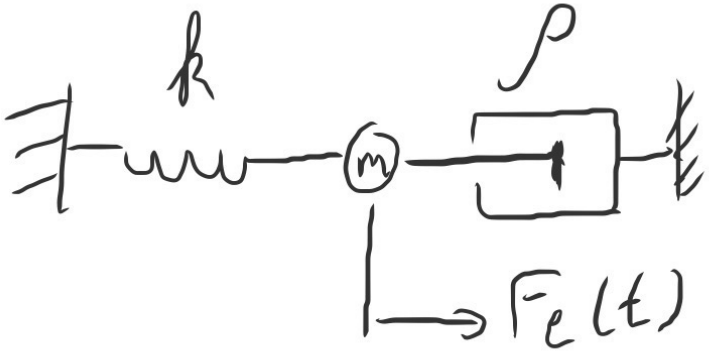
Oscilador Harmônico Amortecido Forçado



$$F = m\ddot{x}(t) = -kx - \rho\dot{x} + F_e(t)$$

en. potencial restauro

Força dissipativa

$$F_e(t) = F_0 \cos(\omega t)$$

E.D.2G.

não-homogênea

$$\boxed{\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos(\omega t)}$$

$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = 0$; $\gamma = \frac{b}{m}$; $\omega_0^2 = \frac{k}{m}$; $\Rightarrow x_1(t)$

$x_2(t)$

$x(t) = x_1 + x_2$ é solução

$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = \frac{F(t)}{m} \Rightarrow x_1 + x_2$ não resolve!

Palpite $F_1(t) + F_2(t) = F(t)$

Resolver $x_1(t) \Rightarrow \ddot{x}_1 + \gamma \dot{x}_1 + \omega_0^2 x_1 = \frac{F_1(t)}{m}$

$+ \; x_2(t) \Rightarrow \ddot{x}_2 + \gamma \dot{x}_2 + \omega_0^2 x_2 = \frac{F_2(t)}{m}$

$x(t) = x_1 + x_2 \Rightarrow \ddot{x} + \gamma \dot{x} + \omega_0^2 x = \frac{F_1 + F_2}{m}$

$F_1 = 0 \Rightarrow$ $x(t)$ é parte da solução (homogênea)

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = \frac{F_0}{m} \cos(\omega t) \Rightarrow$$

$$x(t) = \mathrm{Re}[z(t)] \;; \; z(t) = z_0 e^{i\omega t}$$
$$= A e^{i\varphi} e^{i\omega t}$$

Ansatz

$$\ddot{z} + \gamma \dot{z} + \omega_0^2 z = \frac{F_0}{m} e^{i\omega t} \Rightarrow \mathrm{Re}[\ddot{z} + \gamma \dot{z} + \omega_0^2 z] = \mathrm{Re}\left[\frac{F_0}{m} e^{i\omega t}\right]$$

$$\left((-i\omega)^2 A e^{i\varphi} + i\omega \gamma A e^{i\varphi} + \omega_0^2 A e^{i\varphi}\right) e^{i\omega t} = \frac{F_0}{m} e^{i\omega t}$$

$$\boxed{A e^{i\varphi} = \frac{F_0/m}{(\omega_0^2 - \omega^2) + i\omega\gamma}}$$

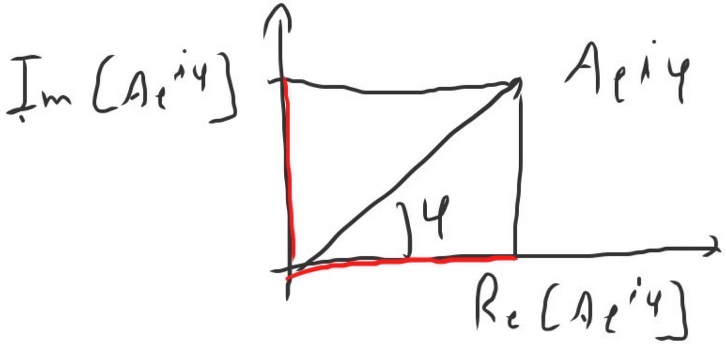
$$|A(\omega)|^2 = A(\omega)\cdot A^*(\omega) = \frac{(F_0/m)^2}{\left[(\omega_0^2-\omega^2)+i\omega\gamma\right]\cdot\left[(\omega_0^2-\omega^2)-i\omega\gamma\right]}$$

$$|A(\omega)|^2 = \frac{(F_0/m)^2}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2};$$

$$\varphi \to e^{i\varphi} = \frac{A}{|A|} = \frac{F_0/m}{\omega_0^2-\omega^2+i\omega\gamma}\cdot\frac{\sqrt{(\omega_0^2-\omega^2+i\omega\gamma)(\omega_0^2-\omega^2-i\omega\gamma)}}{F_0/m}$$

$$e^{i\varphi} = \sqrt{\frac{\omega_0^2-\omega^2-i\omega\gamma}{\omega_0^2-\omega^2+i\omega\gamma}}$$

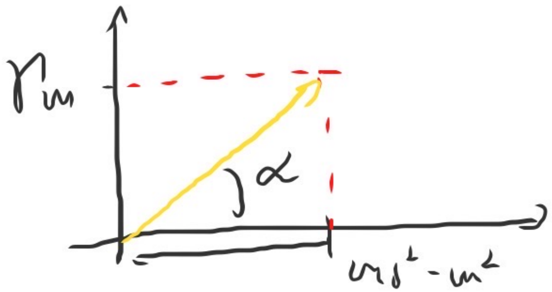
$$\mathrm{tg}\,\varphi = \frac{\gamma m}{m^2 - m_0^2}$$



