Chapter 4
DISTRIBUTED LOADING BENEATH THE SURFACE OF A SEMI-INFINITE MASS

4.1 Vertical Loading on a Horizontal Area

4.1.1 RECTANGULAR AREA

The Mindlin point load equation (Section 2.1.4) for vertical stress σ_z has been integrated over a rectangular area by Skopek (1961). The following expression has been obtained for σ_z beneath the corner of a rectangle $a \times b$ (Fig.4.1):

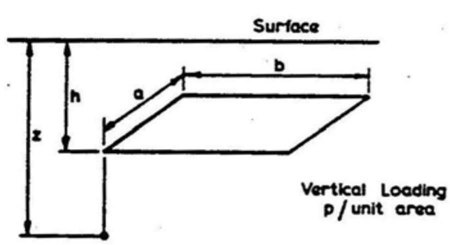

FIG.4.1

$$\sigma_z = \frac{p}{4\pi(1-\nu)}\Big[(1-\nu)\Big(\arctan\frac{ab}{(z-h)R_1} + \arctan\frac{ab}{(z+h)R_2}\Big) + \frac{(z-h)aR_1}{2br_1^2} - \frac{a(z-h)^3}{2br_3^2\,R_1}$$
$$+ \frac{[(3-4\nu)z(z+h)-h(5z-h)]aR_2}{2(z+h)br_2^2}$$
$$- \frac{[(3-4\nu)z(z+h)^2-h(z+h)(5z-h)]a}{2br_4^2\,R_2}$$
$$+ \frac{2hz(z+h)aR_2^3}{b^3r_2^4} + \frac{3hzaR_2r_5^2}{(z+h)b^3r_2^2} - \frac{hz(z+h)^3a}{br_4^4\,R_2}$$
$$\cdot\Big(\frac{2b^2-(z+h)^2}{b^2} - \frac{a^2}{R_2^2}\Big)\Big] \quad \ldots (4.1)$$

where
$$R_1^2 = a^2+b^2+(z-h)^2 \qquad R_2^2 = a^2+b^2+(z+h)^2$$
$$r_1^2 = a^2+(z-h)^2 \qquad r_2^2 = a^2+(z+h)^2$$
$$r_3^2 = b^2+(z-h)^2 \qquad r_4^2 = b^2+(z+h)^2$$
$$r_5^2 = b^2-(z+h)^2.$$

The stresses at other points within the mass may be obtained by use of the principle of superposition.

Influence factors for the vertical displacement of the corner of a rectangle are shown in Fig.4.2 (Groth and Chapman, 1969). The displacement is given by

$$\rho = \frac{q\,a\,I}{E} \quad \ldots (4.2)$$

where a is the shorter side of the rectangle.

The influence factor I is given by

$$I = K_0\Big[K_1\Big\{\beta\,\ln\Big(\frac{1+\sqrt{1+\beta^2}}{\beta}\Big) + \ln\big(\beta+\sqrt{1+\beta^2}\big)\Big\}$$
$$+ K_2\Big\{\ln\Big(\frac{\beta+t}{\sqrt{1+4\alpha^2\beta^2}}\Big) + \beta\,\ln\Big(\frac{1+t}{\beta s}\Big)$$
$$- 2\alpha\beta\tan^{-1}\Big(\frac{1}{2\alpha\beta}\Big) + 4\alpha\beta\tan^{-1}\Big(\frac{(1-s)(\beta s-t)}{2\alpha}\Big)\Big\}$$
$$+ 2\alpha\beta K_1\tan^{-1}\Big(\frac{1}{2\alpha t}\Big) + \frac{8\alpha^4\beta t}{s^2(1+4\alpha^2t^2)}$$
$$\cdot\Big\{2 + \frac{1}{4\alpha^2} - \frac{1}{t^2}\Big\}\Big] \quad \ldots (4.2a)$$

where
$$K_0 = \frac{1+\nu}{8\pi(1-\nu)}$$
$$K_1 = 3-4\nu$$
$$K_2 = 5-12\nu+8\nu^2$$
$$\alpha = h/b$$
$$\beta = b/a$$

$$s = \sqrt{1+4\alpha^2}$$
$$t = \sqrt{1+\beta^2(1+4\alpha^2)}$$

Stresses and displacements beneath a rigid rectangle embedded in a semi-infinite mass are given in Section 7.9.

For the limiting case of a uniformly loaded strip ($b/a=\infty$), Skopek gives the following expression for the vertical stress on the central axis of the strip:

$$\sigma_z = \frac{2}{\pi}\left[\arctan\frac{a}{z-h} + \arctan\frac{a}{z+h} + \frac{a(z-h)}{2[a^2+(z-h)^2](1-\nu)} + \frac{a[(3-4\nu)z+h]}{2[a^2+(z+h)^2](1-\nu)} + \frac{2hz(z+h)a}{[a^2+(z+h)^2]^2(1-\nu)}\right] \quad \ldots (4.1a)$$

