Chapter 1
FUNDAMENTAL DEFINITIONS AND RELATIONSHIPS

1.1 Analysis of Stress

1.1.1 BASIC DEFINITIONS AND SIGN CONVENTION

Since it is often convenient in soil mechanics to consider compressive stresses as positive, this conventional will be adopted here. The normal and shear stresses acting on an element are shown in Fig.1.1, the stresses all being of positive sign.

The normal stresses $\sigma_x, \sigma_y, \sigma_z$ are positive when directed into the surface.

The notation for the shear stress τ_{ij} is as follows:

τ_{ij} is the shear stress acting in the j direction on a plane normal to the i axis.

The sign convention for shear stress is as follows:

The shear stress is *positive* when directed in a *negative* Cartesian direction while acting on a plane whose outward normal points in a *positive* direction, or, when directed in a *positive* Cartesian direction while acting on a plane whose outward normal points in a *negative* Cartesian direction.

Equilibrium requires that

$$\tau_{xy} = \tau_{yx}$$

$$\tau_{yz} = \tau_{zy}$$

$$\tau_{zx} = \tau_{xz}$$

For the definition of stresses in other coordinate systems, see Section 1.3.

1.1.2 STRESS COMPONENTS ON ANY PLANE

Referring to Fig.1.2, the stress components p_{nx}, p_{ny}, p_{nz} on any plane with a directed normal n can be expressed in terms of the stresses in the x, y and z coordinates as

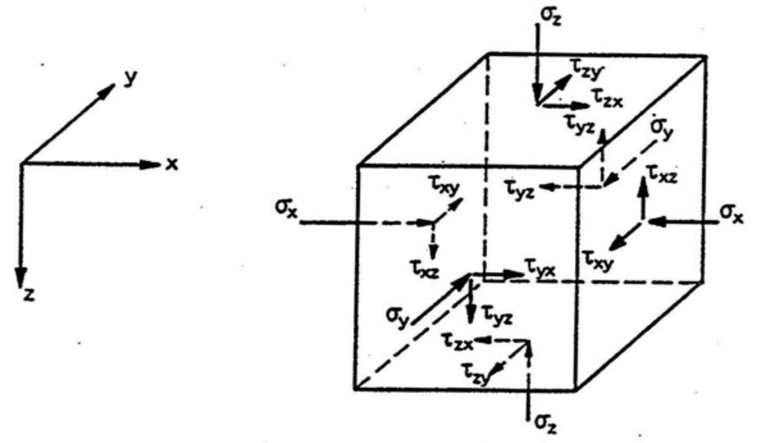


FIG.1.1

$$\begin{vmatrix} p_{nx} \\ p_{ny} \\ p_{nz} \end{vmatrix} = \begin{vmatrix} \sigma_x & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{vmatrix} \begin{vmatrix} cos(n,x) \\ cos(n,y) \\ cos(n,z) \end{vmatrix} \quad \dots (1.1)$$

where $cos(n,x)$ is the cosine of the angle between the n and x directions, and similarly for $cos(n,y)$ and $cos(n,z)$.

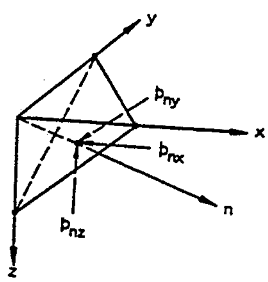


FIG.1.2

1.1.3 TRANSFORMATION OF AXES

If a new set of orthogonal axes x', y', z' are chosen, the stress components in this coordinate system are related to the stress components in the original x, y, z system as follows:

$$S_1 = A\,S\,A^T \quad \dots (1.2)$$

where S_1 is the stress matrix with respect to the $x'y'z'$ axes,

S is the stress matrix with respect to the $x\,y\,z$ axes,

A is the direction cosine matrix, i.e.,

$$A = \begin{vmatrix} cos(x',x) & cos(x',y) & cos(x',z) \\ cos(y',x) & cos(y',y) & cos(y',z) \\ cos(z',x) & cos(z',y) & cos(z',z) \end{vmatrix}$$

A^T is the transpose of A.

1.1.4 PRINCIPAL STRESSES

It is possible to show that there is one set of axes with respect to which all shear stresses are zero and the normal stresses have their extreme values. The three mutually perpendicular planes where this condition exists are called the *principal planes*, and the normal stresses acting on these planes are the principal stresses.

The principal stresses, σ_1, σ_2 and σ_3 (the maximum, intermediate and minimum stresses respectively) may be found as the roots of the equation

$$\sigma_i^3 - J_1\sigma_i^2 + J_2\sigma_i - J_3 = 0 \quad \dots (1.3)$$

where $J_1 = \sigma_x+\sigma_y+\sigma_z = \Theta(bulk\ stress)$ ... (1.4a)

$$J_2 = \sigma_x\sigma_y+\sigma_y\sigma_z+\sigma_z\sigma_x-\tau_{xy}^2-\tau_{yz}^2-\tau_{zx}^2 \quad \dots (1.4b)$$

$$J_3 = \sigma_x\sigma_y\sigma_z-\sigma_x\tau_{yz}^2-\sigma_y\tau_{zx}^2-\sigma_z\tau_{xy}^2 + 2\tau_{xy}\tau_{yz}\tau_{zx} \quad \dots (1.4c)$$

J_1 (or Θ), J_2, J_3 are often known as the first, second and third stress invariants, as they remain constant, independent of the coordinate system.

In terms of the principal stresses,

$$J_1 = \Theta = \sigma_1 + \sigma_2 + \sigma_3 \quad \dots (1.5a)$$

$$J_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 \quad \dots (1.5b)$$

$$J_3 = \sigma_1\sigma_2\sigma_3 \quad \dots (1.5c)$$

The directions of the normals to the principal planes are given by

$$cos(n_i,x) = \frac{A_i}{\sqrt{A_i^2+B_i^2+C_i^2}} \quad \dots (1.6a)$$

$$cos(n_i,y) = \frac{B_i}{\sqrt{A_i^2+B_i^2+C_i^2}} \quad \dots (1.6b)$$

$$cos(n_i,z) = \frac{C_i}{\sqrt{A_i^2+B_i^2+C_i^2}} \quad \dots (1.6c)$$

where $A_i = (\sigma_y-\sigma_i)(\sigma_z-\sigma_i)-\tau_{zy}\tau_{yz}$

$B_i = \tau_{zy}\tau_{xz} - \tau_{xy}(\sigma_z-\sigma_i)$

$C_i = \tau_{xy}\tau_{yz} - \tau_{xz}(\sigma_y-\sigma_i)$

and σ_i are the principal stresses ($i = 1,2,3$).

1.1.5 MAXIMUM SHEAR STRESS

The maximum shear stress occurs on a plane whose normal makes an angle of 45° with the σ_1 and σ_3 directions.

The maximum shear stress, τ_{max} at a point is given by

$$\tau_{max} = \tfrac{1}{2}(\sigma_1-\sigma_3) \qquad \ldots (1.7)$$

1.1.6 OCTAHEDRAL STRESSES

The octahedral normal stress σ_{oct} and the octahedral shear stress τ_{oct} at a point are the stresses acting on the eight planes of an imaginary octahedron surrounding the point, the normals to the faces of the octahedron having direction cosines of $\pm 1/\sqrt{3}$ with the direction of the principal stresses.

The magnitudes of the octahedral stresses are

$$\sigma_{oct} = \frac{\sigma_1+\sigma_2+\sigma_3}{3} = \frac{\Theta}{3} = \frac{\sigma_x+\sigma_y+\sigma_z}{3} \qquad \ldots (1.8)$$

$$\tau_{oct} = \frac{1}{3}\left((\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right)^{\frac{1}{2}}$$

$$= \frac{1}{3}\left((\sigma_x-\sigma_y)^2+(\sigma_y-\sigma_z)^2+(\sigma_z-\sigma_x)^2 + 6(\tau_{xy}^2+\tau_{xz}^2+\tau_{yz}^2)\right)^{\frac{1}{2}} \qquad \ldots (1.9)$$

1.1.7 TWO-DIMENSIONAL STRESS SYSTEMS

Many situations in soil mechanics can be treated as two-dimensional problems in which only the stresses in a single plane need be considered. The most important case is that of *plane strain*, in which the strain (see Section 1.2) in one of the coordinate directions (usually the y direction here) is zero. Another class of problems are those involving *plane stress* conditions, in which the stress in one of the coordinate directions (usually y here) is zero.

