_j}\overline{u_i'u_j'}\right)}=0$$

Hence, the final equation becomes

$$\begin{aligned}&\frac{\partial}{\partial t}\left\{\frac{1}{2}\left(\frac{\partial u_i'}{\partial x_k}\right)^2\right\}+\frac{\partial \overline{U_i}}{\partial x_j}\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}}+\frac{\partial^2\overline{U_i}}{\partial x_k\partial x_j}\overline{u_j'\frac{\partial u_i'}{\partial x_k}}\\&+\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}\frac{\partial u_i'}{\partial x_j}}+\overline{u_j'\frac{\partial}{\partial x_j}\left\{\frac{1}{2}\left(\frac{\partial u_i'}{\partial x_k}\right)^2\right\}}+\frac{\partial \overline{U_j}}{\partial x_k}\overline{\frac{\partial u_j'}{\partial x_k}\frac{\partial u_i'}{\partial x_j}}\\&+\overline{U_j}\overline{\frac{\partial}{\partial x_j}\left\{\frac{1}{2}\left(\frac{\partial u_i'}{\partial x_k}\right)^2\right\}}=-\overline{\frac{1}{\rho}\frac{\partial u_i'}{\partial x_k}\frac{\partial^2 p'}{\partial x_k\partial x_i}}+\nu\overline{\frac{\partial u_i'}{\partial x_k}\left(\frac{\partial^2}{\partial x_j^2}\frac{\partial u_i'}{\partial x_k}\right)}\end{aligned} \qquad \text{(C.4.5a)}$$

Note that $$\overline{\nu\frac{\partial u_i'}{\partial x_k}\left(\frac{\partial^2}{\partial x_j^2}\frac{\partial u_i'}{\partial x_k}\right)}=\nu\frac{\partial}{\partial x_j}\left[\frac{\partial u_i'}{\partial x_k}\frac{\partial}{\partial x_j}\left(\frac{\partial u_i'}{\partial x_k}\right)\right]-\nu\frac{\partial}{\partial x_j}\left(\frac{\partial u_i'}{\partial x_k}\right)\frac{\partial}{\partial x_j}\left(\frac{\partial u_i'}{\partial x_k}\right)$$

$$= \frac{\nu}{2}\frac{\partial}{\partial x_j}\overline{\left[\frac{\partial}{\partial x_j}\left(\frac{\partial u_i'}{\partial x_k}\right)^2\right]} - \nu\overline{\left(\frac{\partial^2}{\partial x_j}\frac{u_i'}{\partial x_k}\right)\left(\frac{\partial^2}{\partial x_j}\frac{u_i'}{\partial x_k}\right)}$$

If we now let $\varepsilon = \nu\overline{\left(\frac{\partial u_i'}{\partial x_j}\right)\left(\frac{\partial u_i'}{\partial x_j}\right)}$, and multiply by 2νρ=2μ, then the ε-equation becomes

$$\begin{aligned}
&\frac{\partial}{\partial t}(\rho\varepsilon) + \rho\overline{U}_j\frac{\partial}{\partial x_j}(\varepsilon) = -2\nu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial^2}{\partial x_k}\frac{p'}{\partial x_i}} + \mu\frac{\partial}{\partial x_j}\left(\frac{\partial\varepsilon}{\partial x_j}\right) \\
&-\overline{u_j'\frac{\partial}{\partial x_j}\left(\mu\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_k}\right)} - 2\nu\mu\overline{\left(\frac{\partial^2}{\partial x_j}\frac{u_i'}{\partial x_k}\right)\left(\frac{\partial^2}{\partial x_j}\frac{u_i'}{\partial x_k}\right)} - 2\mu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}}\frac{\partial\overline{U_i}}{\partial x_j} \\
&-2\mu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_j}}\frac{\partial\overline{U_i}}{\partial x_k} - 2\mu\overline{u_j'\frac{\partial u_i'}{\partial x_k}}\frac{\partial^2\overline{U_i}}{\partial x_k\partial x_j} - 2\mu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}\frac{\partial u_i'}{\partial x_j}}
\end{aligned} \qquad \text{(C.4.5b)}$$

Appendix D: Governing Equations with the Effects of Buoyancy

D.1 Mass

The equation of continuity is given by

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + u_j \frac{\partial \rho}{\partial x_j} = 0 \tag{D.1.1}$$

