EQUAÇÕES DA DINÂMICA DOS FLUIDOS NA FORMA DIFERENCIAL

Equações da Dinâmica dos Fluidos na Forma Integral

$$\begin{cases} \dfrac{d}{dt}\iiint \rho\, d\mathcal{V} = -\iint \rho\vec{V}\cdot\hat{n}\, dS \\ \dfrac{d}{\partial t}\iiint \rho\vec{V}\, d\mathcal{V} = -\iint \rho\vec{V}\left(\vec{V}\cdot\hat{n}\right) dS - \iint p\hat{n}\, dS + \iint \tilde{\tau}\,\hat{n} dS + \iiint \rho\vec{f}\, d\mathcal{V} \\ \dfrac{d}{dt}\iiint \rho E\, d\mathcal{V} = -\iint \rho E\vec{V}\cdot\hat{n}\, dS - \iint p\hat{n}\cdot\vec{V}\, dS + \iint \tilde{\tau}\hat{n}\cdot\vec{V}\, dS + \iiint \rho\vec{f}\cdot\vec{V}\, d\mathcal{V} \\ \qquad\qquad + \iiint \rho\dot{q}\, d\mathcal{V} + \iint \kappa\nabla T\cdot\hat{n}\, dS \end{cases}$$

Operador Nabla

$$\nabla = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}$$

Na forma de gradiente de uma grandeza escalar f:

$$\nabla f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k}$$

Na forma de divergente de uma grandeza vetorial $\vec{g}$:

$$\nabla\cdot\vec{g} = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(g_x\hat{\imath} + g_y\hat{\jmath} + g_z\hat{k}\right) = \frac{\partial g_x}{\partial x} + \frac{\partial g_y}{\partial y} + \frac{\partial g_z}{\partial z}$$

Teorema da Divergência

O Teorema da Divergência relaciona a integral em um volume de controle fechado com a integral na sua superfície. Ele se aplica a escalares, vetores e tensores:

$$\iint f\hat{n}\, dS = \iiint \nabla f\, d\mathcal{V}$$

$$\iint \vec{g}\cdot\hat{n}\, dS = \iiint \nabla\cdot\vec{g}\, d\mathcal{V}$$

$$\iint \tilde{h}\hat{n}\, dS = \iiint \nabla\otimes\tilde{h}\, d\mathcal{V}$$

onde $\nabla\otimes\tilde{h} = \tilde{h}\nabla$.

1) Conservação da massa (continuidade):

$$\frac{d}{dt}\iiint \rho \, d\mathcal{V} = -\iint \rho\vec{V}\cdot\hat{n}\, dS$$

Usando o Teorema da Divergência:

$$\frac{d}{dt}\iiint \rho \, d\mathcal{V} = -\iiint \nabla\cdot\left(\rho\vec{V}\right) d\mathcal{V}$$

$$\iiint \left[\frac{\partial\rho}{\partial t} + \nabla\cdot\left(\rho\vec{V}\right)\right] d\mathcal{V} = 0$$

Para que esse integral seja igual a zero, independentemente da escolha do volume de controle,

$$\boxed{\frac{\partial\rho}{\partial t} + \nabla\cdot\left(\rho\vec{V}\right) = 0}$$

2) Equação da quantidade de movimento (momentum):

$$\frac{d}{dt}\iiint \rho\vec{V}\, d\mathcal{V} = -\iint \rho\vec{V}\left(\vec{V}\cdot\hat{n}\right) dS - \iint p\hat{n}\, dS + \iint \tilde{\tau}\hat{n}\, dS + \iiint \rho\vec{f}\, d\mathcal{V} \quad \Rightarrow$$

$$\Rightarrow \quad \frac{d}{dt}\iiint \rho\vec{V}\, d\mathcal{V} = -\iint \left(\rho\vec{V}\vec{V}^T\right)\hat{n}\, dS - \iint p\hat{n}\, dS + \iint \tilde{\tau}\hat{n}\, dS + \iiint \rho\vec{f}\, d\mathcal{V}$$

Usando o Teorema da Divergência:

$$\boxed{\frac{\partial\rho\vec{V}}{\partial t} = -\nabla\otimes\left(\rho\vec{V}\vec{V}^T\right) - \nabla p + \nabla\otimes\tilde{\tau} + \rho\vec{f}}$$

3) Equação da energia (1ª Lei da Termodinâmica):

$$\frac{d}{dt}\iiint \rho E\, d\mathcal{V} = -\iint \rho E\vec{V}\cdot\hat{n}\, dS - \iint p\hat{n}\cdot\vec{V}\, dS + \iint \tilde{\tau}\hat{n}\cdot\vec{V}\, dS + \iiint \rho\vec{f}\cdot\vec{V}\, d\mathcal{V}$$
$$+ \iiint \rho\dot{q}\, d\mathcal{V} + \iint \kappa\nabla T\cdot\hat{n}\, dS$$

Usando o Teorema da Divergência:

$$\boxed{\frac{\partial\rho E}{\partial t} = -\nabla\cdot\left(\rho E\vec{V}\right) - \nabla\cdot\left(p\vec{V}\right) + \nabla\cdot\left(\tilde{\tau}\,\vec{V}\right) + \rho\vec{f}\cdot\vec{V} + \rho\dot{q} + \nabla\cdot\left(\kappa\nabla \mathrm{T}\right)}$$

