LGN 5822 - Biometrical Genetics

L01b – Matrix Algebra

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2023

Matrix Algebra

What is Algebra?

- The branch of mathematics that helps represent problems or situations in the form of mathematical expressions.

- Have symbols and the arithmetic operations across these symbols.
    - These symbols do not have any fixed values and are called variables.

Motivation

Prediction of Total Genetic Value Using Genome-Wide Dense Marker Maps

T. H. E. Meuwissen,* B. J. Hayes[†] and M. E. Goddard[†,‡]

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Manuscript received August 17, 2000
Accepted for publication January 17, 2001

ABSTRACT

Recent advances in molecular genetic techniques will make dense marker maps available and genotyping many individuals f… especially if combi…

c. Estimate the effects of the haplotypes at the QTL positions simultaneously by the model

$$y = \mu \mathbf{1}_n + \Sigma_i X_i g_i + e,$$

where summation Σ_i is over all QTL positions corresponding to a likelihood peak and g_i was estimated at the peak. All other haplotype effects are assumed to be zero. The overall mean is also arbitrarily set to zero, because its effect cannot be distinguished from that of the fixed haplotype effects.

Meuwissen, Theo HE, Ben J. Hayes, and ME1461589 Goddard. Genetics 157.4 (2001): 1819-1829.

Motivation

BIOMETRICS *31*, 423–447
June 1975

BEST LINEAR UNBIASED ESTIMATION AND PREDICTION UNDER A SELECTION MODEL

C. R. HENDERSON

Department of Animal Science, Cornell University, Ithaca, N. Y. 14850, U.S.A.

SUMMARY

Mixed linear models are assumed in most animal breeding applications. Convenient methods for computing BLUE of the estimable linear functions of the fixed elements of the model and for computing best linear unbiased predictions of the random elements of the model have been available. Most data available to animal breeders, however, do not meet the usual requirements of random sampling, the problem being that the data arise either from selection experiments or from breeders' herds which are undergoing selection. Consequently, the usual methods are likely to yield biased estimates and predictions. Methods for dealing with such data are presented in this paper.

Henderson, C. R. (1975). *Biometrics*, 423-447.

Matrix Algebra

Motivation

Theoretical and Applied Genetics (2019) 132:81–96
https://doi.org/10.1007/s00122-018-3196-1

ORIGINAL ARTICLE

CrossMark

Genomic selection efficiency and a priori estimation of accuracy in a structured dent maize panel

Simon Rio[1] · Tristan Mary-Huard[1,2] · Laurence Moreau[1] · Alain Charcosset[1]

Received: 25 April 2018 / Accepted: 22 September 2018 / Published online: 4 October 2018

Abstract

Key message **Popula ... ast accuracy using standard GBLUP.**

Abstract Genomic pr ... he same population as those to be predict ... ination (CD), were derived from those sa ... ay impact genomic selection efficiency an ... : structure in a dent maize panel ("Amaizi ... For a given training set size, the best accur ... same genetic group. Nevertheless, a divers ... the validation sets, and adding extra-grou ... neric training set for dent maize genomic s ... licitly into account, did not improve the p ... ferent indicators of precision to forecast a ... ; trend of the CD to differentiate scenarios ... he efficiency of this indicator proved to be ... ic structure through group-specific allele diversity at QTLs rather than group-specific allele effects.

Genomic prediction models

All the genomic prediction models used in this study can be written as:

$$y = X\beta + Zg + e$$

where y is the vector of LS-means which will be further referred to as phenotypes, X is the incidence matrix for fixed effects, β is the vector of fixed effects, Z is an incidence matrix linking observations to breeding values, g is the vector of breeding values and e is the vector of errors. All models assume independence between g and e.

Rio, Simon, et al. Theoretical and Applied Genetics 132 (2019): 81-96.

