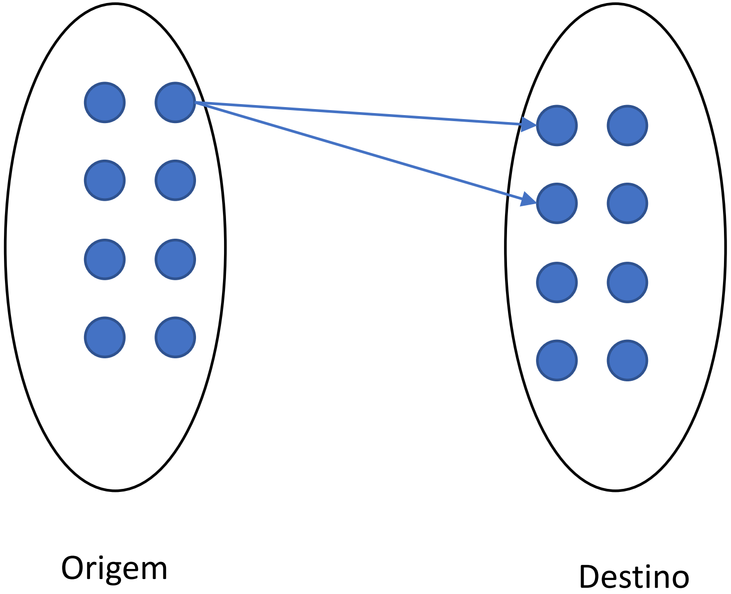
PRO5802 - Programação de Produção Intermitente (2023)

Aula 2 – Teoria de Grafos

Prof. Daniel de Oliveira Mota

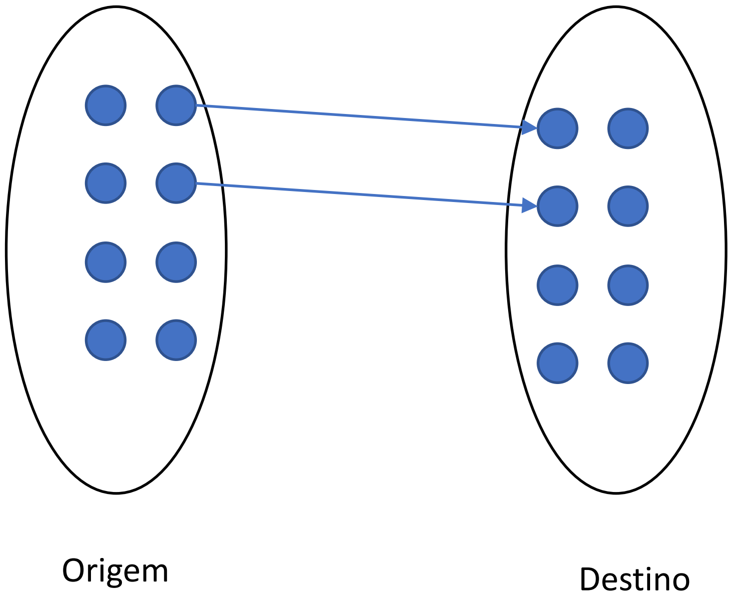
Grafo



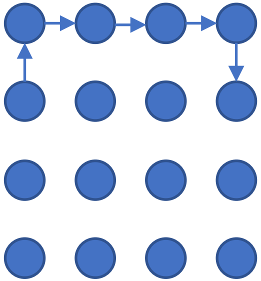
G = (A, V)

Grafo

G = (A, V)

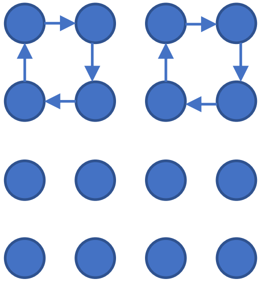


Grafo



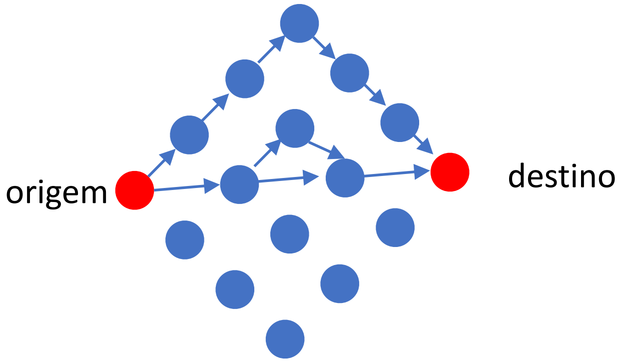
G = (A, V)

Grafo



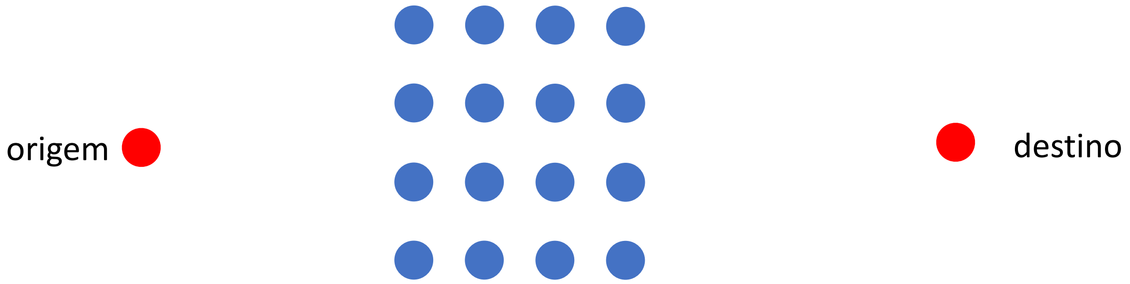
G = (A, V)

Grafo



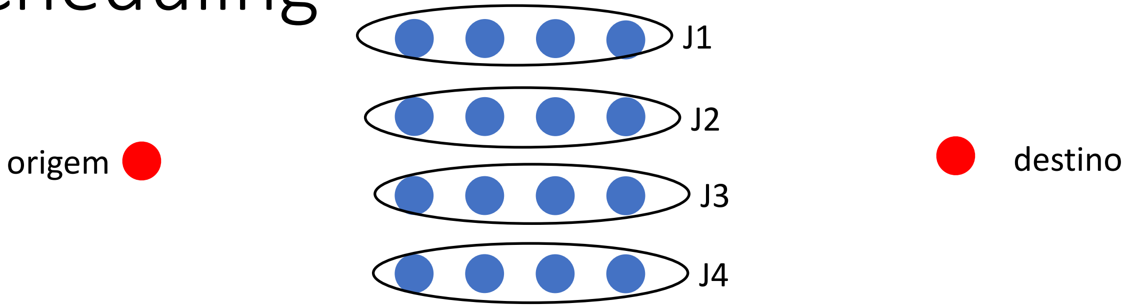
G = (A, V)

Grafo



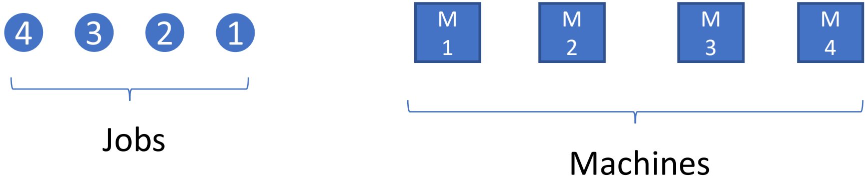
G = (A, V)

Grafo x Scheduling

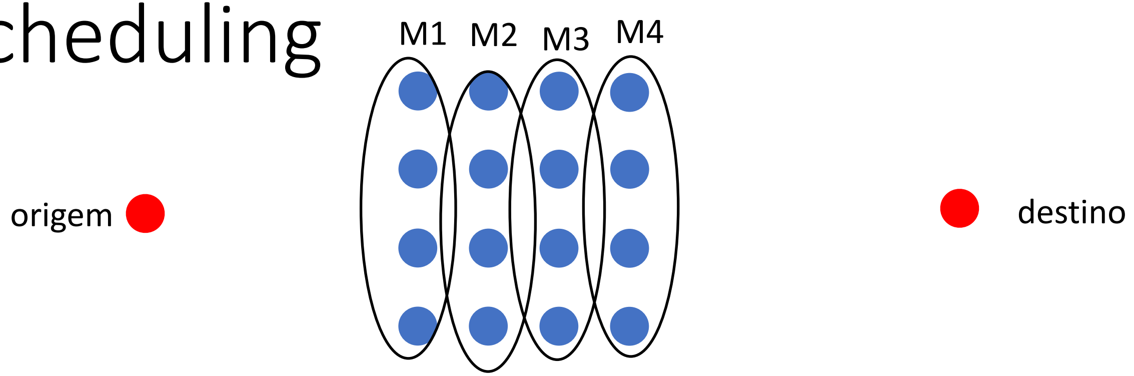


G = (A, V)

Tarefas (jobs) com Recursos (machines)

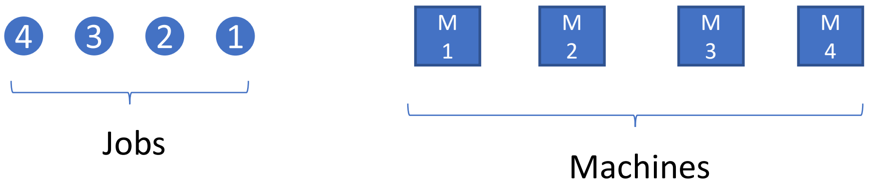


Grafo x Scheduling

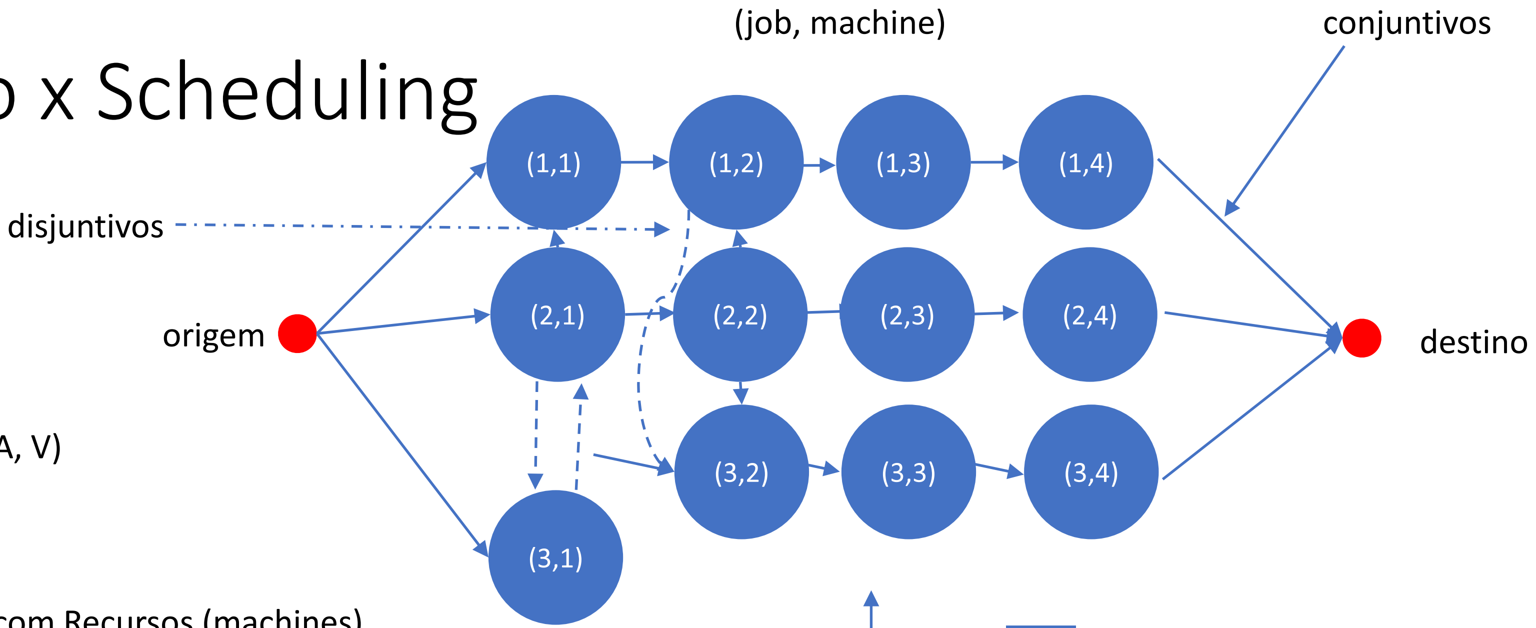


G = (A, V)

Tarefas (jobs) com Recursos (machines)

