Problem 1.1

[Difficulty: 3]

1.1 A number of common substances are

Tar	Sand
"Silly Putty"	Jello
Modeling clay	Toothpaste
Wax	Shaving cream

Some of these materials exhibit characteristics of both solid and fluid behavior under different conditions. Explain and give examples.

Given: Common Substances

Tar	Sand
"Silly Putty"	Jello
Modeling clay	Toothpaste
Wax	Shaving cream

Some of these substances exhibit characteristics of solids and fluids under different conditions.

Find: Explain and give examples.

Solution: Tar, Wax, and Jello behave as solids at room temperature or below at ordinary pressures. At high pressures or over long periods, they exhibit fluid characteristics. At higher temperatures, all three liquefy and become viscous fluids.

Modeling clay and silly putty show fluid behavior when sheared slowly. However, they fracture under suddenly applied stress, which is a characteristic of solids.

Toothpaste behaves as a solid when at rest in the tube. When the tube is squeezed hard, toothpaste "flows" out the spout, showing fluid behavior. Shaving cream behaves similarly.

Sand acts solid when in repose (a sand "pile"). However, it "flows" from a spout or down a steep incline.

Problem 1.2

[Difficulty: 2]

1.2 Give a word statement of each of the five basic conservation laws stated in Section 1.4, as they apply to a system.

Given: Five basic conservation laws stated in Section 1-4.

Write: A word statement of each, as they apply to a system.

Solution: Assume that laws are to be written for a *system*.

a. Conservation of mass — The mass of a system is constant by definition.

b. Newton's second law of motion — The net force acting on a system is directly proportional to the product of the system mass times its acceleration.

c. First law of thermodynamics — The change in stored energy of a system equals the net energy added to the system as heat and work.

d. Second law of thermodynamics — The entropy of any isolated system cannot decrease during any process between equilibrium states.

e. Principle of angular momentum — The net torque acting on a system is equal to the rate of change of angular momentum of the system.

Problem 1.3

[Difficulty: 3]

1.3 The barrel of a bicycle tire pump becomes quite warm during use. Explain the mechanisms responsible for the temperature increase.

Open-Ended Problem Statement: The barrel of a bicycle tire pump becomes quite warm during use. Explain the mechanisms responsible for the temperature increase.

Discussion: Two phenomena are responsible for the temperature increase: (1) friction between the pump piston and barrel and (2) temperature rise of the air as it is compressed in the pump barrel.

Friction between the pump piston and barrel converts mechanical energy (force on the piston moving through a distance) into thermal energy as a result of friction. Lubricating the piston helps to provide a good seal with the pump barrel and reduces friction (and therefore force) between the piston and barrel.

Temperature of the trapped air rises as it is compressed. The compression is not adiabatic because it occurs during a finite time interval. Heat is transferred from the warm compressed air in the pump barrel to the cooler surroundings. This raises the temperature of the barrel, making its outside surface warm (or even hot!) to the touch.

Problem 1.4

[Difficulty: 3]

1.4 Discuss the physics of skipping a stone across the water surface of a lake. Compare these mechanisms with a stone as it bounces after being thrown along a roadway.

Open-Ended Problem Statement: Consider the physics of "skipping" a stone across the water surface of a lake. Compare these mechanisms with a stone as it bounces after being thrown along a roadway.

Discussion: Observation and experience suggest two behaviors when a stone is thrown along a water surface:

1. If the angle between the path of the stone and the water surface is steep the stone may penetrate the water surface. Some momentum of the stone will be converted to momentum of the water in the resulting splash. After penetrating the water surface, the high drag* of the water will slow the stone quickly. Then, because the stone is heavier than water it will sink.
2. If the angle between the path of the stone and the water surface is shallow the stone may not penetrate the water surface. The splash will be smaller than if the stone penetrated the water surface. This will transfer less momentum to the water, causing less reduction in speed of the stone. The only drag force on the stone will be from friction on the water surface. The drag will be momentary, causing the stone to lose only a portion of its kinetic energy. Instead of sinking, the stone may skip off the surface and become airborne again.

When the stone is thrown with speed and angle just right, it may skip several times across the water surface. With each skip the stone loses some forward speed. After several skips the stone loses enough forward speed to penetrate the surface and sink into the water.

Observation suggests that the shape of the stone significantly affects skipping. Essentially spherical stones may be made to skip with considerable effort and skill from the thrower. Flatter, more disc-shaped stones are more likely to skip, provided they are thrown with the flat surface(s) essentially parallel to the water surface; spin may be used to stabilize the stone in flight.

By contrast, no stone can ever penetrate the pavement of a roadway. Each collision between stone and roadway will be inelastic; friction between the road surface and stone will affect the motion of the stone only slightly. Regardless of the initial angle between the path of the stone and the surface of the roadway, the stone may bounce several times, then finally it will roll to a stop.

The shape of the stone is unlikely to affect trajectory of bouncing from a roadway significantly.

Problem 1.5

[Difficulty: 1]

1.5 Make a guess at the order of magnitude of the mass (e.g., 0.01, 0.1, 1.0, 10, 100, or 1000 lbm or kg) of standard air that is in a room 10 ft by 10 ft by 8 ft, and then compute this mass in lbm and kg to see how close your estimate was.

Given: Dimensions of a room

Find: Mass of air

Solution:

Basic equation: $\rho = \dfrac{p}{R_{air} \cdot T}$

Given or available data $p = 14.7 psi$ $\quad T = (59 + 460)R$ $\quad R_{air} = 53.33 \cdot \dfrac{ft \cdot lbf}{lbm \cdot R}$

$V = 10 \cdot ft \times 10 \cdot ft \times 8 \cdot ft$ $\quad V = 800 \cdot ft^3$

Then $\rho = \dfrac{p}{R_{air} \cdot T}$ $\quad \rho = 0.076 \dfrac{lbm}{ft^3}$ $\quad \rho = 0.00238 \dfrac{slug}{ft^3}$ $\quad \rho = 1.23 \dfrac{kg}{m^3}$

$M = \rho \cdot V$ $\quad M = 61.2 \cdot lbm$ $\quad M = 1.90 \cdot slug$ $\quad M = 27.8 kg$

Problem 1.6

[Difficulty: 1]

1.6 A spherical tank of inside diameter 16 ft contains compressed oxygen at 1000 psia and 77°F. What is the mass of the oxygen?

Given: Data on oxygen tank.

Find: Mass of oxygen.

Solution: Compute tank volume, and then use oxygen density (Table A.6) to find the mass.

The given or available data is: $D = 16 \cdot ft$ $\quad p = 1000 \cdot psi$ $\quad T = (77 + 460) \cdot R$ $\quad T = 537 \cdot R$

$$R_{O2} = 48.29 \cdot \frac{ft \cdot lbf}{lbm \cdot R} \quad \text{(Table A.6)}$$

For oxygen the critical temperature and pressure are: $T_c = 279 \cdot R$ $\quad p_c = 725.2 \cdot psi$ $\quad$ (data from NIST WebBook)

so the reduced temperature and pressure are: $T_R = \frac{T}{T_c} = 1.925$ $\quad p_R = \frac{p}{p_c} = 1.379$

Using a compressiblity factor chart: $Z = 0.948$ $\quad$ Since this number is close to 1, we can assume ideal gas behavior.

Therefore, the governing equation is the ideal gas equation $\quad p = \rho \cdot R_{O2} \cdot T$ $\quad$ and $\quad \rho = \frac{M}{V}$

where V is the tank volume $\quad V = \frac{\pi \cdot D^3}{6}$ $\quad V = \frac{\pi}{6} \times (16 \cdot ft)^3$ $\quad V = 2144.7 \cdot ft^3$

Hence $\quad M = V \cdot \rho = \frac{p \cdot V}{R_{O2} \cdot T}$ $\quad M = 1000 \cdot \frac{lbf}{in^2} \times 2144.7 \cdot ft^3 \times \frac{1}{48.29} \cdot \frac{lbm \cdot R}{ft \cdot lbf} \times \frac{1}{537} \cdot \frac{1}{R} \times \left(\frac{12 \cdot in}{ft}\right)^2$

$$M = 11910 \cdot lbm$$

Problem 1.7

[Difficulty: 3]

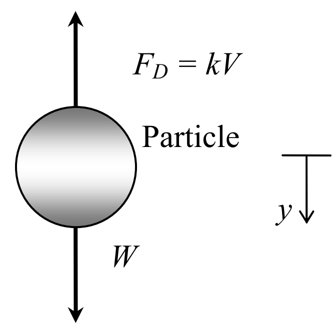
1.7 Very small particles moving in fluids are known to experience a drag force proportional to speed. Consider a particle of net weight W dropped in a fluid. The particle experiences a drag force, $F_D = kV$, where V is the particle speed. Determine the time required for the particle to accelerate from rest to 95 percent of its terminal speed, V_t, in terms of k, W, and g.

Given: Small particle accelerating from rest in a fluid. Net weight is W, resisting force $F_D = kV$, where V is speed.

Find: Time required to reach 95 percent of terminal speed, V_t.

Solution: Consider the particle to be a system. Apply Newton's second law.

Basic equation: $\sum F_y = ma_y$



Assumptions:

1. W is net weight
2. Resisting force acts opposite to V

Then

$$\sum F_y = W - kV = ma_y = m\frac{dV}{dt} = \frac{W}{g}\frac{dV}{dt}$$

or
$$\frac{dV}{dt} = g\left(1 - \frac{k}{W}V\right)$$

Separating variables,
$$\frac{dV}{1 - \frac{k}{W}V} = g\,dt$$

Integrating, noting that velocity is zero initially,
$$\int_0^V \frac{dV}{1 - \frac{k}{W}V} = -\frac{W}{k}\ln\left(1 - \frac{k}{W}V\right) = \int_0^t g\,dt = gt$$

or
$$1 - \frac{k}{W}V = e^{-\frac{kgt}{W}}; \quad V = \frac{W}{k}\left(1 - e^{-\frac{kgt}{W}}\right)$$

But $V \to V_t$ as $t \to \infty$, so $V_t = \frac{W}{k}$. Therefore
$$\frac{V}{V_t} = 1 - e^{-\frac{kgt}{W}}$$

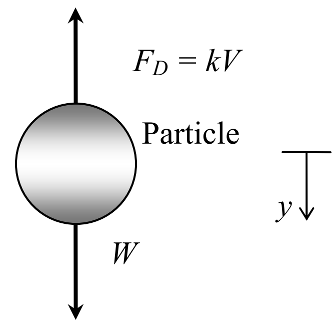
When $\frac{V}{V_t} = 0.95$, then $e^{-\frac{kgt}{W}} = 0.05$ and $\frac{kgt}{W} = 3$. Thus $t = 3W/gk$

Problem 1.8

[Difficulty: 2]

1.8 Consider again the small particle of Problem 1.7. Express the distance required to reach 95 percent of its terminal speed in percent terms of g, k, and W.

Given: Small particle accelerating from rest in a fluid. Net weight is W, resisting force is $F_D = kV$, where V is speed.



Find: Distance required to reach 95 percent of terminal speed, V_t.

Solution: Consider the particle to be a system. Apply Newton's second law.

Basic equation: $\sum F_y = ma_y$

Assumptions:

1. W is net weight.
2. Resisting force acts opposite to V.

Then, $\sum F_y = W - kV = ma_y = m\frac{dV}{dt} = \frac{W}{g}V\frac{dV}{dy}$ or $1 - \frac{k}{W}V = \frac{V}{g}\frac{dV}{dy}$

At terminal speed, $a_y = 0$ and $V = V_t = \frac{W}{k}$. Then $1 - \frac{V}{V_g} = \frac{1}{g}V\frac{dV}{dy}$

Separating variables $\dfrac{V\,dV}{1 - \frac{1}{V_t}V} = g\,dy$

Integrating, noting that velocity is zero initially

$$gy = \int_0^{0.95V_t} \frac{V\,dV}{1 - \frac{1}{V_t}V} = \left[-VV_t - V_t^2 \ln\left(1 - \frac{V}{V_t}\right)\right]_0^{0.95V_t}$$

$$gy = -0.95V_t^2 - V_t^2 \ln(1 - 0.95) - V_t^2 \ln(1)$$

$$gy = -V_t^2\left[0.95 + \ln 0.05\right] = 2.05\,V_t^2$$

$$\therefore y = \frac{2.05}{g}V_t^2 = 2.05\frac{W^2}{gt^2}$$

Problem 1.9

[Difficulty: 2]

1.9 A cylindrical tank must be designed to contain 5 kg of compressed nitrogen at a pressure of 200 atm (gage) and 20°C. The design constraints are that the length must be twice the diameter and the wall thickness must be 0.5 cm. What are the external dimensions?

Given: Mass of nitrogen, and design constraints on tank dimensions.

Find: External dimensions.

Solution: Use given geometric data and nitrogen mass, with data from Table A.6.

The given or available data is: $M = 5 \cdot kg$ $\quad p = (200 + 1) \cdot atm$ $\quad p = 20.4 \cdot MPa$

$T = (20 + 273) \cdot K$ $\quad T = 293 \cdot K$ $\quad R_{N2} = 296.8 \cdot \frac{J}{kg \cdot K}$ (Table A.6)

The governing equation is the ideal gas equation $\quad p = \rho \cdot R_{N2} \cdot T$ and $\rho = \frac{M}{V}$

where V is the tank volume $\quad V = \frac{\pi \cdot D^2}{4} \cdot L$ where $L = 2 \cdot D$

Combining these equations:

Hence

$$M = V \cdot \rho = \frac{p \cdot V}{R_{N2} \cdot T} = \frac{p}{R_{N2} \cdot T} \cdot \frac{\pi \cdot D^2}{4} \cdot L = \frac{p}{R_{N2} \cdot T} \cdot \frac{\pi \cdot D^2}{4} \cdot 2 \cdot D = \frac{p \cdot \pi \cdot D^3}{2 \cdot R_{N2} \cdot T}$$

Solving for D

$$D = \left(\frac{2 \cdot R_{N2} \cdot T \cdot M}{p \cdot \pi}\right)^{\frac{1}{3}} \qquad D = \left(\frac{2}{\pi} \times 296.8 \cdot \frac{N \cdot m}{kg \cdot K} \times 293 \cdot K \times 5 \cdot kg \times \frac{1}{20.4 \times 10^6} \cdot \frac{m^2}{N}\right)^{\frac{1}{3}}$$

$D = 0.239 \cdot m$ $\quad L = 2 \cdot D$ $\quad L = 0.477 \cdot m$

These are internal dimensions; the external ones are 2 x 0.5 cm larger: $\quad L = 0.249 \cdot m$ $\quad D = 0.487 \cdot m$

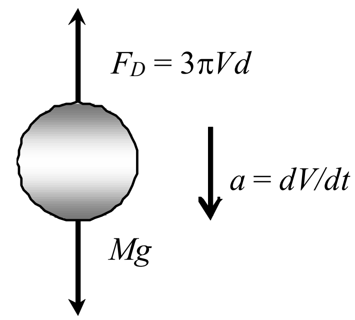
Problem 1.10

[Difficulty: 4]

1.10 In a combustion process, gasoline particles are to be dropped in air at 200°F. The particles must drop at least 10 in. in 1 s. Find the diameter d of droplets required for this. (The drag on these particles is given by $F_D = \pi \mu V d$, where V is the particle speed and μ is the air viscosity. To solve this problem, use *Excel's Goal Seek*.)

NOTE: Drag formula is in error: It should be:

$F_D = 3 \cdot \pi \cdot V \cdot d$



Given: Data on sphere and formula for drag.

Find: Diameter of gasoline droplets that take 1 second to fall 10 in.

Solution: Use given data and data in Appendices; integrate equation of motion by separating variables.

The data provided, or available in the Appendices, are:

$\mu = 4.48 \times 10^{-7} \cdot \frac{\text{lbf} \cdot \text{s}}{\text{ft}^2}$ $\quad \rho_w = 1.94 \cdot \frac{\text{slug}}{\text{ft}^3}$ $\quad SG_{gas} = 0.72$ $\quad \rho_{gas} = SG_{gas} \cdot \rho_w$ $\quad \rho_{gas} = 1.40 \cdot \frac{\text{slug}}{\text{ft}^3}$

Newton's 2nd law for the sphere (mass M) is (ignoring buoyancy effects) $\quad M \cdot \frac{dV}{dt} = M \cdot g - 3 \cdot \pi \cdot \mu \cdot V \cdot d$

so

$$\frac{dV}{g - \frac{3 \cdot \pi \cdot \mu \cdot d}{M} \cdot V} = dt$$

Integrating twice and using limits

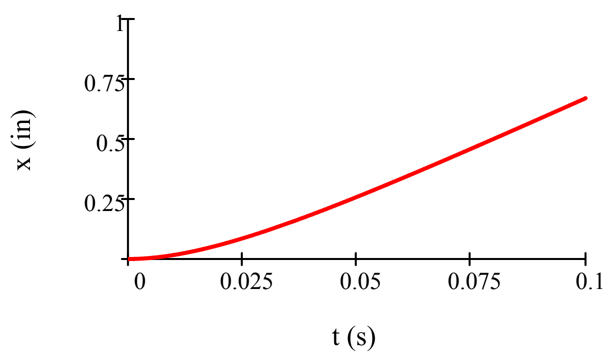
$$V(t) = \frac{M \cdot g}{3 \cdot \pi \cdot \mu \cdot d} \cdot \left(1 - e^{\frac{-3 \cdot \pi \cdot \mu \cdot d}{M} \cdot t}\right) \qquad x(t) = \frac{M \cdot g}{3 \cdot \pi \cdot \mu \cdot d} \cdot \left[t + \frac{M}{3 \cdot \pi \cdot \mu \cdot d} \cdot \left(e^{\frac{-3 \cdot \pi \cdot \mu \cdot d}{M} \cdot t} - 1\right)\right]$$

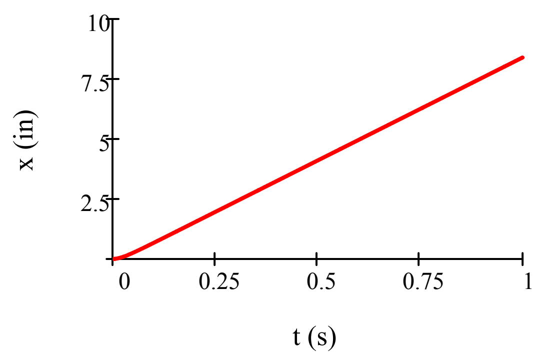
Replacing M with an expression involving diameter d

$$M = \rho_{gas} \cdot \frac{\pi \cdot d^3}{6} \qquad x(t) = \frac{\rho_{gas} \cdot d^2 \cdot g}{18 \cdot \mu} \cdot \left[t + \frac{\rho_{gas} \cdot d^2}{18 \cdot \mu} \cdot \left(e^{\frac{-18 \cdot \mu}{\rho_{gas} \cdot d^2} \cdot t} - 1\right)\right]$$

This equation must be solved for d so that $x(1 \cdot s) = 10 \cdot in$. The answer can be obtained from manual iteration, or by using *Excel's Goal Seek*.

$$d = 4.30 \times 10^{-3} \cdot in$$





Note That the particle quickly reaches terminal speed, so that a simpler approximate solution would be to solve $Mg = 3\pi\mu V d$ for d, with $V = 0.25$ m/s (allowing for the fact that M is a function of d)!