Matrix Algebra

Motivation

Theoretical and Applied Genetics (2021) 134:95–112
https://doi.org/10.1007/s00122-020-03684-z

ORIGINAL ARTICLE

Check for updates

Enviromics in breeding: applications and perspectives on envirotypic-assisted selection

Rafael T. Resende[1] · Hans-Peter Piepho[2] · Guilherme J. M. Rosa[3] · Orzenil B. Silva-Junior[4] · Fabyano F. e Silva[5] · Marcos Deon V. de Resende[6,7] · Dario Grattapaglia[4,8]

Received: 25 June 2019 / Accepted: 10 September 2020 / Published online: 22 September 2020

Abstract

Key message **We prop** … **milarity among sites**
assessed on an "omics" … **iotype performances.**

Abstract Genotype by e … on estimating genetic
parameters over a limite … stem (GIS) techniques
have opened new frontie … easing selection accu-
racy across all sites of int … Here, we introduce the
term enviromics, within … at DNA markers, any
particular site is charact … ding to environmental
variables that may intera … ngs for optimized deci-
sions over different envir … thod (the "GIS–GEI")
which, due to its higher … g of sites to their most
appropriate genotypes; (… ensure selection gains
across environments; an … urther analyses. Envi-
ronmental scenarios can … agement, especially in
the current outlook of dy … etic studies, which are
fairly inexpensive, increa … ure for the integration
of enviromics approach … otyping and powerful
statistical modeling of g …

see Schmidt et al. (2019a). The following model was used for the simulations:

$$y = X\beta + Zg + Wt + \varepsilon, \quad (1)$$

where y is the vector of phenotypic means per genotype and trial; β represents the vector of fixed effects (overall intercept); g represents the vector of random effects of genotypes, assumed $g \sim N\left(0, K\sigma_g^2\right)$; K is a kinship matrix built from pedigree or genomic information; t represents the vector of random effects of trials, assumed $t \sim N\left(0, I\sigma_t^2\right)$; and X, Z and W are known incidence matrices for β, g and t, respectively. The residual vector ε was assumed as $\varepsilon \sim N\left(0, I\sigma_\varepsilon^2\right)$. The relative genetic variance, herein termed trait heritability, is given by $h^2 = \sigma_g^2 / \left(\sigma_g^2 + \sigma_t^2 + \sigma_\varepsilon^2\right)$, where σ_g^2, σ_t^2 and σ_ε^2 are the variance components related to genotypes, trials and residuals, respectively.

Matrix Algebra

Matrix

Matrix is used to compactly represent linear models for large numbers of observations.

Matrix Algebra

Matrix

- A matrix is a rectangular or square array of numbers or variables
  - For example:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 4 & 12 \\ 1 & 8 & 20 \end{bmatrix}$$

$$\boldsymbol{B} = \begin{bmatrix} 5 & 9 \\ -4 & 1 \\ 8 & 0 \end{bmatrix}$$

$$\boldsymbol{C} = \begin{bmatrix} 0 & 15 \\ 20 & 10 \end{bmatrix}$$

Matrix Algebra

Indices can be used to represent individual elements of the matrix:

$$\boldsymbol{A} = (a_{ij}) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix},$$

where subscript i represents the row and j the column

Matrix Algebra

Indices can be used to represent individual elements of the matrix:

$$\boldsymbol{A} = (a_{ij}) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix},$$

where subscript *i* represents the row and *j* the column.

- A matrix with $\boldsymbol{n}$ rows and $\boldsymbol{p}$ columns is of size $\boldsymbol{n\ x\ p}$
- In the example above, $\boldsymbol{A}$ is of size $\boldsymbol{2\ x\ 3}$

Matrix Algebra

Vectors

- A matrix with a single column is denoted a *vector*
- In this case, we can use a single index to represent its elements:

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Matrix Algebra

Scalars

- A scalar is simply a real number;

Matrix Algebra

Scalars

- A **scalar** is simply a real number

$$A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Scalar matrix is a diagonal matrix that has the elements equals

- A $\mathbf{1} \; \boldsymbol{x} \; \mathbf{1}$ matrix **may** sometimes be considered a scalar.

Equality of Matrices

- Matrices are considered to be equal if they have the same number of rows and columns, as well as the same number of elements.
  - For example:

$$\begin{pmatrix} 3 & -2 & 4 \\ 1 & 3 & 7 \end{pmatrix} = \begin{pmatrix} 3 & -2 & 4 \\ 1 & 3 & 7 \end{pmatrix},$$

but

$$\begin{pmatrix} 5 & 2 & -9 \\ 8 & -4 & 6 \end{pmatrix} \neq \begin{pmatrix} 5 & 3 & -9 \\ 8 & -4 & 6 \end{pmatrix}.$$