$$\varphi = -\mathrm{atan}\left(\frac{\gamma m}{m_0^2 - m^2}\right)$$

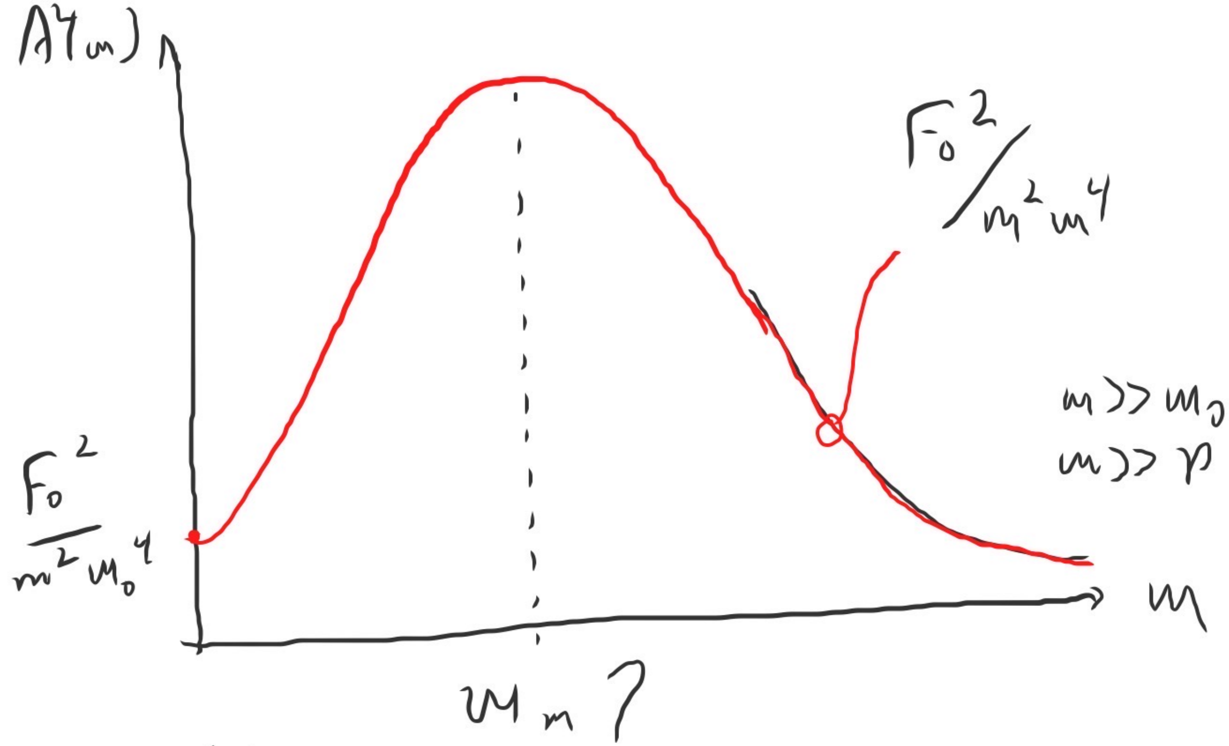
$$e^{i\varphi} = \left[\frac{(m_0^2 - m^2) - i\gamma m}{(m_0^2 - m^2) + i\gamma m}\right]^{1/2} = \left[\frac{\cancel{c}\,e^{-i\alpha}}{\cancel{c}\,e^{i\alpha}}\right]^{1/2} = \left[e^{-i2\alpha}\right]^{1/2} = e^{-i\alpha}$$

$$\alpha = -\varphi$$



$$\mathrm{tg}\,\alpha = \frac{\gamma m}{m_0^2 - m^2}$$

$$\boxed{\mathrm{tg}\,\varphi = \frac{\gamma m}{m^2 - m_0^2}}$$



$$A^2(\omega) = \frac{F_0^2/m^2}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}$$

$$\varphi(\omega) = -tg^{-1}\left(\frac{\gamma\omega}{\omega_0^2-\omega^2}\right)$$

$$\boxed{\frac{d}{d\omega}\left[A^2(\omega)\right] = 0} \Rightarrow \underline{\omega_m}$$

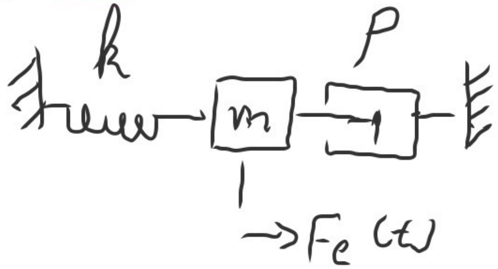
Ex. 4.10

ω_0 ?

$\omega_R \to$ freq natural do osc. harmônico amortecido

$$\omega_R^2 = \omega_0^2 - \left(\frac{\gamma}{2}\right)^2 \ ?$$

Osc. Harmônico Amortecido Forçado



$$m\ddot{x} + \rho\dot{x} + kx = F_e$$

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos(\omega t)$$

Homogênea $\to$ $x_h(t) = \left[A\cos(\omega_n t) + B\,\mathrm{sen}(\omega_n t)\right] e^{-\frac{\gamma}{2}t}$