Problem 1.11

[Difficulty: 3]

1.11 For a small particle of styrofoam (1 lbm/ft^3) (spherical, with diameter $d = 0.3$ mm) falling in standard air at speed V, the drag is given by $F_D = 3\pi\mu V d$, where μ is the air viscosity. Find the maximum speed starting from rest, and the time it takes to reach 95 percent of this speed. Plot the speed as a function of time.

Given: Data on sphere and formula for drag.

Find: Maximum speed, time to reach 95% of this speed, and plot speed as a function of time.

Solution: Use given data and data in Appendices, and integrate equation of motion by separating variables.

The data provided, or available in the Appendices, are:

$$\rho_{air} = 1.17 \cdot \frac{kg}{m^3} \qquad \mu = 1.8 \times 10^{-5} \cdot \frac{N \cdot s}{m^2} \qquad \rho_w = 999 \cdot \frac{kg}{m^3} \qquad SG_{Sty} = 0.016 \qquad d = 0.3 \cdot mm$$

Then the density of the sphere is $\rho_{Sty} = SG_{Sty} \cdot \rho_w \qquad \rho_{Sty} = 16 \frac{kg}{m^3}$

The sphere mass is $M = \rho_{Sty} \cdot \frac{\pi \cdot d^3}{6} = 16 \cdot \frac{kg}{m^3} \times \pi \times \frac{(0.0003 \cdot m)^3}{6} \qquad M = 2.26 \times 10^{-10} kg$

Newton's 2nd law for the steady state motion becomes (ignoring buoyancy effects) $M \cdot g = 3 \cdot \pi \cdot V \cdot d$

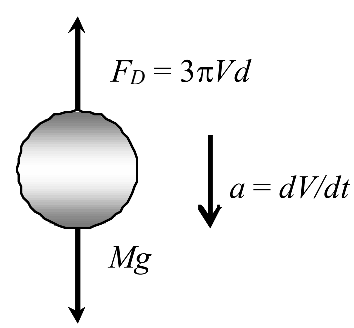
so

$$V_{max} = \frac{M \cdot g}{3 \cdot \pi \cdot \mu \cdot d} = \frac{1}{3 \cdot \pi} \times 2.26 \times 10^{-10} \cdot kg \times 9.81 \cdot \frac{m}{s^2} \times \frac{m^2}{1.8 \times 10^{-5} \cdot N \cdot s} \times \frac{1}{0.0003 \cdot m} \qquad V_{max} = 0.0435 \frac{m}{s}$$

Newton's 2nd law for the general motion is (ignoring buoyancy effects) $M \cdot \frac{dV}{dt} = M \cdot g - 3 \cdot \pi \cdot \mu \cdot V \cdot d$

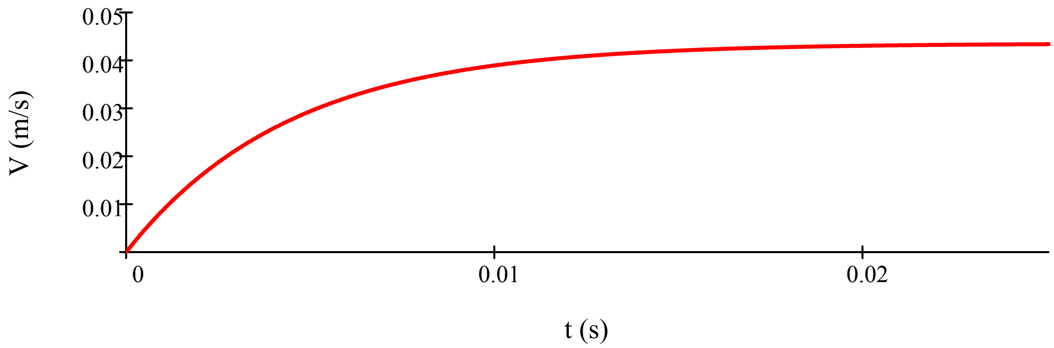
so

$$\frac{dV}{g - \frac{3 \cdot \pi \cdot \mu \cdot d}{M} \cdot V} = dt$$



Integrating and using limits $V(t) = \frac{M \cdot g}{3 \cdot \pi \cdot \mu \cdot d} \cdot \left(1 - e^{\frac{-3 \cdot \pi \cdot \mu \cdot d}{M} \cdot t}\right)$

Using the given data



The time to reach 95% of maximum speed is obtained from

$$\frac{M \cdot g}{3 \cdot \pi \cdot \mu \cdot d} \cdot \left(1 - e^{\frac{-3 \cdot \pi \cdot \mu \cdot d}{M} \cdot t} \right) = 0.95 \cdot V_{max}$$

so $$t = -\frac{M}{3 \cdot \pi \cdot \mu \cdot d} \cdot \ln\left(1 - \frac{0.95 \cdot V_{max} \cdot 3 \cdot \pi \cdot \mu \cdot d}{M \cdot g} \right)$$

Substituting values $t = 0.0133\ \mathrm{s}$

The plot can also be done in *Excel*.

Problem 1.12

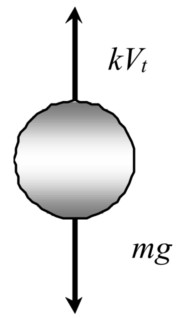
[Difficulty: 3]

1.12 In a pollution control experiment, minute solid particles (typical mass 1×10^{-13} slug) are dropped in air. The terminal speed of the particles is measured to be 0.2 ft/s. The drag of these particles is given by $F_D = kV$, where V is the instantaneous particle speed. Find the value of the constant k. Find the time required to reach 99 percent of terminal speed.

Given: Data on sphere and terminal speed.

Find: Drag constant k, and time to reach 99% of terminal speed.

Solution: Use given data; integrate equation of motion by separating variables.



The data provided are: $M = 1 \times 10^{-13} \cdot \text{slug} \qquad V_t = 0.2 \cdot \frac{\text{ft}}{\text{s}}$

Newton's 2nd law for the general motion is (ignoring buoyancy effects) $M \cdot \frac{dV}{dt} = M \cdot g - k \cdot V \quad (1)$

Newton's 2nd law for the steady state motion becomes (ignoring buoyancy effects) $M \cdot g = k \cdot V_t$ so $k = \frac{M \cdot g}{V_t}$

$$k = 1 \times 10^{-13} \cdot \text{slug} \times 32.2 \cdot \frac{\text{ft}}{\text{s}^2} \times \frac{\text{s}}{0.2 \cdot \text{ft}} \times \frac{\text{lbf} \cdot \text{s}^2}{\text{slug} \cdot \text{ft}} \qquad k = 1.61 \times 10^{-11} \cdot \frac{\text{lbf} \cdot \text{s}}{\text{ft}}$$

To find the time to reach 99% of V_t, we need $V(t)$. From 1, separating variables $\frac{dV}{g - \frac{k}{M} \cdot V} = dt$

Integrating and using limits $t = -\frac{M}{k} \cdot \ln\left(1 - \frac{k}{M \cdot g} \cdot V\right)$

We must evaluate this when $V = 0.99 \cdot V_t \qquad V = 0.198 \cdot \frac{\text{ft}}{\text{s}}$

$$t = -1 \times 10^{-13} \cdot \text{slug} \times \frac{\text{ft}}{1.61 \times 10^{-11} \cdot \text{lbf} \cdot \text{s}} \times \frac{\text{lbf} \cdot \text{s}^2}{\text{slug} \cdot \text{ft}} \cdot \ln\left(1 - 1.61 \times 10^{-11} \cdot \frac{\text{lbf} \cdot \text{s}}{\text{ft}} \times \frac{1}{1 \times 10^{-13} \cdot \text{slug}} \times \frac{\text{s}^2}{32.2 \cdot \text{ft}} \times \frac{0.198 \cdot \text{ft}}{\text{s}} \times \frac{\text{slug} \cdot \text{ft}}{\text{lbf} \cdot \text{s}^2}\right)$$

$t = 0.0286\,\text{s}$

Problem 1.13

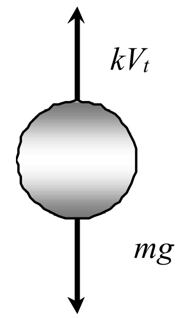
[Difficulty: 5]

1.13 For Problem 1.12, find the distance the particles travel before reaching 99 percent of terminal speed. Plot the distance traveled as a function of time.

Given: Data on sphere and terminal speed from Problem 1.12.

Find: Distance traveled to reach 99% of terminal speed; plot of distance versus time.

Solution: Use given data; integrate equation of motion by separating variables.



The data provided are: $M = 1 \times 10^{-13} \cdot \text{slug}$ $\quad V_t = 0.2 \cdot \frac{\text{ft}}{\text{s}}$

Newton's 2nd law for the general motion is (ignoring buoyancy effects) $\quad M \cdot \frac{dV}{dt} = M \cdot g - k \cdot V \quad (1)$

Newton's 2nd law for the steady state motion becomes (ignoring buoyancy effects) $\quad M \cdot g = k \cdot V_t \quad$ so $\quad k = \frac{M \cdot g}{V_t}$

$$k = 1 \times 10^{-13} \cdot \text{slug} \times 32.2 \cdot \frac{\text{ft}}{\text{s}^2} \times \frac{\text{s}}{0.2 \cdot \text{ft}} \times \frac{\text{lbf} \cdot \text{s}^2}{\text{slug} \cdot \text{ft}} \qquad k = 1.61 \times 10^{-11} \cdot \frac{\text{lbf} \cdot \text{s}}{\text{ft}}$$

To find the distance to reach 99% of V_t, we need $V(y)$. From 1: $\quad M \cdot \frac{dV}{dt} = M \cdot \frac{dy}{dt} \cdot \frac{dV}{dy} = M \cdot V \cdot \frac{dV}{dy} = M \cdot g - k \cdot V$

Separating variables $\quad \frac{V \cdot dV}{g - \frac{k}{M} \cdot V} = dy$

Integrating and using limits $\quad y = -\frac{M^2 \cdot g}{k^2} \cdot \ln\left(1 - \frac{k}{M \cdot g} \cdot V\right) - \frac{M}{k} \cdot V$

We must evaluate this when $\quad V = 0.99 \cdot V_t \qquad V = 0.198 \cdot \frac{\text{ft}}{\text{s}}$

$$y = \left(1 \cdot 10^{-13} \cdot \text{slug}\right)^2 \cdot \frac{32.2 \cdot \text{ft}}{\text{s}^2} \cdot \left(\frac{\text{ft}}{1.61 \cdot 10^{-11} \cdot \text{lbf} \cdot \text{s}}\right)^2 \cdot \left(\frac{\text{lbf} \cdot \text{s}^2}{\text{slug} \cdot \text{ft}}\right)^2 \cdot \ln\left(1 - 1.61 \cdot 10^{-11} \cdot \frac{\text{lbf} \cdot \text{s}}{\text{ft}} \cdot \frac{1}{1 \cdot 10^{-13} \cdot \text{slug}} \cdot \frac{\text{s}^2}{32.2 \cdot \text{ft}} \cdot \frac{0.198 \cdot \text{ft}}{\text{s}} \cdot \frac{\text{slug} \cdot \text{ft}}{\text{lbf} \cdot \text{s}^2}\right) \ldots$$

$$+ 1 \cdot 10^{-13} \cdot \text{slug} \times \frac{\text{ft}}{1.61 \cdot 10^{-11} \cdot \text{lbf} \cdot \text{s}} \times \frac{0.198 \cdot \text{ft}}{\text{s}} \times \frac{\text{lbf} \cdot \text{s}^2}{\text{slug} \cdot \text{ft}}$$

$$y = 4.49 \times 10^{-3} \cdot \text{ft}$$

Alternatively we could use the approach of Problem 1.12 and first find the time to reach terminal speed, and use this time in y(t) to find the above value of y:

From 1, separating variables $\quad \frac{dV}{g - \frac{k}{M} \cdot V} = dt$

Integrating and using limits $\quad t = -\frac{M}{k} \cdot \ln\left(1 - \frac{k}{M \cdot g} \cdot V\right) \quad (2)$

We must evaluate this when $V = 0.99 \cdot V_t$ $\quad V = 0.198 \cdot \frac{ft}{s}$

$$t = 1 \times 10^{-13} \cdot slug \times \frac{ft}{1.61 \times 10^{-11} \cdot lbf \cdot s} \cdot \frac{lbf \cdot s^2}{slug \cdot ft} \cdot \ln\left(1 - 1.61 \times 10^{-11} \cdot \frac{lbf \cdot s}{ft} \times \frac{1}{1 \times 10^{-13} \cdot slug} \times \frac{s^2}{32.2 \cdot ft} \times \frac{0.198 \cdot ft}{s} \times \frac{slug \cdot ft}{lbf \cdot s^2}\right)$$

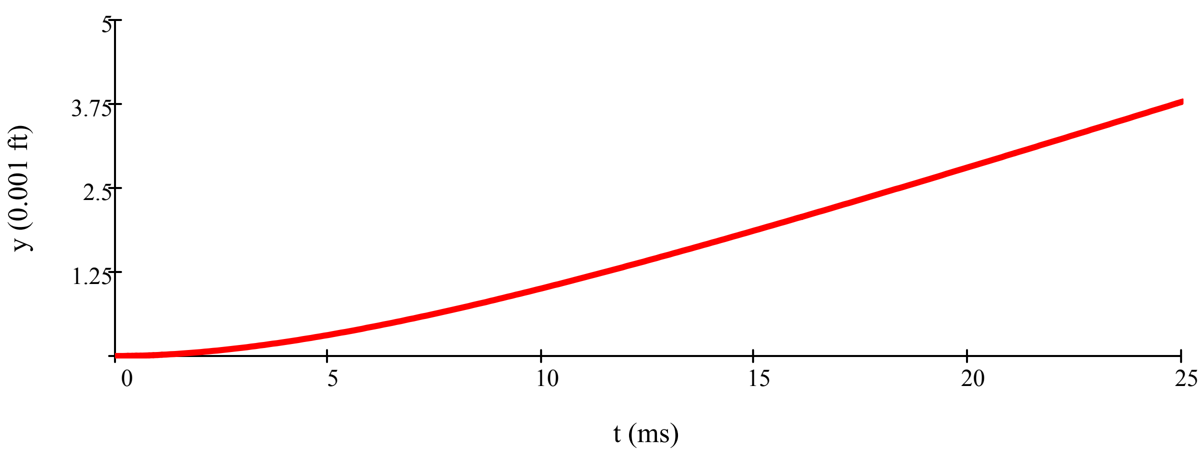
$t = 0.0286\,s$

From 2, after rearranging $\quad V = \frac{dy}{dt} = \frac{M \cdot g}{k} \cdot \left(1 - e^{-\frac{k}{M} \cdot t}\right)$

Integrating and using limits $\quad y = \frac{M \cdot g}{k} \cdot \left[t + \frac{M}{k} \cdot \left(e^{-\frac{k}{M} \cdot t} - 1\right)\right]$

$$y = 1 \times 10^{-13} \cdot slug \times \frac{32.2 \cdot ft}{s^2} \times \frac{ft}{1.61 \times 10^{-11} \cdot lbf \cdot s} \cdot \frac{lbf \cdot s^2}{slug \cdot ft} \cdot \left[0.0291 \cdot s \ldots + 10^{-13} \cdot slug \cdot \frac{ft}{1.61 \times 10^{-11} \cdot lbf \cdot s} \cdot \frac{lbf \cdot s^2}{slug \cdot ft} \cdot \left(e^{-\frac{1.61 \times 10^{-11}}{1 \times 10^{-13}} \cdot .0291} - 1\right)\right]$$

$y = 4.49 \times 10^{-3} \cdot ft$



This plot can also be presented in Excel.

Problem 1.14

[Difficulty: 4]

1.14 A sky diver with a mass of 70 kg jumps from an aircraft. The aerodynamic drag force acting on the sky diver is known to be $F_D = kV^2$, where $k = 0.25\ \mathrm{N \cdot s^2/m^2}$. Determine the maximum speed of free fall for the sky diver and the speed reached after 100 m of fall. Plot the speed of the sky diver as a function of time and as a function of distance fallen.

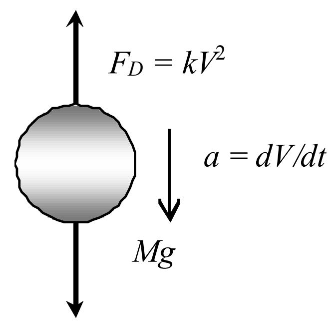
Given: Data on sky diver: $M = 70 \cdot \mathrm{kg}$ $\quad k = 0.25 \cdot \dfrac{\mathrm{N \cdot s^2}}{\mathrm{m^2}}$

Find: Maximum speed; speed after 100 m; plot speed as function of time and distance.

Solution: Use given data; integrate equation of motion by separating variables.

