PME 5411 - Fundamentos de Escoamentos Turbulentos Reativos

Aula 2 - Reynolds Averaged Navier Stokes RANS

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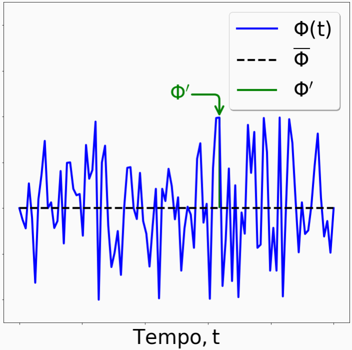
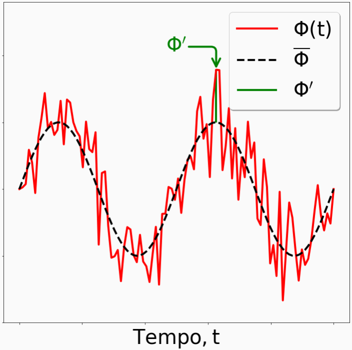
Descrição de escoamentos turbulentos





Decomposição de Reynolds

$$\Phi\left(t\right) = \overline{\Phi} + \Phi'\left(t\right)$$

Média no tempo

$$\overline{\Phi} = \frac{1}{\Delta t}\int_{t_0}^{t_0+\Delta t} \Phi\left(t\right) dt$$

Ergodicidade

Média conjunto = Média no tempo (Média estatística)

Descrição de escoamentos turbulentos

Variância

$$\overline{(\Phi')^2} = \frac{1}{\Delta t}\int_{t_0}^{t_0+\Delta t} (\Phi')^2 dt$$

Desvio Padrão/RMS

$$\Phi_{RMS} = \sqrt{\overline{(\Phi')^2}}$$

Aplicando a média na decomposição de *Reynolds*

$$\frac{1}{\Delta t}\int_{t_0}^{t_0+\Delta t} \Phi\,(t)\ dt = \frac{1}{\Delta t}\int_{t_0}^{t_0+\Delta t} \overline{\Phi}\, dt + \frac{1}{\Delta t}\int_{t_0}^{t_0+\Delta t} \Phi'\,(t)\ dt \longrightarrow$$
$$\overline{\Phi} = \overline{\Phi} + \overline{\Phi'} \longrightarrow$$
$$\overline{\Phi'} = 0 \qquad (1)$$

Portanto, a média da flutuação é nula.

$$\Phi = \overline{\Phi} + \Phi' \; ; \; \Psi = \overline{\Psi} + \Psi' \; ; \; \overline{\Phi'} = \overline{\Psi'} = 0 \; ; \; \boxed{\overline{\Phi'\Psi'} = \frac{1}{\Delta t}\int_{t_0}^{t_0+\Delta t} \Phi'\Psi' dt}$$

$$\text{Para velocidade: } \begin{cases} \overline{u'v'} \\ \overline{u'w'} \\ \overline{v'w'} \end{cases} \text{Se } = 0 \; \Rightarrow \text{ grandezas não correlacionadas}$$

Autocorrelação temporal

$$R_{\Phi'\Phi'(\tau)} = \overline{\Phi'(t)\Phi'(t+\tau)} = \frac{1}{\Delta t}\int_{t_0}^{t_0+\Delta t} \Phi'(t)\Phi'(t+\tau) dt$$

Autocorrelação espacial

$$R_{ij}\left(\overrightarrow{r}, t\right) = \overline{u_i'\left(\overrightarrow{x} + \overrightarrow{r}, t\right) u_j'\left(\overrightarrow{x}, t\right)} = \frac{1}{\Delta t}\int_{t_0}^{t_0+\Delta t} u_i'\left(\overrightarrow{x} + \overrightarrow{r}, t\right) u_j'\left(\overrightarrow{x}, t\right) dt$$

- Média da derivada:

$$\begin{aligned}\overline{\frac{\partial \Phi}{\partial s}} &\equiv \frac{1}{\Delta t}\int_{t_0}^{t_0+\Delta t}\frac{\partial \Phi}{\partial s}dt \longrightarrow \\ \overline{\frac{\partial \Phi}{\partial s}} &= \frac{1}{\Delta t}\frac{\partial}{\partial s}\left(\int_{t_0}^{t_0+\Delta t}\Phi\, dt\right) \longrightarrow \\ \overline{\frac{\partial \Phi}{\partial s}} &= \frac{\partial \overline{\Phi}}{\partial s}\end{aligned} \tag{2}$$

- Média da integral:

$$\begin{aligned}\overline{\int \Phi\, ds} &\equiv \frac{1}{\Delta t}\int_{t_0}^{t_0+\Delta t}\int \Phi\, ds\, dt \longrightarrow \\ \overline{\int \Phi\, ds} &= \frac{1}{\Delta t}\int\left(\int_{t_0}^{t_0+\Delta t}\Phi\, dt\right) ds \longrightarrow \\ \overline{\int \Phi\, ds} &= \int \overline{\Phi}\, ds\end{aligned} \tag{3}$$