Freq. Natural $\omega_n^2 = \omega_0^2 - \left(\frac{\gamma}{2}\right)^2$

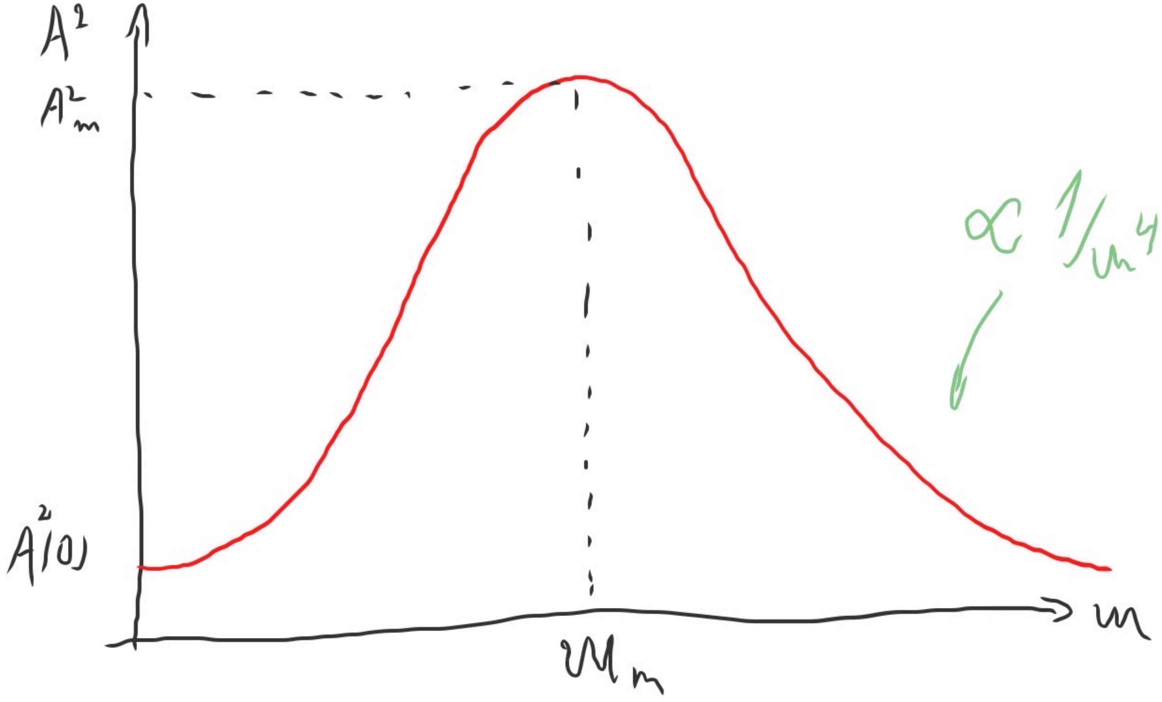
Inomogênea $\to$ $x_i(t) = A(\omega)\cos[\omega t + \varphi(\omega)] = \mathrm{Re}\left[A e^{i\varphi} e^{i\omega t}\right]$

$$x_i(t) = A\cos(\omega t + \varphi)$$

$$A^2 = \frac{F_0^2/m^2}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}$$

$$\varphi = \operatorname{atg}\left(\frac{\gamma\omega}{\omega^2 - \omega_0^2}\right)$$

$$\operatorname{tg}^{-1}$$

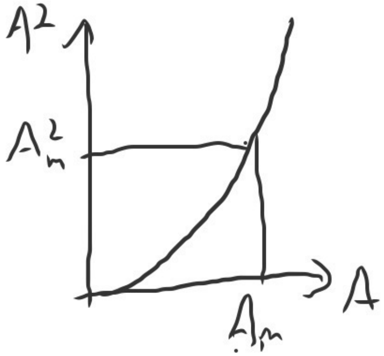


1) $\omega_m = \omega_0$ $\quad \omega_0 = \sqrt{k/m}$

2) $\omega_m = \sqrt{\omega_0^2 - \left(\frac{\gamma}{2}\right)^2} = \omega_m$

3) N.D.A.

$$A^2(\omega) \to \text{máximo}$$



$$\frac{d\, A^2(\omega)}{d\omega} = 0$$

$$A^2 = \frac{F_0^2/m^2}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2} = \frac{F_0^2/m^2}{(\omega_0^2-K)^2+\gamma^2 K}$$

$$\omega^2 = K$$

$$\frac{d}{dK}A^2 = -\frac{F_0^2/m^2}{[(\omega_0^2-K)^2+\gamma^2K]^2}\cdot(-2(\omega_0^2-K)+\gamma^2) = 0$$

$$\frac{d\,A^2(\omega)}{dK}\cdot\frac{dK}{d\omega} = 0$$

$$K = \omega_0^2 - \frac{\gamma^2}{2}$$

$$\frac{dK}{d\omega} = 2\omega$$

$$\Rightarrow -2\omega_0^2 + K + \gamma^2 = 0 \Rightarrow$$

$$\frac{d^2}{d\omega} A(\omega) = 0 \Rightarrow \begin{array}{l} \omega = 0 \\ \omega = \sqrt{\omega_0^2 - \frac{\gamma^2}{2}} \end{array}$$

$$\omega_m < \omega_n < \omega_0$$

$$\sqrt{\omega_0^2 - \frac{\gamma^2}{2}} < \sqrt{\omega_0^2 - \frac{\gamma^2}{4}} < \omega_0$$

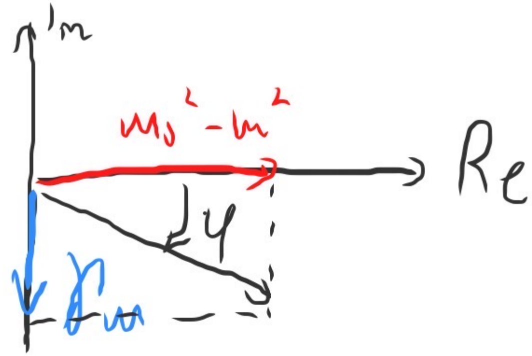
$$A(\omega_m) = \frac{F_0^2/m^2}{\left(\cancel{\omega_0^2} - \cancel{\omega_0^2} + \frac{\gamma^2}{2}\right)^2 + \gamma^2 \cdot \omega_0^2 - \frac{\gamma^2\gamma^2}{2}} = \frac{F_0^2/m^2}{\gamma^2\left(\omega_0^2 - \frac{\gamma^2}{4}\right)}$$

$$A(\omega_m) = \frac{F_0^2}{m^2 \gamma^2 \omega_n^2}$$

$$\varphi = -\tan^{-1}\frac{\gamma\omega}{(\omega_0^2-\omega^2)}$$

fase

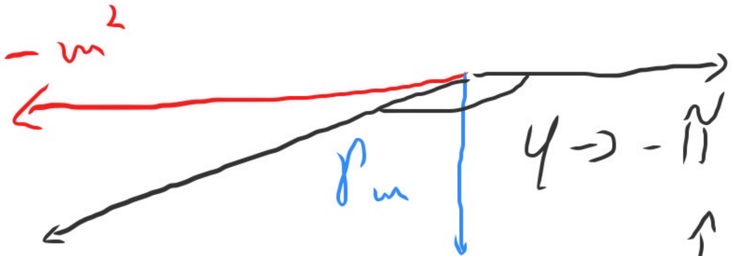
$$Ae^{i\varphi} = \frac{F_0/m}{\omega_0^2-\omega^2+i\gamma\omega}\cdot\frac{\omega_0^2-\omega^2-i\gamma\omega}{\omega_0^2-\omega^2-i\gamma\omega}$$