Treat the sky diver as a system; apply Newton's 2nd law:

Newton's 2nd law for the sky diver (mass M) is (ignoring buoyancy effects): $M \cdot \dfrac{dV}{dt} = M \cdot g - k \cdot V^2$ (1)



(a) For terminal speed V_t, acceleration is zero, so $M \cdot g - k \cdot V^2 = 0$ so $V_t = \sqrt{\dfrac{M \cdot g}{k}}$

$$V_t = \left(70 \cdot \mathrm{kg} \times 9.81 \cdot \frac{\mathrm{m}}{\mathrm{s^2}} \times \frac{\mathrm{m^2}}{0.25 \cdot \mathrm{N \cdot s^2}} \cdot \frac{\mathrm{N \cdot s^2}}{\mathrm{kg \times m}}\right)^{\frac{1}{2}} \qquad V_t = 52.4 \frac{\mathrm{m}}{\mathrm{s}}$$

(b) For V at $y = 100$ m we need to find $V(y)$. From (1) $M \cdot \dfrac{dV}{dt} = M \cdot \dfrac{dV}{dy} \cdot \dfrac{dy}{dt} = M \cdot V \cdot \dfrac{dV}{dt} = M \cdot g - k \cdot V^2$

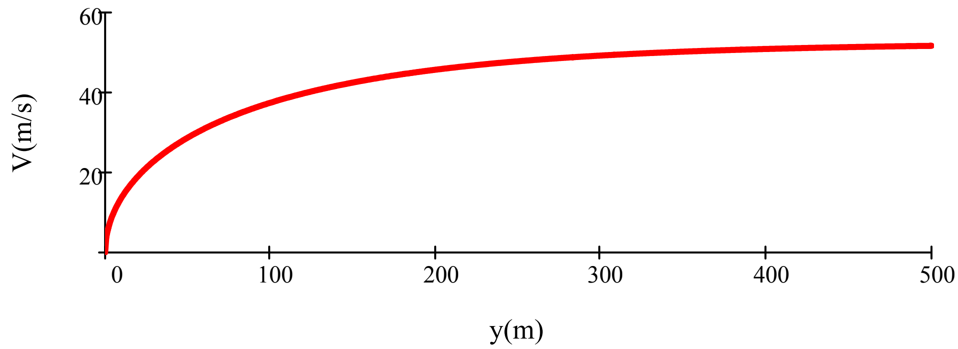
Separating variables and integrating:

$$\int_0^V \frac{V}{1 - \dfrac{k \cdot V^2}{M \cdot g}} \, dV = \int_0^y g \, dy$$

so $\quad \ln\left(1 - \dfrac{k \cdot V^2}{M \cdot g}\right) = -\dfrac{2 \cdot k}{M} y \quad$ or $\quad V^2 = \dfrac{M \cdot g}{k} \cdot \left(1 - e^{-\frac{2 \cdot k \cdot y}{M}}\right)$

Hence $\quad V(y) = V_t \cdot \left(1 - e^{-\frac{2 \cdot k \cdot y}{M}}\right)^{\frac{1}{2}}$

For $y = 100$ m: $\quad V(100 \cdot \mathrm{m}) = 52.4 \cdot \dfrac{\mathrm{m}}{\mathrm{s}} \cdot \left(1 - e^{-2 \times 0.25 \cdot \frac{\mathrm{N \cdot s^2}}{\mathrm{m^2}} \times 100 \cdot \mathrm{m} \times \frac{1}{70 \cdot \mathrm{kg}} \times \frac{\mathrm{kg \cdot m}}{\mathrm{s^2 \cdot N}}}\right)^{\frac{1}{2}} \qquad V(100 \cdot \mathrm{m}) = 37.4 \cdot \dfrac{\mathrm{m}}{\mathrm{s}}$



(c) For $V(t)$ we need to integrate (1) with respect to t:

$$M\cdot\frac{dV}{dt} = M\cdot g - k\cdot V^2$$

Separating variables and integrating:

$$\int_0^V \frac{V}{\frac{M\cdot g}{k} - V^2}\,dV = \int_0^t 1\,dt$$

so

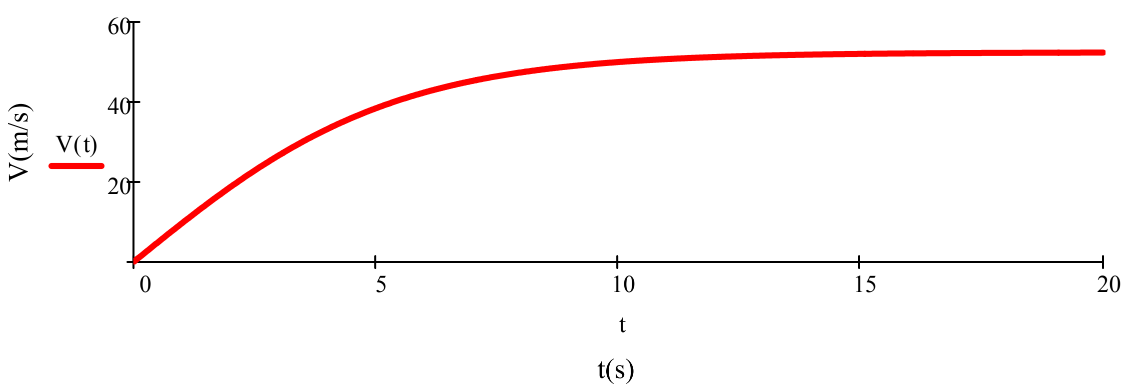
$$t = \frac{1}{2}\cdot\sqrt{\frac{M}{k\cdot g}}\cdot\ln\left(\left|\frac{\sqrt{\frac{M\cdot g}{k}} + V}{\sqrt{\frac{M\cdot g}{k}} - V}\right|\right) = \frac{1}{2}\cdot\sqrt{\frac{M}{k\cdot g}}\cdot\ln\left(\frac{|V_t + V|}{|V_t - V|}\right)$$

Rearranging

$$V(t) = V_t\cdot\frac{\left(e^{2\cdot\sqrt{\frac{k\cdot g}{M}}\cdot t} - 1\right)}{\left(e^{2\cdot\sqrt{\frac{k\cdot g}{M}}\cdot t} + 1\right)}$$

or

$$V(t) = V_t\cdot\tanh\left(V_t\cdot\frac{k}{M}\cdot t\right)$$



The two graphs can also be plotted in Excel.

Problem 1.15

[Difficulty: 5]

1.15 For Problem 1.14, the initial horizontal speed of the sky diver is 70 m/s. As she falls, the k value for the vertical drag remains as before, but the value for horizontal motion is $k = 0.05\ \mathrm{N \cdot s/m^2}$. Compute and plot the 2D trajectory of the sky diver.

Given: Data on sky diver: $M = 70 \cdot kg$ $\quad k_{vert} = 0.25 \cdot \dfrac{N \cdot s^2}{m^2}$ $\quad k_{horiz} = 0.05 \cdot \dfrac{N \cdot s^2}{m^2}$ $\quad U_0 = 70 \cdot \dfrac{m}{s}$

Find: Plot of trajectory.

Solution: Use given data; integrate equation of motion by separating variables.

Treat the sky diver as a system; apply Newton's 2nd law in horizontal and vertical directions:

Vertical: Newton's 2nd law for the sky diver (mass M) is (ignoring buoyancy effects): $M \cdot \dfrac{dV}{dt} = M \cdot g - k_{vert} \cdot V^2 \quad (1)$

For $V(t)$ we need to integrate (1) with respect to t:

Separating variables and integrating:
$$\int_0^V \frac{V}{\frac{M \cdot g}{k_{vert}} - V^2}\, dV = \int_0^t 1\, dt$$

so
$$t = \frac{1}{2} \cdot \sqrt{\frac{M}{k_{vert} \cdot g}} \cdot \ln\left(\left| \frac{\sqrt{\frac{M \cdot g}{k_{vert}}} + V}{\sqrt{\frac{M \cdot g}{k_{vert}}} - V} \right| \right)$$

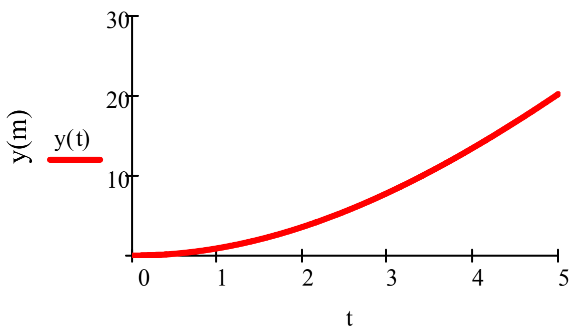
Rearranging
$$V(t) = \sqrt{\frac{M \cdot g}{k_{vert}}} \cdot \frac{\left(e^{2 \cdot \sqrt{\frac{k_{vert} \cdot g}{M}} \cdot t} - 1 \right)}{\left(e^{2 \cdot \sqrt{\frac{k_{vert} \cdot g}{M}} \cdot t} + 1 \right)}$$
so
$$V(t) = \sqrt{\frac{M \cdot g}{k_{vert}}} \cdot \tanh\left(\sqrt{\frac{k_{vert} \cdot g}{M}} \cdot t \right)$$

For $y(t)$ we need to integrate again: $\dfrac{dy}{dt} = V$ or $y = \displaystyle\int V\, dt$

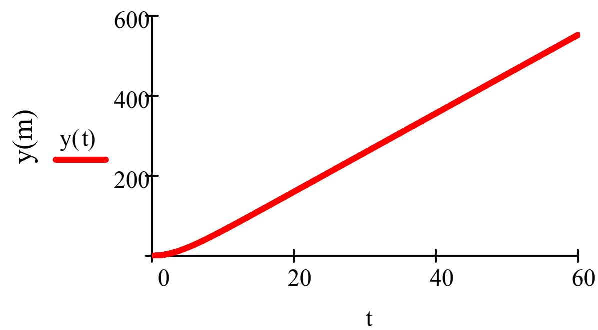
$$y(t) = \int_0^t V(t)\, dt = \int_0^t \sqrt{\frac{M \cdot g}{k_{vert}}} \cdot \tanh\left(\sqrt{\frac{k_{vert} \cdot g}{M}} \cdot t \right) dt = \sqrt{\frac{M \cdot g}{k_{vert}}} \cdot \ln\left(\cosh\left(\sqrt{\frac{k_{vert} \cdot g}{M}} \cdot t \right) \right)$$

$$y(t) = \sqrt{\frac{M \cdot g}{k_{vert}}} \cdot \ln\left(\cosh\left(\sqrt{\frac{k_{vert} \cdot g}{M}} \cdot t \right) \right)$$

After the first few seconds we reach steady state:



t(s)



t(s)

Horizontal: Newton's 2nd law for the sky diver (mass M) is:

$$M \cdot \frac{dU}{dt} = -k_{horiz} \cdot U^2 \qquad (2)$$

For $U(t)$ we need to integrate (2) with respect to t:

Separating variables and integrating: $\displaystyle\int_{U_0}^{U} \frac{1}{U^2}\, dU = \int_0^t -\frac{k_{horiz}}{M}\, dt$ so $\displaystyle -\frac{k_{horiz}}{M} \cdot t = -\frac{1}{U} + \frac{1}{U_0}$

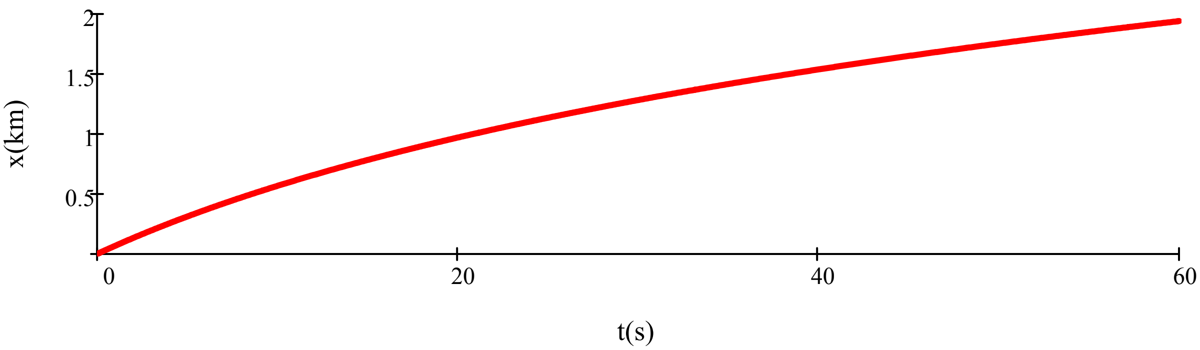
Rearranging

$$U(t) = \frac{U_0}{1 + \frac{k_{horiz} \cdot U_0}{M} \cdot t}$$

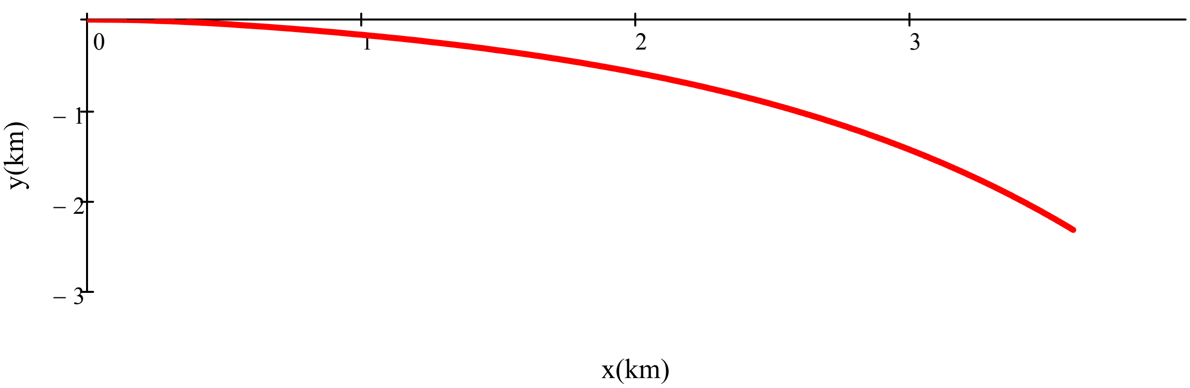
For $x(t)$ we need to integrate again: $\displaystyle\frac{dx}{dt} = U$ or $\displaystyle x = \int U\, dt$

$$x(t) = \int_0^t U(t)\, dt = \int_0^t \frac{U_0}{1 + \frac{k_{horiz} \cdot U_0}{M} \cdot t}\, dt = \frac{M}{k_{horiz}} \cdot \ln\left(\frac{k_{horiz} \cdot U_0}{M} \cdot t + 1\right)$$

$$x(t) = \frac{M}{k_{horiz}} \cdot \ln\left(\frac{k_{horiz} \cdot U_0}{M} \cdot t + 1\right)$$



Plotting the trajectory:



These plots can also be done in Excel.

Problem 1.16

[Difficulty: 3]

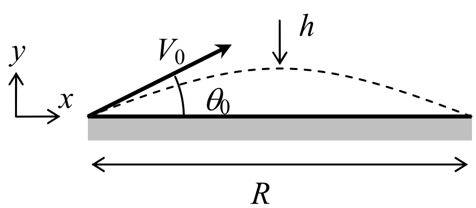
1.16 The English perfected the longbow as a weapon after the Medieval period. In the hands of a skilled archer, the longbow was reputed to be accurate at ranges to 100 m or more. If the maximum altitude of an arrow is less than $h = 10$ m while traveling to a target 100 m away from the archer, and neglecting air resistance, estimate the speed and angle at which the arrow must leave the bow. Plot the required release speed and angle as a function of height h.

Given: Long bow at range, R = 100 m. Maximum height of arrow is h = 10 m. Neglect air resistance.

Find: Estimate of (a) speed, and (b) angle, of arrow leaving the bow.