In two-dimensional stress situations, the stress relationships are considerably simplified in relation to the general three-dimensional case. Referring to Fig.1.3, the stresses on a plane making an angle θ with the z direction are

$$\sigma_\theta = \tfrac{1}{2}(\sigma_x+\sigma_z)+\tfrac{1}{2}(\sigma_x-\sigma_z)\cos 2\theta+\tau_{xz}\sin 2\theta \qquad \ldots (1.10)$$

$$\tau_\theta = \tau_{xz}\cos 2\theta-\tfrac{1}{2}(\sigma_x-\sigma_z)\sin 2\theta \qquad \ldots (1.11)$$

The principal stresses are given by

$$\left.\begin{matrix}\sigma_1\\ \sigma_3\end{matrix}\right\} = \tfrac{1}{2}(\sigma_x+\sigma_z) \pm\tfrac{1}{2}\left((\sigma_x-\sigma_z)^2+4\tau_{xz}^2\right)^{\frac{1}{2}} \qquad \ldots (1.12)$$

The principal planes are inclined at an angle

$$\theta_1 = \tfrac{1}{2}\tan^{-1}\frac{2\tau_{xz}}{\sigma_x-\sigma_z} \qquad \ldots (1.13)$$

and

$\theta_1+90°$ to the z axis.

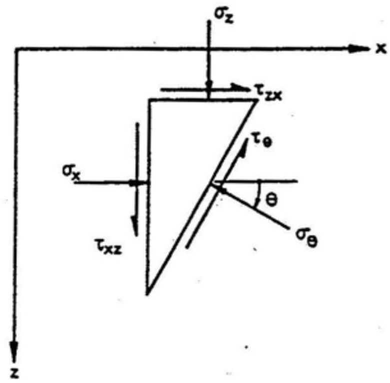


FIG.1.3

The maximum shear stress occurs on planes inclined at $45°$ to the principal planes and is of magnitude

$$\tau_{max} = \tfrac{1}{2}\left((\sigma_x-\sigma_z)^2+4\tau_{xz}^2\right)^{\frac{1}{2}} \qquad \ldots (1.14)$$

(It should be noted that the sign of this maximum shear stress is opposite on the two planes, in order to conform to the sign convention given in Section 1.1.1).

1.1.8 MOHR'S CIRCLE OF STRESS

A geometrical solution for stresses in any direction is provided by Mohr's circle, shown in Fig.1.4 for a two-dimensional stress system. The circle is drawn in relation to a set of orthogonal axes, one for normal stress (σ) and the other for shear stress (τ). The scale of these two axes must be equal.

If the principal stresses σ_1, σ_3 are known, the circle can be drawn with the centre at $\sigma=\tfrac{1}{2}(\sigma_1+\sigma_3)$ and of radius $(\sigma_1-\sigma_3)/2$.

If the normal and shear stresses are known, the circle can be drawn with the centre at $\sigma=\tfrac{1}{2}(\sigma_x+\sigma_y)$ and passing through the points (σ_x, τ_{xz}) and $(\sigma_z, -\tau_{xz})$.

The radius of the circle thus constructed is equal to the maximum shear stress τ_{max} (see Equation 1.14).

The angle $2\theta_1$ is twice the angle between the x-z coordinate axes and the axes corresponding to the directions of principal stress (the 1-3 axes in Fig.1.4). The direction of rotation of the radius from its original constructed position to where the circle intersects the normal stress axis is in the same angular sense as the direction of rotation of the axes for the x-z axes to become the principal 3-1 axes.

The stresses in any other directions x', z' may similarly be determined by drawing a diameter, through the centre of the circle, at an angle $2\theta'$ to the diameter describing the stress conditions on the

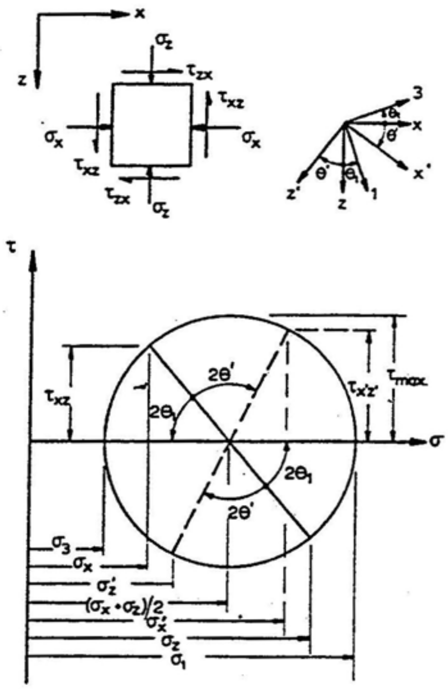


FIG.1.4 Mohr circle of stress.

x-z axes, where θ' is the angle between the x-z axes and the x'-z' axes (see Fig.1.4).

It should be noted that shear stresses are considered positive if they tend to produce a clockwise rotation about a point, outside the element, at the plane on which they act (Fig.1.5). This convention is consistent with that previously developed for three-dimensional conditions.

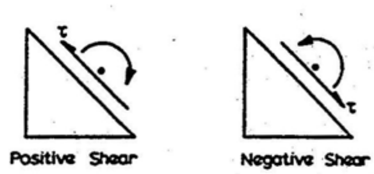


FIG.1.5

1.1.9 POLE CONSTRUCTION

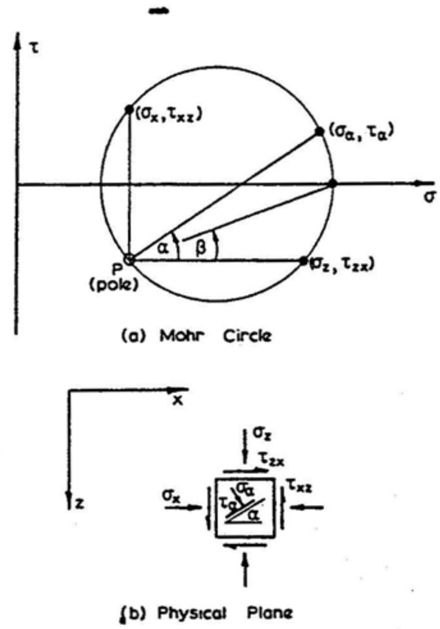


FIG.1.6 Pole construction.

The pole construction is a useful way of linking the stresses at a point in the physical plane (Fig. 1.6b) to the Mohr circle diagram for the stresses (Fig.1.6a). The pole, P, is the point on the circle such that the normal and shear stresses on any plane α (perpendicular to the physical plane) are given by the intersection with the Mohr circle of a line through P parallel with the plane α. For example the stresses on vertical and horizontal planes are as indicated in Fig.1.6a and the major principal plane is inclined at the angle β above the horizontal.

1.2 Analysis of Strain

1.2.1 BASIC DEFINITIONS

Considering first the case of two-dimensional strain (Fig.1.7), the normal strains ε_x and ε_z are defined as

$$\varepsilon_x = -\frac{\partial \rho_x}{\partial x} \qquad \text{... (1.15a)}$$

$$\varepsilon_z = -\frac{\partial \rho_z}{\partial z} \qquad \dots (1.15b)$$

where ρ_x, ρ_z are the displacements in the x and z directions. A positive normal strain corresponds to a decrease in length.

The shear strain γ_{xz} is the angular change in a right angle in a material and is related to the displacements ρ_x and ρ_z as

$$\gamma_{xy} = -\frac{\partial \rho_x}{\partial z} - \frac{\partial \rho_z}{\partial x} \qquad \dots (1.16)$$

A positive shear strain represents an increase in the right angle and a negative shear strain represents a decrease in the right angle.