$\omega = 0$

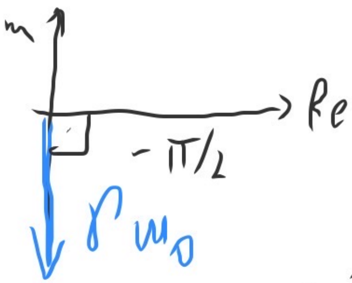
$\varphi = 0$

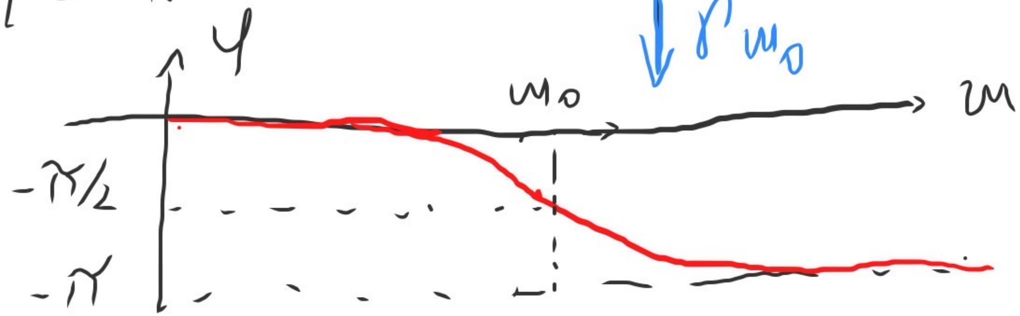
$\omega \gg \omega_0$



$\varphi \to -\pi$

$\omega = \omega_0$





Limite de perdas pequenas $\quad \gamma \ll \omega_0$

$(\omega_0^2 - \omega^2)^2 \longrightarrow$ aproximação?

$|\omega_0 - \omega| \ll \omega_0$

$$(\omega_0^2 - \omega^2) = \underbrace{(\omega_0 + \omega)}(\omega_0 - \omega)$$

$$\simeq 2\omega_0 \, (\omega_0 - \omega) \simeq 2\omega \, (\omega_0 - \omega)$$

$$A^2 \cong \frac{F_0^2/m^2}{[2\,\omega_0(\omega_0 - \omega)]^2 + \gamma^2 \omega_0^2} = \frac{F_0^2/m^2}{4\omega_0^2\left[(\omega_0 - \omega)^2 + \frac{\gamma^2}{4}\right]}$$

$\omega = \omega_0$

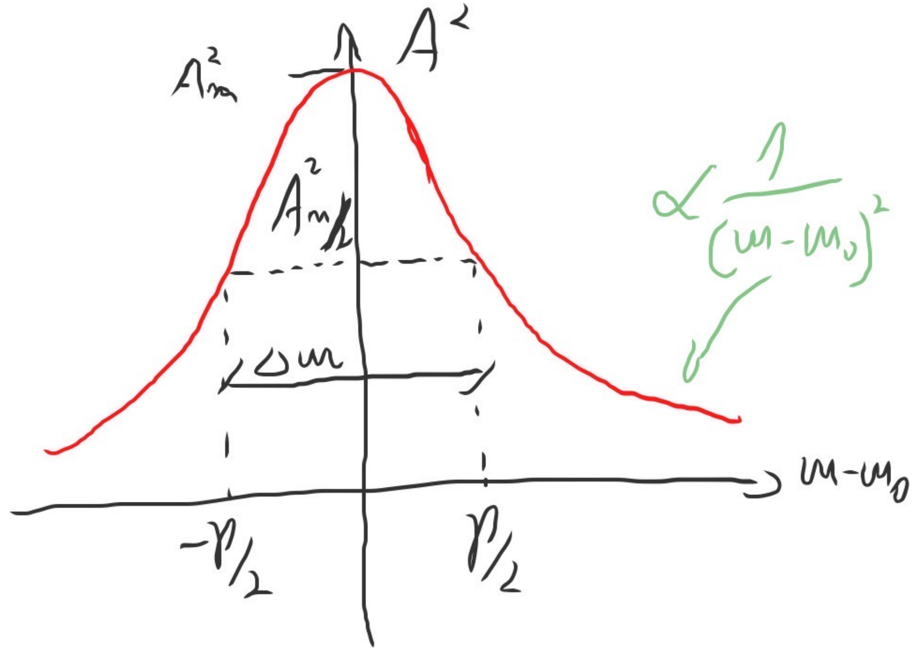
$$A^2 = \left(\frac{F_0}{2m\omega_0}\right)^2 \left[\frac{1}{(\omega_0 - \omega)^2 + \frac{\gamma^2}{4}}\right]$$

$$A^2 = \left(\frac{F_0}{2 m \omega_0}\right)^2 \frac{1}{(\omega_0-\omega)^2 + \frac{\gamma^2}{4}}$$

