Computer Code for Rational Approximation of Frequency Dependent Admittance Matrices

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Abstract—This paper deals with the problem of approximating with rational functions a matrix whose frequency dependent elements have been obtained from calculations or from measurements. Based on a previously developed technique (vector fitting), a set of callable routines have been written in the Matlab language. These routines allow for rational approximation with a common set of stable poles, automatic selection of initial poles, passivity enforcement, and creation of an equivalent electrical network that can be imported into an ATP version of EMTP. Usage of sparse arithmetic permits the computer code to handle large systems. The methodology is demonstrated by application to a frequency dependent network equivalent of a radial distribution network (phase domain), for which the accuracy is validated in both the frequency domain and the time domain. The computer code is in the public domain and is available from the author.

Index Terms—Electromagnetic transients, EMTP, frequency dependence, network synthesization, rational approximation.

I. INTRODUCTION

IN PROGRAMS for simulation of electromagnetic transients (e.g., EMTP), there is sometimes a need for inclusion of network components described by frequency domain responses. The response can be either a scalar function or a matrix of responses. The preferred approach is to approximate the responses with rational functions in order to bring the frequency data into the time domain and in order to achieve high computational efficiency for the time domain simulation [1]. Examples of applications are

- network equivalent for portion of a large network [2];
- high-frequency transformer model based on measured frequency responses [3], [4];
- transfer of voltages from transformer terminals to internal points in winding [5];
- transmission line modeling [1].

The development of vector fitting (VF) [6], [7] has made it possible to calculate a least squares rational approximation of a vector of frequency domain responses using a common set of stable poles. This vector fitting capability has been used for the fitting of matrix columns in transmission line modeling [8], [9] and in transformer modeling [4]. The high-quality approximation has also been useful when fitting scalar functions [8], [10]–[13].

In a Discussion contribution for [4], it was reasoned by Lima *et al.* that the elements of a matrix should have a common set of

poles. The authors suggested, in the Closure, to achieve this by stacking the matrix columns into a single column, which is then subjected to VF.

This paper presents a computer code for achieving this matrix fitting capability with focus on admittance matrices. In particular, five routines are described that together form a platform for matrix fitting. The end result is matrices A, B, C, D, and E, as used in state equations, as well as a file containing an equivalent electrical network. The network can be imported into EMTP-type programs, but only the import into ATP-EMTP has been verified. For other versions, some extra work may be needed. Application of the software is explained and described in some detail. The validity of the approach is demonstrated by application to a network equivalent. The routines (in Matlab) and a user's manual can be obtained from the author.

II. VF

A. Least Squares Rational Approximation

VF [6] takes as input a column $h(s)$ and approximates all of its n elements simultaneously using an identical set of N poles. Each element h_i in h becomes

$$h_i(s) \approx h_{\text{fit}i}(s) = \sum_{m=1}^{N} \frac{c_{im}}{s - a_m} + d_i + se_i, \qquad i = 1 \ldots n \tag{1}$$

where c and a denote the residues and the poles, respectively. Note that the residues depend on i, while the poles do not. The terms d_i and e_i may be specified as zero, if desired. The poles and residues are real or complex conjugate, whereas d_i and e_i are real.

The approximation (1) as produced by VF is a weighted least squares approximation for the entire vector $h(s)$, i.e.,

$$\sum_{i=1}^{n} \sum_{k=1}^{N_s} w_k |h_{\text{fit}i}(s_k) - h_i(s_k)|^2 = \min \tag{2}$$

where w is a weight function specified by the user.

B. Sparse Implementation

The VF approach has been implemented as a callable function `spavectfit.m` in Matlab. This routine fully utilizes the sparse capabilities of Matlab when solving the large but sparse linear problem $Ax = b$ encountered when calculating the poles of long vectors.

III. MATRIX FITTING

A. Least Squares Rational Approximation

A least squares approximation for a matrix $Y(s)$ is to be calculated in a such way that all elements get identical poles. This is achieved by stacking the elements of Y into a single column $h(s)$, which is fitted by VF. As a result, each element i, j of $Y(s)$ gets an approximation

$$Y_{\mathrm{fit},ij} = \sum_{m=1}^{N} \frac{c_{ijm}}{s - a_m} + d_{ij} + se_{ij}. \tag{3}$$

The stacking is done by a routine `mtrxvectfit.m`, which calls the fitter `spavectfit.m`.