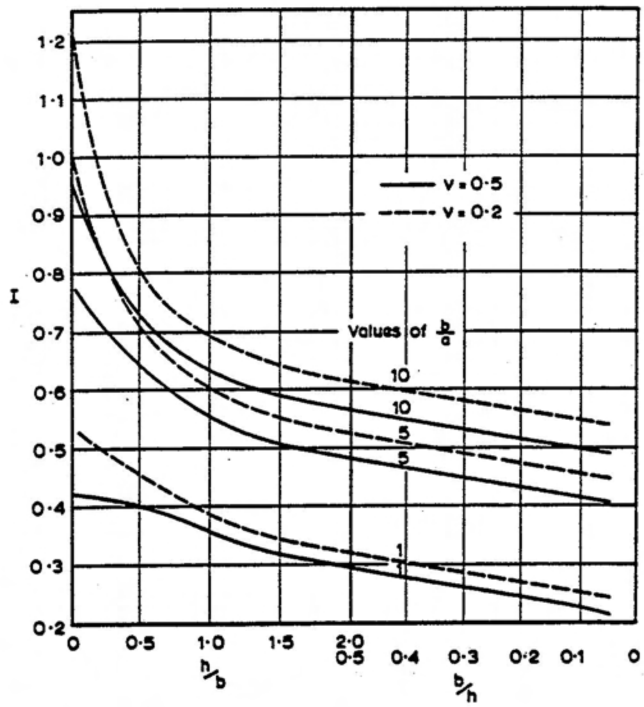

FIG.4.2 Vertical displacement factor for corner of embedded rectangle.

Fox (1948b) has obtained solutions for the relationship between the *mean* vertical displacement ρ_m of a rectangle beneath the surface to the mean displacement ρ_{mo} of a similar rectangle situated at the surface. ρ_m/ρ_{mo} is plotted against $h/\sqrt{ab}$ and $\sqrt{ab}/h$ in Fig.4.3 for various values of b/a and for $\nu=0.5$.

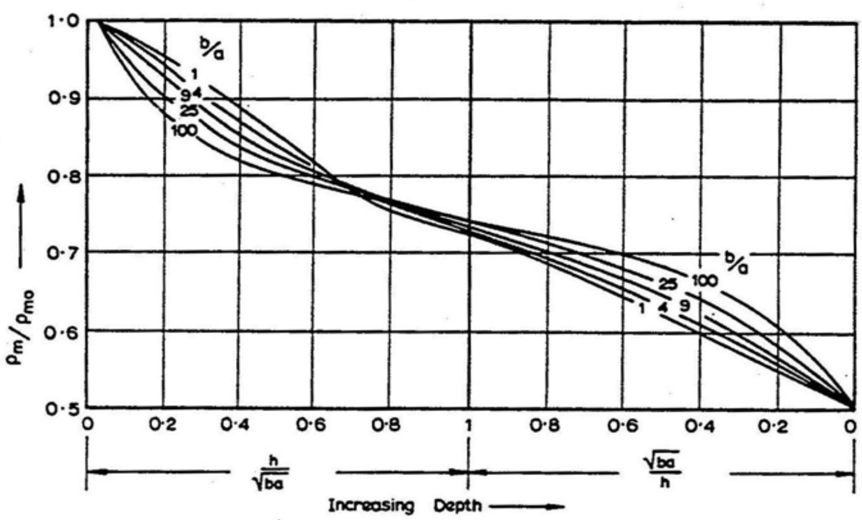

FIG.4.3 Ratio of mean displacement of rectangle at depth h to that of rectangle at surface. ν=0.5. (Fox, 1948).

4.1.2 CIRCULAR AREA (Fig.4.4)

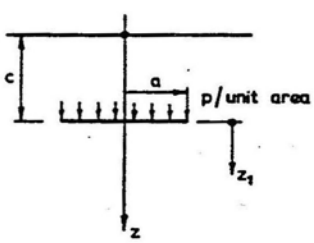

FIG.4.4

Nishida (1966) has derived the following expressions for vertical stress σ_z and vertical displacement ρ_z:

(a) *Beneath the centre*

$$\sigma_{zo} = \frac{p}{4(1-\nu)}\left[(1-2\nu)(z-c)\left\{\frac{1}{\sqrt{a^2+(z+c)^2}} - \frac{1}{\sqrt{a^2+(z-c)^2}} + \frac{1}{z-c} - \frac{1}{z+c}\right\}\right.$$

$$- \left\{\left(\frac{z+c}{\sqrt{a^2+(z+c)^2}}\right)^3 - 1\right\} \cdot \frac{3(3-4\nu)z(z+c)^2 - 3c(z+c)(5z-c)}{3(z+c)^3}$$

$$+ 1 - \left(\frac{z-c}{\sqrt{a^2+(z-c)^2}}\right)^3 + \frac{6cz}{(z+c)^2} \cdot \left.\left\{1 - \left(\frac{z+c}{\sqrt{a^2+(z-c)^2}}\right)^5\right\}\right] \qquad \ldots (4.3)$$

$$\rho_{zo} = \frac{p(1+\nu)}{4E(1-\nu)}\left[(3-4\nu)\left\{\sqrt{a^2+(z-c)^2} - (z-c)\right\}\right.$$

$$+ (5-12\nu+8\nu^2)\left\{\sqrt{a^2+(z+c)^2} - (z+c)\right\} + (z-c)$$

$$- \frac{(z-c)^2}{\sqrt{a^2+(z-c)^2}} + \frac{(3-4\nu)(z+c)^2 - 2cz}{z+c}$$

$$- \frac{2cz(z+c)^2}{\left(\sqrt{a^2+(z+c)^2}\right)^3} + \frac{2cz}{(z+c)}$$

$$\left.- \frac{(3-4\nu)(z+c)^2 - 2cz}{\sqrt{a^2+(z+c)^2}}\right] \qquad \ldots (4.4)$$

i.e., $$\rho_{zo} = \frac{pa\,I_o}{E} \qquad \ldots (4.4a)$$

(b) *Beneath the edge* $(r=a)$

See Nishida (1966) for explicit expressions.

ρ_z beneath the edge is given by

$$\rho_{ze} = \frac{pa\, I_e}{E} \quad \ldots (4.5)$$

Values of σ_z/p beneath the centre and edge of the circle given by Nishida are tabulated in Table 4.1. Influence factors I_o and I_e for the vertical displacement beneath the centre and edge of the circle are tabulated in Table 4.2.

