Cap2 - Aula02

segunda-feira, 28 de agosto de 2023 13:33

Vimos: $\hat{\beta}_1 = \sum c_i y_i \qquad c_i = \dfrac{x_i}{\sum x_i^2} \qquad x_i = X_i - \bar{X}$

$$\hat{\beta}_0 = \sum d_i y_i \qquad d_i = \left(\frac{1}{n} - c_i \bar{X}\right)$$

$$\sum c_i = 0 \qquad \sum c_i X_i = 1$$

$$\sum d_i = 1 \qquad \sum d_i X_i = 0$$

ESTIMADOR NÃO VIESADO $E(\hat{\theta}) = \theta$

1) $E(\hat{\beta}_0) = \beta_0$ e $E(\hat{\beta}_1) = \beta_1$

$$\hat{\beta}_1 = \sum c_i y_i$$

$$E(\hat{\beta}_1) = E\left(\sum c_i y_i\right) = \sum E(c_i y_i) = \sum \left[c_i E(y_i)\right] =$$

$$= \sum c_i (\beta_0 + \beta_1 X_i) = \beta_0 \underbrace{\sum c_i}_{0} + \beta_1 \underbrace{\sum c_i X_i}_{1} = \boxed{\beta_1}$$

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{X}$$

$$E(\hat{\beta}_0) = E(\bar{y} - \hat{\beta}_1 \bar{X}) = E(\bar{y}) - E(\hat{\beta}_1 \bar{X}) = E\left(\frac{\sum y_i}{n}\right) - \bar{X} \overbrace{E(\hat{\beta}_1)}^{\beta_1}$$

$$= \frac{1}{n} \sum E(y_i) - \bar{X}\beta_1$$

$$= \frac{1}{n} \sum (\beta_0 + \beta_1 X_i) - \bar{X}\beta_1$$

$$= \frac{1}{n} \left(n\beta_0 + \beta_1 \sum X_i\right) - \bar{X}\beta_1$$

$$= \beta_0 + \cancel{\beta_1 \bar{X}} - \cancel{\bar{X}\beta_1} = \boxed{\beta_0}$$

---

1) $Var(\hat{\beta}_1) = ?$ $\qquad \hat{\beta}_1 = \sum c_i y_i$ e y_i's independ.

1) $Var(\hat{\beta}_1) = ?$ $\quad \hat{\beta}_1 = \sum c_i Y_i$ e Y_i's independ.

$$Var\left(\sum c_i y_i\right) = Var(c_1 Y_1 + \dots + c_n Y_n)$$

$Var(KY) = K^2 Var(Y)$

$$= Var(c_1 Y_1) + \dots + Var(c_n Y_n)$$
$$= c_1^2 Var(Y_1) + \dots + c_n^2 Var(Y_n)$$
$$= \sum c_i^2 \underbrace{Var(Y_i)}_{\sigma^2} = \sigma^2 \sum c_i^2 = \sigma^2 \sum \left(\frac{x_i}{\sum x_i^2}\right)^2$$
$$= \sigma^2 \sum \frac{x_i^2}{(\sum x_i^2)^2} = \sigma^2 \frac{\cancel{\sum x_i^2}}{(\sum x_i^2)^{\cancel{2}}} = \boxed{\frac{\sigma^2}{\sum x_i^2}}$$

$Var(\hat{\beta}_0) = ?$ $\quad \hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X}$ $\quad \hat{\beta}_1 = \sum c_i Y_i$

$$Var(\bar{Y} - \hat{\beta}_1 \bar{X}) = Var\left(\frac{\sum Y_i}{n} - \sum c_i Y_i \bar{X}\right) = Var\left(\frac{\sum Y_i - n\sum c_i Y_i \bar{X}}{n}\right)$$
$$= Var\left(\frac{\sum Y_i (1 - n c_i \bar{X})}{n}\right) = Var\left(\sum Y_i . \frac{(1 - n c_i \bar{X})}{n}\right)$$
$$= \sum Var\left(Y_i . \frac{(1 - n c_i \bar{X})}{n}\right) = \sum \left(\frac{1}{n} - c_i \bar{X}\right)^2 . Var(Y_i)$$
$$= \sigma^2 \sum \left(\frac{1}{n} - \bar{X} c_i\right)^2 = \sigma^2 \sum \left\{\frac{1}{n^2} - \frac{2\bar{X} c_i}{n} + \bar{X}^2 c_i^2\right\}$$

$\to n/n^2$

$$= \sigma^2 \left[\frac{1}{n} - \frac{2\bar{X}\cancel{\sum c_i}^{\,0}}{n} + \bar{X}^2 \sum c_i^2\right\} =$$