This implies that

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{D.1.2}$$

and

$$\frac{\partial \overline{u_i}}{\partial x_i} = \frac{\partial u_i'}{\partial x_i} = 0 \tag{D.1.3}$$

Then the equation of continuity can be written as

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0 \tag{D.1.4}$$

By dividing the density and velocity into mean and fluctuating components and then time-averaging, Eq. (D.1.1) becomes

$$\overline{\frac{\partial}{\partial t}\left(\overline{\rho} + \rho'\right)} + \overline{\frac{\partial}{\partial x_i}\left[\left(\overline{\rho} + \rho'\right)\left(U_i + u_i'\right)\right]} = 0 \tag{D.1.5}$$

Applying the rules regarding time-averaged variables, the mean continuity equation becomes

$$\boxed{\frac{\partial \overline{\rho}}{\partial t} + \frac{\partial}{\partial x_i}\left(\overline{\rho} U_i\right) + \frac{\partial}{\partial x_i}\left(\overline{\rho' u_i'}\right) = 0} \tag{D.1.6}$$

Also,

$$\frac{\partial \rho}{\partial t} + u_i \frac{\partial \rho}{\partial x_i} - \frac{\partial \overline{\rho}}{\partial t} - U_i \frac{\partial \overline{\rho}}{\partial x_i} - \frac{\partial}{\partial x_i}\left(\overline{\rho' u_i'}\right) = 0 \tag{D.1.7}$$

$$\frac{\partial}{\partial t}\left(\rho - \overline{\rho}\right) + (u_i - U_i)\frac{\partial \overline{\rho}}{\partial x_i} + u_i \frac{\partial \rho'}{\partial x_i} - \frac{\partial}{\partial x_i}\left(\overline{\rho' u_i'}\right) = 0 \tag{D.1.8}$$

$$\frac{\partial \rho'}{\partial t} + u_i' \frac{\partial \overline{\rho}}{\partial x_i} + u_i \frac{\partial \rho'}{\partial x_i} - \frac{\partial}{\partial x_i}\left(\overline{\rho' u_i'}\right) = 0 \tag{D.1.9}$$

D.2 Buoyancy (Mass) Equation

Consider Eq. (D.1.6), which can be rewritten as

$$\frac{\partial}{\partial t}\left(\overline{\rho}-\rho_r\right)+u_i\frac{\partial}{\partial x_i}\left(\overline{\rho}-\rho_r\right)+\frac{\partial}{\partial x_i}\left(\overline{\rho' u_i'}\right)=0 \tag{D.2.1}$$

where ρ_r is some (constant) reference density. We can rewrite Eq. (D.2.1) as

$$\frac{\partial}{\partial t}\left[g\left(\frac{\overline{\rho}-\rho_r}{\rho_r}\right)\right]+u_i\frac{\partial}{\partial x_i}\left[g\left(\frac{\overline{\rho}-\rho_r}{\rho_r}\right)\right]+\frac{1}{\rho_r}\frac{\partial}{\partial x_i}\left[g\left(\overline{\rho' u_i'}\right)\right]=0 \tag{D.2.2}$$

If we now define a buoyancy variable b, given by

$$b=g\frac{\overline{\rho}-\rho_r}{\rho_r} \tag{D.2.3}$$

then we can rewrite Eq. (D.2.2) as

$$\frac{\partial b}{\partial t}+u_i\frac{\partial b}{\partial x_i}=-\frac{g}{\rho_r}\frac{\partial}{\partial x_i}\left(\overline{u_i'\rho'}\right) \tag{D.2.4}$$

where the term

$$-g\frac{\overline{u_i'\rho'}}{\rho_r}\ = \text{Turbulent Buoyancy Flux} \tag{D.2.5}$$

D.3 Momentum

The conservation of momentum equation is given by

$$\rho\frac{\partial u_i}{\partial t}+\rho u_j\frac{\partial u_i}{\partial x_j}=-\frac{\partial P}{\partial x_i}+\frac{\partial}{\partial x_j}\left(\mu\frac{\partial u_i}{\partial x_j}\right)+\rho g_i \tag{D.3.1}$$