Plot: (a) release speed, and (b) angle, as a function of h

Solution: Let $\vec{V_0} = u_0\hat{i} + v_0\hat{j} = V_0(\cos\theta_0\hat{i} + \sin\theta_0\hat{j})$



$$\Sigma F_y = m\tfrac{dv}{dt} = -mg, \text{ so } v = v_0 - gt, \text{ and } t_f = 2t_{v=0} = 2v_0/g$$

Also, $$mv\frac{dv}{dy} = -mg,\ v\,dv = -g\,dy,\ 0 - \frac{v_0^2}{2} = -gh$$

Thus $$h = v_0^2/2g \tag{1}$$

$$\Sigma F_x = m\frac{du}{dt} = 0,\ \text{ so } u = u_0 = \text{const, and } R = u_0 t_f = \frac{2u_0 v_0}{g} \tag{2}$$

From Eq. 1: $$v_0^2 = 2gh \qquad (3)$$

From Eq. 2: $$u_0 = \frac{gR}{2v_0} = \frac{gR}{2\sqrt{2gh}} \qquad \therefore\ u_0^2 = \frac{gR^2}{8h}$$

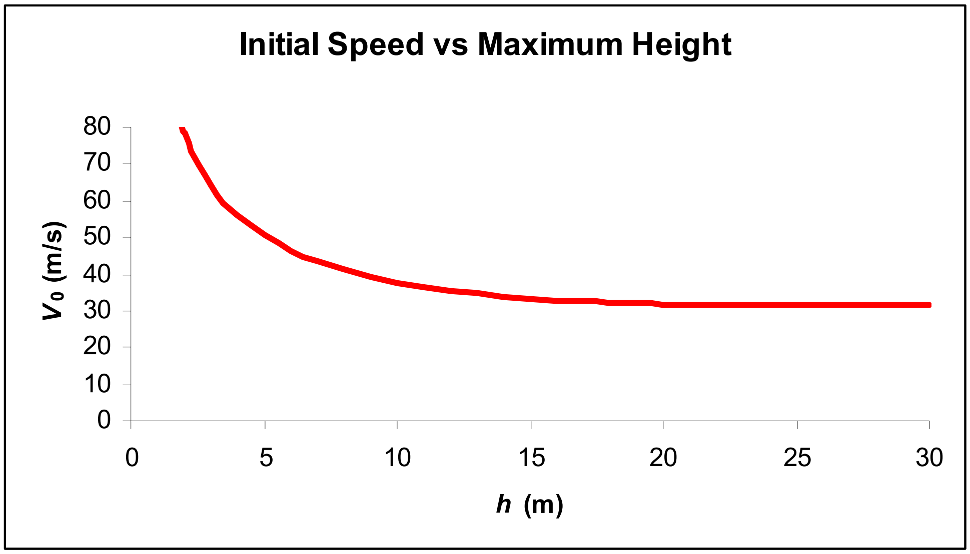
Then $$V_0^2 = u_0^2 + v_0^2 = \frac{gR^2}{8h} + 2gh \quad \text{and} \quad V_0 = \left(2gh + \frac{gR^2}{8h}\right)^{\frac{1}{2}} \tag{4}$$

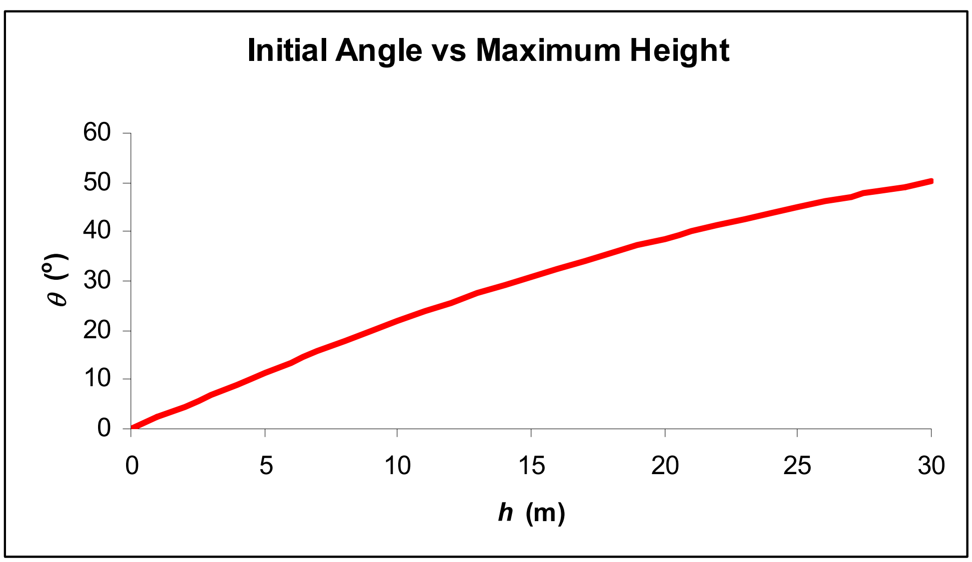
$$V_0 = \left(2\times 9.81\frac{\text{m}}{\text{s}^2}\times 10\text{ m} + \frac{9.81}{8}\frac{\text{m}}{\text{s}^2}\times 100^2\text{ m}^2\times\frac{1}{10\text{ m}}\right)^{\frac{1}{2}} = 37.7\frac{\text{m}}{\text{s}}$$

From Eq. 3: $$v_0 = \sqrt{2gh} = V_0\sin\theta,\ \theta = \sin^{-1}\frac{\sqrt{2gh}}{V_0} \tag{5}$$

$$\theta = \sin^{-1}\left[\left(2\times 9.81\frac{\text{m}}{\text{s}^2}\times 10\text{ m}\right)^{\frac{1}{2}}\times\frac{\text{s}}{37.7\text{ m}}\right] = 21.8°$$

Plots of $V_0 = V_0(h)$ (Eq. 4) and $\theta_0 = \theta_0(h)$ (Eq. 5) are presented below:





Problem 1.17

[Difficulty: 2]

1.17 For each quantity listed, indicate dimensions using mass as a primary dimension, and give typical SI and English units:
(a) Power
(b) Pressure
(c) Modulus of elasticity
(d) Angular velocity
(e) Energy
(f) Moment of a force
(g) Momentum
(h) Shear stress
(i) Strain
(j) Angular momentum

Given: Basic dimensions M, L, t and T.

Find: Dimensional representation of quantities below, and typical units in SI and English systems.

Solution:

(a) Power

$$\text{Power} = \frac{\text{Energy}}{\text{Time}} = \frac{\text{Force} \times \text{Distance}}{\text{Time}} = \frac{F \cdot L}{t}$$

From Newton's 2nd law $\quad \text{Force} = \text{Mass} \times \text{Acceleration} \quad$ so $\quad F = \frac{M \cdot L}{t^2}$

Hence $\quad \text{Power} = \frac{F \cdot L}{t} = \frac{M \cdot L \cdot L}{t^2 \cdot t} = \frac{M \cdot L^2}{t^3} \qquad \frac{kg \cdot m^2}{s^3} \qquad \frac{slug \cdot ft^2}{s^3}$

(b) Pressure

$$\text{Pressure} = \frac{\text{Force}}{\text{Area}} = \frac{F}{L^2} = \frac{M \cdot L}{t^2 \cdot L^2} = \frac{M}{L \cdot t^2} \qquad \frac{kg}{m \cdot s^2} \qquad \frac{slug}{ft \cdot s^2}$$

(c) Modulus of elasticity

$$\text{Pressure} = \frac{\text{Force}}{\text{Area}} = \frac{F}{L^2} = \frac{M \cdot L}{t^2 \cdot L^2} = \frac{M}{L \cdot t^2} \qquad \frac{kg}{m \cdot s^2} \qquad \frac{slug}{ft \cdot s^2}$$

(d) Angular velocity

$$\text{AngularVelocity} = \frac{\text{Radians}}{\text{Time}} = \frac{1}{t} \qquad \frac{1}{s} \qquad \frac{1}{s}$$

(e) Energy

$$\text{Energy} = \text{Force} \times \text{Distance} = F \cdot L = \frac{M \cdot L \cdot L}{t^2} = \frac{M \cdot L^2}{t^2} \qquad \frac{kg \cdot m^2}{s^2} \qquad \frac{slug \cdot ft^2}{s^2}$$

(f) Moment of a force

$$\text{MomentOfForce} = \text{Force} \times \text{Length} = F \cdot L = \frac{M \cdot L \cdot L}{t^2} = \frac{M \cdot L^2}{t^2} \qquad \frac{kg \cdot m^2}{s^2} \qquad \frac{slug \cdot ft^2}{s^2}$$

(g) Momentum

$$\text{Momentum} = \text{Mass} \times \text{Velocity} = M \cdot \frac{L}{t} = \frac{M \cdot L}{t} \qquad \frac{kg \cdot m}{s} \qquad \frac{slug \cdot ft}{s}$$

(h) Shear stress

$$\text{ShearStress} = \frac{\text{Force}}{\text{Area}} = \frac{F}{L^2} = \frac{M \cdot L}{t^2 \cdot L^2} = \frac{M}{L \cdot t^2} \qquad \frac{kg}{m \cdot s^2} \qquad \frac{slug}{ft \cdot s^2}$$

(i) Strain

$$\text{Strain} = \frac{\text{LengthChange}}{\text{Length}} = \frac{L}{L}$$ Dimensionless

(j) Angular momentum

$$\text{AngularMomentum} = \text{Momentum} \times \text{Distance} = \frac{M \cdot L}{t} \cdot L = \frac{M \cdot L^2}{t} \qquad \frac{kg \cdot m^2}{s} \qquad \frac{slugs \cdot ft^2}{s}$$

Problem 1.18

[Difficulty: 2]

1.18 For each quantity listed, indicate dimensions using force as a primary dimension, and give typical SI and English units:
(a) Power
(b) Pressure
(c) Modulus of elasticity
(d) Angular velocity
(e) Energy
(f) Momentum
(g) Shear stress
(h) Specific heat
(i) Thermal expansion coefficient
(j) Angular momentum

Given: Basic dimensions F, L, t and T.

Find: Dimensional representation of quantities below, and typical units in SI and English systems.

Solution:

(a) Power $\quad \text{Power} = \frac{\text{Energy}}{\text{Time}} = \frac{\text{Force} \times \text{Distance}}{\text{Time}} = \frac{F \cdot L}{t} \quad \frac{N \cdot m}{s} \quad \frac{lbf \cdot ft}{s}$

(b) Pressure $\quad \text{Pressure} = \frac{\text{Force}}{\text{Area}} = \frac{F}{L^2} \quad \frac{N}{m^2} \quad \frac{lbf}{ft^2}$

(c) Modulus of elasticity $\quad \text{Pressure} = \frac{\text{Force}}{\text{Area}} = \frac{F}{L^2} \quad \frac{N}{m^2} \quad \frac{lbf}{ft^2}$

(d) Angular velocity $\quad \text{AngularVelocity} = \frac{\text{Radians}}{\text{Time}} = \frac{1}{t} \quad \frac{1}{s} \quad \frac{1}{s}$

(e) Energy $\quad \text{Energy} = \text{Force} \times \text{Distance} = F \cdot L \quad N \cdot m \quad lbf \cdot ft$

(f) Momentum $\quad \text{Momentum} = \text{Mass} \times \text{Velocity} = M \cdot \frac{L}{t}$

From Newton's 2nd law $\quad \text{Force} = \text{Mass} \times \text{Acceleration}$ so $F = M \cdot \frac{L}{t^2}$ or $M = \frac{F \cdot t^2}{L}$

Hence $\quad \text{Momentum} = M \cdot \frac{L}{t} = \frac{F \cdot t^2 \cdot L}{L \cdot t} = F \cdot t \quad N \cdot s \quad lbf \cdot s$

(g) Shear stress $\quad \text{ShearStress} = \frac{\text{Force}}{\text{Area}} = \frac{F}{L^2} \quad \frac{N}{m^2} \quad \frac{lbf}{ft^2}$

(h) Specific heat $\quad \text{SpecificHeat} = \frac{\text{Energy}}{\text{Mass} \times \text{Temperature}} = \frac{F \cdot L}{M \cdot T} = \frac{F \cdot L}{\left(\frac{F \cdot t^2}{L}\right) \cdot T} = \frac{L^2}{t^2 \cdot T} \quad \frac{m^2}{s^2 \cdot K} \quad \frac{ft^2}{s^2 \cdot R}$

(i) Thermal expansion coefficient $\quad \text{ThermalExpansionCoefficient} = \frac{\frac{\text{LengthChange}}{\text{Length}}}{\text{Temperature}} = \frac{1}{T} \quad \frac{1}{K} \quad \frac{1}{R}$

(j) Angular momentum $\quad \text{AngularMomentum} = \text{Momentum} \times \text{Distance} = F \cdot t \cdot L \quad N \cdot m \cdot s \quad lbf \cdot ft \cdot s$

Problem 1.19

[Difficulty: 1]

1.19 Derive the following conversion factors:
(a) Convert a viscosity of 1 m^2/s to ft^2/s.
(b) Convert a power of 100 W to horsepower.
(c) Convert a specific energy of 1 kJ/kg to Btu/lbm.

Given: Viscosity, power, and specific energy data in certain units

Find: Convert to different units

Solution:

Using data from tables (e.g. Table G.2)

(a) $$1 \cdot \frac{m^2}{s} = 1 \cdot \frac{m^2}{s} \times \left(\frac{\frac{1}{12} \cdot ft}{0.0254 \cdot m} \right)^2 = 10.76 \cdot \frac{ft^2}{s}$$

(b) $$100 \cdot W = 100 \cdot W \times \frac{1 \cdot hp}{746 \cdot W} = 0.134 \cdot hp$$

(c) $$1 \cdot \frac{kJ}{kg} = 1 \cdot \frac{kJ}{kg} \times \frac{1000 \cdot J}{1 \cdot kJ} \times \frac{1 \cdot Btu}{1055 \cdot J} \times \frac{0.454 \cdot kg}{1 \cdot lbm} = 0.43 \cdot \frac{Btu}{lbm}$$

Problem 1.20

[Difficulty: 1]

1.20 Derive the following conversion factors:

(a) Convert a pressure of 1 psi to kPa.

(b) Convert a volume of 1 liter to gallons.

(c) Convert a viscosity of 1 $\mathrm{lbf \cdot s/ft^2}$ to $\mathrm{N \cdot s/m^2}$.

Given: Pressure, volume and density data in certain units

Find: Convert to different units

Solution:

Using data from tables (e.g. Table G.2)

(a) $$1 \cdot \mathrm{psi} = 1 \cdot \mathrm{psi} \times \frac{6895 \cdot \mathrm{Pa}}{1 \cdot \mathrm{psi}} \times \frac{1 \cdot \mathrm{kPa}}{1000 \cdot \mathrm{Pa}} = 6.89 \cdot \mathrm{kPa}$$

(b) $$1 \cdot \mathrm{liter} = 1 \cdot \mathrm{liter} \times \frac{1 \cdot \mathrm{quart}}{0.946 \cdot \mathrm{liter}} \times \frac{1 \cdot \mathrm{gal}}{4 \cdot \mathrm{quart}} = 0.264 \cdot \mathrm{gal}$$

(c) $$1 \cdot \frac{\mathrm{lbf} \cdot \mathrm{s}}{\mathrm{ft}^2} = 1 \cdot \frac{\mathrm{lbf} \cdot \mathrm{s}}{\mathrm{ft}^2} \times \frac{4.448 \cdot \mathrm{N}}{1 \cdot \mathrm{lbf}} \times \left(\frac{\frac{1}{12} \cdot \mathrm{ft}}{0.0254 \cdot \mathrm{m}} \right)^2 = 47.9 \cdot \frac{\mathrm{N} \cdot \mathrm{s}}{\mathrm{m}^2}$$

Problem 1.21

[Difficulty: 1]

1.21 Derive the following conversion factors:
(a) Convert a specific heat of 4.18 kJ/kg · K to Btu/lbm · °R.
(b) Convert a speed of 30 m/s to mph.
(c) Convert a volume of 5.0 L to in^3.

Given: Specific heat, speed, and volume data in certain units

Find: Convert to different units

Solution:

Using data from tables (e.g. Table G.2)

(a) $$4.18 \cdot \frac{kJ}{kg \cdot K} = 4.18 \cdot \frac{kJ}{kg \cdot K} \times \frac{1 \cdot Btu}{1.055 \cdot kJ} \times \frac{1 \cdot kg}{2.2046 \cdot lbm} \times \frac{1 \cdot K}{1.8 \cdot R} = 0.998 \cdot \frac{Btu}{lbm \cdot R}$$

(b) $$30 \cdot \frac{m}{s} = 30 \cdot \frac{m}{s} \times \frac{3.281 \cdot ft}{1 \cdot m} \cdot \frac{1 \cdot mi}{5280 \cdot ft} \cdot \frac{3600 \cdot s}{hr} = 67.1 \cdot \frac{mi}{hr}$$

(c) $$5 \cdot L = 5 \cdot L \times \frac{1 \cdot m^3}{1000 \cdot L} \times \left(\frac{100 \cdot cm}{1 \cdot m} \times \frac{1 \cdot in}{2.54 \cdot cm} \right)^3 = 305 \cdot in^3$$

Problem 1.22

[Difficulty: 1]

1.22 Express the following in SI units:
(a) 5 acre · ft
(b) 150 in^3/s
(c) 3 gpm
(d) 3 mph/s

Given: Quantities in English Engineering (or customary) units.

Find: Quantities in SI units.

Solution: Use Table G.2 and other sources (e.g., Machinery's Handbook, Mark's Standard Handbook)

(a) $$3.7\cdot\text{acre}\cdot\text{ft} = 3.7\cdot\text{acre} \times \frac{4047\cdot m^2}{1\cdot\text{acre}} \times \frac{0.3048\cdot m}{1\cdot ft} = 4.56 \times 10^3\cdot m^3$$

(b) $$150\cdot\frac{in^3}{s} = 150\cdot\frac{in^3}{s} \times \left(\frac{0.0254\cdot m}{1\cdot in}\right)^3 = 0.00246\cdot\frac{m^3}{s}$$

(c) $$3\cdot\text{gpm} = 3\cdot\frac{\text{gal}}{\text{min}} \times \frac{231\cdot in^3}{1\cdot\text{gal}} \times \left(\frac{0.0254\cdot m}{1\cdot in}\right)^3\cdot\frac{1\cdot\text{min}}{60\cdot s} = 0.000189\cdot\frac{m^3}{s}$$

(d) $$3\cdot\frac{\text{mph}}{s} = 3\cdot\frac{\text{mile}}{\text{hr}\cdot s} \times \frac{1609\cdot m}{1\cdot\text{mile}} \times \frac{1\cdot\text{hr}}{3600\cdot s} = 1.34\cdot\frac{m}{s^2}$$

Problem 1.23

[Difficulty: 1]

1.23 Express the following in SI units:
(a) 100 cfm (ft^3/min)
(b) 5 gal
(c) 65 mph
(d) 5.4 acres

Given: Quantities in English Engineering (or customary) units.

Find: Quantities in SI units.

Solution: Use Table G.2 and other sources (e.g., Google)

(a)
$$100\cdot\frac{ft^3}{m} = 100\cdot\frac{ft^3}{min} \times \left(\frac{0.0254\cdot m}{1\cdot in} \times \frac{12\cdot in}{1\cdot ft}\right)^3 \times \frac{1\cdot min}{60\cdot s} = 0.0472\cdot\frac{m^3}{s}$$

(b)
$$5\cdot gal = 5\cdot gal \times \frac{231\cdot in^3}{1\cdot gal} \times \left(\frac{0.0254\cdot m}{1\cdot in}\right)^3 = 0.0189\cdot m^3$$

(c)
$$65\cdot mph = 65\cdot\frac{mile}{hr} \times \frac{1852\cdot m}{1\cdot mile} \times \frac{1\cdot hr}{3600\cdot s} = 29.1\cdot\frac{m}{s}$$

(d)
$$5.4\cdot acres = 5.4\cdot acre \times \frac{4047\cdot m^3}{1\cdot acre} = 2.19 \times 10^4\cdot m^2$$

Problem 1.24

[Difficulty: 1]

1.24 Express the following in BG units:
(a) 50 m^2
(b) 250 cc
(c) 100 kW
(d) 5 kg/m^2

Given: Quantities in SI (or other) units.

