Nonlinear Dynamics of structures and mechanical systems

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Lesson 2.1

Elements of Stability Theory

Lagrangian formulation (recalling)

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial q_r} + \frac{\partial V}{\partial q_r} = N_r \;, r = 1, 2, \ldots, n$$

System of second-order differential equations (holonomic constraints)

$$\ddot{\mathbf{q}} = \mathbf{h}\left(\mathbf{q}, \dot{\mathbf{q}}, t\right) \qquad \ddot{q}_r = h_r\left(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n, t\right)$$

Example: SDOF linear oscillator

$$\ddot{q} = \gamma(t) - \omega^2 q - 2\xi\omega\dot{q} \qquad \text{with} \qquad \gamma(t) = \frac{R(t)}{m}, \quad \omega = \sqrt{\frac{k}{m}}, \quad \xi = \frac{c}{2m\omega}$$

Example: MDOF linear system

$$\ddot{\mathbf{q}} = \mathbf{M}^{-1}[\mathbf{R}(t) - \mathbf{K}\mathbf{q} - \mathbf{C}\dot{\mathbf{q}}]$$

Elements of Stability Theory

Hamiltonian formulation (recalling)

Generalized momenta: $$p_r = \frac{\partial T}{\partial \dot{q}_r}$$

Hamiltonian: $$H = \sum_{r=1}^{n} \dot{q}_r p_r - T + V$$

System of first-order differential equations (holonomic constraints)

$$\dot{q}_r = \frac{\partial H}{\partial p_r}$$

$$\dot{p}_r = N_r - \frac{\partial H}{\partial q_r}$$

Elements of Stability Theory

Hamiltonian formulation (recalling)

Example: SDOF linear oscillator

$$p = \frac{\partial T}{\partial \dot{q}} = m\dot{q}$$

$$H = p\dot{q} - T + V = \frac{p^2}{2m} + \frac{kq^2}{2}$$

$$N = R(t) - c\dot{q} = R(t) - c\frac{p}{m}$$

$$\Rightarrow \begin{cases} \dot{q} = \dfrac{1}{m}p \\ \dot{p} = R(t) - kq - \dfrac{c}{m}p \end{cases}$$

Elements of Stability Theory

Lagrangian formulation:
from second- to first-order system of differential equations through change of variables

$$\ddot{q}_r = h_r\left(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n, t\right)$$

$$y_r = q_r \qquad y_{r+n} = \dot{q}_r \quad \Rightarrow \quad \begin{cases} \dot{y}_r = y_{r+n} \\ \dot{y}_{r+n} = h_r\left(y_1, y_2, \ldots, y_{2n}, t\right) \end{cases}$$

$$\dot{\mathbf{y}} = \mathbf{g}\left(\mathbf{y}, t\right)$$

Example: SDOF linear oscillator

$$y_1 = q \qquad y_2 = \dot{q} \quad \Rightarrow \quad \begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = \gamma\left(t\right) - \omega^2 y_1 - 2\xi\omega y_2 \end{cases}$$

