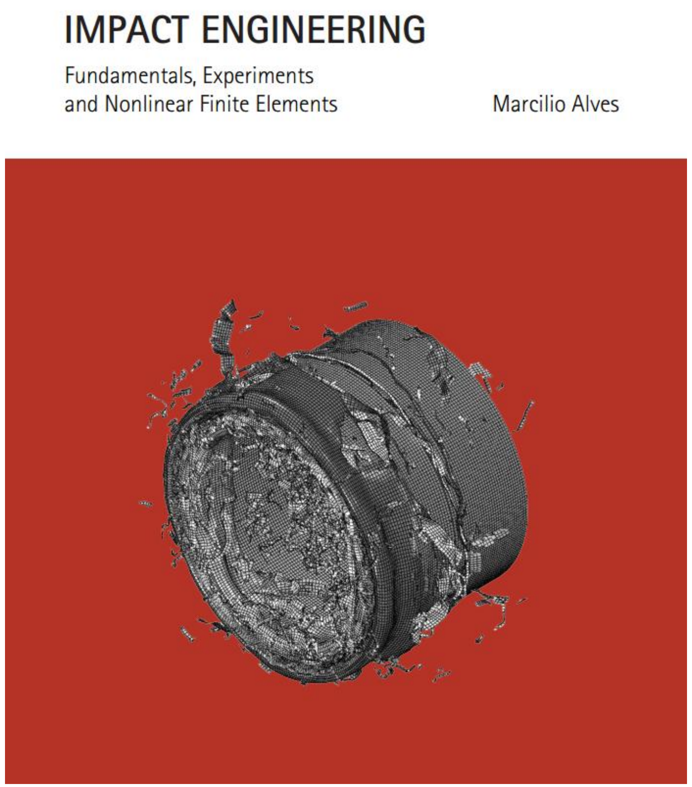
Curso de Vibrações:



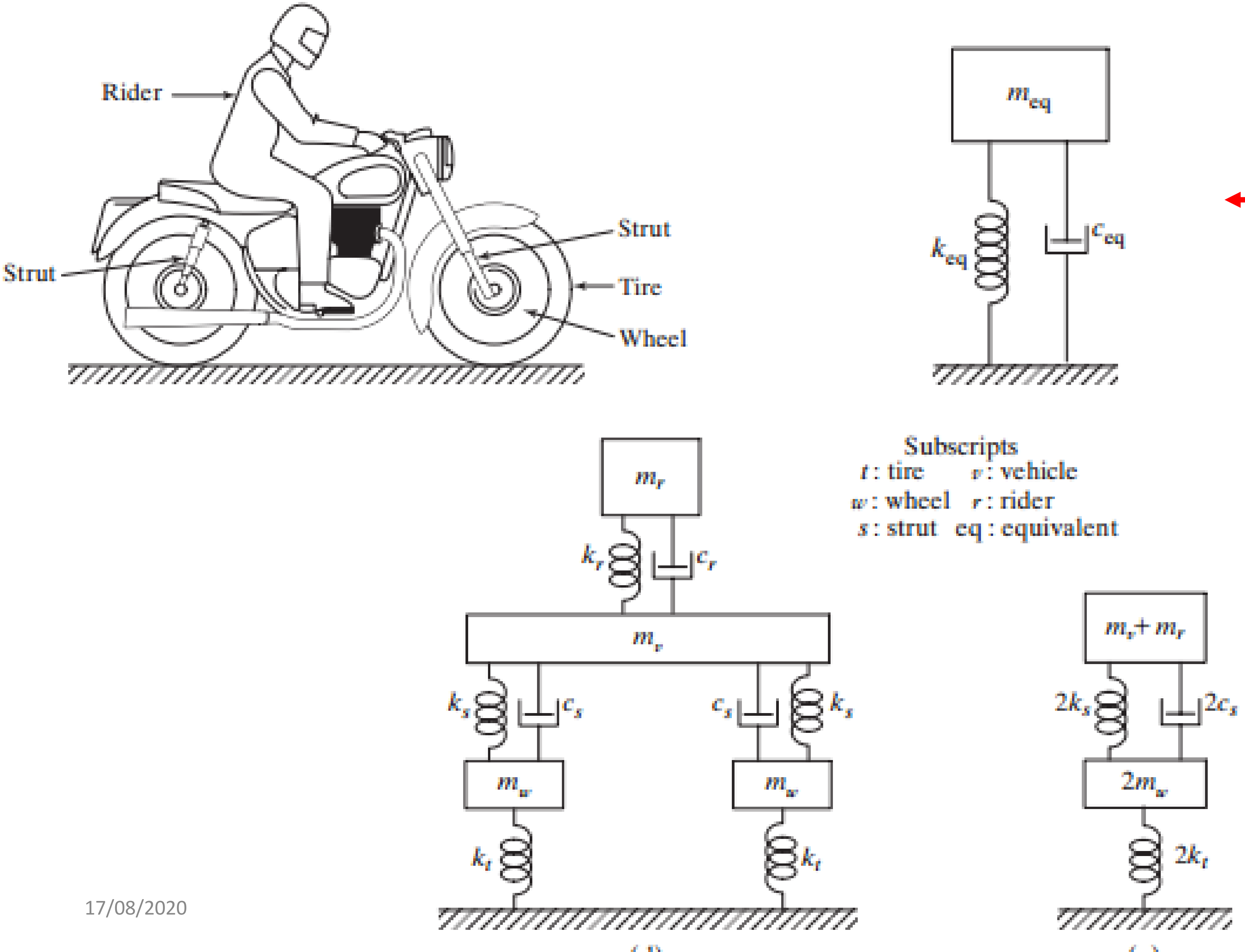
Vibração Livre:
1 Grau de Liberdade

Visão geral

- Vibrações de sistemas de 1 GL
- Vibrações de sistemas com 2 GL
- Vibrações de sistemas com n GL
- Vibrações de sistemas contínuos
- Ilustrações experimentais

- Livres
- Forçadas harmônicas
- Forçadas gerais

- Soluções analíticas [integração e Laplace]
- Soluções numéricas
- Soluções numéricas por Elementos Finitos



Elementos principais usados para modelar um sistema dinâmico

Sistemas de 1 GL: comentários sobre idealização [ou modelagem]

FIGURE 2.3 The space needle (structure).

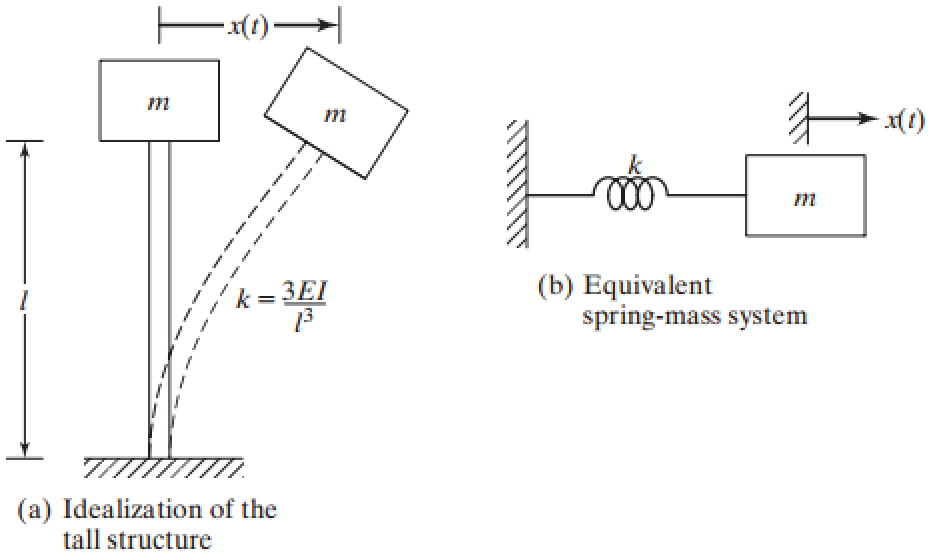
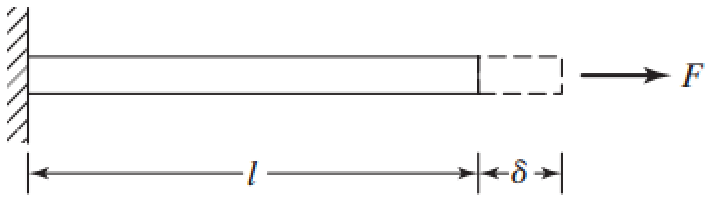


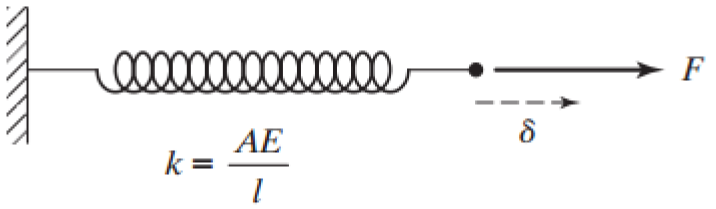
FIGURE 2.4 Modeling of tall structure as spring-mass system.

