Nonlinear dynamics of structures and mechanical systems

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Lesson 1.2

Lagrangian formulation for discrete systems or discretized continua

Holonomic constraints $\mathbf{R} = \mathbf{R}\left(q_1, q_2, \ldots, q_n, t\right) \Rightarrow \dot{\mathbf{R}} = \frac{\partial \mathbf{R}}{\partial q_i}\dot{q}_i + \frac{\partial \mathbf{R}}{\partial t}$

Kinetic energy $T = \frac{1}{2}\int_\Omega \rho\left(\dot{\mathbf{R}} \cdot \dot{\mathbf{R}}\right) d\Omega = \frac{1}{2}A^{ij}\dot{q}_i\dot{q}_j + B^i\dot{q}_i + \frac{1}{2}C$

$$A^{ij}\left(q_1, q_2, \ldots, q_n, t\right) = \int_\Omega \rho\left(\frac{\partial \mathbf{R}}{\partial q_i} \cdot \frac{\partial \mathbf{R}}{\partial q_j}\right) d\Omega$$

$$B^{i}\left(q_1, q_2, \ldots, q_n, t\right) = \int_\Omega \rho\left(\frac{\partial \mathbf{R}}{\partial q_i} \cdot \frac{\partial \mathbf{R}}{\partial t}\right) d\Omega$$

$$C\left(q_1, q_2, \ldots, q_n, t\right) = \int_\Omega \rho\left(\frac{\partial \mathbf{R}}{\partial t} \cdot \frac{\partial \mathbf{R}}{\partial t}\right) d\Omega$$

Total potential energy $V\left(q_1, q_2, \ldots, q_n, t\right)$

Lagrangian formulation…

Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)-\frac{\partial L}{\partial q_i}=N_i$$

$$L=T\left(q_1,q_2,\ldots,q_n,\dot{q}_1,\dot{q}_2,\ldots,\dot{q}_n,t\right)-V\left(q_1,q_2,\ldots,q_n,t\right)$$

$$A^{ij}\ddot{q}_j+\left(\frac{\partial A^{ij}}{\partial q_k}-\frac{1}{2}\frac{\partial A^{jk}}{\partial q_i}\right)\dot{q}_j\dot{q}_k+\frac{\partial A^{ij}}{\partial t}\dot{q}_j$$

$$+\left(\frac{\partial B^{i}}{\partial q_j}-\frac{\partial B^{j}}{\partial q_i}\right)\dot{q}_j+\frac{\partial B^{i}}{\partial t}-\frac{1}{2}\frac{\partial C}{\partial q_i}+\frac{\partial V}{\partial q_i}=N_i$$

Lagrangian formulation…

Gyroscopic force (Coriolis)

$$\left(\frac{\partial B^i}{\partial q_j}-\frac{\partial B^j}{\partial q_i}\right)\dot{q}_j$$

$$\Downarrow$$

Gyroscopic “damping” matrix is anti-symmetric

Scleronomic systems

$$\mathbf{R}=\mathbf{R}\left(q_1,q_2,\ldots,q_n\right)$$

$$\frac{\partial A^{ij}}{\partial t}=0;\quad B^i=0;\quad C=0 \quad\Rightarrow\quad A^{ij}\ddot{q}_j+\left(\frac{\partial A^{ij}}{\partial q_k}-\frac{1}{2}\frac{\partial A^{jk}}{\partial q_i}\right)\dot{q}_j\dot{q}_k+\frac{\partial V}{\partial q_i}=N_i$$

If the mass matrix is constant...

$$\frac{\partial A^{ij}}{\partial q_k}=0 \quad\Rightarrow\quad A^{ij}\ddot{q}_j+\frac{\partial V}{\partial q_i}=N_i$$

Hamiltonian formulation…

Legendre's transformation $$f\left(x,y\right)\Rightarrow g\left(u,y\right)=ux-f\left(x,y\right)$$

with $u=\dfrac{\partial f}{\partial x}$

Observe that $$df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy=udx+\frac{\partial f}{\partial y}dy$$

$$dg=\frac{\partial g}{\partial u}du+\frac{\partial g}{\partial y}dy$$

$$=xdu+\overbrace{udx-\frac{\partial f}{\partial x}dx}^{0}-\frac{\partial f}{\partial y}dy$$

$$=xdu-\frac{\partial f}{\partial y}dy\Rightarrow x=\frac{\partial g}{\partial u}\quad \text{e}\quad \frac{\partial g}{\partial y}=-\frac{\partial f}{\partial y}$$

Hamiltonian formulation…

Duality

"Old"

$$f(x,y) = ux - g(u,y)$$

$$u = \frac{\partial f}{\partial x}$$

"New"

$$g(u,y) = ux - f(x,y)$$

$$x = \frac{\partial g}{\partial u}$$

Application to the Lagrangian

$$L(\mathbf{q},\dot{\mathbf{q}},t) \Rightarrow H(\mathbf{q},\mathbf{p},t) = \mathbf{p}^{\mathrm{T}}\dot{\mathbf{q}} - L(\mathbf{q},\dot{\mathbf{q}},t)$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \Leftarrow$$ with $$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