B. Reduction of Problem Size

When Y is a symmetric matrix, only the elements of the upper triangle of Y are fitted. This reduces the length of the vector $h(s)$ to be fitted, thus reducing the computation time needed by `spavectfit.m`.

C. Initial Poles

Reference [6, Sec. III] and its closure describes how to select the initial poles needed by VF. In `mtrxvectfit.m`, the user can choose between linearly spaced complex poles and logarithmically spaced real poles or specify the poles directly.

D. Weighting

Frequency Dependent Weighting: The accuracy of the least squares approximation can be manipulated by means of a three-dimensional (3-D) weighting array *weight* [w in (2)], This is an $n \times n$ matrix of frequency-dependent elements which defines the weighting of the correspoonding elements in *Y(s)*. Each entry *(i,j,k)* in *weight* multiplies a corresponding row in the least-squares problem for pole identification and subsequent residue identification.

Weighting of Individual Elements: Increased accuracy can be enforced for individual elements by means of an n by n array *elemweight*. The elements in $Y(s)$ are multiplied by the corresponding elements in *elemweight*, prior to pole identification, whereas the unscaled $Y(s)$ is used for the calculation of matrices C, D, and E.

E. Passivity Enforcement

Passivity for Y_{fit} is enforced using the "simplistic approach" described in [14], which has been implemented in a routine `passive.m`. This enforcement is useful to avoid unstable simulations.

IV. STATE EQUATIONS

The output from the matrix fitter are the matrices A, B, C, D, and E, which define a state equation for $Y(s)$ of the form

$$y(s) = Y_{\mathrm{fit}}(s)u(s) = (C(sI - A)^{-1}B + D + sE)u(s). \tag{4}$$

The matrix dimensions are

$$
\begin{array}{lll}
y(s): n \times 1 & u(s): n \times 1 & Y_{\mathrm{fit}}(s): n \times n \\
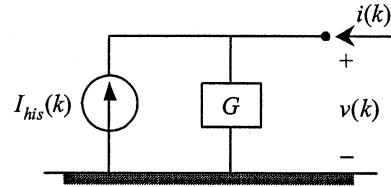C: n \times N & B: N \times n & A: N \times N \text{(diagonal)}
\end{array}
$$

Fig. 1.   Companion network.

The elements of C and A are either real or come in complex conjugate pairs. D and E can be specified as zero.

The structure of the resulting state equations when applying the matrix fitter to $Y(s)$ is shown next. Assuming $Y(s)$ to be of dimension 2×2 and $N = 3$, we get, for matrices A, B, C, D, and E

$$A = \mathrm{diag}(\begin{array}{cccccc} a_1 & a_2 & a_3 & a_1 & a_2 & a_3 \end{array}) \tag{5}$$

$$B^T = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \tag{6}$$

$$C = \begin{bmatrix} c_{111} & c_{112} & c_{113} & c_{121} & c_{122} & c_{123} \\ c_{211} & c_{212} & c_{213} & c_{221} & c_{222} & c_{223} \end{bmatrix} \tag{7}$$

$$D = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \tag{8}$$

$$E = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}. \tag{9}$$

It is noted that the elements in A (poles) are repeated as many times as there are columns in Y.

V. TIME DOMAIN IMPLEMENTATION

The situation is now considered that a component described by a frequency dependent admittance matrix Y is to be included in a time domain simulation host program (EMTP-type program). Voltage now replaces u in (4), and current replaces y.

A. Companion Network

When the source code of the host program is available, it is convenient to include the rational approximation in the form of a companion network (see Fig. 1).

If trapezoidal integration is used with a fixed time step, the following companion network can be formulated which constitutes a conductance matrix in parallel with a vector of current sources.

If Y is assumed to be scalar ($n = 1$), the companion network at time step k can be calculated as follows [13]:

$$x_m(k) = \alpha_m x_m(k-1) + v(k-1), \qquad m = 1 \dots N$$

$$
\begin{aligned}
i(k) &= \sum_{m=1}^{N} \beta_m x_m(k) + Gv(k) \\
&= -I_{his}(k) + Gv(k).
\end{aligned}
\tag{10}
$$

Details on how to calculate the coefficients α, β, and G are given in [13].