$\sum c_i^2 = \frac{1}{\sum x_i^2}$

$$= \sigma^2 \left\{\frac{1}{n} + \frac{\bar{X}^2}{\sum x_i^2}\right\}$$

3) $Cov(\hat{\beta}_0, \hat{\beta}_1) = Cov(\bar{Y} - \hat{\beta}_1 \bar{X}, \hat{\beta}_1) = Cov(\bar{Y}, \hat{\beta}_1) + cov(-\hat{\beta}_1 \bar{X}, \hat{\beta}_1)$

① ②

① $Cov(\bar{Y}, \hat{\beta}_1) = Cov\left(\frac{\sum Y_i}{n}, \sum c_i Y_i\right) = \frac{1}{n} cov\left(\sum Y_i, \sum c_i Y_i\right)$

$$= \frac{1}{n} \operatorname{Cov}\left(Y_1 + \dots + Y_n, c_1 Y_1 + \dots + c_n Y_n\right)$$

$$\operatorname{Cov}(Y_1 + Y_2, c_1 Y_1 + c_2 Y_2)$$
$$= \operatorname{Cov}(Y_1, c_1 Y_1) + \operatorname{Cov}(Y_1, \cancel{c_2 Y_2}) + \operatorname{Cov}(\cancel{Y_2, c_1 Y_1}) + \operatorname{Cov}(Y_2, c_2 Y_2)$$
$$= c_1 \operatorname{Cov}(Y_1, Y_1) + c_2 \operatorname{Cov}(Y_2, Y_2)$$

$$= \frac{1}{n} \sum \operatorname{Cov}(Y_i, c_i Y_i)$$

$$= \frac{1}{n} \sum c_i \underbrace{\operatorname{Cov}(Y_i, Y_i)}_{\operatorname{Var}(Y_i)}$$

$$= \frac{1}{n} \sum c_i \cdot \sigma^2 = \frac{\sigma^2}{n} \cancelto{0}{\sum c_i} = 0$$

② $$\operatorname{Cov}(-\hat{\beta}_1 \bar{X}, \hat{\beta}_1) = -\bar{X} \operatorname{Cov}(\hat{\beta}_1, \hat{\beta}_1) = -\bar{X} \operatorname{Var}(\hat{\beta}_1)$$

$$= \frac{-\bar{X}\sigma^2}{\sum x_i^2} = \operatorname{Cov}(\hat{\beta}_0, \hat{\beta}_1)$$

4) $\operatorname{Var}(\hat{Y}_i) = ?$ $\quad \hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_i \quad \operatorname{Var}(X+Y) = V(X) + V(Y) + 2\operatorname{Cov}(X, Y)$

$$\operatorname{Var}(\hat{\beta}_0 + \hat{\beta}_1 X_i) = \operatorname{Var}(\hat{\beta}_0) + \operatorname{Var}(\hat{\beta}_1 X_i) + 2\operatorname{Cov}(\hat{\beta}_0, \hat{\beta}_1 X_i)$$

$$= \left(\frac{1}{n} + \frac{\bar{X}^2}{\sum x_i^2}\right)\sigma^2 + X_i^2 \frac{\sigma^2}{\sum x_i^2} + 2X_i\left(-\frac{\bar{X}\sigma^2}{\sum x_i^2}\right)$$

$$= \sigma^2 \left[\frac{1}{n} + \frac{\bar{X}^2}{\sum x_i^2} + \frac{X_i^2}{\sum x_i^2} - \frac{\bar{X}\, 2X_i}{\sum x_i^2}\right] \longrightarrow (X_i - \bar{X})^2 = X_i^2 - 2X_i\bar{X} + \bar{X}^2$$

$$= \sigma^2 \left[\frac{1}{n} + \frac{1}{\sum x_i^2} \underbrace{(X_i - \bar{X})^2}_{x_i}\right] = \sigma^2 \left[\frac{1}{n} + \frac{x_i^2}{\sum x_i^2}\right]$$

Teorema de Gauss $\quad Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i \quad$ X e Y linear, Erros homog, independ.