Elements of Stability Theory

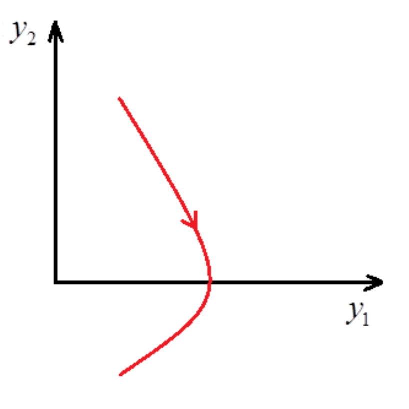
Phase space

Autonomous systems

$$\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y})$$

n-dimensional space

$$y_1 \times y_2 ... \times y_{2n}$$

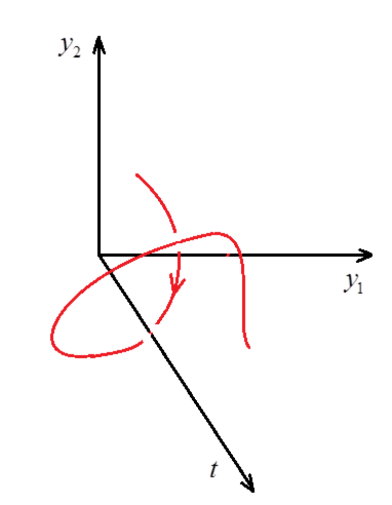


Non-autonomous systems

$$\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y}, t)$$

(n+1)-dimensional space

$$y_1 \times y_2 ... \times y_{2n} \times t$$



Elements of Stability Theory

Phase space properties for SDOF autonomous systems

Singular phase points (equilibrium points) $\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y}) = 0$

Regular phase points $\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y}) \neq 0$

Phase trajectory tangent

$$\frac{dy_2}{dy_1} = \frac{g_2(y_1, y_2)}{g_1(y_1, y_2)} = \frac{g_2(y_1, y_2)}{y_2}$$

Tangent at singular phase points is indeterminate

$$\frac{dy_2}{dy_1} = \frac{g_2(y_1, y_2)}{g_1(y_1, y_2)} = \frac{0}{0}$$

Tangent at regular phase points with $g_1(y_1, y_2) = y_2 = 0$ and $g_2(y_1, y_2) \neq 0$ is orthogonal to the y_1 axis

Through a regular phase point passes just one phase trajectory (Theorem of Cauchy-Lipschitz)

Elements of Stability Theory

Non-perturbed solution: $y_r = y_r^0(t), \quad r = 1, 2, \ldots, 2n$

Perturbed solution: $y_r = y_r^0(t) + \delta y_r(t), \quad r = 1, 2, \ldots, 2n$

$$\delta \dot{y}_r = g_r\left(y_1^0 + \delta y_1, y_2^0 + \delta y_2, \ldots, y_{2n}^0 + \delta y_{2n}, t\right) - \dot{y}_r^0$$

Perturbation equations:

$$\delta \dot{y}_r = f_{0r}(\delta y_1, \delta y_2, \ldots, \delta y_{2n}, t)$$

$$\boldsymbol{\delta}\dot{\mathbf{y}} = \mathbf{f}_0(\boldsymbol{\delta}\mathbf{y}, t)$$

$\boldsymbol{\delta}\dot{\mathbf{y}} = \mathbf{A}(t)\boldsymbol{\delta}\mathbf{y} + \mathbf{N}(\boldsymbol{\delta}\mathbf{y}, t)$ with $\mathbf{A}(t) = \left.\frac{\partial \mathbf{f}_0}{\partial \mathbf{y}}\right|_{\mathbf{0}}$ and $\mathbf{N}(\boldsymbol{\delta}\mathbf{y}, t) = \mathbf{f}_0(\boldsymbol{\delta}\mathbf{y}, t) - \mathbf{A}(t)\boldsymbol{\delta}\mathbf{y}$

Note: the non-perturbed solution corresponds to the trivial solution $\boldsymbol{\delta}\mathbf{y} = \mathbf{0}$ of the perturbation equations

Elements of Stability Theory

Example: SDOF linear oscillator

$$\begin{cases} \delta \dot{y}_1 = \delta y_2 \\ \delta \dot{y}_2 = -\omega^2 \delta y_1 - 2\xi\omega\delta y_2 \end{cases}$$

$$\boldsymbol{\delta}\dot{\mathbf{y}} = \mathbf{f}\left(\boldsymbol{\delta}\mathbf{y}\right) \qquad \text{or} \qquad \boldsymbol{\delta}\dot{\mathbf{y}} = \mathbf{A}\boldsymbol{\delta}\mathbf{y}$$

with

$$\mathbf{A} = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{y}}\right|_{\mathbf{0}} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\xi\omega \end{bmatrix}$$

