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REVIEW ARTICLE

# A survey of structural and multidisciplinary continuum topology optimization: post 2000

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**Abstract** Topology optimization is the process of determining the optimal layout of material and connectivity inside a design domain. This paper surveys topology optimization of continuum structures from the year 2000 to 2012. It focuses on new developments, improvements, and applications of finite element-based topology optimization, which include a maturation of classical methods, a broadening in the scope of the field, and the introduction of new methods for multiphysics problems. Four different types of topology optimization are reviewed: (1) density-based methods, which include the popular Solid Isotropic Material with Penalization (SIMP) technique, (2) hard-kill methods, including Evolutionary Structural Optimization (ESO), (3) boundary variation methods (level set and phase field), and (4) a new biologically inspired method based on cellular division rules. We hope that this survey will provide an update of the recent advances and novel applications of popular methods, provide exposure to lesser known, yet promising, techniques, and serve as a resource for those new to the field. The presentation of each method focuses on new developments and novel applications.

**Keywords** Topology optimization · Density methods · Level set · Evolutionary structural optimization · Phase field · Continuum topology

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## 1 Introduction

Topology optimization is the process of determining the connectivity, shape, and location of voids inside a given design domain. This allows for greater design freedom than size and shape optimization, which deal with variables such as thicknesses or cross-sectional areas of structural members (sizing) and geometric features (shape) of predefined structural configurations. As such, topology optimization has great implications in early conceptual and preliminary design phases where design changes significantly impact the performance of the final structure. The most recent comprehensive surveys regarding topology optimization include those by Rozvany (2001a) and Eschenauer and Olhoff (2001) and the monograph by Bendsøe and Sigmund (2003). Since then, the field has undergone a period of rapid growth in academia, sponsored research, and industrial application. In fact, it has been the most active research area in structural and multidisciplinary optimization in the past two decades. This growth is due to the maturation of some classical topology optimization techniques and the continued development and creation of promising new methods. In addition, the practical scope of topology optimization has increased beyond a few linear structural responses to include combinations of structures, heat transfer, acoustics, fluid flow, aeroelasticity, materials design, and other multiphysics disciplines.

This survey seeks to consolidate and highlight the advancements in topology optimization of continuum structures from 2000 to 2012. Some references prior to the year 2000 are included for important topics and we have considered primarily journal publications. This review also identifies each method's unique strengths for certain classes of problems. Section 2 focuses on density-based methods for continuum topology optimization, which have historically been the most prominent due to the widespread acceptance of the Solid Isotropic Material with Penalization (SIMP) method. Section 3 discusses so called "hard-kill" methods, the most popular of which is known as Evolutionary Structural Optimization (ESO). Boundary variation methods, which implicitly represent the boundary surface of the structure and include level set and phase field representations, are discussed in Section 4. A recently developed, biologically-inspired method based on cellular division rules is explored in Section 5. Finally, Section 6 contains concluding remarks, including recommendations and perceptions for the future. Effort has been made to include relevant background information, to highlight the recent developments in critical aspects for each method, and to survey recent applications.

## 2 Density-based methods

The most widely used methodologies for structural topology optimization can be broadly classified as density-based methods, which include the popular Solid Isotropic Material with Penalization (SIMP) method. Density-based methods operate on a fixed domain of finite elements with the basic goal of minimizing an objective function by identifying whether each element should consist of solid material or void. In structural topology optimization, this objective is often compliance, and constraints are placed on the amount of material that may be utilized. Fundamentally, this poses an extremely challenging large-scale integer programming problem. As a result, it is desirable to replace the discrete variables with continuous variables and identify a means to iteratively steer the solution towards a discrete solid/void solution. This is accomplished with an interpolation function, where the continuous design variables are explicitly interpreted as the material density of each element. Penalty methods are then utilized to force solutions to suitable "0/1", "black/white", or "solid/void" topologies.

The fundamental mathematical statement of a density-based topology optimization problem contains an objective function, set of constraints (that likely includes an upper limit on material usage), and a discretized representation of the physical system. A general formulation based on linear static finite element analysis may be given as:

$$\begin{aligned} \min: \quad & f(\boldsymbol{\rho}, \mathbf{U}) \\ \text{subject to}: \quad & \mathbf{K}(\boldsymbol{\rho})\mathbf{U} = \mathbf{F}(\boldsymbol{\rho}) \\ & g_i(\boldsymbol{\rho}, \mathbf{U}) \leq 0 \\ & 0 \leq \boldsymbol{\rho} \leq 1 \end{aligned} \tag{1}$$

where $f$ is the objective function, $\boldsymbol{\rho}$ is the vector of density design variables, $\mathbf{U}$ is the displacement vector, $\mathbf{K}$ is the global stiffness matrix, $\mathbf{F}$ is the force vector, and $g_i$ are the constraints. We note that the stiffness matrix $\mathbf{K}$, and sometimes load vector $\mathbf{F}$, are explicitly dependent upon the density design variables at the element level. Within this generalized statement, a number of problems can be formulated considering a variety of objectives and constraints, including compliance, stresses, frequency, displacements, and alternative physics such as eigenvalue problems, fluid flow, and nonlinear systems as discussed later. As an example, the popular (and likely overused at this time) compliance problem can be setup by minimizing an objective of structural compliance as $f = c = \mathbf{U}^T\mathbf{K}\mathbf{U}$ and constraining the amount of material usage as $g = V/V_0 - V_f \leq 0$. In the previous equations, $c$ denotes the compliance, $V$ and $V_0$ are the material volume and design domain volume, respectively, and $V_f$ is the allowable volume fraction.

For developments and basic discussion of density-based topology optimization prior to 2003, especially related to the SIMP technique, the reader is referred to the well-known monograph by Bendsøe and Sigmund (2003). In the following subsections, we highlight recent developments for several important topics in density-based topology optimization.

### 2.1 Density interpolation/penalization

A critical aspect of density-based methods is the selection of an appropriate interpolation function and penalization technique to express the physical quantities of the problem as a function of continuous design variables. As previously stated, the distributed function of design variables in density-based topology optimization is interpreted as the material density of each finite element, $\rho_e$. The values of density range as $0 \leq \rho_e \leq 1$ or $0 < \rho_{min} \leq \rho_e \leq 1$ where 0 corresponds to a void element, 1 to a solid element, and $\rho_{min}$ is the minimum value of density, which is required with some formulations to prevent difficulties associated with zero values. These difficulties include singularity in finite element matrices and issues with the inability of material to reappear in an area with zero density in some cases. With the choice of this parameterization comes the need to steer the problem toward a solid/void solution. This is typically accomplished by using implicit penalization techniques, the most common of which is the Solid Isotropic Material (originally Microstructure) with Penalization (SIMP) method. This method was originally developed independently by Bendsøe (1989) and Zhou and Rozvany (1991), see also Rozvany et al. (1992) in which the term "SIMP" was coined. In the SIMP method, also referred to as the power law or fictitious material model, density variables are penalized with a basic power law (whose value is finite) and multiplied onto physical quantities such as material stiffness, cost, or conductivity (Bendsøe and Sigmund 1999, 2003). It is likely that the simplicity of the SIMP method has led to its widespread use and acceptance in both industry and academia. Some theoretical convergence properties of the SIMP method have been discussed by Rietz (2001), Martinez (2005), and Stolpe and Svanberg (2001b). Sigmund (2007) discusses the advantages of a slightly modified version of SIMP, which includes a minimum stiffness (or other material parameter) that is independent of penalization. For additional references concerning both the origins and theoretical mechanisms behind the SIMP method the reader is directed to the early review (Rozvany 2001a; Eschenauer and Olhoff 2001) and forum (Rozvany 2009a) articles and the references therein.

Stolpe and Svanberg (2001a) proposed an alternative interpolation scheme known as the Rational Approximation of Material Properties (RAMP). A desirable feature of the RAMP model is that, unlike SIMP, it has nonzero sensitivity at zero density. As a result, the RAMP material model has been shown to remedy some numerical difficulties in problems related to very low density values in the presence of design dependent loading. Bruns (2005) discusses another alternative interpolation scheme known as the SINH (pronounced "cinch") method, SINH is an inverted version of cost penalization suggested by Zhou and Rozvany (1991), in which the specified cost can represent material weight. This scheme differs from others in that usually material parameters are penalized, whereas in the SINH formulation the volume is penalized. As such, intermediate density material consumes more volume with respect to its load-carrying capability than solid or void material. A comparison of the SIMP, RAMP, and SINH penalization schemes is shown in Fig. 1. In the figure, $\rho$ is the density variable, $p$ is the penalization parameter for the SIMP and SINH methods, and $q$ is the penalization parameter for RAMP. The range of penalization parameters shown in the figure is representative of the parameters used in practice, with the actual value depending on the underlying physics of the problem.

In a different technique, Fuchs et al. (2005) obtain a solid/void layout from a linear function of densities and a new constraint they refer to as the sum of the reciprocal variables (SRV). The constraint stipulates that the SRV must be larger or equal to its value at a discrete design for a specified amount of material. This technique proves useful on benchmark problems in the paper, but has yet to be further explored in the literature.

For topology optimization problems involving alternative physics, including eigenvalue problems, heat transfer, and fluid flow among others, a number of works have proposed variations of these interpolation schemes to handle particular nuances. These are highlighted in Section 2.6. Finally, since penalized topology optimization problems are nonconvex, and often have a large number of local minima, continuation methods are frequently used in the literature to increase the chance of obtaining a global optimal solution (Bendsøe and Sigmund 2003). Continuation methods slowly increase the effect of penalization (by increasing penalization parameters) over the course of several iterations in optimization. While it is shown by Stolpe and Svanberg (2001b) that continuation methods are not guaranteed to give fully solid/void results in every situation, they nonetheless perform very well in practical applications, especially when used with a regularization scheme (Bendsøe and Sigmund 2003), discussed next.

## 2.2 Regularization techniques

Regularization in topology optimization refers to the process of controlling the density values (or sensitivities) to prevent numerical issues and to control the quality of final results. These issues include checkerboarding, which refers to the formation of adjacent solid-void elements arranged in a checkerboard pattern, and mesh dependency, which concerns the phenomenon that without special consideration, different topologies result from identical design domains of different discretization sizes. The two primary

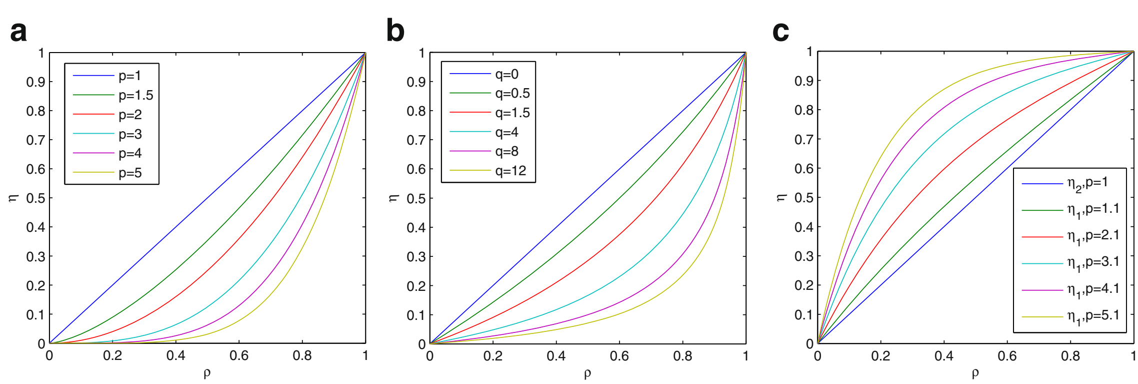



**Fig. 1** Comparison of **a** $\eta(\rho) = \rho^p$ SIMP, **b** $\eta(\rho) = \frac{\rho}{1+(1-\rho)}$ RAMP, **c** $\eta_1(\rho) = 1 - \frac{sinh(p(1-\rho))}{sinh(p)}$, $\eta_2(\rho) = \rho$ SINH intermediate density penalization models for density-based topology optimization

methods of regularization are filtering and constraint techniques. Filtering methods are applied via direct modification of density variables or sensitivities while constraint methods utilize localized or global-level constraints added to the optimization problem. It is noted here that mesh independency algorithms tend to prevent checkerboards because a checkerboard pattern is a small feature in topology that can be removed by enforcing minimum length scale. On the other hand, checkerboard-prevention algorithms may not necessarily alleviate mesh dependency.

#### *2.2.1 Mesh-independence and checkerboard alleviation*

The origin of constraint methods lies in pioneering works prior to the year 2000. However, since that time extensions have been proposed to some methods including global gradient (Borrvall 2001), slope control (Schury et al. 2012), local gradient control (Zhou et al. 2001), and regularized penalty methods (Borrvall and Petersson 2001b). In addition, a number of new constraint methods have been developed such as patch control (Poulsen 2002) and integral methods (Poulsen 2003).

Filter methods remain the most popular regularization methods due largely to their ease of application. These include the sensitivity filter (Sigmund and Petersson 1998) and the density filter (Bourdin 2001; Bruns and Tortorelli 2001), which modify either the sensitivity or the density value of an element based on the sensitivity or density of elements in a localized neighborhood. Recently, Sigmund and Maute (2012) published a brief note addressing the perception of the sensitivity filter as "heuristic" or "inconsistent" despite its widespread popularity in the field. They demonstrate the underlying concepts of the sensitivity filter may be rigorously derived from principles in continuum mechanics and nonlocal elasticity. For basic regularization methods prior to 2003, the reader is referred to either the review article by Sigmund and Petersson (1998) or the monograph by Bendsøe and Sigmund (2003). In addition, Almeida et al. (2009) proposed an inverse filter scheme to control the size of void regions in topology. Regularization was also suggested by design parameterization in wavelet space (Kim and Yoon 2000; Poulsen 2002), and Pomezanski et al. (2005) proposed an extension of SIMP to penalize corner contact directly. In addition, while not explicitly a regularization technique, Jang et al. (2003) eliminated checkerboards by using non-conforming finite elements in density-based topology optimization. Rahmatalla and Swan (2004) and Matsui and Terada (2004) proposed nodal based design variables which allowed for a continuous approximation of material density (CAMD) that naturally alleviates checkerboards.

Recently, Lazarov and Sigmund (2011) developed more efficient variations of the basic sensitivity and density filters based on the solution of Helmholtz-type differential equations. A similar Helmholtz-based density filter was independently formulated by Kawamoto et al. (2011). These PDE-based filters reduce computational requirements by eliminating both the need to compute and the need to store neighborhood information as required by their conventional counterparts. It is shown that the computational advantage of the PDE-based filters increases with filter radius, dimensionality, and parallelization capacity.

To force designs from utilizing excessively large structural members, Guest (2009a) recently proposed a constraint method that can impose a maximum length scale on structural features.