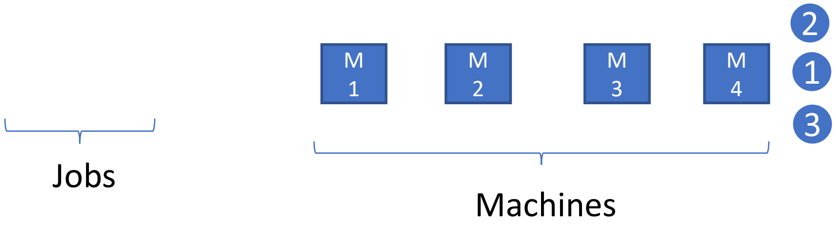


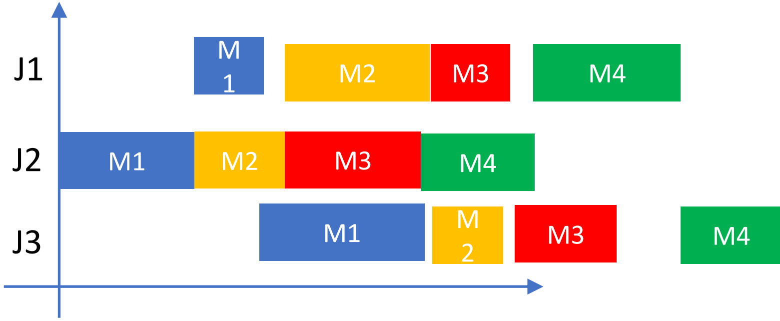
Grafo x Scheduling



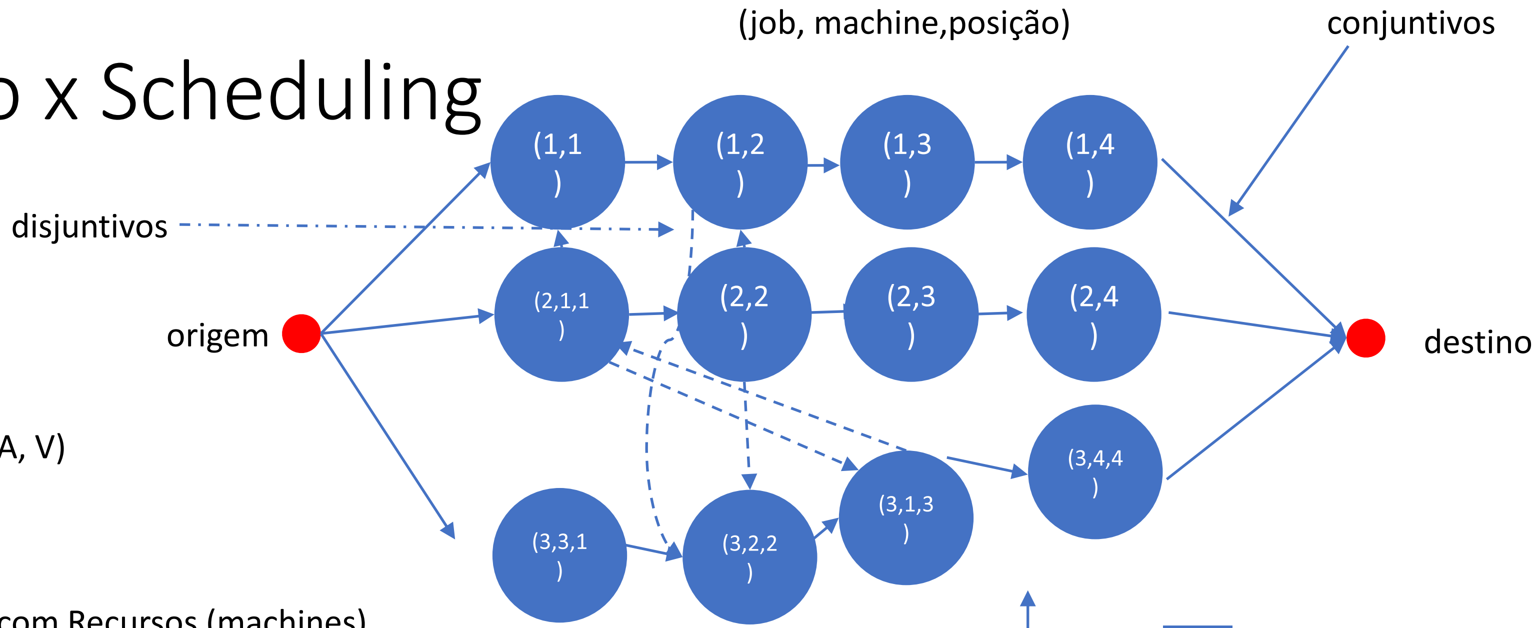
G = (A, V)

Tarefas (jobs) com Recursos (machines)



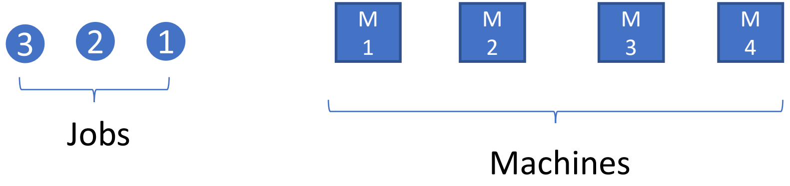


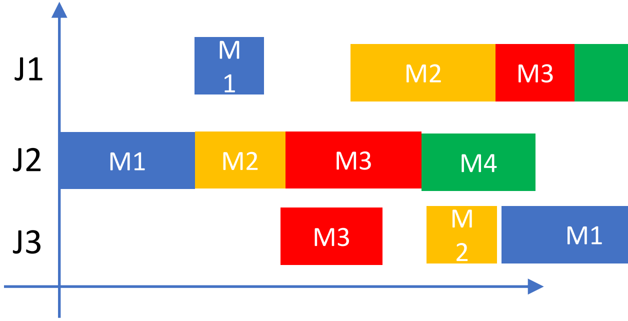
Grafo x Scheduling



G = (A, V)

Tarefas (jobs) com Recursos (machines)



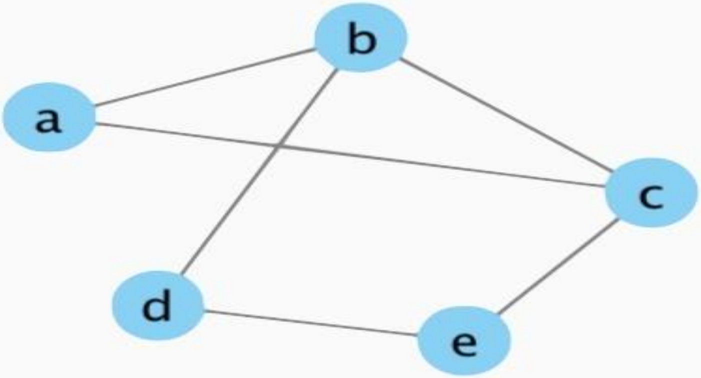


WHAT ARE GRAPHS?

Graphs in Graph Theory

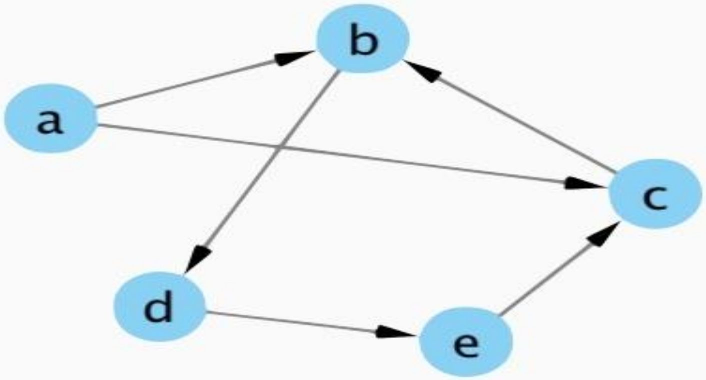
- G = (V, E)
  - V := Set of vertices (nodes)
  - E := Set of edges
- Model **relationships** between pairs of objects

Undirected Graphs



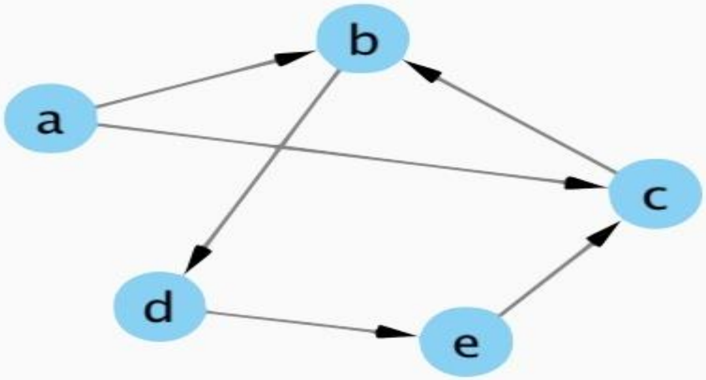
- V = {a, b, c, d, e} |V| = 5
- E = { {a, b}, {a, c}, {b, c}, {b, d}, {d, e}, {b, e} } |E| = 6

Directed Graphs



- V = {a, b, c, d, e} |V| = 5
- E = { (a, b), (a, c), (c, b), (b, d), (d, e), (e, c) } |E| = 6

Directed Graphs



- Can we traverse this graph from **a** to **e**?
- Can we traverse this graph from **e** to **a**?