Matrix Algebra

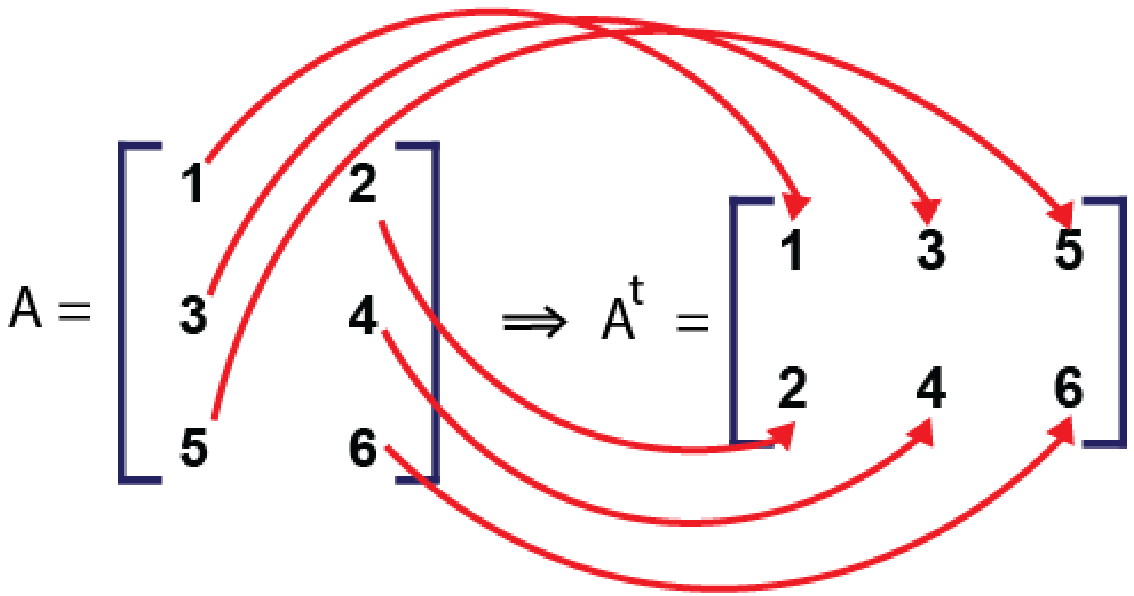
Transpose

- Exchanging rows and columns of a matrix results in its *transpose;*
- The transpose of matrix $\boldsymbol{X}$ can be denoted as $\boldsymbol{X}^{\boldsymbol{T}}$ or $\boldsymbol{X}^{\prime}$:

Matrix Algebra

Transpose of a Matrix

- The **Transpose of a Matrix** is obtained by changing its rows into columns (or equivalently, its columns into rows)
  - Can be denoted as $\boldsymbol{X^T}$ or $\boldsymbol{X}'$



Matrix Algebra

Transpose of a Matrix

$$\boldsymbol{B} = \begin{bmatrix} 5 & 9 \\ -4 & 1 \\ 8 & 0 \end{bmatrix}, \qquad \boldsymbol{B}' = \begin{bmatrix} 5 & -4 & 8 \\ 9 & 1 & 0 \end{bmatrix}$$

Matrix Algebra

Transpose of a Matrix

If $\boldsymbol{X} = (\boldsymbol{x_{ij}})$ then

$\boldsymbol{X}^{\prime} = (\boldsymbol{x_{ij}})^{\prime} = (\boldsymbol{x_{ij}})$

If X is of size $\boldsymbol{n\ x\ p}$, then $X^{\prime}$ is $\boldsymbol{p\ x\ n}$

Matrix Algebra

Transpose of a Matrix

- The transpose of a (column) vector is a row vector

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \qquad \boldsymbol{x}' = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}$$

Matrix Algebra

Special Matrices: Symmetric Matrix

- If $\boldsymbol{A} = \boldsymbol{A}'$, i.e., $(a_{ij}) = (a_{ij})$, then $\boldsymbol{A}$ is **symmetric**