- Média da soma:

$$\overline{\Phi + \Psi} \equiv \frac{1}{\Delta t} \int_{t_0}^{t_0+\Delta t} (\Phi + \Psi)\, dt \longrightarrow$$

$$\overline{\Phi + \Psi} = \frac{1}{\Delta t} \int_{t_0}^{t_0+\Delta t} \Phi \, dt + \frac{1}{\Delta t} \int_{t_0}^{t_0+\Delta t} \Psi \, dt \longrightarrow$$

$$\overline{\Phi + \Psi} = \overline{\Phi} + \overline{\Psi} \tag{4}$$

- Média do produto:

$$\overline{\Phi\ \Psi} = \overline{\left(\overline{\Phi} + \Phi'\right)\left(\overline{\Psi} + \Psi'\right)} \longrightarrow$$

$$\overline{\Phi\ \Psi} = \overline{\left(\overline{\Phi}\ \overline{\Psi} + \overline{\Phi}\ \Psi' + \Phi'\ \overline{\Psi} + \Phi'\ \Psi'\right)} \longrightarrow$$

$$\overline{\Phi\ \Psi} = \overline{\overline{\Phi}\ \overline{\Psi}} + \overline{\overline{\Phi}\ \Psi'} + \overline{\Phi'\ \overline{\Psi}} + \overline{\Phi'\ \Psi'} \longrightarrow$$

$$\overline{\Phi\ \Psi} = \overline{\Phi}\ \overline{\Psi} + \underbrace{\overline{\Phi}\ \overline{\Psi'}}_{=0} + \underbrace{\overline{\Phi'}\ \overline{\Psi}}_{=0} + \overline{\Phi'\ \Psi'} \longrightarrow$$

$$\overline{\Phi\ \Psi} = \overline{\Phi}\ \overline{\Psi} + \overline{\Phi'\ \Psi'} \tag{5}$$

Para um vetor $\vec{a}$:

$$\vec{a} = \vec{\overline{a}} + \vec{a'} \text{ ou } \mathbf{a} = \overline{\mathbf{a}} + \mathbf{a}' \text{ ou } a_i = \overline{a_i} + a_i'$$

- Média do divergente de um vetor ($\vec{a}$):

$$\overline{\frac{\partial a_i}{\partial x_i}} = \overline{\frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial x_3}} \xrightarrow{(4)}$$

$$\overline{\frac{\partial a_i}{\partial x_i}} = \overline{\frac{\partial a_1}{\partial x_1}} + \overline{\frac{\partial a_2}{\partial x_2}} + \overline{\frac{\partial a_3}{\partial x_3}} \xrightarrow{(2)}$$

$$\overline{\frac{\partial a_i}{\partial x_i}} = \frac{\partial \overline{a}_1}{\partial x_1} + \frac{\partial \overline{a}_2}{\partial x_2} + \frac{\partial \overline{a}_3}{\partial x_3} \longrightarrow$$

$$\overline{\frac{\partial a_i}{\partial x_i}} = \frac{\partial \overline{a}_i}{\partial x_i} \longrightarrow$$

$$\overline{\frac{\partial a_i}{\partial x_i}} = \frac{\partial \overline{a}_i}{\partial x_i} \text{ ou } \overline{\vec{\nabla} \cdot \vec{a}} = \vec{\nabla} \cdot \vec{\overline{a}} \tag{6}$$

- Média do divergente do produto de um escalar por um vetor $(\Phi \overrightarrow{a})$:

$$\overline{\frac{\partial\left(\Phi a_i\right)}{\partial x_i}} = \overline{\frac{\partial\left(\Phi a_1\right)}{\partial x_1} + \frac{\partial\left(\Phi a_2\right)}{\partial x_2} + \frac{\partial\left(\Phi a_3\right)}{\partial x_3}} \xrightarrow{(4)}$$

$$\overline{\frac{\partial\left(\Phi a_i\right)}{\partial x_i}} = \overline{\frac{\partial\left(\Phi a_1\right)}{\partial x_1}} + \overline{\frac{\partial\left(\Phi a_2\right)}{\partial x_2}} + \overline{\frac{\partial\left(\Phi a_3\right)}{\partial x_3}} \xrightarrow{(2)}$$

$$\overline{\frac{\partial\left(\Phi a_i\right)}{\partial x_i}} = \frac{\partial\left(\overline{\Phi a_1}\right)}{\partial x_1} + \frac{\partial\left(\overline{\Phi a_2}\right)}{\partial x_2} + \frac{\partial\left(\overline{\Phi a_3}\right)}{\partial x_3} \xrightarrow{(5)}$$

$$\overline{\frac{\partial\left(\Phi a_i\right)}{\partial x_i}} = \frac{\partial\left(\overline{\Phi}\,\overline{a}_1 + \overline{\Phi' a_1'}\right)}{\partial x_1} + \frac{\partial\left(\overline{\Phi}\,\overline{a}_2 + \overline{\Phi' a_2'}\right)}{\partial x_2} + \frac{\partial\left(\overline{\Phi}\,\overline{a}_3 + \overline{\Phi' a_3'}\right)}{\partial x_3} \longrightarrow$$

$$\overline{\frac{\partial\left(\Phi a_i\right)}{\partial x_i}} = \frac{\partial\left(\overline{\Phi}\,\overline{a}_1\right)}{\partial x_1} + \frac{\partial\left(\overline{\Phi' a_1'}\right)}{\partial x_1}$$

$$+ \frac{\partial\left(\overline{\Phi}\,\overline{a}_2\right)}{\partial x_2} + \frac{\partial\left(\overline{\Phi' a_2'}\right)}{\partial x_2}$$

$$+ \frac{\partial\left(\overline{\Phi}\,\overline{a}_3\right)}{\partial x_3} + \frac{\partial\left(\overline{\Phi' a_3'}\right)}{\partial x_3} \longrightarrow$$

$$\frac{\overline{\partial\left(\Phi \mathrm{a_i}\right)}}{\partial \mathrm{x_i}} = \frac{\partial\left(\overline{\Phi}\;\overline{\mathrm{a}}_1\right)}{\partial \mathrm{x_1}} + \frac{\partial\left(\overline{\Phi}\;\overline{\mathrm{a}}_2\right)}{\partial \mathrm{x_2}} + \frac{\partial\left(\overline{\Phi}\;\overline{\mathrm{a}}_3\right)}{\partial \mathrm{x_3}}$$