↳ Breit-Wigner

Lorentziana

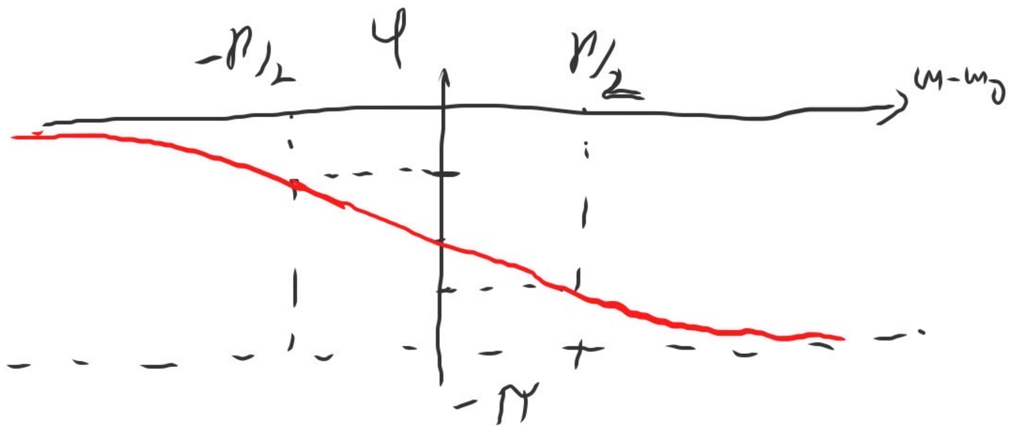
Sistemas com fraca dissipação



$$A_m^2 = \left(\frac{F_0}{m \gamma \omega_0}\right)^2$$

$|\omega-\omega_0| \ll \omega_0$

↳ ressonância



$$\varphi = -\tan^{-1}\left[\frac{\gamma\omega}{(\omega_0^2-\omega^2)}\right] = -\tan^{-1}\left(\frac{\gamma/2}{\omega_0-\omega}\right)$$

Efeito da Ressonância: "Amplificação" da resposta

$$\frac{A(\omega_0)}{A(\omega=0)} = \frac{F_0}{m\gamma\omega_0} \cdot \frac{m\omega_0^2}{F_0} = \frac{\omega_0}{\gamma} = Q \rightarrow \text{Fator de qualidade}$$

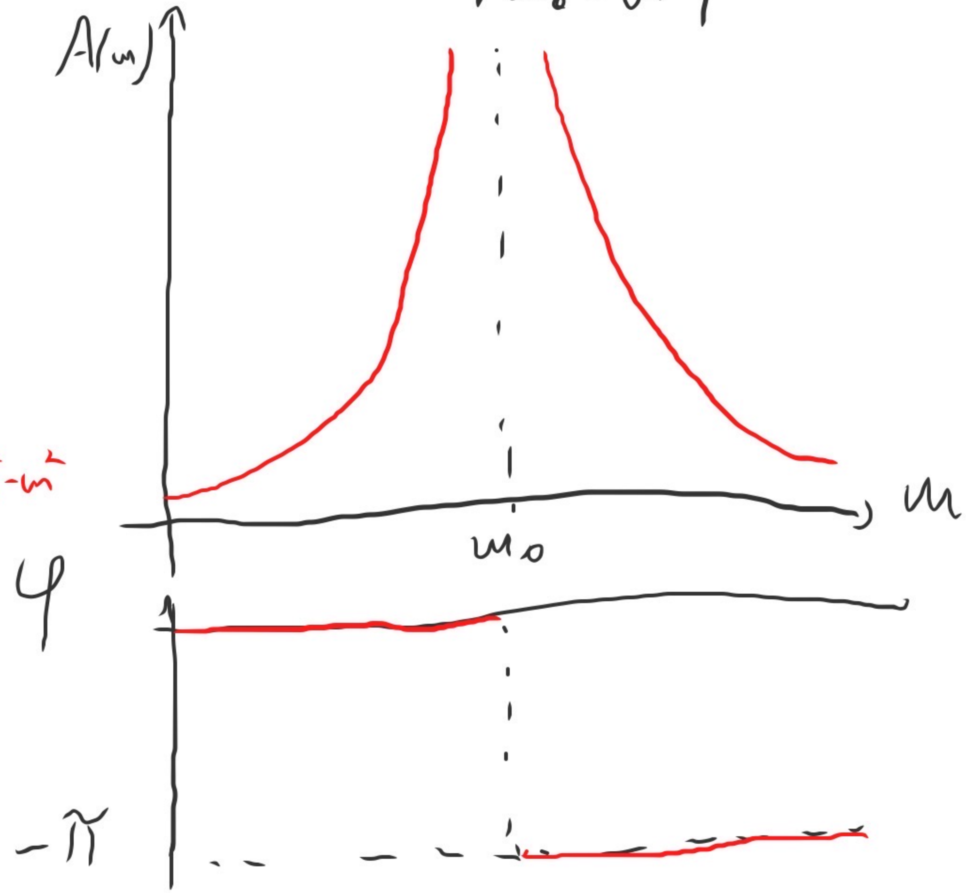
$$Q = 2\pi \cdot \frac{T_d}{T} \rightarrow \text{osc. harmônico amortecido}$$

$$Q = \frac{A(\omega_0)}{A(0)} \rightarrow \text{amplificação da resposta (osc. h. a. forçado)}$$

$$\gamma = \frac{1}{T_d} \rightarrow \Delta\omega \rightarrow \text{largura de frequência / banda}$$

Caso Particular: Perdas Nulas $\Rightarrow \Gamma = 0$

$$A^2(\omega) = \left(\frac{F_0}{m}\right)^2 \frac{1}{(\omega_0^2-\omega^2)^2} \Rightarrow A(\omega) = \frac{F_0}{m} \frac{1}{|\omega_0^2-\omega^2|}$$



$$\operatorname{tg} \varphi = \frac{\Gamma\omega}{\omega_0^2-\omega^2} \begin{cases} \to -\infty & \omega < \omega_0 \\ \to +\infty & \omega = 0 \end{cases}$$

$\omega < \omega_0 \to \varphi = 0$

$\Gamma\omega$ $\quad \omega_0^2-\omega^2$

$\omega > \omega_0 \to \varphi = -\pi$

$\omega_0^2-\omega^2$ $\quad \Gamma\omega$

Resposta do sistema → fase : $m\ddot{x} + kx = F_e$

$$-\omega^2 x + \omega_0^2 x = \frac{F_e}{m}$$

$\omega < \omega_0$ → Dominância da Força de restouro

$$\omega_0^2 x = \frac{F_e}{m} \qquad x(t) \simeq \frac{F_0 \cos(\omega t)}{m\omega_0}$$