TABLE 4.1

VERTICAL STRESS σ_z BENEATH CIRCULAR AREA (Nishida, 1966)

		σ_{zo}/p (centre)					σ_{ze}/p (edge)				
ν	z_1/a \ c/a	0	1	2	3	∞	0	1	2	3	∞
0.00	0	1.00	0.70	0.56	0.54	0.50	0.50	0.33	0.30	0.28	0.25
	1	0.64	0.35	0.30	0.27	0.25	0.34	0.21	0.18	0.17	0.13
	2	0.28	0.17	0.13	0.12	0.10	0.20	0.12	0.10	0.09	0.07
	4	0.09	0.06	0.05	0.04	0.03	0.12	0.05	0.04	0.03	0.01
0.25	0	1.00	0.71	0.57	0.53	0.50	0.50	0.38	0.31	0.28	0.25
	1	0.64	0.46	0.39	0.29	0.26	0.34	0.24	0.18	0.15	0.13
	2	0.28	0.18	0.15	0.13	0.11	0.20	0.13	0.11	0.09	0.08
	4	0.09	0.07	0.06	0.04	0.03	0.12	0.06	0.05	0.03	0.02
0.50	0	1.00	0.75	0.58	0.54	0.50	0.50	0.40	0.32	0.28	0.25
	1	0.64	0.45	0.38	0.35	0.34	0.34	0.29	0.21	0.19	0.16
	2	0.28	0.22	0.18	0.15	0.14	0.20	0.17	0.13	0.11	0.10
	4	0.09	0.08	0.07	0.04	0.04	0.12	0.07	0.06	0.05	0.04

The centre displacement may also be obtained from

$$\rho_{zo} = F_R \cdot (\rho_z)_s \quad \ldots (4.6)$$

where $(\rho_z)_s$ is the surface displacement given in Section 3.3.1,

and F_R is a reduction factor, plotted in Fig.4.7. For this case, $r_s=a$.

TABLE 4.2

INFLUENCE FACTORS FOR VERTICAL DISPLACEMENT OF CIRCLE (Nishida, 1966)

	I_o (centre)			I_e (edge)		
c/a \ ν	0.50	0.25	0.00	0.50	0.25	0.00
0.00	1.500	1.875	2.000	0.955	1.194	1.273
3.50	0.908	0.995	0.909			
5.00	0.862	0.947	0.862	0.586	0.640	0.585
100	0.750	0.833	0.750	0.478	0.530	0.478
1000	0.750	0.833	0.750	0.478	0.530	0.478

$$\rho_z = \frac{pa}{E} . I$$

4.1.3 GENERAL AREAS

Sector curves for bulk stress Θ are shown in Figs. 4.5 and 4.6 for $\nu=0$ and 0.5 (Poulos, 1967a).

Sector curves for the ratio F_R of the vertical displacement of a sector at depth d below the surface to the vertical displacement of the same sector situated at the surface, are shown in Fig.4.7 (Poulos, 1967a).

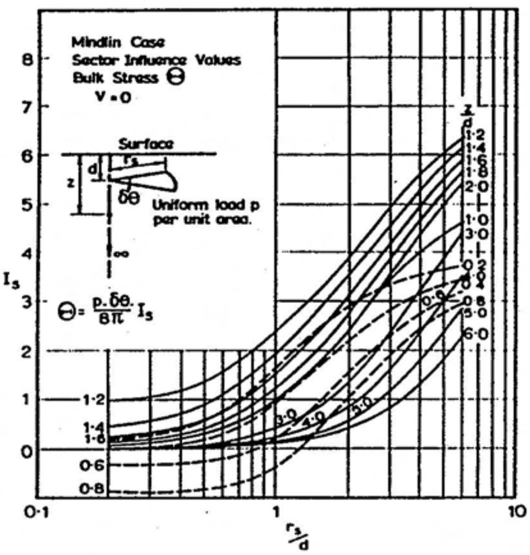

FIG.4.5 Sector curves for bulk stress Θ. ν=0.

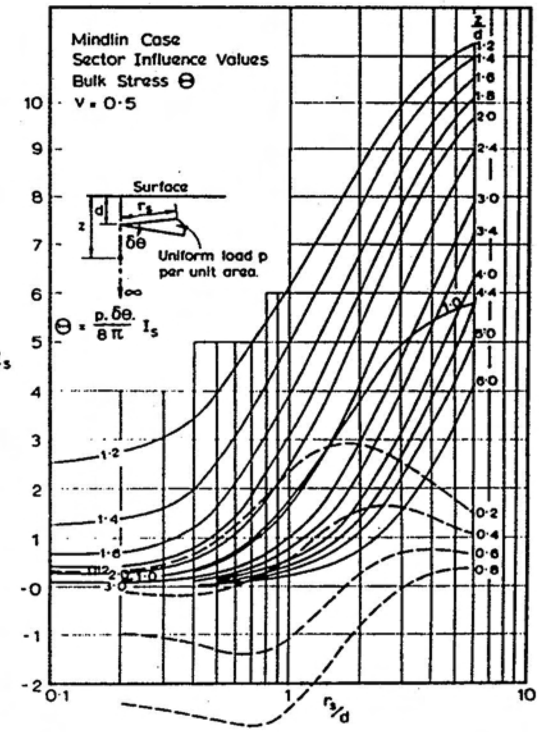

FIG.4.6 Sector curves for bulk stress Θ. ν=0.5.