Find: Quantities in BG units.

Solution: Use Table G.2.

(a)
$$50 \cdot m^2 = 50 \cdot m^2 \times \left(\frac{1 \cdot in}{0.0254 m} \times \frac{1 \cdot ft}{12 \cdot in}\right)^2 = 538 \cdot ft^2$$

(b)
$$250 \cdot cc = 250 \cdot cm^3 \times \left(\frac{1 \cdot m}{100 \cdot cm} \times \frac{1 \cdot in}{0.0254 m} \times \frac{1 \cdot ft}{12 \cdot in}\right)^3 = 8.83 \times 10^{-3} \cdot ft^3$$

(c)
$$100 \cdot kW = 100 \cdot kW \times \frac{1000 \cdot W}{1 \cdot kW} \times \frac{1 \cdot hp}{746 \cdot W} = 134 \cdot hp$$

(d)
$$5 \cdot \frac{kg}{m^2} = 5 \cdot \frac{kg}{m^2} \times \left(\frac{0.0254 m}{1 \cdot in} \times \frac{12 \cdot in}{1 \cdot ft}\right)^2 \times \frac{1 \cdot slug}{14.95 kg} = 0.0318 \frac{slug}{ft^2}$$

Problem 1.25

[Difficulty: 1]

1.25 Express the following in BG units:
(a) 180 cc/min
(b) 300 kW · hr
(c) 50 N · s/m^2
(d) 40 m^2 · hr

Given: Quantities in SI (or other) units.

Find: Quantities in BG units.

Solution: Use Table G.2.

(a)
$$180\cdot cc = 180\cdot cm^3 \times \left(\frac{1\cdot m}{100\cdot cm} \times \frac{1\cdot in}{0.0254\,m} \times \frac{1\cdot ft}{12\cdot in}\right)^3 = 6.36\times 10^{-3}\cdot ft^3$$

(b)
$$300\cdot kW = 300\cdot kW \times \frac{1000\cdot W}{1\cdot kW} \times \frac{1\cdot hp}{746\cdot W} = 402\cdot hp$$

(c)
$$50\cdot\frac{N\cdot s}{m^2} = 50\cdot\frac{N\cdot s}{m^2} \times \frac{1\cdot lbf}{4.448\,N} \times \left(\frac{0.0254\,m}{1\cdot in} \times \frac{12\cdot in}{1\cdot ft}\right)^2 = 1.044\,\frac{lbf\cdot s}{ft^2}$$

(d)
$$40\cdot m^2\cdot hr = 40\cdot m^2 \times \left(\frac{1\cdot in}{0.0254\,m} \times \frac{1\cdot ft}{12\cdot in}\right)^2\cdot hr = 431\cdot ft^2\cdot hr$$

Problem 1.26

[Difficulty: 2]

1.26 While you're waiting for the ribs to cook, you muse about the propane tank of your barbecue. You're curious about the volume of propane versus the actual tank size. Find the liquid propane volume when full (the weight of the propane is specified on the tank). Compare this to the tank volume (take some measurements, and approximate the tank shape as a cylinder with a hemisphere on each end). Explain the discrepancy.

Given: Geometry of tank, and weight of propane.

Find: Volume of propane, and tank volume; explain the discrepancy.

Solution: Use Table G.2 and other sources (e.g., Google) as needed.

The author's tank is approximately 12 in in diameter, and the cylindrical part is about 8 in. The weight of propane specified is 17 lb.

The tank diameter is $D = 12 \cdot in$

The tank cylindrical height is $L = 8 \cdot in$

The mass of propane is $m_{prop} = 17 \cdot lbm$

The specific gravity of propane is $SG_{prop} = 0.495$

The density of water is $\rho = 998 \cdot \frac{kg}{m^3}$

The volume of propane is given by

$$V_{prop} = \frac{m_{prop}}{\rho_{prop}} = \frac{m_{prop}}{SG_{prop} \cdot \rho}$$

$$V_{prop} = 17 \cdot lbm \times \frac{1}{0.495} \times \frac{m^3}{998 \cdot kg} \times \frac{0.454 \cdot kg}{1 \cdot lbm} \times \left(\frac{1 \cdot in}{0.0254 \cdot m}\right)^3 \qquad V_{prop} = 953 \cdot in^3$$

The volume of the tank is given by a cylinder diameter D length L, $\pi D^2 L/4$ and a sphere (two halves) given by $\pi D^3/6$

$$V_{tank} = \frac{\pi \cdot D^2}{4} \cdot L + \frac{\pi \cdot D^3}{6}$$

$$V_{tank} = \frac{\pi \cdot (12 \cdot in)^2}{4} \cdot 8 \cdot in + \pi \cdot \frac{(12 \cdot in)^3}{6} \qquad V_{tank} = 1810 \cdot in^3$$

The ratio of propane to tank volumes is $\frac{V_{prop}}{V_{tank}} = 53 \cdot \%$

This seems low, and can be explained by a) tanks are not filled completely, b) the geometry of the tank gave an overestimate of the volume (the ends are not really hemispheres, and we have not allowed for tank wall thickness).

Problem 1.27

[Difficulty: 1]

1.27 A farmer needs 4 cm of rain per week on his farm, with 10 hectares of crops. If there is a drought, how much water (L/min) will have to be supplied to maintain his crops?

Given: Acreage of land, and water needs.

Find: Water flow rate (L/min) to water crops.

Solution: Use Table G.2 and other sources (e.g., Machinery's Handbook, Mark's Standard Handbook) as needed.

The volume flow rate needed is $Q = \frac{4 \cdot \text{cm}}{\text{week}} \times 10 \cdot \text{hectare}$

Performing unit conversions $Q = \frac{4 \cdot \text{cm} \times 10 \cdot \text{hectare}}{\text{week}} = \frac{0.04 \cdot \text{m} \times 10 \cdot \text{hectare}}{\text{week}} \times \frac{1 \times 10^4 \cdot \text{m}^2}{1 \cdot \text{hectare}} \times \frac{1000 \cdot \text{L}}{\text{m}^3} \times \frac{1 \cdot \text{week}}{7 \cdot \text{day}} \times \frac{1 \cdot \text{day}}{24 \cdot \text{hr}} \times \frac{1 \cdot \text{hr}}{60 \cdot \text{min}}$

$Q = 397 \cdot \frac{\text{L}}{\text{min}}$

Problem 1.28

[Difficulty: 1]

1.28 Derive the following conversion factors:

(a) Convert a volume flow rate in cubic inches per minute to cubic millimeters per minute.

(b) Convert a volume flow rate in cubic meters per second to gallons per minute (gpm).

(c) Convert a volume flow rate in liters per minute to gpm.

(d) Convert a volume flow rate of air in standard cubic feet per minute (SCFM) to cubic meters per hour. A standard cubic foot of gas occupies one cubic foot at standard temperature and pressure ($T = 15°\text{C}$ and $p = 101.3$ kPa absolute).

Given: Data in given units

Find: Convert to different units

Solution:

(a) $$1\cdot\frac{\text{in}^3}{\text{min}} = 1\cdot\frac{\text{in}^3}{\text{min}} \times \left(\frac{0.0254\,\text{m}}{1\cdot\text{in}} \times \frac{1000\,\text{mm}}{1\cdot\text{m}}\right)^3 \times \frac{1\cdot\text{min}}{60\cdot\text{s}} = 273\cdot\frac{\text{mm}^3}{\text{s}}$$

(b) $$1\cdot\frac{\text{m}^3}{\text{s}} = 1\cdot\frac{\text{m}^3}{\text{s}} \times \frac{1\cdot\text{gal}}{4\times 0.000946\text{m}^3} \times \frac{60\cdot\text{s}}{1\cdot\text{min}} = 15850\,\text{gpm}$$

(c) $$1\cdot\frac{\text{liter}}{\text{min}} = 1\cdot\frac{\text{liter}}{\text{min}} \times \frac{1\cdot\text{gal}}{4\times 0.946\cdot\text{liter}} \times \frac{60\cdot\text{s}}{1\cdot\text{min}} = 0.264\cdot\text{gpm}$$

(d) $$1\cdot\text{SCFM} = 1\cdot\frac{\text{ft}^3}{\text{min}} \times \left(\frac{0.0254\cdot\text{m}}{\frac{1}{12}\cdot\text{ft}}\right)^3 \times \frac{60\cdot\text{min}}{1\cdot\text{hr}} = 1.70\cdot\frac{\text{m}^3}{\text{hr}}$$

Problem 1.29

[Difficulty: 1]

1.29 The density of mercury is given as 26.3 slug/ft^3. Calculate the specific gravity and the specific volume in m^3/kg of the mercury. Calculate the specific weight in lbf/ft^3 on Earth and on the moon. Acceleration of gravity on the moon is 5.47 ft/s^2.

Given: Density of mercury.

Find: Specific gravity, volume and weight.

Solution: Use basic definitions

$SG = \frac{\rho}{\rho_w}$ From Appendix A $\rho_w = 1.94 \frac{slug}{ft^3}$ so $SG = \frac{26.3}{1.94}$ $SG = 13.6$

$v = \frac{1}{\rho}$ so $v = \frac{1}{26.3} \cdot \frac{ft^3}{slug} \times \left(\frac{0.3048 m}{1 \cdot ft}\right)^3 \times \frac{1 \cdot slug}{32.2 \cdot lbm} \times \frac{1 \cdot lbm}{0.4536 kg}$ $v = 7.37 \times 10^{-5} \frac{m^3}{kg}$

$\gamma = \rho \cdot g$

Hence on earth $\gamma_E = 26.3 \cdot \frac{slug}{ft^3} \times 32.2 \cdot \frac{ft}{s^2} \times \frac{1 \cdot lbf \cdot s^2}{1 \cdot slug \cdot ft}$ $\gamma_E = 847 \frac{lbf}{ft^3}$

On the moon $\gamma_M = 26.3 \cdot \frac{slug}{ft^3} \times 5.47 \cdot \frac{ft}{s^2} \times \frac{1 \cdot lbf \cdot s^2}{1 \cdot slug \cdot ft}$ $\gamma_M = 144 \frac{lbf}{ft^3}$

Note that mass-based quantities are independent of gravity

Problem 1.30

[Difficulty: 1]

1.30 The kilogram force is commonly used in Europe as a unit of force. (As in the U.S. customary system, where 1 lbf is the force exerted by a mass of 1 lbm in standard gravity, 1 kgf is the force exerted by a mass of 1 kg in standard gravity.) Moderate pressures, such as those for auto or truck tires, are conveniently expressed in units of kgf/cm^2. Convert 32 psig to these units.

Given: Definition of kgf.

Find: Conversion from psig to kgf/cm^2.

Solution: Use Table G.2.

Define kgf $\quad \text{kgf} = 1\cdot\text{kg} \times 9.81\cdot\frac{\text{m}}{\text{s}^2} \qquad \text{kgf} = 9.81\text{N}$

Then

$$32\cdot\frac{\text{lbf}}{\text{in}^2} \times \frac{4.448\,\text{N}}{1\cdot\text{lbf}} \times \frac{1\cdot\text{kgf}}{9.81\cdot\text{N}} \times \left(\frac{12\cdot\text{in}}{1\cdot\text{ft}} \times \frac{1\cdot\text{ft}}{0.3048\,\text{m}} \times \frac{1\cdot\text{m}}{100\cdot\text{cm}}\right)^2 = 2.25\frac{\text{kgf}}{\text{cm}^2}$$

Problem 1.31

[Difficulty: 3]

1.31 In Section 1.6 we learned that the Manning equation computes the flow speed V (m/s) in a canal made from unfinished concrete, given the hydraulic radius R_h (m), the channel slope S_0, and a Manning resistance coefficient constant value $n \approx 0.014$. For a canal with $R_h = 7.5$ m and a slope of 1/10, find the flow speed. Compare this result with that obtained using the same n value, but with R_h first converted to ft, with the answer assumed to be in ft/s. Finally, find the value of n if we wish to *correctly* use the equation for BG units (and compute V to check!).

Given: Information on canal geometry.

Find: Flow speed using the Manning equation, correctly and incorrectly!

Solution: Use Table G.2 and other sources (e.g., Google) as needed.

The Manning equation is $$V = \frac{R_h^{\frac{2}{3}} \cdot S_0^{\frac{1}{2}}}{n}$$ which assumes R_h in meters and V in m/s.

The given data is $R_h = 7.5 \cdot m$ $\quad S_0 = \frac{1}{10}$ $\quad n = 0.014$

Hence $$V = \frac{7.5^{\frac{2}{3}} \cdot \left(\frac{1}{10}\right)^{\frac{1}{2}}}{0.014} \qquad V = 86.5 \cdot \frac{m}{s}$$

(Note that we don't cancel units; we just write m/s next to the answer! Note also this is a very high speed due to the extreme slope S_0.)

Using the equation incorrectly: $$R_h = 7.5 \cdot m \times \frac{1 \cdot in}{0.0254 \cdot m} \times \frac{1 \cdot ft}{12 \cdot in} \qquad R_h = 24.6 \cdot ft$$

Hence $$V = \frac{24.6^{\frac{2}{3}} \cdot \left(\frac{1}{10}\right)^{\frac{1}{2}}}{0.014} \qquad V = 191 \cdot \frac{ft}{s}$$

(Note that we again don't cancel units; we just write ft/s next to the answer!)

This incorrect use does not provide the correct answer $$V = 191 \cdot \frac{ft}{s} \times \frac{12 \cdot in}{1 \cdot ft} \times \frac{0.0254 \cdot m}{1 \cdot in} \qquad V = 58.2 \frac{m}{s}$$ which is wrong!

This demonstrates that for this "engineering" equation we must be careful in its use!

To generate a Manning equation valid for R_h in ft and V in ft/s, we need to do the following:

$$V\left(\frac{ft}{s}\right) = V\left(\frac{m}{s}\right) \times \frac{1 \cdot in}{0.0254 \cdot m} \times \frac{1 \cdot ft}{12 \cdot in} = \frac{R_h(m)^{\frac{2}{3}} \cdot S_0^{\frac{1}{2}}}{n} \times \left(\frac{1 \cdot in}{0.0254 \cdot m} \times \frac{1 \cdot ft}{12 \cdot in}\right)$$

$$V\left(\frac{ft}{s}\right) = \frac{R_h(ft)^{\frac{2}{3}} \cdot S_0^{\frac{1}{2}}}{n} \times \left(\frac{1 \cdot in}{0.0254 \cdot m} \times \frac{1 \cdot ft}{12 \cdot in}\right)^{-\frac{2}{3}} \times \left(\frac{1 \cdot in}{0.0254 \cdot m} \times \frac{1 \cdot ft}{12 \cdot in}\right) = \frac{R_h(ft)^{\frac{2}{3}} \cdot S_0^{\frac{1}{2}}}{n} \times \left(\frac{1 \cdot in}{0.0254 \cdot m} \times \frac{1 \cdot ft}{12 \cdot in}\right)^{\frac{1}{3}}$$

In using this equation, we ignore the units and just evaluate the conversion factor $\left(\frac{1}{.0254} \cdot \frac{1}{12}\right)^{\frac{1}{3}} = 1.49$

Hence $V\left(\frac{ft}{s}\right) = \frac{1.49 \cdot R_h(ft)^{\frac{2}{3}} \cdot S_0^{\frac{1}{2}}}{n}$

Handbooks sometimes provide this form of the Manning equation for direct use with BG units. In our case we are asked to instead define a new value for n:

$n_{BG} = \frac{n}{1.49}$ $\quad n_{BG} = 0.0094$ $\quad$ where $\quad V\left(\frac{ft}{s}\right) = \frac{R_h(ft)^{\frac{2}{3}} \cdot S_0^{\frac{1}{2}}}{n_{BG}}$

Using this equation with Rh = 24.6 ft: $V = \frac{24.6^{\frac{2}{3}} \cdot \left(\frac{1}{10}\right)^{\frac{1}{2}}}{0.0094}$ $\quad V = 284 \frac{ft}{s}$

Converting to m/s $\quad V = 284 \cdot \frac{ft}{s} \times \frac{12 \cdot in}{1 \cdot ft} \times \frac{0.0254 \cdot m}{1 \cdot in}$ $\quad V = 86.6 \frac{m}{s}$ $\quad$ which is the correct answer!

Problem 1.32

[Difficulty: 2]

1.32 From thermodynamics, we know that the coefficient of performance of an ideal air conditioner (COP_{ideal}) is given by

$$COP_{ideal} = \frac{T_L}{T_H - T_L}$$

where T_L and T_H are the room and outside temperatures (absolute). If an AC is to keep a room at 20°C when it is 40°C outside, find the COP_{ideal}. Convert to an EER value, and compare this to a typical Energy Star–compliant EER value.

Given: Equation for COP_{ideal} and temperature data.

Find: COP_{ideal}, EER, and compare to a typical Energy Star compliant EER value.

Solution: Use the COP equation. Then use conversions from Table G.2 or other sources (e.g., www.energystar.gov) to find the EER.

The given data is $T_L = (20 + 273)\cdot K$ $\quad T_L = 293\cdot K$ $\quad T_H = (40 + 273)\cdot K$ $\quad T_H = 313\cdot K$

The COP_{Ideal} is

$$COP_{Ideal} = \frac{293}{313 - 293} = 14.65$$

The EER is a similar measure to COP except the cooling rate (numerator) is in BTU/hr and the electrical input (denominator) is in W:

$$EER_{Ideal} = COP_{Ideal} \times \frac{\frac{BTU}{hr}}{W} \qquad EER_{Ideal} = 14.65 \times \frac{2545\cdot\frac{BTU}{hr}}{746\cdot W} = 50.0\cdot\frac{BTU}{hr\cdot W}$$

This compares to Energy Star compliant values of about 15 BTU/hr/W! We have some way to go! We can define the isentropic efficiency as

$$\eta_{isen} = \frac{EER_{Actual}}{EER_{Ideal}}$$

Hence the isentropic efficiency of a very good AC is about 30%.