EXAMPLES OF GRAPHS

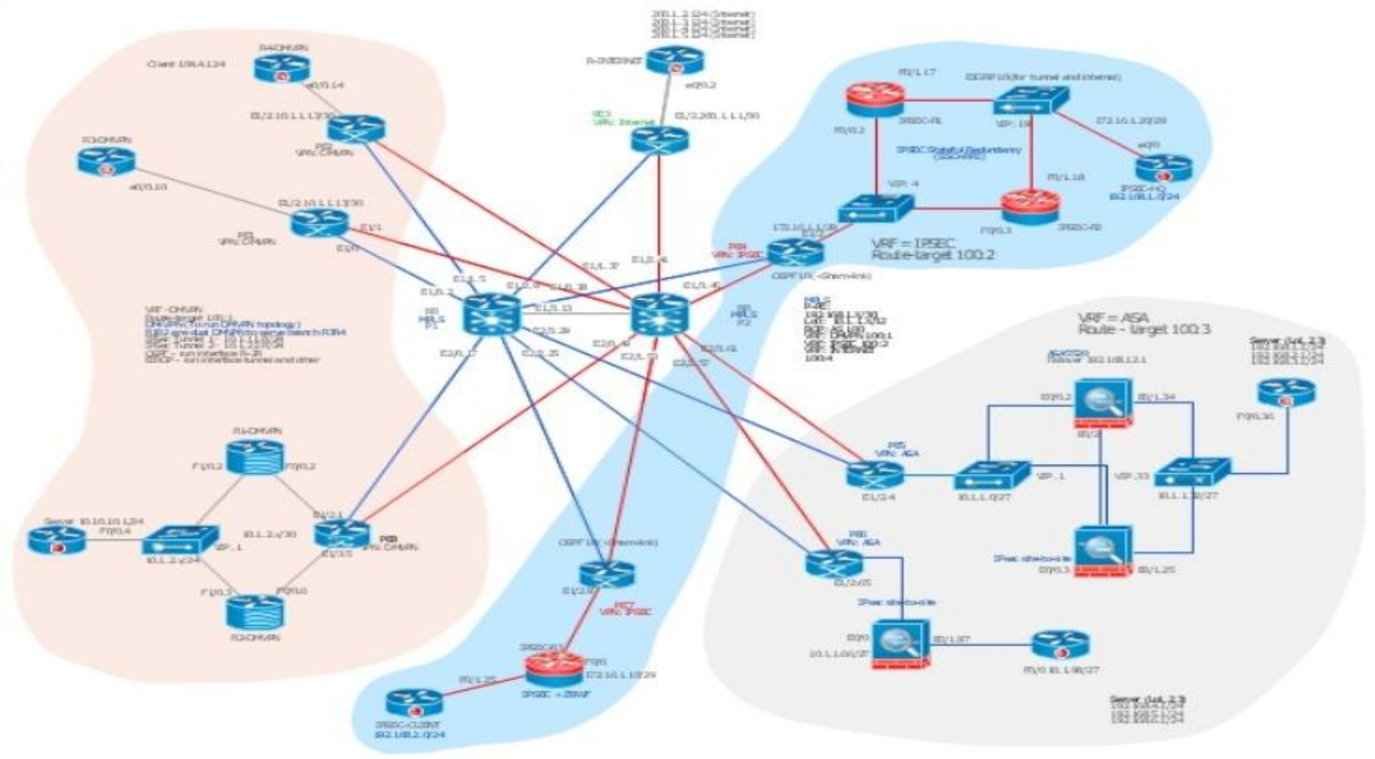
Rail Map

https://www.reddit.com/r/WTF/comments/1ng9py/a_simple_map_of_the_tokyo_metro/

Road Map

http://ontheworldmap.com/usa/state/north-carolina/north-carolina-highway-map.jpg

Network Devices

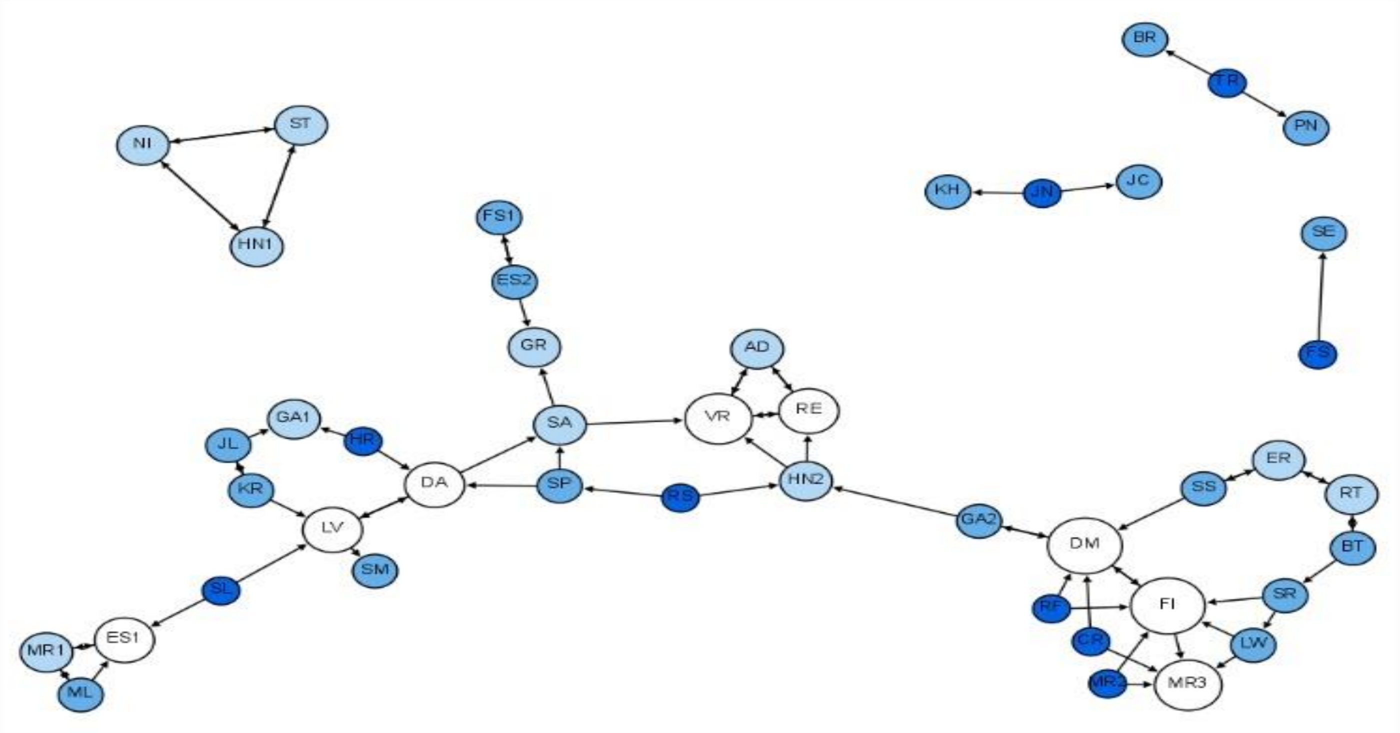


https://neo4j.com/business-edge/managing-network-operations-with-graphs/

Social Network

https://www.flickr.com/photos/clintjcl/4126188232/

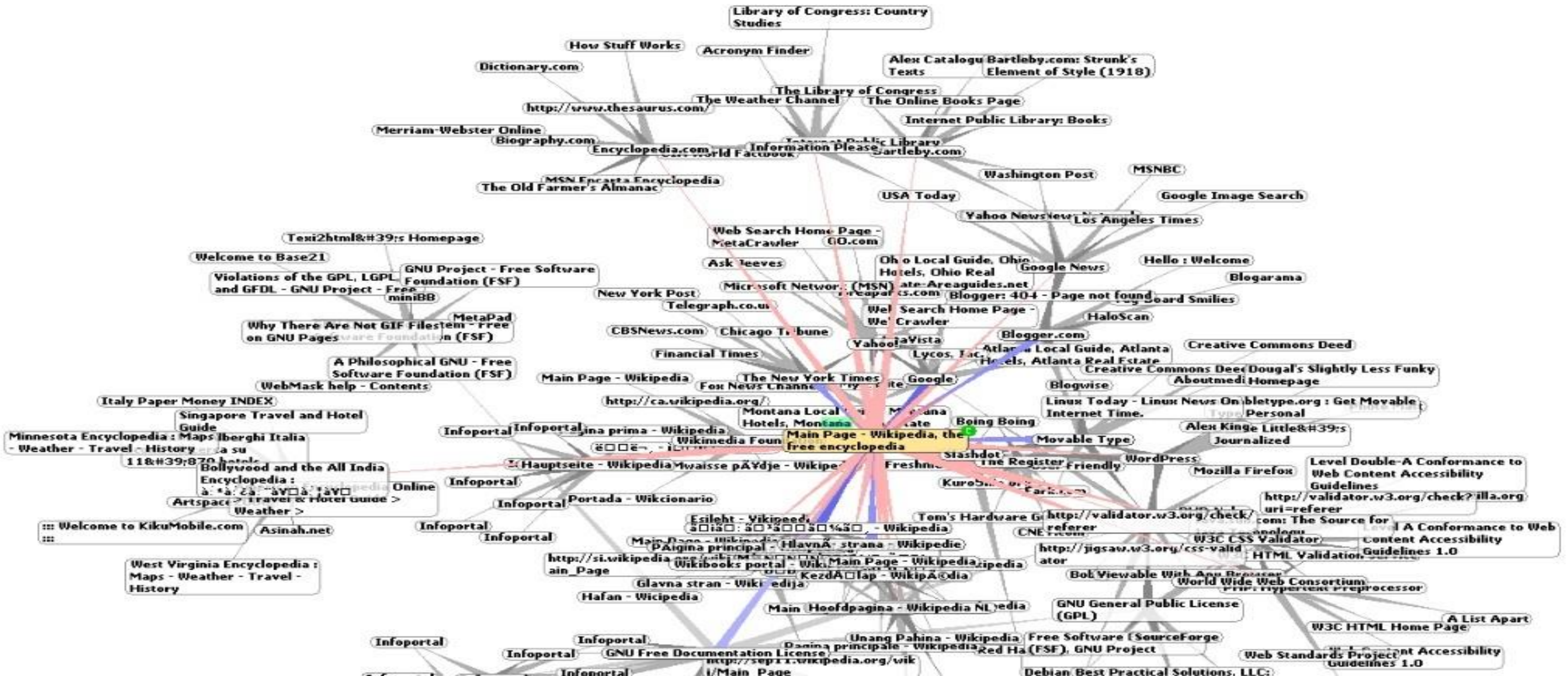
Moreno's Sociogram



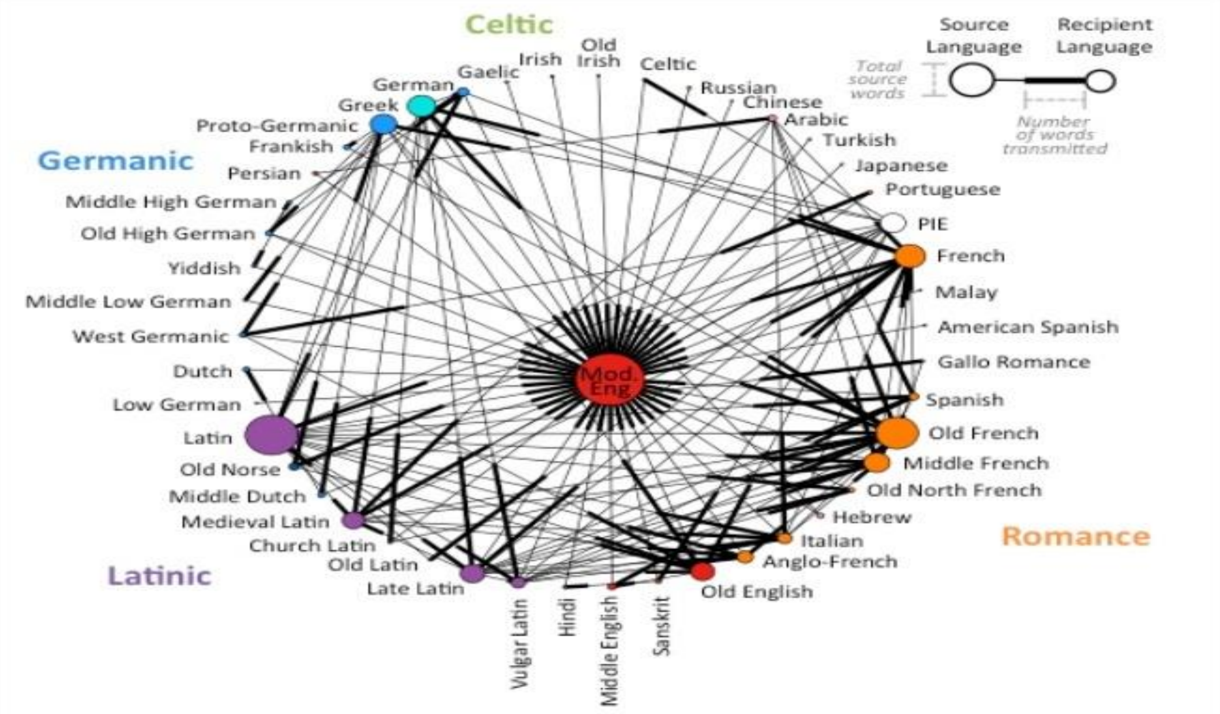
http://www.martingrandjean.ch/social-network-analysis-visualization-morenos-sociograms-revisited/

World Wide Web

https://en.wikipedia.org/wiki/World_Wide_Web



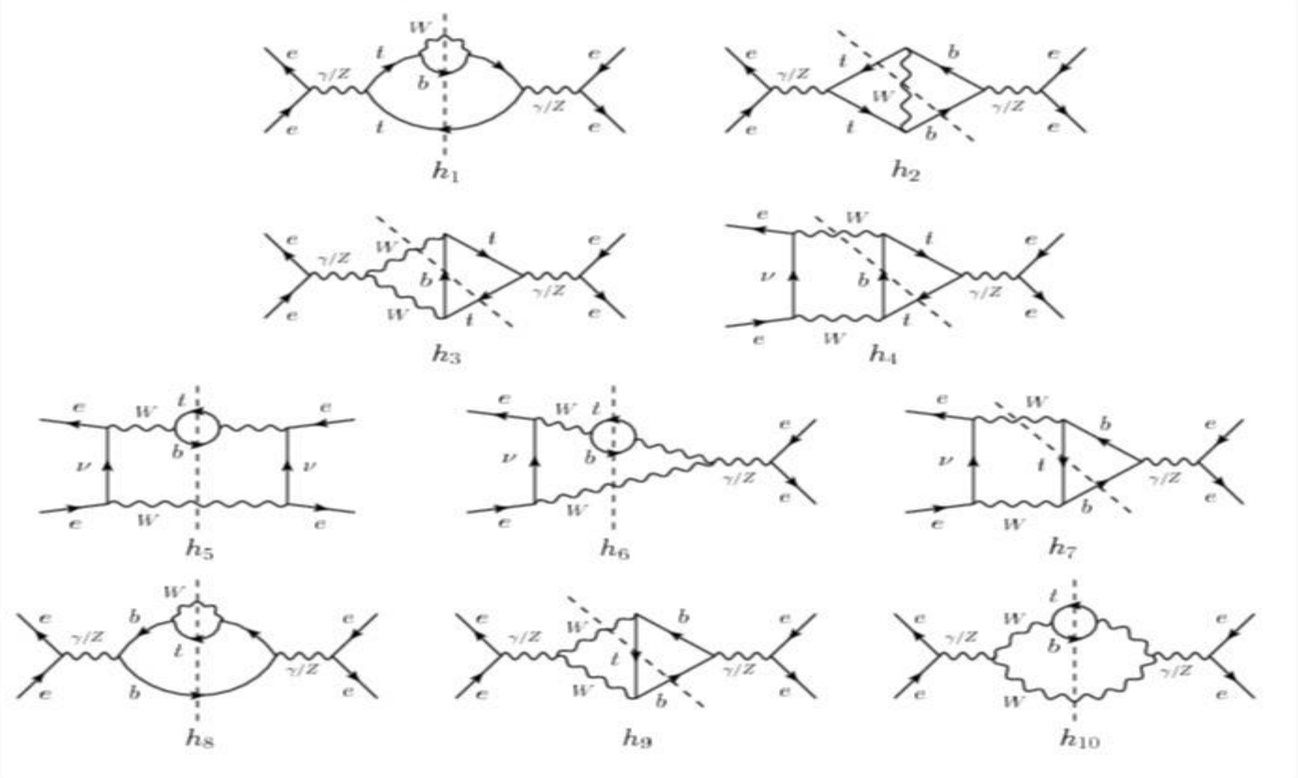
Linguistics



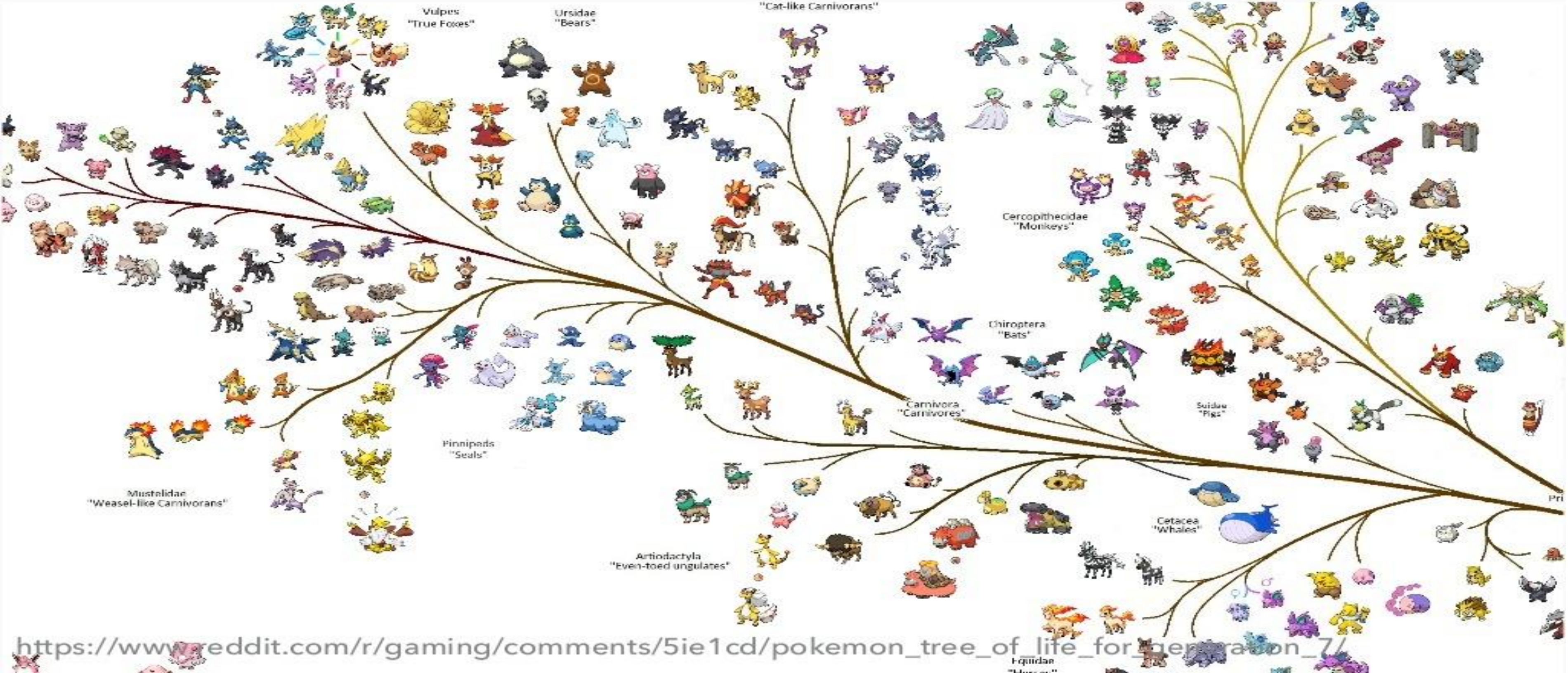
http://www.replicatedtypo.com/causality-in-linguistics-nodes-and-edges-in-causal-graphs/10518.html