Elements of Stability Theory

Stability concept (Leipholz)

A non-perturbed solution $\mathbf{y}^0(t)$ is stable if the distance $\delta\mathbf{y}(t)$ to the perturbed solutions remains within prescribed bounds for all times and arbitrarily defined perturbations

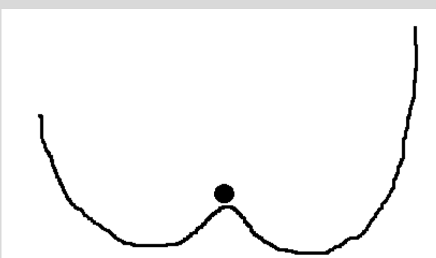
Non-perturbed solution
- Equilibrium $\mathbf{y}^0 = const.$
- Motion $\mathbf{y}^0(t)$

"Type" of perturbation
- Kinematical (initial conditions): $\delta\mathbf{y}(0) \neq \mathbf{0}$
- Topological (perturbation of parameters or perturbation of mathematical model)

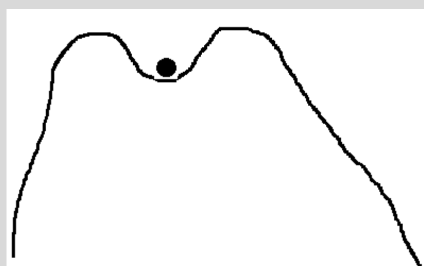
Elements of Stability Theory

Stability concept (Leipholz)

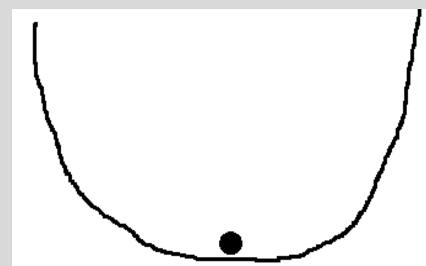
Perturbation “size”
- Local $\left\|\boldsymbol{\delta}\mathbf{y}(0)\right\| < \delta$
- Global



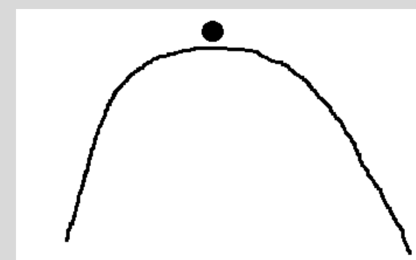
Global S
Local I



Global I
Local S



Global S
Local S



Global I
Local I

Elements of Stability Theory

Stability concept (Leipholz)

"Character" of perturbation
- Deterministic
- Stochastic

Example: definition of stability in the quadratic mean:

$$\lim_{\tau \to \infty} E_{\tau} \left\| \boldsymbol{\delta y}(t) \right\|^2 < \varepsilon \qquad \sigma^2_{\boldsymbol{\delta y}} = \int_{-\infty}^{\infty} S_{\boldsymbol{\delta y}}(\omega)\, d\omega < \varepsilon$$

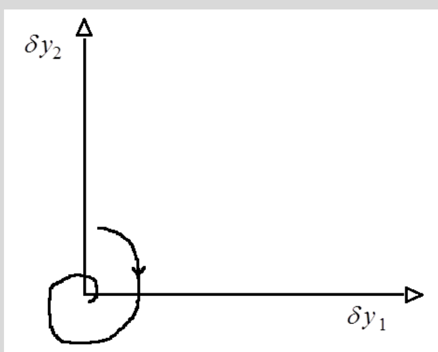
Stability x Reliability x Integrity

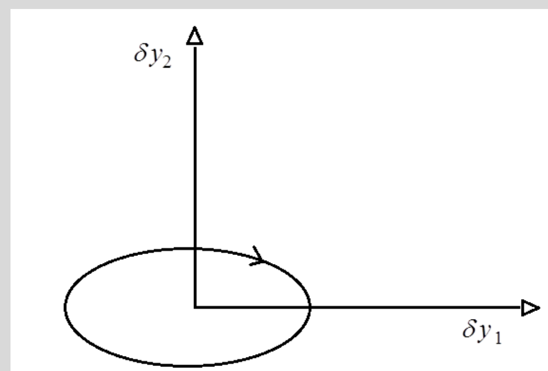
Elements of Stability Theory

Stability concept (Leipholz)