Problem 1.33

[Difficulty: 2]

1.33 The maximum theoretical flow rate (slug/s) through a supersonic nozzle is

$$\dot{m}_{max} = 2.38\frac{A_t p_0}{\sqrt{T_0}}$$

where A_t (ft^2) is the nozzle throat area, p_0 (psi) is the tank pressure, and T_0 (°R) is the tank temperature. Is this equation dimensionally correct? If not, find the units of the 2.38 term. Write the equivalent equation in SI units.

Given: Equation for maximum flow rate.

Find: Whether it is dimensionally correct. If not, find units of 2.38 coefficient. Write a SI version of the equation

Solution: Rearrange equation to check units of 0.04 term. Then use conversions from Table G.2 or other sources (e.g., Google)

"Solving" the equation for the constant 2.38: $$2.38 = \frac{m_{max}\sqrt{T_0}}{A_t \cdot p_0}$$

Substituting the units of the terms on the right, the units of the constant are

$$\frac{slug}{s} \times R^{\frac{1}{2}} \times \frac{1}{ft^2} \times \frac{1}{psi} = \frac{slug}{s} \times R^{\frac{1}{2}} \times \frac{1}{ft^2} \times \frac{in^2}{lbf} \times \frac{lbf \cdot s^2}{slug \cdot ft} = \frac{R^{\frac{1}{2}} \cdot in^2 \cdot s}{ft^3}$$

Hence the constant is actually $$c = 2.38\frac{R^{\frac{1}{2}} \cdot in^2 \cdot s}{ft^3}$$

For BG units we could start with the equation and convert each term (e.g., A_t), and combine the result into a new constant, or simply convert c directly:

$$c = 2.38\frac{R^{\frac{1}{2}} \cdot in^2 \cdot s}{ft^3} = 2.38\frac{R^{\frac{1}{2}} \cdot in^2 \cdot s}{ft^3} \times \left(\frac{K}{1.8 \cdot R}\right)^{\frac{1}{2}} \times \left(\frac{1 \cdot ft}{12 \cdot in}\right)^2 \times \frac{1 \cdot ft}{0.3048 m}$$

$$c = 0.04 \cdot \frac{K^{\frac{1}{2}} \cdot s}{m} \quad \text{so} \quad m_{max} = 0.04 \cdot \frac{A_t \cdot p_0}{\sqrt{T_0}}$$

with A_t in m^2, p_0 in Pa, and T_0 in K.

Problem 1.34

[Difficulty: 1]

1.34 The mean free path λ of a molecule of gas is the average distance it travels before collision with another molecule. It is given by

$$\lambda = C \frac{m}{\rho d^2}$$

where m and d are the molecule's mass and diameter, respectively, and ρ is the gas density. What are the dimensions of constant C for a dimensionally consistent equation?

Given: Equation for mean free path of a molecule.

Find: Dimensions of C for a diemsionally consistent equation.

Solution: Use the mean free path equation. Then "solve" for C and use dimensions.

The mean free path equation is $\lambda = C \cdot \frac{m}{\rho \cdot d^2}$

"Solving" for C, and using dimensions $C = \frac{\lambda \cdot \rho \cdot d^2}{m}$

$$C = \frac{L \times \frac{M}{L^3} \times L^2}{M} = 0$$

The constant C is dimensionless.

Problem 1.35

[Difficulty: 1]

1.35 In Chapter 9 we will study aerodynamics and learn that the drag force F_D on a body is given by

$$F_D = \frac{1}{2}\rho V^2 A C_D$$

Hence the drag depends on speed V, fluid density ρ, and body size (indicated by frontal area A) and shape (indicated by drag coefficient C_D). What are the dimensions of C_D?

Given: Equation for drag on a body.

Find: Dimensions of C_D.

Solution: Use the drag equation. Then "solve" for C_D and use dimensions.

The drag equation is
$$F_D = \frac{1}{2}\cdot\rho\cdot V^2\cdot A\cdot C_D$$

"Solving" for C_D, and using dimensions
$$C_D = \frac{2\cdot F_D}{\rho\cdot V^2\cdot A}$$

$$C_D = \frac{F}{\frac{M}{L^3}\times\left(\frac{L}{t}\right)^2\times L^2}$$

But, From Newton's 2nd law
$$\text{Force} = \text{Mass}\cdot\text{Acceleration} \qquad \text{or} \qquad F = M\cdot\frac{L}{t^2}$$

Hence
$$C_D = \frac{F}{\frac{M}{L^3}\times\left(\frac{L}{t}\right)^2\times L^2} = \frac{M\cdot L}{t^2}\times\frac{L^3}{M}\times\frac{t^2}{L^2}\times\frac{1}{L^2} = 0$$

The drag coefficient is dimensionless.

Problem 1.36

[Difficulty: 1]

1.36 A container weighs 3.5 lbf when empty. When filled with water at 90°F, the mass of the container and its contents is 2.5 slug. Find the weight of water in the container, and its volume in cubic feet, using data from Appendix A.

Given: Data on a container and added water.

Find: Weight and volume of water added.

Solution: Use Appendix A.

For the empty container $W_c = 3.5 \cdot lbf$

For the filled container $M_{total} = 2.5 \cdot slug$

The weight of water is then $W_w = M_{total} \cdot g - W_c$

$$W_w = 2.5 \cdot slug \times 32.2 \cdot \frac{ft}{s^2} \times \frac{1 \cdot lbf \cdot s^2}{1 \cdot slug \cdot ft} - 3.5 \cdot lbf \qquad W_w = 77.0\,lbf$$

The temperature is $90°F = 32.2°C$ and from Table A.7 $\rho = 1.93 \cdot \frac{slug}{ft^3}$

Hence $V_w = \frac{M_w}{\rho}$ or $V_w = \frac{W_w}{g \cdot \rho}$

$$V_w = 77.0 \cdot lbf \times \frac{1}{32.2} \cdot \frac{s^2}{ft} \times \frac{1}{1.93} \cdot \frac{ft^3}{slug} \times \frac{1 \cdot slug \cdot ft}{1 \cdot lbf \cdot s^2} \qquad V_w = 1.24 ft^3$$

Problem 1.37

[Difficulty: 1]

1.37 An important equation in the theory of vibrations is

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = f(t)$$

where m (kg) is the mass and x (m) is the position at time t (s). For a dimensionally consistent equation, what are the dimensions of c, k, and f? What would be suitable units for c, k, and f in the SI and BG systems?

Given: Equation for vibrations.

Find: Dimensions of c, k and f for a dimensionally consistent equation. Also, suitable units in SI and BG systems.

Solution: Use the vibration equation to find the diemsions of each quantity

The first term of the equation is $m\cdot\frac{d^2x}{dt^2}$

The dimensions of this are $M \times \frac{L}{t^2}$

Each of the other terms must also have these dimensions.

Hence $c\cdot\frac{dx}{dt} = \frac{M\cdot L}{t^2}$ so $c \times \frac{L}{t} = \frac{M\cdot L}{t^2}$ and $c = \frac{M}{t}$

$k\cdot x = \frac{M\cdot L}{t^2}$ so $k \times L = \frac{M\cdot L}{t^2}$ and $k = \frac{M}{t^2}$

$f = \frac{M\cdot L}{t^2}$

Suitable units for c, k, and f are c: $\frac{kg}{s}$ $\frac{slug}{s}$ k: $\frac{kg}{s^2}$ $\frac{slug}{s^2}$ f: $\frac{kg\cdot m}{s^2}$ $\frac{slug\cdot ft}{s^2}$

Note that c is a damping (viscous) friction term, k is a spring constant, and f is a forcing function. These are more typically expressed using F (force) rather than M (mass). From Newton's 2nd law:

$F = M\cdot\frac{L}{t^2}$ or $M = \frac{F\cdot t^2}{L}$

Using this in the dimensions and units for c, k, and f we find $c = \frac{F\cdot t^2}{L\cdot t} = \frac{F\cdot t}{L}$ $k = \frac{F\cdot t^2}{L\cdot t^2} = \frac{F}{L}$ $f = F$

c: $\frac{N\cdot s}{m}$ $\frac{lbf\cdot s}{ft}$ k: $\frac{N}{m}$ $\frac{lbf}{ft}$ f: N lbf

Problem 1.38

[Difficulty: 1]

1.38 A parameter that is often used in describing pump performance is the specific speed, $N_{S_{cu}}$, given by

$$N_{S_{cu}} = \frac{N(\text{rpm})[Q(\text{gpm})]^{1/2}}{[H(\text{ft})]^{3/4}}$$

What are the units of specific speed? A particular pump has a specific speed of 2000. What will be the specific speed in SI units (angular velocity in rad/s)?

Given: Specific speed in customary units

Find: Units; Specific speed in SI units

Solution:

The units are $\dfrac{\text{rpm}\cdot\text{gpm}^{\frac{1}{2}}}{\text{ft}^{\frac{3}{4}}}$ or $\dfrac{\text{ft}^{\frac{3}{4}}}{\text{s}^{\frac{3}{2}}}$

Using data from tables (e.g. Table G.2)

$$N_{Scu} = 2000\cdot\frac{\text{rpm}\cdot\text{gpm}^{\frac{1}{2}}}{\text{ft}^{\frac{3}{4}}}$$

$$N_{Scu} = 2000\times\frac{\text{rpm}\cdot\text{gpm}^{\frac{1}{2}}}{\text{ft}^{\frac{3}{4}}}\times\frac{2\cdot\pi\cdot\text{rad}}{1\cdot\text{rev}}\times\frac{1\cdot\text{min}}{60\cdot\text{s}}\times\left(\frac{4\times 0.000946\cdot\text{m}^3}{1\cdot\text{gal}}\cdot\frac{1\cdot\text{min}}{60\cdot\text{s}}\right)^{\frac{1}{2}}\times\left(\frac{\frac{1}{12}\cdot\text{ft}}{0.0254\cdot\text{m}}\right)^{\frac{3}{4}}$$

$$N_{Scu} = 4.06\cdot\frac{\frac{\text{rad}}{\text{s}}\cdot\left(\frac{\text{m}^3}{\text{s}}\right)^{\frac{1}{2}}}{\text{m}^{\frac{3}{4}}}$$

Problem 1.39

[Difficulty: 1]

1.39 A particular pump has an "engineering" equation form of the performance characteristic equation given by H (ft) = $1.5 - 4.5 \times 10^{-5}$ $[Q\ (\text{gpm})]^2$, relating the head H and flow rate Q. What are the units of the coefficients 1.5 and 4.5×10^{-5}? Derive an SI version of this equation.

Given: "Engineering" equation for a pump

Find: SI version

Solution:

The dimensions of "1.5" are ft.

The dimensions of "4.5×10^{-5}" are ft/gpm^2.

Using data from tables (e.g. Table G.2), the SI versions of these coefficients can be obtained

$$1.5 \cdot \text{ft} = 1.5 \cdot \text{ft} \times \frac{0.0254\,\text{m}}{\frac{1}{12} \cdot \text{ft}} = 0.457 \cdot \text{m}$$

$$4.5 \times 10^{-5} \cdot \frac{\text{ft}}{\text{gpm}^2} = 4.5 \cdot 10^{-5} \cdot \frac{\text{ft}}{\text{gpm}^2} \times \frac{0.0254\,\text{m}}{\frac{1}{12} \cdot \text{ft}} \times \left(\frac{1 \cdot \text{gal}}{4 \cdot \text{quart}} \cdot \frac{1\text{quart}}{0.000946\text{m}^3} \cdot \frac{60 \cdot \text{s}}{1\text{min}} \right)^2$$

$$4.5 \cdot 10^{-5} \cdot \frac{\text{ft}}{\text{gpm}^2} = 3450 \cdot \frac{\text{m}}{\left(\frac{\text{m}^3}{\text{s}} \right)^2}$$

The equation is

$$H(\text{m}) = 0.457 - 3450 \cdot \left(Q \left(\frac{\text{m}^3}{\text{s}} \right) \right)^2$$

Problem 1.40

[Difficulty: 2]

1.40 Calculate the density of standard air in a laboratory from the ideal gas equation of state. Estimate the experimental uncertainty in the air density calculated for standard conditions (29.9 in. of mercury and 59°F) if the uncertainty in measuring the barometer height is ±0.1 in. of mercury and the uncertainty in measuring temperature is ±0.5°F. (Note that 29.9 in. of mercury corresponds to 14.7 psia.)

Given: Air at standard conditions – p = 29.9 in Hg, T = 59°F

Uncertainty in p is ± 0.1 in Hg, in T is ± 0.5°F

Note that 29.9 in Hg corresponds to 14.7 psia

Find: Air density using ideal gas equation of state; Estimate of uncertainty in calculated value.

Solution:

$$\rho = \frac{p}{RT} = 14.7\,\frac{\text{lbf}}{\text{in}^2} \times \frac{\text{lb}\cdot{}^{\circ}\text{R}}{53.3\ \text{ft}\cdot\text{lbf}} \times \frac{1}{519^{\circ}\text{R}} \times 144\,\frac{\text{in}^2}{\text{ft}^2}$$

The uncertainty in density is given by

$$u_\rho = \left[\left(\frac{p}{\rho}\frac{\partial \rho}{\partial p}u_p\right)^2 + \left(\frac{T}{\rho}\frac{\partial \rho}{\partial T}u_T\right)^2\right]^{\frac{1}{2}}$$

$$\frac{p}{\rho}\frac{\partial \rho}{\partial p} = RT\frac{1}{RT} = \frac{RT}{RT} = 1; \qquad u_p = \frac{\pm 0.1}{29.9} = \pm 0.334\%$$

$$\frac{T}{\rho}\frac{\partial \rho}{\partial T} = \frac{T}{\rho}\cdot -\frac{p}{RT^2} = -\frac{p}{\rho RT} = -1; \qquad u_T = \frac{\pm 0.5}{460+59} = \pm 0.0963\%$$

Then

$$u_\rho = \left[u_p^2 + (-u_T)^2\right]^{\frac{1}{2}} = \pm\left[0.334\%^2 + (-0.0963\%)^2\right]^{\frac{1}{2}}$$

$$u_\rho = \pm 0.348\% = \pm 2.66\times 10^{-4}\,\frac{\text{lbm}}{\text{ft}^3}$$

Problem 1.41

[Difficulty: 2]

1.41 Repeat the calculation of uncertainty described in Problem 1.40 for air in a hot air balloon. Assume the measured barometer height is 759 mm of mercury with an uncertainty of ±1 mm of mercury and the temperature is 60°C with an uncertainty of ±1°C. [Note that 759 mm of mercury corresponds to 101 kPa (abs).]

Given: Air in hot air balloon

$$p = 759 \pm 1 \text{ mm Hg} \quad T = 60 \pm 1 °\text{C}$$

Find: (a) Air density using ideal gas equation of state
(b) Estimate of uncertainty in calculated value

Solution: We will apply uncertainty concepts.

Governing Equations:

$$\rho = \frac{p}{R \cdot T}$$ (Ideal gas equation of state)

$$u_R = \pm\left[\left(\frac{x_1}{R}\frac{\partial R}{\partial x_1}u_{x_1}\right)^2 + \cdots\right]^{\frac{1}{2}}$$ (Propagation of Uncertainties)

We can express density as:

$$\rho = 101 \cdot 10^3 \times \frac{\text{N}}{\text{m}^2} \times \frac{\text{kg} \cdot \text{K}}{287 \cdot \text{N} \cdot \text{m}} \times \frac{1}{333 \cdot \text{K}} = 1.06 \frac{\text{kg}}{\text{m}^3} \qquad \rho = 1.06 \frac{\text{kg}}{\text{m}^3}$$

So the uncertainty in the density is:

$$u_\rho = \pm\left[\left(\frac{p}{\rho}\frac{\partial \rho}{\partial p}u_p\right)^2 + \left(\frac{T}{\rho}\frac{\partial \rho}{\partial T}u_T\right)^2\right]^{\frac{1}{2}}$$

Solving each term separately:

$$\frac{p}{\rho}\frac{\partial \rho}{\partial p} = RT\frac{1}{RT} = 1 \qquad u_p = \frac{1}{759} = 0.1318\%$$

$$\frac{T}{\rho}\frac{\partial \rho}{\partial T} = \frac{T}{\rho}\left(\frac{-p}{RT^2}\right) = -\frac{p}{RT} = -1 \qquad u_T = \frac{1}{333} = 0.3003\%$$

Therefore:

$$u_\rho = \pm\left[(u_p)^2 + (-u_T)^2\right]^{\frac{1}{2}} = \pm\left[(0.1318\%)^2 + (-0.3003\%)^2\right]^{\frac{1}{2}}$$

$$u_\rho = \pm 0.328\% \left(\pm 3.47 \times 10^{-3} \frac{\text{kg}}{\text{m}^3}\right)$$

Problem 1.42

[Difficulty: 2]

1.42 The mass of the standard American golf ball is 1.62 ± 0.01 oz and its mean diameter is 1.68 ± 0.01 in. Determine the density and specific gravity of the American golf ball. Estimate the uncertainties in the calculated values.

Given: Standard American golf ball:

$$m = 1.62 \pm 0.01\,\text{oz} \quad (20\text{ to }1)$$
$$D = 1.68 \pm 0.01\,\text{in.} \quad (20\text{ to }1)$$

Find: Density and specific gravity; Estimate uncertainties in calculated values.