Physics – Feynman Diagrams

https://www.quora.com/What-is-the-most-complex-Feynman-Diagram

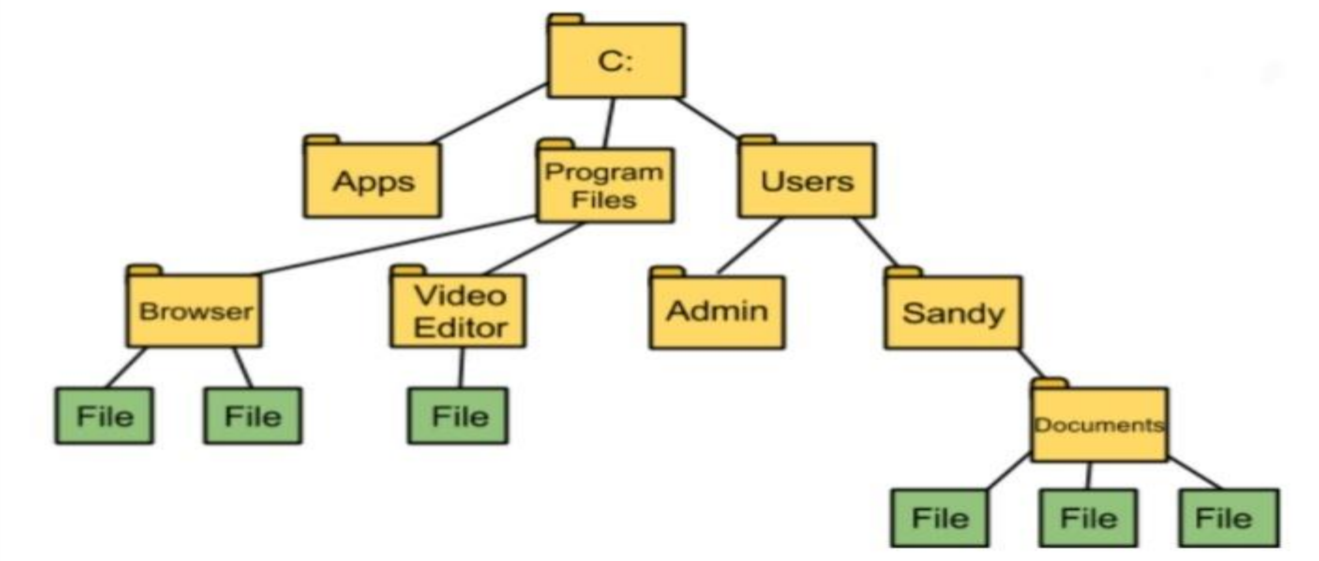


Evolution Trees

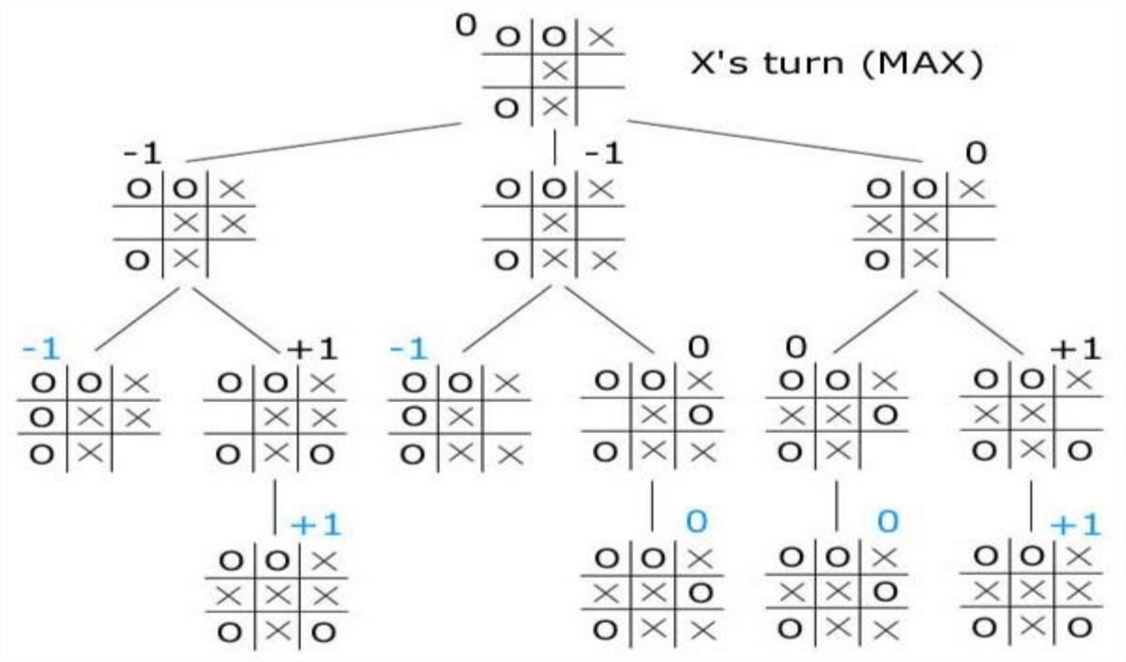


https://www.reddit.com/r/gaming/comments/5ie1cd/pokemon_tree_of_life_for_generation_7/

File Systems

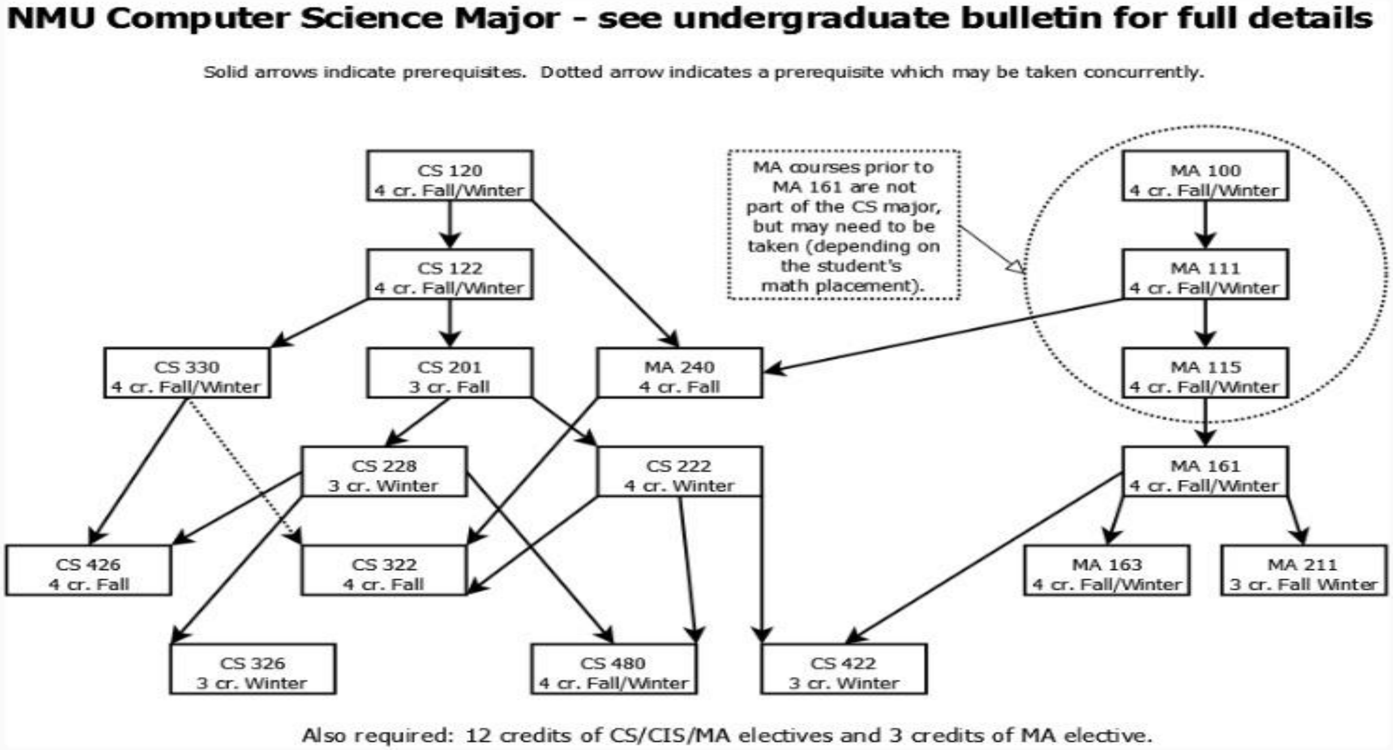


Game Trees



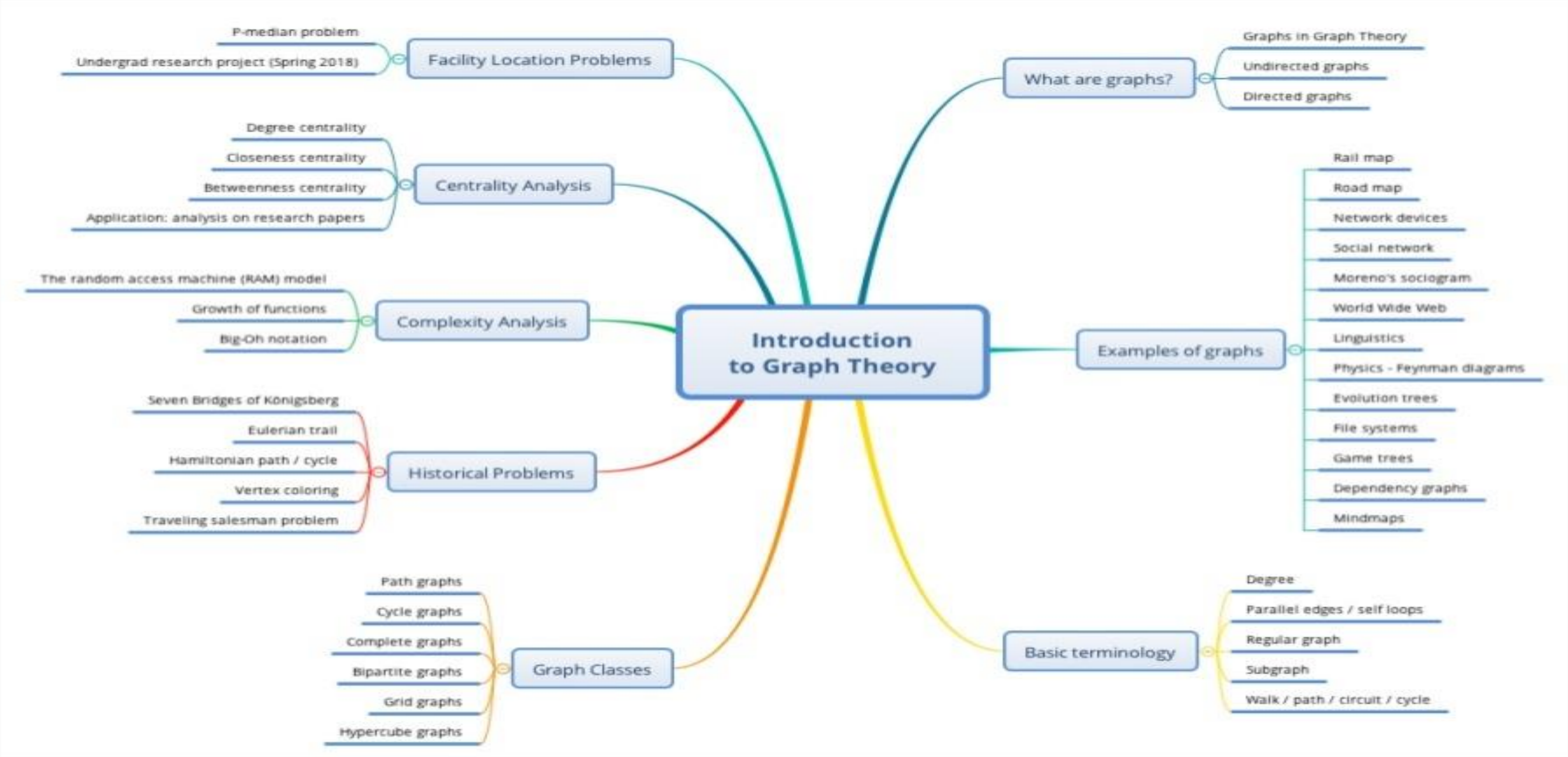
https://www.ocf.berkeley.edu/~yosenl/extras/alphabeta/alphabeta.html

Dependency Graphs



http://euclid.nmu.edu/~mkowalcz/

Mindmaps


Introduction to Graph Theory
Facility Location Problems
P-median problem
Undergrad research project (Spring 2018)
Centrality Analysis
Degree centrality
Closeness centrality
Betweenness centrality
Application: analysis on research papers
Complexity Analysis
The random access machine (RAM) model
Growth of functions
Big-Oh notation
Historical Problems
Seven Bridges of Königsberg
Eulerian trail
Hamiltonian path / cycle
Vertex coloring
Traveling salesman problem
Graph Classes
Path graphs
Cycle graphs
Complete graphs
Bipartite graphs
Grid graphs
Hypercube graphs
What are graphs?
Graphs in Graph Theory
Undirected graphs
Directed graphs
Examples of graphs
Rail map
Road map
Network devices
Social network
Moreno's sociogram
World Wide Web
Linguistics
Physics - Feynman diagrams
Evolution trees
File systems
Game trees
Dependency graphs
Mindmaps
Basic terminology
Degree
Parallel edges / self loops
Regular graph
Subgraph
Walk / path / circuit / cycle

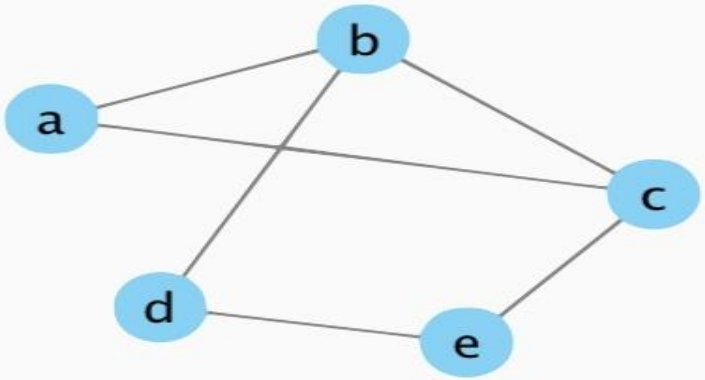
BASIC TERMINOLOGY

Degree

- Total number of edges connected to a vertex
- **Total degree** of a graph:
  - Sum of the degrees of all of the vertices.

Degree

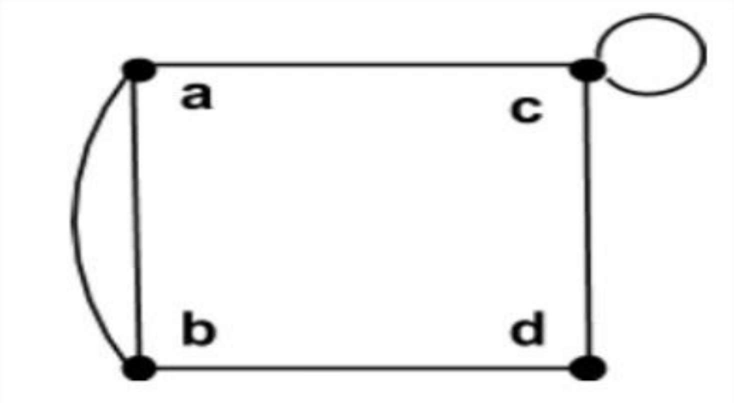
- What is the total degree of this graph?



|V| = 5

|E| = 6

Parallel Edges / Self Loop



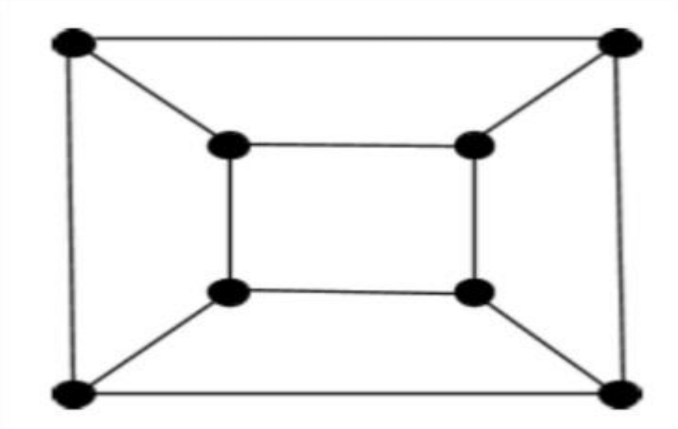
- Parallel Edges:
  - Multiple edges between the same pair of vertices
- Self Loop:
  - Edge between a vertex and itself

Regular Graph

- All the vertices have the same degree.
- ***D*-regular graph**:
  - All the vertices have degree d.
- [Question] Draw a 3-regular undirected graph with 8 vertices.