Spring Constant of a Rod

Find the equivalent spring constant of a uniform rod of length l, cross-sectional area A, and Young's modulus E subjected to an axial tensile (or compressive) force F as shown in Fig. 1.24(a).



(a)



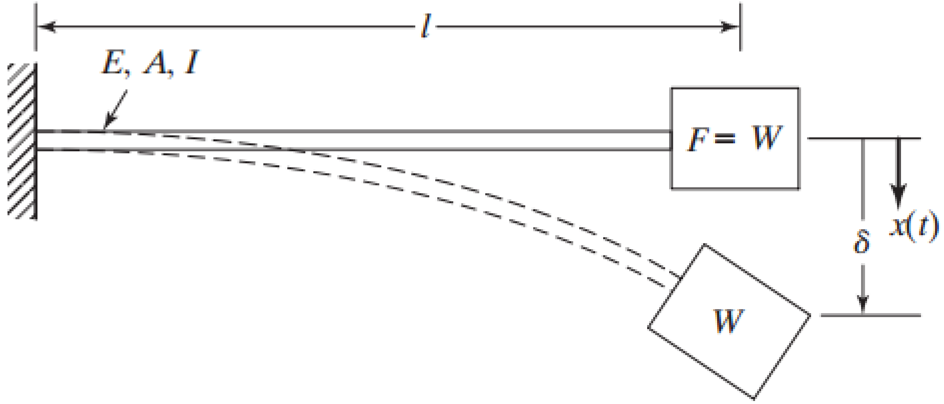
(b)

Solution: The elongation (or shortening) δ of the rod under the axial tensile (or compressive) force F can be expressed as

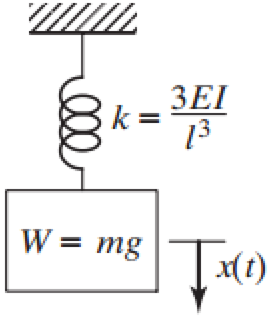
$$\delta = \frac{\delta}{l} l = \varepsilon l = \frac{\sigma}{E} l = \frac{Fl}{AE} \tag{E.1}$$

where $\varepsilon = \frac{\text{change in length}}{\text{original length}} = \frac{\delta}{l}$ is the strain and $\sigma = \frac{\text{force}}{\text{area}} = \frac{F}{A}$ is the stress induced in the rod. Using the definition of the spring constant k, we obtain from Eq. (E.1):

$$k = \frac{\text{force applied}}{\text{resulting deflection}} = \frac{F}{\delta} = \frac{AE}{l} \tag{E.2}$$



(a) Cantilever with end force



(b) Equivalent spring

Spring Constant of a Cantilever Beam

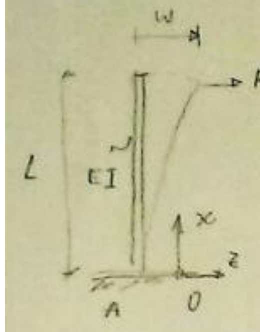
Find the equivalent spring constant of a cantilever beam subjected to a concentrated load F at its end as shown in Fig. 1.25(a).

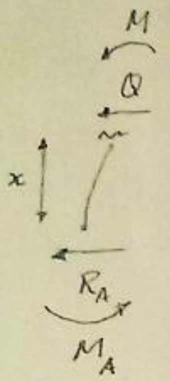
Solution: We assume, for simplicity, that the self weight (or mass) of the beam is negligible and the concentrated load F is due to the weight of a point mass ($W = mg$). From strength of materials [1.26], we know that the end deflection of the beam due to a concentrated load $F = W$ is given by

$$\delta = \frac{Wl^3}{3EI} \tag{E.1}$$

where E is the Young's modulus and I is the moment of inertia of the cross section of the beam about the bending or z-axis (i.e., axis perpendicular to the page). Hence the spring constant of the beam is (Fig. 1.25(b)):

$$k = \frac{W}{\delta} = \frac{3EI}{l^3} \tag{E.2}$$





$\Sigma F_z = 0$

$R_A - F = 0$

$R_A = F$

$\Sigma M_A = 0$

$-M_A + FL = 0$

$M_A = FL$

$$\begin{cases} -R_A - Q = 0 \\ Q = -R_A = -F \end{cases} \qquad \begin{cases} -M_A - Qx - M = 0 \\ M = -FL - -Fx \end{cases}$$

$M = F(x - L)$

$$\varepsilon = \frac{dx' - dx}{dx} = \frac{(R+z)\,d\theta - R\,d\theta}{R\,d\theta} = \frac{z}{R} = zk = -z\frac{\partial^2 w}{\partial x^2}$$

$dM = \sigma \, dA \, z$

$M = \int_A E \varepsilon z \, dA$

$M = E\int_A zkz \, dA = Ek\int z^2 dA = EIk$

$$\frac{M}{EI} = -w''$$

$$\frac{F}{EI}(x - L) = -w''$$

$$\frac{F}{EI}\left(\frac{x^2}{2} - Lx\right) = -w' \qquad C_1 = 0 \quad (w'(0) = 0)$$

$$w'(x = L) = \frac{FL^2}{2EI}$$

$$\frac{F}{EI}\left(\frac{x^3}{6} - \frac{Lx^2}{2}\right) = -w \qquad C_2 = 0 \quad (w(0) = 0)$$

$$x = L \qquad w = \frac{FL^3}{3EI}$$

$F = kx$

$$k = \frac{3EI}{L^3}$$

In many practical applications, several linear springs are used in combination. These springs can be combined into a single equivalent spring as indicated below.

Case 1: Springs in Parallel. To derive an expression for the equivalent spring constant of springs connected in parallel, consider the two springs shown in Fig. 1.27(a). When a load W is applied, the system undergoes a static deflection δ_{st} as shown in Fig. 1.27(b). Then the free-body diagram, shown in Fig. 1.27(c), gives the equilibrium equation

$$W = k_1\delta_{st} + k_2\delta_{st} \tag{1.8}$$

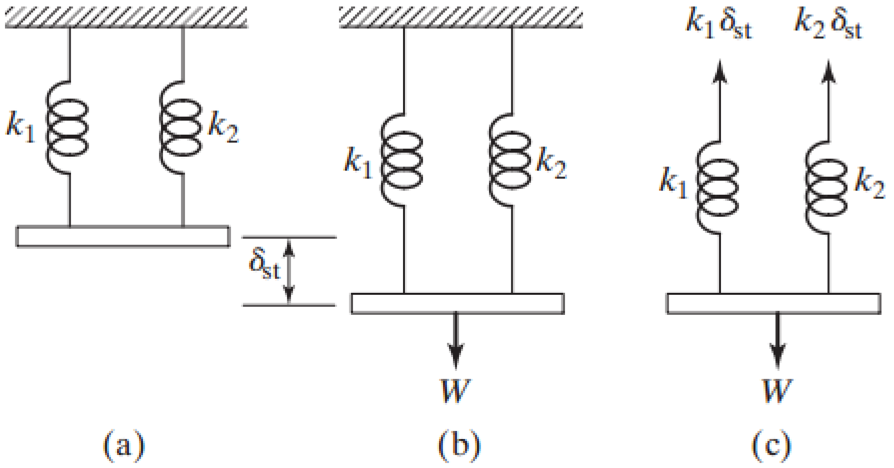


FIGURE 1.27 Springs in parallel.