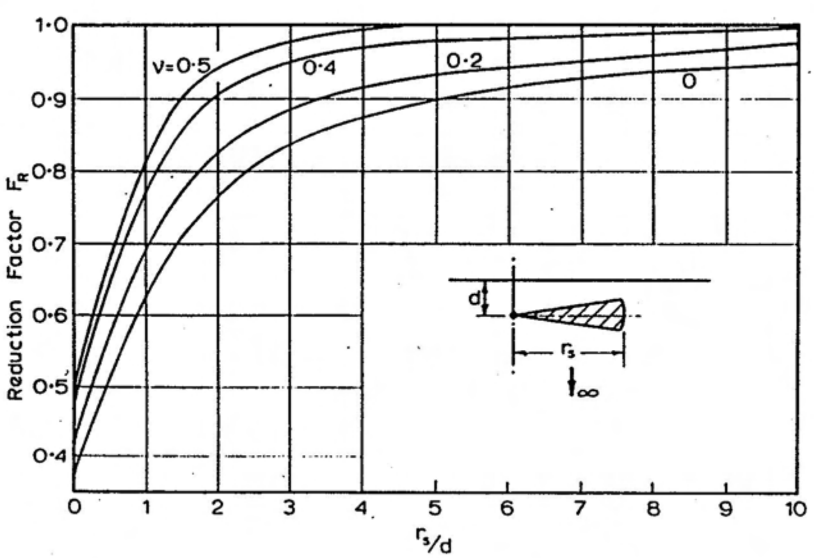

FIG.4.7 Ratio F_R of displacement at apex of sector at depth d to that of sector at surface.

4.2 Horizontal Loading on a Vertical Rectangle

The horizontal displacement ρ_x at the upper and lower corner of a rectangular area (Fig.4.8) has been obtained by Douglas and Davis (1964).

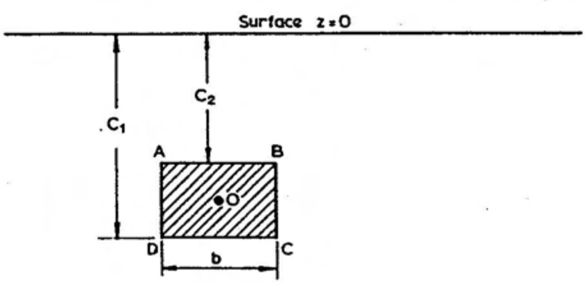

FIG.4.8

At the upper corners A and B, for a uniform horizontal pressure p,

$$\rho_x = \frac{pb}{32\pi G(1-\nu)} \{(3-4\nu)F_1+F_4+4(1-2\nu)(1-\nu)F_5\} \qquad \ldots (4.7)$$

At the lower corners D and C,

$$\rho_x = \frac{pb}{32\pi G(1-\nu)} \{(3-4\nu)F_1+F_2+4(1-2\nu)(1-\nu)F_3\} \qquad \ldots (4.8)$$

where $K_1 = \dfrac{2c_1}{b}$ $\quad K_2 = \dfrac{2c_2}{b}$

$$F_1 = -(K_1-K_2)\ \ln\left(\frac{(K_1-K_2)}{2+\sqrt{4+(K_1-K_2)^2}}\right) - 2\ \ln\left(\frac{2}{(K_1-K_2)+\sqrt{4+(K_1-K_2)^2}}\right)$$

$$F_2 = 2\ \ln\left(\frac{2(K_1+\sqrt{1+K_1^2})}{(K_1+K_2)+\sqrt{4+(K_1+K_2)^2}}\right) + (K_1-K_2) \times \ln\left(\frac{2+\sqrt{4+(K_1+K_2)^2}}{(K_1+K_2)}\right) - K_1^2\left(\frac{\sqrt{4+(K_1+K_2)^2}}{(K_1+K_2)} - \frac{\sqrt{1+K_1^2}}{K_1}\right)$$

$$F_3 = -2K_1\ \ln\left(\frac{K_1}{1+\sqrt{1+K_1^2}}\right) + (K_1+K_2) \times \ln\left(\frac{(K_1+K_2)}{2+\sqrt{4+(K_1+K_2)^2}}\right) - \ln\left(\frac{(K_1+K_2)+\sqrt{4+(K_1+K_2)^2}}{2(K_1+\sqrt{1+K_1^2})}\right) + \frac{(K_1+K_2)}{4} \times [\sqrt{4+(K_1+K_2)^2} - (K_1+K_2)] - K_1(\sqrt{1+K_1^2} - K_1)$$

$$F_4 = -2\ \ln\left(\frac{2(K_2+\sqrt{1+K_2^2})}{(K_1+K_2)+\sqrt{4+(K_1+K_2)^2}}\right) + (K_1-K_2)\ \ln\left(\frac{2+\sqrt{4+(K_1+K_2)^2}}{(K_1+K_2)}\right) + K_2^2\left(\frac{\sqrt{4+(K_1+K_2)^2}}{(K_1+K_2)} - \frac{\sqrt{1+K_2^2}}{K_2}\right)$$

$$F_5 = 2K_2\ \ln\left(\frac{K_2}{1+\sqrt{1+K_2^2}}\right) - (K_1+K_2) \times \ln\left(\frac{K_1+K_2}{2+\sqrt{4+(K_1+K_2)^2}}\right) + \ln\left(\frac{(K_1+K_2)+\sqrt{4+(K_1+K_2)^2}}{2(K_2+\sqrt{1+K_2^2})}\right) - \left(\frac{K_1+K_2}{4}\right) \times [\sqrt{4+(K_1+K_2)^2} - (K_1+K_2)] - K_2(K_2-\sqrt{1+K_2^2})$$

For the displacement at other points in the same plane, the principle of superposition may be employed.