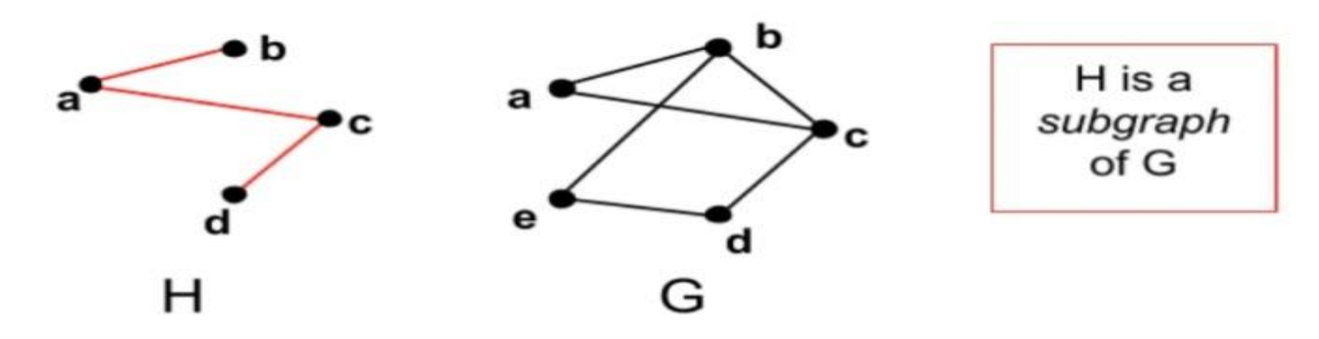
Regular Graph

- 3-regular graph

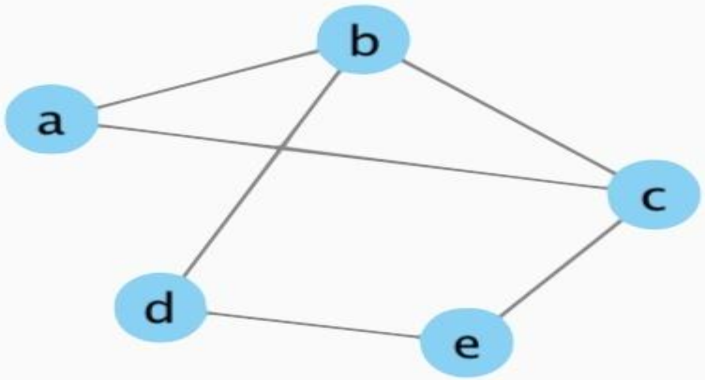


Subgraph

- G' = (V', E') is a subgraph of a graph G = (V, E) if
  - V' ⊆ V and E' ⊆ E

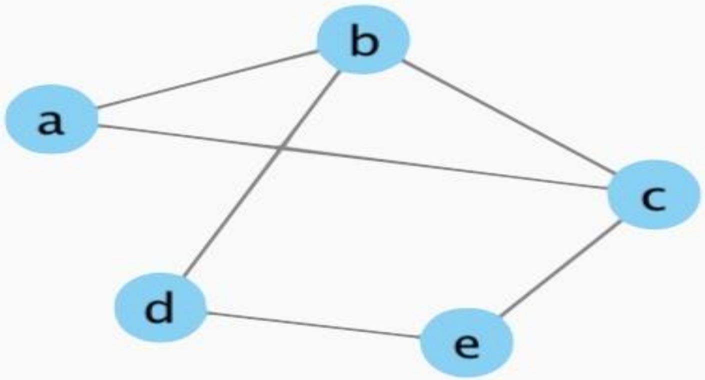


Walk



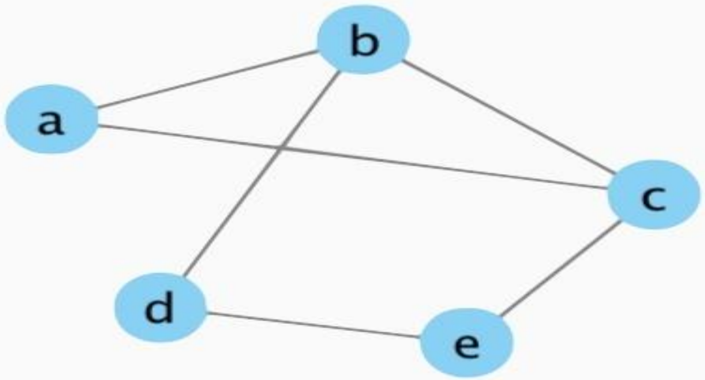
- From v_0 to v_l
- Ex. <a, b, d, b, c>
- Can repeat a vertex or edge

Path



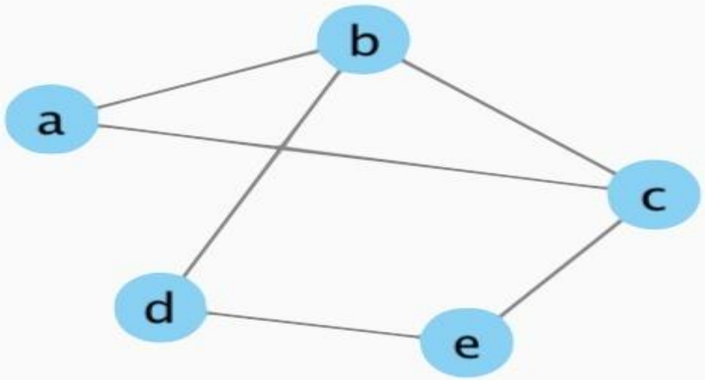
- From v_0 to v_l
- Ex. <a, b, c>
- No vertex is repeated

Circuit



- From v_0 to v_0
- Ex. <a, b, d, e, c, b, a>
- Can repeat a vertex or edge

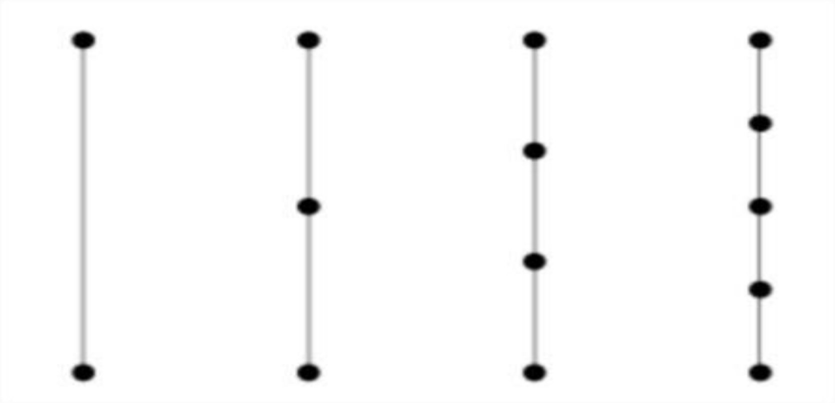
Cycle



- From v_0 to v_0
- Ex. <a, b, c, a>
- Length: at least three
- No vertex is repeated except the first and last

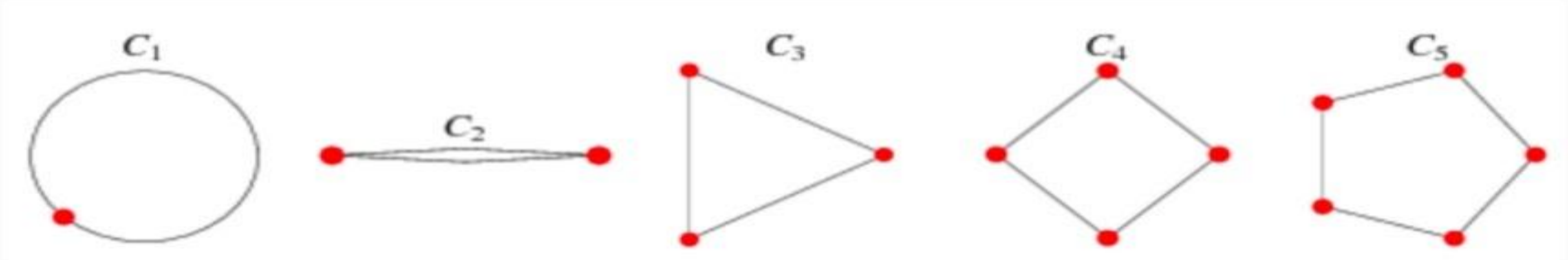
GRAPH CLASSES

Path Graphs (P_n)



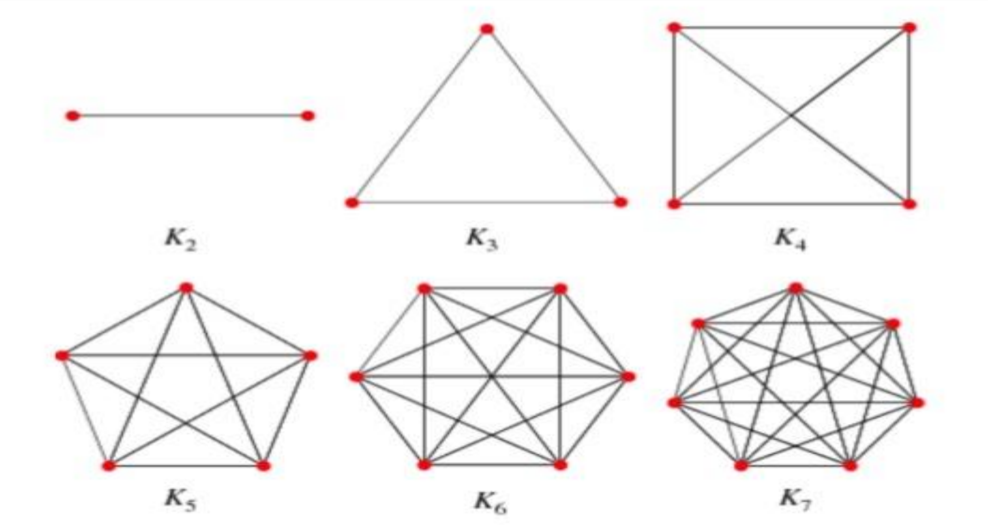
- Two vertices of degree 1
- n-2 vertices of degree 2

Cycle Graphs (C_n)



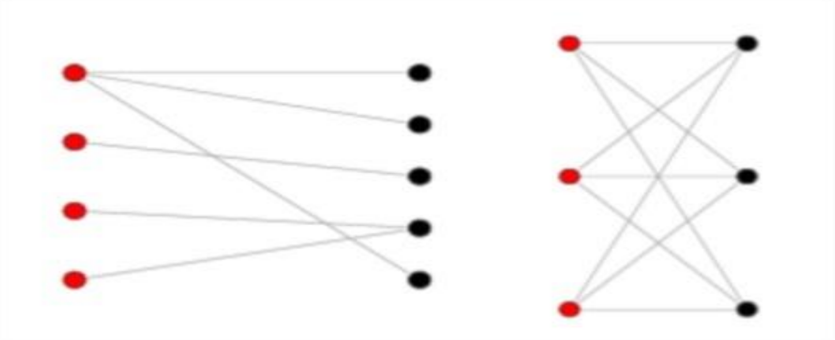
- Single cycle through all vertices.

Complete Graphs (K_n)



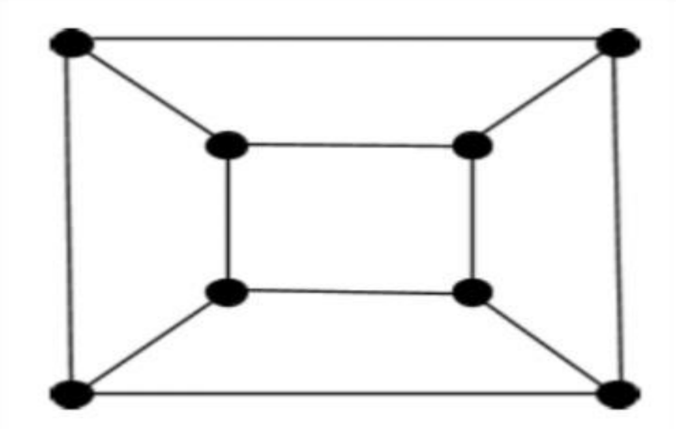
- Each vertex pair is connected by an edge.
- [Question] How many edges does K_n have?

Bipartite Graphs



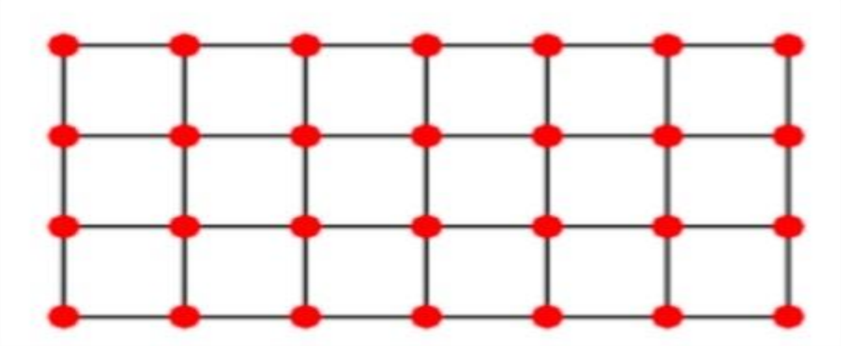
- Vertices: two disjoint sets.
- No two vertices within the same set are connected.

Bipartite Graphs



- Is this a bipartite graph?