For a matrix Y, the following pseudo code can be used for updating the history current source. It follows from the code that when $n \gg 1$, the number of floating point operations for each time step becomes $O(2Nn^2)$.

```
for  col = 1 : n
  for  m = 1 : N
    x(m, col) = α(m) * x(m, col) + V(col)
  end
end
for  col = 1 : n
  Ihis(col) = 0
  for  row = 1 : n
    for  m = 1 : N
        Ihis(col) = Ihis(col) − β(m, row, col) * x(m, col)
    end
  end
end
```

B. Electrical Network

When the source code of the host program is not available, it is convenient to represent the fitting of Y in the form of an electrical network which can be read in by the host program.

The network has branches between all nodes and ground and between all nodes. Each branch is calculated as follows [2]:

Branch between node i and ground:

$$y_i = \sum_{j=1}^{n} Y_{\text{fit}ij}. \tag{11}$$

Branch between node i and node j:

$$y_{ij} = -Y_{\text{fit}ij}. \tag{12}$$

This procedure is straightforward when matrix fitting has been applied: All elements of Y_{fit} have the same poles, and these become the poles for the network branches when calculated by (11) and (12). The Appendix shows how the rational functions for the network branches are converted into basic network elements (R, L, C).

The routine `netgen.m` converts the state equation produced by the matrix fitter (defined by matrices A, B, C, D, and E) into an electrical network that can be imported directly into ATP-EMTP.

VI. COMPUTER CODE

A modularized computational platform for Matrix Fitting has been developed using the following five routines:

- `spavectfit.m`
- `mtrxvectfit.m`
- `passive.m`
- `netgen.m`
- `mainfit.m`

Each routine is a fully documented standalone function that can be called from the Matlab environment. For the purpose of matrix fitting, the routines have been integrated within the routine `mainfit.m`. Fig. 2 shows the hierarchy of the matrix fitting platform and the most important features. Several other capabilities are built in, including plotting of results.

The following gives a brief description of the respective routines.
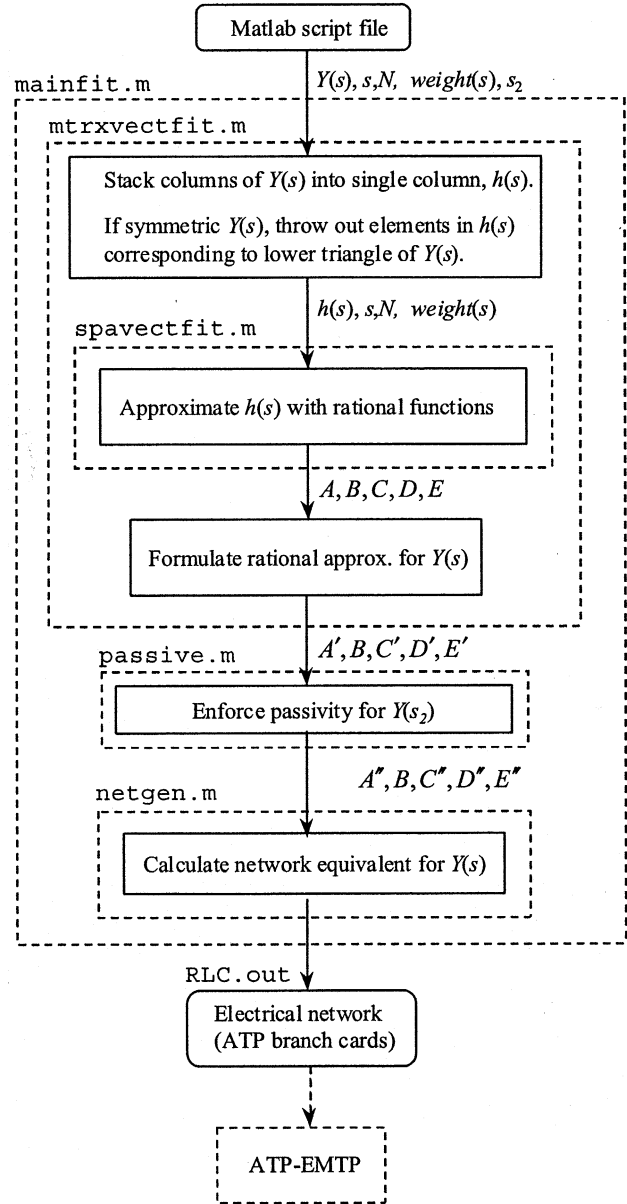
`spavectfit.m`

Fig. 2.   Flowchart of matrix fitting.