$$+ \frac{\partial\left(\overline{\Phi' \mathrm{a}'_1}\right)}{\partial \mathrm{x_1}} + \frac{\partial\left(\overline{\Phi' \mathrm{a}'_2}\right)}{\partial \mathrm{x_2}} + \frac{\partial\left(\overline{\Phi' \mathrm{a}'_3}\right)}{\partial \mathrm{x_3}} \longrightarrow$$

$$\frac{\overline{\partial\left(\Phi \mathrm{a_i}\right)}}{\partial \mathrm{x_i}} = \frac{\partial\left(\overline{\Phi}\;\overline{\mathrm{a}}_\mathrm{i}\right)}{\partial \mathrm{x_i}} + \frac{\partial\left(\overline{\Phi' \mathrm{a}'_\mathrm{i}}\right)}{\partial \mathrm{x_i}} \longrightarrow$$

$$\frac{\overline{\partial(\Phi \mathrm{a_i})}}{\partial \mathrm{x_i}} = \frac{\partial}{\partial \mathrm{x_i}}\left(\overline{\Phi}\;\overline{\mathrm{a}}_\mathrm{i}\right) + \frac{\partial}{\partial \mathrm{x_i}}\left(\overline{\Phi' \mathrm{a}'_\mathrm{i}}\right) \tag{7}$$

ou $\overline{\vec{\nabla} \cdot (\Phi \vec{\mathrm{a}})} = \vec{\nabla} \cdot \left(\overline{\Phi}\,\overline{\vec{\mathrm{a}}}\right) + \vec{\nabla} \cdot \left(\overline{\Phi' \vec{\mathrm{a}'}}\right)$

- Média do divergente do gradiente de um vetor ($\overrightarrow{a}$) (média do divergente de um tensor):

$$\overline{\frac{\partial}{\partial x_j}\left(\frac{\partial a_i}{\partial x_j}\right)} = \overline{\frac{\partial}{\partial x_j}\underbrace{\left(\frac{\partial a_i}{\partial x_j}\right)}_{\text{tensor}}} \longrightarrow$$

$$\overline{\frac{\partial}{\partial x_j}\left(\frac{\partial a_i}{\partial x_j}\right)} = \overline{\begin{bmatrix} \frac{\partial a_1}{\partial x_1} & \frac{\partial a_1}{\partial x_2} & \frac{\partial a_1}{\partial x_3} \\ \frac{\partial a_2}{\partial x_1} & \frac{\partial a_2}{\partial x_2} & \frac{\partial a_2}{\partial x_3} \\ \frac{\partial a_3}{\partial x_1} & \frac{\partial a_3}{\partial x_2} & \frac{\partial a_3}{\partial x_3} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{bmatrix}} \longrightarrow$$

Descrição de escoamentos turbulentos

$$\overline{\frac{\partial}{\partial \mathrm{x_j}}\left(\frac{\partial \mathrm{a_i}}{\partial \mathrm{x_j}}\right)} = \overline{\begin{bmatrix} \frac{\partial}{\partial \mathrm{x_1}}\left(\frac{\partial \mathrm{a_1}}{\partial \mathrm{x_1}}\right) + \frac{\partial}{\partial \mathrm{x_2}}\left(\frac{\partial \mathrm{a_1}}{\partial \mathrm{x_2}}\right) + \frac{\partial}{\partial \mathrm{x_3}}\left(\frac{\partial \mathrm{a_1}}{\partial \mathrm{x_3}}\right) \\ \frac{\partial}{\partial \mathrm{x_1}}\left(\frac{\partial \mathrm{a_2}}{\partial \mathrm{x_1}}\right) + \frac{\partial}{\partial \mathrm{x_2}}\left(\frac{\partial \mathrm{a_2}}{\partial \mathrm{x_2}}\right) + \frac{\partial}{\partial \mathrm{x_3}}\left(\frac{\partial \mathrm{a_2}}{\partial \mathrm{x_3}}\right) \\ \frac{\partial}{\partial \mathrm{x_1}}\left(\frac{\partial \mathrm{a_3}}{\partial \mathrm{x_1}}\right) + \frac{\partial}{\partial \mathrm{x_2}}\left(\frac{\partial \mathrm{a_3}}{\partial \mathrm{x_2}}\right) + \frac{\partial}{\partial \mathrm{x_3}}\left(\frac{\partial \mathrm{a_3}}{\partial \mathrm{x_3}}\right) \end{bmatrix}} \longrightarrow$$

$$\overline{\frac{\partial}{\partial \mathrm{x_j}}\left(\frac{\partial \mathrm{a_i}}{\partial \mathrm{x_j}}\right)} = \begin{bmatrix} \overline{\frac{\partial}{\partial \mathrm{x_1}}\left(\frac{\partial \mathrm{a_1}}{\partial \mathrm{x_1}}\right) + \frac{\partial}{\partial \mathrm{x_2}}\left(\frac{\partial \mathrm{a_1}}{\partial \mathrm{x_2}}\right) + \frac{\partial}{\partial \mathrm{x_3}}\left(\frac{\partial \mathrm{a_1}}{\partial \mathrm{x_3}}\right)} \\ \overline{\frac{\partial}{\partial \mathrm{x_1}}\left(\frac{\partial \mathrm{a_2}}{\partial \mathrm{x_1}}\right) + \frac{\partial}{\partial \mathrm{x_2}}\left(\frac{\partial \mathrm{a_2}}{\partial \mathrm{x_2}}\right) + \frac{\partial}{\partial \mathrm{x_3}}\left(\frac{\partial \mathrm{a_2}}{\partial \mathrm{x_3}}\right)} \\ \overline{\frac{\partial}{\partial \mathrm{x_1}}\left(\frac{\partial \mathrm{a_3}}{\partial \mathrm{x_1}}\right) + \frac{\partial}{\partial \mathrm{x_2}}\left(\frac{\partial \mathrm{a_3}}{\partial \mathrm{x_2}}\right) + \frac{\partial}{\partial \mathrm{x_3}}\left(\frac{\partial \mathrm{a_3}}{\partial \mathrm{x_3}}\right)} \end{bmatrix} \longrightarrow$$