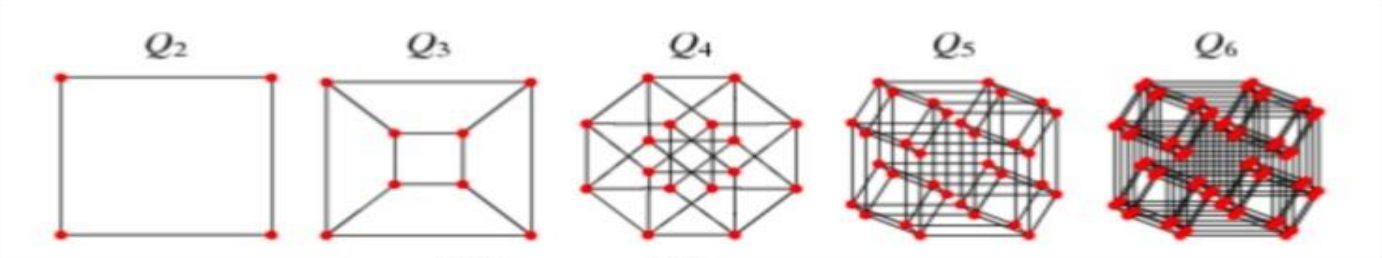
Grid Graphs ($G_{m,n}$)

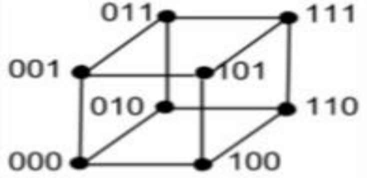


$G_{7,4}$

- $P_m \times P_n$

Hypercube Graphs (Q_n)



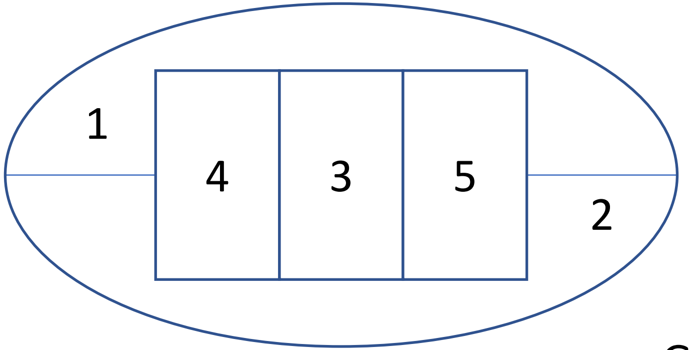


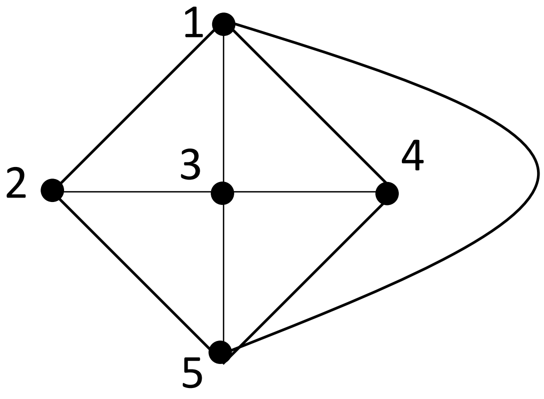
- Has 2^n vertices.
- Each vertex is labeled with an n-bit string.
- Two vertices are connected by an edge if their corresponding labels differ by only one bit.

HISTORICAL PROBLEMS

O problema dos cinco principes

Houve um rei na Índia que tinha um grande reino e cinco filhos. Em seu último testamento, o rei disse que após sua morte os filhos deveriam dividir o reino entre si de tal forma que a região pertencente a cada filho deveria ter uma fronteira (não apenas um ponto) em comum com as quatro regiões restantes. Como o reino deve ser dividido?

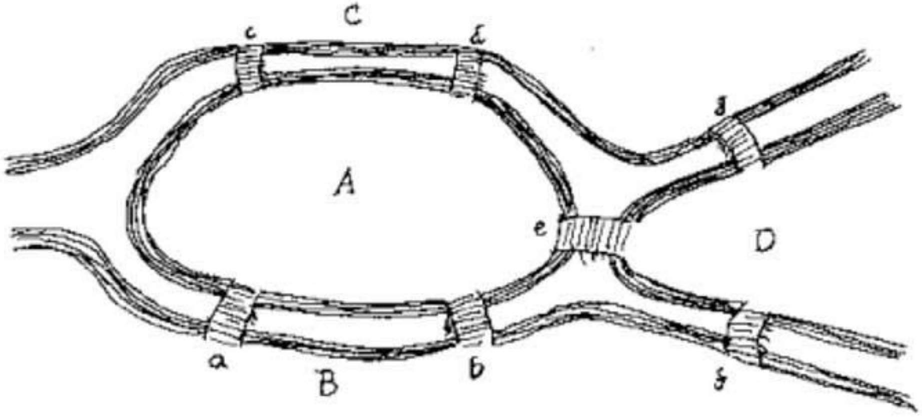


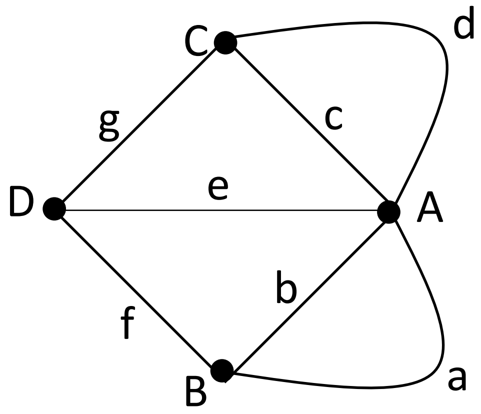


Grafo K_5 completo

Pontes de Königsberg

No início do século 18, os cidadãos de Königsberg passavam os dias caminhando no intrincado arranjo de pontes sobre as águas do rio Pregel (Pregolya), que circundava duas massas de terra centrais conectadas por uma ponte (3). Além disso, a primeira massa de terra (uma ilha) foi conectada por duas pontes (5 e 6) à margem inferior do Pregel e também por duas pontes (1 e 2) à margem superior, enquanto a outra massa de terra (que dividiu o Pregel em dois ramos) estava ligado à margem inferior por uma ponte (7) e à margem superior por uma ponte (4), num total de sete pontes. Segundo o folclore, surgiu a questão de saber se um cidadão poderia passear pela cidade de forma que cada ponte fosse atravessada uma única vez.

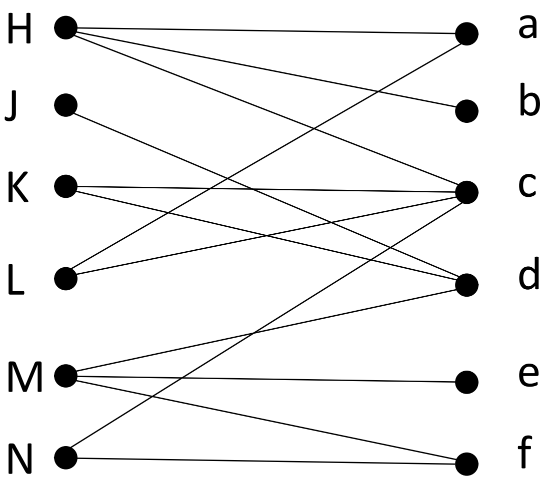




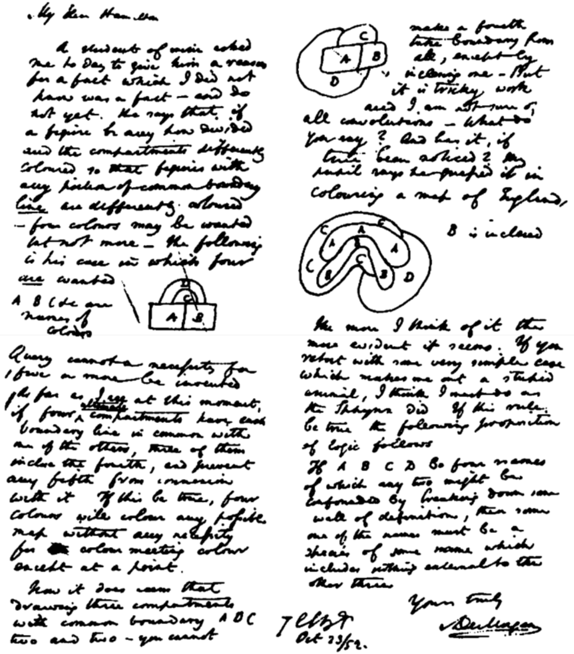
Problema da busca por emprego

Seis alunos em seu ultimo ano buscam por estágio em seis áreas diferentes, como arranjá-los de forma a garantir maior empregabilidade à turma?

Aluno	Interesse
Harry	Arquitetura, banco, construção
Jack	Design, eletrônica, financeiro
Ken	Arquitetura, banco, construção, design
Linda	Arquitetura, banco, construção
Maureen	Design, eletrônica, finanças
Nancy	Arquitetura, banco, construção



Problema das quatro cores



My dear Hamilton

A student of mine asked me to day to give him a reason for a fact which I did not know was a fact — and do not yet. He says that if a figure be any how divided and the compartments differently coloured so that figures with any portion of common boundary line are differently coloured — four colours may be wanted but not more — the following is his case in which four are wanted

A B C D are names of colours

Query cannot a necessity for five or more be invented. As far as I see at this moment, if four ultimate compartments have each boundary line in common with one of the others, three of them inclose the fourth, and prevent any fifth from connexion with it. If this be true, four colours will colour any possible map without any necessity for colour meeting colour except at a point.

Now it does seem that drawing three compartments with common boundary A B C two and two — you cannot make a fourth take boundary from all, except by inclosing one — But it is tricky work and I am not sure of all convolutions — What do you say? And has it, if true been noticed? My pupil says he guessed it in colouring a map of England.

B is inclosed

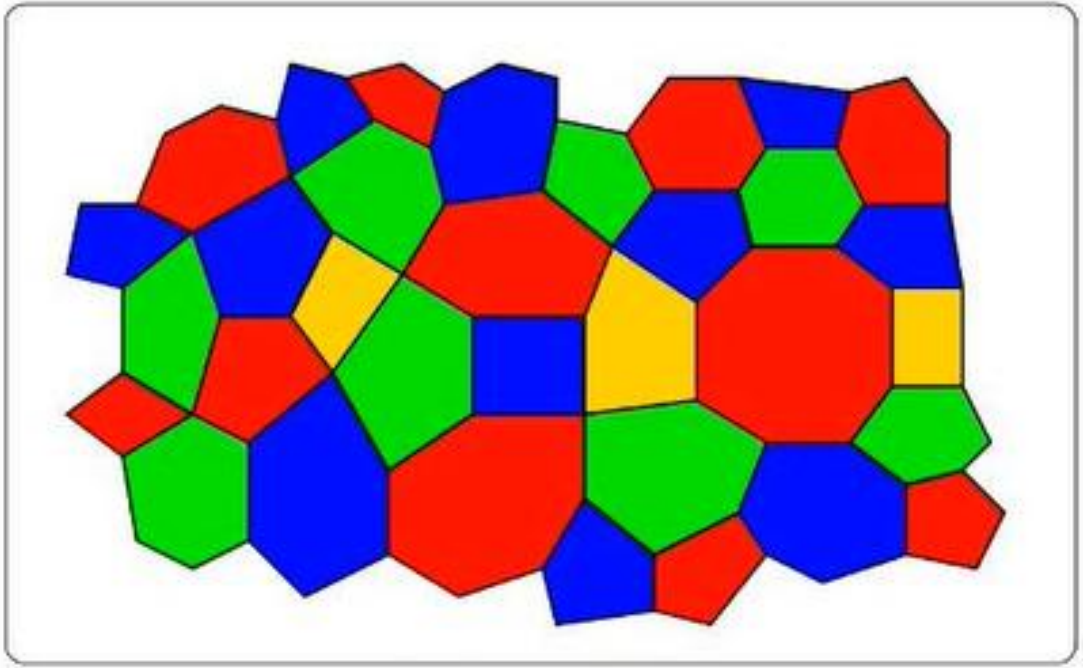
The more I think of it the more evident it seems. If you retort with some very simple case which makes me out a stupid animal, I think I must do as the Sphynx did. If this rule be true the following proposition of logic follows

If A B C D be four names of which any two might be confounded by breaking down some wall of definition, then some one of the names must be a species of some name which includes nothing external to the other three.

Yours truly
A De Morgan

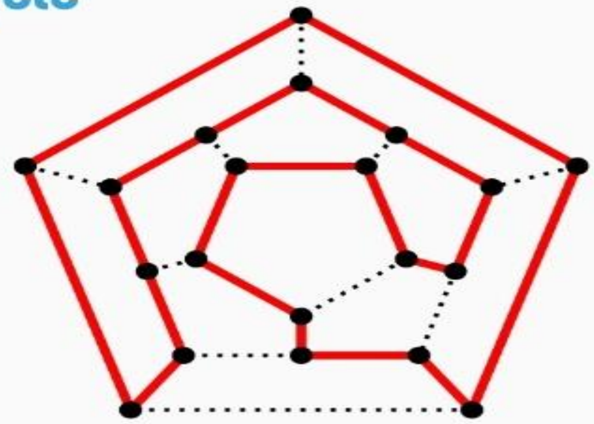
7 Camden St
Oct 23/52

Letter of De Morgan to Hamilton, 23 Oct. 1852



4–Colored Map

Hamiltonian Path / Cycle



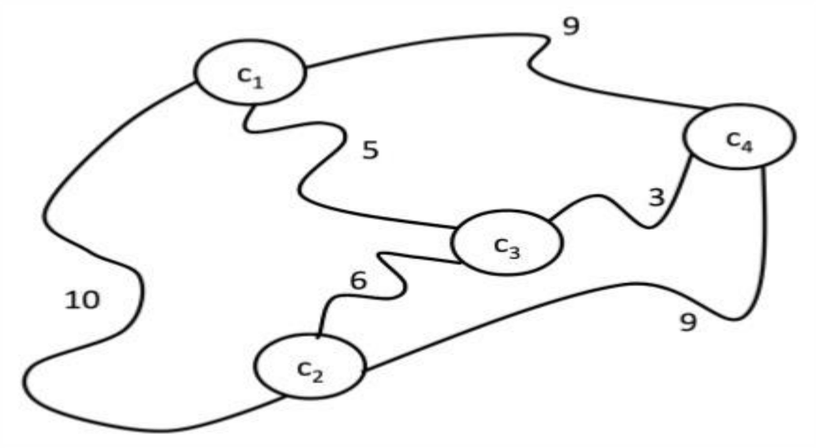
- Path / cycle which visits every vertex exactly once.
- No effective algorithm to construct Hamiltonian paths.
- [Question] Draw a graph that does not have any Hamiltonian paths.