#### *2.2.2 Projection methods and morphology filters*

A basic consequence of sensitivity and density filtering is the formation of gray transition material between solid and void regions. To combat this for situations where crisp boundary definition is important, several schemes have recently been developed to project the filtered densities into 0/1 (void/solid) space via a relaxed Heaviside function (Guest et al. 2004, 2011; Kawamoto et al. 2011) and morphology-based operators (Sigmund 2007; Wang et al. 2011c). These schemes are called projection methods in the literature and are also able to enforce length scale control. A volume-preserving projection based on a Heaviside function was also introduced by Xu et al. (2010), but while it does produce crisp designs, it cannot control length scale. Projection schemes have also been proposed for length scale control of void regions (holes) and both solid and void regions simultaneously where conventional filters only ensure minimum length scale of the solid phase (Sigmund 2007; Guest 2009b). Wang et al. (2011c) proposed a modified formulation to remedy the acknowledged deficiency that while projection methods provide global mesh-independence (overall topology will converge with mesh refinement), they will not guarantee local mesh-independence such as the formation of hinges or narrow gaps. Figure 2 shows characteristic results of density-based topology optimization using filtering and projection. Note the gray transition material along structural boundaries for (a) sensitivity and (b) density filters. This region is eliminated using the (c) Heaviside projection. In addition, Zhou et al. (2001) also proposed a process using density slope control that is removed in the last iteration to obtain crisp 0/1 results.

### 2.3 Nonlinear responses

While the roots of topology optimization lie in stiffness design of linear elastic structures, density-based topology

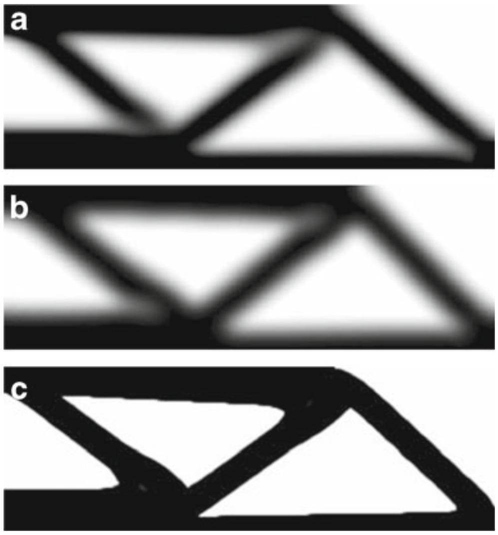



**Fig. 2** Density-based topology optimization results for the MBB benchmark problem (100 × 300 mesh size, filter radius of 16, $p = 3$, and 50 % volume fraction) with **a** sensitivity filter, **b** density filter, and **c** Heaviside projection filter (from Andreassen et al. 2011)

optimization is now being applied in situations with geometric and/or material nonlinearity. When applying density-based methods to nonlinear problems, numerical instabilities are typically faced due to low density elements in incremental and iterative nonlinear finite element analysis (FEA). This occurs because regions of low density often experience extremely large deformations, which causes their tangent stiffness matrices to lose positive definiteness. These issues may be overcome in a number of ways including convergence criteria relaxation in the nonlinear FEA as suggested by Buhl et al. (2000) or by an element removal and reintroduction technique proposed by Bruns and Tortorelli (2003). Cho and Jung (2003) proposed the use of a displacement-loaded formulation, which contrasts most force-loaded systems, to alleviate the problem of excessive displacements. Yoon and Kim (2005) suggested an element connectivity parameterization, where structural elements remain solid throughout optimization, but are connected by 1-D elastic link elements. The stiffness of the link elements, which are more numerically stable even at low stiffnesses, are taken as design variables to allow for topology optimization. Recently, Kawamoto (2009) proposed the use of the Levenberg-Marquardt method as an alternative to the usual iterative Newton-Raphson nonlinear solver because it is less susceptible to low density induced anomalies. Another challenge to utilizing nonlinear modeling is that the problem becomes dependent on loading magnitude with different load amounts leading to different optimal topologies. As such, problem setup becomes much more critical.

The design of compliant mechanisms and micro-actuators is an important area for nonlinear topology design because their displacement response may be large and the slender members that often result in optimum designs may be prone to buckling. Pedersen et al. (2001), Bruns et al. (2002), and Bruns and Tortorelli (2003) optimized compliant mechanisms with geometric nonlinearity while Jung and Gea (2004) did so with hyperelastic material nonlinearity. Sigmund (2001a, b) demonstrated the effects of geometric nonlinearity in multiphysics actuators. Several papers have also focused on stiffness design under nonlinearity. Gea and Luo (2001) investigated geometric nonlinearity for minimum compliance in 2D structures while Kemmler et al. (2005) did so with consideration of buckling stability. Jung and Gea (2004) studied the problem of both geometric and material nonlinearity together where material nonlinearity is hyperelastic and Klarbring and Strömberg (2013) did so for a number of different material models. Results from Jung and Gea (2004), which demonstrate the effects of geometric, material, and combinations of both nonlinearities on optimum designs for maximum stiffness are shown in Fig. 3. Yoon and Kim (2007) studied material nonlinearity, including both elastic-plastic and hyperelastic behavior, using a density-based element connectivity method. Earlier,

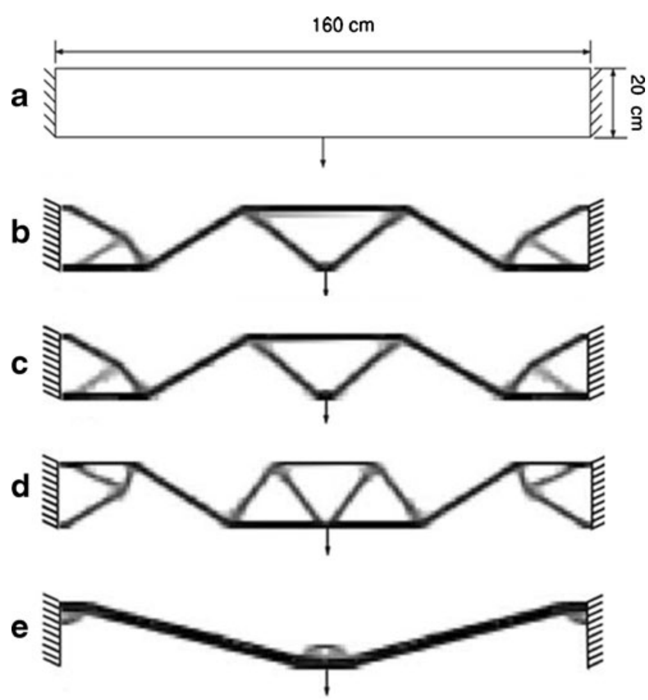



**Fig. 3** **a** Design domain and loading for maximum stiffness designs for **b** linear, **c** geometric nonlinear, **d** material nonlinear, and **e** both geometric and material nonlinear (from Jung and Gea 2004)

Schwarz et al. (2001) visited the topic of topology optimization with elastoplastic material. Also, Stegmann and Lund (2005) designed anisotropic shell structures for maximum stiffness with geometric nonlinearity. Finally, Jung and Cho (2004) and Kang and Luo (2009) included geometric nonlinearity in reliability-based topology optimization with a density method.

As an alternative to directly solving the topology optimization problem using nonlinear analysis, the Equivalent Static Loads method for nonlinear structural optimization can be utilized (Lee and Park 2012). The ESL method begins by computing equivalent linear loads that produce identical responses to nonlinear analysis. Optimization is performed using the equivalent linear loads with basic linear topology methods. At this point, an updated nonlinear analysis is performed on the design resulting from the inner linear analysis-based optimization, and subsequently new equivalent linear loads are determined. This process is iterated until convergence in design variables.

## 2.4 Stress-based topology design

Stress is an essential consideration in the design of any mechanical system (Duysinx et al. 2008). Despite this, the overwhelming majority of developments in structural topology optimization using density methods are related to minimum compliance and other global level design criteria. This is primarily due to three challenges posed by stress-based topology optimization. These are the singularity phenomenon, the local nature of stress constraints, and the highly non-linear stress behavior (Bendsøe and Sigmund 2003). The singularity phenomenon occurs when the optimum solution is in a degenerate subspace of the design space, which arises in density-based topology optimization as elements tend towards zero density. This problem, whose roots lie in truss optimization, has been studied extensively prior to 2000 and is resolved by stress constraint relaxation methods (Rozvany 2001b; Bruggi 2008). The references contained in the previously mentioned works include many of the early works in stress-based topology optimization.

Techniques to address the local nature of stress constraints in density-based topology optimization can be grouped into local methods, global methods, and regional or block aggregation techniques. Local methods place a constraint on each element in the design model (Bendsøe and Sigmund 2003; Pereira et al. 2004; Navarrina et al. 2005; Bruggi and Venini 2008). However, when local constraints are subjected to multiple load cases an intractable design problem often results due to the large number of constraints. Global methods remedy this issue by combining the local stress values into a single combined relationship. This can be accomplished using variations of the *Kresselmeier-Steinhauser* (KS) function, *p-norm* measures, or global $L^q$ constraints (Bendsøe and Sigmund 2003; Guilherme and Fonseca 2007; París et al. 2007, 2009; Qiu and Li 2010). An evident tradeoff between global and local stress methods is that while globalization formulations streamline the optimization problem, they cannot guarantee that maximum stresses are indeed maintained locally. Regional and block aggregation (also called clustering) techniques seek to help restore control of maximum stress levels by using several localized regions that cover the design space. The elements in each region are then aggregated to a single constraint value computed via a global formulation. Therefore, rather than one overall global constraint to account for stresses, multiple constraints are used corresponding to each of the regions, and the aggregation errors are reduced somewhat. Several aggregation methods have been proposed in the literature (París et al. 2010a, b; Le et al. 2010; Holmberg et al. 2013). To date, the regionalized aggregation techniques appear to be the most efficient and robust methods to incorporate stress-based design criteria. Le et al. (2010) also introduced a method to adaptively update the aggregated constraints such that they exactly match maximum stress values, which overcomes one of the primary drawbacks to aggregated methods. A recent alternative by (Luo et al. 2013) is an adaptive aggregation method that treats potentially active and non-active local constraints separately while updating aggregation metrics throughout optimization.

For an excellent overall comparison of several methods previously discussed for density-based topology optimization with stress criteria, see the recent article by Le et al. (2010). Also, París et al. (2010c) presented derivations of first- and directional second-order sensitivities for local, global, and block aggregated stress constraints. Applications to solid/void structures using stress-based criteria can be found in nearly all of the references cited previously in this section. Stump et al. (2007) designed functionally graded structures using stress constraints and density-based topology optimization. Luo and Kang (2012) used Drucker-Prager yield stress constraints (as opposed to von Mises stresses in other works) to account for different stress limits in tension and compression and Rozvany and Sokol (2012) treated different stress limits in an analytical benchmark example. Lee et al. (2012) used regional stress-constraints along with design dependent loading. Finally, Bruggi and Duysinx (2012) explored the case of minimum weight design with both compliance and stress constraints, which combines the strengths of both stiffness and stress-based design.

Related to stress-based criteria, fatigue and damage criteria are also often considered in the design of mechanical components. Sherif et al. (2010) demonstrate a density-based topology optimization that systematically reinforces a structure to achieve a desired damage level. The basic problem solved is a minimum mass problem with a constraint on the upper bound of damage. Their method relies on the use of the Equivalent Static Load method for transient dynamic systems, which is similar to the ESL method described previously for nonlinear structures optimization.

## 2.5 Design dependent loading, supports, and integrated domains

Design dependent loads in topology optimization refer to loads whose location, direction, or magnitude vary along with changes in the design during the optimization process. Examples of these loads include self-weight loading due to gravity, transmissible or pressure loading, and thermal (temperature) loads. Regardless of design dependency type, it is important to capture the dependency during the sensitivity analysis. In addition, we also discuss works that focus on the optimal placement of structural supports and connections between multi-component systems in this section.

### *2.5.1 Transmissible or pressure loads*

Transmissible loads have a constant direction and magnitude, but their location will change throughout topology optimization. Pressure loads demonstrate both direction and location dependency and are commonly used to represent hydrostatic loading from a fluid on the structure. As such, a primary challenge with design dependent transmissible or pressure type loading in a density-based topology optimization problem is determining, at every iteration, the material boundary upon which the loading should be applied. There exist several techniques to do so in the literature, but they can primarily be arranged into two groups.

The first group seeks to identify a fluid/structure boundary and apply loading directly on the finite elements. Hammer and Olhoff (2000) captured the fluid/structure boundary by using iso-density curves along with Bézier splines. Fuchs and Shemesh (2004) also used Bézier curves, but define control points that are independent of density and controlled by the optimizer. Du and Olhoff (2004a, b) extended the work by Hammer and Olhoff (2000) by connecting iso-density points directly along element boundaries without splines to define a loading surface and used finite differences to obtain design-dependent load sensitivities. This idea was further extended by Lee et al. (2012) and Lee and Martins (2012), who used analytical sensitivities and incorporated predefined void regions from which pressure loading originates, which is useful when symmetry conditions cannot be exploited. Similarly, Gao and Zhang (2009) developed a pressure updating model for solid weight pressure for contact problems with a solid object. Finally, Zhang et al. (2008) proposed a simple boundary search scheme where the sensitivity of loading to density can be disregarded since loads are determined from real element boundaries rather than an iso-line. However, the beginning and end locations for the search procedure must be provided and the method may not be readily extended into three-dimensions.

The second group of methods do not explicitly identify a loading surface, but model pressure loading with alternative physics or utilize mixed formulations. Chen and Kikuchi (2001) simulated pressure loading by approximating it as an equivalent thermal load that is penalized with the design and a “dryness coefficient” to identify fluid vs. solid regions. Zheng et al. (2009) introduced a potential function based on the electric potential and applied a fictitious electric field to model pressure loading. Sigmund and Clausen (2007) expressed the void phase as an incompressible hydrostatic fluid and introduced an additional variable for each element. With two variables in an element to determine its phase, pressure loads can be applied without surface parameterization. A similar approach was presented by Bruggi and Cinquini (2009).

### *2.5.2 Self-weight or thermal loads*

Self-weight (body forces) and thermal (temperature) loads represent another type of design dependent loading where the magnitude of the loading is density dependent. The literature related to these types of topology problems is much more limited when compared to the transmissible and pressure loads of the previous section. Bruyneel and Duysinx (2001, 2005) outline the challenges of self-weight loads in density-based topology optimization with a minimum compliance objective. The first challenge relates to non-monotonous behavior of the compliance objective that results from the additional density-dependency of the force vector in the objective function. In addition, the optimum topology is often not found with an active volume constraint, which can preclude the formation of solid/void designs. Finally, the parasitic effects of SIMP interpolation are explored. A modified SIMP formulation is suggested to overcome these effects and it is also proposed that an alternative interpolation scheme (RAMP, Stolpe and Svanberg 2001a) may be suitable as well. Gao and Zhang (2010) identified similar issues in compliance minimization of structures subjected to thermal loading. In both cases, the relative amount of externally applied density-independent loading versus the amount of density-dependent self-weight or thermal loading is very important in determining the optimum design.

### *2.5.3 Design of supports and multicomponent structures*

Most work in structural topology optimization is performed on a design domain with fixed boundary and load conditions. However, it was demonstrated by Buhl (2002) that by allowing the optimizer to simultaneously design structural topology and support locations, superior performance could be achieved when compared to the conventional formulation. This was accomplished by introducing new support variables into the optimization problem and constraining the amount of allowable support. More recently, this particular problem has also been explored by Zhu and Zhang (2010) who allowed for the placement of integrated support members. Rozvany and Sokol (2012) also investigated support placement costs in an analytical benchmark problem.