Descrição de escoamentos turbulentos

Então,

$$\overline{\frac{\partial}{\partial \mathrm{x_j}}\left(\frac{\partial \mathrm{a_i}}{\partial \mathrm{x_j}}\right)} = \begin{bmatrix} \frac{\partial}{\partial \mathrm{x_1}}\left(\frac{\partial \overline{\mathrm{a}}_1}{\partial \mathrm{x_1}}\right) + \frac{\partial}{\partial \mathrm{x_2}}\left(\frac{\partial \overline{\mathrm{a}}_1}{\partial \mathrm{x_2}}\right) + \frac{\partial}{\partial \mathrm{x_3}}\left(\frac{\partial \overline{\mathrm{a}}_1}{\partial \mathrm{x_3}}\right) \\ \frac{\partial}{\partial \mathrm{x_1}}\left(\frac{\partial \overline{\mathrm{a}}_2}{\partial \mathrm{x_1}}\right) + \frac{\partial}{\partial \mathrm{x_2}}\left(\frac{\partial \overline{\mathrm{a}}_2}{\partial \mathrm{x_2}}\right) + \frac{\partial}{\partial \mathrm{x_3}}\left(\frac{\partial \overline{\mathrm{a}}_2}{\partial \mathrm{x_3}}\right) \\ \frac{\partial}{\partial \mathrm{x_1}}\left(\frac{\partial \overline{\mathrm{a}}_3}{\partial \mathrm{x_1}}\right) + \frac{\partial}{\partial \mathrm{x_2}}\left(\frac{\partial \overline{\mathrm{a}}_3}{\partial \mathrm{x_2}}\right) + \frac{\partial}{\partial \mathrm{x_3}}\left(\frac{\partial \overline{\mathrm{a}}_3}{\partial \mathrm{x_3}}\right) \end{bmatrix} \longrightarrow$$

$$\overline{\frac{\partial}{\partial \mathrm{x_j}}\left(\frac{\partial \mathrm{a_i}}{\partial \mathrm{x_j}}\right)} = \begin{bmatrix} \frac{\partial \overline{\mathrm{a}}_1}{\partial \mathrm{x_1}} & \frac{\partial \overline{\mathrm{a}}_1}{\partial \mathrm{x_2}} & \frac{\partial \overline{\mathrm{a}}_1}{\partial \mathrm{x_3}} \\ \frac{\partial \overline{\mathrm{a}}_2}{\partial \mathrm{x_1}} & \frac{\partial \overline{\mathrm{a}}_2}{\partial \mathrm{x_2}} & \frac{\partial \overline{\mathrm{a}}_2}{\partial \mathrm{x_3}} \\ \frac{\partial \overline{\mathrm{a}}_3}{\partial \mathrm{x_1}} & \frac{\partial \overline{\mathrm{a}}_3}{\partial \mathrm{x_2}} & \frac{\partial \overline{\mathrm{a}}_3}{\partial \mathrm{x_3}} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial \mathrm{x_1}} \\ \frac{\partial}{\partial \mathrm{x_2}} \\ \frac{\partial}{\partial \mathrm{x_3}} \end{bmatrix} \longrightarrow$$

Logo,

$$\begin{gathered}\overline{\frac{\partial}{\partial x_j}\left(\frac{\partial a_i}{\partial x_j}\right)} = \frac{\partial}{\partial x_j}\left(\frac{\partial \overline{a}_i}{\partial x_j}\right) \longrightarrow \\ \overline{\frac{\partial}{\partial x_j}\left(\frac{\partial a_i}{\partial x_j}\right)} = \frac{\partial}{\partial x_j}\left(\frac{\partial \overline{a}_i}{\partial x_j}\right) = \frac{\partial^2 \overline{a}_i}{\partial x_j \partial x_j} \\ \text{ou } \overline{\vec{\nabla} \cdot \left(\vec{\nabla} \vec{a}\right)} = \vec{\nabla} \cdot \left(\vec{\nabla} \overline{\vec{a}}\right) = \nabla^2 \overline{\vec{a}}\end{gathered} \tag{8}$$

Reynolds Averaged Navier-Stokes

A equação da continuidade para escoamento com massa específica constante é dado por:

$$\frac{\partial \mathrm{u_i}}{\partial \mathrm{x_i}} = 0 \tag{9}$$

Aplicando a média:

$$\overline{\frac{\partial \mathrm{u_i}}{\partial \mathrm{x_i}}} = 0$$

Logo, pela equação (6):

Equação Média da Continuidade

$$\frac{\partial \overline{\mathrm{u}}_\mathrm{i}}{\partial \mathrm{x_i}} = 0 \tag{10}$$

$$\nabla \cdot \overline{\mathbf{u}} = 0$$

Reynolds Averaged Navier-Stokes

As equações da quantidade de movimento (*Navier-Stokes*) para massa específica constante é dada por:

$$\frac{\partial \mathrm{u_i}}{\partial \mathrm{t}} + \frac{\partial}{\partial \mathrm{x_j}}\left(\mathrm{u_i u_j}\right) = -\frac{1}{\rho}\frac{\partial \mathrm{p}}{\partial \mathrm{x_i}} + \nu\frac{\partial^2 \mathrm{u_i}}{\partial \mathrm{x_j}\partial \mathrm{x_j}} \tag{11}$$