Special Matrices: Symmetric Matrix

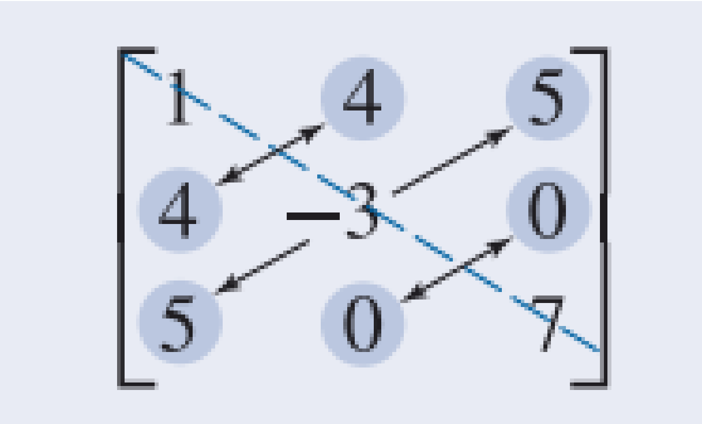
- If $\boldsymbol{A} = \boldsymbol{A}'$, i.e., $(a_{ij}) = (a_{ij})$, then $\boldsymbol{A}$ is **symmetric**

Example of a Symmetric Matrix

$$\boldsymbol{A} = \begin{bmatrix} 12 & -3 & 7 \\ -3 & 1 & 0 \\ 7 & 0 & 8 \end{bmatrix}$$

*All symmetric matrices are square

Special Matrices: Symmetric Matrix



Special Matrices: Diagonal Matrix

- If a matrix contains zeros in all off-diagonal positions, it is said to be a diagonal matrix
- For example:

Diagonal Matrix

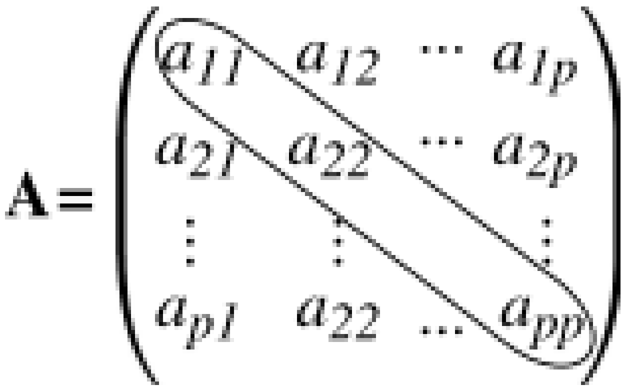
- A matrix with all off-diagonal elements equal to zero

$$\boldsymbol{A} = \begin{bmatrix} 12 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

Matrix Algebra

Special Matrices: Diagonal Matrix

- What is diagonal matrix?
  - The diagonal of a $\boldsymbol{p}\ \boldsymbol{x}\ \boldsymbol{p}$ square matrix $\boldsymbol{A} = (\boldsymbol{a_{ij}})$ consists of the elements $a_{11}, a_{22,\ldots,}a_{pp}$



Special Matrices: Identity Matrix

- An identity matrix is a square matrix in which all the elements of principal diagonals are one, and all other elements are zeros

The Identity Matrix

$$\boldsymbol{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Matrix Algebra

Special Matrices: Identity Matrix

- A matrix multiplied by its inverse equals the identity matrix

$$A \; x \; A^{-1} = I$$

Special Matrices: Upper Triangular Matrix

- A square matrix with zeros below the diagonal
- For example:

$$\mathbf{T} = \begin{pmatrix} 7 & 2 & 3 & -5 \\ 0 & 0 & -2 & 6 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 8 \end{pmatrix}$$

Matrix Algebra

Special Matrices: Lower Triangular Matrix

- A square matrix with zeros above the diagonal
- For example:

$$\begin{pmatrix} 4 & 0 & 0 \\ -1 & 2 & 0 \\ 5 & -8 & 6 \end{pmatrix}$$

Matrix Algebra

Sum of Two Matrices

If $\boldsymbol{A}$ is $\boldsymbol{n\ x\ p}$ and $\boldsymbol{B}$ is $\boldsymbol{n\ x\ p}$, then:

$\boldsymbol{C} = \boldsymbol{A} + \boldsymbol{B}$ is also $\boldsymbol{n\ x\ p}$

$$\boldsymbol{C} = (\boldsymbol{c_{ij}}) = (\boldsymbol{a_{ij}} + \boldsymbol{b_{ij}})$$

Sum of Two Matrices

If $\boldsymbol{A}$ is $\boldsymbol{n\ x\ p}$ and $\boldsymbol{B}$ is $\boldsymbol{n\ x\ p}$, then:

$\boldsymbol{C} = \boldsymbol{A} + \boldsymbol{B}$ is also $\boldsymbol{n\ x\ p}$

$$\boldsymbol{C} = (\boldsymbol{c_{ij}}) = (\boldsymbol{a_{ij}} + \boldsymbol{b_{ij}})$$