If k_{eq} denotes the equivalent spring constant of the combination of the two springs, then for the same static deflection δ_{st}, we have

$$W = k_{eq}\delta_{st} \tag{1.9}$$

Equations (1.8) and (1.9) give

$$k_{eq} = k_1 + k_2 \tag{1.10}$$

In general, if we have n springs with spring constants $k_1, k_2, \ldots, k_n$ in parallel, then the equivalent spring constant k_{eq} can be obtained:

$$k_{eq} = k_1 + k_2 + \cdots + k_n \tag{1.11}$$

Case 2: Springs in Series. Next we derive an expression for the equivalent spring constant of springs connected in series by considering the two springs shown in Fig. 1.28(a). Under the action of a load W, springs 1 and 2 undergo elongations δ_1 and δ_2, respectively, as shown in Fig. 1.28(b). The total elongation (or static deflection) of the system, δ_{st}, is given by

$$\delta_{st} = \delta_1 + \delta_2 \tag{1.12}$$

Since both springs are subjected to the same force W, we have the equilibrium shown in Fig. 1.28(c):

$$\begin{aligned} W &= k_1\delta_1 \\ W &= k_2\delta_2 \end{aligned} \tag{1.13}$$

If k_{eq} denotes the equivalent spring constant, then for the same static deflection,

$$W = k_{eq}\delta_{st} \tag{1.14}$$

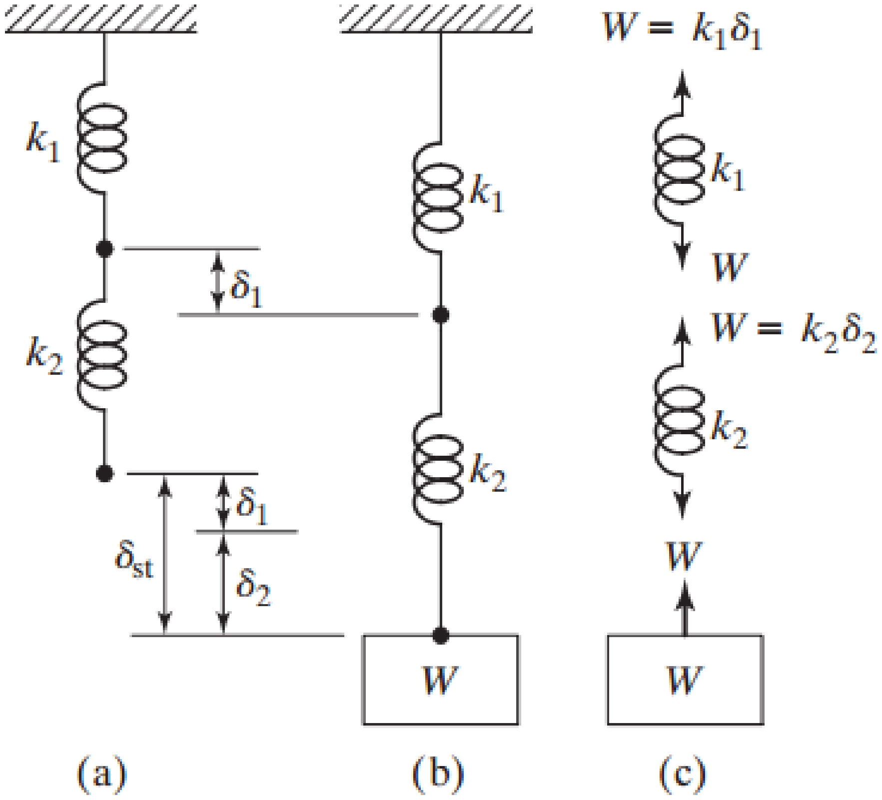


FIGURE 1.28 Springs in series.

Equations (1.13) and (1.14) give

$$k_1\delta_1 = k_2\delta_2 = k_{eq}\delta_{st}$$

or

$$\delta_1 = \frac{k_{eq}\delta_{st}}{k_1} \quad \text{and} \quad \delta_2 = \frac{k_{eq}\delta_{st}}{k_2} \tag{1.15}$$

Substituting these values of δ_1 and δ_2 into Eq. (1.12), we obtain

$$\frac{k_{eq}\delta_{st}}{k_1} + \frac{k_{eq}\delta_{st}}{k_2} = \delta_{st}$$

—that is,

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} \tag{1.16}$$

Equation (1.16) can be generalized to the case of n springs in series:

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \cdots + \frac{1}{k_n} \tag{1.17}$$

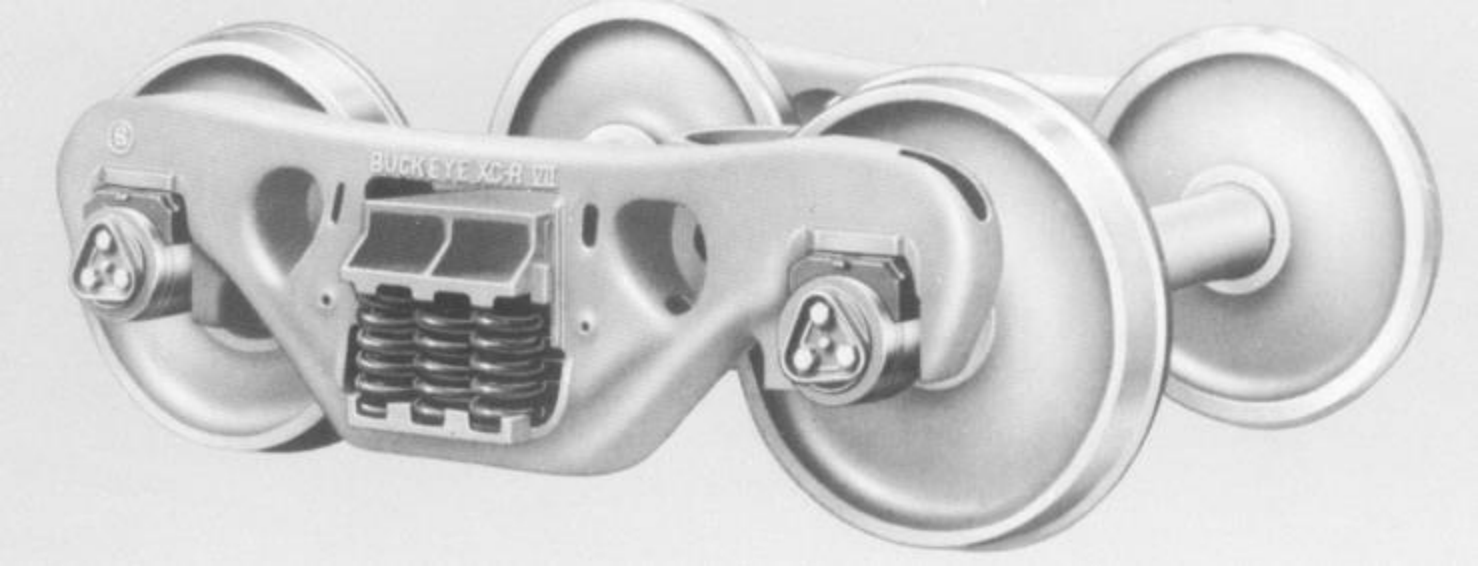


Figure 1.29 shows the suspension system of a freight truck with a parallel-spring arrangement. Find the equivalent spring constant of the suspension if each of the three helical springs is made of steel with a shear modulus $G = 80 \times 10^9$ N/m^2 and has five effective turns, mean coil diameter $D = 20$ cm, and wire diameter $d = 2$ cm.

Solution: The stiffness of each helical spring is given by

$$k = \frac{d^4 G}{8 D^3 n} = \frac{(0.02)^4 (80 \times 10^9)}{8(0.2)^3 (5)} = 40{,}000.0 \text{ N/m}$$

(See inside front cover for the formula.)

Since the three springs are identical and parallel, the equivalent spring constant of the suspension system is given by

$$k_{eq} = 3k = 3(40{,}000.0) = 120{,}000.0 \text{ N/m}$$

Torsional Spring Constant of a Propeller Shaft

Determine the torsional spring constant of the steel propeller shaft shown in Fig. 1.30.

Solution: We need to consider the segments 12 and 23 of the shaft as springs in combination. From Fig. 1.30 the torque induced at any cross section of the shaft (such as *AA* or *BB*) can be seen to be equal to the torque applied at the propeller, *T*. Hence the elasticities (springs) corresponding to the two segments 12 and 23 are to be considered as series springs. The spring constants of segments 12 and 23 of the shaft ($k_{t_{12}}$ and $k_{t_{23}}$) are given by

$$k_{t_{12}} = \frac{GJ_{12}}{l_{12}} = \frac{G\pi(D_{12}^4 - d_{12}^4)}{32 l_{12}} = \frac{(80 \times 10^9)\pi(0.3^4 - 0.2^4)}{32(2)}$$