$\omega > \omega_0$ → Dominância da inércia

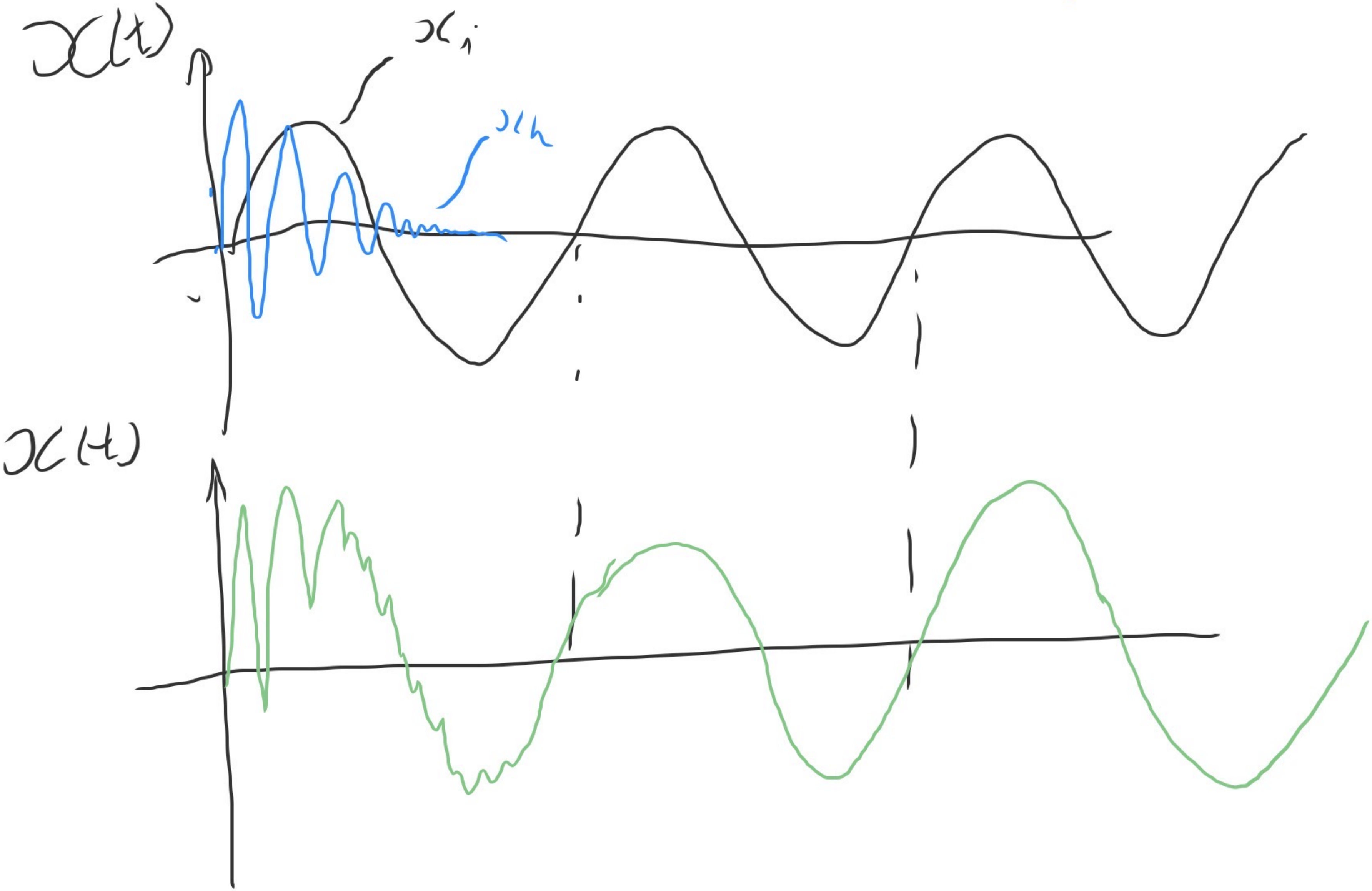
$$-\omega^2 x = \frac{F_e}{m}$$

oposição de Fase

$$x(t) \simeq -\frac{F_0 \cos(\omega t)}{m\omega^2}$$

Solução Homogênea

$$x(t) = x_h(t) + x_i(t)$$

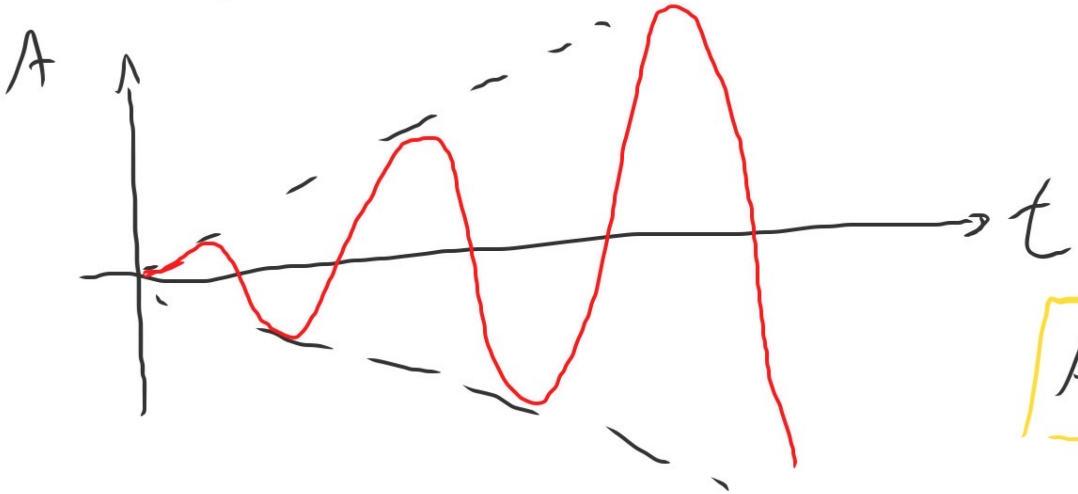


Dissipação Nula: $x(t) = A(\omega)\cos(\omega t + \varphi) + B\cos(\omega_0 t + \varphi_0)$

inomogênea — homogênea $\omega_0 = \sqrt{k/m}$

$$x(t) = \frac{F_0}{m(\omega_0+\omega)}\left[\frac{\cos(\omega_0 t) - \cos(\omega t)}{\omega_0 - \omega}\right] \quad ; \quad \begin{matrix} x(0)=0 \\ v(0)=0 \end{matrix}$$

$$\lim_{\omega \to \omega_0} x(t) = \frac{F_0}{2m\omega_0}\left[\frac{d}{d\omega}\cos(\omega t)\right] = -t\,\frac{F_0}{2m\omega_0}\,\mathrm{sen}(\omega t)$$



$\exists$ limite?