EXAMPLE

$$\begin{bmatrix} 0 & 7 \\ 2 & -4 \\ 5 & 9 \end{bmatrix} + \begin{bmatrix} -7 & 3 \\ 0 & 2 \\ -3 & 11 \end{bmatrix} = \begin{bmatrix} -7 & 10 \\ 2 & -2 \\ 2 & 20 \end{bmatrix}$$

Difference between Two Matrices

Similarly, for the two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$:

$\boldsymbol{D} = \boldsymbol{A} - \boldsymbol{B}$ is also $n\ x\ p$

$$\boldsymbol{D} = \left(\boldsymbol{d_{ij}}\right) = (\boldsymbol{a_{ij}} - \boldsymbol{b_{ij}})$$

Difference between Two Matrices

Similarly, for the two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$:

$\boldsymbol{D} = \boldsymbol{A} - \boldsymbol{B}$ is also $n\ x\ p$

$\boldsymbol{D} = (\boldsymbol{d}_{ij}) = (\boldsymbol{a}_{ij} - \boldsymbol{b}_{ij})$

EXAMPLE

$$\begin{bmatrix} 0 & 7 \\ 2 & -4 \\ 5 & 9 \end{bmatrix} - \begin{bmatrix} -7 & 3 \\ 0 & 2 \\ -3 & 11 \end{bmatrix} = \begin{bmatrix} 7 & 4 \\ 2 & -6 \\ 8 & -2 \end{bmatrix}$$

Matrix Algebra

Properties

Sum or difference of two matrices is only possible for matrices of the same size!

- $\mathrm{A} + \mathrm{B} = \mathrm{B} + \mathrm{A}$
- $(A + B)' = A' + B'$

Product of Two Matrices

- We denote the product of two matrices A and B as:

$$\boldsymbol{C} = \boldsymbol{AB}$$

- This product only exists if the number of **columns** in **A** is equal to the number of **rows** in **B**

- We need to look at the first row for matrix A and the first column for matrix B

Matrix Algebra

Product of Two Matrices

- Sum of products of the elements in the i_{th} row of **A** and elements in the j_{th} column of **B**
- Multiplication of every row of **A** by every column of **B**
- If $\boldsymbol{A}$ is $n\ x\ m$ and $\boldsymbol{B}$ is $m\ x\ p$, then $\boldsymbol{C}$ is $n\ x\ p$.

MATRIX MULTIPLICATION

- $\boldsymbol{C} = \boldsymbol{AB} = (c_{ij})$
- $c_{ij} = \sum_k a_{ik} b_{kj}$

Product of Two Matrices

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 3 \\ 4 & 6 & 5 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 1 & 4 \\ 2 & 6 \\ 3 & 8 \end{pmatrix}$$

$$\mathbf{AB} = \begin{pmatrix} 2\cdot 1 + 1\cdot 2 + 3\cdot 3 & 2\cdot 4 + 1\cdot 6 + 3\cdot 8 \\ 4\cdot 1 + 6\cdot 2 + 5\cdot 3 & 4\cdot 4 + 6\cdot 6 + 5\cdot 8 \end{pmatrix} = \begin{pmatrix} 13 & 38 \\ 31 & 92 \end{pmatrix},$$

$$\mathbf{BA} = \begin{pmatrix} 18 & 25 & 23 \\ 28 & 38 & 36 \\ 38 & 51 & 49 \end{pmatrix}$$

$AB \neq BA$

Matrix Algebra

Product involving scalars

- If $\mathbf{A}$ is an $n\ x\ p$ matrix and $\mathbf{c}$ is a scalar, then:

$$cA = (ca_{ij}) = \begin{bmatrix} ca_{11} & ca_{12} & \dots & ca_{1p} \\ ca_{21} & ca_{22} & \dots & ca_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ ca_{n1} & ca_{n2} & \dots & ca_{np} \end{bmatrix} =$$

- It is true that $cA = Ac$

Matrix Algebra

Product involving scalars

- If $\mathbf{A}$ is an $n\ x\ p$ matrix and $\mathbf{c}$ is a scalar, then:

$$2 \cdot \begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 \cdot 5 & 2 \cdot 2 \\ 2 \cdot 3 & 2 \cdot 1 \end{bmatrix}$$