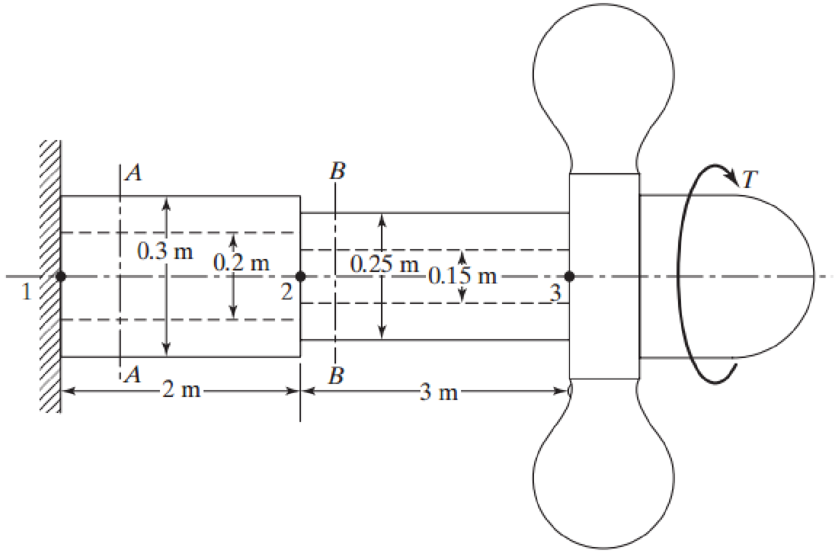
$$= 25.5255 \times 10^6 \text{ N-m/rad}$$

$$k_{t_{23}} = \frac{GJ_{23}}{l_{23}} = \frac{G\pi(D_{23}^4 - d_{23}^4)}{32 l_{23}} = \frac{(80 \times 10^9)\pi(0.25^4 - 0.15^4)}{32(3)}$$

$$= 8.9012 \times 10^6 \text{ N-m/rad}$$

Since the springs are in series, Eq. (1.16) gives

$$k_{t_{eq}} = \frac{k_{t_{12}} k_{t_{23}}}{k_{t_{12}} + k_{t_{23}}} = \frac{(25.5255 \times 10^6)(8.9012 \times 10^6)}{(25.5255 \times 10^6 + 8.9012 \times 10^6)} = 6.5997 \times 10^6 \text{ N-m/rad}$$



Equivalent k of Hoisting Drum

A hoisting drum, carrying a steel wire rope, is mounted at the end of a cantilever beam as shown in Fig. 1.31 (a). Determine the equivalent spring constant of the system when the suspended length of the wire rope is l. Assume that the net cross-sectional diameter of the wire rope is d and the Young's modulus of the beam and the wire rope is E.

Solution: The spring constant of the cantilever beam is given by

$$k_b = \frac{3EI}{b^3} = \frac{3E}{b^3}\left(\frac{1}{12}at^3\right) = \frac{Eat^3}{4b^3} \tag{E.1}$$

The stiffness of the wire rope subjected to axial loading is

$$k_r = \frac{AE}{l} = \frac{\pi d^2 E}{4l} \tag{E.2}$$

Since both the wire rope and the cantilever beam experience the same load W, as shown in Fig. 1.31 (b), they can be modeled as springs in series, as shown in Fig. 1.31 (c). The equivalent spring constant k_{eq} is given by

$$\frac{1}{k_{eq}} = \frac{1}{k_b} + \frac{1}{k_r} = \frac{4b^3}{Eat^3} + \frac{4l}{\pi d^2 E}$$

or

$$k_{eq} = \frac{E}{4}\left(\frac{\pi a t^3 d^2}{\pi d^2 b^3 + l a t^3}\right) \tag{E.3}$$

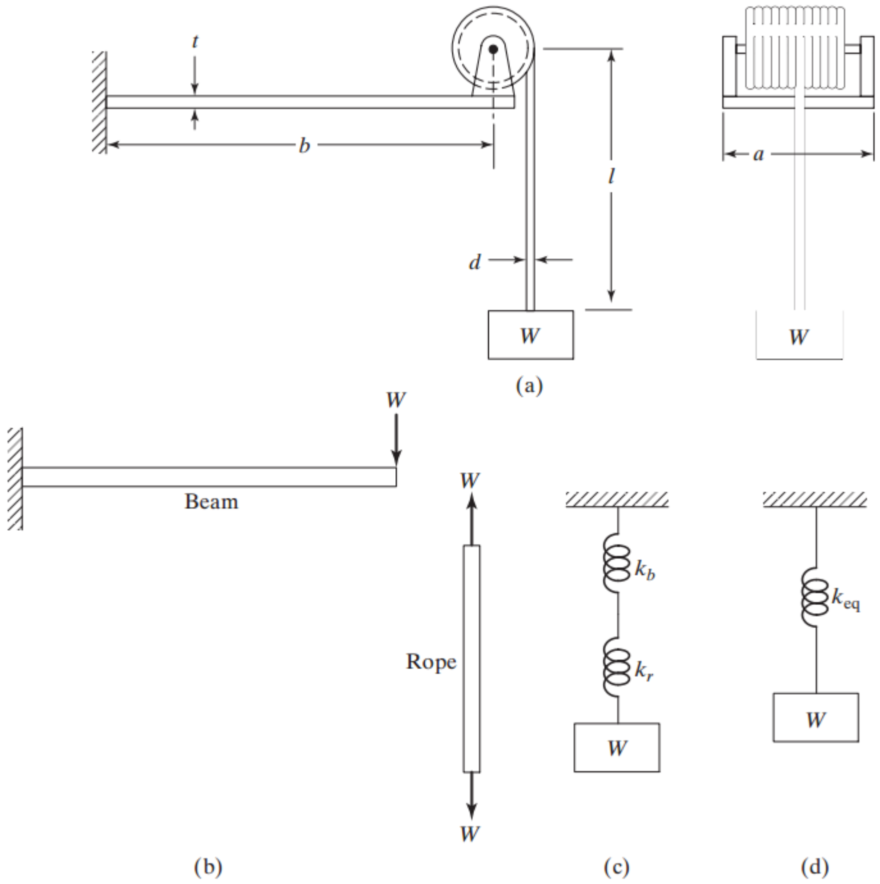
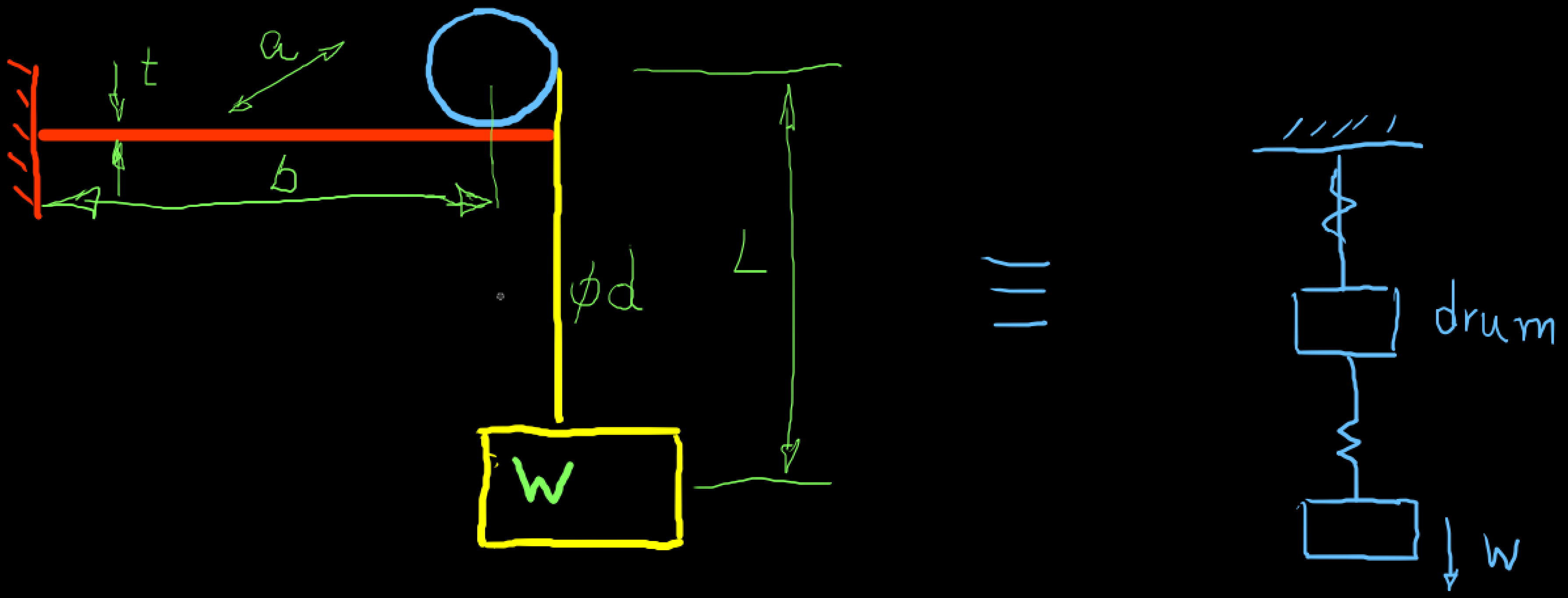