Considering the xy and yz planes similarly, the six strain components are related to the displacements ρ_x, ρ_y, ρ_z in the x, y and z directions as

$$\varepsilon_x = -\frac{\partial \rho_x}{\partial x} \qquad \gamma_{xy} = -\frac{\partial \rho_x}{\partial y} - \frac{\partial \rho_y}{\partial x} \qquad \dots (1.17a)$$

$$\varepsilon_y = -\frac{\partial \rho_y}{\partial y} \qquad \gamma_{yz} = -\frac{\partial \rho_y}{\partial z} - \frac{\partial \rho_z}{\partial y} \qquad \dots (1.17b)$$

$$\varepsilon_z = -\frac{\partial \rho_z}{\partial z} \qquad \gamma_{zx} = -\frac{\partial \rho_z}{\partial x} - \frac{\partial \rho_x}{\partial z} \qquad \dots (1.17c)$$

As for shear stresses $\gamma_{ij} = \gamma_{ji}$

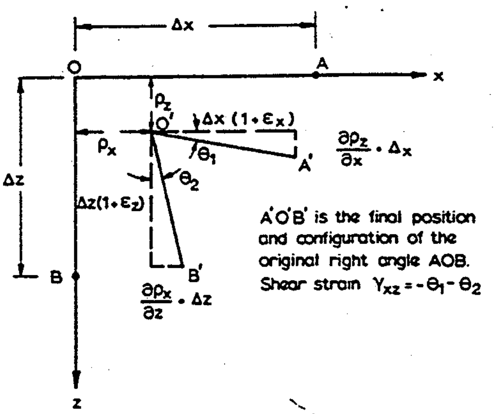


FIG.1.7

1.2.2 STRAIN IN A PLANE

Considering again a two-dimensional strain situation, the normal strain ε_θ in a plane inclined at θ to the x axis is

$$\varepsilon_\theta = \frac{\varepsilon_x+\varepsilon_z}{2} + \frac{\varepsilon_x-\varepsilon_z}{2}\cos 2\theta + \frac{\gamma_{xz}}{2}\sin 2\theta \qquad \dots (1.18)$$

and the shear strain is

$$\gamma_\theta = \gamma_{xz}\cos 2\theta - (\varepsilon_x-\varepsilon_z)\sin 2\theta \qquad \dots (1.19)$$

(Note that the above expressions correspond to those for the normal and shear stresses (Section 1.1), except for a factor of ½ in the last term).

1.2.3 TRANSFORMATION OF AXES

If a new set of orthogonal axes x', y', z' are chosen, the strain components in this coordinate system are related to the strain components in the original x, y, z system as

$$D_1 = A\,D\,A^T \qquad \dots (1.20)$$

where D is the strain matrix in the x, y, z system,

i.e.,

$$D = \begin{vmatrix} \varepsilon_x & \tfrac{1}{2}\gamma_{xy} & \tfrac{1}{2}\gamma_{xz} \\ \tfrac{1}{2}\gamma_{yx} & \varepsilon_y & \tfrac{1}{2}\gamma_{yz} \\ \tfrac{1}{2}\gamma_{zx} & \tfrac{1}{2}\gamma_{zy} & \varepsilon_z \end{vmatrix} \qquad \dots (1.21)$$

D_1 is the strain matrix in the x', y', z' system.

A is the direction cosine matrix defined in Section 1.1.3.

A^T is the transpose of A.

In matrix operations, it is convenient to use the double suffix notation and to define $\tfrac{1}{2}\gamma_{ij}$ as ε_{ij}. The strain matrix is then

$$D = \begin{vmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{yz} & \varepsilon_{zz} \end{vmatrix} \qquad \dots (1.22)$$

1.2.4 PRINCIPAL STRAINS

Analogous to the principal planes of stress, there are three principal planes of strain. The shear strains in these planes are zero and the normal strains are the principal strains. The major and minor principal strains are respectively, the greatest and least normal strains at the point. For an isotropic elastic material, the principal planes of strain can be shown to coincide with the principal planes of stress.

The principal strains are determined, in a similar manner to principal stresses, as the roots of the equation

$$\varepsilon_i^3 - I_1\varepsilon_i^2 + I_2\varepsilon_i - I_3 = 0 \quad \text{... (1.23)}$$

$$\text{where } I_1 = \varepsilon_x + \varepsilon_y + \varepsilon_z \quad \text{... (1.24a)}$$

$$I_2 = \varepsilon_x\varepsilon_y + \varepsilon_y\varepsilon_z + \varepsilon_z\varepsilon_x - \frac{\gamma_{xy}^2}{4} - \frac{\gamma_{yz}^2}{4} - \frac{\gamma_{zx}^2}{4} \quad \text{... (1.24b)}$$

$$I_3 = \varepsilon_x\varepsilon_y\varepsilon_z - \frac{\varepsilon_x\gamma_{yz}^2}{4} - \frac{\varepsilon_y\gamma_{zx}^2}{4} - \frac{\varepsilon_z\gamma_{xy}^2}{4} + \frac{\gamma_{xy}\,\gamma_{xz}\,\gamma_{yz}}{4} \quad \text{... (1.24c)}$$

I_1, I_2, I_3 are the strain invariants, analogous to the stress invariants.

In two-dimensional systems, the principal strains ε_1, ε_3 are as follows:

$$\left.\begin{matrix}\varepsilon_1\\ \varepsilon_3\end{matrix}\right\} = \frac{\varepsilon_x+\varepsilon_z}{2} \pm \tfrac{1}{2}\sqrt{(\varepsilon_x-\varepsilon_z)^2 + \gamma_{xz}^2} \quad \text{... (1.25)}$$

and the principal planes are inclined at an angle θ_1 to the x and z axes, where

$$\theta_1 = \tfrac{1}{2}\tan^{-1}\frac{\gamma_{xz}}{\varepsilon_x-\varepsilon_z} \quad \text{... (1.26)}$$

1.2.5 MAXIMUM SHEAR STRAIN

$$\gamma_{max} = \varepsilon_1 - \varepsilon_3 \quad \text{... (1.27)}$$

where ε_1 = maximum principal normal strain,
ε_3 = minimum principal normal strain.

γ_{max} occurs on a plane whose normal makes an angle of $45°$ with the ε_1 and ε_3 directions.

1.2.6 MOHR'S CIRCLE OF STRAIN

A geometrical solution for strains in any direction is provided by Mohr's circle of strain (Fig.1.8). The only difference between the circle of strain and the circle of stress is that, in the circle of strain, the ordinate represents only one-half the shear strain (i.e. the ordinate axis is $\gamma/2$). As in Fig.1.4, the axes *1-3* represent the principal axes, x-z the horizontal and vertical space axes and x'-z' the axes in direction at an angle θ' to the x-z axes.

The diameter of the circle is equal to the maximum shear strain

$$\gamma_{max} = \sqrt{(\varepsilon_x-\varepsilon_z)^2 + \gamma_{xz}^2}$$

The pole construction as described for the Mohr circle of stress may be adapted for the Mohr circle of strain.

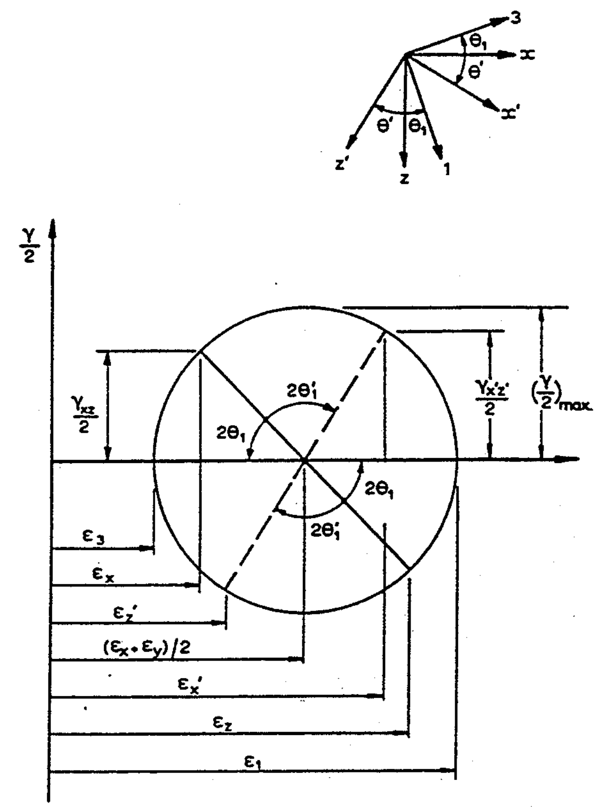


FIG.1.8 Mohr circle of strain.

