Nonlinear Dynamics of structures and mechanical systems

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Lesson 3.1

Elements of Stability Theory

Example: stability analysis for the solution $\boldsymbol{\delta y} = \mathbf{0}$ of a SDOF oscillator

$$\begin{cases} \delta \dot{y}_1 = \delta y_2 \\ \delta \dot{y}_2 = -\omega^2 \delta y_1 - 2\xi\omega\,\delta y_2 \end{cases}$$

$$\boldsymbol{\delta \dot{y}} = \mathbf{A}\boldsymbol{\delta y}$$

$$\mathbf{A} = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{y}}\right|_{\mathbf{0}} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\xi\omega \end{bmatrix}$$

$$2\xi\omega \to b \qquad \omega^2 \to c$$
$$b \in \mathbb{R} \qquad c \in \mathbb{R}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -c & -b \end{bmatrix}$$

characteristic equation $\lambda^2 + b\lambda + c = 0$ ⇨ $\lambda = \dfrac{-b \pm \sqrt{b^2 - 4c}}{2}$

Elements of Stability Theory

Example: stability analysis for the solution $\boldsymbol{\delta y} = \mathbf{0}$ of a SDOF oscillator

Let $\boldsymbol{\delta x} = \mathbf{B}\boldsymbol{\delta y}$ such that $\boldsymbol{\delta \dot{y}} = \mathbf{A}\boldsymbol{\delta y} \Rightarrow \boldsymbol{\delta \dot{x}} = \mathbf{C}\boldsymbol{\delta x}$

with $\mathbf{C}$ being a Jordan canonical form

Remark: $\mathbf{B}$ must be such that $\mathbf{CB} = \mathbf{BA} \Rightarrow \mathbf{C} = \mathbf{BAB}^{-1}$

Case (a): $\lambda_1 \in \mathbb{R},\ \lambda_2 \in \mathbb{R},\ \lambda_1 \neq \lambda_2 \rightarrow$ $\mathbf{C} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

$$b^2 - 4c > 0$$

Case (b): $\lambda_1 = \lambda_2 = \lambda \in \mathbb{R} \rightarrow$ $\mathbf{C} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$ ou $\mathbf{C} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$

$$b^2 - 4c = 0$$

Case (c): $\lambda_1 = \lambda = \alpha + i\beta \in \mathbb{C},\ \lambda_2 = \bar{\lambda} = \alpha - i\beta \in \mathbb{C} \rightarrow$ $\mathbf{C} = \begin{bmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{bmatrix}$

$$b^2 - 4c < 0$$

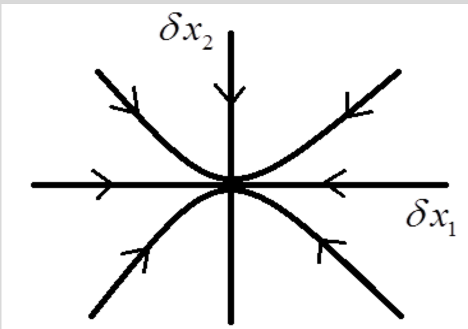
Elements of Stability Theory

Example: stability analysis for the solution $\boldsymbol{\delta y} = \mathbf{0}$ of a SDOF oscillator

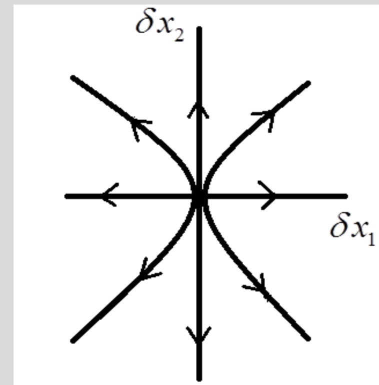
Case (a)

$$\delta x_i = \delta x_i^0 e^{\lambda_i t}$$

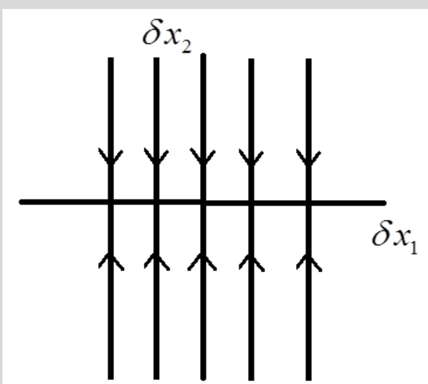
$$\frac{d(\delta x_2)}{d(\delta x_1)} = \left(\frac{\lambda_2}{\lambda_1}\right)\left(\frac{\delta x_2^0}{\delta x_1^0}\right) e^{(\lambda_2-\lambda_1)t}$$