In a somewhat similar variation of structural topology optimization, the design problem is not only to design structural topology, but also define the location of embedded rigid components or subsystems within the domain. This design formulation is often referred to as the design of multicomponent structures in the literature and effectively places mobile non-design regions inside the domain that may represent functional components that have void, elastic, or rigid properties. Pioneering work on this topic was performed prior to the year 2000, but recently new formulations have been proposed that include location variables for integrated components in the optimization problem. In the literature, these integrated components may be either rigid (Qian and Ananthasuresh 2004) or elastic (Zhu et al. 2008, 2009, 2010). Recently, formulations have been introduced that alleviate the need for semi-analytic sensitivities in favor of analytical sensitivities for location variables with superelement techniques (Xia et al. 2012a, b).

## 2.6 Alternate physics, multiphysics, and applications

One of the primary developments in density-based topology optimization over the last decade has been its application to design problems with physics outside of linear structural responses with stiffness objectives. The following subsections review the application to a variety of engineering disciplines.

### *2.6.1 Heat transfer & thermoelasticity*

While there have been several early applications of topology optimization to heat conduction problems, where the material's thermal conductivity is parameterized via penalization, further developments have occurred more recently. Gersborg-Hansen et al. (2006) developed a SIMP topology optimization technique which utilized the finite volume method to solve the thermal system, as opposed to the usual finite element method. Zhou and Li (2008b, c), investigated topology optimization for obtaining extremal conductivity microstructures. Bruns (2007) proposed a technique that allowed for the application of design dependent convection boundaries to be applied to a heat transfer domain via a new penalization/interpolation scheme to correctly formulate the convection portions of the finite element matrices. This method can also be utilized to model external radiation, where an equivalent nonlinear convection coefficient can be determined. However, to date no topology optimization publications have included the physics of internal or enclosure radiation, which is still required to solve many modern full physics heat transfer design problems. Finally, in an innovative application, Ryu et al. (2012) demonstrated a mobile robot control algorithm that relies on an analogous heat conduction topology optimization problem to determine the optimal robot path throughout a domain with z obstacles.

Thermoelastic problems have been solved using density-based topology optimization techniques as well. Some of the earliest work in this area was related to the design of multiphysics actuators, where deformation due to thermal expansion is controlled in the design of compliant mechanisms (Sigmund 2001a, b). In this work, an efficient coupled field adjoint sensitivity analysis that can capture both the topological dependency of the temperature field and its effect on a sequential thermoelastic response is utilized. This process was also demonstrated by Cho and Choi (2005). As it turns out, compliance minimization problems in the presence of thermal loads pose several additional challenges related to the penalization of the design dependent loads and their contribution to the compliance objective as discussed previously. It is often difficult to eliminate intermediate material density because the volume constraint in the usual minimum compliance problem is rarely active for an optimal design with thermal loads. This was recently explored by Gao and Zhang (2010), who penalized a thermal stress coefficient (TSC) to effectively control both the structural stiffness and design dependent loading. They also demonstrated superior performance of the RAMP interpolation over the SIMP interpolation in their problems. An example of troublesome intermediate density material when using SIMP with thermal loading that was eliminated in their work using the RAMP method and TSC formulation is given in Fig. 4. Pedersen and Pedersen (2010, 2012) demonstrated that minimum compliance designs do not lead to maximum strength designs in the presence of thermoelastic loading for 2D and 3D structures and suggest an alternative problem based on obtaining uniform energy density. They also investigated different interpolation schemes and their effects on the optimization process. Wang et al. (2011a) utilized thermoelastic topology optimization to design optimal bi-material structures with low thermal directional expansion and high stiffness. In a similar design problem, Deaton and Grandhi (2013a, b) investigated various density-based topology optimization formulations for stiffening thin structures with restrained thermal

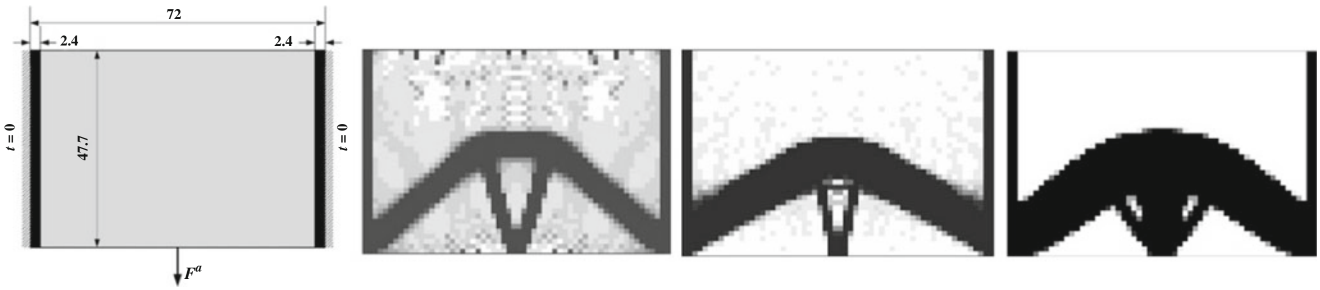



**Fig. 4** **a** Design Domain Benchmark thermoelastic topology optimization problem where SIMP methods exhibit intermediate gray material (**b**) SIMP, $\triangle T = 1$ and (**c**) SIMP, $\triangle T = 3$ that may be eliminated via (**d**) RAMP, $\triangle T = 1$ RAMP interpolation (from Gao and Zhang 2010)

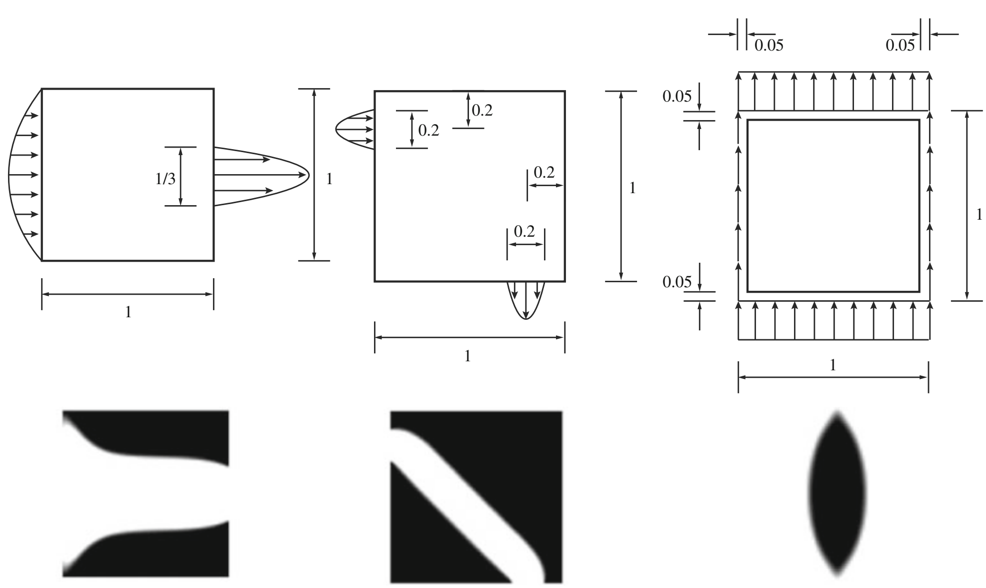
expansion and also investigate stress-constrained thermoelastic problems.

*2.6.2 Fluid flow*

By modeling a potential solid/fluid domain as a region of porous media, topology optimization has been applied to various fluid flows using techniques similar to density-based topological design of elastic continua. In general, both internal flows for channel-like geometry and external flows for optimizing airfoil-like shapes have been studied with the majority of work in the former area. This is demonstrated in Fig. 5. By continuously varying the permeability in a region from nearly completely permeable (fluid) to nearly impermeable (solid) it has been shown in the literature that efficient fluid flow paths may be generated. This was first demonstrated by Borrvall and Petersson (2003), who minimized the dissipated power in a flow domain according to a fluid volume constraint in Stokes flow. The approach was generalized by Evgrafov (2005) to allow the extremal cases of the porous materials, both pure solid and pure flow regions, to appear in the optimization procedure. Aage et al. (2008) also demonstrated the methods on large-scale problems with up to 1,125,000 and 128,000 elements in 2D and 3D, respectively. Alternative formulations for topology optimization of Stokes flow have been proposed by Guest and Prévost (2006b), who utilized a Darcy-Stokes representation, and Wiker et al. (2007), who allowed for discrete regions controlled by Stokes equations and other regions controlled by Darcy's equation.

Many flows encountered in engineering problems cannot be represented under the assumptions of Stokes flow. Thus, some researchers modified the previous works to allow for topology optimization of Navier-Stokes flow. By including inertial effects, Gersborg-Hansen et al. (2005) and Kreissl et al. (2011) focused on topology optimization of incom-

**Fig. 5** Types of fluid topology problems: Minimum power dissipation for **a** Diffuser an optimal diffuser and **b** Pipe Bend optimal pipe bend and **c** Drag Shape minimum drag design for an aerodynamic shape (from Borrvall and Petersson 2003)



pressible Navier-Stokes flow at low Reynolds numbers for channels while Oleson et al. (2006) utilized a similar formulation along with the commercial tool FEMLAB. Evgrafov (2006) also demonstrated topology optimization of Navier-Stokes flow in addition to using filters to improve results and provide stronger convergence. We note that in these applications of topology optimization to fluid domains, system equations are stilled solved via the finite element method (FEM), as opposed to the finite difference method or others used in the computational fluid dynamics (CFD) field. Recently, Yoon (2010c) proposed a monolithic fluid-structure interaction (FSI) formulation capable of performing topology optimization where fluid and solid regions are distinctly represented by the governing Navier-Stokes and linear elasticity equations, respectively. Similarly, Kreissl et al. (2010) utilized topology optimization in an FSI problem to design micro-fluidic devices where the structure with optimal embedded flow paths deforms in response to changes in fluid loading.

### *2.6.3 Dynamics, acoustics, and wave propagation*

While a significant amount of literature exists regarding the topology optimization of vibrating structures and dynamic effects prior to 2000 using homogenization and density-based methods, a number of developments have been made since. These include improvements in basic frequency optimization, extensions to acoustic responses and coupled acoustic structures, and design for control of wave propagation in a variety of media.

A fundamental engineering design problem is that of frequency maximization, which involves the solution of an eigenvalue problem formulated using the mass and stiffness matrices of a structure. When attempting to apply density-based topology optimization to this type of problem, spurious modes are often computed for localized regions of low density material due to the accompanying low stiffness. In reality, these modes are completely artificial since low density elements represent void regions in the structure. In fact, this challenge is also encountered when solving the eigenvalue problem for bifurcation buckling in density-based topology optimization where low density regions may exhibit artificial buckling modes because of their reduced stiffness (Rahmatalla and Swan 2003). To remedy this, Pedersen (2000) proposed a solution in which the degrees of freedom associated with low density elements be removed from the system when numerically solving the eigenvalue problem. Alternatively, Tcherniak (2002) avoided localized modes by setting the element mass to zero in subregions of low material density. Du and Olhoff (2007b) extended this idea with a formulation that places a heavy penalty, when compared to the penalization of stiffness, on the mass of elements with density below 0.1. A number of works have investigated various frequency control problems including frequency maximization (Pedersen 2000; Du and Olhoff 2007b), including geometric nonlinearity (Yoon 2010a), the control of gaps between two frequencies (Du and Olhoff 2007b; Jensen and Pedersen 2006), and the tailoring of structures for specified eigenfrequencies and eigenmodes (Maeda et al. 2006). In addition, Tcherniak (2002) utilized density-based topology optimization to maximize the steady-state dynamic response for a given excitation frequency, which effectively maximizes the resonant response of a structure. Recently, (Yoon 2010b) demonstrated the use of model reduction techniques for increased computational efficiency in topology optimization of dynamic problems with the SIMP method.

Minimum compliance and stiffness design problems have also been addressed for dynamic and transient loads by density-based topology optimization beginning with the introduction of “dynamic compliance” by Jog (2002). Recently, Jang et al. (2012) demonstrated the use of the equivalent static load (ESL) method for compliance minimization of dynamically loaded structures in the time domain. In their work, an equivalent static load, that is a load case which produces equal displacement to the dynamic response, is computed for each time step in the transient response and topology optimization is performed using the statics loads.

Density-based topology optimization has also recently seen applications in wave propagation problems beginning with Sigmund and Jensen (2003) and Halkjaer et al. (2006), who maximized the bandwidth of phononic band-gap materials and structures. Similarly, other works have focused on the optimal material distribution in photonic crystal waveguides, which transmit electromagnetic waves for a variety of objectives (Jensen and Sigmund 2011). These include maximum power transmission (minimum signal loss) for waveguides with various bends (Jensen and Sigmund 2004; Borel et al. 2004), T-junctions (Jensen and Sigmund 2005), and Y-splitters (Borel et al. 2005), matching desired dispersion properties (Stainko and Sigmund 2007; Wang et al. 2011b), and obtaining desirable emission characteristics (Frei et al. 2005). In addition, work by Larsen et al. (2009) demonstrate similar topology optimization for wave propagation; however, the goal is energy transport through elastic media in addition to vibration suppression objectives.

In recent years, density-based topology optimization has also been applied to acoustic design problems. These can be generally grouped into three categories: (i) optimization of a purely acoustic domain described by a Helmholtz equation, (ii) minimization of sound radiation via structural design, and (iii) coupled acoustic-structures. In the first category, Wadbro and Berggren (2006) utilized topology optimization to design an acoustic horn by controlling geometric features inside an acoustic domain. Later, Duhring et al. (2008) optimized room acoustics by geometry control and placement of absorbing/reflective boundary material. In the second category, a number of researchers have utilized topology optimization to reduce acoustic radiation. Olhoff and Du (2006) and Du and Olhoff (2007a, 2010) minimized the total sound power radiated from structural surfaces of vibrating bi-material structures. It is important to note that with this selection of objective function, acoustic analysis is not required. Similarly, (Nandy and Jog (2012) recently studied a dynamic compliance objective for the reduction of radiated noise, which also does not require a dedicated acoustic analysis. On the other hand, the third category of acoustic topology optimization problems necessitate a coupled acoustic and structural analysis domain. Yoon et al. (2006, 2007) utilized a mixed finite element formulation to solve pressure and displacements to minimize the pressure integral (a representation of noise) in a particular region of an enclosed structure.

#### *2.6.4 Aerospace design and aeroelasticity*

As noted by Stanford and Ifju (2009a), the aerospace applications of topology optimization are still relatively rare from a literature standpoint, but applications to flight vehicle structures can be mostly grouped into two categories. The first category uses an aerodynamic solver to compute the pressure distribution over a wing (during typical flight maneuvers). This load is subsequently applied to the topology optimization model to consider classic topology optimization objectives such as compliance. The effect on the loading due to elastic deformation of the structure is not considered. Another category of applications explicitly utilizes aerodynamic forces computed on a flexible wing, which inherently includes aeroelastic coupling between the aerodynamic and structural analysis.