Values of F_1 to F_5 are plotted in Figures 4.9 to 4.11.

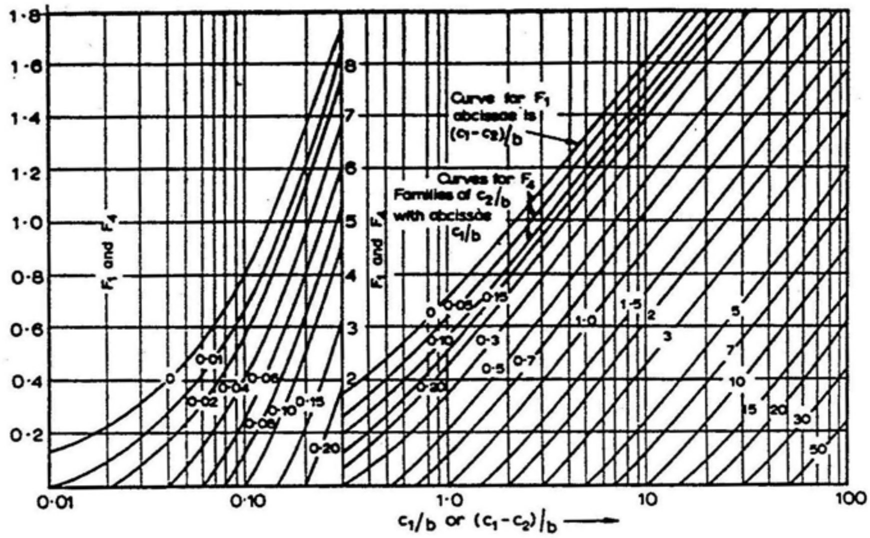

FIG.4.9 Factors F_1 and F_4 (Douglas and Davis, 1964).

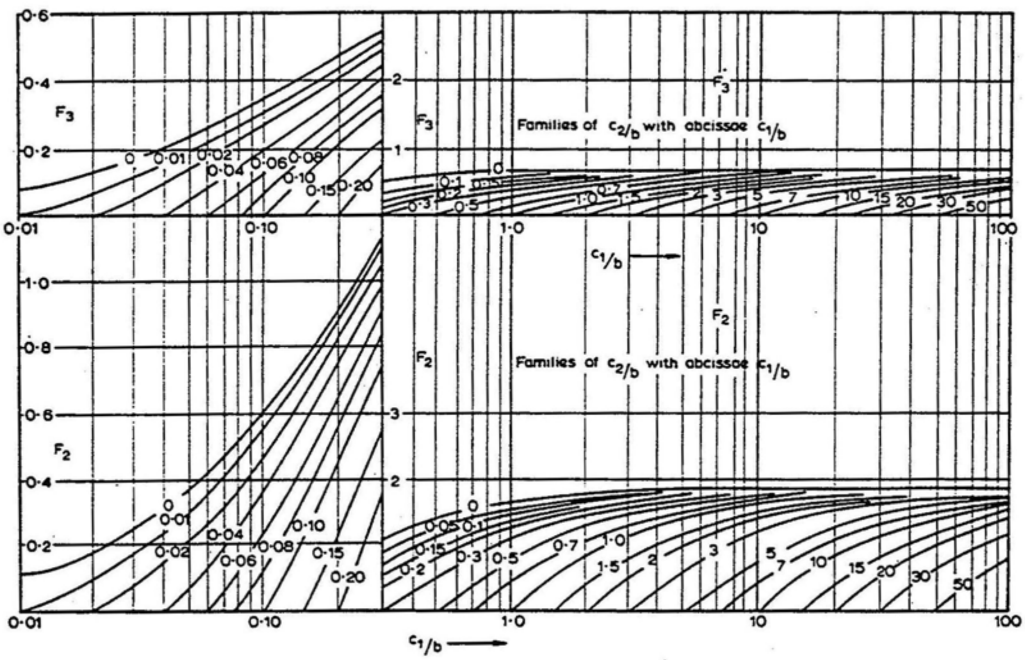

FIG.4.10 Factors F_2 and F_3 (Douglas and Davis, 1964).

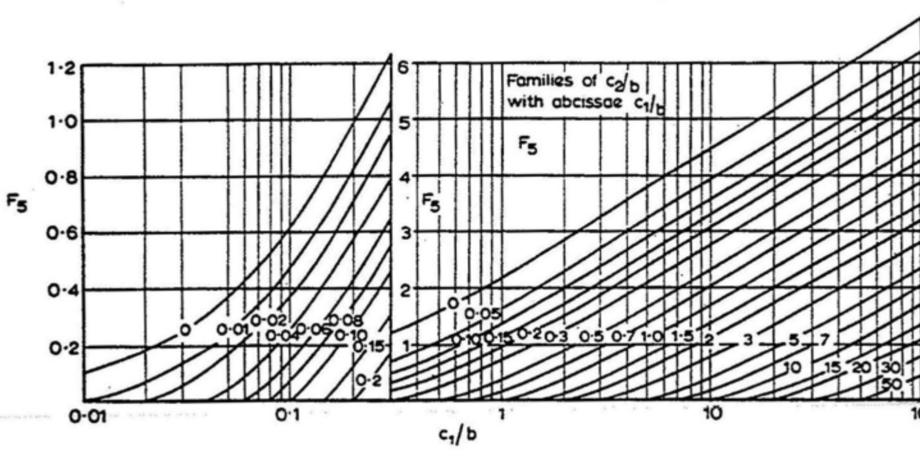

FIG.4.11 Factor F_5 (Douglas and Davis, 1964).