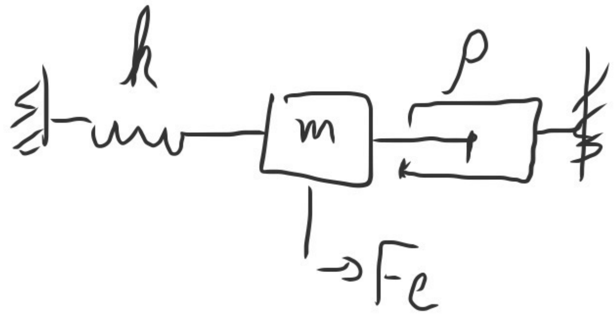
Linearidade $\Rightarrow F_k = -kx$

$U = -\frac{kx^2}{2}$

$\boxed{\text{Anarmonicidade}}$

Jogo da Energia: Oscilador Harmônico

$E = V[x(t)] + K[v(t)]$ Forçado Amortecido



$$E = \frac{k x^2}{2} + \frac{m}{2} v^2$$

$\gamma = \rho/m$

$\omega_0^2 = k/m$

Potência: $\frac{d}{dt} E(t) = \frac{k}{2} \cdot \frac{d}{dt} x^2(t) + \frac{m}{2} \frac{d}{dt} \dot{x}^2(t)$

$$= \frac{k}{2} 2 \cdot x(t) \cdot \dot{x}(t) + \frac{m}{2} \cdot 2 \dot{x}(t) \cdot \ddot{x}(t)$$

$$= (kx + m\ddot{x})\, \dot{x} = (F_e - m\gamma \dot{x})\dot{x}$$

$$\frac{d}{dt}E(t) = -m\gamma\dot{x}^2(t) + F_e(t)\cdot\dot{x}(t)$$

Potência dissipada

$P_d \geq 0$

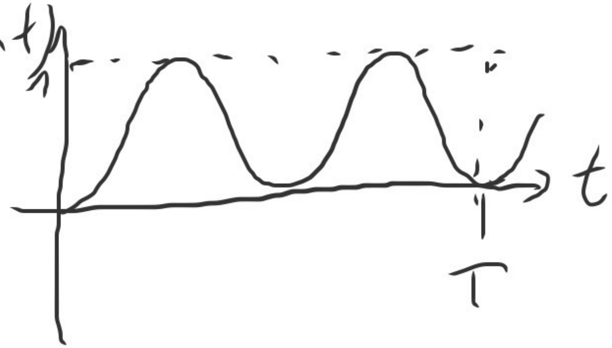
Potência "fornecida"

(Positiva ou negativa)

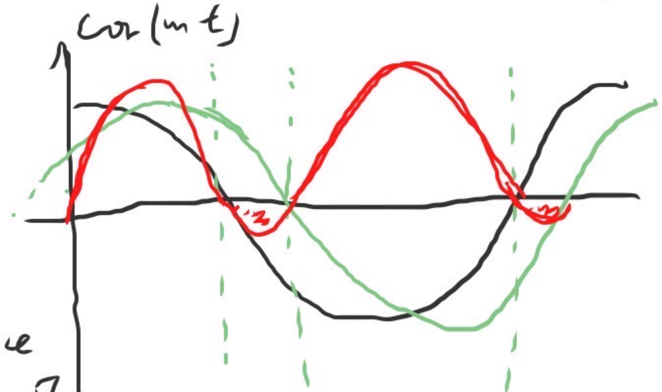
$$F_e(t) = F_0\cos(\omega t)\,;\quad x(t) = A\cos(\omega t+\varphi)$$

$$\Rightarrow \dot{x}(t) = -\omega A\,\text{sen}(\omega t+\varphi)$$

$$\frac{d}{dt}E(t) = -m\gamma\omega^2A^2\,\text{sen}^2(\omega t+\varphi) - \omega A F_0\cos(\omega t)\,\text{sen}(\omega t+\varphi)$$



$-\pi < \varphi \leq 0$



hora fornece

hora retira!

Estudando melhor

$$\frac{d}{dt}E = m\ddot{x}\dot{x} + kx\dot{x}$$

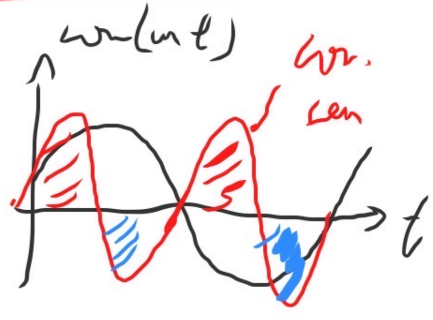
$$x(t) = A\cos(\omega t+\varphi)$$

$$\frac{dE}{dt} = m\left[-\omega^2 A\cos(\omega t+\varphi)\right]\cdot\left[-\omega A^2 \operatorname{sen}(\omega t+\varphi)\right]$$
$$+\, m\,\omega_0^2\left[A\cos(\omega t+\varphi)\right]\left[-\omega A^2 \operatorname{sen}(\omega t+\varphi)\right]$$

$$k = m\omega_0^2$$

$$\frac{d}{dt}E = m\left(\omega^2-\omega_0^2\right)\omega A^2 \underline{\cos(\omega t+\varphi)\operatorname{sen}(\omega t+\varphi)}$$

$$\frac{d}{dt}E = \frac{1}{2}m\left(\omega^2-\omega_0^2\right)\omega A^2 \operatorname{sen}(2\omega t+\varphi)$$



$$\frac{d}{dt}\bar{E} = m(\omega^2-\omega_0^2)\omega A^2\left[\int_t^{t+T}\frac{1}{2}\,\mathrm{sen}(2\omega t+\varphi)\,dt\right]\frac{1}{T}=0$$

Regime Estacionário !