Equações da Dinâmica dos Fluidos na Forma Diferencial

$$\begin{cases}\dfrac{\partial}{\partial t}\rho = -\nabla\cdot\left(\rho\vec{V}\right)\\ \dfrac{\partial}{\partial t}\left(\rho\vec{V}\right) = -\nabla\otimes\left(\rho\vec{V}\vec{V}^T\right) - \nabla p + \nabla\otimes\tilde{\tau} + \rho\vec{f}\\ \dfrac{\partial}{\partial t}(\rho E) = -\nabla\cdot\left(\rho E\vec{V}\right) - \nabla\cdot\left(p\vec{V}\right) + \nabla\cdot\left(\tilde{\tau}\,\vec{V}\right) + \rho\vec{f}\cdot\vec{V} + \rho\dot{q} + \nabla\cdot(\kappa\nabla\mathrm{T})\end{cases}$$

Expandindo a equação de momentum:

$$\frac{\partial\rho\vec{V}}{\partial t} = -\nabla\otimes\left(\rho\vec{V}\vec{V}^T\right) - \nabla p + \nabla\otimes\tilde{\tau} + \rho\vec{f}$$

$$\frac{\partial}{\partial t}\left(\rho\begin{Bmatrix}u\\ v\\ w\end{Bmatrix}\right) = -\left(\rho\begin{Bmatrix}u\\ v\\ w\end{Bmatrix}[u\ v\ w]\right)\begin{Bmatrix}\dfrac{\partial}{\partial x}\\ \dfrac{\partial}{\partial y}\\ \dfrac{\partial}{\partial z}\end{Bmatrix} - \begin{Bmatrix}\dfrac{\partial p}{\partial x}\\ \dfrac{\partial p}{\partial y}\\ \dfrac{\partial p}{\partial z}\end{Bmatrix} + \begin{bmatrix}\tau_{xx} & \tau_{yx} & \tau_{zx}\\ \tau_{xy} & \tau_{yy} & \tau_{zy}\\ \tau_{xz} & \tau_{yz} & \tau_{zz}\end{bmatrix}\begin{Bmatrix}\dfrac{\partial}{\partial x}\\ \dfrac{\partial}{\partial y}\\ \dfrac{\partial}{\partial z}\end{Bmatrix} + \rho\begin{Bmatrix}f_x\\ f_y\\ f_z\end{Bmatrix}$$

$$\frac{\partial}{\partial t}\begin{Bmatrix}\rho u\\ \rho v\\ \rho w\end{Bmatrix} = -\begin{Bmatrix}\dfrac{\partial\rho u^2}{\partial x} + \dfrac{\partial\rho vu}{\partial y} + \dfrac{\partial\rho wu}{\partial z}\\ \dfrac{\partial\rho uv}{\partial x} + \dfrac{\partial\rho v^2}{\partial y} + \dfrac{\partial\rho wv}{\partial z}\\ \dfrac{\partial\rho uw}{\partial x} + \dfrac{\partial\rho vw}{\partial y} + \dfrac{\partial\rho w^2}{\partial z}\end{Bmatrix} - \begin{Bmatrix}\dfrac{\partial p}{\partial x}\\ \dfrac{\partial p}{\partial y}\\ \dfrac{\partial p}{\partial z}\end{Bmatrix} + \begin{Bmatrix}\dfrac{\partial\tau_{xx}}{\partial x} + \dfrac{\partial\tau_{yx}}{\partial y} + \dfrac{\partial\tau_{zx}}{\partial z}\\ \dfrac{\partial\tau_{xy}}{\partial x} + \dfrac{\partial\tau_{yy}}{\partial y} + \dfrac{\partial\tau_{zy}}{\partial z}\\ \dfrac{\partial\tau_{xz}}{\partial x} + \dfrac{\partial\tau_{yz}}{\partial y} + \dfrac{\partial\tau_{zz}}{\partial z}\end{Bmatrix} + \rho\begin{Bmatrix}f_x\\ f_y\\ f_z\end{Bmatrix}$$

$$\boxed{\frac{\partial}{\partial t}\begin{Bmatrix}\rho u\\ \rho v\\ \rho w\end{Bmatrix} = -\frac{\partial}{\partial x}\begin{Bmatrix}\rho u^2 + p - \tau_{xx}\\ \rho uv - \tau_{xy}\\ \rho uw - \tau_{xz}\end{Bmatrix} - \frac{\partial}{\partial y}\begin{Bmatrix}\rho vu - \tau_{yx}\\ \rho v^2 + p - \tau_{yy}\\ \rho vw - \tau_{yz}\end{Bmatrix} - \frac{\partial}{\partial z}\begin{Bmatrix}\rho wu - \tau_{zx}\\ \rho wv - \tau_{zy}\\ \rho w^2 + p - \tau_{zz}\end{Bmatrix} + \begin{Bmatrix}\rho f_x\\ \rho f_y\\ \rho f_z\end{Bmatrix}}$$

Escoamento Potencial

Para escoamento irrotacional (sem vorticidade), incompressível, isentrópico e invíscido pode-se definir um potencial de velocidade ϕ tal que $\nabla\phi = \vec{V}$ e a equação da continuidade

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\left(\rho\vec{V}\right) = 0$$

é reduzida a

$$\rho\nabla\cdot(\nabla\phi) = 0$$

obtendo a equação potencial, muito utilizada em simulações aerodinâmicas,

$$\nabla^2\phi = 0$$