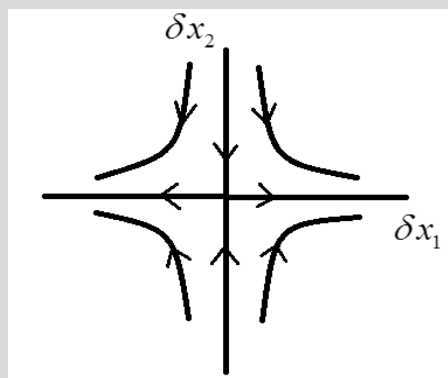
$\lambda_2 < \lambda_1 < 0$



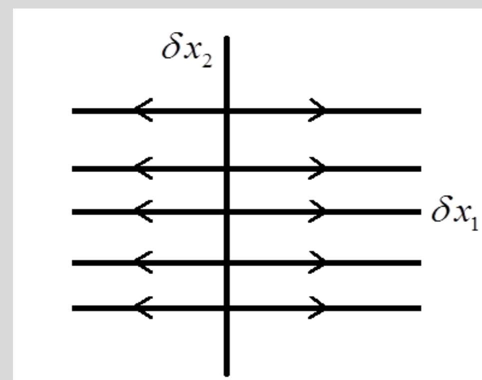
$0 < \lambda_2 < \lambda_1$



$\lambda_2 < \lambda_1 = 0$



$\lambda_2 < 0 < \lambda_1$



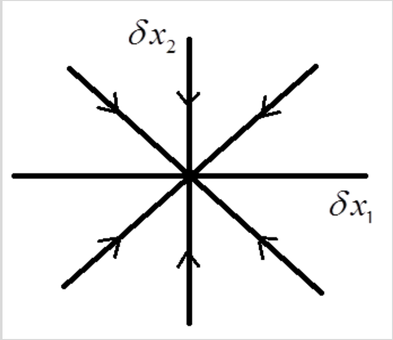
$0 = \lambda_2 < \lambda_1$

Elements of Stability Theory

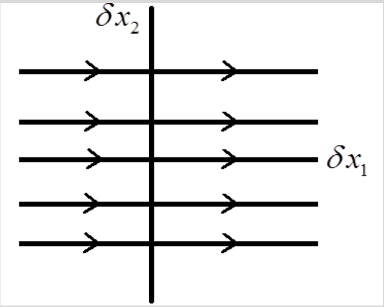
Example: stability analysis for the solution $\boldsymbol{\delta y = 0}$ of a SDOF oscillator

Case (b1)

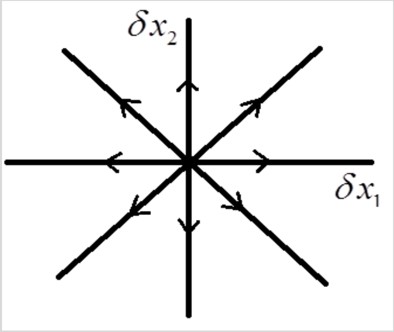
$$\delta x_i = \delta x_i^0 e^{\lambda t} \Rightarrow \frac{d(\delta x_2)}{d(\delta x_1)} = \left(\frac{\delta x_2^0}{\delta x_1^0}\right)$$



$\lambda_2 = \lambda_1 < 0$

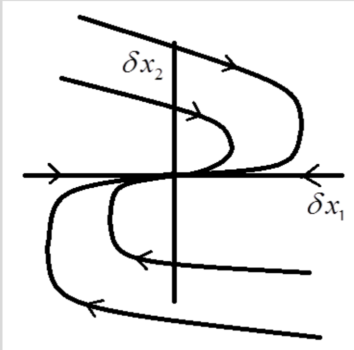


$\lambda_2 = \lambda_1 = 0$



$0 < \lambda_2 = \lambda_1$

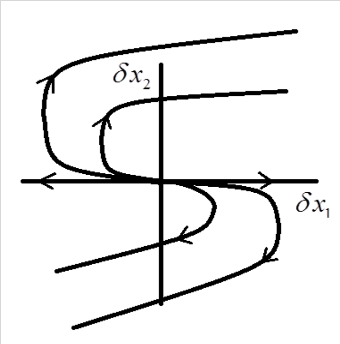
Case (b2)



$\lambda_2 = \lambda_1 < 0$

$$\delta x_1 = (\delta x_1^0 + t\delta x_2^0)e^{\lambda t} \quad \delta x_2 = \delta x_2^0 e^{\lambda t}$$

$$\frac{d(\delta x_2)}{d(\delta x_1)} = \frac{\delta x_2^0}{\delta x_1^0 + \left(t + \frac{1}{\lambda}\right)\delta x_2^0} = \frac{1}{\frac{\delta x_1^0}{\delta x_2^0} + \left(t + \frac{1}{\lambda}\right)}$$



$0 < \lambda_2 = \lambda_1$

Elements of Stability Theory

Example: stability analysis for the solution $\boldsymbol{\delta y} = \mathbf{0}$ of a SDOF oscillator

Case (c)

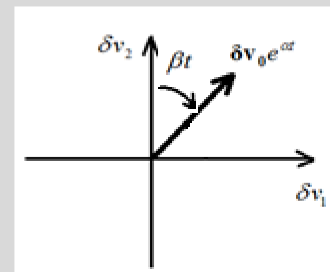
Change variables...