Considere o **Modelo I** estabelecido e suas pressuposições. Sejam $\hat{\beta}_0$ e $\hat{\beta}_1$ os estimadores não viesados de mínimos quadrados de β_0 e β_1 e $\tau = a_1\beta_0 + a_2\beta_1$ uma combinação linear de β_0 e β_1. Então, dentre todos os estimadores imparciais de τ ①, lineares em Y ②, o estimador

$$\hat{\tau} = a_1\hat{\beta}_0 + a_2\hat{\beta}_1$$

de β_0 e β_1. Então, dentre todos os estimadores imparciais de τ, lineares em Y, o estimador

1 2

$$\hat{\tau} = a_1\hat{\beta}_0 + a_2\hat{\beta}_1$$

3

tem variância mínima, isto é, se $T = \sum_{i=1}^{n} l_i Y_i$, em que l_i são constantes arbitrárias e $E(T) = \tau$, então,

4

$$\text{Var}(\hat{\tau}) \leq \text{Var}(T).$$

5

$$\tau = a_1\beta_0 + a_2\beta_1$$

$\hat{\tau} = a_1\hat{\beta}_0 + a_2\hat{\beta}_1 \Rightarrow Var(\hat{\tau})$

EST: não viesados de τ e lineares em y_i

$T = \underbrace{\sum l_i y_i} \Rightarrow Var(T)$

$Var(\hat{\tau}) \leq Var(T)$

1 ESTIMADORES NÃO VIESADO

$$E(\hat{\tau}) = \tau \Rightarrow E(a_1\hat{\beta}_0 + a_2\hat{\beta}_1) = a_1 E(\hat{\beta}_0) + a_2 E(\hat{\beta}_1)$$
$$= a_1\beta_0 + a_2\beta_1 = \tau //$$

$$E(T) = \tau \Rightarrow E(\sum l_i y_i) = \sum E(l_i y_i) = \sum l_i E(y_i) =$$
$$= \sum l_i(\beta_0 + \beta_1 x_i) = \sum l_i \beta_0 + \sum l_i \beta_1 x_i$$
$$= \beta_0 \underbrace{\sum l_i}_{a_1} + \beta_1 \underbrace{\sum l_i x_i}_{a_2}$$
$$= a_1\beta_0 + a_2\beta_1 = \tau //$$

2 $\hat{\tau} = a_1\hat{\beta}_0 + a_2\hat{\beta}_1$ em que: $\hat{\beta}_0 = \sum d_i y_i$ e $\hat{\beta}_1 = \sum c_i y_i$

Então: $\hat{\tau} = a_1 \sum d_i y_i + a_2 \sum c_i y_i$

$$= \sum a_1 d_i y_i + \sum a_2 c_i y_i$$
$$= \sum y_i \underbrace{(a_1 d_i + a_2 c_i)}_{k_i}$$
$$= \sum k_i y_i \Rightarrow \text{comb linear } y_i$$

Por imposição $T = \sum l_i y_i$

$V(ky) = k^2 V(y)$

3 $Var(\hat{\tau}) = var(a_1\hat{\beta}_0 + a_2\hat{\beta}_1)$

$$= Var(a_1\hat{\beta}_0) + var(a_2\hat{\beta}_1) + 2cov(a_1\hat{\beta}_0, a_2\hat{\beta}_1)$$

$$= Var(a_1\hat{\beta}_0) + Var(a_2\hat{\beta}_1) + 2Cov(a_1\hat{\beta}_0, a_2\hat{\beta}_1)$$

$$= a_1^2 Var(\hat{\beta}_0) + a_2^2 Var(\hat{\beta}_1) + 2a_1a_2 Cov(\hat{\beta}_0, \hat{\beta}_1)$$

$$= a_1^2\left[\sigma^2\left(\frac{1}{n} + \frac{\bar{X}^2}{\Sigma x_i^2}\right)\right] + a_2^2\left[\frac{\sigma^2}{\Sigma x_i^2}\right] + 2a_1a_2\left[-\frac{\bar{X}\sigma^2}{\Sigma x_i^2}\right]$$

$$= \sigma^2\left[\frac{a_1^2}{n} + \frac{a_1^2\bar{X}^2}{\Sigma x_i^2} + \frac{a_2^2}{\Sigma x_i^2} - \frac{2\bar{X}a_1a_2}{\Sigma x_i^2}\right]$$

$$= \sigma^2\left[\frac{a_1^2}{n} + \frac{1}{\Sigma x_i^2}(a_1\bar{X} - a_2)^2\right]$$

$$Var(T) = Var(\Sigma l_i Y_i) = \Sigma l_i^2 Var(Y_i)$$

$$= \sigma^2 \Sigma l_i^2$$

$Cov(T, \hat{\tau}) \overset{?}{=}$ $= Var(\hat{\tau})$

$$Cov(\Sigma l_i Y_i, \Sigma k_i Y_i) = \Sigma\left[l_i k_i \overbrace{Cov(Y_i, Y_i)}^{Var(Y_i)}\right] = \Sigma l_i k_i \sigma^2 = \sigma^2 \Sigma l_i k_i$$