Early work related to density-based topology application of the first category of problems was performed by Balabanov and Haftka (1996), who optimized the internal wing structure of a high speed civil transport (HSCT) aircraft using a ground structure approach. Eschenauer and Olhoff (2001) also demonstrated the conceptual design of aircraft wing ribs via topology optimization using air loads. More recently, Krog et al. (2004) optimized wing box ribs both locally and globally for a variety of design criteria. Luo et al. (2006) performed topological design on a missile body for static pressure loads and natural frequencies. Recently, Wang et al. (2011d) demonstrated a subset simulation-based topology optimization method for the design of wing leading-edge ribs. Finally, Choi et al. (2011) utilized the equivalent static load method (ESL) along with CFD simulation to obtain aerodynamic loading for both path and topology optimization of flapping wings. It is the authors' opinion that the limited work reflected in the literature concerning these applications of topology optimization, where various aircraft components are designed to meet conventional performance measures such as maximum stiffness or minimum frequency given a particular set of loading conditions, is not indicative of the impact of topology optimization in the aerospace industry. We believe it is quite the contrary, where topology optimization is commonly used to develop conceptual designs of a variety of components including wing ribs/spars, landing gears, and various attachments.

The first work to explore aeroelastic design as described in the second category above was that of Maute and Allen (2004), who designed wing stiffeners using a 3D Euler solver, linear finite element model, and adjoint sensitivity analysis to minimize mass with constraints on lift, drag, and wing displacement. This work was later extended by Maute and Reich (2006) for compliant morphing mechanisms inside an airfoil for both active and passive shape control. Stanford and Ifju (2009a, b) utilized topology optimization for the design of membrane aircraft wings for a variety of objectives including load augmentation, load alleviation, and efficiency. In addition, Stanford and Beran (2011) also utilized density-based topology optimization to develop optimal compliant drive mechanisms for flapping wings. Recently, Leon et al. (2012) demonstrated aeroelastic tailoring with combined composite fiber orientation and topology optimization to minimum mass design of thin wings subjected to flutter velocity constraints using the commercial tool ZAERO for aeroelastic stability analysis. While not a density-based method, Gomes and Suleman (2008) utilizes level-set topology optimization (discussed in Section 4) to optimize a reinforced wing box for enhanced roll maneuvers. We have chosen to include this reference here for completeness of aerospace and aeroelastic applications of topology optimization.

#### *2.6.5 Multifunctional materials*

The area of multifunctional materials and structures is one field where topology optimization shows great promise. In these types of design problems, objectives include obtaining desirable component properties or characteristics, including mechanical, thermal, electromagnetic, chemical, flow, and weight, that span a range of engineering disciplines. Problems are typically set up using multiobjective techniques for the competing physics and the basic methods in the proceeding references are readily extendable to include other properties. Torquato et al. (2003) demonstrated this concept via topology optimization for both maximum thermal and electrical conductivity. Later, Guest and Prévost (2006a, 2007) and Chen et al. (2009) designed periodic structures for both stiffness and fluid permeability while Chen et al. (2010b) and de Kruijf et al. (2007) designed for stiffness and thermal conductivity. With continued advances in the computational simulation of alternative physics, with the appropriate modifications, topology optimization should continue to prove valuable to the development of components with innovate multifunctional capabilities.

### *2.6.6 Biomedical design*

Recently, density-based topology optimization was demonstrated for the design of patient-specific facial bone replacements (Sutradhar et al. 2010). According to this work, a conventional surgical procedure to repair severe facial injuries is an extremely time intensive process with significant ad hoc effort, in real-time, on the part of the surgeon using tissue from bones that are dissimilar from those of the face. In order to improve this process, a custom fabrication procedure is envisioned where topology optimization is utilized to tailor a bone replacement to an individual by determining a functional load-carrying structure according to the injured skull geometry and particular type of injury. An example of this is demonstrated in Fig. 6, where the particularities of the optimized facial structure have been tailored to fit the specific injury. This field has tremendous potential to use topology optimization in the design of artificial limbs, hip and knee joint replacements, and medical implant devices. References in the area of biomechanics and topology optimization of bone mechanics prior to 2000 may be found in the review by Eschenauer and Olhoff (2001). In addition, significant work in the area of tissue engineering scaffold structures has been performed in the last decade. In these problems, which have similar objectives to multifunctional materials previously discussed, a material microstructure is sought with desirable stiffness and porosity properties to enable rapid skeletal tissue regeneration once implanted in the human body (Lin et al. 2004; Hollister 2005; Kang et al. 2010; Chen et al. 2011a; Sturm et al. 2010). Chen et al. (2011c) also incorporated a degradation model to investigate biodegradable scaffold design. Further applications of topology optimization in the area of biomedical design are discussed in later sections using the ESO and level set methods.

## 2.7 Reliability-based and robust topology optimization

While reliability-based design optimization (RBDO) techniques have long been applied to sizing and shape optimization problems (see Choi et al. 2007), it was not until the mid-2000s that these statistical and probabilistic design methods were introduced in topology optimization. Early reliability-based topology optimization (RBTO) papers utilized first-order reliability methods (FORM) or performance measure approaches, both of which are suited to topology optimization problems with large numbers of design variables. Both methods account for uncertainties related to material, loading, and geometric dimensions in the topology optimization for minimum compliance (Kharmanda et al. 2004), micro-electromechanical systems (MEMS) (Maute and Frangopol 2003; Wang et al. 2006a), and geometrically nonlinear structures (Jung and Cho 2004) under a probabilistic failure constraint. It is evident from Fig. 7, that with the addition of such a constraint, we obtain different optimal topologies that are statistically less likely to fail. Kang et al. (2004) performed RBTO of electromagnetic systems where permeability, coercive force, and applied current density were taken as normally distributed uncertain variables. Kim et al. (2007a) developed a RBTO method for applica-

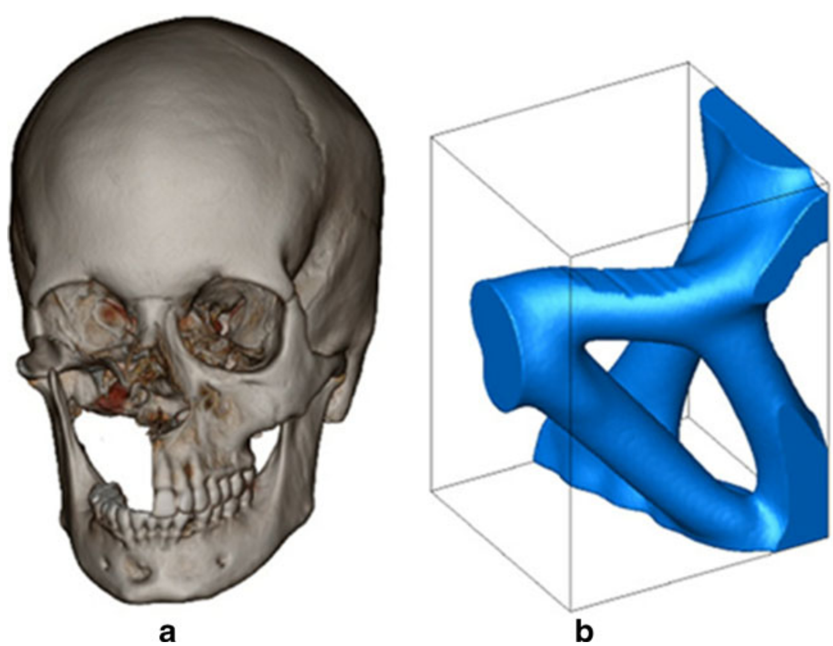



**Fig. 6** **a** Skull with craniofacial injury and **b** patient specific bone replacement designed via topology optimization (from Sutradhar et al. 2010)

tion to MEMS devices that utilizes parallel computing and an advanced response surface method. Rozvany and Maute (2011) developed an analytical solution to an elementary reliability-based topology optimization problem and compared results obtained using a SIMP method. An important aspect of the previous references is that they use a double-looped procedure where an inner reliability analysis loop is required to determine the probability of failure for each configuration of the outer optimization loop. More recently, Silva et al. (2010) demonstrated a single-loop system level reliability-based topology optimization method. Nguyen et al. (2011) also used a single-loop technique to capture statistical dependence of multiple limit-state functions. Finally, Kang and Luo (2009) proposed a non-probabilistic reliability-based topology optimization method using convex models for cases where probability distribution data is unavailable.

In a similar sense, robust topology optimization formulations have received much attention of late. Rather than stating a probabilistic failure metric, uncertainties are introduced into the topology optimization problem to study the sensitivity (or robustness) of a resulting design to uncertainties or variation in problem parameters based strictly on stochastic moments of a system response. In the literature, design robustness against several engineering parameters using density-methods have been studied including loading

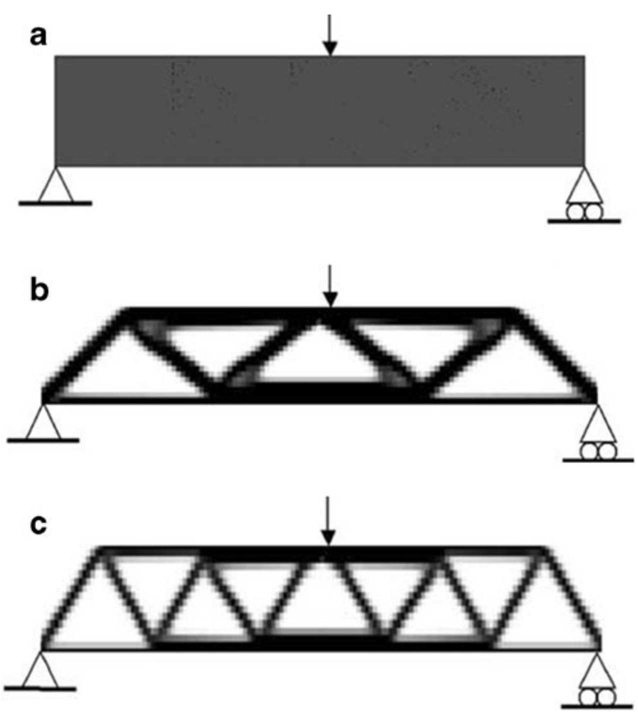



**Fig. 7** Topology optimization and RBTO of the MBB-beam: **a** full design domain, **b** resulting deterministic topology optimized beam, and **c** resulting RBTO structure (from Kharmanda et al. 2004)

(Guest and Igusa 2008; Lógó et al. 2009), manufacturing errors (Sigmund 2009; Wang et al. 2011c; Schevenels et al. 2011; Qian and Sigmund 2013), and material properties (Tootkaboni et al. 2012; Lazarov et al. 2012). Further discussion for uncertainties in other topology methods in later sections.

## 2.8 Additional comments

### *2.8.1 Educational resources*

Several educational resources that demonstrate density-based topology optimization are publicly available. These include two MATLAB implementations using the SIMP formulation: the original 99 line code by Sigmund (2001a) and a more recent 88 line variant by Andreassen et al. (2011) that demonstrates several of the developments discussed in this paper. Of merit in Andreassen et al. (2011) is the application of a Heaviside filter to achieve crisp black-white topology results and discussions related to the solution of very large topology optimization problems in MATLAB. It is a recommended resource for both new and experienced practitioners of topology optimization. Talischi et al. (2012) also provide a MATLAB implementation for general topology optimization that uses unstructured polygonal finite element meshes. In addition, a web-based version of SIMP topology optimization (Tcherniak and Sigmund 2001) and a mobile application (Aage et al. 2013), for the iPhone or Android for example, using SIMP have been developed by the same research group. Finally, the monograph by Bendsøe and Sigmund (2003) remains as perhaps the most useful tool for a new researcher in the area of topology optimization.

### *2.8.2 Choice of optimizer*

Several benchmark, academic, and early works in density-based topology optimization were solved using optimality criteria methods. However, general optimization codes (e.g. CONLIN, DOT, MMA, SNOPT, etc.) offer black-box solutions to general constrained optimization problems. In addition, the Method of Moving Asymptotes (MMA) optimizer (Svanberg 1987) has become the de facto standard tool for density-based topology optimization, especially in multiphysics applications. Public MATLAB and FORTRAN implementations of MMA, along with a more recent, more robust globally convergent implementation (GCMMA, Svanberg 2002), can be obtained by contacting Professor Svanberg.

### *2.8.3 Efficiency & multiresolution methods*

The primary challenge to expanding topology optimization to large-scale problems is the computational requirement of iteratively solving a large system of state equations to feed the optimizer system responses and sensitivities. In the literature, attempts to address this can be broadly divided into three groups: (i) employing large-scale computing resources, (ii) introducing efficient solution procedures for finite element analysis, and (iii) reducing the total number of degrees of freedom in the analysis model. Several researchers have applied the first method via parallel or distributed computing to more rapidly compute system responses (Borrvall and Petersson 2001a; Kim et al. 2004; Vemeganti and Lawrence 2005; Evgrafov et al. 2008; Mahdavi et al. 2006; Aage et al. 2008). Aage and Lazarov (2013) also demonstrated parallel topology optimization where the popular MMA optimizer is parallelized as well.

Related to the second method, Andreassen et al. (2011) specify several simple concepts for computational efficiency in MATLAB implementations of topology optimization. Wang et al. (2007a) introduced an efficient solver utilizing preconditioning and subspace recycling. Amir et al. (2009, 2010), demonstrated more efficient use of iterative solvers in nested topology optimization in addition to an approximate reanalysis procedure where FEA is performed only at certain intervals of iteration and approximate reanalysis is used otherwise. Finally, Oded and Sigmund (2011) contributed additional discussion regarding methods that fall into the second category.

The third method has seen a wider variety of techniques developed in the literature. As previously cited, Yoon (2010b) proposed model reduction techniques for dynamic problems. Several studies have also investigated the use of multiple mesh sizes where optimization begins on a coarse mesh and continues later on a finer mesh by utilizing numerical continuation methods (Kim and Kwak 2002). In a different multi-mesh concept, Nguyen et al. (2010) proposed a multiresolution formulation based on a coarse finite element mesh and fine density and design variable meshes as shown in Fig. 8. Figure 8a shows a Q4 displacement element commonly used in topology optimization, Fig. 8b presents the multiple meshes, and Fig. 8c shows the density mesh with 25 density elements (25 design variables). In this way, high resolution designs can be obtained with lower computational cost, due to a smaller analysis model, compared to a uniform mesh at all levels. Recently, improvements to this method were suggested and an adaptive mesh refinement scheme was incorporated to further increase efficiency (Nguyen et al. 2012). The mechanisms at work in these multiresolution methods are similar to previous papers that effectively allow for variation of density within finite elements and are analogous to earlier work based on CAMD approaches (Rahmatalla and Swan 2004; Matsui and Terada 2004; Paulino and Le 2009). A number of other adaptive mesh refinement techniques for density-based topology optimization have also been demonstrated in the literature (Kim et al. 2003; Stainko 2006; Guest and Genut 2010).