1.3 Equilibrium Equations

1.3.1 CARTESIAN COORDINATES

By considering the equilibrium of the element shown in Fig.1.1 in the Cartesian coordinate system, the following equilibrium equations are obtained:

$$\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{yx}}{\partial y} + \frac{\partial\tau_{xz}}{\partial z} - X = 0 \quad \text{... (1.28a)}$$

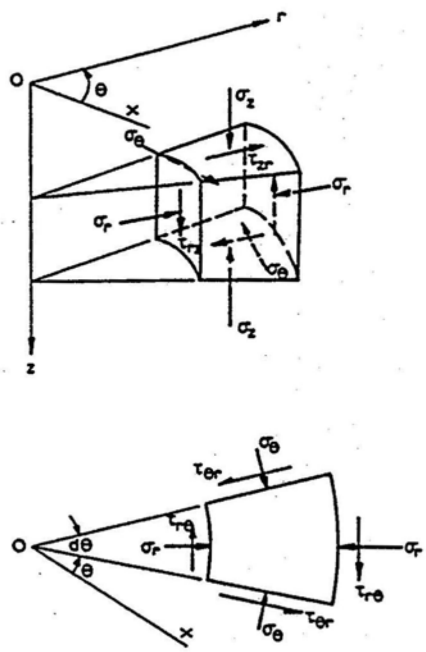
$$\frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{yz}}{\partial z} - Y = 0 \quad \text{... (1.28b)}$$

$$\frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} + \frac{\partial\sigma_z}{\partial z} - Z = 0$$

where X, Y, Z are the body forces, per unit volume, in the x, y and z directions.

With an ordinary gravity field and the z direction vertically downwards, X and Y are zero and Z is the unit weight, γ, of the material.

1.3.2 CYLINDRICAL COORDINATES



Plan of element in rθ plane.

FIG.1.9

Considering the equilibrium of the element in the cylindrical (r,z,θ) coordinate system shown in Fig. 1.9, the equilibrium equations are (neglecting body forces)

$$\frac{\partial\sigma_r}{\partial r} + \frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{\partial\tau_{zr}}{\partial z} + \frac{\sigma_r-\sigma_\theta}{r} = 0 \qquad \ldots (1.29a)$$

$$\frac{\partial\tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta} + \frac{\partial\tau_{\theta z}}{\partial z} + \frac{2\tau_{r\theta}}{r} = 0 \qquad \ldots (1.29b)$$

$$\frac{\partial\tau_{zr}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{\theta z}}{\partial\theta} + \frac{\partial\sigma_z}{\partial z} + \frac{\tau_{zr}}{r} = 0 \qquad \ldots (1.29c)$$

With axial symmetry, these become

$$\frac{\partial\sigma_r}{\partial r} + \frac{\partial\tau_{zr}}{\partial z} + \frac{\sigma_r-\sigma_\theta}{r} = 0 \qquad \ldots (1.30a)$$

$$\frac{\partial\tau_{zr}}{\partial r} + \frac{\partial\sigma_z}{\partial z} + \frac{\tau_{zr}}{r} = 0 \qquad \ldots (1.30b)$$

1.3.3. SPHERICAL COORDINATES (Fig.1.10)

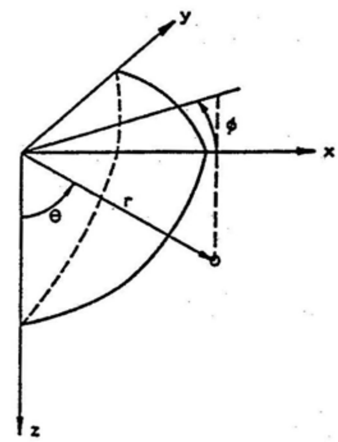


FIG.1.10

$$\frac{\partial\sigma_r}{\partial r} + \frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\tau_{r\phi}}{\partial\phi} + \frac{2\sigma_r-\sigma_\theta-\sigma_\phi+\tau_{r\theta}\cot\theta}{r} = 0 \qquad \ldots (1.31a)$$

$$\frac{\partial\tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\tau_{\theta\phi}}{\partial\phi} + \frac{3\tau_{r\theta}+(\sigma_\theta-\sigma_\phi)\cot\theta}{r} = 0 \qquad \ldots (1.31b)$$

$$\frac{\partial\tau_{r\phi}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{\theta\phi}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\sigma_\phi}{\partial\phi} + \frac{3\tau_{r\phi}+2\tau_{\theta\phi}\cot\theta}{r} = 0 \qquad \ldots (1.31c)$$

For complete spherical symmetry these become

$$\frac{\partial\sigma_r}{\partial r} + \frac{2(\sigma_r-\sigma_\theta)}{r} = 0 \qquad \ldots (1.32)$$

1.4 Strain-Displacement and Compatibility Equations

1.4.1 CARTESIAN COORDINATES

The strain - displacement relationships are given in equation (1.17). Since six strain components are derived from only three displacements, the strains are not independent of each other. Six further relationships, known as the compatibility equations, can be derived. These are as follows:

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \quad \dots (1.33a)$$

$$\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z} \quad \dots (1.33b)$$

$$\frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial z \partial x} \quad \dots (1.33c)$$

$$2\left(\frac{\partial^2 \varepsilon_x}{\partial y \partial z}\right) = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right) \quad \dots (1.33d)$$

$$2\left(\frac{\partial^2 \varepsilon_y}{\partial z \partial x}\right) = \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right) \quad \dots (1.33e)$$

$$2\left(\frac{\partial^2 \varepsilon_z}{\partial x \partial y}\right) = \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right) \quad \dots (1.33f)$$

1.4.2 CYLINDRICAL COORDINATES

The strain - displacement equations are

$$\varepsilon_r = -\frac{\partial \rho_r}{\partial r} \qquad \gamma_{r\theta} = -\frac{1}{r}\frac{\partial \rho_r}{\partial \theta} - \frac{\partial \rho_\theta}{\partial r} + \frac{\rho_\theta}{r} \quad \dots (1.34a)$$

$$\varepsilon_\theta = -\frac{\rho_r}{r} - \frac{1}{r}\frac{\partial \rho_\theta}{\partial \theta} \qquad \gamma_{\theta z} = -\frac{\partial \rho_\theta}{\partial z} - \frac{1}{r}\frac{\partial \rho_z}{\partial \theta} \quad \dots (1.34b)$$

$$\varepsilon_z = -\frac{\partial \rho_z}{\partial z} \qquad \gamma_{zr} = -\frac{\partial \rho_z}{\partial r} - \frac{\partial \rho_r}{\partial z} \quad \dots (1.34c)$$

The corresponding compatability equations are quoted by L'ure (1964).

1.4.3. SPHERICAL COORDINATES

The strain - displacement equations are:

$$\varepsilon_r = -\frac{\partial \rho_r}{\partial r} \qquad \gamma_{r\theta} = -\frac{1}{r}\frac{\partial \rho_r}{\partial \theta} - \frac{\partial \rho_\theta}{\partial r} + \frac{\rho_\theta}{r} \quad \dots (1.35a)$$

$$\varepsilon_\theta = -\frac{\rho_r}{r} - \frac{1}{r}\frac{\partial \rho_\theta}{\partial \theta} \qquad \gamma_{\theta\phi} = -\frac{1}{r\sin\theta}\frac{\partial \rho_\theta}{\partial \phi} - \frac{1}{r}\frac{\partial \rho_\phi}{\partial \theta} + \frac{\rho_\phi \cot\theta}{r} \quad \dots (1.35b)$$

$$\varepsilon_\phi = -\frac{\rho_r}{r} - \frac{\rho_\theta}{r}\cot\theta - \frac{1}{r\sin\theta}\frac{\partial \rho_\phi}{\partial \phi} \qquad \gamma_{\phi r} = -\frac{\partial \rho_\phi}{\partial r} + \frac{\rho_\phi}{r} - \frac{1}{r\sin\theta}\frac{\partial \rho_r}{\partial \phi} \quad \dots (1.35c)$$

The compatibility equations, for the case of axial symmetry, are quoted by L'ure (1964).