Traveling Salesman Problem (TSP)

- Given a list of cities and the distances between each pair of cities, what is the shortest possible route that visits each city and returns to the origin city?

http://mathgifs.blogspot.com/2014/03/the-traveling-salesman.html

Traveling Salesman Problem (TSP)

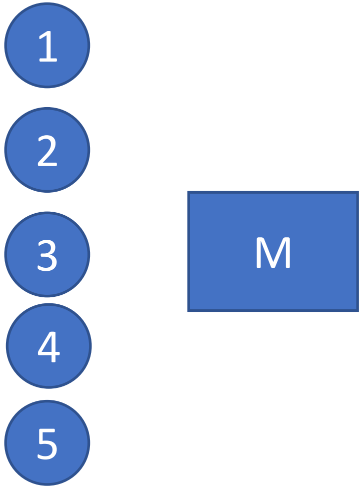


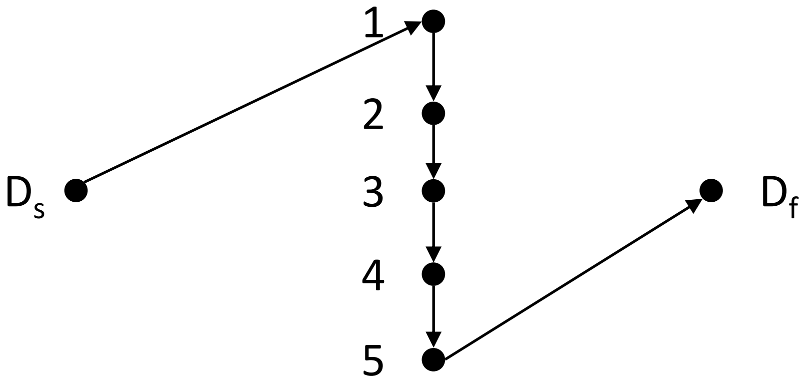
- What is the shortest TSP route starting from c1?
- How much time would you take to compute the route if the number of cities is 10 / 100 / 1000?

Problema de sequenciamento

- Máquina simples

Job	Tempo de processamento
1	5
2	4
3	3
4	1
5	6

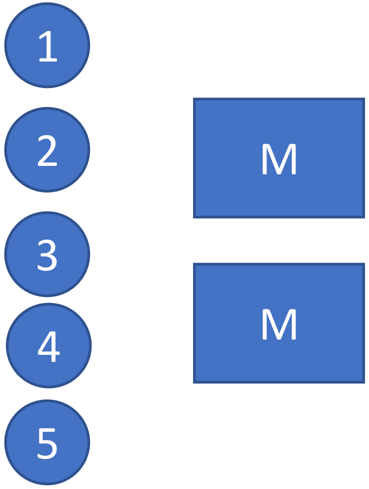


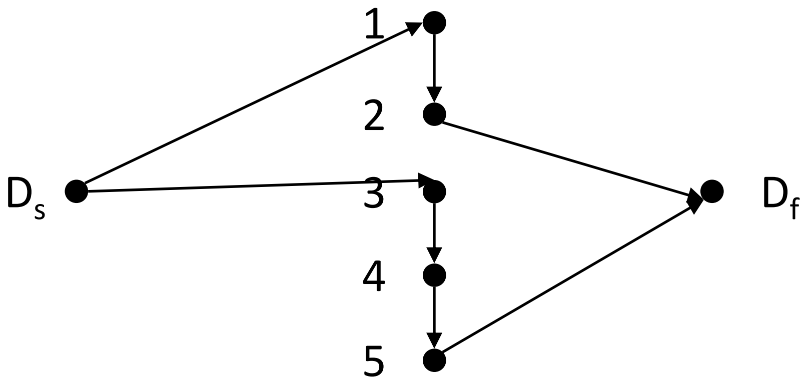


Problema de sequenciamento

- Máquina paralelas

Job	Tempo de processamento
1	5
2	4
3	3
4	1
5	6

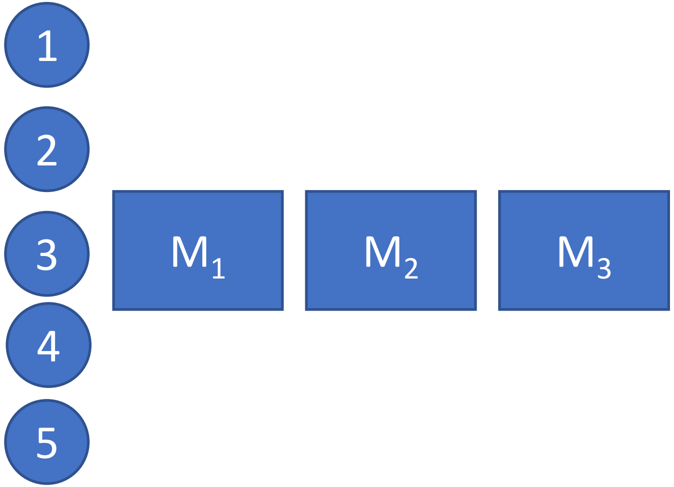


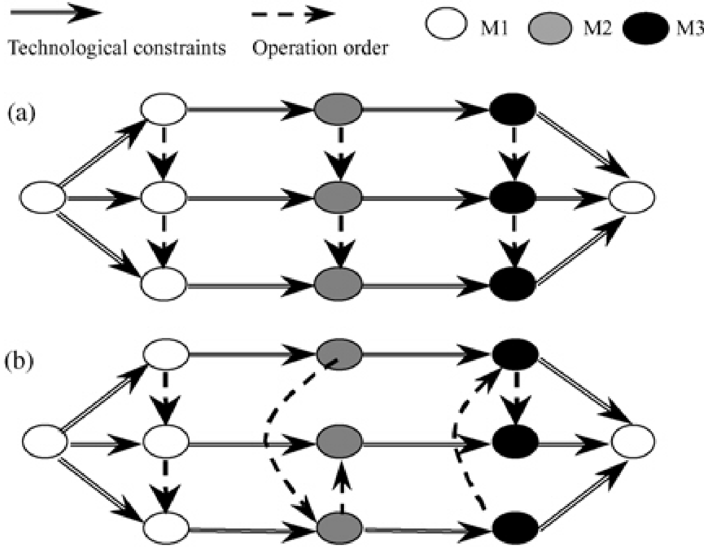


Problema de sequenciamento

- FlowShop

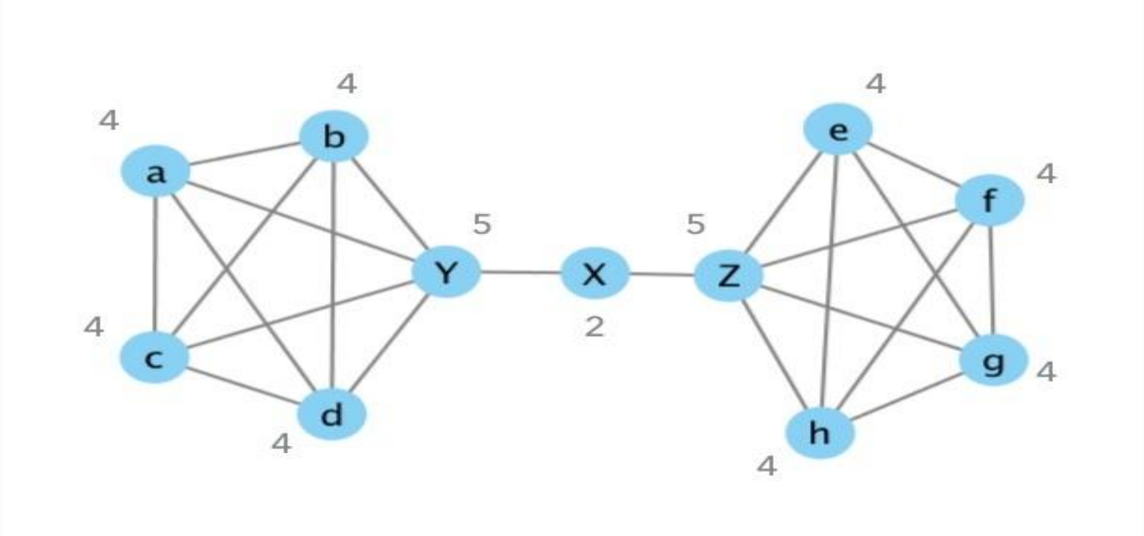
Job	P_1	P_2	P_3
1	5	2	1
2	4	6	2
3	3	1	3
4	1	3	5
5	6	2	1





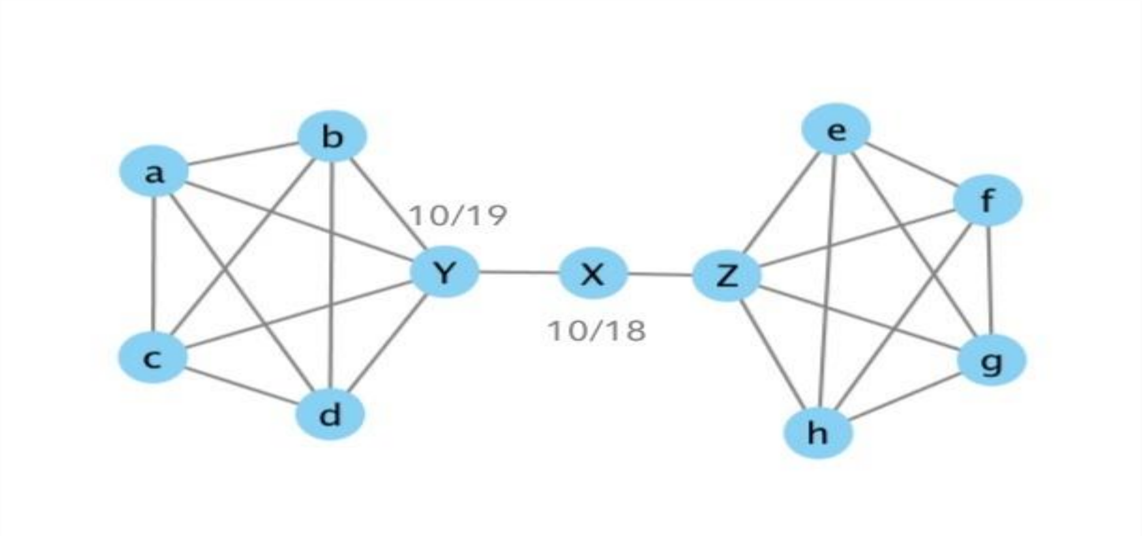
CENTRALITY ANALYSIS

Degree Centrality



- Degree of the vertex (very simple!)

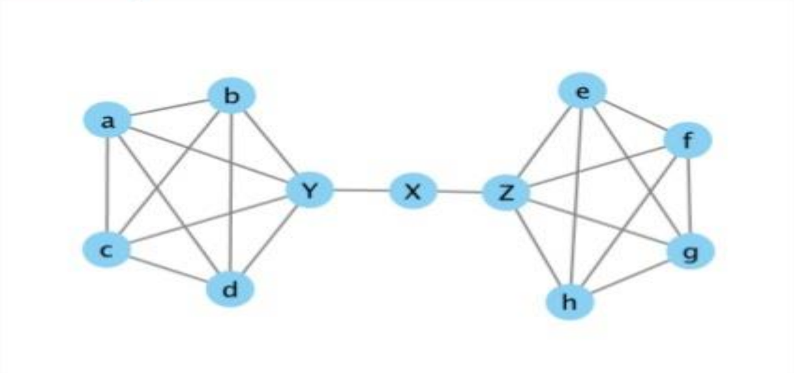
Closeness Centrality



- Reciprocal of the average of the shortest distances from the vertex to all other vertices.

$$C(x) = \frac{N-1}{\sum_y d(y,x)}$$

Betweenness Centrality



- Calculate all pairs shortest paths.
- Count how many times the vertex is in between the shortest paths.
- Useful for finding "mediators"

$$C_B(v) = \sum_{s \neq v \neq t \in V} \frac{\sigma_{st}(v)}{\sigma_{st}}$$

Application: Analysis on Research Papers (2015)

- Joined a research project of the Institute of Statistical Mathematics in Japan.
- They do institutional research (IR) using governmental budgets.
- They wanted to develop new indices to measure researchers' productivity other than existing ones (i.e. the impact factor)

Application: Analysis on Research Papers (2015)

Data Source

- Thomson Reuters
  "Web of Science Core Collection"
- We surveyed all articles published in 2014.
  - 1,786,791 articles
  - 5,462,190 authors
  - 273,013 organizations
- We identified authors
  by their full name + main affiliation.