This routine is identical to the previous `vectfit.m` developed in [6], except that the matrix problem in [6, (A.7)] is now solved using sparse arithmetic. This greatly reduces the memory requirements and increases the computational efficiency for matrices of big dimensions. All poles produced by the routine are stable. The user can specify E, or D and E, to be zero.

`mtrxvectfit.m`

This routine essentially stacks the columns of $Y(s)$ into a single column, and calls `spavectfit.m`. The symmetry of $Y(s)$ is taken advantage of by fitting only the upper triangle of $Y(s)$, which leads to a shorter column to be fitted by `spavectfit.m`.

`passive.m`

This routine enforces passivity by the Simplistic Approach in [14] at frequency samples s_2 provided in the input. These frequency samples should in general be chosen to cover a wider frequency range than the samples s used in the fitting because
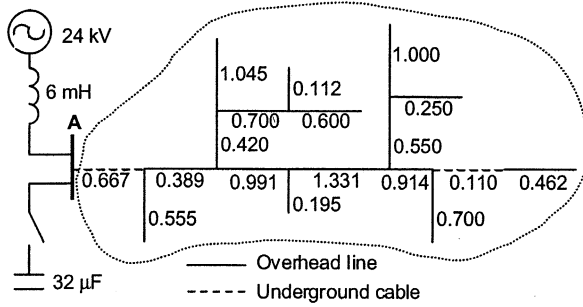
Fig. 3. Electrical three-phase network (all distances in kilometers).

Fig. 4. Magnitude functions.

nonpassivity may occur outside the fitting range. Upon output, modified matrices A, C, D, E are produced if nonpassivity was detected.

netgen.m

This routine takes as input the rational approximation of Y_{fit} as given by matrices A, B, C, D, E and calculates an equivalent network as described in Section V-B and the Appendix. The equivalent network is written to file (RLC.out) as branch cards for ATP. Predefined node names are used so that the file can be imported using the $INCLUDE feature of ATP.

mainfit.m

This routine integrates the above routines into a complete procedure for Matrix Fitting. It takes as input the frequency domain data, does the rational function approximation, enforces passivity, and generates an electrical network which can be included in ATP-EMTP. The user can choose between logarithmically spaced real starting poles, linearly spaced complex starting poles, or the poles can be provided by the user.

VII. CALCULATED RESULTS

A. Electrical System

As an example, a 24-kV rural distribution system will be considered, as shown in Fig. 3. Each line section consists of a flat untransposed overhead line without ground wires, whereas each cable section consists of three single core coaxial cables.

B. Network Equivalent

A frequency-dependent network equivalent (fdne) is to be calculated in the phase domain for the radial network connected to bus A, as indicated by the dotted line in Fig. 3. An exact Pi-equivalent (frequency dependent) was established in the phase domain for each of the line and cable stubs in the range 50 Hz–150 kHz. The total admittance matrix was formed and reduced with respect to bus A, thus giving a 3 by 3 $Y(s)$.