### 2.8.4 Non-zero prescribed boundary conditions

The majority of research in density-based topology optimization is based on problems where structures are subjected to external forces and zero-prescribed displacement boundary conditions (or the appropriate analogs in alternative physics problems). However, sometimes structures are subjected to prescribed loading conditions and non-zero prescribed displacements. These instances have been recently been investigated for compliance, stiffness, and strength considerations (Cho and Jung 2003; Pedersen and Pedersen 2011; Niu et al. 2011; Klarbring and Strömberg 2012, 2013). Analogies to this problem are also seen in many of the fluid optimization problems previously referenced.

### 2.8.5 Manufacturing constraints

The results of a topology optimization are only valuable if the design can be manufactured at acceptable cost. Thus, in practice, feasibility in topology optimization is maintained using additional constraints on the problem that are representative of the limitations for a particular manufacturing process, for example casting or tooling tolerances. Examples of this include Zhou et al. (2002), Harzheim and Graf (2002, 2006), and Leiva et al. (2004).

### 2.8.6 Commercial software implementation

To the authors' best knowledge, all commercial structural optimization or finite element analysis tools that include topology optimization capabilities are based on some variant of the SIMP density-based method. These include Vanderplaats Research & Development GENESIS, MSC.Software Nastran, Altair OptiStruct, Abaqus, Fe-design TOSCA, ANSYS Workbench, and COMSOL Multiphysics. It is noted that compared to the scope of multidisciplinary applications highlighted in this paper and the other multiphysics analysis options in the commercial packages mentioned, their topology optimization capabilities are generally limited to structural problems with global responses, including stiffness and frequency, with linear physics. Some packages do provide provisions for constraints of local nature; however, details of their implementation are not readily available in the literature. In addition, some success has been documented utilizing the equivalent static loads (ESL) method with

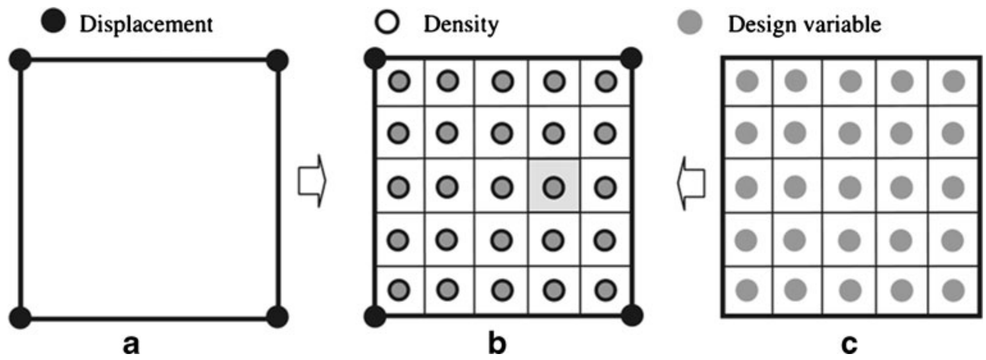



**Fig. 8** Multiresolution Topology Optimization (MTOP) element: **a** Q4 displacement element, **b** superimposed meshes, **c** finer design variable mesh (Nguyen et al. 2010)

commercial tools for topology optimization of nonlinear structures. The growing use of commercial optimization software with topology capabilities in automotive, aerospace, and other engineering industries is a primary impetus for continuing research in topology optimization methods (Pedersen and Allinger 2006; Schramm and Zhou 2006).

*2.8.7 Extended optimality*

Extended optimality involves a increasing the solution space of the optimization problem to simultaneously consider volume fraction, thickness (for 2D problems), and topology and was introduced by Rozvany et al. (2002), Rozvany (2009b). A 3D analogue considers a set of materials whose density is proportional to their elastic modulus or strength rather than thickness. In the references, extended optimality was shown to result potentially in a much lower structural weight when compared to traditional topology optimization.

## 3 Hard-kill methods

Hard-kill methods of topology optimization work by gradually removing (or adding) a finite amount of material from the design domain. The choice of the material to be removed or added is based on heuristic criteria, which may or may not be based on sensitivity information, and in contrast to density-based methods, the discrete design space is not relaxed. The most well known hard-kill method of topology optimization is known as Evolutionary Structural Optimization (ESO) originally proposed by Xie and Steven (1993, 1997). A recent text by Huang and Xie (2010b) provides an excellent overview of new developments in topology optimization using the ESO methods.

One of the most attractive features of hard-kill methods such as ESO is the simplicity with which they can be utilized with commercial finite element packages. Often times, the integration of the algorithms with FEA solvers requires only simple pre- or post-processing steps. In addition, hard-kill methods for topology optimization result in a design with crisply defined structural boundaries that are free of intermediate or gray material because finite elements are explicitly defined as existent or absent. The basic optimization problem for ESO of the common minimum compliance problem is given as:

$$\begin{aligned} \min :\quad & c = \mathbf{U}^T\mathbf{K}\mathbf{U} \\ \text{subject to} :\quad & \frac{V}{V_0} \leq V_f \\ & \mathbf{K}\mathbf{U} = \mathbf{F} \\ & \boldsymbol{x} = [0, 1] \end{aligned} \tag{2}$$

Here variables are defined identically to those in (1) with the exception of **x**, which is the vector of element design variables. We note the primary difference in basic formulation between a hard-kill method and density-based method that design variables are now taken as the existence ($x_e = 1$) or absence ($x_e = 0$) of finite elements rather than their associated physical or material properties.

### 3.1 Element removal and addition

In all the variants of ESO-type methods, a criterion function is calculated for each element, which is often called the sensitivity number, and element removal/addition is applied to elements with low criteria values. The following sections highlight the progression of ESO methods and the recent developments regarding how the algorithmic decision of whether to retain, add, or delete elements is made.

*3.1.1 Early ESO algorithms*

The original versions of the ESO method (Xie and Steven 1997) allow for only the removal of material and were based on the idea that an efficient structural component is one where all of the stresses are nearly uniform at some safe level. This notion leads to a natural criterion function of local elemental stress, where low-stressed elements are removed iteratively according to (3) where $\sigma_e^{vm}$ is the von Mises stress in element $e$, $\sigma_{max}^{vm}$ is the maximum von Mises stress in the structure and $RR_i$ is the rejection ratio for the current iteration.

$$\frac{\sigma_e^{vm}}{\sigma_{max}^{vm}} < RR_i \tag{3}$$

Additional formulations were created to solve maximum stiffness and displacement optimization problems where the element removal criterion is based on a sensitivity number (Liang et al. 2000a) as given by (4) for a mean compliance objective and $\alpha_i^e$ is the sensitivity number of element $i$, $\mathbf{u}_i$ is the displacement vector and $\mathbf{K}_i$ is the stiffness matrix for element $i$.

$$\alpha_i^e = \frac{1}{2}\mathbf{u}_i^T\mathbf{K}_i\mathbf{u}_i \tag{4}$$

Since ESO is susceptible to checkerboarding similar to density-based methods, Li et al. (2001a) introduced a simple algorithm to alleviate these numerical difficulties. In addition, performance-based methods were proposed where an ESO-type element removal procedure was performed and optimal topologies were selected from iteration histories (Liang et al. 2001; Liang and Steven 2002).

### 3.1.2 Bi-directional ESO (BESO)

After the proposal of an additive ESO procedure (AESO), whereby elements were added to a very simple base kernel structure (Querin et al. 2000a), early versions of bi-directional ESO (BESO) were developed in which elements could be both added and removed (Querin et al. 1998). BESO with element addition adjacent to those with low criterion values was introduced for both stress-based (Querin et al. 2000b) and stiffness/displacement (Yang et al. 1999) criterion functions using sensitivity numbers. It is important to note that the sensitivity numbers for void elements, which do not exist in the FEA model, are computed using extrapolation methods that are not consistent with those utilized for the solid elements. Yang et al. (2003) developed BESO including the perimeter constraint and Kim et al. (2000b, 2002a) introduced cavity control techniques, both of which afford geometric control of structural complexity. Fixed grids for ESO/BESO were also proposed to reduce the computational time associated with building the finite element matrices (Kim et al. 2000a, 2002b). These works also utilize nodal sensitivities and boundaries are not optimized elementally. More recently, Huang and Xie (2007b), Huang et al. (2006) proposed a modified BESO for compliance minimization that uses nodal sensitivity numbers (as opposed to elemental measures) with a mesh dependency filter that exhibits stable convergence to mesh independent and checkerboard free solutions. In this scheme, nodal sensitivity numbers are determined by first averaging elemental sensitivity numbers by (5) where $M$ denotes the total number of elements connected to the $j$th node.

$$\alpha_j^n = \sum_{i=1}^{M} w_i \alpha_i^e \tag{5}$$

$w_i$ is the weight factor of the $i$th element and is defined by

$$w_i = \frac{1}{M-1}\left(1 - \frac{r_{ij}}{\sum_{i=1}^{M} r_{ij}}\right) \tag{6}$$

where $r_{ij}$ is the distance between the center of the $i$th element and the $j$th node. The mesh dependency filter is given by (7)

$$\alpha_i = \frac{\sum_{j=1}^{K} w(r_{ij})\alpha_j^n}{\sum_{j=1}^{K} w(r_{ij})} \tag{7}$$

where $K$ is the total number of nodes to inside the filter domain with radius $r_{min}$ and $w(r_{ij})$ is the linear weight factor determined as

$$w(r_{ij}) = r_{min} - r_{ij} \quad (j = 1, 2, \ldots, K) \tag{8}$$

The sensitivity numbers are then sorted and elements are removed if their sensitivity number is less than a deletion threshold value, $\alpha_{del}^{th}$, and added if their sensitivity number is greater than addition threshold, $\alpha_{add}^{th}$. The values of the thresholds are updated throughout optimization according to predefined algorithm parameters. A significant improvement in this work over legacy BESO is that the sensitivity numbers for both solid and void elements are computed consistently, which proved to increase the robustness of the algorithm. In subsequent sections of this paper we refer to the former development as the improved hard-kill BESO method.

### 3.1.3 "Soft-kill" BESO

As an alternative to conventional ESO and other "hard-kill" methods, wherein elements are completely removed from the computational model, a number of early works advocate using a very soft element. This is highlighted in the Sequential Element Rejection and Admission (SERA) method proposed by Rozvany and Querin (2002). Doing so allows for the computation of admission criteria based directly on the void elements, rather than extrapolating criteria from nearby solid elements.

Recently, with void elements being retained by some means in the computational model, BESO methods have been introduced that utilize analytical sensitivities with respect to density values to determine the element rejection/admission criteria. Zhu et al. (2007) and Huang and Xie (2009) independently proposed methods that employ a penalized density measure similar to the SIMP density-based method. In these works, which will be henceforth referred to as soft-kill penalty-based BESO, the sensitivity number for element rejection/admission is computed using sensitivity analysis of compliance with respect to the penalized density measure as opposed to strain energy criterion. The new penalty based sensitivity number for a compliance objective is given by (9) where $p$ is the penalty, $x_i$ is the density, $x_{min}$ is the minimum allowable density, and $\mathbf{K}_i^0$ is the stiffness matrix of the solid element. The educated reader will note the resemblance to the compliance sensitivity in the SIMP formulation.

$$\alpha_i = -\frac{1}{p}\frac{\partial C}{\partial x_i} = \begin{cases} \frac{1}{2}\mathbf{u}_i^T\mathbf{K}_i^0\mathbf{u}_i & x_i = 1 \\ \frac{x_{min}^{p-1}}{2}\mathbf{u}_i^T\mathbf{K}_i^0\mathbf{u}_i & x_i = x_{min} \end{cases} \tag{9}$$

Similar to previously proposed "soft-kill" methods, elements are directly interpreted as solid/void with no intermediate values. Finally, it was demonstrated by Huang and Xie (2009) that legacy "hard-kill" BESO methods are a special case of this soft-kill penalty-based BESO method with infinite penalty. In addition, in a recent paper Huang and Xie (2010c) extended the soft-kill penalty-based BESO to include local displacement constraints.

### 3.2 Nonlinear responses

Since elements are explicitly added or removed from the finite element analysis model when utilizing ESO/BESO techniques, optimization with nonlinear responses is relatively simple compared to that for the non-discrete, density-based methods where low density elements create numerical difficulties. Huang and Xie first demonstrated BESO with both material and geometric nonlinearity in structural analysis using both the conventional BESO (2007a) and the improved hard-kill BESO method with filtering (2008b) for stiffness optimization. In a separate work, Huang et al. (2007) demonstrate BESO with nonlinear material and geometry for energy absorption objectives typically found in the automotive industry for crashworthiness design.

### 3.3 Design dependent loading

Just as in density-based methods, the design dependency of loads requires special consideration in ESO/BESO. By modifying the sensitivity number, Yang et al. (2005) demonstrated that the conventional BESO procedure can accommodate transmissible loads, surface loading with fixed load direction, and self-weight body loads. Ansola et al. (2006) also proposed a modified sensitivity number for self-weight loads in ESO. Recently, Huang and Xie (2011) presented a formulation for self-weight loading using the soft-kill penalty-based BESO method. In this work, the sensitivity analysis used to compute the sensitivity number was modified to capture self-weight load dependency consistent with density methods and results were compared to those obtained via improved hard-kill BESO.

### 3.4 Genetic ESO/BESO

While genetic algorithms (GAs) have been directly applied to topology optimization problems in the literature where their binary nature intuitively lends itself well to the determination of solid/void material, they have not experienced widespread acceptance due in part because it is difficult to ensure structural connectivity because of GA's stochastic search procedure in addition to extra computational expense. For some successful applications of topology optimization using GAs, the reader is directed to the work by Wang et al. (2006b), Guest and Genut (2010), Bureerat and Limtragool (2006), which utilize bit array representations of the design domain. On a different path, Liu et al. (2008a) overcome conventional challenges of GAs by treating each element in a structural domain as an individual of the GA population rather than generating a large number of individual structural designs to form a population. In doing so, the usual ESO sensitivity number is used as the fitness function and less fit elements die off throughout the evolutionary procedure just as less fit designs die off in a conventional GA design problem. This method is termed genetic evolutionary structural optimization (GESO). In theory, GESO holds an advantage over conventional ESO because the probabilistic criteria involved in the GA helps to avoid the convergence of ESO towards only local optima. Additional examples of the application of GESO are given in the recent paper by Liu and Yi (2010). Finally, Zuo et al. (2009) developed a genetic BESO method that is similar to GESO and utilizes a BESO formulation much like the improved hard-kill BESO.