Hamiltonian formulation…

$$H(\mathbf{q},\mathbf{p},t) = \mathbf{p}^{\mathrm{T}}\dot{\mathbf{q}} - L(\mathbf{q},\dot{\mathbf{q}},t)$$

$$dH = \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial t} dt =$$

$$= \dot{q}_i dp_i + \underbrace{p_i d\dot{q}_i - \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i}_{0} - \frac{\partial L}{\partial q_i} dq_i - \frac{\partial L}{\partial t} dt$$

$$\Rightarrow \begin{cases} \dot{q}_i = \dfrac{\partial H}{\partial p_i} \\[2ex] \dfrac{\partial H}{\partial q_i} = -\dfrac{\partial L}{\partial q_i} \\[2ex] \dfrac{\partial H}{\partial t} = -\dfrac{\partial L}{\partial t} \end{cases}$$

Hamiltonian formulation…

Lagrange's equation with $N_i = 0$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \Rightarrow \dot{p}_i = \frac{\partial L}{\partial q_i} = -\frac{\partial H}{\partial q_i}$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$

First-order system

Hamilton's canonical equations

$$\dot{H} = \frac{dH}{dt} = \frac{\partial H}{\partial q_i}\dot{q}_i + \frac{\partial H}{\partial p_i}\dot{p}_i + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial H}{\partial q_i} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}$$

Hamiltonian formulation…

Lagrange's equation with $N_i \neq 0$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)-\frac{\partial L}{\partial q_i}=N_i \Rightarrow \dot{p}_i = N_i + \frac{\partial L}{\partial q_i} = N_i - \frac{\partial H}{\partial q_i}$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$

Hamilton's canonical equations

$$\dot{H}=\frac{dH}{dt}=\frac{\partial H}{\partial q_i}\dot{q}_i+\frac{\partial H}{\partial p_i}\dot{p}_i+\frac{\partial H}{\partial t}=\frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i}+\frac{\partial H}{\partial p_i}\left(N_i-\frac{\partial H}{\partial q_i}\right)+\frac{\partial H}{\partial t}=N_i\frac{\partial H}{\partial p_i}+\frac{\partial H}{\partial t}$$

Hamiltonian formulation…

Scleronomic conservative system $\quad \mathbf{R} = \mathbf{R}(\mathbf{q}) \quad$ e $\quad N_i = 0$

$$H = H(\mathbf{p},\mathbf{q}) \therefore \frac{\partial H}{\partial t} = 0 \Rightarrow \dot{H} = 0 \Rightarrow H = const.$$

$$L = T - V = \frac{1}{2} A^{ij} \dot{q}_i \dot{q}_j - V(\mathbf{q}) \Rightarrow \frac{\partial L}{\partial \dot{q}_j} = A^{ij} \dot{q}_i$$

$$H(\mathbf{q},\mathbf{p}) = \mathbf{p}^{\mathrm{T}} \dot{\mathbf{q}} - L(\mathbf{q},\dot{\mathbf{q}}) = p_j \dot{q}_j - L = \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j - L$$

$$= A^{ij} \dot{q}_i \dot{q}_j - \frac{1}{2} A^{ij} \dot{q}_i \dot{q}_j + V = \frac{1}{2} A^{ij} \dot{q}_i \dot{q}_j + V$$

$$= T + V$$

Hamiltonian formulation…

Rheonomic system $\mathbf{R}=\mathbf{R}(\mathbf{q},t)$ with $N_i=0$ or $N_i\neq 0$

$$H=H(\mathbf{p},\mathbf{q},t) \text{ and } \dot{H}=N_i\frac{\partial H}{\partial p_i}+\frac{\partial H}{\partial t}\Rightarrow H\neq const.$$

$$L=T-V=\frac{1}{2}A^{ij}\dot{q}_i\dot{q}_j+B^i\dot{q}_i+\frac{1}{2}C-V(\mathbf{q})\Rightarrow\frac{\partial L}{\partial\dot{q}_j}=A^{ij}\dot{q}_i+B^j$$

$$H(\mathbf{q},\mathbf{p})=\mathbf{p}^{\mathrm{T}}\dot{\mathbf{q}}-L(\mathbf{q},\dot{\mathbf{q}})=p_j\dot{q}_j-L=\frac{\partial L}{\partial\dot{q}_j}\dot{q}_j-T+V$$

$$=\left(A^{ij}\dot{q}_i\dot{q}_j+B^j\dot{q}_j\right)-T+V=\left(2T-C-B^j\dot{q}_j\right)-T+V$$

$$=\underset{\text{mechanic energy}}{T+V}-\underset{\text{pseudo-potential energy}}{C}-\underset{\text{Coriolis 'energy'}}{B^j\dot{q}_j}$$