Evolution of perturbed solution

- Asymptotic
- Non-asymptotic



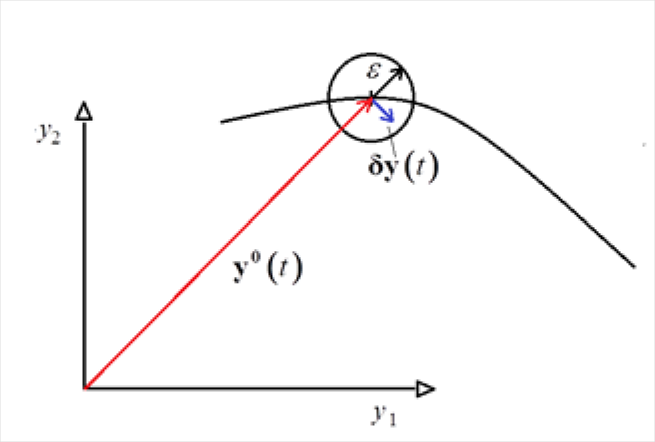


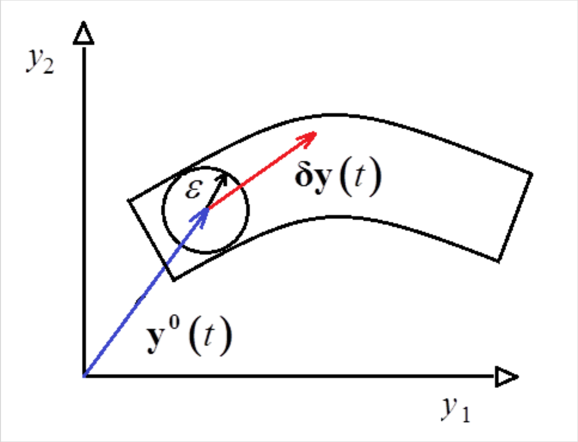
Elements of Stability Theory

Stability concept (Leipholz)

Admissible region for perturbed solution

- Kinetic
- Geometric





Elements of Stability Theory

Stability definitions

Liapunov

Stability of equilibrium of autonomous systems in the sense:
kinematical, local, deterministic, non-asymptotic, kinetic

Poincaré

Stability of motion of autonomous systems in the sense:
kinematical, local, deterministic, non-asymptotic,geometric

Particular case: orbital stability of periodic motions

Structural

Stability of equilibrium or motion in the sense:
topological, local, deterministic, asymptotic

Particular cases: parametric stability; Mathieu stability

Elements of Stability Theory

Liapunov stability

Given $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$, such that,

if $\|\mathbf{\delta y}(0)\| < \delta(\varepsilon)$ then $\|\mathbf{\delta y}(t)\| < \varepsilon$ for $t > 0$

Liapunov's methods

First method (indirect)

Second method (direct)

Elements of Stability Theory

Liapunov's first method

Perturbation equation for the analysis of the stability of equilibrium of the trivial solution $\boldsymbol{\delta y} = \mathbf{0}$

$$\boldsymbol{\delta\dot{y}} = \mathbf{f}_0(\boldsymbol{\delta y}) = \mathbf{A}\boldsymbol{\delta y} + \mathbf{N}(\boldsymbol{\delta y})$$

with $\mathbf{A} = \left.\dfrac{\partial \mathbf{f}_0}{\partial \mathbf{y}}\right|_{\mathbf{0}}$ and $\mathbf{N}(\boldsymbol{\delta y}) = \mathbf{f}_0(\boldsymbol{\delta y}) - \mathbf{A}\boldsymbol{\delta y}$