### 3.5 Multicriteria methods

In ESO/BESO, element rejection/admission rules are based on the sensitivity of a single objective functional. As such, multiple design criteria, from single or multiple physics, cannot be solved as readily as compared to density-based topology optimization where additional constraints can simply be added to the optimization problem. Therefore to solve multicriteria design problems using ESO/BESO, all responses of interest must be combined into a single function, in effect forming a multiobjective problem. This is also true for multiple load cases. Proos et al. (2001a) demonstrated both a weighting and a global method for multicriteria ESO and demonstrated them for combinations of von Mises stress and frequency. In the weighting method, the normalized sensitivity numbers for each criteria are assigned a weighting factor and are summed to form a single new objective criteria as given by

$$F^i_{multicrit} = \sum_{j=1}^{N} w_j R^i_j \tag{10}$$

where $F^i_{multicrit}$ is the new multiple criteria function, $w_j$ is the $j$th criteria weight factor with $w_j \geq 0$, and $R_j = \alpha^i_j/\alpha^*_j$ is the ratio of the $j$th criteria sensitivity number for each element $i$ to the maximum value of the $j$th criteria sensitivity number, and $N$ is the total number of criteria. Note that the criteria weights must sum to unity. On the other hand, the global criteria method is based on the formulation of a metric function that represents the distance between the ideal solution (minimum value of each criterion) and the optimum solution as given by (11).

$$G^i_{multicrit} = \left[\sum_{j=1}^{N} \left(R^i_j - S^i_j\right)^p\right]^{1/p} \tag{11}$$

Here, $G^i_{multicrit}$ is the metric for the multiple criterion function for element $i$, $S^i_j = \alpha^{min}_j/\alpha^*_j$ is the ratio of the minimum value of the $j$th criterion sensitivity number to the maximum value of the $j$th criterion sensitivity number, and $p$ is a constant constrained by the condition $1 \leq p \leq \infty$. In other work, these methods were investigated for stiffness and inertial design criteria (Proos et al. 2001b). Similarly, Kim et al. (2006) proposed a combined static/dynamic control parameter for use in the ESO for thermal stress and frequency criteria. Their method is demonstrated by application to a spacecraft thermal protection component, which demonstrates both lower thermal stresses and higher fundamental frequency than baseline designs.

## 3.6 Alternate physics and applications

Just as with density-based topology optimization, ESO has been applied to a number of different problems outside of structural stiffness design. An important decision in the application of ESO to alternative physics problems is the identification of the appropriate sensitivity number to guide element addition or removal. In fact, a number of different alternative physics examples were presented by Steven et al. (2000) related to general field problems including heat conduction, elastic torsion, inviscid incompressible fluid flow, electrostatics, and magneto-statics.

### *3.6.1 Heat transfer and thermoelasticity*

Prior to the year 2000, the original formulations of ESO were applied to a number of simple heat transfer and thermoelastic design problems. References can be located in the works by Li et al., who investigate heat conduction (2004), combined thermal and structural criteria (2000), and thermoelasticity due to various temperature distributions (2001c) with ESO. Patil et al. (2008) employed ESO for topology optimization of extremal conductivity microstructures. Gao et al. (2008) utilized BESO to design heat conducting structures with design dependent heat loads. Most recently, Ansola et al. (2010) demonstrated a BESO method for the design of thermal compliant actuators subjected to a uniform temperature distribution and non-uniform heating including conduction and convection effects (Ansola et al. 2012).

### *3.6.2 Buckling and vibrations*

Similar to heat transfer and thermoelasticity, some research related to vibrations and buckling (eigenvalue-type problems) in structural mechanics using the original ESO was performed prior to the year 2000. References may be found in Rong et al. (2001), which also presents an application of ESO for critical buckling loads. More recently, Huang et al. (2010) demonstrated frequency optimization using the new soft-kill penalty-based BESO method. One potential advantage to ESO/BESO for these problems is that the algorithm is not susceptible to the numerical instabilities when solving eigenvalue problems related to low density elements as encountered in density-based methods.

### *3.6.3 Biomedical design*

Limited work in the area of biomedical design has been performed utilizing the BESO method. For example, Chen et al. (2011b) utilized BESO to study tissue scaffolds using a wall shear stress criterion and the effect of flow induced erosion.

### *3.6.4 Applications*

While ESO/BESO has not enjoyed the same level of vast acceptance in industrial topology optimization and commercial software, it has found a number of interesting applications. Several civil engineering applications include reinforced concrete strut-and-tie models (Liang et al. 2000b), connection patterns for joints (Li et al. 2001b), bridge design (Guan et al. 2003), underground mining cavity shape (Ren et al. 2005), and tunneling engineering (Liu et al. 2008b). In addition, several real world civil applications on existing structures including bridges, buildings, and supports designed using ESO methods are highlighted in the recent text by Huang and Xie (2010b). Das and Jones (2011) utilized conventional ESO to design aircraft bulkheads for strength and low weight and Naceur et al. (2004) demonstrated ESO for sheet metal blank optimization. Ansola et al. (2007) demonstrated the use of ESO for compliant mechanism design and were able to develop topologies analogous to those obtained via SIMP. Recently, Huang et al. (2012) utilized BESO to design multifunctional periodic composites with both extremal magnetic permeability and electrical permittivity and Yang et al. (2011) did so for stiffness and thermal conductivity. A number recent works have investigated BESO methods for topology optimization of material microstructures (Yang et al. 2013; Huang et al. 2013; Zuo et al. 2013).

### 3.7 RBTO with BESO

Compared to reliability-based topology optimization utilizing density-based methods, RBTO applications with ESO/BESO are currently much more scarce. Kim et al. (2007b) demonstrated the use of ESO with first-order reliability methods in RBTO. Recently, Eom et al. (2011) demonstrated RBTO using the improved hard-kill BESO along with a response surface method (RSM) to compute the reliability index using a first-order reliability method. In this work uncertainties included material properties and applied loading. Also, Cho et al. (2011) used BESO and the performance measure approach (PMA) to find the reliability index where uncertainties included stiffness, applied load, and dimensions for a multi-objective problem.

### 3.8 Additional comments

#### *3.8.1 Educational resources*

For resources related to the most recent developments in both hard-kill and soft-kill ESO/BESO methods for topology optimization, the reader is again referred to the manuscript by Huang and Xie (2010b). In addition, a short MATLAB code for soft-kill BESO and details for a publicly available BESO program from the authors can be located in the appendix of that text. The original developments of ESO formulations are reviewed in the book by Xie and Steven (1997).

#### *3.8.2 Reviews & critiques of ESO/BESO*

Early versions of the ESO/BESO algorithms have faced criticism beginning with Zhou and Rozvany (2001), who demonstrated a breakdown of the methods on a simple cantilever tie-beam example. Further discussion is offered by Rozvany (2001c) where the ESO method is compared against classical Fully Stressed Design (FSD) and Minimum Compliance (MC) sizing optimization methods. In an effort to explain with technical rigor the workings of ESO, Tanskanen (2002) investigated the theoretical aspects of the method. More recently, in a brief technical note, Huang and Xie (2008a) offer potential solutions to the long-standing concerns raised in Zhou and Rozvany (2001). In addition, Edwards et al. (2007) independently offered a comparison study in support of the ESO methods. Finally, Rozvany (2009a) presents a summary of the previous criticisms in addition to critical opinions regarding the supporting papers in a forum article. Most recently, Huang and Xie (2010a) responded in friendly discussion to these constructive criticisms in a forum article that highlights the newest fundamental developments for ESO-type methods including several of the filtering and soft-kill methodologies cited previously in this review. It is advised to first develop a soft-kill BESO method for a new topology optimization problem and then investigate the application of hard-killing elements to increase computational efficiency. Additional works have also commented on various aspects regarding the application of ESO/BESO methods (Abolbashari and Keshavarzmanesh 2006).

## 4 Boundary variation methods

Boundary variation methods are a most recent development in structural and multidisciplinary topology optimization with their roots lying in shape optimization techniques. In contrast to density-based methods, they are based on implicit functions that define structural boundaries rather than an explicit parameterization of the design domain. Figure 9a shows an explicit representation where the domain, $\Omega$, exists as an explicit parameterization of variables $x$ between 0 and 1. The structural boundary $d\Omega$ then exists at the interface of regions 0 and 1. Figure 9b demonstrates an implicit representation where the structural boundary is implicitly specified as a contour line of the field $\Phi$, which is a function of $x$.

Two boundary variation techniques currently undergoing development in the literature are the level set method and the phase-field method. These methods produce results in the design domain with crisp and smooth edges that require little post-processing effort to interpret results. In addition, these methods are fundamentally different than shape optimization techniques because they allow for not only the movement of structural boundaries, but also the formation, disappearance, and

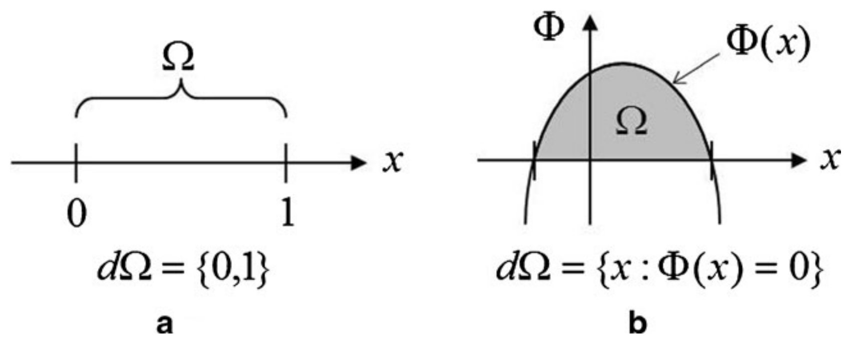



**Fig. 9** **a** Explicit versus **b** Implicit representation of a design domain and boundaries

merger of void regions, which defines true topological design.

## 4.1 Level set topology optimization

In the level set method, boundaries are represented as the zero level curve (or contour) of a scalar function Φ (the level set function) as shown simply for 2D topologies in Fig. 10. Boundary motion and merging, as well as the necessary introduction of new holes, are performed on this scalar function. The shape of the geometric boundary is modified by controlling the motion of the level set according to the physical problem and optimization conditions. It is also important to note here that while a smooth boundary representation is realized in the design domain as shown in Fig. 10, most level-set formulations rely on finite elements. Thus, boundaries are still represented by a discretized, likely unsmooth, mesh in the analysis domain unless alternative techniques are utilized to map the geometry to the analysis model.

Level sets for moving interface problems in physics were first developed by Osher and Sethian (1988) with the fundamental goal of tracking the motion of curves and surfaces and have since been applied in a wide variety of research areas (Sethian 1999; Osher and Fedkiw 2002). The level set method was first applied to topology optimization in the early 2000s by Sethian and Wiegmann (2000), where it was used to capture the free boundary of a structure in linear elasticity, and Osher and Santosa (2001), who combined level sets with a shape sensitivity analysis framework for optimization of structural frequencies. For more comprehensive discussion of level set methods for topology optimization, we refer interested readers to the review articles of Burger and Osher (2005) and most recently van Dijk et al. (2013).

### *4.1.1 Conventional level set topology design*

The development of what are considered conventional level set methods in the modern day began when Wang et al. (2003) identified the velocity of points on the structural boundary and the design sensitivity as a critical link between the general structural optimization process and the level set method for boundary definition. First let the structural

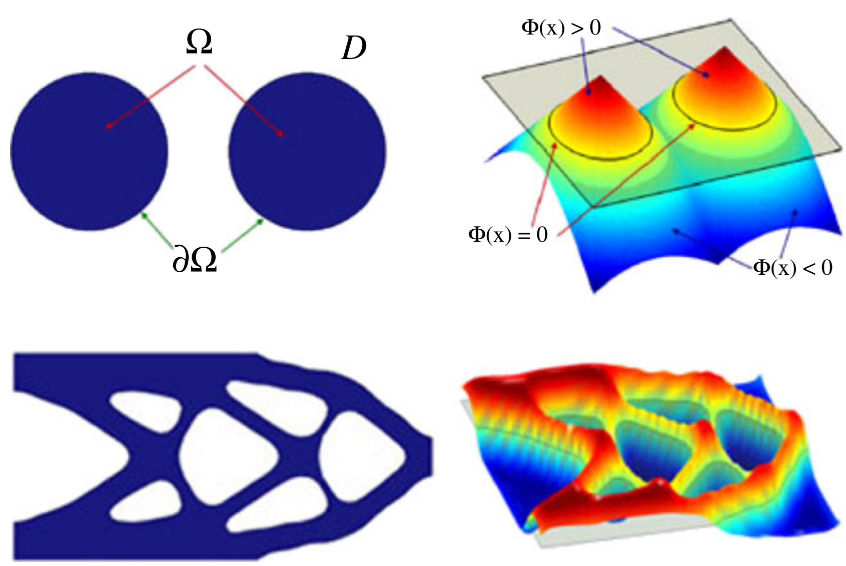



**Fig. 10** Level set representations: **a** 2D topology with (**b**) corresponding level set function along with a more complicated representation of benchmark structure (**c**) & (**d**) (from Luo et al. 2012)

boundary be specified as a level set in implicit form as an iso-surface of a scalar function in 3D as (12)

$$S = \{x : \Phi(x) = k\} \tag{12}$$

where $k$ is the iso-value and is arbitrary, and $x$ is a point in space on the iso-surface. Structural optimization can be performed by letting the level set model vary in time, yielding (13).

$$S(t) = \{x(t) : \Phi(x(t), t) = k\} \tag{13}$$

Taking the time derivative of (13) and applying the chain rule yields the following "Hamilton-Jacobi-type" equation

$$\frac{\partial \Phi(x,t)}{\partial t} + \nabla \Phi(x,t)\frac{dx}{dt} = 0, \quad \Phi(x,0) = \Phi_0(x) \tag{14}$$

which defines an initial value problem for the time dependent function $\Phi$. In the solution process, let $dx/dt$ be the movement of a point driven by the objective of optimization such that it can be expressed in terms of the position of $x$ and the geometry of the surface at that point. The optimal structural boundary then becomes the solution of a partial differential equation on $\Phi$ given by (15)

$$\frac{\partial \Phi(x)}{\partial t} = -\nabla \Phi(x)\frac{dx}{dt} \equiv -\nabla \Phi(x)\Gamma(x,\Phi)\Phi(x,0) = \Phi_0(x) \tag{15}$$

where $\Gamma(x, \Phi)$ is the "speed vector" of the level set and depends on the objective of optimization. A proper vector is obtained as a descent direction of the objective via sensitivity analysis. With the level set formulation characterized, a general minimum compliance optimization problem may be written as (Dunning and Kim 2013):

$$\begin{aligned}
\text{min}: \quad & C(u,\Phi) = \int_\Omega E\varepsilon(u)\varepsilon(u)H(\Phi)d\Omega \\
\text{subject to}: \quad & \int_\Omega H(\Phi)d\Omega \le V_f \\
& \int_\Omega E\varepsilon(u)\varepsilon(v)H(\Phi)d\Omega = \int_\Omega bvH(\Phi)d\Omega \\
& + \int_{\Gamma_s} fv\Gamma_s u|_{\Gamma_D} = 0\,\forall v \in U
\end{aligned} \tag{16}$$

where $\Omega$ is a domain larger than $\Omega_S$ such that $\Omega_S \in \Omega$, $V_f$ is the limit on material volume, $E$ is the material property tensor, $\varepsilon(u)$ is the strain tensor for displacement field $u$, $U$ is the space of permissible displacement fields, $v$ is any permissible displacement field, $b$ are body forces, $f$ are surface tractions, and $H(\Phi)$ is a Heaviside function that equals 1 when $\Phi \ge 0$ and zero otherwise.