1.5 Stress-Strain Relationships

1.5.1 LINEAR HOMOGENEOUS ISOTROPIC MATERIAL

Strains in terms of stress:

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] \quad \dots (1.36a)$$

$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)] \quad \dots (1.36b)$$

$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)] \quad \dots (1.36c)$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy} \quad \dots (1.36d)$$

$$\gamma_{yz} = \frac{1}{G}\tau_{yz} \quad \dots (1.36e)$$

$$\gamma_{zx} = \frac{1}{G}\tau_{zx} \quad \dots (1.36f)$$

where E = Young's modulus
ν = Poisson's ratio
G = shear modulus

$$= \frac{E}{2(1+\nu)} \quad \dots (1.37)$$

Also, volume strain

$$\varepsilon_v = \frac{(1-2\nu)}{E}\Theta = \frac{\Theta}{3K} \quad \dots (1.37)$$

where $\varepsilon_v = \varepsilon_x + \varepsilon_y + \varepsilon_z$

$\Theta = \sigma_x + \sigma_y + \sigma_z$ = bulk stress

K = bulk modulus.

Stresses in terms of strains:

$$\sigma_x = \lambda\varepsilon_v + 2G\varepsilon_x \quad \ldots (1.38a)$$

$$\sigma_y = \lambda\varepsilon_v + 2G\varepsilon_y \quad \ldots (1.38b)$$

$$\sigma_z = \lambda\varepsilon_v + 2G\varepsilon_z \quad \ldots (1.38c)$$

$$\tau_{xz} = G\gamma_{xz} \quad \text{etc.} \quad \ldots (1.38d)$$

where λ, G are Lame's parameters

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \quad \ldots (1.39a)$$

$$G = \frac{E}{2(1+\nu)} = \text{shear modulus} \quad \ldots (1.39b)$$

(G is also often denoted as μ).

For the special case of plane stress e.g. in the x-z plane, $\sigma_y=0$ in the above equations.

For the special case of plane strain in the x-z plane, $\varepsilon_y=0$ and hence

$$\sigma_y = \nu(\sigma_x + \sigma_z) \quad \ldots (1.40)$$

Equations (1.36) then reduce to

$$\varepsilon_x = \frac{(1+\nu)}{E}[\sigma_x(1-\nu) - \nu\sigma_z] \quad \ldots (1.41a)$$

$$\varepsilon_y = 0 \quad \ldots (1.41b)$$

$$\varepsilon_z = \frac{(1+\nu)}{E}[\sigma_z(1-\nu) - \nu\sigma_x] \quad \ldots (1.41c)$$

$$\gamma_{xz} = \frac{1}{G}\tau_{xz} \quad \ldots (1.41d)$$

Solutions for a plane strain problem can be used for the corresponding plane stress problem provided that the following equivalent values of E and ν are used in the plane strain problem:

$$E_e = \frac{(1+2\nu)E}{(1+\nu)^2} \quad \ldots (1.42a)$$

$$\nu_e = \frac{\nu}{1+\nu} \quad \ldots (1.42b)$$

Conversely, to use solutions for a plane stress problem for the corresponding plane strain problem, the equivalent moduli are

$$E_T = \frac{E}{1-\nu^2} \quad \ldots (1.43a)$$

$$\nu_T = \frac{\nu}{1-\nu} \quad \ldots (1.43b)$$

Plane stress solutions which do not involve the elastic parameters are therefore identical with the corresponding plane strain solutions e.g. stresses within a semi-infinite plate and stresses due to line loading on a semi-infinite mass.

Summary of Relationships Between Elastic Parameters

$$G \text{ (or } \mu) = \frac{E}{2(1+\nu)} \quad \ldots (1.44)$$

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad \ldots (1.45)$$

$$K = \frac{E}{3(1-2\nu)} = \frac{2(1+\nu)G}{3(1-2\nu)} \quad \ldots (1.46)$$

$$E = \frac{9KG}{3K+G} \quad \ldots (1.47)$$

$$\nu = \frac{(3K-2G)}{2(3K+G)} \quad \ldots (1.48)$$

$$\frac{\lambda}{G} = \frac{2\nu}{1-2\nu} \quad \ldots (1.49)$$

Constrained modulus ($1/m_v$ in Soil Mechanics)

$$= \frac{(1-\nu)E}{(1+\nu)(1-2\nu)} \quad \ldots (1.50)$$

1.5.2 CROSS ANISOTROPIC MATERIAL

Stresses in terms of strains:

Cartesian coordinates:

$$\sigma_x = a\varepsilon_x + b\varepsilon_y + c\varepsilon_z \quad \ldots (1.51a)$$

$$\sigma_y = b\varepsilon_x + a\varepsilon_y + c\varepsilon_z \quad \ldots (1.51b)$$

$$\sigma_z = c\varepsilon_x + c\varepsilon_y + d\varepsilon_z \quad \ldots (1.51c)$$

$$\tau_{xz} = f\varepsilon_{xz} \quad \ldots (1.51d)$$

$$\tau_{zy} = f\varepsilon_{zy} \quad \ldots (1.51e)$$

$$\tau_{xy} = (a-b)\varepsilon_{xy} \quad \ldots (1.51f)$$

Cylindrical coordinates:

$$\sigma_r = a\varepsilon_r + b\varepsilon_\theta + c\varepsilon_z \quad \ldots (1.52a)$$

$$\sigma_\theta = b\varepsilon_r + a\varepsilon_\theta + c\varepsilon_z \quad \ldots (1.52b)$$

$$\sigma_z = c\varepsilon_r + c\varepsilon_\theta + d\varepsilon_z \quad \ldots (1.52c)$$

$$\tau_{rz} = f\varepsilon_{rz} \quad \ldots (1.52d)$$

$$\tau_{\theta z} = f\varepsilon_{\theta z} \quad \ldots (1.52e)$$

$$\tau_{r\theta} = (a-b)\varepsilon_{r\theta} \quad \ldots (1.52f)$$

where
$$a = \frac{E_h(1-\nu_{hv}\nu_{vh})}{(1+\nu_h)(1-\nu_h-2\nu_{hv}\nu_{vh})} \quad \ldots (1.53a)$$

$$b = \frac{E_h(\nu_h+\nu_{hv}\nu_{vh})}{(1+\nu_h)(1-\nu_h-2\nu_{hv}\nu_{vh})} \quad \ldots (1.53b)$$

$$c = \frac{E_h\,\nu_{vh}}{1-\nu_h-2\nu_{hv}\nu_{vh}} \quad \ldots (1.53c)$$

$$d = \frac{E_v(1-\nu_h)}{1-\nu_h-2\nu_{hv}\nu_{vh}} \quad \ldots (1.53d)$$

and E_h = modulus of elasticity in the horizontal direction

E_v = modulus of elasticity in the vertical direction

ν_h = Poisson's ratio for effect of horizontal stress on complementary horizontal strain

ν_{hv} = Poisson's ratio for effect of horizontal stress on vertical strain

ν_{vh} = Poisson's ratio for effect of vertical stress on horizontal strain.

It can be shown that

$$\frac{E_h}{E_v} = \frac{\nu_{hv}}{\nu_{vh}} \quad \ldots (1.54)$$

The elastic constant f is a shear modulus and cannot be expressed in terms of the Young's moduli or Poisson's ratios. f is often denoted as G_v.

Strains in terms of stresses:

$$\varepsilon_x = \frac{\sigma_x}{E_h} - \frac{\nu_h\sigma_y}{E_h} - \frac{\nu_{vh}\sigma_z}{E_v} \quad \ldots (1.55a)$$

$$\varepsilon_y = -\frac{\nu_h\sigma_x}{E_h} + \frac{\sigma_y}{E_h} - \frac{\nu_{vh}\sigma_z}{E_v} \quad \ldots (1.55b)$$

$$\varepsilon_z = -\frac{\nu_{hv}\sigma_x}{E_h} - \frac{\nu_{hv}\sigma_y}{E_h} + \frac{\sigma_z}{E_v} \quad \ldots (1.55c)$$

$$\varepsilon_{xz} = \frac{\tau_{xz}}{f} = \frac{\tau_{xz}}{G_v} \quad \ldots (1.55d)$$

$$\varepsilon_{yz} = \frac{\tau_{yz}}{f} = \frac{\tau_{yz}}{G_v} \quad \ldots (1.55e)$$

$$\varepsilon_{xy} = \frac{\tau_{xy}}{(a-b)} = \frac{(1+\nu_h)\tau_{xy}}{E_h} = \frac{\tau_{xy}}{G_h} \quad \ldots (1.55f)$$

In some works (e.g. Urena et al, 1966) ν_{hv} is denoted merely as ν_v and the use of ν_{vh} is avoided by using equation (1.54).