If we now use a reasoning similar to a Boussinesq approximation, by dividing the density into mean and fluctuating components, and then dividing both sides by the mean density, Eq. (D.3.1) can be rewritten as

$$\left(1+\frac{\rho'}{\overline{\rho}}\right)\frac{\partial u_i}{\partial t}+\left(1+\frac{\rho'}{\overline{\rho}}\right)u_j\frac{\partial u_i}{\partial x_j}=-\frac{1}{\overline{\rho}}\frac{\partial P}{\partial x_i}+\frac{\partial}{\partial x_j}\left(\frac{\mu}{\overline{\rho}}\frac{\partial u_i}{\partial x_j}\right)+\left(1+\frac{\rho'}{\overline{\rho}}\right)g_i \tag{D.3.2}$$

If we then neglect the contribution of $\rho'/\overline{\rho}$ in the inertial terms, and drop the overbars for the mean quantities, then Eq. (D.3.2) becomes

$$\rho\frac{\partial u_i}{\partial t}+\rho u_j\frac{\partial u_i}{\partial x_j}=-\frac{\partial P}{\partial x_i}+\frac{\partial}{\partial x_j}\left(\mu\frac{\partial u_i}{\partial x_j}\right)+(\rho+\rho')g_i \tag{D.3.3}$$

If we now divide the velocity terms into mean and fluctuating components by

$$u_i=U_i+u_i' \tag{D.3.4}$$

so that the mean velocity is given by

$$U_i=u_i-u_i' \tag{D.3.5}$$

and time-average Eq. (D.3.3), then this yields the mean momentum equation:

$$\rho\frac{\partial U_i}{\partial t}+\rho U_j\frac{\partial U_i}{\partial x_j}=-\frac{\partial P}{\partial x_i}+\frac{\partial}{\partial x_j}\left(\mu\frac{\partial U_i}{\partial x_j}-\rho\overline{u_i'u_j'}\right)+\rho g_i \tag{D.3.6}$$

D.4 Mean Flow Energy Equation

The mean flow energy equation can be written as

$$\rho U_i\frac{\partial U_i}{\partial t}+\rho U_jU_i\frac{\partial U_i}{\partial x_j}=-U_i\frac{\partial P_i}{\partial x_i}+U_i\frac{\partial}{\partial x_j}\left(\mu\frac{\partial U_i}{\partial x_j}-\rho\overline{u_i'u_j'}\right)+\rho U_ig_i \tag{D.4.1}$$

With the definition of the kinetic energy,

$$K=\frac{1}{2}U_iU_i \tag{D.4.2}$$

and knowing that

$$A\frac{\partial A}{\partial x}=\frac{\partial}{\partial x}\left(\frac{A^2}{2}\right) \tag{D.4.3}$$

then Eq. (D.4.1) can be rewritten as

$$\rho\frac{\partial K}{\partial t}+\rho U_j\frac{\partial K}{\partial x_j}=-U_i\frac{\partial P_i}{\partial x_i}+\mu\frac{\partial}{\partial x_j}\left(U_i\frac{\partial U_i}{\partial x_j}-\rho\overline{u_i'u_j'}\right)-\mu\frac{\partial U_i}{\partial x_j}\frac{\partial U_i}{\partial x_j}+\rho U_i\overline{u_i'u_j'}+\rho U_ig_i \tag{D.4.4}$$

which becomes

$$\rho\frac{\partial K}{\partial t}+\rho U_j\frac{\partial K}{\partial x_j}=\frac{\partial}{\partial x_j}\left(\mu\frac{\partial K}{\partial x_j}-\rho U_i\overline{u_i'u_j'}-U_jP_i\right)+\rho\overline{u_i'u_j'}\frac{\partial U_i}{\partial x_j}-\mu\frac{\partial U_i}{\partial x_j}\frac{\partial U_i}{\partial x_j}+\rho U_ig_i \tag{D.4.5}$$

The fourth term on the RHS of Eq. (D.4.5) represents the production of turbulent kinetic energy

$$\rho\overline{u_i'u_j'}\frac{\partial U_i}{\partial x_j}=-P_k \tag{D.4.6}$$

and the fifth term on the RHS of Eq. (D.4.6) represents viscous dissipation, which is usually small

$$\mu\frac{\partial U_i}{\partial x_j}\frac{\partial U_i}{\partial x_j}=\Phi \tag{D.4.7}$$

Substituting Eqs. (D.4.6) and (D.4.7) into Eq. (D.4.5) yields

$$\rho\frac{\partial K}{\partial t}+\rho U_j\frac{\partial K}{\partial x_j}=-\Phi-P_k+\rho U_ig_i+\frac{\partial}{\partial x_j}\left(\mu\frac{\partial K}{\partial x_j}-\rho U_i\overline{u_i'u_j'}-U_jP_i\right) \tag{D.4.8}$$