Aplicando a média:

$$\overline{\frac{\partial \mathrm{u_i}}{\partial \mathrm{t}} + \frac{\partial}{\partial \mathrm{x_j}}\left(\mathrm{u_i u_j}\right)} = \overline{-\frac{1}{\rho}\frac{\partial \mathrm{p}}{\partial \mathrm{x_i}} + \nu\frac{\partial^2 \mathrm{u_i}}{\partial \mathrm{x_j}\partial \mathrm{x_j}}} \xrightarrow{(4)}$$

$$\overline{\frac{\partial \mathrm{u_i}}{\partial \mathrm{t}}} + \overline{\frac{\partial}{\partial \mathrm{x_j}}\left(\mathrm{u_i u_j}\right)} = -\frac{1}{\rho}\overline{\frac{\partial \mathrm{p}}{\partial \mathrm{x_i}}} + \nu\overline{\frac{\partial^2 \mathrm{u_i}}{\partial \mathrm{x_j}\partial \mathrm{x_j}}} \xrightarrow{(2)\text{ e }(8)}$$

$$\frac{\partial \overline{\mathrm{u}}_\mathrm{i}}{\partial \mathrm{t}} + \frac{\partial}{\partial \mathrm{x_j}}\left(\overline{\mathrm{u_i u_j}}\right) = -\frac{1}{\rho}\frac{\partial \overline{\mathrm{p}}}{\partial \mathrm{x_i}} + \nu\frac{\partial^2 \overline{\mathrm{u}}_\mathrm{i}}{\partial \mathrm{x_j}\partial \mathrm{x_j}} \xrightarrow{(5)}$$

$$\frac{\partial \overline{\mathrm{u}}_\mathrm{i}}{\partial \mathrm{t}} + \frac{\partial}{\partial \mathrm{x_j}}\left(\overline{\mathrm{u}}_\mathrm{i}\overline{\mathrm{u}}_\mathrm{j} + \overline{\mathrm{u}'_\mathrm{i}\mathrm{u}'_\mathrm{j}}\right) = -\frac{1}{\rho}\frac{\partial \overline{\mathrm{p}}}{\partial \mathrm{x_i}} + \nu\frac{\partial^2 \overline{\mathrm{u}}_\mathrm{i}}{\partial \mathrm{x_j}\partial \mathrm{x_j}} \rightarrow$$

$$\frac{\partial \overline{\mathrm{u}}_\mathrm{i}}{\partial \mathrm{t}} + \frac{\partial}{\partial \mathrm{x_j}}\left(\overline{\mathrm{u}}_\mathrm{i}\overline{\mathrm{u}}_\mathrm{j}\right) + \frac{\partial}{\partial \mathrm{x_j}}\left(\overline{\mathrm{u}'_\mathrm{i}\mathrm{u}'_\mathrm{j}}\right) = -\frac{1}{\rho}\frac{\partial \overline{\mathrm{p}}}{\partial \mathrm{x_i}} + \nu\frac{\partial^2 \overline{\mathrm{u}}_\mathrm{i}}{\partial \mathrm{x_j}\partial \mathrm{x_j}} \rightarrow$$

Reynolds Averaged Navier-Stokes

Desse modo, supondo que a média não varie no tempo, as equações da quantidade de movimento ficam:

Equações Médias da Quantidade de Movimento

$$\frac{\partial}{\partial \mathrm{x_j}}\left(\overline{\mathrm{u}}_\mathrm{i}\overline{\mathrm{u}}_\mathrm{j}\right)+\frac{\partial}{\partial \mathrm{x_j}}\left(\overline{\mathrm{u}'_\mathrm{i}\mathrm{u}'_\mathrm{j}}\right)=-\frac{1}{\rho}\frac{\partial\overline{\mathrm{p}}}{\partial \mathrm{x_i}}+\nu\frac{\partial^2\overline{\mathrm{u}}_\mathrm{i}}{\partial \mathrm{x_j}\partial \mathrm{x_j}} \tag{12}$$

$$\nabla\cdot\left(\overline{\mathbf{u}}\,\overline{\mathbf{u}}\right)+\nabla\cdot\left(\overline{\mathbf{u}'\mathbf{u}'}\right)=-\frac{1}{\rho}\nabla\overline{\mathrm{p}}+\nu\nabla^2\overline{\mathbf{u}}$$

Portanto, são 4 equações (10) e (12) e 10 incógnitas $\overline{\mathrm{p}}$, $\overline{\mathrm{u}}_1$, $\overline{\mathrm{u}}_2$, $\overline{\mathrm{u}}_3$, $\overline{\mathrm{u}'_1\mathrm{u}'_1}$, $\overline{\mathrm{u}'_2\mathrm{u}'_2}$, $\overline{\mathrm{u}'_3\mathrm{u}'_3}$, $\overline{\mathrm{u}'_1\mathrm{u}'_2}\left(=\overline{\mathrm{u}'_2\mathrm{u}'_1}\right)$, $\overline{\mathrm{u}'_1\mathrm{u}'_3}\left(=\overline{\mathrm{u}'_3\mathrm{u}'_1}\right)$ e $\overline{\mathrm{u}'_2\mathrm{u}'_3}\left(=\overline{\mathrm{u}'_3\mathrm{u}'_2}\right)$.

$$-\rho\overline{u_i'u_j'} = -\rho\begin{bmatrix}\overline{u_1'u_1'} & \overline{u_1'u_2'} & \overline{u_1'u_3'}\\ \overline{u_2'u_1'} & \overline{u_2'u_2'} & \overline{u_2'u_3'}\\ \overline{u_3'u_1'} & \overline{u_3'u_2'} & \overline{u_3'u_3'}\end{bmatrix} \Rightarrow \text{Tensor de Reynolds}$$

$$\underbrace{\mathrm{Tr}\begin{pmatrix}\text{Tensor de}\\ \text{Reynolds}\end{pmatrix}}_{\text{traço}} = -\rho\left(\overline{u_1'^2}+\overline{u_2'^2}+\overline{u_3'^2}\right) = -2K$$

$$K \Rightarrow \text{Energia cinética turbulenta}$$

Energia Cinética Turbulenta

$$k = \frac{1}{2}\left(\overline{u_1'^2}+\overline{u_2'^2}+\overline{u_3'^2}\right)$$