The Kronecker Product

- The Kronecker product is an operation that transforms two matrices into a larger matrix that contains all the possible products of the entries of the two matrices

Matrix Algebra

The Kronecker Product

- Let $\mathbf{A} = (a_{ij})$ be an $n\ x\ p$ matrix and $\mathbf{B} = (b_{ij})$ an $r\ x\ s$ matrix

- The Kronecker product of A and B, denoted by $A \otimes B$:

$$\boldsymbol{A} \otimes \boldsymbol{B} = \begin{bmatrix} a_{11}\boldsymbol{B} & \ldots & a_{1p}\boldsymbol{B} \\ \vdots & \ddots & \vdots \\ a_{n1}\boldsymbol{B} & \ldots & a_{np}\boldsymbol{B} \end{bmatrix}$$

Matrix Algebra

The Kronecker Product

$$\begin{pmatrix} 1 & 3 & 2 \\ 1 & 0 & 0 \\ 1 & 2 & 2 \end{pmatrix} \otimes \begin{pmatrix} 0 & 5 \\ 5 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 \cdot \begin{pmatrix} 0 & 5 \\ 5 & 0 \\ 1 & 1 \end{pmatrix} & 3 \cdot \begin{pmatrix} 0 & 5 \\ 5 & 0 \\ 1 & 1 \end{pmatrix} & 2 \cdot \begin{pmatrix} 0 & 5 \\ 5 & 0 \\ 1 & 1 \end{pmatrix} \\ 1 \cdot \begin{pmatrix} 0 & 5 \\ 5 & 0 \\ 1 & 1 \end{pmatrix} & 0 \cdot \begin{pmatrix} 0 & 5 \\ 5 & 0 \\ 1 & 1 \end{pmatrix} & 0 \cdot \begin{pmatrix} 0 & 5 \\ 5 & 0 \\ 1 & 1 \end{pmatrix} \\ 1 \cdot \begin{pmatrix} 0 & 5 \\ 5 & 0 \\ 1 & 1 \end{pmatrix} & 2 \cdot \begin{pmatrix} 0 & 5 \\ 5 & 0 \\ 1 & 1 \end{pmatrix} & 2 \cdot \begin{pmatrix} 0 & 5 \\ 5 & 0 \\ 1 & 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 & 5 & 0 & 15 & 0 & 10 \\ 5 & 0 & 15 & 0 & 10 & 0 \\ 1 & 1 & 3 & 3 & 2 & 2 \\ 0 & 5 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 10 & 0 & 10 \\ 5 & 0 & 10 & 0 & 10 & 0 \\ 1 & 1 & 2 & 2 & 2 & 2 \end{pmatrix}$$

Matrix inverse

- If the A_{nxm} matrix is square, that is, the same number of rows and columns, the inverse matrix for A is given by A_{nxm}^{-1}, such that:

- The inverse matrix is denoted by A^{-1}, such that:

$$\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$$

Matrix inverse

- How to know if a matrix is invertible?

    - To know if a matrix is invertible we need to find its **determinant**.

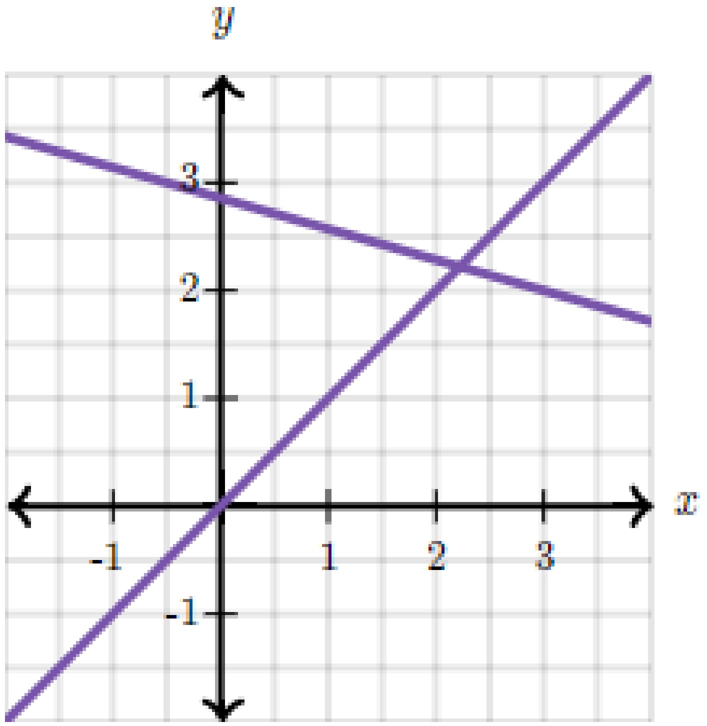
    - If the determinant of a matrix is nonzero, then the matrix is invertible. Otherwise it does not have an inverse matrix.