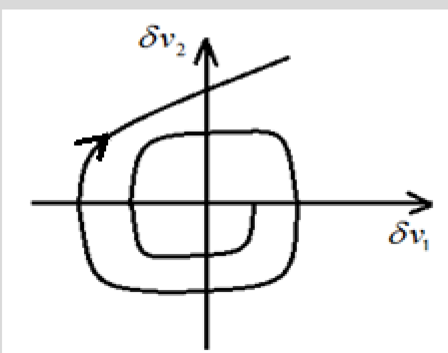
$$\boldsymbol{\delta\dot{x}} = \begin{bmatrix} \alpha + i\beta & 0 \\ 0 & \alpha - i\beta \end{bmatrix} \boldsymbol{\delta x} \qquad \boldsymbol{\delta v} = \begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix} \boldsymbol{\delta x}$$

$$\boldsymbol{\delta\dot{v}} = \begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix} \boldsymbol{\delta\dot{x}} = \begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix} \begin{bmatrix} \alpha + i\beta & 0 \\ 0 & \alpha + i\beta \end{bmatrix} \begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}^{-1} \boldsymbol{\delta v} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \boldsymbol{\delta v}$$

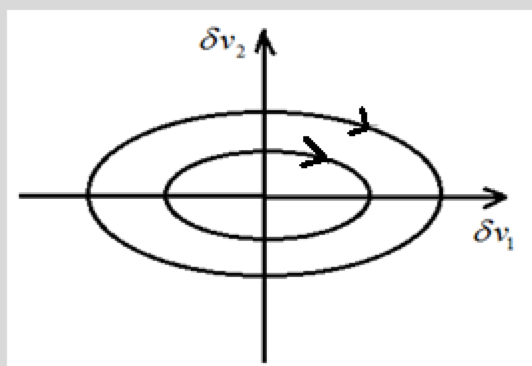
Define vector $\boldsymbol{\delta v} = \delta v_1 + i\delta v_2$ in Argand's plane ...

$$\boldsymbol{\delta\dot{v}} = (\alpha + i\beta)\boldsymbol{\delta v} \Rightarrow \boldsymbol{\delta v} = \boldsymbol{\delta v_0} e^{\alpha t} e^{i\beta t}$$