C. Matrix Fitting

A rational approximation for $Y(s)$ was obtained in Matlab by making a call to the routine mainfit.m. 80 linearly spaced complex poles were used, and all of the elements were weighted with the inverse of the sum of the elements of $Y(s)$ (absolute value). This weighting was chosen in order to obtain an approximation with a small relative deviation (see the Closure of [6] for an explanation).
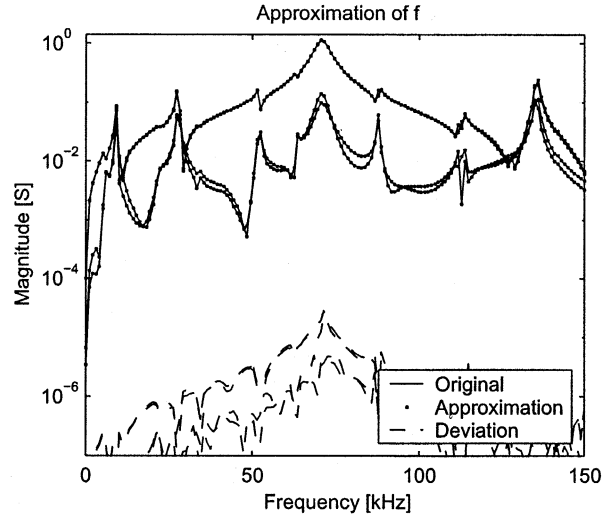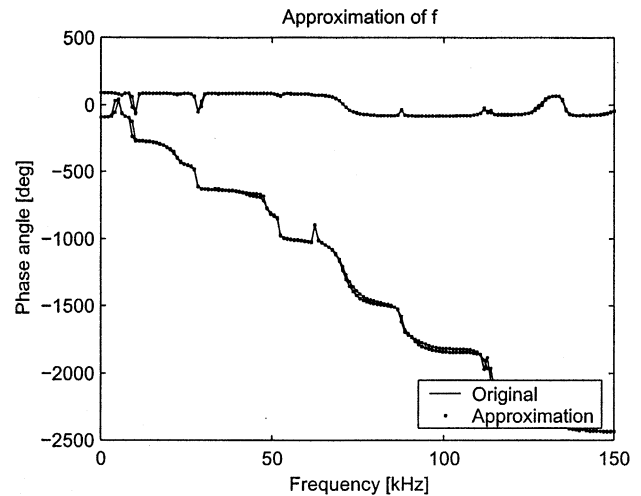

Fig. 5. Phase angles.

Figs. 4–6 show the resulting plots produced by mainfit.m. Fig. 4 shows the magnitude of the nine elements of $Y(s)$ and the magnitude of the complex fitting error, whereas Fig. 5 shows the corresponding phase angles. The accuracy of the approximation is seen to be very good. Fig. 6 shows the three eigenvalues of $G(s_2) = \mathrm{Re}\{Y(s_2)\}$ in the range 0 Hz–400 kHz. In this case, all eigenvalues were positive so that no passivity enforcement was needed.

D. Time Domain Simulation

Next, we calculate the resulting current flowing into one of the outer phases of the network equivalent when energizing the capacitor bank in Fig. 3 at voltage maximum in this phase.

An ATP source file was created for the circuit in Fig. 3 with the network equivalent read in from RLC.out, as produced by mainfit.m. Using ATP, the energization currents were calculated at the terminals of the network equivalent.

In addition, the circuit in Fig. 3 was modeled in EMTDC, where each of the line stubs were modeled explicitly using a phase domain transmission line model [12], [13].
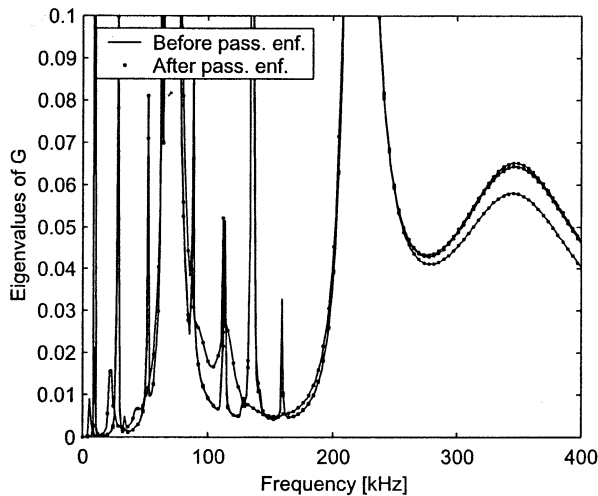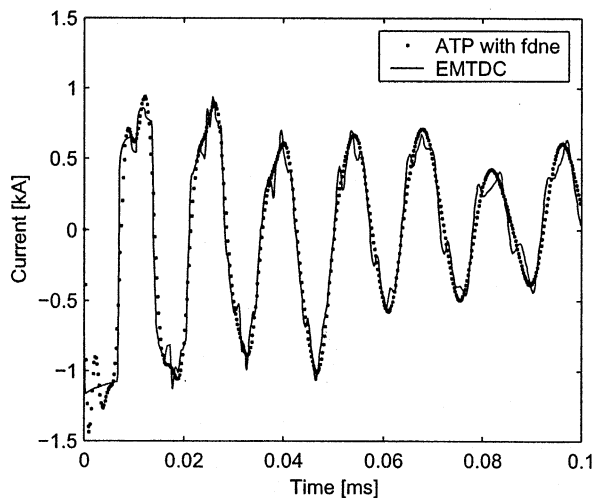
Fig. 6.   Eigenvalues of $G(s)$.
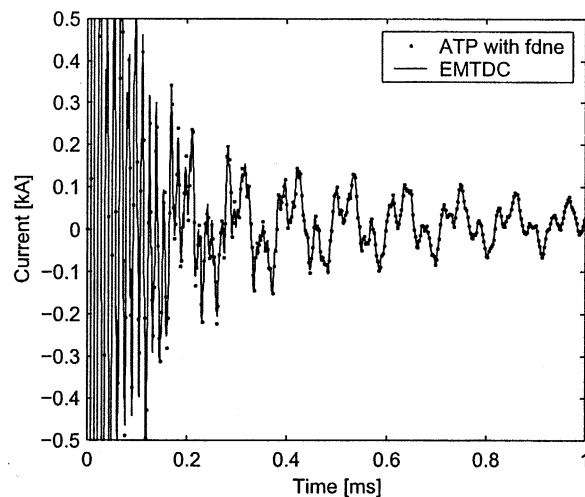