FIGURE 1.31 Hoisting drum.



$$\frac{1}{K_{eq}} = \frac{1}{K_b} + \frac{1}{K_d}$$

$$K_b = \frac{3EI}{b^3}$$

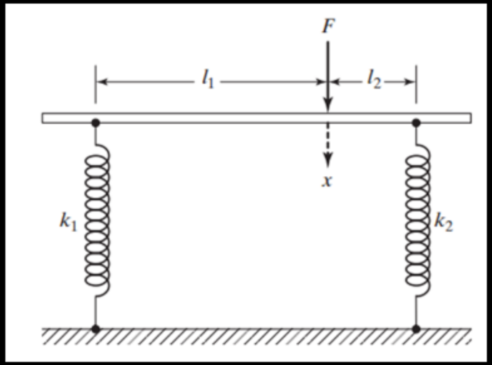
$$K_d = \frac{EA}{L}$$

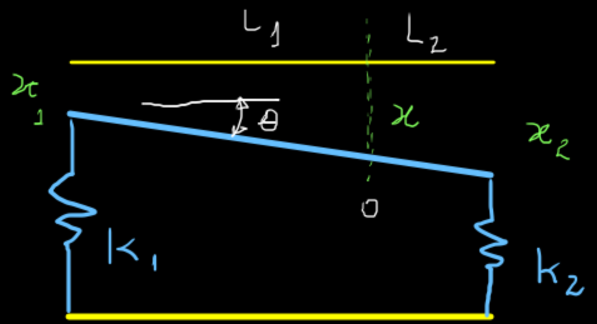
$$I = \frac{a t^3}{12}$$

$$A = \frac{\pi d^2}{4}$$

$$K_{eq} = \frac{E}{4} \cdot \frac{\pi a t^3 d^2}{\pi d^2 b^3 + L a t^3}$$

Exercícios: Keq?





$$tg\,\theta = \frac{x - x_1}{L_1}$$

$$x = \frac{l_2}{l} x_1 + \frac{l_1}{l} x_2$$

$$\left.\begin{aligned} \Sigma F = 0 \quad & F_1 + F_2 = 0 \\ \Sigma M = 0 \quad & F_1 l_1 - F_2 l_2 = 0 \end{aligned}\right\} \quad F_1 = F\frac{l_2}{l} \qquad F_2 = F\frac{l_1}{l}$$

$$x_1 = \frac{F_1}{K_1} = \frac{F}{K_1}\frac{l_2}{l} \qquad x_2 = \frac{F_2}{K_2} = \frac{F}{K_2}\frac{l_1}{l}$$

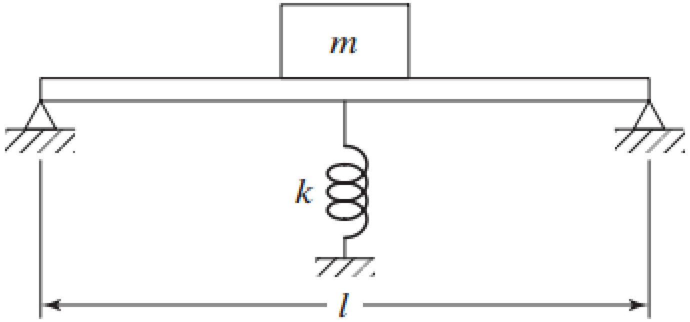
$$x = \frac{F}{K_{eq}} = \frac{l_2}{l}\frac{F}{k_1}\frac{l_2}{l} + \frac{l_1}{l}\frac{F}{K_2}\frac{l_1}{l}$$

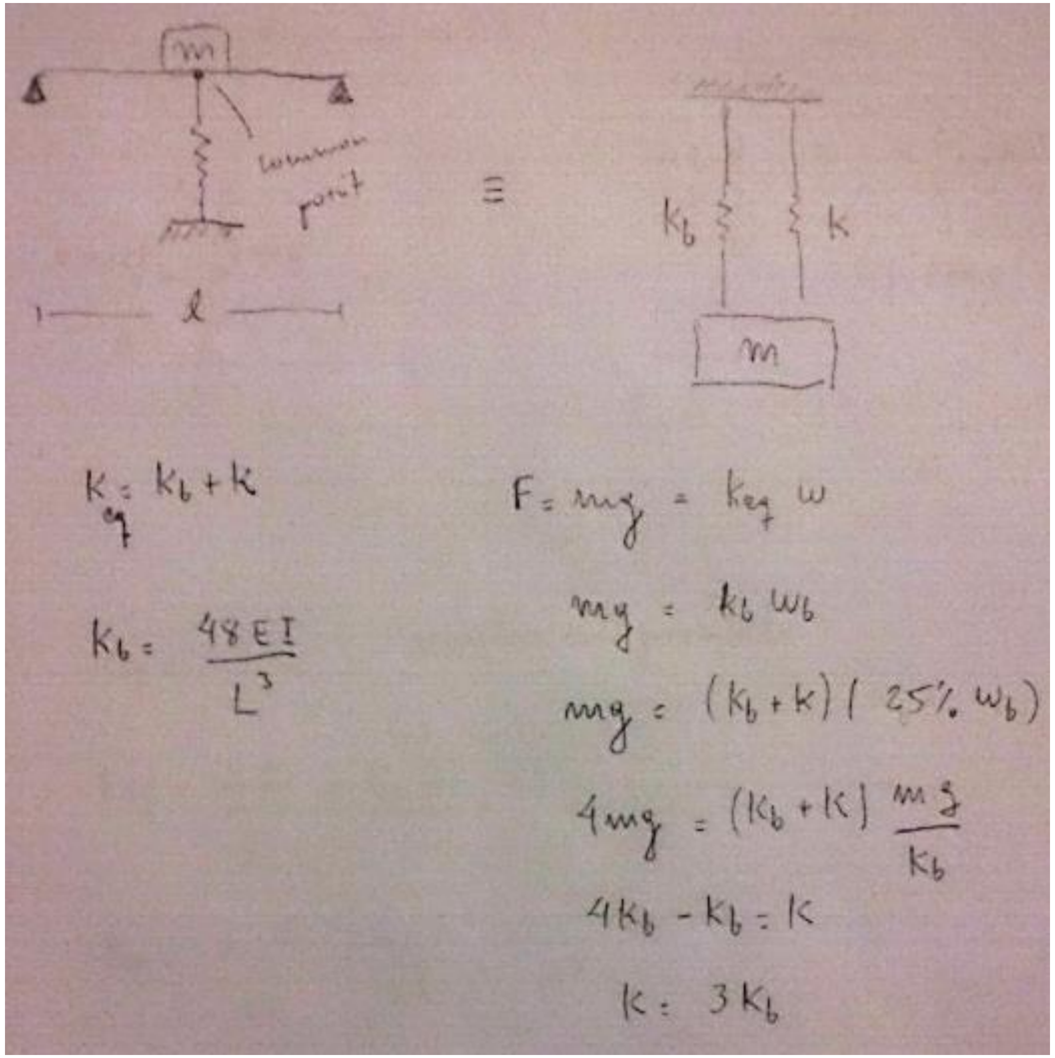
$$K_{eq} = \frac{l^2 K_1 K_2}{l_1^2 K_1 + l_2^2 K_2}$$

1.11 A machine of mass $m = 500$ kg is mounted on a simply supported steel beam of length $l = 2$ m having a rectangular cross section (depth = 0.1 m, width = 1.2 m) and Young's modulus $E = 2.06 \times 10^{11}$ N/m^2. To reduce the vertical deflection of the beam, a spring of stiffness k is attached at mid-span, as shown in Fig. 1.71. Determine the value of k needed to reduce the deflection of the beam by

a. 25 percent of its original value.
b. 50 percent of its original value.
c. 75 percent of its original value.

Assume that the mass of the beam is negligible.





1.18 The static equilibrium position of a massless rigid bar, hinged at point O and connected with springs k_1 and k_2, is shown in Fig. 1.74. Assuming that the displacement (x) resulting from the application of a force F at point A is small, find the equivalent spring constant of the system, k_e, that relates the applied force F to the displacement x as $F = k_e x$.