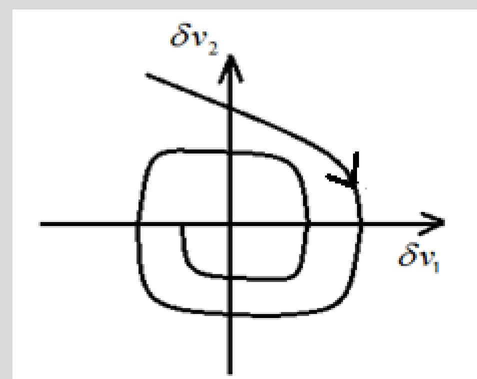




$\alpha > 0$



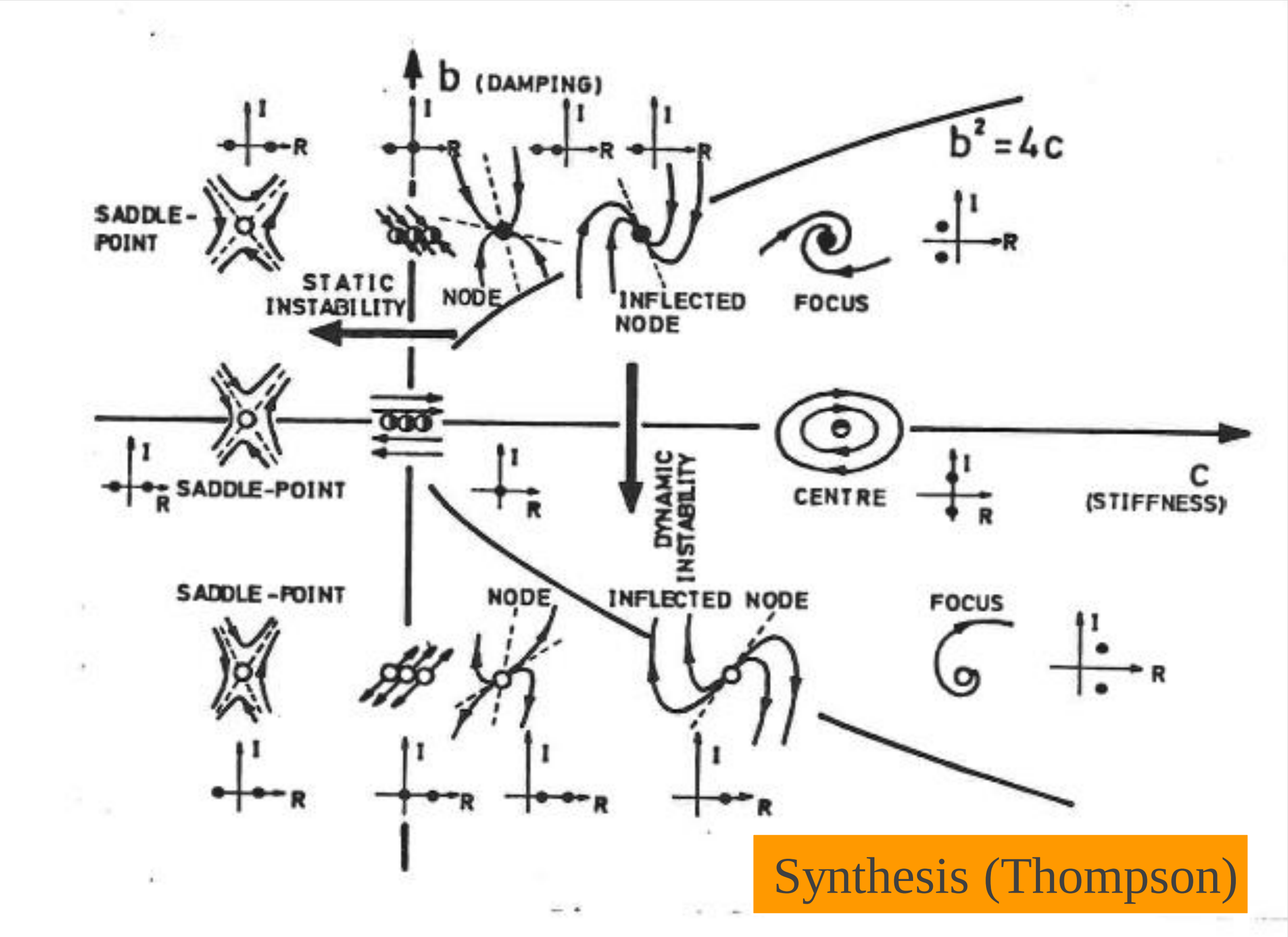
$\alpha = 0$



$\alpha < 0$

Elements of Stability Theory

Example: stability analysis for the solution $\boldsymbol{\delta y} = \mathbf{0}$ of a SDOF oscillator



Synthesis (Thompson)

Elements of Stability Theory

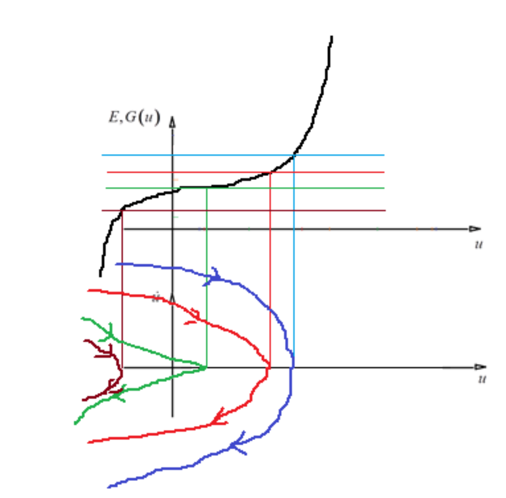
Conservative SDOF oscillator

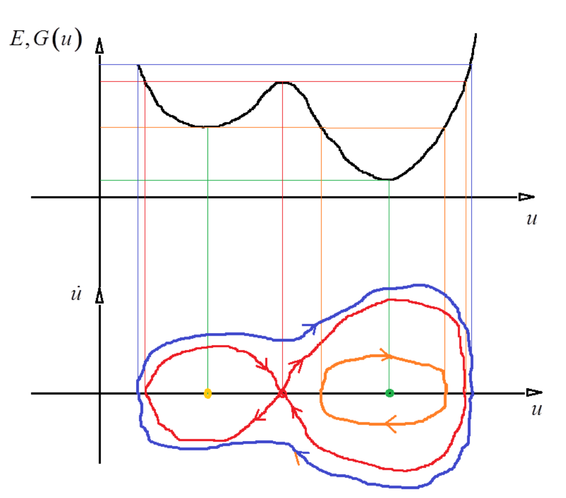
$$\ddot{u}+g(u)=0 \Rightarrow \ddot{u}\,du+g(u)\,du=0 \Rightarrow \ddot{u}\dot{u}dt+g(u)\,du=0$$