The fact that the strain energy must be positive imposes restrictions on the values of the elastic parameters. For a cross-anisotropic material with a vertical axis of elastic symmetry, Hearmon (1961) gives these restrictions as

$$a > 0 \quad \ldots (1.56a)$$

$$d > 0 \quad \ldots (1.56b)$$

$$f > 0 \quad \ldots (1.56c)$$

$$a^2 > b^2 \quad \ldots (1.56d)$$

$$(a+b)d > 2c^2 \quad \ldots (1.56e)$$

$$ad > c^2 \quad \ldots (1.56f)$$

In terms of the Poisson's ratios, these restrictions impose the limits

$$1 - \nu_h - 2\nu_{hv}\,\nu_{vh} > 0 \quad \ldots (1.57a)$$

$$1 - \nu_h > 0 \quad \ldots (1.57b)$$

$$1 + \nu_h > 0 \quad \ldots (1.57c)$$

1.6 Differential Equations of Isotropic Elasticity

1.6.1 EQUATIONS IN TERMS OF STRESSES

Cartesian Coordinates

$$\nabla^2\sigma_x + \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial x^2} = \frac{\nu}{1-\nu}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right) + 2\frac{\partial X}{\partial x} \quad \ldots (1.58a)$$

$$\nabla^2\sigma_y + \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial y^2} = \frac{\nu}{1-\nu}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right) + 2\frac{\partial Y}{\partial y} \quad \ldots (1.58b)$$

$$\nabla^2\sigma_z + \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial z^2} = \frac{\nu}{1-\nu}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right) + 2\frac{\partial Z}{\partial z} \quad \ldots (1.58c)$$

$$\nabla^2\tau_{yz} + \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial y\,\partial z} = \frac{\partial Y}{\partial z} + \frac{\partial Z}{\partial y} \quad \ldots (1.58d)$$

$$\nabla^2\tau_{zx} + \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial z\,\partial x} = \frac{\partial Z}{\partial x} + \frac{\partial X}{\partial z} \quad \ldots (1.58e)$$

$$\nabla^2\tau_{xy} + \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial x\partial y} = \frac{\partial X}{\partial y} + \frac{\partial Y}{\partial x} \qquad \ldots (1.58f)$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

$\Theta = \sigma_x + \sigma_y + \sigma_z$

For constant or zero body forces, the first three equations of (1.58) reduce to the Laplace equation

$$\nabla^2\Theta = 0 \qquad \ldots (1.59)$$

For the special case of *plane stress*, the equations are the equilibrium equations

$$\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xz}}{\partial z} - X = 0 \qquad \ldots (1.60a)$$

$$\frac{\partial\sigma_z}{\partial z} + \frac{\partial\tau_{xz}}{\partial x} - Z = 0 \qquad \ldots (1.60b)$$

and

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)(\sigma_x + \sigma_z) = (1+\nu)\left(\frac{\partial X}{\partial x} + \frac{\partial Z}{\partial z}\right) \qquad \ldots (1.60c)$$

For *plane strain*, the first two of the above three equations are again applicable. The third equation is

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)(\sigma_x + \sigma_z) = \frac{1}{1-\nu}\left(\frac{\partial X}{\partial x} + \frac{\partial Z}{\partial z}\right) \qquad \ldots (1.61)$$

If body forces are constant, the equations for plane stress and plane strain conditions are identical.

Cylindrical Coordinates

With zero or constant body forces:

$$\nabla^2\sigma_z + \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial z^2} = 0 \qquad \ldots (1.62a)$$

$$\nabla^2\sigma_r + \frac{2}{r^2}(\sigma_\theta - \sigma_r) - \frac{4}{r^2}\frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial r^2} = 0 \qquad \ldots (1.62b)$$

$$\nabla^2\sigma_\theta - \frac{2}{r^2}(\sigma_\theta - \sigma_r) + \frac{4}{r^2}\frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{1}{1+\nu}\left(\frac{1}{r}\frac{\partial\Theta}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Theta}{\partial\theta^2}\right) = 0 \qquad \ldots (1.62c)$$

$$\nabla^2\tau_{r\theta} - \frac{2}{r^2}\frac{\partial}{\partial\theta}(\sigma_\theta - \sigma_r) - \frac{4}{r^2}\tau_{r\theta} + \frac{1}{1+\nu}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Theta}{\partial\theta}\right) = 0 \qquad \ldots (1.62d)$$

$$\nabla^2\tau_{rz} - \frac{\tau_{rz}}{r^2} - \frac{2}{r^2}\frac{\partial\tau_{\theta z}}{\partial\theta} + \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial r\partial z} = 0 \qquad \ldots (1.62e)$$

$$\nabla^2\tau_{z\theta} - \frac{\tau_{z\theta}}{r^2} + \frac{2}{r}\frac{\partial\tau_{rz}}{\partial\theta} + \frac{1}{1+\nu}\frac{1}{r}\frac{\partial^2\Theta}{\partial\theta\partial z} = 0 \qquad \ldots (1.62f)$$

For the general case of non-constant body forces, the corresponding equations, in tensor form, are given by L'ure (1964).

1.6.2 EQUATIONS IN TERMS OF STRESS FUNCTION ϕ

Cartesian Coordinates

$$\nabla^4\phi = 0 = \nabla^2\nabla^2\phi \qquad \ldots (1.63)$$

where $\nabla^2 \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$

For plane stress or plane strain,

$$\frac{\partial^4\phi}{\partial x^4} + \frac{2\partial^4\phi}{\partial x^2\partial z^2} + \frac{\partial^4\phi}{\partial z^4} = 0 \qquad \ldots (1.64)$$

and the stresses are related to ϕ as follows:

$$\sigma_z = \frac{\partial^2\phi}{\partial z^2} \qquad \ldots (1.65a)$$

$$\sigma_x = \frac{\partial^2\phi}{\partial x^2} \qquad \ldots (1.65b)$$

$$\tau_{xz} = \frac{-\partial^2\phi}{\partial x\partial z} \qquad \ldots (1.65c)$$

Cylindrical Coordinates

For axial symmetry,

$$\nabla^4\phi = 0 = \nabla^2\nabla^2\phi \qquad \ldots (1.66)$$

where $\nabla^2 \equiv \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)$

The stresses are related to ϕ as

$$\sigma_r = \frac{\partial}{\partial z}\left(\nu\nabla^2\phi - \frac{\partial^2\phi}{\partial r^2}\right) \qquad \ldots (1.67a)$$

$$\sigma_\theta = \frac{\partial}{\partial z}\left(\nu\nabla^2\phi - \frac{1}{r}\frac{\partial\phi}{\partial r}\right) \qquad \ldots (1.67b)$$

$$\sigma_z = \frac{\partial}{\partial z}\left[(2-\nu)\nabla^2\phi - \frac{\partial^2\phi}{\partial z^2}\right] \qquad \ldots (1.67c)$$

$$\tau_{rz} = \frac{\partial}{\partial r}\left[(1-\nu)\,\nabla^2\phi - \frac{\partial^2\phi}{\partial z^2} \right] \qquad \ldots (1.67d)$$

1.6.3 EQUATIONS IN TERMS OF DISPLACEMENTS

Cartesian Coordinates

$$(\lambda+G)\,\frac{\partial\varepsilon_v}{\partial x} + G\nabla^2\rho_x - X = 0 \qquad \ldots (1.68a)$$

$$(\lambda+G)\,\frac{\partial\varepsilon_v}{\partial y} + G\nabla^2\rho_y - Y = 0 \qquad \ldots (1.68b)$$

$$(\lambda+G)\,\frac{\partial\varepsilon_v}{\partial z} + G\nabla^2\rho_z - Z = 0 \qquad \ldots (1.68c)$$

where λ, G are Lame's parameters

ε_v = volume strain

$= \varepsilon_x + \varepsilon_y + \varepsilon_z$

Cylindrical Coordinates

For axial symmetry,

$$(\lambda+2G)\left(\frac{\partial^2\rho_r}{\partial r^2} + \frac{1}{r}\frac{\partial\rho_r}{\partial r} - \frac{\rho_r}{r^2}\right) + G\,\frac{\partial^2\rho_r}{\partial z^2} + (\lambda+G)\,\frac{\partial^2\rho_z}{\partial r\partial z} = R \qquad \ldots (1.69a)$$

$$(\lambda+2G)\,\frac{\partial^2\rho_z}{\partial z^2} + \frac{G}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial\rho_z}{\partial r}\right) + \frac{\rho_r}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\rho_r}{\partial z}\right) + \frac{\lambda}{r}\frac{\partial}{\partial z}\left(\frac{\partial}{\partial r}(\rho_r\, r)\right) = Z \qquad \ldots (1.69b)$$

where R, Z are the body forces in the r and z directions.