D.5 TKE Equation with the Effects of Buoyancy

If we divide the velocities, density, and pressure into mean and fluctuating components, and multiply both sides by (U_i + u'$_i$), the conservation of momentum equation can be written as

$$\begin{aligned}&(U_i+u_i')\rho\frac{\partial}{\partial t}(U_i+u_i')+(U_i+u_i')\rho(U_j+u_j')\frac{\partial}{\partial x_j}(U_i+u_i')=-(U_i+u_i')\frac{\partial}{\partial x_i}(P+p')\\&+(U_i+u_i')\frac{\partial}{\partial x_j}\left[\mu\frac{\partial}{\partial x_j}(U_i+u_i')\right]+(U_i+u_i')(\rho+\rho')g_i\end{aligned} \tag{D.5.1}$$

Time-averaging Eq. (D.5.1) yields

$$\begin{aligned}&\rho\frac{\partial}{\partial t}\left(\frac{U_iU_i}{2}\right)+\rho\frac{\partial}{\partial t}\left(\frac{\overline{u_i'u_i}}{2}\right)+\rho\frac{\partial}{\partial x_j}\overline{\left[\frac{1}{2}(U_i+u_i')^2(U_j+u_j')\right]}=-\frac{\partial}{\partial x_j}\left[\overline{(U_j+u_j')(P+p')}\right]\\&+\overline{(P+p')\frac{\partial}{\partial x_j}(U_j+u_j')}+\overline{\frac{\partial}{\partial x_j}\left[(U_i+u_i')\mu\frac{\partial}{\partial x_j}(U_i+u_i')\right]}-\overline{\mu\left[\frac{\partial}{\partial x_j}(U_i+u_i')\right]\left[\frac{\partial}{\partial x_j}(U_i+u_i')\right]}\\&+U_i\rho g_i+\overline{u_i'\rho'}g_i\end{aligned} \tag{D.5.2}$$

The second term on the RHS of Eq. (D.5.2) is zero from the incompressibility condition. This yields

$$\begin{array}{c}(1)\\ \rho\frac{\partial}{\partial t}\left(\frac{U_iU_i}{2}\right)+\rho\frac{\partial}{\partial t}\left(\frac{\overline{u_i'u_i}}{2}\right)+\rho\frac{\partial}{\partial x_j}\overline{\left[\frac{1}{2}(U_i+u_i')^2(U_j+u_j')\right]}=-\frac{\partial}{\partial x_j}\overline{\left[(U_j+u_j')(P+p')\right]}\\ +\overline{\frac{\partial}{\partial x_j}\left[(U_i+u_i')\mu\frac{\partial}{\partial x_j}(U_i+u_i')\right]}-\mu\overline{\left[\frac{\partial}{\partial x_j}(U_i+u_i')\right]\left[\frac{\partial}{\partial x_j}(U_i+u_i')\right]}+U_i\rho g_i+\overline{u_i'\rho'}g_i\\ (2)\end{array} \tag{D.5.3}$$

Consider the third term on the LHS of Eq. (D.5.3), which can be expanded to give

$$(1)=\frac{\rho}{2}\frac{\partial}{\partial x_j}\left[\overline{U_iU_iU_j}+2\overline{U_iU_ju_i'}+\overline{u_i'u_i'U_j}+\overline{U_iU_iu_j'}+2\overline{U_iu_i'u_j'}+\overline{u_i'u_i'u_j'}\right] \tag{D.5.4}$$

But the second and fourth terms on the RHS of Eq. (D.5.4) are zero. Then Eq. (D.5.4) becomes

$$(1)=\frac{\rho}{2}\frac{\partial}{\partial x_j}\left[\overline{U_iU_iU_j}+\overline{u_i'u_i'U_j}+2\overline{U_iu_i'u_j'}+\overline{u_i'u_i'u_j'}\right] \tag{D.5.5}$$

Applying the definition for the turbulent kinetic energy, Eq. (D.5.5) becomes

$$(1)=\rho\frac{\partial}{\partial x_j}\left[U_jK+U_jk+U_i\overline{u_i'u_j'}+\overline{u_i'u_i'u_j'}\right] \tag{D.5.6}$$

or

$$(1)=\rho U_j\frac{\partial K}{\partial x_j}+\rho U_j\frac{\partial k}{\partial x_j}+\rho\frac{\partial}{\partial x_j}\left[U_i\overline{u_i'u_j'}+\overline{u_i'u_i'u_j'}\right] \tag{D.5.7}$$