Intensidade Turbulenta

$$I = \frac{\left(\frac{2}{3}k\right)^{1/2}}{U_{ref}}$$

RANS - exemplo: Placa plana c/ $\rho =$ cte (Turns cap. 11)

Quantidade de movimento em x $\left(\frac{\partial p}{\partial x} = 0\right)$

$$\underbrace{\frac{\partial v_x}{\partial t}}_{1} + \underbrace{\frac{\partial}{\partial x}\left(v_x v_x\right)}_{2} + \underbrace{\frac{\partial}{\partial y}\left(v_x v_y\right)}_{3} = \underbrace{\nu\frac{\partial^2 v_x}{\partial y \partial y}}_{4}$$

$$1 : \overline{\frac{\partial}{\partial t}\left(\overline{v}_x + v'_x\right)} = \frac{\partial \overline{v}_x}{\partial t} + \frac{\partial \overline{v}'_x}{\partial t} = 0$$

$$2 : \overline{\frac{\partial}{\partial x}\left(\overline{v}_x + v'_x\right)\left(\overline{v}_x + v'_x\right)} = \frac{\partial}{\partial x}\left(\overline{v}_x\overline{v}_x\right) + \frac{\partial}{\partial x}\left(\overline{v'_x v'_x}\right)$$

$$3 : \overline{\frac{\partial}{\partial y}\left(\overline{v}_x + v'_x\right)\left(\overline{v}_y + v'_y\right)} = \frac{\partial}{\partial y}\left(\overline{v}_x\overline{v}_y\right) + \frac{\partial}{\partial y}\left(\overline{v'_x v'_y}\right)$$

$$4 : \nu\overline{\frac{\partial^2 v_x}{\partial y \partial y}} = \nu\frac{\partial^2 \overline{v}_x}{\partial y \partial y}$$

RANS - exemplo: Placa plana c/ $\rho =$ cte (Turns cap. 11)

Quantidade de movimento em x

$$\frac{\partial}{\partial \mathrm{x}}\left(\overline{\mathrm{v}}_\mathrm{x}\overline{\mathrm{v}}_\mathrm{x}\right) + \frac{\partial}{\partial \mathrm{y}}\left(\overline{\mathrm{v}}_\mathrm{x}\overline{\mathrm{v}}_\mathrm{y}\right) + \frac{\partial}{\partial \mathrm{y}}\left(\overline{\mathrm{v}'_\mathrm{x}\mathrm{v}'_\mathrm{y}}\right) + \frac{\partial}{\partial \mathrm{x}}\left(\overline{\mathrm{v}'_\mathrm{x}\mathrm{v}'_\mathrm{x}}\right) = \nu\frac{\partial^2\overline{\mathrm{v}}_\mathrm{x}}{\partial \mathrm{y}^2}$$

Continuidade

$$\frac{\partial\overline{\mathrm{v}}_\mathrm{x}}{\partial \mathrm{x}} + \frac{\partial\overline{\mathrm{v}}_\mathrm{y}}{\partial \mathrm{y}} = 0$$

Para jato livre (análogo à placa plana):

$$\overline{\mathrm{v}}_\mathrm{x}\frac{\partial\overline{\mathrm{v}}_\mathrm{x}}{\partial \mathrm{x}} + \overline{\mathrm{v}}_\mathrm{r}\frac{\partial\overline{\mathrm{v}}_\mathrm{x}}{\partial \mathrm{r}} = \nu\frac{1}{\mathrm{r}}\frac{\partial}{\partial \mathrm{r}}\left(\mathrm{r}\frac{\partial\overline{\mathrm{v}}_\mathrm{x}}{\partial \mathrm{r}}\right) - \frac{1}{\mathrm{r}}\frac{\partial}{\partial \mathrm{r}}\left(\mathrm{r}\overline{\mathrm{v}'_\mathrm{x}\mathrm{v}'_\mathrm{r}}\right) \tag{13}$$

Eddy viscosity ou hipótese de Boussinesq (1877)

Hipótese de Boussinesq (1877)

$$\frac{1}{\mathrm{r}}\frac{\partial}{\partial \mathrm{r}}\left[\mathrm{r}\left(\tau_{\mathsf{lam}}+\tau_{\mathsf{turb}}\right)\right]$$

$$\tau_{\mathsf{lam}} = \mu\frac{\partial \overline{\mathrm{v}}_{\mathrm{x}}}{\partial \mathrm{r}}$$

$$\tau_{\mathsf{turb}} = -\rho\nu_{\mathrm{t}}\frac{\partial \overline{\mathrm{v}}_{\mathrm{x}}}{\partial \mathrm{r}}\ ;\ \nu_{\mathrm{t}} = \frac{\mu_{\mathrm{t}}}{\rho}$$

$$\underbrace{\mu_{\mathsf{ef}}}_{\text{efetivo}} = \underbrace{\mu}_{\text{molecular}} + \underbrace{\mu_{\mathrm{t}}}_{\text{turbulento}}$$

1. Como determinar μ_{t}?
2. μ é propriedade termodinâmica de transporte;
3. μ_{t} é dependente do "padrão"do escoamento, nem sempre é função apenas do gradiente da velocidade.