$\Rightarrow k_i = a_1 d_i + a_2 c_i$; $d_i = \frac{1}{n} - c_i\bar{X}$

$$k_i = a_1\left(\frac{1}{n} - c_i\bar{X}\right) + a_2 c_i$$

$$Cov(T, \hat{\tau}) = \Sigma l_i k_i \sigma^2$$

$$= \Sigma l_i\left[a_1\left(\frac{1}{n} - c_i\bar{X}\right) + a_2 c_i\right]\sigma^2$$

$$= \Sigma l_i\left[\frac{a_1}{n} - a_1\bar{X}c_i + a_2 c_i\right]\sigma^2$$

$$= \Sigma l_i\left[\frac{a_1}{n} + (a_2 - a_1\bar{X})c_i\right]\sigma^2$$

$$= \left[\frac{a_1}{n}.\Sigma l_i + (a_2 - a_1\bar{X})\Sigma l_i c_i\right]\sigma^2$$

$* \; c_i = \frac{x_i}{\Sigma x_i^2}$

$a_1 = \Sigma l_i$

$a_2 = \Sigma l_i X_i$

$$= \left[\frac{a_1}{n}.a_1 + (a_2 - a_1\bar{X})\Sigma l_i . \frac{x_i}{\Sigma x_i^2}\right]\sigma^2$$

$a_2 = \sum l_i X_i$

$$= \left[\frac{a_1}{n}\cdot a_1 + (a_2 - a_1\bar{X})\sum l_i \cdot \frac{x_i}{\sum x_i^2}\right]\sigma^2$$

$$= \left[\frac{a_1^2}{n} + (a_2 - a_1\bar{X})\,\frac{\sum l_i (X_i - \bar{X})}{\sum x_i^2}\right]\sigma^2$$

$$= \left[\frac{a_1^2}{n} + (a_2 - a_1\bar{X})\cdot\frac{\overbrace{\sum l_i X_i}^{a_2} - \bar{X}\overbrace{\sum l_i}^{a_1}}{\sum x_i^2}\right]\sigma^2$$

$$= \underbrace{\left[\frac{a_1^2}{n} + \frac{(a_2 - a_1\bar{X})^2}{\sum x_i^2}\right]\sigma^2}_{Var(\hat{\tau})}$$

(5) $Var(\hat{\tau}) \leq Var(T)$

$var(X+Y) = V(X) + V(Y) + 2C(X,Y)$

$var(X-Y) = V(X) + V(Y) - 2cov(X,Y)$

$$Var(T - \hat{\tau}) \geq 0$$

$$Var(T) + Var(\hat{\tau}) - 2\,\overbrace{cov(T,\hat{\tau})}^{Var(\hat{\tau})} \geq 0$$

$$Var(T) + Var(\hat{\tau}) - 2\,Var(\hat{\tau}) \geq 0$$

$$Var(T) - Var(\hat{\tau}) \geq 0$$

$$\underline{Var(T) \geq Var(\hat{\tau})}$$

$$\hat{\tau} = a_1\hat{\beta}_0 + a_2\hat{\beta}_1$$

$$Var(\hat{\tau}) = \sigma^2\left[\frac{a_1^2}{n} + \frac{1}{\sum x_i^2}(a_1\bar{X} - a_2)^2\right]$$

→ Se $a_1 = 1$ e $a_2 = 0$ então

$$\hat{\tau} = \hat{\beta}_0 \qquad Var(\hat{\tau}) = \sigma^2\left[\frac{1}{n} + \frac{\bar{X}^2}{\sum x_i^2}\right]$$

→ Se $a_1 = 0$ e $a_2 = 1$ então

$$\hat{T} = \hat{B}_1 \qquad Var(\hat{\tau}) = \sigma^2\left[\frac{1}{\sum x_i^2}\right] = \frac{\sigma^2}{\sum x_i^2}$$

→ Se a_1 ...

$$\hat{\tau} = \hat{\beta}_1 \qquad Var(\hat{\tau}) = \sigma^2 \left[\frac{1}{\sum x_i^2} \right] = \frac{\sigma^2}{\sum x_i^2}$$

→ Se $a_1 = 1$ e $a_2 = X_i$

$$\hat{\tau} = \hat{\beta}_0 + \hat{\beta}_1 X_i = \hat{Y}_i$$

$$Var(\hat{\tau}) = Var(\hat{Y}_i) = \sigma^2 \left[\frac{1}{n} + \frac{(\bar{x} - X_i)^2}{\sum x_i^2} \right]$$

$$= \sigma^2 \left[\frac{1}{n} + \frac{x_i^2}{\sum x_i^2} \right]$$