Systems of Equations

A system of equations can:

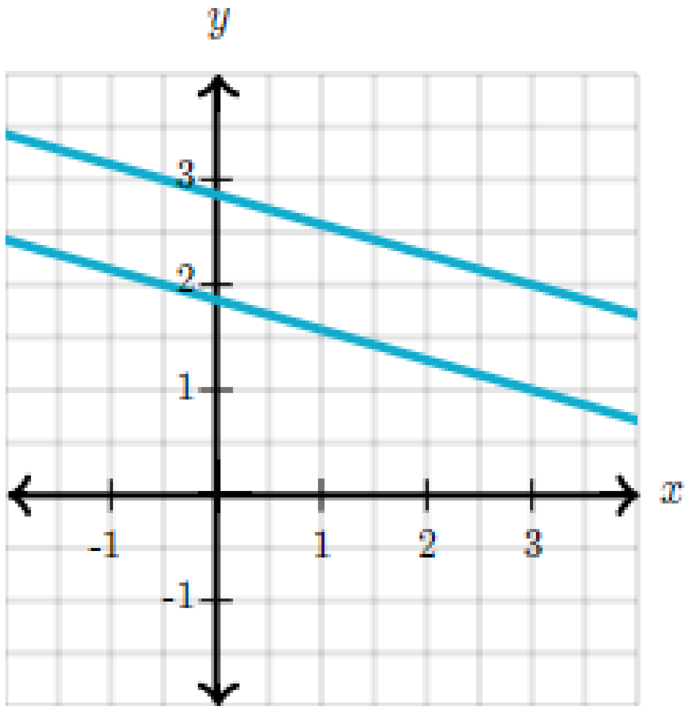
- No solution
- A single solution
- More than one solution

Systems of Equations: One solution



A system of linear equations has one solution when the graphs intersect at a point

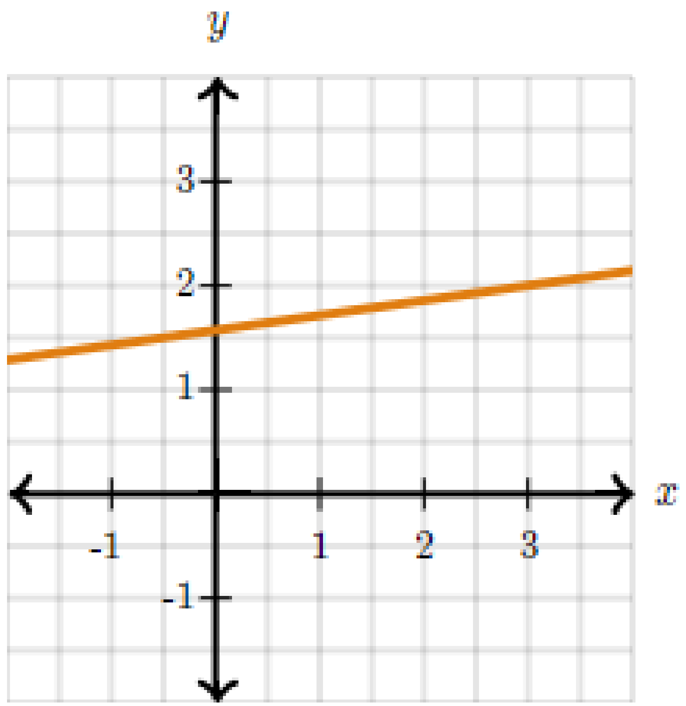
Systems of Equations: No solution



A system of linear equations has no solution when the graphs are parallel

Matrix Algebra

Systems of Equations: More than one solution



A system of linear equations has infinite solutions
when the graphs are the exact same line

Systems of Equations

- A system of equations is a set of **one or more equations** involving a number of same **variables**

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1p}x_p &= c_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2p}x_p &= c_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{np}x_p &= c_n
\end{aligned}
$$