Next, consider the third term on the RHS of Eq. (D.5.3), which can be written as

$$(2)=-\mu\overline{\left[\frac{\partial}{\partial x_j}(U_i+u_i')\right]\left[\frac{\partial}{\partial x_j}(U_i+u_i')\right]}=-\mu\overline{\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial u_i'}{\partial x_j}\right)^2}=-\mu\frac{\partial U_i}{\partial x_j}\frac{\partial U_i}{\partial x_j}-\mu\overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}} \tag{D.5.8}$$

Substituting for terms (1) and (2), and applying the definition of the turbulent kinetic energy to the first two terms in Eq. (D.5.3) yields

$$\begin{array}{l}\rho\frac{\partial K}{\partial t}+\rho\frac{\partial k}{\partial t}+\rho U_j\frac{\partial K}{\partial x_j}+\rho U_j\frac{\partial k}{\partial x_j}=-\frac{\partial}{\partial x_j}(U_jP)-\frac{\partial}{\partial x_j}(\overline{u_j'p'})-\rho\frac{\partial}{\partial x_j}(U_i\overline{u_i'u_j'})\\ -\rho\frac{\partial}{\partial x_j}(\overline{u_i'u_i'u_j'})+\frac{\partial}{\partial x_j}\left[\mu\frac{\partial}{\partial x_j}(K+k)\right]-\mu\left(\frac{\partial U_i}{\partial x_j}\right)^2-\mu\left(\frac{\partial u_i'}{\partial x_j}\right)^2+U_i\rho g_i+\overline{u_i'\rho'}g_i\end{array} \tag{D.5.9}$$

Adding and subtracting the term $\rho\overline{u_i'u_j'}\frac{\partial U_i}{\partial x_j}$ from Eq. (D.5.9) and rearranging yields

$$\rho\frac{\partial k}{\partial t}+\rho U_j\frac{\partial k}{\partial x_j}+\left[\rho\frac{\partial K}{\partial t}+\rho U_j\frac{\partial K}{\partial x_j}\right]=\left[\frac{\partial}{\partial x_j}\left\{\mu\frac{\partial K}{\partial x_j}-\rho\left(U_i\overline{u_i'u_j'}\right)-U_jP\right\}+\rho\overline{u_i'u_j'}\frac{\partial U_i}{\partial x_j}\right.$$
$$\left.-\mu\left(\frac{\partial U_i}{\partial x_j}\right)^2+U_i\rho g_i\right]-\rho\overline{u_i'u_j'}\frac{\partial U_i}{\partial x_j}-\mu\left(\frac{\partial u_i'}{\partial x_j}\right)^2+\frac{\partial}{\partial x_j}\left(-\overline{u_j'p'}-\overline{u_i'u_i'u_j'}+\mu\frac{\partial k}{\partial x_j}\right)+\overline{u_i'\rho'}g_i \qquad \text{(D.5.10)}$$

Subtracting the mean flow equation, represented by the bracketed terms in Eq. (D.5.10), yields

$$\rho\frac{\partial k}{\partial t}+\rho U_j\frac{\partial k}{\partial x_j}=\frac{\partial}{\partial x_j}\left(\mu\frac{\partial k}{\partial x_j}-\overline{u_j'p'}-\overline{u_i'u_i'u_j'}\right)-\rho\overline{u_i'u_j'}\frac{\partial U_i}{\partial x_j}-\mu\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}+\overline{u_i'\rho'}g_i \qquad \text{(D.5.11)}$$

which is the mean TKE equation where the Boussinesq approximation has been employed.