Eddy viscosity ou hipótese de Boussinesq (1877)

Comprimento de mistura de Prandtl $\mathrm{l_m}$

$$\mu_\mathrm{t} = \rho\,\nu_\mathrm{t} = \rho\,\mathrm{l_m}\,\mathrm{v_{turb}} = \rho \mathrm{l_m^2} \left| \frac{\partial \mathrm{v_x}}{\partial \mathrm{x_r}} \right|$$

Para jatos livres *Prandtl* propôs que $\mathrm{v_{turb}} \propto \overline{\mathrm{v}}_\mathrm{x,max} - \overline{\mathrm{v}}_\mathrm{x,min}$, então viscosidade turbulenta fica:

$$\mu_\mathrm{t} = \underbrace{0,1365}_{\text{experimental}}\ \rho\,\mathrm{l_m}\left(\overline{\mathrm{v}}_\mathrm{x,max} - \overline{\mathrm{v}}_\mathrm{x,min}\right) \tag{14}$$

Eddy viscosity ou hipótese de Boussinesq (1877)

Aplicando a hipótese de *Boussinesq* para jatos livres na equação da quantidade de movimento (13) e a equação média da continuidade ficam:

Continuidade

$$\frac{\partial}{\partial \mathrm{x}}\left(\overline{\mathrm{v}}_{\mathrm{x}}\mathrm{r}\right)+\frac{\partial}{\partial \mathrm{r}}\left(\overline{\mathrm{v}}_{\mathrm{r}}\mathrm{r}\right)=0 \tag{15}$$

Quantidade de movimento

$$\rho\left(\overline{\mathrm{v}}_{\mathrm{x}}\frac{\partial \overline{\mathrm{v}}_{\mathrm{x}}}{\partial \mathrm{x}}+\overline{\mathrm{v}}_{\mathrm{r}}\frac{\partial \overline{\mathrm{v}}_{\mathrm{x}}}{\partial \mathrm{r}}\right)=\frac{1}{\mathrm{r}}\frac{\partial}{\partial \mathrm{r}}\left[\mathrm{r}(\mu+\mu_{\mathrm{t}})\frac{\partial \overline{\mathrm{v}}_{\mathrm{x}}}{\partial \mathrm{r}}\right] \tag{16}$$

Eddy viscosity ou hipótese de Boussinesq (1877)

Note que o problema apenas mudou de variável, porém experimentos em jatos livres indicam que:

$$\mathrm{l_m} = 0,075\ \delta_{99\%}$$

Onde, $\delta_{99\%}$ é a largura do jato para queda de 99% da velocidade axial na linha central.

$$\delta_{99\%} \rightarrow \text{raio no qual } \frac{\overline{\mathrm{v}}_\mathrm{x}(\mathrm{r})}{\overline{\mathrm{v}}_{\mathrm{x},0}} = 1\%$$

Dessa forma, o comprimento de mistura possui características semelhantes à meia largura de jato laminar. Logo, cada estação axial possui um comprimento e não há dependência radial, assim a equação (14) fica:

$$\frac{\mu_\mathrm{t}}{\rho} = 0,0102\ \delta_{99\%}(\mathrm{x})\ \overline{\mathrm{v}}_{\mathrm{x,max}}(\mathrm{x}) \tag{17}$$

Eddy viscosity ou hipótese de Boussinesq (1877)

Entretanto, resultados experimentais indicam que $\delta_{99\%} \approx \frac{5}{2}\,\mathrm{r}_{1/2}$ e sabe-se que a velocidade máxima decai inversamente com x, logo:

$$\delta_{99\%}\left(\mathrm{x}\right) \approx \frac{5}{2}\,\mathrm{r}_{1/2}\left(\mathrm{x}\right) \propto \mathrm{x}^{+1} \tag{18}$$

$$\overline{\mathrm{v}}_{\mathrm{x,max}}\left(\mathrm{x}\right) = \overline{\mathrm{v}}_{\mathrm{x,0}}\left(\mathrm{x}\right) \propto \mathrm{x}^{-1} \tag{19}$$

Então, a equação (17) fica:

$$\nu_{\mathrm{t}} = \frac{\mu_{\mathrm{t}}}{\rho} = 0,0102\;\delta_{99\%}\left(\mathrm{x}\right)\;\overline{\mathrm{v}}_{\mathrm{x,max}}\left(\mathrm{x}\right) = \mathrm{constante} \tag{20}$$

Portanto, adotando a hipótese de *Boussinesq* e o comprimento de mistura de *Prandtl*, a viscosidade (cinemática) turbulenta é constante em jatos livres turbulentos.

Eddy viscosity ou hipótese de Boussinesq (1877)

A equação da quantidade de movimento para jatos livres (16) fica:

$$\overline{v}_x \frac{\partial \overline{v}_x}{\partial x} + \overline{v}_r \frac{\partial \overline{v}_x}{\partial r} = \nu_{ef} \frac{1}{r} \frac{\partial}{\partial r} \left[r \frac{\partial \overline{v}_x}{\partial r} \right] \tag{21}$$

Relembrando, a equação da quantidade de movimento para jatos livres laminares é dada por:

$$v_x \frac{\partial v_x}{\partial x} + v_r \frac{\partial v_x}{\partial r} = \nu \frac{1}{r} \frac{\partial}{\partial r} \left[r \frac{\partial v_x}{\partial r} \right] \tag{22}$$

como as condições de contorno são semelhantes, os resultados das velocidades <u>médias</u> do escoamento turbulento é o mesmo das velocidades do escoamento laminar, então:

$$\overline{v}_x (r, x) = \frac{3}{8\pi} \frac{J_e}{\rho \nu_t x} \left[1 + \frac{\xi^2}{4} \right]^{-2} \qquad \overline{v}_r (r, x) = \left[\frac{3 J_e}{16 \pi \rho_e} \right]^{1/2} \frac{1}{x} \frac{\xi - \xi^3/4}{[1 + \xi^2/4]^2}$$

Eddy viscosity ou hipótese de Boussinesq (1877)