Fig. 7.   Energization current.

Fig. 8.   Energization current (expanded time scale).

Figs. 7 and 8 show the energization current in an outer phase of the cable at bus A, as calculated by ATP using the frequency-dependent network equivalent (fdne), and by EMTDC. The figures show the current in the time range 0–0.1 ms and 0–1 ms, respectively. It is seen that a very good agreement has been
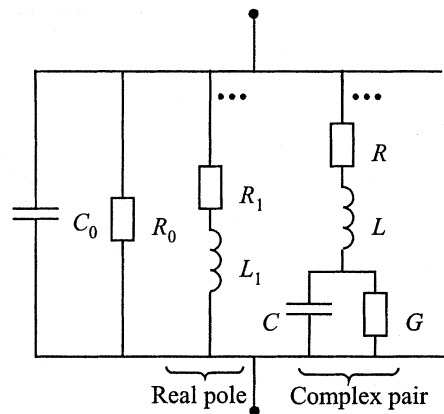

Fig. 9.   Synthesization by electrical network.

achieved. As expected, the very highest frequency components in Fig. 7 are missing in the ATP simulation, as $Y(s)$ was fitted up to only 150 kHz.

VIII. CONCLUSIONS

This paper has presented a public domain calculation package for rational approximation of frequency dependent admittance matrices. The main features are the following.

- All matrix elements are fitted simultaneously with an identical set of stable poles using the vector-fitting approach. Automatic generation of starting poles is available.
- The computational efficiency is high due to the use of sparse arithmetic during pole identification.
- Frequency-dependent weighting is available for controlling the accuracy of the rational approximation.
- Passivity is enforced by the "Simplistic Approach" in [14], which ensures a stable simulation.
- The resulting rational approximation is expressed as *A, B, C, D, E* matrices as used in state equations, and in the form of an equivalent electrical network which is written to file as branch cards for ATP-EMTP.
- User friendliness is achieved by modularization using callable routines in the Martlab environment. Graphics as in Figs. 4–6 are automatically produced.

The validity of the produced rational approximation was verified in Section VII by comparing a time domain simulation with a network equivalent imported in ATP with the result from a fully detailed representation in EMTDC.

APPENDIX

A. Calculation of Electrical Network

Each branch in (11) and (12) is given as a rational function

$$y(s) = \sum_{m=1}^{N} \frac{c_m}{s - a_m} + d + se. \qquad (A.1)$$

The branch can be represented by an electrical network as shown in Fig. 9. R_0 and C_0 are calculated as

$$C_0 = e, \quad R_0 = 1/d. \qquad (A.2)$$

Each real pole gives an RL-branch

$$R_1 = -a/c, \quad L_1 = 1/c. \tag{A.3}$$

Each complex conjugate pair

$$\frac{c' + jc''}{s - (a' + ja'')} + \frac{c' - jc''}{s - (a' - ja'')} \tag{A.4}$$

gives an RLC-branch

$$L = 1/(2c') \tag{A.5}$$

$$R = (-2a' + 2(c'a' + c''a'')L)L \tag{A.6}$$

$$1/C = (a'^2 + a''^2 + 2(c'a' + c''a'')R)L \tag{A.7}$$

$$G = -2(c'a' + c''a'')CL. \tag{A.8}$$

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