The capability for topology design for stiffness under a volume constraint was demonstrated (Wang et al. 2003). Additional discussion can be found in Wang and Wang (2004b, 2005), Wang et al. (2004b) in addition to a multiple material level set method (Wang and Wang 2004a; Wang et al. 2005). Independently, Allaire et al. (2002), Allaire et al. (2004) also developed a numerical framework for boundary design that linked the velocity of level set boundaries to adjoint shape sensitivity analysis for stiffness design and frequency maximization (Allaire and Jouve 2005). A limitation of these direct methods are that algorithms cannot create new holes in the level set function away from free boundaries (typically the outside of a design domain) and resulting solutions are heavily dependent on the initial state of the design problem. In response a number of works, including Burger et al. (2004), Allaire et al. (2005), Allaire and Jouve (2006), Wang et al. (2004a), and He et al. (2007), proposed different mechanisms to include topological derivatives in the level set problem. Topological derivatives represent the change in objective functional with the introduction of infinitesimally small holes and allow for the nucleation of new holes anywhere in the design domain. This enables a true topological design capability. Dunning and Kim (2013) present a new method for the introduction of holes without the topological derivative using a secondary level set function. Recently, James and Martins (2012) proposed an extension of the conventional level set method for use with a body-fitted finite element mesh, which is useful when the design domain is nonrectangular or irregularly shaped.

#### *4.1.2 Alternative level set formulations*

In the conventional level set methods, the Hamilton-Jacobi partial differential equation (PDE) controlling the structural boundary is solved explicitly. This places time step size restrictions for convergence stability and also often requires the reinitialization of level set functions when they become too flat or steep, both of which decrease the computational efficiency of the schemes. To alleviate this, a number of alternative formulations, which do not require the explicit solution process, have been proposed to circumvent these issues. Luo et al. (2008a, b) proposed a new formulation where the Hamilton-Jacobi PDE is solved using a semi-implicit additive operator splitting scheme. Another family of parameterization techniques have been explored that convert the Hamilton-Jacobi PDE into a simpler set of ordinary differential equations (ODEs) using radial basis functions (RBFs) (Wang and Wang 2006a, b, Wang et al. 2007b) or system of algebraic equations using compactly supported RBFs (CSRBFs) (Luo et al. 2007, 2008c, 2009a). In another alternative, the spectral level set method, Fourier coefficients of the level set function are taken as design variables, which serves to reduce the design space (Gomes and Suleman 2008). Recently, Luo et al. (2012) proposed a meshless Galerkin level set method using the CSRBFs.

In addition, piecewise constant level set methods have been successful for multiphase material problems (Luo et al. 2009b). In yet another alternative level set method, Yamada et al. (2010) abandon the Hamilton-Jacobi PDE in favor of a reaction-diffusion equation similar to phase-field methods described later. Other alternatives were proposed by Amstutz and Andrä (2006) and Norato et al. (2007), who utilized an evolution equation based directly on topological gradients. Belytschko et al. (2003) described the level set function in a narrow band of the zero level set according to nodal variables, C0 continuous shape functions, and a heuristic updating scheme. Finally, Haber (2004) used a sequential quadratic programming (SQP) method in conjunction with multilevel continuation schemes to advance the implicit shape boundary rather than solving the Hamilton-Jacobi PDE. It is important to note that a number of the alternative level set formulations highlighted in this section can accommodate new hole formation without the use of a topological derivative.

## 4.2 Phase-field topology optimization

The phase-field method for topology optimization is based on theories originally developed as a way to represent the surface dynamics of phase-transition phenomena such as solid-liquid transitions (Chen 2002; Macfadden 2002). The methods have been utilized in a number of different surface dynamics simulations, especially in materials science, including diffusion, solidification, crack-propagation, and multiphase flow in addition to phase transitions. In these theories, a phase field function $\phi$ is specified over the design domain $\Omega$ that is composed of two phases, A and B, which are represented by values $\alpha$ and $\beta$ of $\phi$, respectively, as shown in Fig. 11. The boundary region between phases is a continuously varying region of thin finite thickness $\xi$.

In phase-field topology optimization this region defines structural boundaries and is modified via dynamic evolution equations of the phase field function $\phi$. A primary difference between the level set and phase-field methods lies in the fact that in the phase-field method the boundary interface between phases is not tracked throughout optimization as is done when using level sets. That is, the governing equations of phase transition are solved over the complete design domain without prior information about the location of the phase interface. In addition, phase-field methods do not require the reinitialization step of level set functions.

The use of phase-field methods for topology optimization was first proposed by Bourdin and Chambolle (2003, 2006). Wang and Zhou (2004a) further explored the idea by using van der Waals-Cahn-Hilliard phase transition theory and a numerical method based on the theory of $\Gamma$-convergence to solve the variational system for mean compliance minimization of bi-material phases of solids and later for three-phase systems (Wang and Zhou 2004b). They later investigated the phase-field method for compliance minimization based this time on the generalized Cahn-Hilliard equations for multiphase transitions (Zhou and Wang 2006, 2007). Wallin et al. (2012) incorporated an adaptive finite element formulation into a Cahn-Hilliard based phase-field method. Recently, Takezawa et al. (2010) used a time dependent reaction-diffusion equation, called the Allen-Cahn equation, to evolve the phase function. They present structural topology optimization examples of compliance minimization, compliant mechanism design, and eigenfrequency maximization. It was demonstrated that with a suitable choice of double well potential function, the evolution equation can be approximately represented as a conventional steepest descent method where phase evolves in the direction of negative gradient of the objective function. This work was later extended by Gain and Paulino (2012) to non-Cartesian solution domains using polygonal elements. In each paper on the phase-field method, the ability to generate structural results without the need to explicitly define an interface between material phases was demonstrated. A recent article by Blank et al. (2010) provides a comparison between the Cahn-Hilliard and Allen-Cahn transition methods for evolving the phase function in structural topology design on a benchmark cantilever beam problem. Results demonstrate that utilizing the Allen-Cahn equation reduces the computational cost and increases efficiency of the topology optimization when compared to the Cahn-Hilliard equation for linear elasticity.

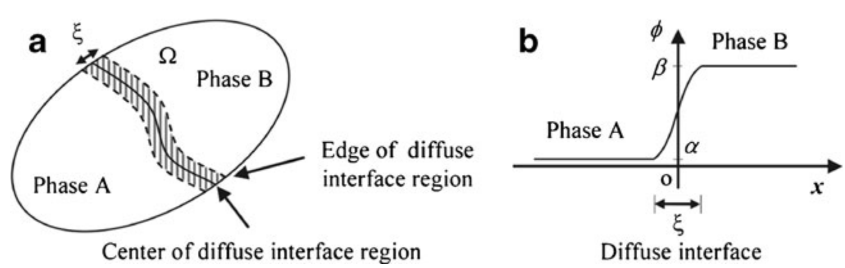



**Fig. 11** **a** A 2D domain represented by the phase field function and **b** a 1D illustration of the phase field function (from Takezawa et al. (2010))

### 4.3 Nonlinear responses

Studies involving topology optimization with nonlinear responses indicate that the numerical difficulties and convergence issues experienced by density-based methods, which are due primarily to low density elements in the nonlinear solution procedure, are avoided with the level set method. In their introduction to level set topology design, Allaire et al. (2004) discussed a generalization of their method to nonlinear elasticity problems. In similar work, Kwak and Cho (2005) introduced a level set method for compliance minimization with geometric nonlinearity. More recently, Ha and Cho (2008) used an unstructured mesh along with level set topology for compliance minimization with geometric nonlinearity and hyperelastic material. Luo and Tong (2008) demonstrated a formulation for level set topology optimization for the design of large-displacement compliant mechanisms. In their work, radial basis functions are utilized to increase solution efficiency. Kim et al. (2009) utilized level set topology for steady-state nonlinear heat conduction where material properties are temperature dependent. Finally, while no publications utilizing the phase-field method with nonlinear responses were found, no significant barriers to its application appear evident.

### 4.4 Stress-based boundary variation topology design

Boundary variation methods have also been demonstrated for topology optimization with stress considerations. In fact, due to the fundamental formulation of these methods with purely black and white designs, several of the challenges of stress-based topology optimization described for density-based methods are naturally avoided. Allaire and Jouve (2008) demonstrate level set topology optimization of several benchmark problems including a cantilever, L-bracket, gripping mechanism, and 3D mast structure for both minimum compliance and minimum global stress objectives. Guo et al. (2011) studied two different stress-based objectives for minimum stress design in the level set method. The first objective was the integral of von Mises stress over the whole structure and the second objective was the maximum value on von Mises stress in the structure, which was dealt with by an active-set strategy for numerical stability. More recently, Xia et al. (2012c, 2013) and Wang and Li (2013) have introduced solutions to the stress problems using level sets methods. In addition, Burger and Stainko (2006) demonstrated the phase-field method for minimum volume structures with local stress constraints on benchmark beam examples. Numerical results appeared consistent with those obtained with stress-based design and other topology optimization methods. While not directly stress-based design, Challis et al. (2008b) provides interesting discussion regarding optimization against fracture resistance using a level set method.

### 4.5 Alternate physics, multiphysics, and application

#### *4.5.1 Heat transfer and thermoelasticity*

Ha and Cho (2005) first applied the level set method to heat conduction problems via a weak variational form of the heat conduction equation and variational level set calculus to determine the velocity field of the level set equation. Later, Zhuang et al. (2007) included both shape and topological derivatives in the level set formulation and performed topology optimization for optimal heat dissipation with multiple load cases for steady state conduction. In an extension to thermoelasticity, Xia and Wang (2008) demonstrated the use of the level set method for stiffness design of structures subjected to uniform temperature loading. One significant advantage of the level set method in thermoelasticity is the lack of intermediate density material, which causes several issues in density-based topology optimization as discussed earlier. Recently, Luo et al. (2009c) used level sets along with CSRBFs for optimization of thermomechanical and electrothermomechanical microactuators, where actuation results from temperature increases due to Joule heating.

#### *4.5.2 Fluid flow*

Duan et al. (2008c) used a modified Hamilton-Jacobi level set, called variational level set, for topological design of two-dimensional Stokes and Navier-Stokes flow (Duan et al. 2008a, b) for minimum power dissipation channel flows. Zhou and Li (2008a) also utilized a variational level set method for both 2D and 3D Navier-Stokes flows for low-loss junctions and maximum permeability problems. Challis and Guest (2009) utilized a conventional level set method for optimization of minimum power dissipation flow paths in Stokes flow and Abdelwahed and Hassine (2009) also proposed a level set formulation for Stokes flow with 2D and 3D examples. Finally, Pingen et al. (2007, 2010) formulated a parametric level set method with RBFs that utilized a hydrodynamic Lattice Boltzmann Method (LBM) rather than the Navier-Stokes finite element methods utilized by the other cited works.

#### *4.5.3 Biomedical design & multifunctional materials*

Level set methods have also shown to be suitable in the design of tissue scaffolds for biomedical design. Challis et al. (2012) utilized level set topology optimization for both stiffness and permeability objectives to generate a first estimate at cross-property bounds for porous materials. Challis et al. (2010) also experimentally studied some topologically derived bone scaffold designs manufactured using freeform techniques. The level set methods have also be successfully applied to other multifunctional domains including optimal stiffness and thermal conductivity (Challis et al. 2008a), electromagnetic meta materials for permittivity and electrical permeability (Zhou et al. 2010b), and electric dipole antenna design (Zhou et al. 2010a).

### 4.6 RBTO with level set method

Just as with ESO, the number of publications related to reliability-based topology optimization using the level set method is much more limited when compared to the more widely accepted density-based methods. In fact, the limited publications have actually been applied to robust optimization, since a meaningful failure state is not given. For example, de Gournay et al. (2008) developed a robust compliance minimization using level set topology to identify a set of worst case loading perturbations and perform optimization. More recently, Dunning et al. (2011) presented an efficient method for considering both loading magnitude and directional uncertainty in topology optimization in order to produce robust solutions. Chen et al. (2010a) and Chen and Chen (2011) also developed robust topology optimization capabilities using the level set method for both random field and geometric uncertainties.

### 4.7 Additional comments

#### *4.7.1 Educational resources*

Inspired by the 99 line SIMP topology optimization code, Challis (2010) presents a simple 129 line MATLAB implementation of discrete level set topology optimization. The paper provides insight into the main facets of the level set method for topology optimization and provides concrete implementation details. In addition, another publicly available implementation of the level set method is provided by Allaire and Pantz (2006).

## 5 Bio-inspired cellular division-based method

Recently, an innovative biologically-inspired layout and topology optimization method capable of generating discrete and continuum-like structures was proposed by Kobayashi (2010). In the method, which is inspired by the cellular division processes of living organisms, topological layout is implicitly governed by a developmental program that when executed completes a sequence of tasks that develop the topology in stages. When driven by a genetic algorithm (GA), the set of rules, called a Lindenmayer or map-L system, that define the tasks of the developmental program become the design variables in the optimization problem. With control over these developmental rules, a strikingly diverse set of topological designs may be generated with relatively few design variables.

One advantage of this method is the straightforward nature with which it can be coupled onto existing finite element (or other) analysis tools to develop a topology optimization capability. Just as with the ESO-type methods, it is readily accomplished using simple pre/post-processing operations, especially for multiphysics designs. The method is also capable of generating potential topological layouts that are immediately ready for subsequent or even simultaneous sizing and shape optimization. A potential drawback is that the method is driven via genetic algorithm, which regardless of innovative application, is still often more computationally expensive than gradient-based techniques. The following subsections briefly summarize the important aspects of the biologically-inspired method and highlight novel features and applications. For more in depth explanation, the reader is referred to the publications by Kobayashi (2010) and Pedro and Kobayashi (2011).

### 5.1 Map-L system

*L* systems are a type of grammar system originally introduced by biologist Aristid Lindenmayer to model branched topology in plants. Informally, these systems are rewriting methods that can generate developmental programs to describe the construction of a natural or engineered system (Nakamura et al. 1986). A map is defined as finite set of regions with each region bounded by a sequence of edges that intersect at vertices. The maps are analogous to cellular layers, where the regions represent the cells and the edges their walls. By using a series of production rules, an example of which is given in (17), which govern the processes that construct the map, interpretations of complex topology can be obtained.

$$\begin{aligned} &A \rightarrow B[+A]x[-A]B \\ &B \rightarrow A \\ &x \rightarrow x \end{aligned} \tag{17}$$

Execution of the production rules in (17) and the axiom $\omega = ABAB$, which indicates the initial edge labeling, for the first four steps in the process are shown by Fig. 12.

By utilizing additional rules apart from the simple division process demonstrated in the figure, more complex features, for example adding radii to edges, can be obtained. In addition, the geometry can be superimposed or stretched onto non-rectangular domains that can also change shape.