Application: Analysis on Research Papers (2015)

Co-authors Network

- Vertices: 5,223,375
- Edges: 87,094,183
- Clusters: 368,917
  - The largest cluster contains 3,287,768 (63%) vertices and 81,904,506 (94%) edges.
- Degree
  - median: 6
  - average: 33.35

Application: Analysis on Research Papers (2015)

The Mediators Found by Gateway BC

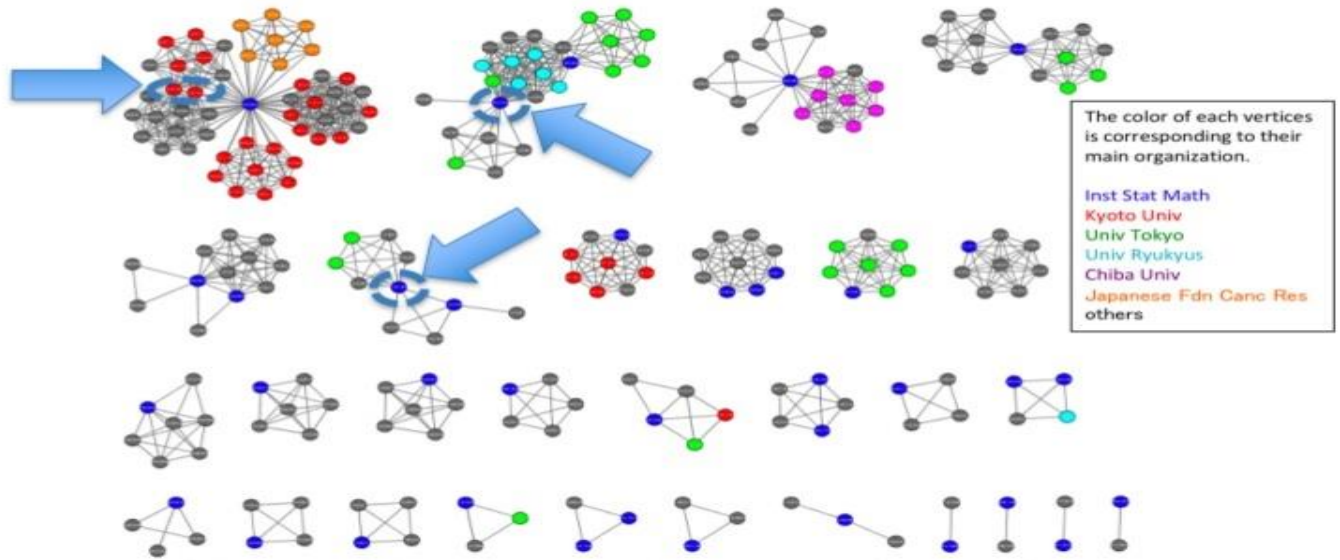
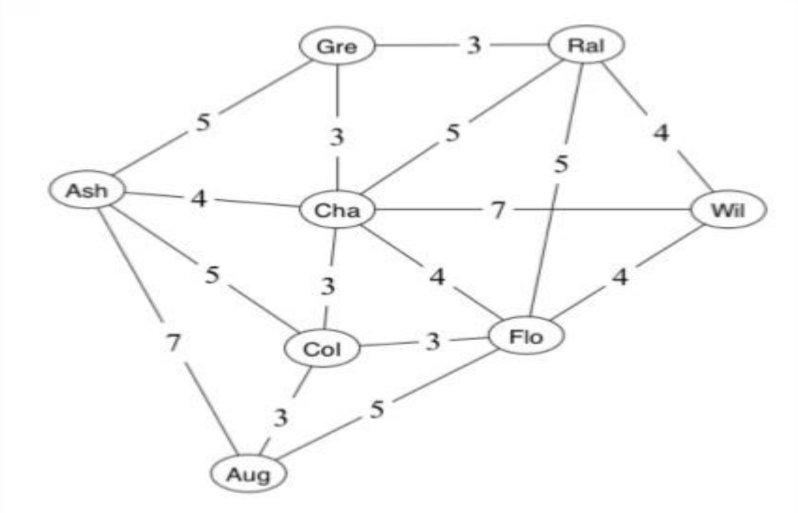


Fig. 3: Network diagram of the authors around the Institute of Statistical Mathematics (Made by Cytoscape version 3.2.1)

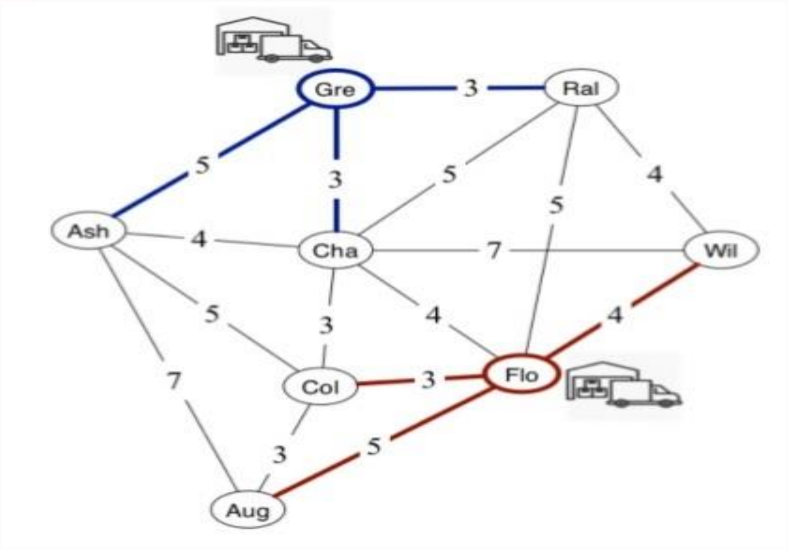
FACILITY LOCATION PROBLEMS

P-median Problem



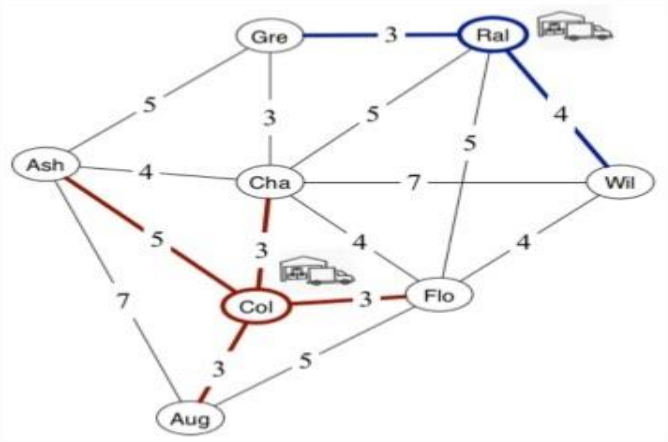
- Numbers represent transportation costs.
- We want to minimize total transportation costs with a fixed number of warehouses (*p*).

P–median Problem



- A possible solution. Total cost = (5+3+3) + (3+4+5) = 23
- Can you find a better solution?

P–median Problem

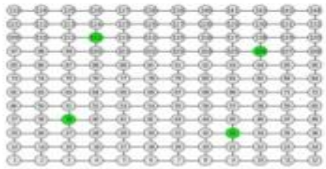


- Complexity
  - $O(n^p)$ in general, making this NP-hard
  - $O(p^2n^2)$ for trees
  - $O(pn)$ for path graphs

Undergrad Research Project (Spring 2018)

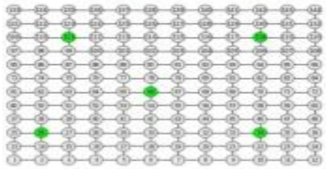
- Developed research framework
  - Graph generation
  - Solving (for smaller input)
  - Visualization
- Found interesting properties on trees / grids
- Applied linear programming to help approximation

4-median Fractional Solutions



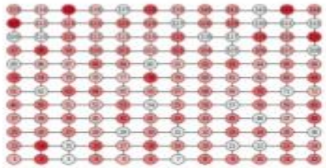
Node	Value
33	1.00
39	1.00
106	1.00
112	1.00

5-median Fractional Solutions



Node	Value
26	1.00
34	1.00
66	1.00
111	1.00
118	1.00

34-median Fractional Solutions



COMPLEXITY ANALYSIS

The Random Access Machine (RAM) Model

- A computer consists of:
  - A CPU
  - A bank with an unlimited number of memory cells
- Assumptions
  - Accessing any cell in memory takes constant time
  - Each "simple" operation takes 1 time step

The Random Access Machine (RAM) Model

- Example: one "simple" operation takes 1 ns (10^{-9} seconds)
- How much time would it take if the number of "simple" operations are:
  - 10^9
  - 10^{10}
  - 10^{11}

Growth of Functions

- The larger input size, the longer the computation would take.
- Let $T(n)$ be the computation time of an algorithm for some input n.
- Changing hardware/software environment affects $T(n)$ by a constant factor. (At least for today's computers...)

Growth of Functions

When T(1) = 1 ns

	n=10	n=20	n=100	n=1000	$n=10^6$	$n=10^9$	$n=10^{12}$
$T(\log_2 n)$	3 ns	4 ns	7 ns	10 ns	20 ns	30 ns	40 ns
$T(\sqrt{n})$	3 ns	4 ns	10 ns	32 ns	1 μs	32 μs	1 ms
T(n)	10 ns	20 ns	100 ns	1 μs	1 ms	1 s	17 min
$T(n \log_2 n)$	33 ns	86 ns	660 ns	10 μs	20 ms	30 s	11 h
$T(n^2)$	100 ns	400 ns	10 μs	1 ms	17 min	32 yrs	3×10^7 yrs
$T(n^3)$	1 μs	8 μs	1 ms	1 s	32 yrs	3×10^{10} yrs	3×10^{19} yrs
$T(2^n)$	1 μs	1 ms	4×10^{13} yrs	3×10^{284} yrs	-	-	-
T(n!)	4 ms	77 yrs	9×10^{157} yrs	-	-	-	-

Our universe: 1.3×10^{10} yrs old

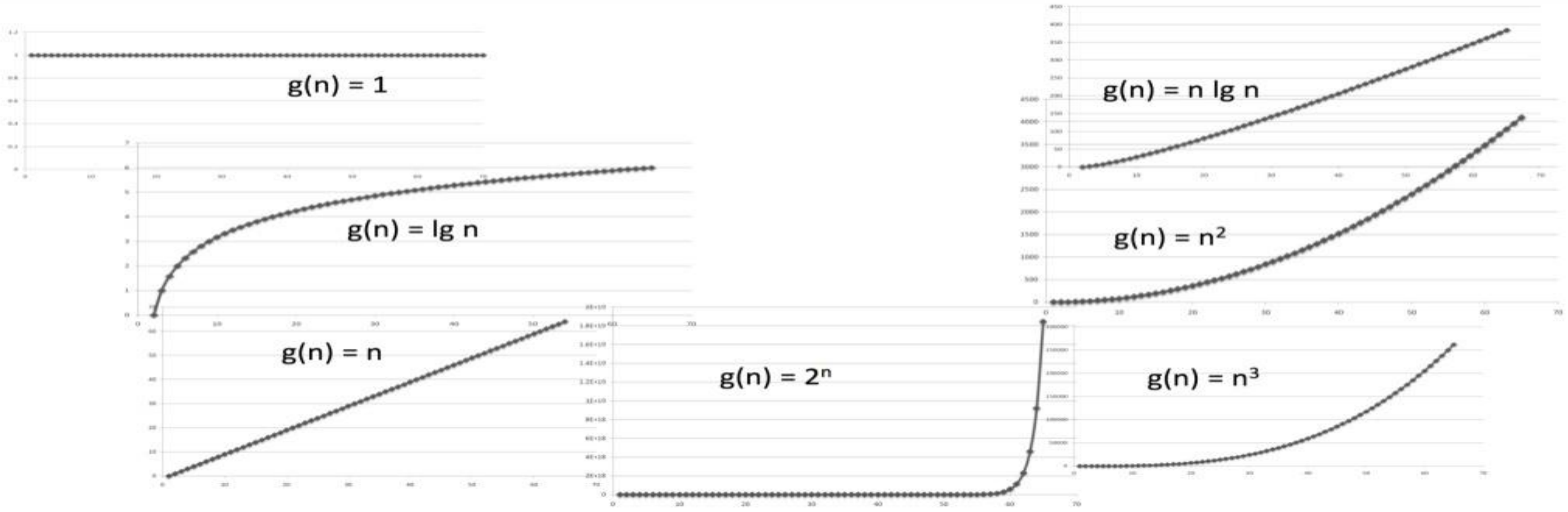
Growth of Functions

When we have a supercomputer and assume $T(1) = 1 \times 10^{-14}$ s

	n=10	n=20	n=100	n=1000	$n=10^6$	$n=10^9$	$n=10^{12}$
$T(\log_2 n)$	<1 ns	<1 ns	<1 ns	<1 ns	<1 ns	<1 ns	<1 ns
$T(\sqrt{n})$	<1 ns	<1 ns	<1 ns	<1 ns	<1 ns	<1 ns	<1 ns
$T(n)$	<1 ns	<1 ns	<1 ns	<1 ns	10 ns	10 μs	10 ms
$T(n \log_2 n)$	<1 ns	<1 ns	<1 ns	<1 ns	200 ns	300 μs	400 ms
$T(n^2)$	<1 ns	<1 ns	<1 ns	10 ns	10 ms	3 h	300 yrs
$T(n^3)$	<1 ns	<1 ns	10 ns	10 μs	3 h	3×10^5 yrs	3×10^{14} yrs
$T(2^n)$	<1 ns	10 ns	4×10^8 yrs	3×10^{279} yrs	-	-	-
$T(n!)$	40 ns	7 h	9×10^{152} yrs	-	-	-	-

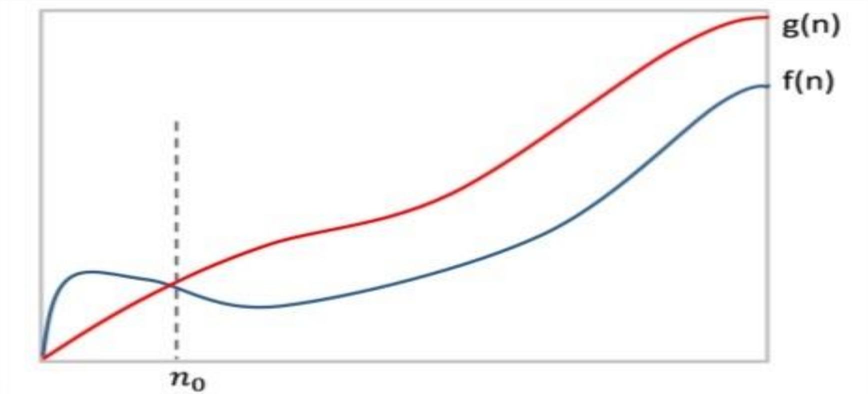
Our universe: 1.3×10^{10} yrs old

Growth of Functions



Big-Oh Notation

- Defines asymptotic upper bound
- Given functions f(n) and g(n), we say that f(n) is O(g(n)) if there are positive constants c and n_0 such that $0 \leq f(n) \leq c\, g(n)$ for all $n \geq n_0$



Dr. Jason King, CSC 316 Course Material

Big-Oh Notation

- Example:
  - $f(n) = 3n^2 + 2n + 10000$ is $O(n^2)$
  - $f(n) = n^{100} + 1.01^n$ is $O(1.01^n)$
- Finding Eulerian trails: $O(|E|)$
- Solving the traveling salesman problem: $O(n^2 \cdot 2^n)$
  - Categorized as
    NP (Non-deterministic Polynomial-time) **-hard**

Thank you!

- References
  - Wikipedia

    - https://en.wikipedia.org/wiki/Graph_theory
    - https://en.wikipedia.org/wiki/Eulerian_path
    - https://en.wikipedia.org/wiki/Hamiltonian_path
    - https://en.wikipedia.org/wiki/Graph_coloring

  - Wolfram MathWorld

    - http://mathworld.wolfram.com/PathGraph.html
    - http://mathworld.wolfram.com/CycleGraph.html
    - http://mathworld.wolfram.com/CompleteGraph.html
    - http://mathworld.wolfram.com/BipartiteGraph.html
    - http://mathworld.wolfram.com/GridGraph.html
    - http://mathworld.wolfram.com/HypercubeGraph.html

  - zyBooks: Discrete Mathematics

    - https://www.zybooks.com/catalog/discrete-mathematics/
