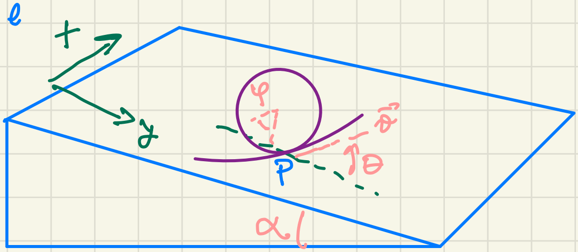
II.6 Alguns exemplos

.6.3 Moeda Rolando; [distância do CM ao plano inclinado cte = R]

Se a massa da moeda é M e seu raio R



$$\underbrace{I_1 = I_2}_{\text{no plano da moed.}} = \underbrace{\frac{I_3}{2}}_{I \perp \text{ à moeda pelo centro}} = \frac{MR^2}{4}$$

(¿Por que $I_3 = I_1 + I_2$?)

Em geral, $T = \underbrace{\frac{M}{2}(\dot{x}^2 + \dot{y}^2)}_{T_{CM}} + \frac{1}{2}(I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2)$

$\omega_3 = \dot{\varphi}$ $\qquad \omega_1^2 + \omega_2^2 = \dot{\theta}^2$

$V = -mgy \sin\alpha$

Logo, $L = \frac{M}{2}(\dot{x}^2 + \dot{y}^2) + \frac{MR^2}{4}\dot{\varphi}^2 + \frac{MR^2}{8}\dot{\theta}^2 + mgy \operatorname{sen}\alpha$

Isso não é tudo, hipótese: rola sem escorregar, i.e, $\vec{v}_P = 0$

$$\Rightarrow \begin{cases} \dot{x} - R\dot{\varphi} \operatorname{sen}\theta = 0 & (i) \\ \dot{y} - R\dot{\varphi} \cos\theta = 0 & (ii) \end{cases}$$

vínculos

Variáveis: $1 \to x$ $\quad 2 \to y$ $\quad 3 \to \theta$ $\quad 4 \to \varphi$

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Usando os multiplicadores de Lagrange, os a_{ij} não nulos são

$a_{11} = 1 \qquad a_{13} = -R\,\text{sen}\,\theta$

$a_{22} = 1 \qquad a_{23} = -R\cos\theta$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = \sum_{l=1}^{2} a_{lk}\lambda_l \Rightarrow$$

$M\ddot{x} = \lambda_1 \qquad$ (iii)

$M\ddot{y} - mg\sin\alpha = \lambda_2 \qquad$ (iv)

$\frac{MR^2}{4}\ddot{\theta} = 0 \Rightarrow \frac{MR^2}{4}\dot{\theta}$ é c^{te} movimento (v)

$M\frac{R^2}{2}\ddot{\varphi} = -\lambda_1 R\sin\theta - \lambda_2 R\cos\theta \qquad$ (vi)

Agora

$\frac{d}{dt}(i) \Rightarrow \ddot{x} = R\ddot{\varphi}\,\text{sen}\,\theta + R\dot{\varphi}\dot{\theta}\cos\theta \overset{iii}{\Longrightarrow} \lambda_1 = MR\ddot{\varphi}\,\text{sen}\,\theta + MR\dot{\varphi}\dot{\theta}\cos\theta$

$\frac{d}{dt}(ii) \Rightarrow \ddot{y} = R\ddot{\varphi}\cos\theta - R\dot{\varphi}\dot{\theta}\,\text{sen}\,\theta \overset{iv}{\Longrightarrow} \lambda_2 = MR\ddot{\varphi}\cos\theta - MR\dot{\varphi}\dot{\theta}\,\text{sen}\,\theta$
$- mg\sin\alpha$

Subst. λ_1 e λ_2 em (vi)

$$M\frac{R^2}{2}\ddot{\varphi} = -MR^2\text{sen}^2\theta\,\ddot{\varphi} - \cancel{MR^2\dot{\varphi}\dot{\theta}\,\text{sen}\,\theta\cos\theta} - MR^2\cos^2\theta\,\ddot{\varphi} + \cancel{MR^2\dot{\varphi}\dot{\theta}\,\text{sen}\,\theta\cos\theta}$$
$$+ MRg\,\text{sen}\,\alpha\cos\theta \Rightarrow$$

$$\ddot{\varphi} = \frac{2g\sin\alpha}{3R}\cos\theta$$

MCU $\theta = \Omega t + \theta_0$

$$\ddot{\varphi} = \frac{2g\sin\alpha}{3R}\cos(\theta_0 + \Omega t) \quad (*)$$

A serem determinadas

$\Omega \neq 0$

$$(*) \Rightarrow \varphi = \varphi_0 + \omega t - \frac{2g\sin\alpha}{3\Omega^2 R}\cos(\theta_0 + \Omega t)$$