Representation of Systems of Equations

- Can be written in matrix form as

$$\boldsymbol{Ax} = \boldsymbol{c}$$

Where $\boldsymbol{A}$ is $n\ x\ p$, $\boldsymbol{x}$ is $p\ x\ 1$ and $\boldsymbol{c}$ is $n\ x\ 1$

- If $\boldsymbol{A}$ is square $(n = p)$ and nonsingular, there exists a unique solution vector $\boldsymbol{x}$ obtained as:

$$\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{c}$$

Matrix Algebra

Representation of Systems of Equations

$$\begin{cases} 2x + 3y + 5z = 10 \\ x - y + 10z = 20 \\ -x + y - z = 5 \end{cases}$$

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 1 & -1 & 10 \\ -1 & 1 & -1 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 10 \\ 20 \\ 5 \end{bmatrix}$$

$$\mathbf{Ax} = \mathbf{B}$$

Matrix Algebra

Determinant

- The **determinant of a Matrix** is defined as a special number that is defined only for **square matrices** (matrices that have the same number of rows and columns)

- It is denoted by $\det(A) = |A|$

Matrix Algebra

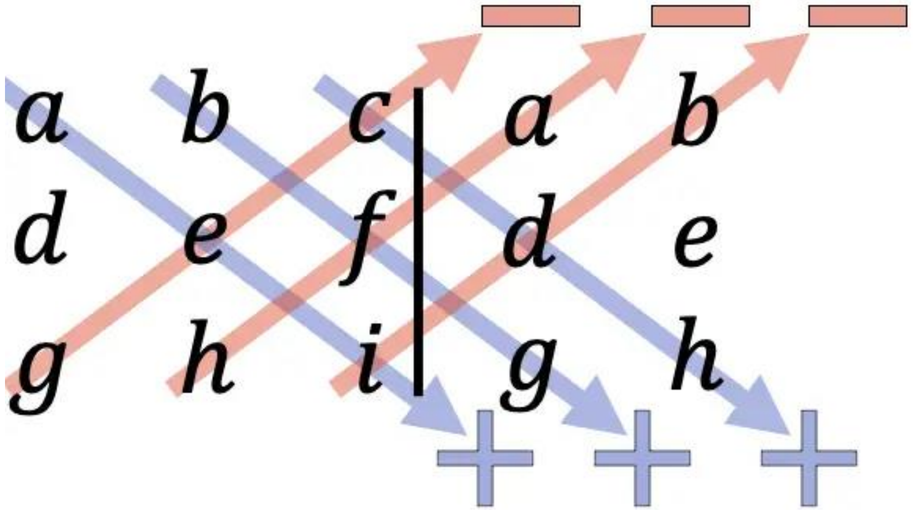
Determinant of 2x2 Matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$|A| = ad - bc$$

Determinant of 3x3 Matrix

- Rule of Sarrus



Matrix Algebra

Lets Practice!

1)

- If $\boldsymbol{A} = \begin{bmatrix} 10 & -3 & 2 \\ 5 & 0 & 7 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ -1 & 1 \end{bmatrix}$
- $\boldsymbol{AB} =$
- $\boldsymbol{BA} =$

2)

- Let $\boldsymbol{X} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

Obtain

- $\boldsymbol{X}'$
- $\boldsymbol{X}'\boldsymbol{X}$

Matrix Algebra

3)

- Let $\boldsymbol{B} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$
- Can we find $\boldsymbol{B}^{-1}$?

4)

$$\text{Let } \boldsymbol{A} = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 0 & 0 \\ 1 & 1 & 2 \\ 1 & 1 & -1 \end{bmatrix} \text{ and } \boldsymbol{b} = \begin{bmatrix} 10 \\ 3 \\ -2 \end{bmatrix}$$

Calculate

- $\boldsymbol{Ab}$
- $\boldsymbol{b'b}$
- $\boldsymbol{bb'}$

Matrix Algebra

5)

- Let $\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix}$
- Calculate $\boldsymbol{A} \otimes \boldsymbol{B}$

Matrix Algebra

6)

Find the determinant

- $\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}$

- $\boldsymbol{B} = \begin{bmatrix} -1 & 0 & 3 \\ 4 & 2 & 1 \\ -2 & 0 & 1 \end{bmatrix}$

References

1. Rencher, A. Linear Models in Statistics. 5–61 (1999). Chapter 2 - Matrix Algebra.