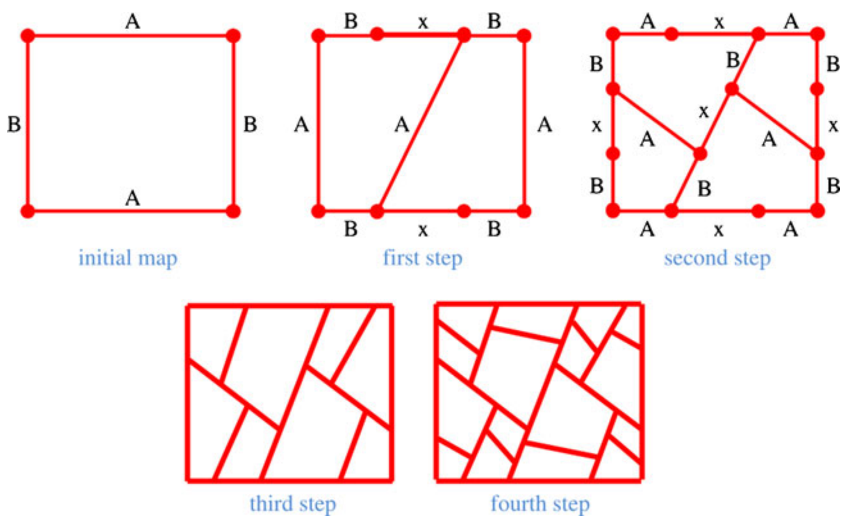



**Fig. 12** First four steps in a cellular division process (from Stanford et al. 2012a)

It is also important to note that the topological layout generated by the map-*L* system in itself has no physical or structural meaning attached to it. Thus, the geometry must be interpreted into structural elements, which can often be done in a pre-processing step. This is demonstrated in Fig. 13 where structural topology is defined by the map-*L* system, projected to a non-rectangular planform, and interpreted into a conceptual 3D wing and spar layout for an aircraft component (Kolonay and Kobayashi 2010).

## 5.2 Encoding rules to GA

The map-*L* system described above is encoded into a binary representation for use with a GA to perform optimization. The optimizer is given control of a number of parameters that affect the resulting topology. These include those that control the growth and dynamics of the development, developmental rules, and also the definition of the overall geometry or physical properties of the system. This enables the GA to modify not only the initial map, but also adjust the rules that create the topology according to a fitness function. In an aircraft optimization for example, the fitness function may be minimum mass with constraints on flutter speed, efficiency, and stresses. The parameters of the map-*L* system are "designed" by the GA to layout the topology of internal wing structure.

We note this implicit representation of topology differs from the explicit representation (where one design variable for each finite element corresponds to the genome) that is utilized in other GA-driven topology optimization work (Wang et al. 2006b). In fact, it is this representation that allows not only for fewer design variables and more design freedom, but also avoids other issues such as maintaining domain connectivity throughout evolution.

## 5.3 Applications

While the number of applications of the cellular-division based topology optimization are limited compared to other methods, it has proven suitable on both benchmark problems and practical applications. Pedro and Kobayashi (2011) demonstrated the method on benchmark cantilever beam problem and Stanford et al. (2012a) demonstrated the method on a number of applications to flapping wing

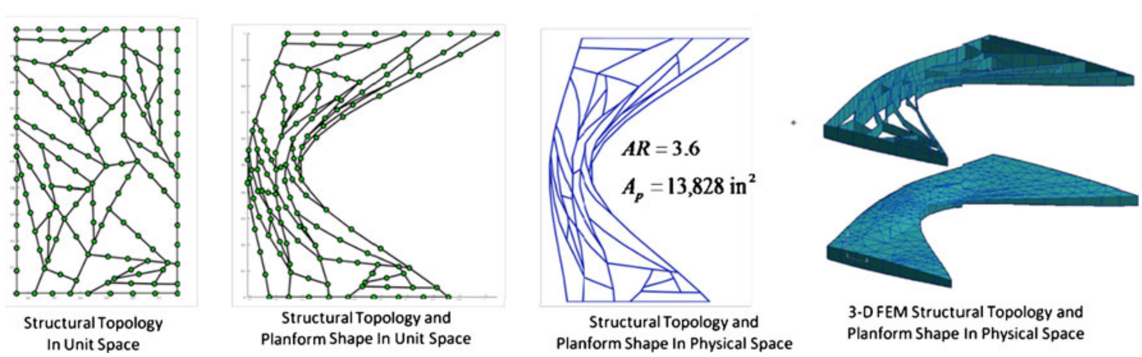



**Fig. 13** Planar shape and topology mapped to a 3D wing box rib and spar layout for a conceptual aircraft component where 3-D FEM is ready for preliminary-level physics-based analysis (from Kolonay and Kobayashi 2010)

flight including wing venation design and compliant mechanism design for actuation (Stanford et al. 2012b). Figure 14 shows a typical solution for a flapping wing flier where both the actuation mechanism to produce the flapping motion (whose topology layout is in black) and the stiffening or venation topology of the wings (layout in red) have been simultaneously optimized using the cellular-based method.

Kobayashi et al. (2009b) performed topological substructure layout for a concept fighter aircraft wing and demonstrated the ability to design around an internal component such as a landing gear or other subsystem. In a similar study, Kolonay and Kobayashi (2010) performed simultaneous size, shape, and topology design for aircraft wings. In this work, the map-L system is utilized to create a topological layout and a conventional shape and sizing optimization is performed on the configuration; however, the system responses utilized for the overall fitness function in the GA are from optimal shape and size designs. As such, the overall optimization algorithm is afforded control at all levels for simultaneous size, shape, and topological layout freedom in a manner different from other works on simultaneous structural optimization using more conventional methods (Zhou et al. 2004). This also allows for the coupling of topology design onto established shape and sizing capabilities for alternative physics. For example, in the area of aeroelasticity, it is straightforward to couple cellular-based topology optimization onto a dedicated aerospace design tool such as ASTROS or Nastran and utilize industry-accepted methods for size and shape variables. The method has also been applied to non-structural topology optimization problems as well. Kobayashi et al. (2009a), Kobayashi (2010) and Pedro et al. (2008) demonstrated the method on several dendritic transport and heat conduction field problems.

## 6 Summary, recommendations, and perspectives

In an effort to aggregate research works in the field, we have highlighted advancements and outlined the strengths and challenges of four different topology optimization methods: (1) density-based methods, (2) hard-kill methods, (3) boundary variation methods, and (4) a new biologically inspired cellular-division method.

The time period from 2000 to the present day has been one of rapid growth in the topology optimization field. We have seen the maturation of existing methods, including SIMP and other density-based techniques, with the introduction of advanced filters to increase the quality of topological results, new engineering applications to disciplines beyond structural mechanics, and the integration of topology optimization capabilities in several commercial software packages. In addition, more rigorous design metrics including stress constraints have extended the scope of structural applications of density methods beyond compliance, frequency, and micro-actuation. Lively discussions related to the theory behind hard-kill methods such as ESO have led to more stable formulations for those techniques. Finally, new formulations utilizing implicit representations of the design domain have shown promise as alternative methods that do not rely on idealizations of intermediate material and thus avoid many of the numerical difficulties associated with such idealizations. The results obtained by these methods also require little post-processing effort to obtain feasible designs.

### 6.1 Recommendations

In the following subsections, we provide some short recommendations related to new algorithm development and code validation for topology optimization.

#### *6.1.1 Algorithm development*

Too often new topology optimization algorithms are tested on problems of pure mechanical compliance (or equivalent energy-based analogue in alternative physics). It is well known that such an optimization problem is exceptionally

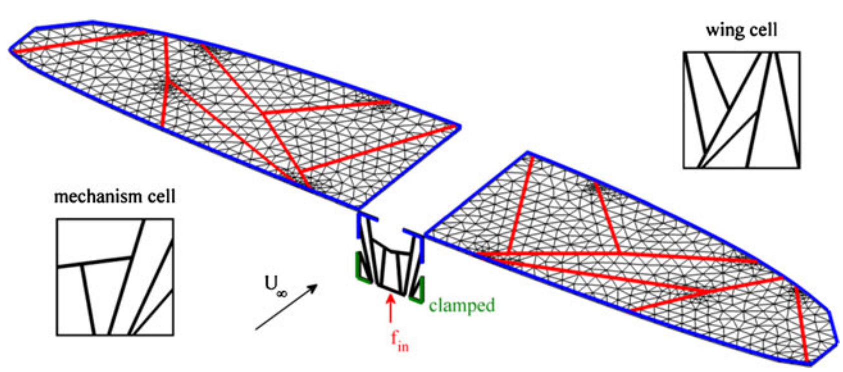



**Fig. 14** Simultaneous optimal topology designs for membrane wing venation (*red*) and compliant mechanism actuator (*black*) for a flapping wing obtained using the cellular-division method (from Stanford et al. 2012b)

well-behaved. This occurs because in minimum compliance topology optimization, the sensitivity of material addition, by any technique, is strictly negative and the problem is self-adjoint. These unique features are lost for most practical engineering problems that are multiply constrained in nature or have more complex responses. Thus, algorithm developers should focus on methods and test cases within the general setting of real world problems. For example, in the compliant force inverter problem, sensitivities may be both positive and negative for material addition, which tests both the topology methodology and also the choice of optimizer. Other scenarios include minimum compliance with design dependent body or thermal loads, which contain contributions to compliance sensitivity from both stiffness/deformation and loading that generally have opposite magnitudes, and problems with multiple displacement constraints.

#### *6.1.2 Analytical benchmarks*

Analytical solutions to a number benchmark problems for topology optimization exist in the literature. The community should actively utilize these to test new algorithm developments and validate code. In fact, Rozvany, Lewiński, and colleagues have published several of these works among other authors (Lewinski and Rozvany 1994, 2007, 2008a, b; Rozvany and Maute 2011; Rozvany et al. 2006; Sokol and Rozvany 2012). For brevity, we only reference a few recent papers but many more exist including those prior to the year 2000. In the same sense, as the topology optimization field continues to expand past structural responses, analytical benchmarks covering alternative physics and multi-physics problems would be an excellent contribution to the field. In fact, rigorous analytical solutions related to any of the non-structural topics covered in this review would certainly be well received.

### 6.2 Future perspectives

In short, the future of topology optimization is most exciting and a number of innovative applications await practitioners in the field. The following subsections, though certainly not comprehensive, identify some important future directions and areas that require attention from the field. Many of these build upon sections previously described in this document, which we hope may serve as a stepping-stone for future development.

#### *6.2.1 Design dependent physics*

Topology optimization, especially related to compliance minimization, performs best when applied to problems where loading is independent of the design. However, there exist a number of situations where the capabilities of topology optimization are desirable, but design dependent physics, including loads, makes implementation challenging. For example, in the authors' work related to thermoelastic design for thermal stresses, which is in itself inherently a material layout problem, the current methods not only lead to ill-behaved optimization problems, but also produce solutions that are directly opposed to design practices related to thermal stresses. Another example includes fluid flow problems where some methods use different flow models in the void and solid domains, which leads to different governing equations for each material phase. These cases demonstrate the need exists not only for techniques that can resolve additional challenges from design dependent physics, but also for new problem formulations that are more appropriate for the engineering design problems at hand.

#### *6.2.2 Stress-based topology optimization*

Despite the fact that stress is a critical parameter in almost all structural designs, the overwhelming majority of applications in structural topology optimization remain concerned with stiffness criteria, which do not always lead to strength-optimal designs. While the reasons for this are obvious, we would be remiss if we did not continue to challenge the field for innovation in this area. Recent works highlighted for each method have made considerable progress using novel applications of techniques deeply rooted in structural optimization such as constraint aggregation. Despite this success, we should continue to pursue break-through techniques that can make designing a structure while directly considering stresses as straightforward as obtaining a minimum compliance solution.

#### *6.2.3 Multidisciplinary and multiphysics applications*

The number of topology optimization applications in fields outside of conventional structures in the last ten years is impressive. We should continue this trend in not only alternative applications, but also multiphysics applications where the design freedom afforded by a coupled design domain can be best exploited by topology optimization. This ultimately requires the continued development of efficient coupled sensitivity analysis methods. In addition, when attempting to extend a topology solution to a new physical domain, researchers should actively attempt to address the accepted design criteria for that field rather than isolating a response that presents the best optimization problem. Above all, we should remember the most useful alternative physics applications are those that directly address the specific engineering design issues in that area.

#### *6.2.4 Biomedical design and medical applications*

Through the completion of this review we have made the (somewhat unsettling) observation that more optimization is performed on aircraft, automobiles, and micro-devices than biomedical tools and implants that are placed in the human body! Not to be taken as critical to anyone, this observation is a compliment to those who have introduced topology solutions to the medical industry and a comment on the tremendous potential of optimization in this field. Despite economic circumstances, the healthcare industry including medical research, is ripe with opportunity. The authors' are optimistic regarding innovative solutions that topology optimization may deliver in biomedical design and encourage researchers to explore applications outside of the "structural-inspired" orthopedic areas highlighted in this review.

#### *6.2.5 Robust & reliability-based topology*

It is well understood that real world systems are not deterministic due to a number of uncertainties that may or may not be stochastic in nature. The quantification of these uncertainties, and their inclusion in design, is especially important in topology optimization because of the high sensitivity of results to design criteria such as loading and boundary conditions. The field is beginning to address this issue with the application of robust and reliability-based design to topology optimization as highlighted in this review. However, the potential for continued research in this area is still large to help determine best-practice techniques for identifying risks and uncertainties in results.

The field may also begin to consider the uncertainties introduced by the selection of the physics-based models utilized for analysis. Multiple models usually exist to obtain a given system response, but differ in the assumptions taken in their development or solution methods. As a result, different models are more appropriate in some regions of a design space, some are more computationally expensive than others, and it is often unknown which is the best to use. In the general uncertainty quantification (UQ) field, these types of algorithmic-uncertainties and those associated with model selection are known as "model-form" or "model" uncertainties and are currently an active area of research in the UQ community (Park and Grandhi 2011; Riley and Grandhi 2011).

#### *6.2.6 Topology to shape & size transition*

In practice, the results of a topology optimization often yield only a conceptual idealization of a component and there is significant overhead associated with creating more refined geometric representations and engineering models that are suitable for subsequent shape and sizing design. While some methods including Heaviside projections and level set or phase-field methods, all of which lead to smooth boundary representation, help to more clearly define topology results, new methods are required to streamline the transition from conceptual topology optimization to later stages of design and analysis. The authors' imagine that this may be accomplished by algorithms that can identify engineering features in topology results, such as fillets, extrusions, planes, and beam sections among others. Such a capability, or some other solution, that can help produce engineering models for later design stages and high fidelity analysis in an automated fashion would be extremely desirable in industrial applications.

#### *6.2.7 Advanced manufacturing capabilities*

Advances in additive and freeform manufacturing technologies offer significant promise in alleviating some of the challenges related to the practical realization of designs obtained via topology optimization. Potential benefits that may soon be realized include increased complexity and structural features not obtainable by conventional manufacturing methods at lower cost. The topology optimization field would be wise to investigate the additional considerations that are required to efficiently translate topology optimized designs to additive and freeform manufacturing processes using different materials. These include improved modeling for residual stresses and interface conditions in addition to process limitation constraints such as overhang angle and minimum deposition tolerances.

#### *6.2.8 HPC and GPU implementation*

As the scope of topology optimization problems continue to grow to in terms of the number of element and design variables, included physics, and number of systems/components, so do the demand for computational resources. One solution to address this growth includes continued research in the area of high performance computing (HPC) and the utilization of graphics processing units (GPU). In particular, the capabilities of GPUs are very interesting. In comparison to conventional processors that include only a few cores designed for serial operations, GPUs contain thousands of smaller cores optimized for parallel computing. Thus, new algorithms and solvers for topology optimization that are compatible with this massively parallel capability certainly hold the potential for performance increases measured in orders of magnitude.

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