Solution: Density is mass per unit volume, so

$$\rho = \frac{m}{\forall} = \frac{m}{\frac{4}{3}\pi R^3} = \frac{3}{4\pi}\frac{m}{(D/2)^3} = \frac{6}{\pi}\frac{m}{D^3}$$

$$\rho = \frac{6}{\pi} \times 1.62\,\text{oz} \times \frac{1}{(1.68)^3\,\text{in.}^3} \times \frac{0.4536\,\text{kg}}{16\,\text{oz}} \times \frac{\text{in.}^3}{(0.0254)^3\,\text{m}^3} = 1130\,\text{kg/m}^3$$

and

$$\text{SG} = \frac{\rho}{\rho_{H_2O}} = 1130\,\frac{\text{kg}}{\text{m}^3} \times \frac{\text{m}^3}{1000\,\text{kg}} = 1.13$$

The uncertainty in density is given by

$$u_\rho = \left[\left(\frac{m}{\rho}\frac{\partial \rho}{\partial m}u_m\right)^2 + \left(\frac{D}{\rho}\frac{\partial \rho}{\partial D}u_D\right)^2\right]^{\frac{1}{2}}$$

$$\frac{m}{\rho}\frac{\partial \rho}{\partial m} = \frac{m}{\rho}\frac{1}{\forall} = \frac{\forall}{\forall} = 1; \qquad u_m = \frac{\pm 0.01}{1.62} = \pm 0.617\%$$

$$\frac{D}{\rho}\frac{\partial \rho}{\partial D} = \frac{D}{\rho}\cdot\left(-3\frac{6m}{\pi D^4}\right) = -3\frac{6}{\pi}\frac{m}{\rho D^4} = -3; \qquad u_D = \frac{\pm 0.1}{1.68} = \pm 0.595\%$$

Thus

$$u_\rho = \pm\left[u_m^2 + (-3u_D)^2\right]^{\frac{1}{2}} = \pm\left[0.617\%^2 + (-3\times 0.595\%)^2\right]^{\frac{1}{2}} \qquad u_\rho = \pm 1.89\% = \pm 21.4\,\frac{\text{kg}}{\text{m}^3}$$

$$u_{SG} = u_\rho = \pm 1.89\% = \pm 0.0214$$

Finally,

$$\rho = 1130 \pm 21.4\,\text{kg/m}^3 \quad (20\text{ to }1)$$
$$\text{SG} = 1.13 \pm 0.0214 \quad (20\text{ to }1)$$

Problem 1.43

[Difficulty: 2]

1.43 A can of pet food has the following internal dimensions: 102 mm height and 73 mm diameter (each ± 1 mm at odds of 20 to 1). The label lists the mass of the contents as 397 g. Evaluate the magnitude and estimated uncertainty of the density of the pet food if the mass value is accurate to ± 1 g at the same odds.

Given: Pet food can

$$H = 102 \pm 1 \text{ mm (20 to 1)}$$
$$D = 73 \pm 1 \text{ mm} \quad \text{(20 to 1)}$$
$$m = 397 \pm 1 \text{ g} \quad \text{(20 to 1)}$$

Find: Magnitude and estimated uncertainty of pet food density.

Solution: Density is

$$\rho = \frac{m}{\forall} = \frac{m}{\pi R^2 H} = \frac{4}{\pi}\frac{m}{D^2 H} \quad \text{or} \quad \rho = \rho\,(m, D, H)$$

From uncertainty analysis:

$$u_\rho = \pm\left[\left(\frac{m}{\rho}\frac{\partial \rho}{\partial m}u_m\right)^2 + \left(\frac{D}{\rho}\frac{\partial \rho}{\partial D}u_D\right)^2 + \left(\frac{H}{\rho}\frac{\partial \rho}{\partial H}u_H\right)^2\right]^{\frac{1}{2}}$$

Evaluating:

$$\frac{m}{\rho}\frac{\partial \rho}{\partial m} = \frac{m}{\rho}\frac{4}{\pi}\frac{1}{D^2 H} = \frac{1}{\rho}\frac{4\,m}{\pi D^2 H} = 1; \qquad u_m = \frac{\pm 1}{397} = \pm 0.252\%$$

$$\frac{D}{\rho}\frac{\partial \rho}{\partial D} = \frac{D}{\rho}(-2)\frac{4m}{\pi D^3 H} = (-2)\frac{1}{\rho}\frac{4m}{\pi D^2 H} = -2; \qquad u_D = \frac{\pm 1}{73} = \pm 1.37\%$$

$$\frac{H}{\rho}\frac{\partial \rho}{\partial H} = \frac{H}{\rho}(-1)\frac{4m}{\pi D^2 H^2} = (-1)\frac{1}{\rho}\frac{4m}{\pi D^2 H} = -1; \qquad u_H = \frac{\pm 1}{102} = \pm 0.980\%$$

Substituting:

$$u_\rho = \pm\left[(1 \times 0.252)^2 + (-2 \times 1.37)^2 + (-1 \times 0.980)^2\right]^{\frac{1}{2}}$$
$$u_\rho = \pm 2.92\%$$

$$\forall = \frac{\pi}{4}D^2 H = \frac{\pi}{4} \times (73)^2 \text{ mm}^2 \times 102 \text{ mm} \times \frac{\text{m}^3}{10^9 \text{ mm}^3} = 4.27 \times 10^{-4} \text{ m}^3$$

$$\rho = \frac{m}{\forall} = \frac{397 \text{ g}}{4.27 \times 10^{-4} \text{ m}^3} \times \frac{\text{kg}}{1000 \text{ g}} = 930 \text{ kg/m}^3$$

Thus:

$$\rho = 930 \pm 27.2 \text{ kg/m}^3 \text{ (20 to 1)}$$

Problem 1.44 [Difficulty: 2]

1.44 The mass flow rate in a water flow system determined by collecting the discharge over a timed interval is 0.2 kg/s. The scales used can be read to the nearest 0.05 kg and the stopwatch is accurate to 0.2 s. Estimate the precision with which the flow rate can be calculated for time intervals of (a) 10 s and (b) 1 min.

Given: Mass flow rate of water determine by collecting discharge over a timed interval is 0.2 kg/s.

Scales can be read to nearest 0.05 kg.

Stopwatch can be read to nearest 0.2 s.

Find: Estimate precision of flow rate calculation for time intervals of (a) 10 s, and (b) 1 min.

Solution: Apply methodology of uncertainty analysis, Appendix F:

Computing equations:

$$\dot{m} = \frac{\Delta m}{\Delta t}$$

$$u_{\dot{m}} = \pm\left[\left(\frac{\Delta m}{\dot{m}}\frac{\partial \dot{m}}{\partial \Delta m}u_{\Delta m}\right)^2 + \left(\frac{\Delta t}{\dot{m}}\frac{\partial \dot{m}}{\partial \Delta t}u_{\Delta t}\right)^2\right]^{\frac{1}{2}}$$

Thus

$$\frac{\Delta m}{\dot{m}}\frac{\partial \dot{m}}{\partial \Delta m} = \Delta t\frac{1}{\Delta t} = 1 \quad \text{and} \quad \frac{\Delta t}{\dot{m}}\frac{\partial \dot{m}}{\partial \Delta t} = \frac{\Delta t^2}{\Delta m}\cdot -\frac{\Delta m}{\Delta t^2} = -1$$

The uncertainties are expected to be ± half the least counts of the measuring instruments.

Tabulating results:

Time Interval, Δt (s)	**Error in Δt (s)**	**Uncertainty in Δt (%)**	**Water Collected, Δm (kg)**	**Error in Δm (kg)**	**Uncertainty in Δm (%)**	**Uncertainty in $\dot{m}$ (%)**
10	± 0.10	± 1.0	2.0	± 0.025	± 1.25	± 1.60
60	± 0.10	± 0.167	12.0	± 0.025	± 0.208	± 0.267

A time interval of about 15 seconds should be chosen to reduce the uncertainty in results to ± 1 percent.

Problem 1.45

[Difficulty: 3]

1.45 The mass flow rate of water in a tube is measured using a beaker to catch water during a timed interval. The nominal mass flow rate is 100 g/s. Assume that mass is measured using a balance with a least count of 1 g and a maximum capacity of 1 kg, and that the timer has a least count of 0.1 s. Estimate the time intervals and uncertainties in measured mass flow rate that would result from using 100, 500, and 1000 mL beakers. Would there be any advantage in using the largest beaker? Assume the tare mass of the empty 1000 mL beaker is 500 g.

Given: Nominal mass flow rate of water determined by collecting discharge (in a beaker) over a timed interval is $\dot{m} = 100\ \mathrm{g/s}$; Scales have capacity of 1 kg, with least count of 1 g; Timer has least count of 0.1 s; Beakers with volume of 100, 500, 1000 mL are available – tare mass of 1000 mL beaker is 500 g.

Find: Estimate (a) time intervals, and (b) uncertainties, in measuring mass flow rate from using each of the three beakers.

Solution: To estimate time intervals assume beaker is filled to maximum volume in case of 100 and 500 mL beakers and to maximum allowable mass of water (500 g) in case of 1000 mL beaker.

Then $\dot{m} = \dfrac{\Delta m}{\Delta t}$ and $\Delta t = \dfrac{\Delta m}{\dot{m}} = \dfrac{\rho \Delta \forall}{\dot{m}}$

Tabulating results

$\Delta\forall$ = 100 mL 500 mL 1000 mL

Δt = 1 s 5 s 5 s

Apply the methodology of uncertainty analysis, Appendix E. Computing equation:

$$u_{\dot{m}} = \pm\left[\left(\frac{\Delta m}{\dot{m}}\frac{\partial \dot{m}}{\partial \Delta m}u_{\Delta m}\right)^2 + \left(\frac{\Delta t}{\dot{m}}\frac{\partial \dot{m}}{\partial \Delta t}u_{\Delta t}\right)^2\right]^{\frac{1}{2}}$$

The uncertainties are ± half the least counts of the measuring instruments: $\delta\Delta m = \pm 0.5\ \mathrm{g}$ $\delta\Delta t = 0.05\ \mathrm{s}$

$$\frac{\Delta m}{\dot{m}}\frac{\partial \dot{m}}{\partial \Delta m} = \Delta t\frac{1}{\Delta t} = 1 \quad \text{and} \quad \frac{\Delta t}{\dot{m}}\frac{\partial \dot{m}}{\partial \Delta t} = \frac{\Delta t^2}{\Delta m}\cdot - \frac{\Delta m}{\Delta t^2} = -1 \qquad \therefore u_{\dot{m}} = \pm\left[u_{\Delta m}{}^2 + \left(-u_{\Delta t}\right)^2\right]^{\frac{1}{2}}$$

Tabulating results:

Beaker Volume $\Delta\forall$ (mL)	Water Collected Δm(g)	Error in Δm (g)	Uncertainty in Δm (%)	Time Interval Δt (s)	Error in Δt (s)	Uncertainty in Δt (%)	Uncertainty in $\dot{m}$ (%)
100	100	± 0.50	± 0.50	1.0	± 0.05	± 5.0	± 5.03
500	500	± 0.50	± 0.10	5.0	± 0.05	± 1.0	± 1.0
1000	500	± 0.50	± 0.10	5.0	± 0.05	± 1.0	± 1.0

Since the scales have a capacity of 1 kg and the tare mass of the 1000 mL beaker is 500 g, there is no advantage in using the larger beaker. The uncertainty in $\dot{m}$ could be reduced to ± 0.50 percent by using the large beaker if a scale with greater capacity the same least count were available

Problem 1.46

[Difficulty: 2]

1.46 The mass of the standard British golf ball is 45.9 ± 0.3 g and its mean diameter is 41.1 ± 0.3 mm. Determine the density and specific gravity of the British golf ball. Estimate the uncertainties in the calculated values.

Given: Standard British golf ball:

$$m = 45.9 \pm 0.3\ \mathrm{g} \quad (20\ \text{to}\ 1)$$
$$D = 41.1 \pm 0.3\ \mathrm{mm} \quad (20\ \text{to}\ 1)$$

Find: Density and specific gravity; Estimate of uncertainties in calculated values.

Solution: Density is mass per unit volume, so

$$\rho = \frac{m}{\forall} = \frac{m}{\frac{4}{3}\pi R^3} = \frac{3}{4\pi}\frac{m}{(D/2)^3} = \frac{6}{\pi}\frac{m}{D^3}$$

$$\rho = \frac{6}{\pi} \times 0.0459\ \mathrm{kg} \times \frac{1}{(0.0411)^3}\mathrm{m^3} = 1260\ \mathrm{kg/m^3}$$

and

$$SG = \frac{\rho}{\rho H_2O} = 1260\,\frac{\mathrm{kg}}{\mathrm{m^3}} \times \frac{\mathrm{m^3}}{1000\ \mathrm{kg}} = 1.26$$

The uncertainty in density is given by

$$u_\rho = \pm\left[\left(\frac{m}{\rho}\frac{\partial\rho}{\partial m}u_m\right)^2 + \left(\frac{D}{\rho}\frac{\partial\rho}{\partial D}u_D\right)^2\right]^{\frac{1}{2}}$$

$$\frac{m}{\rho}\frac{\partial\rho}{\partial m} = \frac{m}{\rho}\frac{1}{\forall} = \frac{\forall}{\forall} = 1; \qquad u_m = \pm\frac{0.3}{45.9} = \pm 0.654\%$$

$$\frac{D}{\rho}\frac{\partial D}{\partial m} = \frac{D}{\rho}\left(-3\frac{6}{\pi}\frac{m}{D^4}\right) = -3\left(\frac{6m}{\pi D^4}\right) = -3; \qquad u_D = \pm\frac{0.3}{41.1} = \pm 0.730\%$$

Thus

$$u_\rho = \pm\left[u_m^{\ 2} + \left(-3u_D\right)^2\right]^{\frac{1}{2}} = \pm\left[0.654^2 + \left(-3\times 0.730\right)^2\right]^{\frac{1}{2}}$$

$$u_\rho = \pm 2.29\% = \pm 28.9\ \mathrm{kg/m^3}$$

$$u_{SG} = u_\rho = \pm 2.29\% = \pm 0.0289$$

Summarizing

$$\rho = 1260 \pm 28.9\ \mathrm{kg/m^3}\ (20\ \text{to}\ 1)$$

$$SG = 1.26 \pm 0.0289\ (20\ \text{to}\ 1)$$

Problem 1.47

[Difficulty: 3]

1.47 The estimated dimensions of a soda can are $D = 66.0 \pm 0.5$ mm and $H = 110 \pm 0.5$ mm. Measure the mass of a full can and an empty can using a kitchen scale or postal scale. Estimate the volume of soda contained in the can. From your measurements estimate the depth to which the can is filled and the uncertainty in the estimate. Assume the value of SG = 1.055, as supplied by the bottler.

Given: Soda can with estimated dimensions D = 66.0 ± 0.5 mm, H = 110 ± 0.5 mm. Soda has SG = 1.055

Find: Volume of soda in the can (based on measured mass of full and empty can); Estimate average depth to which the can is filled and the uncertainty in the estimate.

Solution: Measurements on a can of coke give

$$m_f = 386.5 \pm 0.50\ \text{g}, \quad m_e = 17.5 \pm 0.50\ \text{g} \therefore m = m_f - m_e = 369 \pm u_m\ \text{g}$$

$$u_m = \left[\left(\frac{m_f}{m}\frac{\partial m}{\partial m_f}u_{m_f}\right)^2 + \left(\frac{m_e}{m}\frac{\partial m}{\partial m_e}u_{m_e}\right)^2\right]^{\frac{1}{2}}$$

$$u_{m_f} = \pm\frac{0.5\ \text{g}}{386.5\ \text{g}} = \pm 0.00129, \quad u_{m_e} = \pm\frac{0.50}{17.5} = 0.0286$$

$$u_m = \pm\left[\left(\frac{386.5}{369}\times 1\times 0.00129\right)^2 + \left(\frac{17.5}{369}\times 1\times 0.0286\right)^2\right]^{\frac{1}{2}} = \pm 0.0019$$

Density is mass per unit volume and SG = $\rho/\rho H_2O$ so

$$\forall = \frac{m}{\rho} = \frac{m}{\rho H_2O\ SG} = 369\ \text{g} \times \frac{\text{m}^3}{1000\ \text{kg}} \times \frac{1}{1.055} \times \frac{\text{kg}}{1000\ \text{g}} = 350 \times 10^{-6}\ \text{m}^3$$

The reference value ρH_2O is assumed to be precise. Since SG is specified to three places beyond the decimal point, assume $u_{SG} = \pm 0.001$. Then

$$u_v = \left[\left(\frac{m}{v}\frac{\partial v}{\partial m}u_m\right)^2 + \left(\frac{SG}{v}\frac{\partial v}{\partial SG}u_{SG}\right)^2\right]^{\frac{1}{2}}$$

$$u_v = \pm\left[(1\times 0.0019)^2 + (-1\times 0.001)^2\right]^{\frac{1}{2}} = \pm 0.0021 = \pm 0.21\%$$

$$\forall = \frac{\pi D^2}{4}L \quad \text{or} \quad L = \frac{4\forall}{\pi D^2} = \frac{4}{\pi}\times\frac{350\times 10^{-6}\,\text{m}^3}{0.066^2\,\text{m}^2}\times\frac{10^3\,\text{mm}}{\text{m}} = 102\,\text{mm}$$

$$u_L = \left[\left(\frac{\forall}{L} \frac{\partial L}{\partial \forall} u_\forall \right)^2 + \left(\frac{D}{L} \frac{\partial L}{\partial D} u_D \right)^2 \right]^{\frac{1}{2}}$$

$$\frac{\forall}{L} \frac{\partial L}{\partial \forall} = \frac{4}{\pi D^2} \frac{\pi D^2}{4} = 1$$

$$\frac{D}{L} \frac{\partial L}{\partial D} = \frac{D}{L} \cdot -2 \frac{4\forall}{\pi D^3} = -2 \frac{4\forall}{\pi D^2 L} = -2; \quad u_D = \pm \frac{0.5}{66} = \pm 0.0076$$

$$u_L = \pm \left[(1 \times 0.0021)^2 + (-2 \times 0.0076)^2 \right]^{\frac{1}{2}} = \pm 0.0153 = \pm 1.53\%$$

Notes:

1. Printing on the can states the content as 355 ml. This suggests that the implied accuracy of the SG value may be over stated.
2. Results suggest that over seven percent of the can height is void of soda.

Problem 1.48

[Difficulty: 3]

1.48 From Appendix A, the viscosity μ ($N \cdot s/m^2$) of water at temperature T (K) can be computed from $\mu = A10^{B/(T-C)}$, where $A = 2.414 \times 10^{-5}\ N \cdot s/m^2$, $B = 247.8$ K, and $C = 140$ K. Determine the viscosity of water at 30°C, and estimate its uncertainty if the uncertainty in temperature measurement is ±0.5°C.