---

*Note: The term $\overline{u_i'\rho'}g$ is generally modeled as

$$\overline{u_i'\rho'}g=g\cdot\frac{\nu_\tau}{\sigma_p}\cdot\nabla\rho \qquad \text{(D.5.12)}$$

where

$$\nu_\tau=C_\mu\frac{k^2}{\varepsilon} \qquad \text{(D.5.13)}$$

and $\nabla\rho$ is used since $\partial\rho/\partial z$ = negative is a sink usually, and $\partial\rho/\partial z$ = positive is a source usually, of TKE. Then

$$\overline{u_i'\rho'}\propto-\frac{\partial\rho}{\partial z} \qquad \text{(D.5.14)}$$

where

$\overline{u_i'\rho'}$ = positive (*) Unstable stratification

$\overline{u_i'\rho'}$ = negative (*) Stable stratification

Appendix E: Reynolds-Averaged Equations

E.1 Reynolds-Averaged Momentum Equation

The conservation of momentum equation is given by

$$\frac{\partial}{\partial t}(\rho u_i)+u_j\frac{\partial}{\partial x_j}(\rho u_i)=-\frac{\partial p}{\partial x_i}+\frac{\partial}{\partial x_j}(2\mu S_{ji})+\rho g_i \tag{E.1.1}$$

From continuity (for incompressible flow), the strain-rate tensor S_{ji} can be written as

$$S_{ji}=\frac{1}{2}\left(\frac{\partial u_j}{\partial x_i}+\frac{\partial u_i}{\partial x_j}\right) \tag{E.1.2}$$

If we divide the velocities, pressure, and strain-rate tensor into mean and fluctuating components, given by

$$u_i=U_i+u_i' \qquad\qquad u_j=U_j+u_j' \tag{E.1.3a,b}$$

$$p=P+p' \qquad\qquad S_{ji}=\overline{S_{ji}}+s_{ji}' \tag{E.1.3c,d}$$

and further assume that density fluctuations are negligible, so that

$$\rho=\overline{\rho}+\rho'=\overline{\rho} \tag{E.1.3e}$$

then substituting into Eq. (E.1.1) and time-averaging yields

$$\overline{\rho\frac{\partial}{\partial t}(U_i+u_i')}+\overline{\rho\frac{\partial}{\partial x_j}\left[(U_i+u_i')(U_j+u_j')\right]}=-\overline{\frac{\partial}{\partial x_i}(P+p')}+\overline{\rho g_i}+\frac{\partial}{\partial x_j}\left[2\mu\left(\overline{S_{ji}}+s_{ji}'\right)\right] \tag{E.1.4}$$

Next, we will consider the terms in Eq. (E.1.4) individually.

Transient term:

$$\rho\frac{\partial}{\partial t}\left(\overline{U_i+u_i'}\right)=\rho\frac{\partial U_i}{\partial t} \tag{E.1.5}$$

Convective term:

$$\begin{aligned}\rho\frac{\partial}{\partial x_j}\left[\overline{(U_i+u_i')(U_j+u_j')}\right]&=\rho\frac{\partial}{\partial x_j}\left[\overline{U_iU_j+U_iu_j'+U_ju_i'+u_i'u_j'}\right]\\&=\rho\left[\frac{\partial}{\partial x_j}(U_iU_j)+\frac{\partial}{\partial x_j}\left(\overline{u_i'u_j'}\right)\right]\end{aligned} \tag{E.1.6}$$

Pressure term:

$$-\frac{\partial}{\partial x_i}\left(\overline{P+p'}\right)=-\frac{\partial P}{\partial x_i} \tag{E.1.7}$$

Viscous term:

$$\frac{\partial}{\partial x_j}\left[\overline{2\mu\left(S_{ji}+s'_{ji}\right)}\right]=\frac{\partial}{\partial x_j}\left(2\mu S_{ji}\right) \tag{E.1.8}$$

In the above relations, we have used the identity that the time-average of a fluctuating component is zero. Collecting all of the above terms together yields the Reynolds-averaged momentum equation, given by

$$\rho\frac{\partial U_i}{\partial t}+\rho U_j\frac{\partial U_i}{\partial x_j}=-\frac{\partial P}{\partial x_i}+\frac{\partial}{\partial x_j}\left[\left(2\mu S_{ji}\right)-\rho\overline{u'_i u'_j}\right]+\rho g_i \tag{E.1.9}$$

where the terms given by $-\rho\overline{u'_i u'_j}$ are typically called the Reynolds stresses.