Integrating: $$\underbrace{\frac{\dot{u}^2}{2}}_{\text{kinetic energy}} + \underbrace{\int_0^u g(\eta)d\eta}_{\text{potential energy}} = \underbrace{E}_{\text{mechanical energy}} = const.$$

Define: $$G(u)=\int_0^u g(\eta)d\eta \Rightarrow \dot{u}=\pm\sqrt{2[E-G(u)]} \Rightarrow \underbrace{T=2\int_{u(0)}^{u(T/2)}\frac{du}{\sqrt{2[E-G(u)]}}}_{\text{period of motion}}$$

saddle-node





saddle
&
centres

Elements of Stability Theory

Liapunov's second method

$$\boldsymbol{\delta}\dot{\mathbf{y}} = \mathbf{f}_0(\boldsymbol{\delta}\mathbf{y}) = \mathbf{A}\boldsymbol{\delta}\mathbf{y} + \mathbf{N}(\boldsymbol{\delta}\mathbf{y})$$

where $\mathbf{A} = \left.\dfrac{\partial \mathbf{f}_0}{\partial \mathbf{y}}\right|_{\mathbf{0}}$ and $\mathbf{N}(\boldsymbol{\delta}\mathbf{y}) = \mathbf{f}_0(\boldsymbol{\delta}\mathbf{y}) - \mathbf{A}\boldsymbol{\delta}\mathbf{y}$

Theorem 6 (Liapunov): if there exists a function $F(\boldsymbol{\delta}\mathbf{y}): E \to \mathbb{R}$ such that:

$$F \geq 0 \quad \forall \boldsymbol{\delta}\mathbf{y}$$
$$F = 0 \Leftrightarrow \boldsymbol{\delta}\mathbf{y} = \mathbf{0}$$
$$\dot{F} = \frac{\partial F}{\partial \delta y_r}\delta\dot{y}_r = \frac{\partial F}{\partial \delta y_r} f_{0r} \leq 0$$

then $\boldsymbol{\delta}\mathbf{y} = \mathbf{0}$ is L-stable

Elements of Stability Theory

Liapunov's second method

Theorem 7 (Liapunov): if there exists a function $F(\boldsymbol{\delta}\mathbf{y}): E \to \mathbb{R}$ such that:

$$F \geq 0 \quad \forall \boldsymbol{\delta}\mathbf{y}$$
$$F = 0 \Leftrightarrow \boldsymbol{\delta}\mathbf{y} = \mathbf{0}$$
$$\dot{F} = \frac{\partial F}{\partial \delta y_r} \delta \dot{y}_r = \frac{\partial F}{\partial \delta y_r} f_{0r} < 0$$

then $\boldsymbol{\delta}\mathbf{y} = \mathbf{0}$ is asymptotically stable in Liapunov's sense

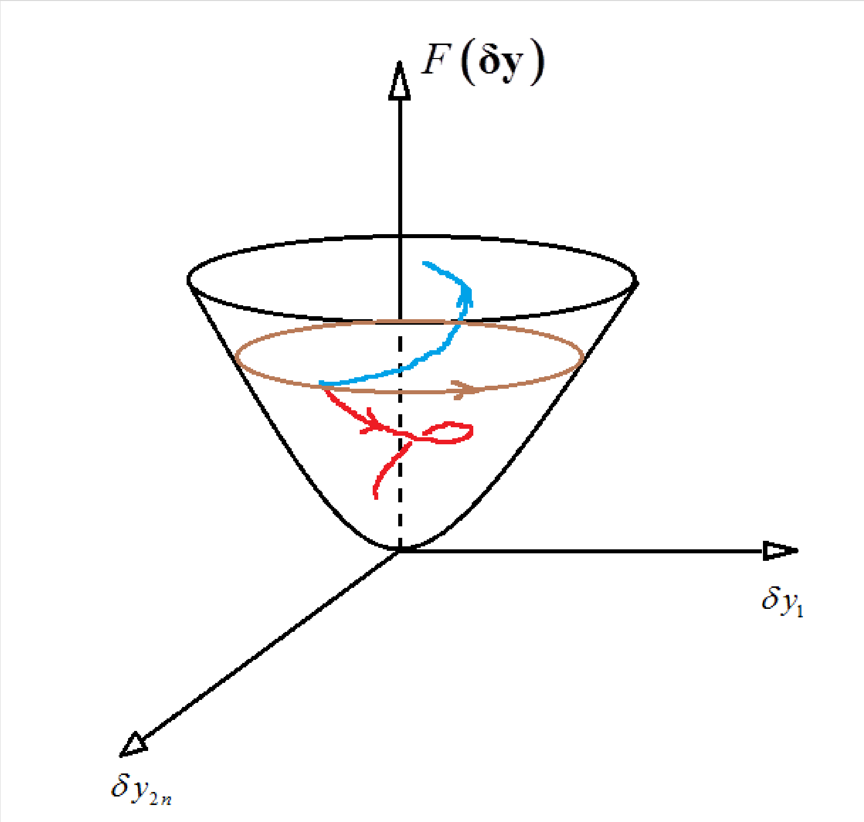
Theorem 8 (Chetayev): if there exists a function $F(\boldsymbol{\delta}\mathbf{y}): E \to \mathbb{R}$ such that:

$$F \geq 0 \quad \forall \boldsymbol{\delta}\mathbf{y}$$
$$F = 0 \Leftrightarrow \boldsymbol{\delta}\mathbf{y} = \mathbf{0}$$
$$\dot{F} = \frac{\partial F}{\partial \delta y_r} \delta \dot{y}_r = \frac{\partial F}{\partial \delta y_r} f_{0r} > 0$$

then $\boldsymbol{\delta}\mathbf{y} = \mathbf{0}$ is L-unstable

Elements of Stability Theory

Liapunov's second method



$F(\boldsymbol{\delta y})$ is called Liapunov's function

Elements of Stability Theory

Attractor

Subset of the phase space to which a solution of the dynamical system tends when $t \to \infty$ for initial conditions in a non-localized subset of the phase space (basin of attraction)

- Fixed point (stable equilibrium point): asymptotically stable singularity
- Stable limit cycle (periodic attractor): asymptotically stable orbit in the phase space with one dominating frequency or more than one commensurate dominating frequencies
- Limit torus: asymptotically stable manifold in the phase space, with more than one non-commensurate dominating frequency
- Strange attractor (chaos): coexistence of some of the previous attractors with non-compact (fractal) basins of attraction

Elements of Stability Theory

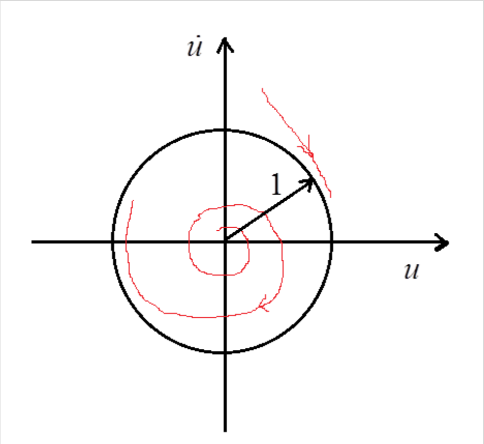
Periodic attractor in autonomous dynamical system $\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y})$

Example: van der Pol equation

$$\ddot{u} - \dot{u} + u + \left(u^2 + \dot{u}^2\right)\dot{u} = 0$$

Trivial solution $u(t) = 0$ is unstable

Periodic attractor $u(t) = \sin t$ is stable



Elements of Stability Theory

Dynamical Systems

Hirsch & Smale: Differential Equations, Dynamical Systems
and Linear Algebra
Guckenheimer & Holmes: Nonlinear Oscillations, Dynamical Systems
And Bifurcation of Vector Fields

Elements of Stability Theory

Orbital stability of autonomous SDOF oscillators

- First Poincaré-Bendixson's Theorem:

If a phase trajectory C remains within a finite region without approaching a singularity, then C is a limit cycle or it tends to one.

- Second Poincaré-Bendixson's Theorem:

Given a region D of the phase space, bounded by two curves C' and C", without a singularity in D, C' e C", if all phase trajectories enter (exit) in D through the boundaries C' e C", then there exists at least a stable (unstable) limit cycle in D.

Elements of Stability Theory

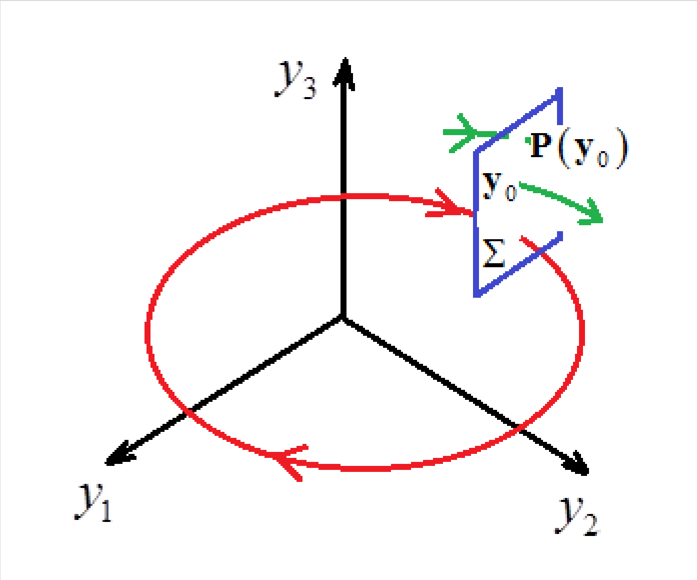
Poincaré’s section (map)