4.3 Rectangles Subjected to Shear Loading

Three cases have been considered by Groth and Chapman (1969), as shown in Fig.4.12, of the corner displacements in the direction of uniform loading applied to subsurface rectangles. Influence factors for the displacements are shown in Figs. 4.13 to 4.17. In all cases, the displacement is expressed as

$$\rho = \frac{q\,a\,I}{E} \qquad \ldots (4.9)$$

Case 1

Top Corner A:

$$I = K_0\Big\{(K_1+1)\,\ell n\big(\beta+\sqrt{1+\beta^2}\big) + \beta K_1\,\ell n\Big(\frac{1+\sqrt{1+\beta^2}}{\beta}\Big)$$
$$+ (K_1+K_2)\,\ell n\Big(\frac{s+\beta(1+2\alpha)}{t+2\alpha\beta}\Big)$$
$$+ \beta(1+2\alpha)K_2\,\ell n\Big(\frac{s+1}{\beta(1+2\alpha)}\Big)$$
$$- 2\alpha\beta K_2\,\ell n\Big(\frac{t+1}{2\alpha\beta}\Big) + \alpha\beta\,\ell n\Big(\frac{t+1}{t-1}\Big)$$
$$-\alpha\beta\,\ell n\Big(\frac{s+1}{s-1}\Big) + 2\alpha\beta\Big(\frac{1+2\alpha^2\beta^2}{t}$$
$$- \frac{1+\alpha\beta^2(1+2\alpha)}{s}\Big) + \alpha\Big(\frac{2\alpha\beta s}{1+2\alpha} - \beta t\Big)\Big\}$$

where
$$\alpha = \frac{h}{b}$$
$$\beta = \frac{b}{a}$$
$$s = \sqrt{1+\beta^2(1+2\alpha)^2}$$
$$t = \sqrt{1+4\alpha^2\beta^2}$$
$$K_0 = \frac{1+\nu}{8\pi(1-\nu)}$$
$$K_1 = 3-4\nu$$

Bottom Corner B:

$$I = K_0\Big\{\beta K_1\,\ell n\Big(\frac{1+\sqrt{1+\beta^2}}{\beta}\Big) - (K_1+1)\,\ell n\big(\sqrt{1+\beta^2}-\beta\big)$$
$$+(K_1+K_2)\,\ell n\Big(\frac{2\beta(1+\alpha)+s}{\beta(1+2\alpha)+t}\Big)$$
$$+ 2K_2\beta(1+\alpha)\,\ell n\Big(\frac{1+s}{2\beta(1+\alpha)}\Big)$$
$$- K_2\beta(1+2\alpha)\,\ell n\Big(\frac{1+t}{\beta(1+2\alpha)}\Big)$$
$$+ \beta(1+\alpha)\Big[\,\ell n\Big(\frac{t+1}{t-1}\Big) - \ell n\Big(\frac{s+1}{s-1}\Big) + s$$
$$- \frac{2(1+\alpha)t}{1+2\alpha} - \frac{2}{s}\big(1+2\beta^2(1+\alpha)^2\big)$$
$$+ \frac{2}{t}\big(1+\beta^2(1+\alpha)(1+2\alpha)\big)\Big]\Big\}$$

where α, β, K_0, K_1 as above
$$s = \sqrt{1+4\beta^2(1+\alpha)^2}$$
$$t = \sqrt{1+\beta^2(1+2\alpha)^2}$$

Case 2

Top Corner A:

$$I = K_0\Big\{(K_1+1)\Big[\beta\,\ell n\Big(\frac{1+\sqrt{1+\beta^2}}{\beta}\Big)$$
$$+ \beta(1+2\alpha)\,\ell n\Big(\frac{s+1}{\beta(1+2\alpha)}\Big) - 2\alpha\beta\,\ell n\Big(\frac{t+1}{2\alpha\beta}\Big)\Big]$$
$$+ \ell n\Big(\frac{s+\beta(1+2\alpha)}{t+2\alpha\beta}\Big) + K_1\,\ell n\big(\beta+\sqrt{1+\beta^2}\big)$$
$$- c\Big(\frac{1+\alpha\beta^2(1+2\alpha)}{s} - \frac{1+2\alpha^2\beta^2}{t}\Big)$$
$$+ 2(1-\nu)(1-2\nu)\Big[\beta(1+2\alpha)\big(s-\beta(1+2\alpha)\big)$$
$$- 2\alpha\beta(t-2\alpha\beta) + \ell n\Big(\frac{s+\beta(1+2\alpha)}{t+2\alpha\beta}\Big)\Big]\Big\}$$

where α, β, K_0, K_1 as above
$$s = \sqrt{1+\beta^2(1+2\alpha)^2}$$
$$t = \sqrt{1+4\alpha^2\beta^2}$$
$$c = 2\alpha\beta$$

Bottom Corner B:

$$I = K_0\Big\{\beta(K_1+1)\Big[\,\ell n\Big(\frac{1+\sqrt{1+\beta^2}}{\beta}\Big)$$
$$+ 2(1+\alpha)\,\ell n\Big(\frac{s+1}{2\beta(1+\alpha)}\Big)$$
$$- (1+2\alpha)\,\ell n\Big(\frac{t+1}{\beta(1+2\alpha)}\Big)\Big]$$
$$- K_1\,\ell n\big(\sqrt{1+\beta^2}-\beta\big) + \ell n\Big(\frac{s+2\beta(1+\alpha)}{t+\beta(1+2\alpha)}\Big)$$
$$-2\beta(1+\alpha)\Big(\frac{1}{s} - \frac{1}{t}\Big) - 2\beta^2(1+\alpha)\,\cdot$$
$$\Big(\frac{2\beta(1+\alpha)}{s} - \frac{\beta(1+2\alpha)}{t}\Big)$$
$$+ 2(1-\nu)(1-2\nu)\big[2\beta(1+\alpha)s-\beta(1+2\alpha)t$$