E.2 Reynolds-Averaged Thermal Energy Equation

The equation of thermal energy is given by

$$\rho\frac{\partial h}{\partial t}+\rho\frac{\partial}{\partial x_i}\left(hu_i\right)=\frac{\partial}{\partial x_i}\left(k\frac{\partial T}{\partial x_i}\right)+S \tag{E.2.1}$$

Assuming again that the density fluctuations are negligible, and dividing the velocity, enthalpy, and temperature into mean and fluctuating components given by

$$u_i=U_i+u'_i \qquad\qquad h=H+h' \tag{E.2.2a,b}$$

$$T=\overline{T}+T' \tag{E.2.2c}$$

then the time-average of Eq. (E.2.1) becomes

$$\rho\overline{\frac{\partial}{\partial t}\left(H+h'\right)}+\rho\overline{\frac{\partial}{\partial x_i}\left[\left(H+h'\right)\left(U_i+u'_i\right)\right]}=\frac{\partial}{\partial x_i}\left[k\overline{\frac{\partial}{\partial x_i}\left(\overline{T}+T'\right)}\right]+S \tag{E.2.3}$$

Again, we will consider each term individually.
Transient term:

$$\rho\overline{\frac{\partial}{\partial t}(H+h')} = \rho\frac{\partial H}{\partial t} \tag{E.2.4}$$

Convective term:

$$\rho\overline{\frac{\partial}{\partial x_i}[(H+h')(U_i+u_i')]} = \rho\frac{\partial}{\partial x_i}\left[\overline{HU_i + Hu_i' + h'U_i + h'u_i'}\right] = \rho\frac{\partial}{\partial x_i}(U_iH) + \rho\frac{\partial}{\partial x_i}\left(\overline{u_i'h'}\right) \tag{E.2.5}$$

Diffusive term:

$$\frac{\partial}{\partial x_i}\left[k\overline{\frac{\partial}{\partial x_i}\left(\overline{T}+T'\right)}\right] = \frac{\partial}{\partial x_i}\left(k\frac{\partial T}{\partial x_i}\right) \tag{E.2.6}$$

In the above relations, we have used the identity that the time-average of a fluctuating quantity is zero. Collecting all of the above terms and substituting into Eq. (E.2.3) yields

$$\rho\frac{\partial H}{\partial t} + \rho\frac{\partial}{\partial x_i}(U_iH) = \frac{\partial}{\partial x_i}\left[k\frac{\partial T}{\partial x_i} - \rho\overline{u_i'h'}\right] + S \tag{E.2.7}$$

And realizing that for an ideal fluid, $h = c_pT$, then Eq. (E.2.7) becomes

$$\frac{\partial\overline{T}}{\partial t} + \frac{\partial}{\partial x_i}\left(U_i\overline{T}\right) = \frac{\partial}{\partial x_i}\left[\alpha\frac{\partial\overline{T}}{\partial x_i} - \overline{u_i'T'}\right] + \frac{S}{\rho c_p} \tag{E.2.8}$$

E.3 Reynolds-Averaged Scalar Transport Equation

The generic transport equation, including a source, for some scalar quantity ϕ is given by

$$\frac{\partial}{\partial t}(\rho\phi) + \frac{\partial}{\partial x_i}(\rho u_i\phi) = \Gamma\frac{\partial}{\partial x_i}\left(\frac{\partial\phi}{\partial x_i}\right) + S_C + S_P\phi \tag{E.3.1}$$

Assuming again that the density fluctuations are negligible, we can divide the velocity and scalar into mean and fluctuating components. Time-averaging the resulting equation yields

$$\overline{\frac{\partial}{\partial t}[\rho(\Phi+\phi')]} + \overline{\frac{\partial}{\partial x_i}[\rho(U_i+u_i')(\Phi+\phi')]} = \Gamma\overline{\frac{\partial}{\partial x_i}\left[\frac{\partial}{\partial x_i}(\Phi+\phi')\right]} + \overline{S_C + S_P(\Phi+\phi')} \tag{E.3.2}$$

Again, we will consider each term separately.

Transient term:

$$\overline{\frac{\partial}{\partial t}\left[\rho\left(\Phi+\phi'\right)\right]}=\frac{\partial}{\partial t}\left(\rho\Phi\right) \tag{E.3.3}$$

Convective term:

$$\overline{\frac{\partial}{\partial x_i}\left[\rho\left(U_i+u_i'\right)\left(\Phi+\phi'\right)\right]}=\overline{\frac{\partial}{\partial x_i}\left(\rho U_i\Phi\right)}+\overline{\frac{\partial}{\partial x_i}\left(\rho U_i\phi'\right)}+\overline{\frac{\partial}{\partial x_i}\left(\rho u_i'\Phi\right)}+\overline{\frac{\partial}{\partial x_i}\left(\rho u_i'\phi'\right)} \tag{E.3.4}$$