Usando os vínculos temos

$$x = x_0 + \frac{g\sin\alpha}{3\Omega} t - \left[\frac{\omega R}{\Omega} + \frac{g\sin\alpha}{3\Omega^2}\sin(\theta_0 + \Omega t)\right]\cos(\theta_0 + \Omega t)$$

A serem determinados

$$y = y_0 + \left[\frac{\omega R}{\Omega} + \frac{g\sin\alpha}{3\Omega^2}\sin(\theta_0 + \Omega t)\right]\sin(\theta_0 + \Omega t)$$

Exercício: Ache a solução para $\Omega = 0$

Desafio: Considere que a distância do CM ao plano inclinado não seja fixa. Quais os graus de liberdade? Quais os vínculos? Quais as eq's de movimento?

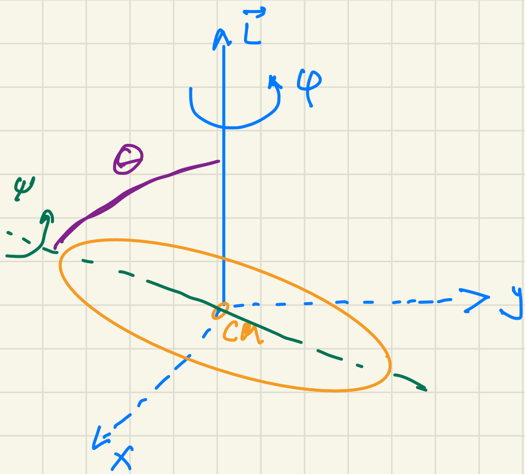
II.7 Pião simétrico

Consideremos um pião simétrico ($I_1 = I_2$) sem a ação de torques. Consideremos o referencial S' dado pelos eixos principais de inércia do pião. Vimos que a energia cinética é dada por

$$T = \frac{1}{2}\sum_{j=1}^{3} \tilde{\omega}_j^2 I_j$$

onde $\tilde{\omega}_j$ é a expressão da velocidade angular no referencial S'.

Usemos os ângulos de Euler c/ o eixo z na direção do momento angular total $\vec{L}$ que é conservado:



Sabemos que:

$$\tilde{\omega}_1 = \dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi$$

$$\tilde{\omega}_2 = \dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi$$

$$\tilde{\omega}_3 = \dot{\psi} + \dot{\varphi}\cos\theta$$

Desprezando o movimento do centro de massa

$$T = \frac{I_1}{2}\left(\dot{\theta}^2 + \dot{\varphi}^2 \operatorname{sen}^2\theta\right) + \frac{I_3}{2}\left(\dot{\psi} + \dot{\varphi}\cos\theta\right)^2$$

e $L = T$. Logo, as equações do movimento são

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\psi}}\right) = \frac{\partial L}{\partial \psi} \Rightarrow \frac{d}{dt}\left(\underbrace{I_3\left(\dot{\psi} + \dot{\varphi}\cos\theta\right)}_{P_\psi}\right) = 0 \leftarrow \text{var. cíclica}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) = \frac{\partial L}{\partial \varphi} \Rightarrow \frac{d}{dt}\left(\underbrace{I_1 \dot{\varphi} \operatorname{sen}^2\theta + I_3\left(\dot{\psi} + \dot{\varphi}\cos\theta\right)\cos\theta}_{P_\varphi}\right) = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta} \Rightarrow I_1 \ddot{\theta} = I_1 \dot{\varphi}^2 \operatorname{sen}\theta\cos\theta - I_3\left(\dot{\psi} + \dot{\varphi}\cos\theta\right)\operatorname{sen}\theta\,\dot{\varphi}$$

$$P_\psi = \left(\dot{\psi} + \dot{\varphi}\cos\theta\right) I_3$$

$$P_\varphi = I_1 \dot{\varphi}\operatorname{sen}^2\theta + P_\psi \cos\theta \Rightarrow \dot{\varphi} = \frac{P_\varphi - P_\psi\cos\theta}{I_1 \operatorname{sen}^2\theta}$$

Logo, $$I_1 \ddot{\theta} = \frac{\left(P_\varphi - P_\psi\cos\theta\right)^2\cos\theta}{I_1 \operatorname{sen}^3\theta} - \frac{P_\psi\left(P_\varphi - P_\psi\cos\theta\right)}{I_1 \operatorname{sen}\theta}$$

$$= -\frac{d}{d\theta}\left[\frac{\left(P_\varphi - P_\psi\cos\theta\right)^2}{2 I_1 \operatorname{sen}^2\theta}\right]$$