$$- 4\beta^2(1+\alpha)^2 + \beta^2(1+2\alpha)^2$$

$$+ \ln\left(\frac{s+2\beta(1+\alpha)}{t+\beta(1+2\alpha)}\right)]\}$$

where α, β, K_0, K_1 as above

$$s = \sqrt{1+4\beta^2(1+\alpha)^2}$$

$$t = \sqrt{1+\beta^2(1+2\alpha)^2}$$

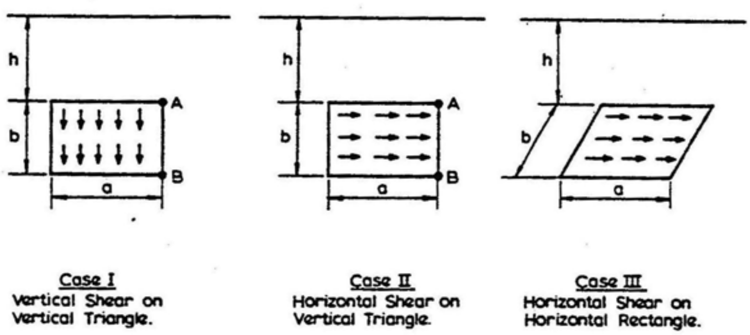

FIG.4.12 Loading cases for horizontal loading on a rectangle.

Case 3

$$I = K_0\{K_1[\ln(\beta+\sqrt{1+\beta^2}) + \tfrac{1}{2}\beta \ln\left(\frac{s+1}{s-1}\right)]$$

$$+ \beta(K_1+1)\ln\left(\frac{1+\sqrt{1+\beta^2}}{\beta}\right)$$

$$+ (K_2+1)\ln\left(\frac{\beta+s}{t}\right) + \beta\ln\left(\frac{1+s}{u}\right)$$

$$+ 2\alpha\beta\,[2\tan^{-1}\left(\frac{\beta(t-1)}{2\alpha\beta(t+s)}\right) - \tan^{-1}\frac{1}{2\alpha}$$

$$+ \frac{\alpha(s^2-1)}{s(1+4\alpha^2 s^2)} - K_1\tan^{-1}\left(\frac{1}{2\alpha s}\right)$$

$$+ K_2\left(2\tan^{-1}\left(\frac{u-\beta}{2\alpha\beta(u+s)}\right) - \tan^{-1}\left(\frac{1}{2\alpha\beta}\right)\right.$$

$$\left.+ 2\tan^{-1}\left(\frac{u-2\alpha\beta}{\beta(u+s)}\right)\right)]\}$$

where α, β, K_0, K_1 as above

$$K_2 = 4(1-\nu)(1-2\nu)$$

$$s = \sqrt{1+\beta^2(1+4\alpha^2)}$$

$$t = \sqrt{1+4\alpha^2\beta^2}$$

$$u = \beta\sqrt{1+4\alpha^2}$$

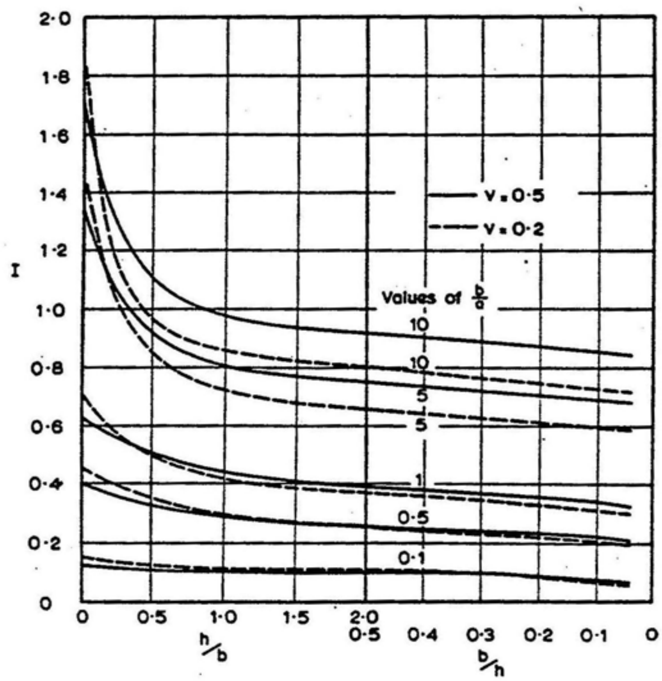

FIG.4.13 Vertical displacement factor for upper corner A. Case I.

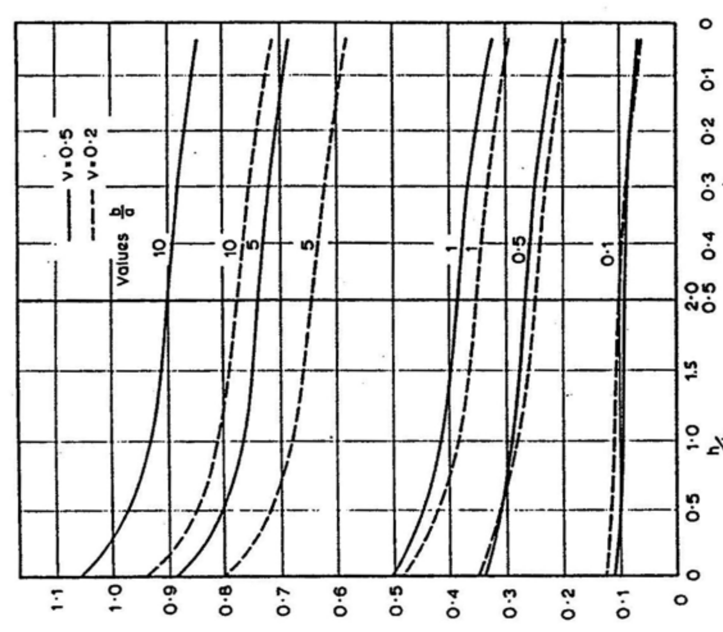

FIG.4.14 Vertical displacement factor for lower corner B. Case I.