<u>Potência Dissipada:</u> $P_d = -\rho\dot{x}\cdot\dot{x} = F_d\cdot\dot{x}$

$$P_d = m\gamma\omega^2A^2\,\mathrm{sen}^2(\omega t+\varphi)$$

$$\overline{P_d} = m\gamma\omega^2A^2\,\frac{\int_t^{t+T}\mathrm{sen}^2(\omega t+\varphi)\,dt}{T} = \frac{m\gamma\omega^2A^2}{2}$$

$$\overline{P}_d(\omega) = \frac{\gamma F_0^2 \omega^2}{2m\left[(\omega^2-\omega_0^2)^2+\gamma^2\omega^2\right]} \; ; \; A^2 = \frac{(F_0/m)^2}{(\omega^2-\omega_0^2)^2+\gamma^2\omega^2}$$

Máximo $\overline{P}_d(\omega)$? Antes: Máximo $A^2(\omega) \Rightarrow \omega^2 = \omega_0^2 - \frac{\gamma^2}{2}$

Mostre! Máximo em $\omega = \omega_0$

$\frac{d}{dt}E(t) = 0 \rightarrow$ Toda energia fornecida é perdida na dissipação

Força externa compensa exatamente o atrito viscoso!

$$\varphi = -\operatorname{tg}^{-1}\left(\frac{\rho\omega}{\omega^2-\omega_0^2}\right); \quad \omega=\omega_0 \Rightarrow \varphi = -\pi/2$$

$$\cos(\omega t)\,\operatorname{sen}(\omega t+\varphi)$$

$$-\cos(\omega t)\cos(\omega t)$$

$$\frac{d}{dt}E = \underbrace{-m\rho\omega A^2 \underbrace{\operatorname{sen}^2\left(\omega t-\frac{\pi}{2}\right)}_{\cos^2(\omega t)}}_{\text{dissipação}} + \underbrace{\omega A F_0 \cos^2(\omega t)}_{\text{fornecimento}}$$

$$\overline{P(\omega)} = \frac{F_0^2}{2m\omega_0 Q}\left[\frac{1}{\left(\alpha - \frac{1}{\alpha}\right)^2 + \frac{1}{Q^2}}\right]$$

Fator $Q = \frac{\omega_0}{\gamma}$

$\alpha = \omega/\omega_0$

máxima em $\alpha = 1$

se $Q \gg 1$, $|\omega - \omega_0| \sim \gamma$

$$\overline{P(\omega)} = \frac{\gamma F_0^2}{8m} \frac{1}{\left[(\omega - \omega_0)^2 + \left(\frac{\gamma}{2}\right)^2\right]}$$

Lorentziana; Breit-Wigner $\Rightarrow$ dissipação pequena

Fator de Potência

$$P_d = F_e \cdot \dot{x} = -\omega A F_0 \cos(\omega t)\,\mathrm{sen}(\omega t + \varphi)$$

$$\mathrm{sen}(\omega t)\cos\varphi + \mathrm{sen}\,\varphi \cos(\omega t)$$

$$\overline{P_d} = -\omega A F_0 \left[\overline{\cos(\omega t)\,\mathrm{sen}(\omega t)}\right]\cos\varphi \rightarrow 0$$

$$-\omega A F_0 \left[\overline{\cos^2(\omega t)}\right]\mathrm{sen}\,\varphi \quad 1/2$$

$$\overline{P_d} = -\frac{\omega_0 F_0 A}{2}\,\mathrm{sen}\,\varphi$$

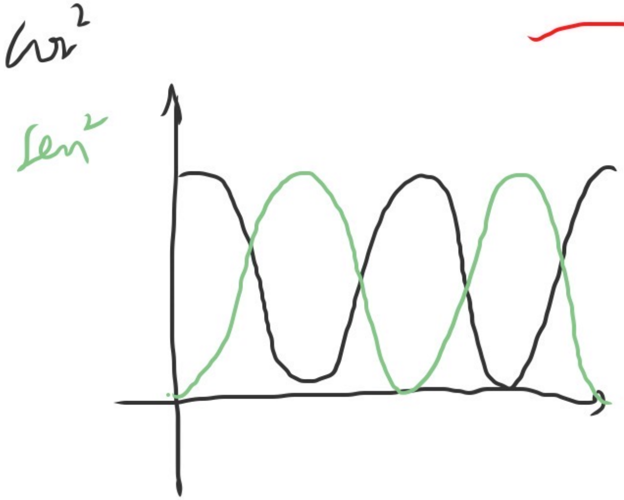
Fator de Potência

Máximo de Potência dissipada

Energia no Oscilador → Dependência em ω

$$E = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} \underbrace{m \omega_0^2}_{k} x^2$$

$$= \underbrace{\frac{1}{2} m \cdot \omega^2 A^2 \operatorname{sen}^2(\omega t + \varphi)}_{K} + \underbrace{\frac{1}{2} m \omega_0^2 A^2 \cos^2(\omega t + \varphi)}_{U}$$



Média $\bar{K} = \frac{1}{4} m A^2 \omega^2$; $\bar{U} = \frac{1}{4} m A^2 \omega_0^2$

Desbalanceio entre $\bar{K}$ e $\bar{U}$!

Fator de Qualidade revisitado

$$Q = 2\pi \left(\frac{E_{armazenada}}{E_{dissipada}} \right)_{ciclo}$$

$$E_{dissipada} = \bar{P} \cdot T = \frac{2\pi}{\omega} \bar{P} = \pi \gamma m \omega A^2$$

$$Q = 2\pi \cdot \frac{mA^2}{4} \cdot \frac{(\omega^2 + \omega_0^2)}{\pi \gamma m \omega A^2} = \frac{1}{2} \frac{\omega_0}{\gamma} \left(\frac{\omega}{\omega_0} + \frac{\omega_0}{\omega} \right)$$

$$= \frac{1}{2} \frac{\omega_0}{\gamma} \left(\alpha + \frac{1}{\alpha} \right)$$

$$Q_L = \frac{\omega_0}{\gamma} \rightarrow \text{Osc. Livre}$$