- Let $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$ be a flow of an autonomous system in $\mathbb{R}^{2n}$ and $\Sigma : \mathbf{f}(\mathbf{y}) \cdot \mathbf{N} \neq 0$ a section with normal $\mathbf{N}$. Consider the mapping $\mathbf{y}_0 \rightarrow \mathbf{P}(\mathbf{y}_0)$ defined by the intersection of the flow $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$ with Σ. $\mathbf{P}(\mathbf{y}_0)$ is termed a “Poincaré’s section” of the flow $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$ through $\mathbf{y}_0$

- If the system is non-autonomous, defined by the flow $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, t)$, an associated autonomous one $\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y})$ defined in $\mathbb{R}^{2n+1}$ can be proposed with the addition of $\dot{y}_{2n+1} = 1$, so that the Poincaré’s sections can be defined orthogonally to the axis $y_{2n+1} = t$ at $t = t_0 + iT, \quad i = 1,2, \ldots$

Elements of Stability Theory

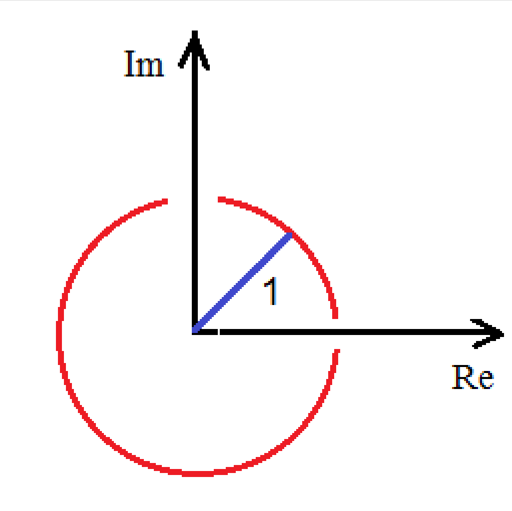
Poincaré's section (map)



Analyse the complex eigenvalues $\lambda_j = \mathrm{Re}_j + i\,\mathrm{Im}_j$ of linearized mapping $\mathbf{DP}(\mathbf{y}_0)$ to test stability.

Stability for $|\lambda_j| < 1$

Instability for $|\lambda_j| > 1$



Elements of Stability Theory

Example of Poincaré's section (map)

$$\ddot{u}+\left(-1+u^2+\dot{u}^2\right)\dot{u}+u=0$$

$$\left.\begin{aligned} y_1 &= u \\ y_2 &= \dot{u} \end{aligned}\right\} \Rightarrow \dot{\mathbf{y}} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}\mathbf{y} + \begin{Bmatrix} 0 \\ -\left(y_1^2+y_2^2\right)y_2 \end{Bmatrix}$$

In polar co-ordinates

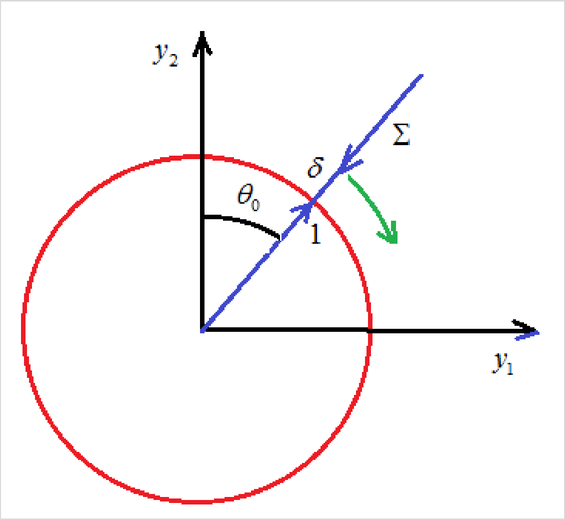
$$\left.\begin{aligned} y_1 &= r\sin\theta \\ y_2 &= r\cos\theta \end{aligned}\right\} \Rightarrow r=0 \text{ corresponds to an unstable focus}$$

$$\text{for } r\neq 0 \Rightarrow \begin{cases} \dot{r} = -r\left(r^2-1\right)\cos^2\theta \\ \dot{\theta} = 1+\left(r^2-1\right)\sin\theta\cos\theta \end{cases}$$

It is readily seen that $r=1$ and $\theta=t$ are a limit cycle

Elements of Stability Theory

Example of Poincaré's section (map)



Poincaré's section: $\theta = \theta_0$

$r_0 = 1 + \varepsilon_0 \to r_j = 1 + \varepsilon_j$ for $\theta = \theta_0 + 2\pi j \quad j = 1,2,\ldots$