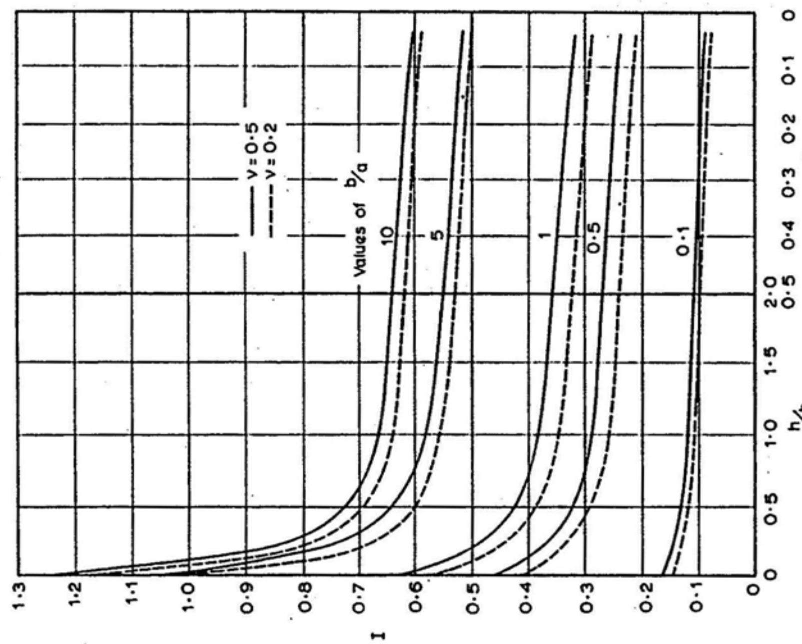

FIG.4.15 Horizontal displacement factor for upper corner A. Case II.

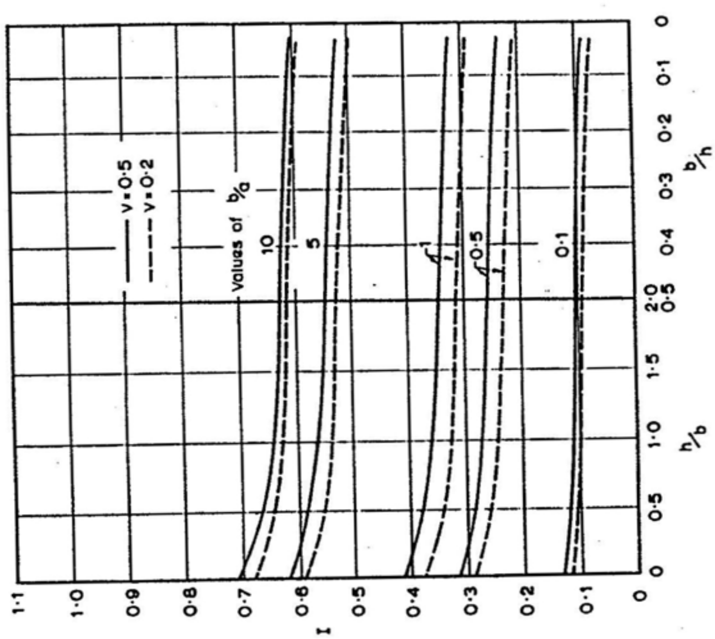

FIG.4.16 Horizontal displacement factor for lower corner B. Case II.

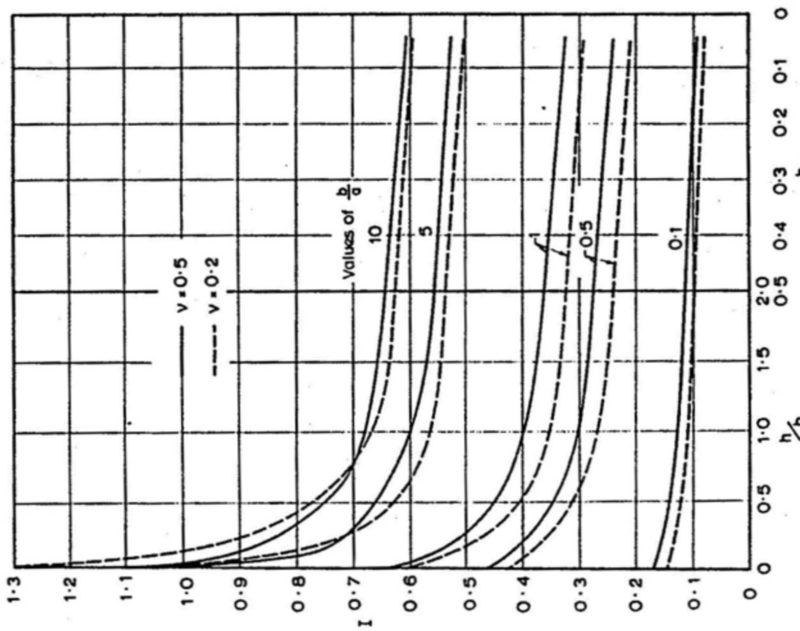

FIG.4.17 Horizontal displacement factor for corners. Case III.