Consider the associated linearized problem

$$\boldsymbol{\delta\dot{y}} = \mathbf{A}\boldsymbol{\delta y}$$

Solução geral

$$\boldsymbol{\delta y} = \boldsymbol{\delta y}_{\mathbf{0}} e^{\lambda t}$$

Elements of Stability Theory

Liapunov's first method

$$(\mathbf{A} - \lambda\mathbf{I})\boldsymbol{\delta}\mathbf{y_0} = \mathbf{0}$$

For non-trivial solutions it is required that

$$|\mathbf{A} - \lambda\mathbf{I}| = 0$$

It is the classic eigenvalue problem for matrix $\mathbf{A}$

$$b_0\lambda^{2n} + b_1\lambda^{2n-1} + \ldots + b_{2n-1}\lambda + b_{2n} = 0$$

In the general case, there exists 2*n* complex roots for the characteristic equation

$$\lambda_k = R_k + iI_k, \quad R_k \ and \ I_k \ are \ real \ numbers$$

Elements of Stability Theory

Liapunov's first method

Theorem 1 (Liapunov): If $R_k < 0 \quad \forall k = 1, 2 \ldots 2n \Rightarrow \boldsymbol{\delta y} = \mathbf{0}$ is L-stable

Theorem 2 (Liapunov): If $\exists R_k > 0 \quad \Rightarrow \boldsymbol{\delta y} = \mathbf{0}$ is L-unstable

Definition of L-critical case: there exists at least one eigenvalue with zero real part $R_k = 0$, yet none of them with positive real part.

Theorem 3 (Leipholz): In the critical case, if the multiplicity p_k of all the eigenvalues with null real part $(R_k = 0)$ is equal to the rank decrement d_k of the matrix $\mathbf{A} - \lambda_k \mathbf{I}$, then the solution $\boldsymbol{\delta y} = \mathbf{0}$ is L-stable for the linear system. If $p_k > d_k$, then the solution $\boldsymbol{\delta y} = \mathbf{0}$ is L-unstable for the linear system.

Elements of Stability Theory

Liapunov's first method

Theorem 4 (Routh-Hurwitz): If all principal minors of the matrix **B** (below) are positive, then the solution $\boldsymbol{\delta y} = \mathbf{0}$ is L-stable. The reciprocal is also true.

$$\mathbf{B} = \begin{bmatrix} b_1 & b_0 & 0 & 0 & 0 & 0 & \dots & 0 \\ b_3 & b_2 & b_1 & b_0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & b_{2n} & b_{2n-1} & b_{2n-2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_{2n} \end{bmatrix}$$

$$b_{r>2n} = 0 \quad \text{and} \quad b_{r<0} = 0$$

Elements of Stability Theory

Liapunov's first method

Theorem 5 (Liapunov): Except for the L-critical case, the conclusions drawn from Theorems 1 and 2 for the linearized system $\boldsymbol{\delta\dot{y}} = \mathbf{A}\boldsymbol{\delta y}$ can be extended to the non-linear system $\boldsymbol{\delta\dot{y}} = \mathbf{A}\boldsymbol{\delta y} + \mathbf{N}(\boldsymbol{\delta y})$

Dynamical systems theory

Theorem 5' (Hartman-Grobman): If a singularity of the linear system $\boldsymbol{\delta\dot{y}} = \mathbf{A}\boldsymbol{\delta y}$ is hyperbolic , then the linearized system is topologically equivalent to the non-linear system $\boldsymbol{\delta\dot{y}} = \mathbf{A}\boldsymbol{\delta y} + \mathbf{N}(\boldsymbol{\delta y})$ in the singularity neighbourhood, that is, between the phase space flows of the non-linear and the linear systems there exists a diffeomorphism (transformation that is continuous with continuous derivative)