On the Z axis $(r=0)$ the relevant equation is

$$(\lambda+2G)\,\frac{\partial^2\rho_z}{\partial z^2} + 2G\,\frac{\partial^2\rho_z}{\partial r^2} + 2(\lambda+G)\,\frac{\partial^2\rho_r}{\partial r\partial z} = Z \qquad \ldots (1.69c)$$

1.7 Convenient Methods of Considering Loaded Areas

1.7.1 SUPERPOSITION OF RECTANGLES

If the loaded area can be approximated by a rectangle, or by a series of rectangles, and appropriate influence factors for stress or displacement beneath the corner of a rectangle are available, the stress or displacement at any point may be determined by superposition of rectangles.

For the simple case of a single rectangle, the stress beneath an interior point O_I (see Fig.1.11) may simply be calculated as

$$\sigma = p(K_1 + K_2 + K_3 + K_4) \qquad \ldots (1.70)$$

where K_1, K_2, K_3, K_4 are the appropriate influence factors for areas 1,2,3 and 4, for the appropriate geometry of each rectangle.

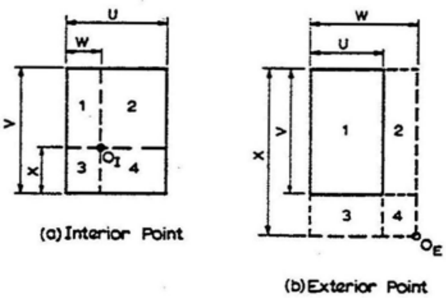


FIG.1.11

For an exterior point O_E (Fig.1.11)

$$\sigma = p(K_{1+2+3+4} - K_{2+4} - K_{3+4} + K_4) \qquad \ldots (1.71)$$

For computer calculations, the superposition principle can be stated as (see Fig.1.11):

$$\sigma = J(W,X) - J(W-U,X) - J(W,X-V) + J(W-U,X-V) \qquad \ldots (1.72)$$

where $J(m,n) = \text{sign}\,(mn)\,\sigma(|m|,|n|)$

$\text{sign}\,(mn) = 1$ when $mn > 0$

$\qquad\qquad -1$ $\quad mn < 0$

$\sigma(|m|,|n|)$ = stress beneath corner of a rectangle $m \times n$.

Displacements are calculated similarly.

For horizontal and shear stresses, care must be taken to take account of the sign of K for each rectangle.

1.7.2 NEWMARK'S METHOD

This method was developed by Newmark (1935) and is a graphical method involving the use of an influence chart, examples of which are shown in Figs.3.68-3.78. A drawing is made of the loaded area to a scale which is marked on the chart, and this drawing is so placed on the chart that the origin of the chart coincides with the point at or beneath which the stress or displacement is required.

The number of blocks covered by the loaded area is then counted and multiplied by an appropriate factor (shown on the chart) and the applied loading to give the required stress or displacement.

When the area is not uniformly loaded, the charts can still be used by considering the non-uniform loading to be made up of several sets of uniformly loaded areas.

In using the charts, parts of blocks may be estimated with sufficient accuracy for practical purposes. In general, the loaded area will be drawn on tracing paper and laid upon the chart.

Several "Newmark Charts" for stresses and displacements in a semi-infinite mass are given in Section 3.6. For a finite layer, Burmister (1956) has prepared charts, but the use of these charts is more complicated as they must be used in conjunction with a table of influence values (see Section 5.4.1).

1.7.3 SECTOR METHOD

This method has been described by Poulos (1967a). For any particular problem, a set of curves relating the stress or displacement influence factor beneath the apex of a uniformly loaded sector to the sector radius may be obtained by integration of the appropriate point load influence factors over a sector. Such sets of curves are referred to as "sector curves", and typical examples are given in Sections 3.6.2 and 5.4.2.

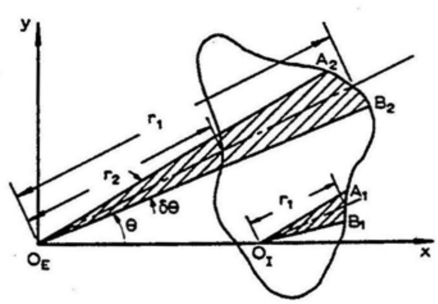


FIG.1.12 Division of loaded area into sectors.

In order to use the sector curves for calculation of the required influence factors for a loaded area of any shape, a scale diagram of the area is drawn, and a number of relatively small-angled sectors are drawn to cut the loaded area, each sector emanating from an apex which lies on the same vertical line as the point at which the influence factor is required. The point on the surface of the elastic solid through which this vertical line passes will be termed the "surface origin". In Fig.1.12, typical sectors $O_E\,A_2B_2$ and $O_I\,A_1B_1$ are shown in plan for surface origins O_E outside and O_I inside the loaded area.

For invariant stresses such as the bulk stress Θ, and for the stress σ_z and the displacement ρ_z, the influence of the typical sector $O_E\,A_2B_2$ for the external surface origin O_E is

$$(I_{s_1} - I_{s_2})\,.\delta\theta$$

where I_{s_1} is the sector influence value at the required depth for a mean sector radius r_1, and similarly for I_{s_2}.

The influence factor at O_E for the whole loaded area is

$$I = \sum(I_{s_1} - I_{s_2})\,.\delta\theta \qquad \ldots (1.73)$$

For the surface origin O_I within the loaded area, the influence factor for the whole loaded area is

$$I = \sum I_{s_1}.\delta\theta \qquad \ldots (1.74)$$

When evaluating the influence factor for a stress or displacement which is in a direction other than the z direction, the summation of sector influence factors must be vectorial. To calculate the horizontal stress σ_x in the x direction at O_E, both the tangential and radial stress influence factors for each sector are required, the influence value for the whole loaded area being given by

$$I_{\sigma_x} = \sum\{(\sigma_r I_{s_1} - \sigma_r I_{s_2}).\delta\theta.\cos^2\theta + (\sigma_\theta I_{s_1} - \sigma_\theta I_{s_2})\delta\theta.\sin^2\theta\} \qquad \ldots (1.75)$$

where $I\sigma_x$ is the influence factor for due to the loaded area,

$\sigma_r I_{s_1}$, $\sigma_r I_{s_2}$ are the sector influence factors for the radial stress, for sector radii of r_1 and r_2 respectively.

$\sigma_\theta I_{s_1}$, $\sigma_\theta I_{s_2}$ are the sector influence factors for the tangential stress, for sector radii of r_1, r_2.

The influence factors for horizontal stress σ_y in the y direction may be obtained similarly,

$$I_{\sigma_y} = \sum\{(\sigma_r I_{s_1} - \sigma_r I_{s_2}).\delta\theta.\sin^2\theta + (\sigma_\theta I_{s_1} - \sigma_\theta I_{s_2})\delta\theta.\cos^2\theta\} \qquad \ldots (1.76)$$

In the same manner, it may be shown that the influence factors for the three shear stresses in the Cartesian coordinate system are as follows:

$$I_{\tau_{yx}} = I_{\tau_{xy}} = \sum\{(\sigma_r I_{s_1} - \sigma_r I_{s_2} - \sigma_\theta I_{s_1} + \sigma_\theta I_{s_2}).\delta\theta.\sin\theta.\cos\theta\} \qquad \ldots (1.77a)$$

$$I_{\tau_{zx}} = I_{\tau_{xz}} = \sum\{({}_{\tau_{rz}}I_{s_1} - {}_{\tau_{rz}}I_{s_2}).\cos\theta.\delta\theta \qquad \text{... (1.77b)}$$

$$I_{\tau_{zy}} = I_{\tau_{yz}} = \sum({}_{\tau_{rz}}I_{s_1} - {}_{\tau_{rz}}I_{s_2}) . \sin\theta.\delta\theta \qquad \text{... (1.77c)}$$

where ${}_{\tau_{rz}}I_{s_1}$, ${}_{\tau_{rz}}I_{s_2}$ are the sector influence factors for radial shear stress, for sector radii of r_1, r_2.