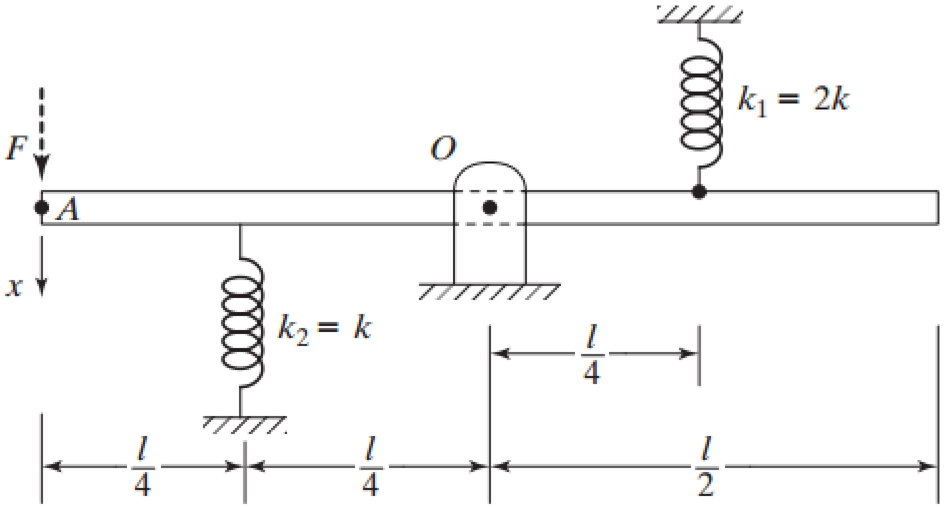


FIGURE 1.74 Rigid bar connected by springs.

K_eq=3k/4

1.20 Figure 1.76 shows a uniform rigid bar of mass m that is pivoted at point O and connected by springs of stiffnesses k_1 and k_2. Considering a small angular displacement θ of the rigid bar about the point O, determine the equivalent spring constant associated with the restoring moment.

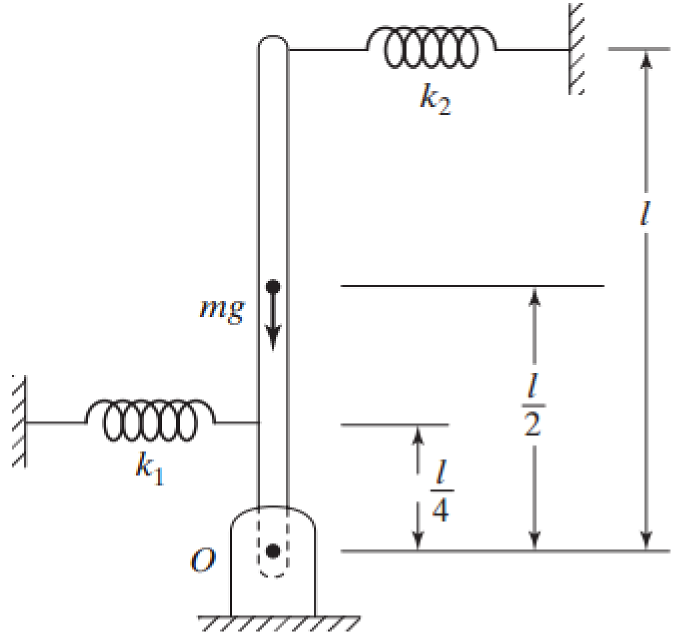
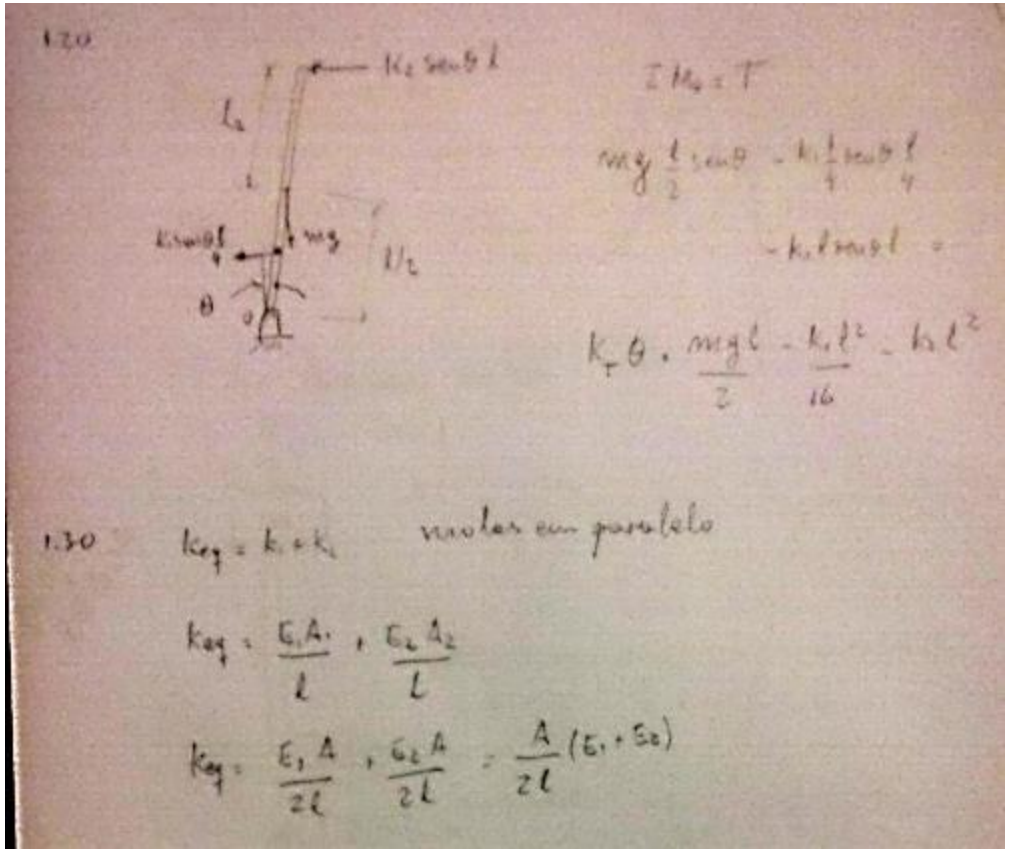
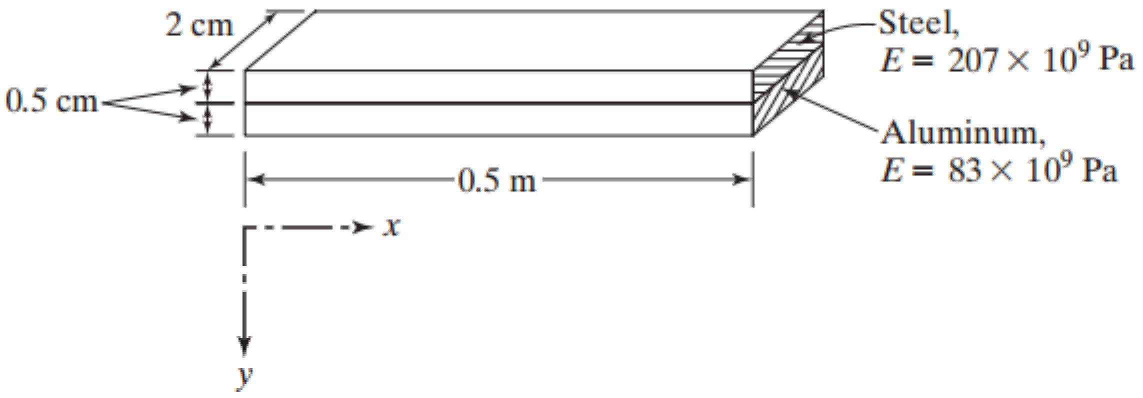


FIGURE 1.76 Rigid bar connected by springs.