Onde, $\mathrm{J_e} = \rho_\mathrm{e} \mathrm{v_e^2} \pi \mathrm{R}^2$ é o fluxo da quantidade de movimento na saída do jato e ξ contém a variável de similaridade $\mathrm{r/x}$.

$$\xi = \left(\frac{3\rho_\mathrm{e} \mathrm{J_e}}{16\pi} \right)^{1/2} \frac{1}{\mu} \frac{\mathrm{r}}{\mathrm{x}} \tag{23}$$

Substituindo $\mathrm{J_e}$ na solução $\overline{\mathrm{v_x}}$ e rearranjando tem-se:

$$\frac{\overline{\mathrm{v}}_\mathrm{x}\left(\mathrm{r},\mathrm{x}\right)}{\mathrm{v_e}} = \frac{3}{8} \left(\frac{\mathrm{v_e R}}{\nu_\mathrm{t}} \right) \left(\frac{\mathrm{R}}{\mathrm{x}} \right) \left[1 + \frac{\xi^2}{4} \right]^{-2} \tag{24}$$

Na linha de centro $\mathrm{r} = 0 \rightarrow (\xi = 0)$:

$$\frac{\overline{\mathrm{v}}_{\mathrm{x},0}\left(\mathrm{x}\right)}{\mathrm{v_e}} = \frac{3}{8} \left(\frac{\mathrm{v_e R}}{\nu_\mathrm{t}} \right) \left(\frac{\mathrm{R}}{\mathrm{x}} \right) \tag{25}$$

que é a equação (19) de forma explícita.

Eddy viscosity ou hipótese de Boussinesq (1877)

A expressão para $\mathrm{r}_{1/2}$ é encontrada fazendo:

$$\frac{\overline{\mathrm{v}}_{\mathrm{x}}(\mathrm{r},\mathrm{x})}{\overline{\mathrm{v}}_{\mathrm{x},0}(\mathrm{x})} = \frac{1}{2} = \left[1 + \frac{\xi^2}{4}\right]^{-2} \longrightarrow$$
$$\xi = 1,287$$

Usando a equação (23)

$$1,287 = \left(\frac{3}{16}\right)^{1/2} \left[\frac{\rho_{\mathrm{e}} \mathrm{v}_{\mathrm{e}}^2 \pi \mathrm{R}^2}{\rho_{\mathrm{e}} \pi}\right]^{1/2} \frac{1}{\nu_{\mathrm{t}}} \frac{\mathrm{r}_{1/2}}{\mathrm{x}} \longrightarrow$$
$$1,287 = \left(\frac{3}{16}\right)^{1/2} \mathrm{v}_{\mathrm{e}} \mathrm{R} \frac{1}{\nu_{\mathrm{t}}} \frac{\mathrm{r}_{1/2}}{\mathrm{x}} \longrightarrow$$
$$\frac{\mathrm{r}_{1/2}}{\mathrm{x}} = 2,97 \left(\frac{\nu_{\mathrm{t}}}{\mathrm{v}_{\mathrm{e}} \mathrm{R}}\right) \qquad (26)$$

O que mostra a validade da equação (19), além de ser um resultado semelhante ao encontrado no escoamento laminar.

Eddy viscosity ou hipótese de Boussinesq (1877)

Assim, substituindo a equação (26) em (18) e então em (20) juntamente com (25)

$$\nu_t = 0,0102\,\frac{5}{2}\,2,97\left(\frac{\nu_t}{v_e R}\right) x\,\frac{3}{8}\left(\frac{v_e R}{\nu_t}\right)\left(\frac{R}{x}\right) v_e \longrightarrow$$
$$\nu_t = 0,028\, v_e R \longrightarrow$$
$$\frac{\nu_t}{v_e R} = 0,028 \tag{27}$$

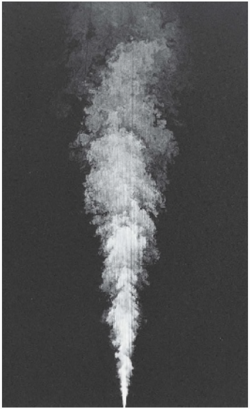
Então, retornando em (26) tem-se:

$$\frac{r_{1/2}}{x} = 0,08468$$
$$\alpha = 4,84^\circ$$

Portanto, a taxa de espalhamento $\left(\frac{r_{1/2}}{x}\right)$ e o ângulo de espalhamento $\alpha = \tan^{-1}\left(\frac{r_{1/2}}{x}\right)$ são constantes.

Eddy viscosity ou hipótese de Boussinesq (1877)

Lembrando que no caso laminar $\frac{\mathrm{r}_{1/2}}{\mathrm{x}} = \frac{\nu}{\mathrm{v_e R}} = \mathrm{Re_j^{-1}}$, mas no caso turbulento não há dependência do número de Reynolds.



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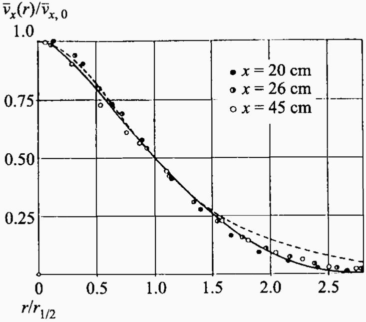

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Lista de exercícios 2

Próxima aula: Equação de Transporte do Tensor de Reynolds

Leituras recomendadas:

Básica: capítulo 11 Turns;

Complementar: capítulo 1 Poinsot;

Referências Bibliográficas

- Turns, Stephen R. **An introduction to combustion: concepts and applications.** 3rd ed. New York: McGraw-Hill, 2012.
- Poinsot, T.; Veynante, D. **Theoretical and Numerical Combustion.** 2nd ed. Philadelphia: Edwards, 2005.
- Pope, Stephen B. **Turbulent flows.** 10th ed. New York: Cambridge University Press, 2013.