Mapping: $\dot{r}_j = \dot{\varepsilon}_j = -(1+\varepsilon_j)\left[(1+\varepsilon_j)^2 - 1\right]\cos^2\theta_0$

$$\dot{\varepsilon}_j = -\left(2\varepsilon_j + 3\varepsilon_j^2 + \varepsilon_j^3\right)\cos^2\theta_0$$

Linearizing: $\dot{\varepsilon}_j = -(2\cos^2\theta_0)\varepsilon_j \Rightarrow \varepsilon_j = \varepsilon_0 e^{-4\pi j \cos^2\theta_0}$

Mapping in $\mathbb{R}^1$: $r_j \to r_{j+1} = P(r_j)$

$= 1 + (r_j - 1)e^{-4\pi\cos^2\theta_0}$

$\mathbf{DP} = \dfrac{dP(r_j)}{dr_j} = e^{-4\pi\cos^2\theta_0}$

asymptotic stability for $\theta_0 \neq \dfrac{\pi}{2}$ or $\dfrac{3\pi}{2}$, since $|\lambda| < 1$

stability for $\theta_0 = \dfrac{\pi}{2}$ or $\dfrac{3\pi}{2}$, since $\dot{\varepsilon}_j = 0 \Rightarrow \varepsilon_j = \varepsilon_0$

Elements of Stability Theory

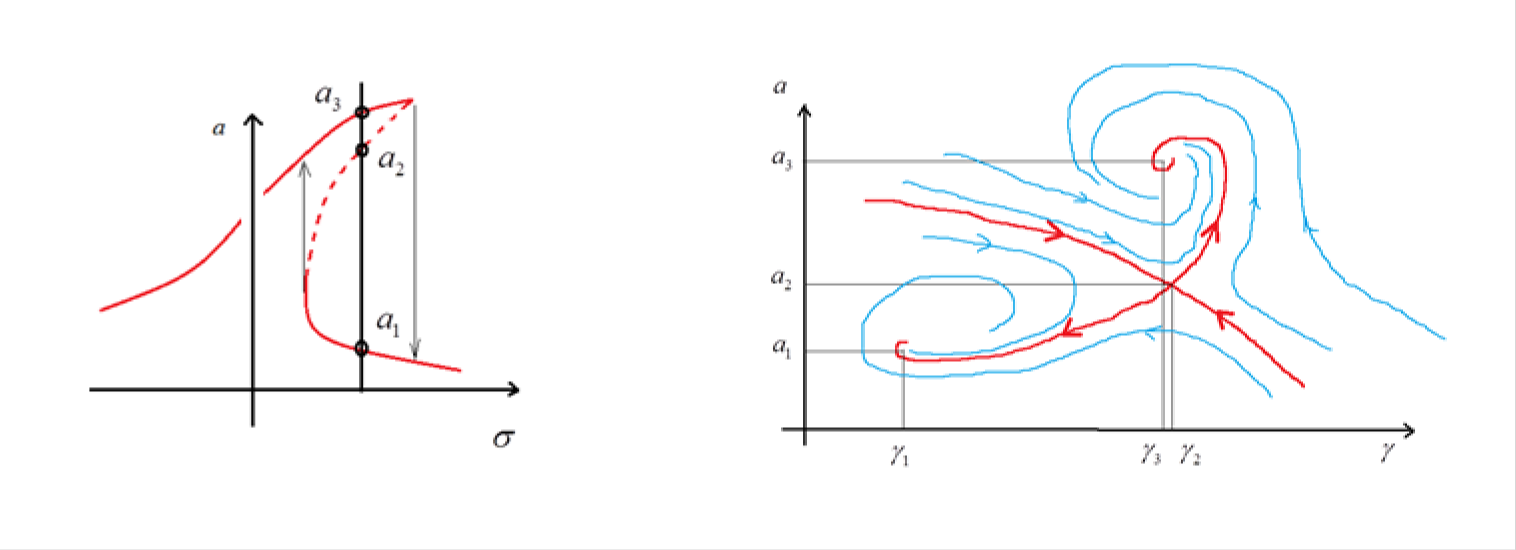
Periodic attractor in non-autonomous dynamical system $\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y}, t)$

Example: forced Duffing's equation

$$\ddot{u} + 2\varepsilon\mu\dot{u} + \omega_0^2 u + \varepsilon\alpha u^3 = \varepsilon k \cos(\omega_0 + \varepsilon\sigma)t \quad \text{with } 0 < \varepsilon << 1$$

There exist periodic attractors

$$u(t) = a\cos\left[(\omega_0 + \varepsilon\sigma)t + \gamma\right] + O(\varepsilon)$$



Falls within stability of singularities...