1.39 Find the spring constant of the bimetallic bar shown in Fig. 1.89 in axial motion.



Movimento harmônico

hence the mass m of the spring-mass system are displaced from their middle positions by an amount x (in time t) given by

$$x = A \sin \theta = A \sin \omega t \tag{1.30}$$

This motion is shown by the sinusoidal curve in Fig. 1.46. The velocity of the mass m at time t is given by

$$\frac{dx}{dt} = \omega A \cos \omega t \tag{1.31}$$

and the acceleration by

$$\frac{d^2x}{dt^2} = -\omega^2 A \sin \omega t = -\omega^2 x \tag{1.32}$$

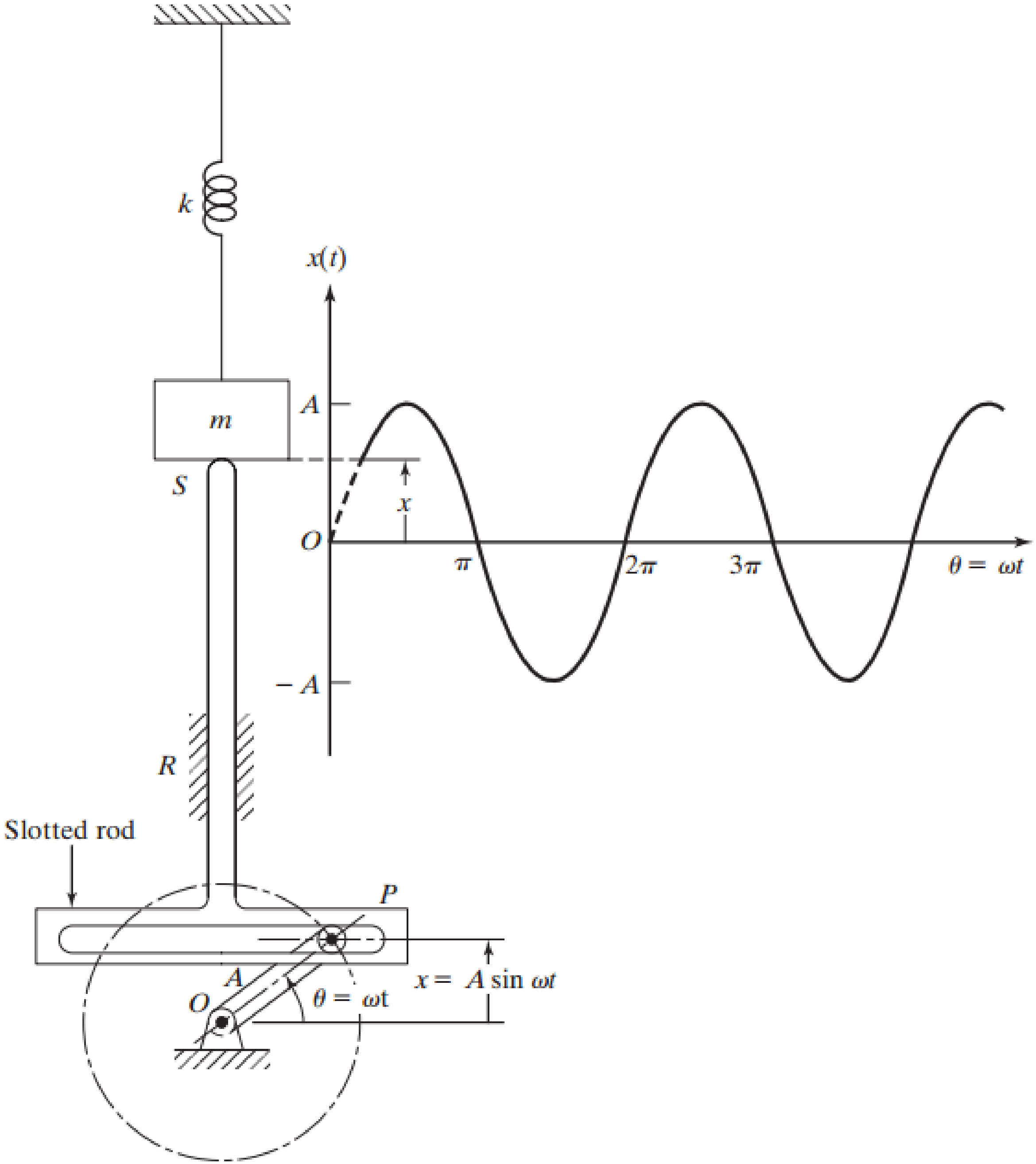


FIGURE 1.46 Scotch yoke mechanism.

Harmonic motion can be represented conveniently by means of a vector $\overrightarrow{OP}$ of magnitude A rotating at a constant angular velocity ω. In Fig. 1.47, the projection of the tip of the vector $\vec{X} = \overrightarrow{OP}$ on the vertical axis is given by

$$y = A \sin \omega t \tag{1.33}$$

and its projection on the horizontal axis by

$$x = A \cos \omega t \tag{1.34}$$

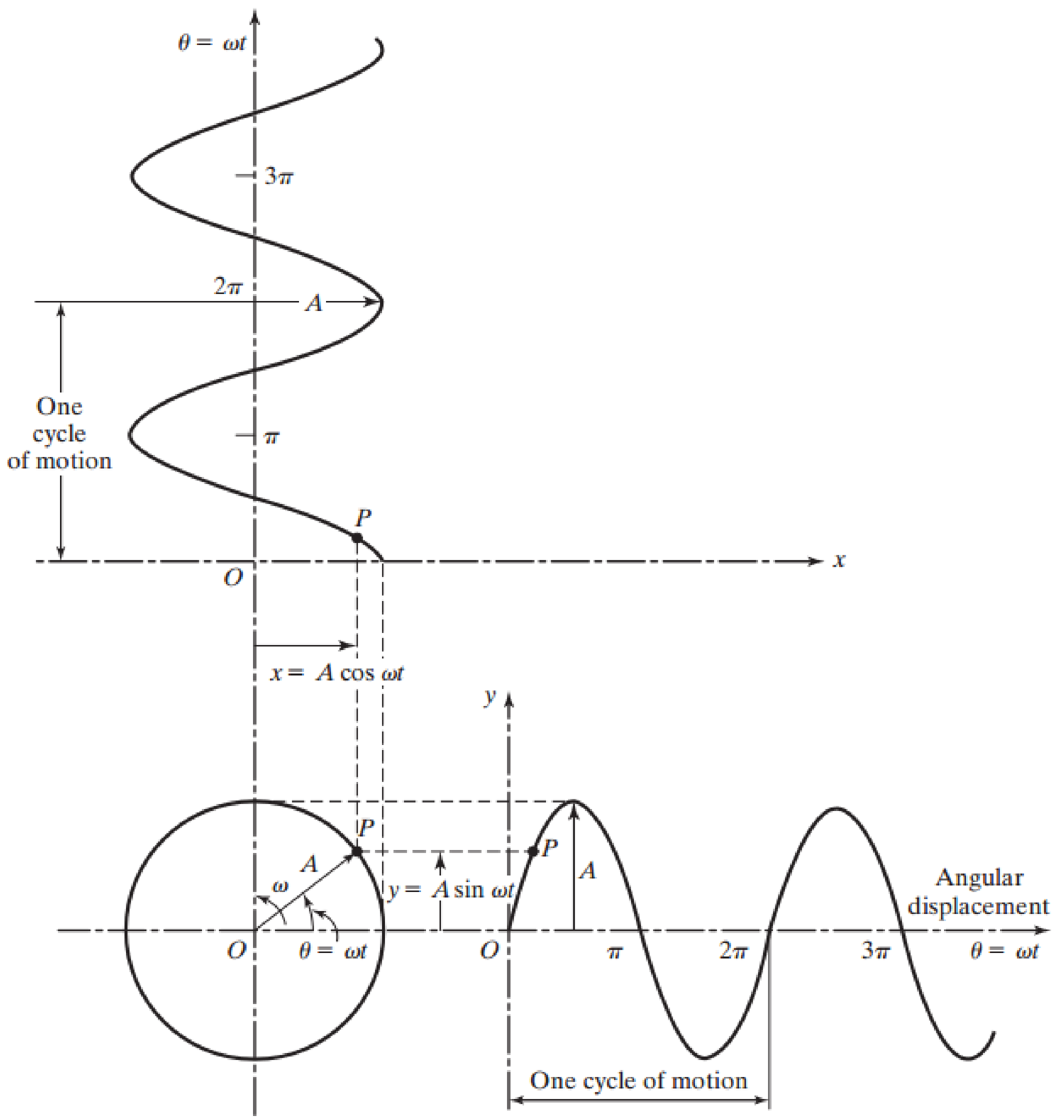


FIGURE 1.47 Harmonic motion as the projection of the end of a rotating vector.