The second and third terms on the RHS of Eq. (E.3.4) are zero, because the average of a fluctuating component times a mean quantity is zero. Then Eq. (E.3.4) becomes

$$\overline{\frac{\partial}{\partial x_i}\left[\rho\left(U_i+u_i'\right)\left(\Phi+\phi'\right)\right]}=\overline{\frac{\partial}{\partial x_i}\left(\rho U_i\Phi\right)}+\overline{\frac{\partial}{\partial x_i}\left(\rho u_i'\phi'\right)} \tag{E.3.5}$$

Diffusive term:

$$\Gamma\overline{\frac{\partial}{\partial x_i}\left[\frac{\partial}{\partial x_i}\left(\Phi+\phi'\right)\right]}=\Gamma\frac{\partial}{\partial x_i}\left(\frac{\partial\Phi}{\partial x_i}\right) \tag{E.3.6}$$

Source:

$$\overline{S_C+S_P\left(\Phi+\phi'\right)}=S_C+S_P\Phi \tag{E.3.7}$$

In all of the above relations, we have used the identity that the time-average of a fluctuating quantity is zero. Collecting all of the above terms and substituting yields

$$\frac{\partial}{\partial t}\left(\rho\Phi\right)+\frac{\partial}{\partial x_i}\left(\rho U_i\Phi\right)=\frac{\partial}{\partial x_i}\left[\Gamma\frac{\partial}{\partial x_i}\left(\Phi\right)-\rho\overline{u_i'\phi'}\right]+S_C+S_P\Phi \tag{E.3.8}$$

In a similar manner, an equation can be derived for the variance of the fluctuation of a conserved scalar quantity, i.e. $\eta=\overline{\phi'^2}$

$$\frac{\partial}{\partial t}\left(\rho\eta\right)+\frac{\partial}{\partial x_i}\left(\rho U_i\eta\right)=\frac{\partial}{\partial x_i}\left[-\rho\overline{u_i'\eta'}\right]+G_\eta-D_\eta \tag{E.3.9}$$

where G_η and D_η represent the generation and destruction rate of η, respectively.

APPENDIX F: TURBULENCE MODELS

The following is a list of models from Haase et al., 1997. A short description of the models can be found in the same reference.

1 Algebraic Turbulence Models
- 1.1 Introduction
- 1.2 The Baldwin-Lomax [BL] Model
- 1.3 The Granville Model for Use in Baldwin-Lomax [GRB]
- 1.4 The Cebeci-Smith [CS] Model
- 1.5 The Granville Model for Use in Cebeci-Smith [GRC]
- 1.6 The Goldberg Backflow [GB] Model
- 1.7 The Algebraic Yakhot-Orszag [YA] Model

2 Half-Equation Turbulence Models
- 2.1 Introduction
- 2.2 The Johnson-King [JK] Model
- 2.3 The Johnson-Coakley [JC] Model
- 2.4 The 3D Extension of the JK Model by Abid-Vatsa-Johnson-Wedan [AVJW]
- 2.5 The Horton [HH] Model
- 2.6 The Le Balleur 2-half-equation k-u'v' [LBA.KUV] Model

3 One-Equation Models
- 3.1 Introduction
- 3.2 The Wolfshtein [W] Model
- 3.3 The Hassid-Poreh [HP] Model
- 3.4 The Baldwin-Barth Model [BB]

4 Two-Equation Models
- 4.1 Introduction
- 4.2 The Jones/Launder and Launder/Sharma [JL] Model
- 4.3 the Chien [C] Model
- 4.4 The Lam-Bremhorst [LB] Model
- 4.5 The Lien-Leschziner [LL] Model
- 4.6 The 'Standard' High-Reynolds-number Jones-Launder [SJL]. Model
- 4.7 The Huang-Coakley [HC] Model
- 4.8 The Yakhot/Orszag [YO] RNG Model
- 4.9 The Wilcox [WX] Model
- 4.10 The Kalitzin-Gould k-τ [KG] Model

5 Non-Linear Eddy Viscosity (k-ε) Models
- 5.1 Introduction
- 5.2 The Speziale Model [S]
- 5.3 The Rubenstein-Barton [RB] Model
- 5.4 The Shih-Lumley-Zhu [SLZ] Model

6 Reynolds-Stress Models
- 6.1 Introduction