For the displacements ρ_z and ρ_y in the x and y directions for the surface origin O_E, the influence factors are

$$I_{\rho_x} = \sum({}_{\rho_r}I_{s_1} - {}_{\rho_r}I_{s_2}).\delta\theta.\cos\theta \qquad \text{... (1.78a)}$$

$$I_{\rho_y} = \sum({}_{\rho_r}I_{s_1} - {}_{\rho_r}I_{s_2}).\delta\theta.\sin\theta \qquad \text{... (1.78b)}$$

where $I\rho_z$ and $I\rho_y$ are influence factors for the displacements in the x and y directions due to the whole loaded area,

${}_{\rho_r}I_{s_1}$, ${}_{\rho_r}I_{s_2}$ are the sector influence factors for radial displacement, for sector radii of r_1, r_2.

Having found the influence factor I for the whole area, the stresses and displacements for O_E and O due to uniform loading are given in all cases by

$$\sigma = \frac{p}{2\pi} . I \qquad \text{... (1.79)}$$

$$\text{and, } \rho = \frac{pX}{2\pi E} . I \qquad \text{... (1.80)}$$

The accuracy of the influence factors calculated by the sector method increases with the number of sectors used, and the more irregular the shape of the loaded area, the greater is the desirable number of sectors. For the calculation of influence factors for stresses and displacements which are neither invariant nor in the z direction, the sector angle $\delta\theta$ must be small in order to preserve the accuracy of both the magnitude and direction of the calculated influence factor.

Stresses and Displacements Beneath the Centre of a Uniformly Loaded Circle

Beneath the centre of a uniformly loaded circle, the expressions for stresses and displacements reduce to very simple forms.

For σ_z, ρ_z and the invariant stresses,

$$I = 2\pi I_{sa} \qquad \text{... (1.81)}$$

where I_{sa} is the sector influence factor for a sector radius equal to the radius of the circle.

For the horizontal stresses σ_x and σ_y,

$$I_{\sigma_x} = I_{\sigma_y} = \pi({}_{\sigma_r}I_{sa} + {}_{\sigma_\theta}I_{sa}) \qquad \text{... (1.82)}$$

where ${}_{\sigma_r}I_{sa}$, ${}_{\sigma_\theta}I_{sa}$ are sector influence factors for σ_θ and σ_r for a sector radius equal to the radius of the circle.

The influence factors for all shear stresses and for the horizontal displacements ρ_x and ρ_y are zero in this case.

1.8 Superposition of Solutions for Various Loadings

Solutions are usually only available for relatively simple types of loading. If the loading pattern is complicated, superposition of solutions for simple loadings may frequently be employed. Examples of the decomposition of complicated loadings into simpler loads have been given by Giroud (1968) in terms of four simple loading types, uniform vertical load, linearly varying vertical load, uniform horizontal load and linearly varying horizontal load. Fig.1.13 shows the examples given by Giroud, the signs in each case referring to the signs of the four simple loading types.

The foregoing is exact for generalized linear loading. The approach can be extended approximately to completely general non-linear loading by division of this loading into a series of general linear loadings.

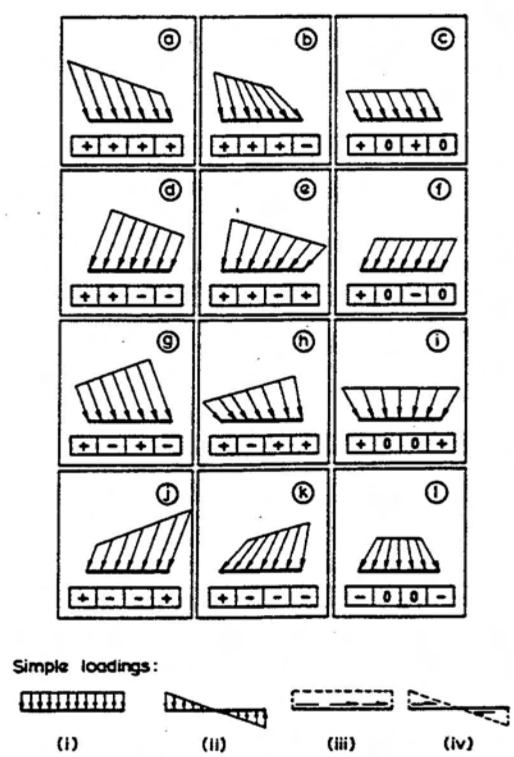


FIG.1.13 Decomposition of loadings (Giroud,1968).

1.9 Equations of Simple Bending Theory

1.9.1 HORIZONTAL BEAM

$$EI \frac{d^2\rho}{dx^2} = -M \qquad \text{... (1.83)}$$

where EI = flexural rigidity
ρ = deflection (positive downwards)
x = distance along beam
M = bending moment ("sagging" moments positive, "hogging" moments negative)

$$\text{Slope} \quad \theta = \frac{d\rho}{dx} \qquad \text{... (1.84)}$$

$$\text{Shear force} \quad V = -\frac{dM}{dx} \qquad \text{... (1.85a)}$$

$$= EI \frac{d^3\rho}{dx^3} \quad \text{(for constant } EI\text{)} \qquad \text{... (1.85b)}$$

$$\text{Load per unit length} \quad p = -\frac{d^2M}{dx^2} \qquad \text{... (1.86a)}$$

$$= EI \frac{d^4\rho}{dx^4} \quad \text{(for constant } EI\text{)} \qquad \text{... (1.86b)}$$

1.9.2 CIRCULAR PLATE

For axially-symmetrical loading,

$$\frac{d^4\rho}{dr^4} + \frac{2}{r}\frac{d^3\rho}{dr^3} - \frac{1}{r^2}\frac{d^2\rho}{dr^2} + \frac{1}{r^3}\frac{d\rho}{dr} = \frac{q}{D} \qquad \text{... (1.87)}$$

where ρ = deflection (positive downwards)
r = radial distance from centre load intensity
q = load intensity
D = flexural rigidity of plate

$$= \frac{Et^3}{12(1-\nu^2)}$$

E = Young's modulus of plate
ν = Poisson's ratio of plate
t = plate thickness

The bending moments M_r and M_θ per unit length in the radial and tangential directions are given by

$$M_r = -D\left(\frac{d^2\phi}{dr^2} + \frac{\nu}{r}\frac{d\rho}{dr}\right) \qquad \text{... (1.88a)}$$

$$\text{and} \quad M_\theta = -D\left(\frac{d\rho}{dr} + \nu\frac{d^2\rho}{dr^2}\right) \qquad \text{... (1.88b)}$$

1.9.3 RECTANGULAR PLATE

$$\frac{\partial^4\rho}{\partial x^4} + 2\frac{\partial^4\rho}{\partial x^2\partial y^2} + \frac{\partial^4\rho}{\partial y^4} = \frac{q}{D} \qquad \text{... (1.89a)}$$

$$\text{i.e.,} \quad \nabla^4\rho = q/D \qquad \text{... (1.89b)}$$

where q = intensity of load
D = flexural rigidity of plate as before.

The moments per unit length, M_x and M_y, in the x and y directions are

$$M_x = -D\left(\frac{\partial^2\rho}{\partial x^2} + \nu\frac{\partial^2\rho}{\partial y^2}\right) \qquad \text{... (1.90a)}$$

$$M_y = -D\left(\frac{\partial^2\rho}{\partial y^2} + \nu\frac{\partial^2\rho}{\partial x^2}\right) \qquad \text{... (1.90b)}$$