As seen above, the vectorial method of representing harmonic motion requires the description of both the horizontal and vertical components. It is more convenient to represent harmonic motion using a complex-number representation. Any vector $\vec{X}$ in the xy-plane can be represented as a complex number:

$$\vec{X} = a + ib \tag{1.35}$$

where $i = \sqrt{-1}$ and a and b denote the x and y components of $\vec{X}$, respectively (see Fig. 1.48). Components a and b are also called the *real* and *imaginary* parts of the vector $\vec{X}$. If A denotes the modulus or absolute value of the vector $\vec{X}$, and θ represents the argument or the angle between the vector and the x-axis, then $\vec{X}$ can also be expressed as

$$\vec{X} = A\cos\theta + iA\sin\theta \tag{1.36}$$

with

$$A = (a^2 + b^2)^{1/2} \tag{1.37}$$

and

$$\theta = \tan^{-1}\frac{b}{a} \tag{1.38}$$

Noting that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, ..., $\cos\theta$ and $i\sin\theta$ can be expanded in a series as

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots = 1 + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^4}{4!} + \cdots \tag{1.39}$$

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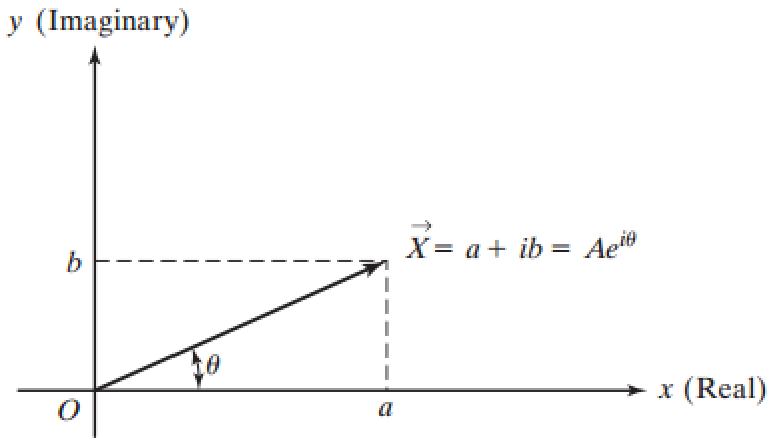

FIGURE 1.48 Representation of a complex number.

$$i\sin\theta = i\left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right] = i\theta + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^5}{5!} + \cdots \tag{1.40}$$

Equations (1.39) and (1.40) yield

$$(\cos\theta + i\sin\theta) = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots = e^{i\theta} \tag{1.41}$$

and

$$(\cos\theta - i\sin\theta) = 1 - i\theta + \frac{(i\theta)^2}{2!} - \frac{(i\theta)^3}{3!} + \cdots = e^{-i\theta} \tag{1.42}$$

Thus Eq. (1.36) can be expressed as

$$\vec{X} = A(\cos\theta + i\sin\theta) = Ae^{i\theta} \tag{1.43}$$

Using complex-number representation, the rotating vector $\vec{X}$ of Fig. 1.47 can be written as

$$\vec{X} = Ae^{i\omega t} \tag{1.51}$$

where ω denotes the circular frequency (rad/sec) of rotation of the vector $\vec{X}$ in counter-clockwise direction. The differentiation of the harmonic motion given by Eq. (1.51) with respect to time gives

$$\frac{d\vec{X}}{dt} = \frac{d}{dt}(Ae^{i\omega t}) = i\omega Ae^{i\omega t} = i\omega\vec{X} \tag{1.52}$$

$$\frac{d^2\vec{X}}{dt^2} = \frac{d}{dt}(i\omega Ae^{i\omega t}) = -\omega^2 Ae^{i\omega t} = -\omega^2\vec{X} \tag{1.53}$$

Thus the displacement, velocity, and acceleration can be expressed as[4]

$$\text{displacement} = \mathrm{Re}[Ae^{i\omega t}] = A\cos\omega t \tag{1.54}$$

$$\begin{aligned}\text{velocity} &= \mathrm{Re}[i\omega Ae^{i\omega t}] = -\omega A\sin\omega t \\ &= \omega A\cos(\omega t + 90^\circ)\end{aligned} \tag{1.55}$$

$$\begin{aligned}\text{acceleration} &= \mathrm{Re}[-\omega^2 Ae^{i\omega t}] = -\omega^2 A\cos\omega t \\ &= \omega^2 A\cos(\omega t + 180^\circ)\end{aligned} \tag{1.56}$$

where Re denotes the real part. These quantities are shown as rotating vectors in Fig. 1.49. It can be seen that the acceleration vector leads the velocity vector by 90°, and the latter leads the displacement vector by 90°.

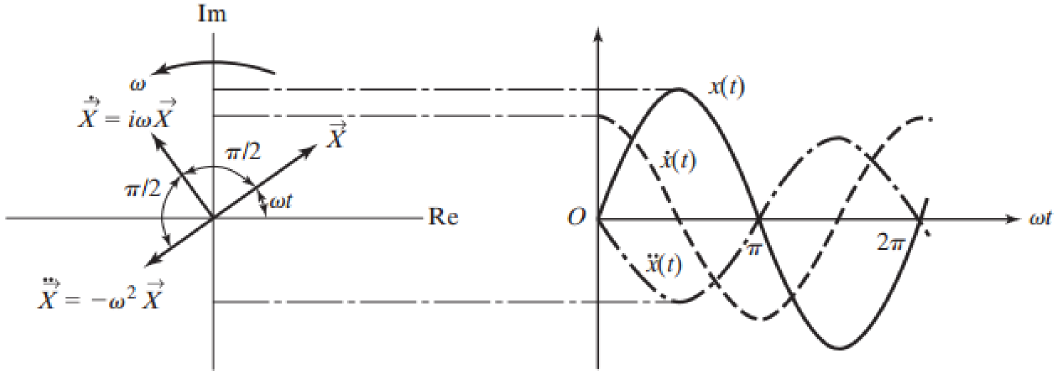

FIGURE 1.49 Displacement, velocity, and accelerations as rotating vectors.

Beats. When two harmonic motions, with frequencies close to one another, are added, the resulting motion exhibits a phenomenon known as beats. For example, if

$$x_1(t) = X \cos \omega t \tag{1.63}$$

$$x_2(t) = X \cos(\omega + \delta)t \tag{1.64}$$

where δ is a small quantity, the addition of these motions yields

$$x(t) = x_1(t) + x_2(t) = X[\cos \omega t + \cos(\omega + \delta)t] \tag{1.65}$$

Using the relation

$$\cos A + \cos B = 2 \cos\left(\frac{A + B}{2}\right) \cos\left(\frac{A - B}{2}\right) \tag{1.66}$$

Eq. (1.65) can be rewritten as

$$x(t) = 2X \cos \frac{\delta t}{2} \cos\left(\omega + \frac{\delta}{2}\right)t \tag{1.67}$$

This equation is shown graphically in Fig. 1.53. It can be seen that the resulting motion, $x(t)$, represents a cosine wave with frequency $\omega + \delta/2$, which is approximately equal to ω, and with a varying amplitude of $2X \cos \delta t/2$. Whenever the amplitude reaches a maximum, it is called a beat. The frequency (δ) at which the amplitude builds up and dies down between 0 and $2X$ is known as beat frequency. The phenomenon of beats is often observed in machines, structures, and electric power houses. For example, in machines and structures, the beating phenomenon occurs when the forcing frequency is close to the natural frequency of the system (see Section 3.3.2).

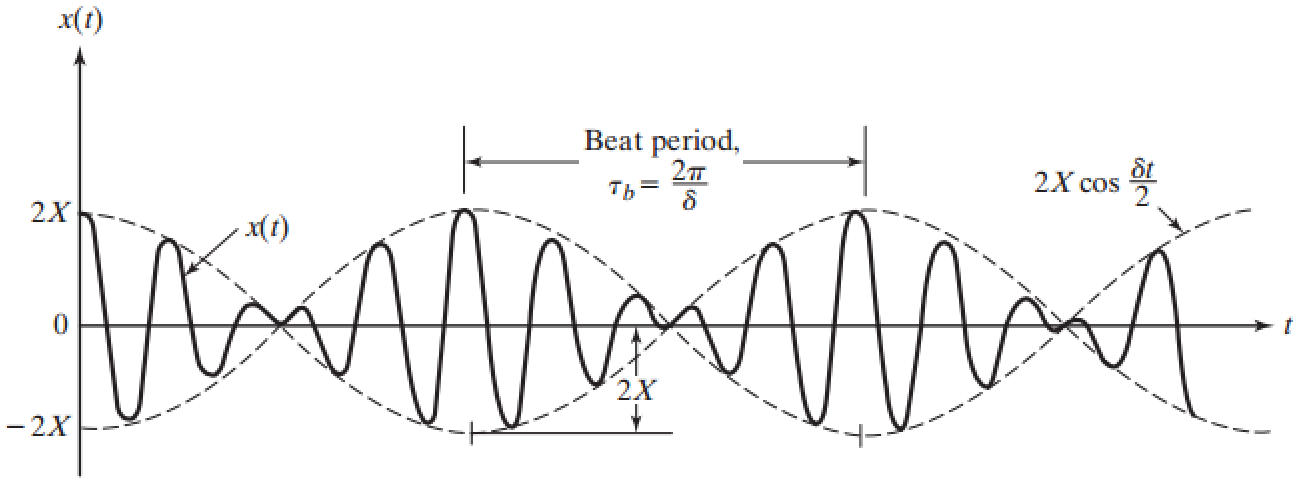

FIGURE 1.53 Phenomenon of beats.

https://en.wikipedia.org/wiki/Beat_(acoustics)

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