Given: Data on water

Find: Viscosity; Uncertainty in viscosity

Solution:

The data is: $A = 2.414 \times 10^{-5} \cdot \frac{N \cdot s}{m^2}$ $\quad B = 247.8K$ $\quad C = 140 \cdot K$ $\quad T = 303 \cdot K$

The uncertainty in temperature is $u_T = \frac{0.5 \cdot K}{293 \cdot K}$ $\quad u_T = 0.171 \cdot \%$

Also $\mu(T) = A \cdot 10^{\frac{B}{(T-C)}}$ $\quad$ Evaluating $\quad \mu(293 \cdot K) = 1.005 \times 10^{-3} \cdot \frac{N \cdot s}{m^2}$

For the uncertainty

$$\frac{d}{dT}\mu(T) = -\frac{A \cdot B \cdot \ln(10)}{10^{\frac{B}{C-T}} \cdot (C-T)^2}$$

Hence $u_\mu(T) = \left| \frac{T}{\mu(T)} \cdot \frac{d}{dT}\mu(T) \cdot u_T \right| = \frac{\ln(10) \cdot |B \cdot T \cdot u_T|}{(|C - T|)^2}$ $\quad$ Evaluating $\quad u_\mu(T) = 1.11 \cdot \%$

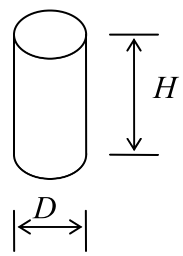
Problem 1.49

[Difficulty: 4]

1.49 Using the nominal dimensions of the soda can given in Problem 1.47, determine the precision with which the diameter and height must be measured to estimate the volume of the can within an uncertainty of ±0.5 percent.

Given: Dimensions of soda can: D = 66 mm, H = 110 mm

Find: Measurement precision needed to allow volume to be estimated with an uncertainty of ± 0.5 percent or less.



Solution: Use the methods of Appendix F:

Computing equations:

$$\forall = \frac{\pi D^2 H}{4}$$

$$u_\forall = \pm\left[\left(\frac{H}{\forall}\frac{\partial \forall}{\partial H}u_H\right)^2 + \left(\frac{D}{\forall}\frac{\partial \forall}{\partial D}u_D\right)^2\right]^{\frac{1}{2}}$$

Since $\forall = \frac{\pi D^2 H}{4}$, then $\frac{\partial \forall}{\partial H} = \frac{\pi D^2}{4}$ and $\frac{\partial \forall}{\partial D} = \frac{\pi DH}{2}$. Letting $u_D = \pm\frac{\delta x}{D}$ and $u_H = \pm\frac{\delta x}{H}$, and substituting,

$$u_\forall = \pm\left[\left(\frac{4H}{\pi D^2 H}\frac{\pi D^2}{4}\frac{\delta x}{H}\right)^2 + \left(\frac{4D}{\pi D^2 H}\frac{\pi DH}{2}\frac{\delta x}{D}\right)^2\right]^{\frac{1}{2}} = \pm\left[\left(\frac{\delta x}{H}\right)^2 + \left(\frac{2\delta x}{D}\right)^2\right]^{\frac{1}{2}}$$

Solving,

$$u_\forall{}^2 = \left(\frac{\delta x}{H}\right)^2 + \left(\frac{2\delta x}{D}\right)^2 = (\delta x)^2\left[\left(\frac{1}{H}\right)^2 + \left(\frac{2}{D}\right)^2\right]$$

$$\delta x = \pm\frac{u_\forall}{\left[\left(\frac{1}{H}\right)^2 + \left(\frac{2}{D}\right)^2\right]^{\frac{1}{2}}} = \pm\frac{0.005}{\left[\left(\frac{1}{110\text{ mm}}\right)^2 + \left(\frac{2}{66\text{ mm}}\right)^2\right]^{\frac{1}{2}}} = \pm 0.158\text{ mm}$$

Check:

$$u_H = \pm\frac{\delta x}{H} = \pm\frac{0.158\text{ mm}}{110\text{ mm}} = \pm 1.44\times 10^{-3}$$

$$u_D = \pm\frac{\delta x}{D} = \pm\frac{0.158\text{ mm}}{66\text{ mm}} = \pm 2.39\times 10^{-3}$$

$$u_\forall = \pm[(u_H)^2 + (2u_D)^2]^{\frac{1}{2}} = \pm[(0.00144)^2 + (0.00478)^2]^{\frac{1}{2}} = \pm 0.00499$$

If δx represents half the least count, a minimum resolution of about 2 $\delta x \approx$ 0.32 mm is needed.

Problem 1.50

[Difficulty: 3]

1.50 An enthusiast magazine publishes data from its road tests on the lateral acceleration capability of cars. The measurements are made using a 150-ft-diameter skid pad. Assume the vehicle path deviates from the circle by ±2 ft and that the vehicle speed is read from a fifth-wheel speed-measuring system to ±0.5 mph. Estimate the experimental uncertainty in a reported lateral acceleration of 0.7 *g*. How would you improve the experimental procedure to reduce the uncertainty?

Given: Lateral acceleration, a = 0.70 g, measured on 150-ft diameter skid pad; Uncertainties in Path deviation ±2 ft; vehicle speed ±0.5 mph

Find: Estimate uncertainty in lateral acceleration; ow could experimental procedure be improved?

Solution: Lateral acceleration is given by a = V^2/R.

From Appendix F, $u_a = \pm[(2u_v)^2 + (u_R)^2]^{1/2}$

From the given data, $V^2 = aR; \quad V = \sqrt{aR} = \sqrt{0.70 \times 32.2\,\frac{\text{ft}}{\text{s}^2} \times 75\text{ ft}} = 41.1\,\frac{\text{ft}}{\text{s}}$

Then $u_v = \pm\frac{\delta V}{V} = \pm 0.5\,\frac{\text{mi}}{\text{hr}} \times \frac{\text{s}}{41.1\text{ ft}} \times 5280\,\frac{\text{ft}}{\text{mi}} \times \frac{\text{hr}}{3600\text{ s}} = \pm 0.0178$

and $u_R = \pm\frac{\delta R}{R} = \pm 2\text{ ft} \times \frac{1}{75\text{ ft}} = \pm 0.0267$

so

$$u_a = \pm\left[(2 \times 0.0178)^2 + (0.0267)^2\right]^{1/2} = \pm 0.0445$$

$$u_a = \pm 4.45\text{ percent}$$

Experimental procedure could be improved by using a larger circle, assuming the absolute errors in measurement are constant.

For

$$D = 400\text{ ft}; \quad R = 200\text{ ft}$$

$$V^2 = aR; \quad V = \sqrt{aR} = \sqrt{0.70 \times 32.2\,\frac{\text{ft}}{\text{s}^2} \times 200\text{ ft}} = 67.1\,\frac{\text{ft}}{\text{s}} = 45.8\text{ mph}$$

$$u_V = \pm\frac{0.5}{45.8} = \pm 0.0109; \quad u_R = \pm\frac{2}{200} = \pm 0.0100$$

$$u_a = \pm\left[(2 \times 0.0109)^2 + 0.0100^2\right] = \pm 0.0240 = \pm 2.4\%$$

Problem 1.51

[Difficulty: 4]

1.51 The height of a building may be estimated by measuring the horizontal distance to a point on the ground and the angle from this point to the top of the building. Assuming these measurements are $L = 100 \pm 0.5$ ft and $\theta = 30 \pm 0.2°$, estimate the height H of the building and the uncertainty in the estimate. For the same building height and measurement uncertainties, use *Excel's Solver* to determine the angle (and the corresponding distance from the building) at which measurements should be made to minimize the uncertainty in estimated height. Evaluate and plot the optimum measurement angle as a function of building height for $50 \le H \le 1000$ ft.

Given: Data on length and angle measurements

Find: Height; Angle for minimum uncertainty in height; Plot

Solution:

The data is: $L = 100 \cdot ft$ $\quad \delta L = 0.5 \cdot ft$ $\quad \theta = 30 \cdot deg$ $\quad \delta\theta = 0.2 \cdot deg$

Uncertainties: $u_L = \dfrac{\delta L}{L}$ $\quad u_L = 0.5 \cdot \%$ $\quad u_\theta = \dfrac{\delta\theta}{\theta}$ $\quad u_\theta = 0.667 \cdot \%$

The height is: $H = L \cdot \tan(\theta)$ $\quad H = 57.7 \cdot ft$ $\quad$ with uncertainty $\quad u_H = \sqrt{\left(\dfrac{L}{H}\cdot\dfrac{\partial}{\partial L}H \cdot u_L\right)^2 + \left(\dfrac{\theta}{H}\cdot\dfrac{\partial}{\partial \theta}H \cdot u_\theta\right)^2}$

Hence with $\quad \dfrac{\partial}{\partial L}H = \tan(\theta)$ $\quad \dfrac{\partial}{\partial \theta}H = L \cdot \left(1 + \tan(\theta)^2\right)$ $\quad u_H = \sqrt{\left(\dfrac{L}{H}\cdot\tan(\theta)\cdot u_L\right)^2 + \left[\dfrac{L\cdot\theta}{H}\cdot\left(1 + \tan(\theta)^2\right)\cdot u_\theta\right]^2}$

Evaluating $\quad u_H = 0.949 \cdot \%$ $\quad$ and $\quad \delta H = u_H \cdot H$ $\quad \delta H = 0.548 \cdot ft$

The height is then $H = 57.7 \cdot ft$ +/- $\delta H = 0.548 \cdot ft$

To plot u_H versus θ for a given H we need to replace L, u_L and u_θ with functions of θ. Doing this and simplifying

$$u_H(\theta) = \sqrt{\left(\tan(\theta)\cdot\frac{\delta L}{H}\right)^2 + \left[\frac{\delta\theta}{\tan(\theta)}\cdot\left(1 + \tan(\theta)^2\right)\right]^2}$$

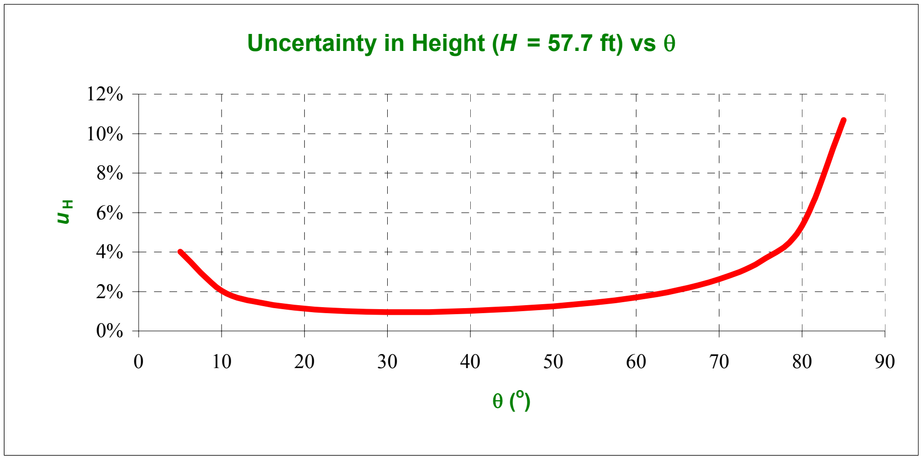
Given data:

H =	57.7	ft
δL =	0.5	ft
$\delta\theta$ =	0.2	deg

For this building height, we are to vary θ (and therefore L) to minimize the uncertainty u_H.

Plotting u_H vs θ

θ (deg)	u_H
5	4.02%
10	2.05%
15	1.42%
20	1.13%
25	1.00%
30	0.95%
35	0.96%
40	1.02%
45	1.11%
50	1.25%
55	1.44%
60	1.70%
65	2.07%
70	2.62%
75	3.52%
80	5.32%
85	10.69%

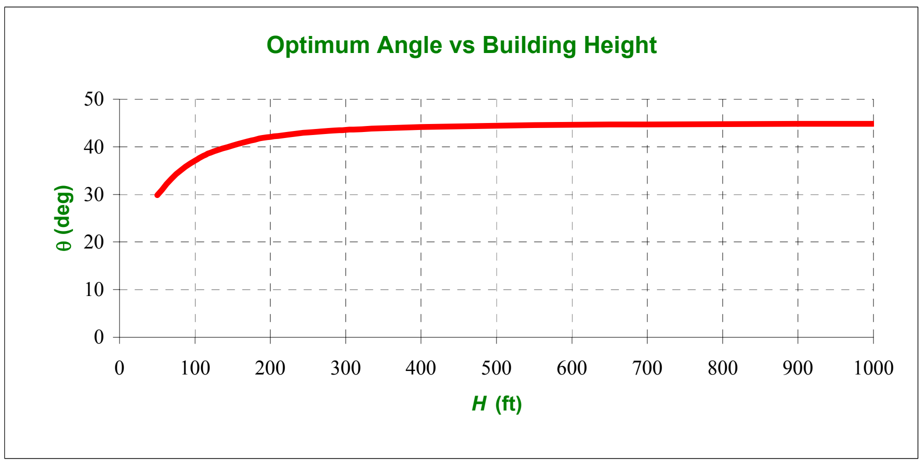


Optimizing using *Solver*

θ (deg)	u_H
31.4	0.947%

To find the optimum θ as a function of building height *H* we need a more complex *Solver*

H (ft)	θ (deg)	u_H
50	29.9	0.992%
75	34.3	0.877%
100	37.1	0.818%
125	39.0	0.784%
175	41.3	0.747%
200	42.0	0.737%
250	43.0	0.724%
300	43.5	0.717%
400	44.1	0.709%
500	44.4	0.705%
600	44.6	0.703%
700	44.7	0.702%
800	44.8	0.701%
900	44.8	0.700%
1000	44.9	0.700%



Use *Solver* to vary ALL θ's to minimize the total u_H!

Total u_H's: 11.3%

Problem 1.52

[Difficulty: 4]

1.52 An American golf ball is described in Problem 1.42 Assuming the measured mass and its uncertainty as given, determine the precision to which the diameter of the ball must be measured so the density of the ball may be estimated within an uncertainty of ±1 percent.

Given: American golf ball, m = 1.62 ± 0.01 oz, D = 1.68 in.

Find: Precision to which D must be measured to estimate density within uncertainty of ± 1percent.

Solution: Apply uncertainty concepts

Definition: Density, $\rho \equiv \frac{m}{\forall}$ $\quad \forall = \frac{4}{3}\pi R^3 = \frac{\pi D^3}{6}$

Computing equation:
$$u_R = \pm\left[\left(\frac{x_1}{R}\frac{\partial R}{\partial x_1}u_{x_1}\right)^2 + \cdots\right]^{\frac{1}{2}}$$

From the definition,

$$\rho = \frac{m}{\pi D^{3/6}} = \frac{6\,m}{\pi D^3} = \rho(m, D)$$

Thus $\frac{m}{\rho}\frac{\partial\rho}{\partial m} = 1$ and $\frac{D}{\rho}\frac{\partial\rho}{\partial D} = 3$, so

$$u_\rho = \pm[(1\,u_m)^2 + (3\,u_D)^2]^{\frac{1}{2}}$$
$$u_\rho^2 = u_m{}^2 + 9\,u_D^2$$

Solving,

$$u_D = \pm\tfrac{1}{3}[u_\rho{}^2 - u_m^2]^{\frac{1}{2}}$$

From the data given,

$$u_\rho = \pm 0.0100$$
$$u_m = \frac{\pm 0.01\text{ oz}}{1.62\text{ oz}} = \pm 0.00617$$

$$u_D = \pm\frac{1}{3}[(0.0100)^2 - (0.00617)^2]^{\frac{1}{2}} = \pm 0.00262 \text{ or } \pm 0.262\%$$

Since $u_D = \pm\frac{\delta D}{D}$, then

$$\delta D = \pm D\;u_D = \pm 1.68\text{ in.}_x\;0.00262 = \pm 0.00441\text{ in.}$$

The ball diameter must be measured to a precision of ± 0.00441 in.(± 0.112 mm) or better to estimate density within ± 1percent. A micrometer or caliper could be used.

Problem 1.53

[Difficulty: 5]

1.53 A syringe pump is to dispense liquid at a flow rate of 100 mL/min. The design for the piston drive is such that the uncertainty of the piston speed is 0.001 in./min, and the cylinder bore diameter has a maximum uncertainty of 0.0005 in. Plot the uncertainty in the flow rate as a function of cylinder bore. Find the combination of piston speed and bore that minimizes the uncertainty in the flow rate.

Given: Syringe pump to deliver 100 mL/min $\quad \delta V = 0.001 \cdot \dfrac{in}{min} \quad \delta D = 0.0005 \cdot in$

Find: (a) Plot uncertainty in flow rate as a function of bore.
(b) Find combination of piston speed and bore resulting in minimum uncertainty in flow rate.

Solution: We will apply uncertainty concepts.

Governing Equations:

$$Q = \frac{\pi}{4} \cdot D^2 \cdot V \quad \text{(Flow rate in syringe pump)}$$

$$u_R = \pm\left[\left(\frac{x_1}{R}\frac{\partial R}{\partial x_1}u_{x_1}\right)^2 + \cdots\right]^{\frac{1}{2}} \quad \text{(Propagation of Uncertainties)}$$

Now solving for the piston speed in terms of the bore: $\quad V(D) = \dfrac{4 \cdot Q}{\pi \cdot D^2}$

So the uncertainty in the flow rate is:

$$u_Q = \pm\left[\left(\frac{D}{Q}\frac{\partial Q}{\partial D}u_D\right)^2 + \left(\frac{V}{Q}\frac{\partial Q}{\partial V}u_V\right)^2\right]^{\frac{1}{2}} = \pm\left[\left(\frac{D}{Q}\frac{2Q}{D}u_D\right)^2 + \left(\frac{V}{Q}\frac{Q}{V}u_V\right)^2\right]^{\frac{1}{2}}$$

$$u_Q = \pm\left[(2u_D)^2 + (u_V)^2\right]^{\frac{1}{2}} \quad \text{where} \quad u_D = \frac{\delta D}{D} \quad u_V = \frac{\delta V}{V}$$

The uncertainty is minimized when $\dfrac{\partial u_Q}{\partial D} = 0$

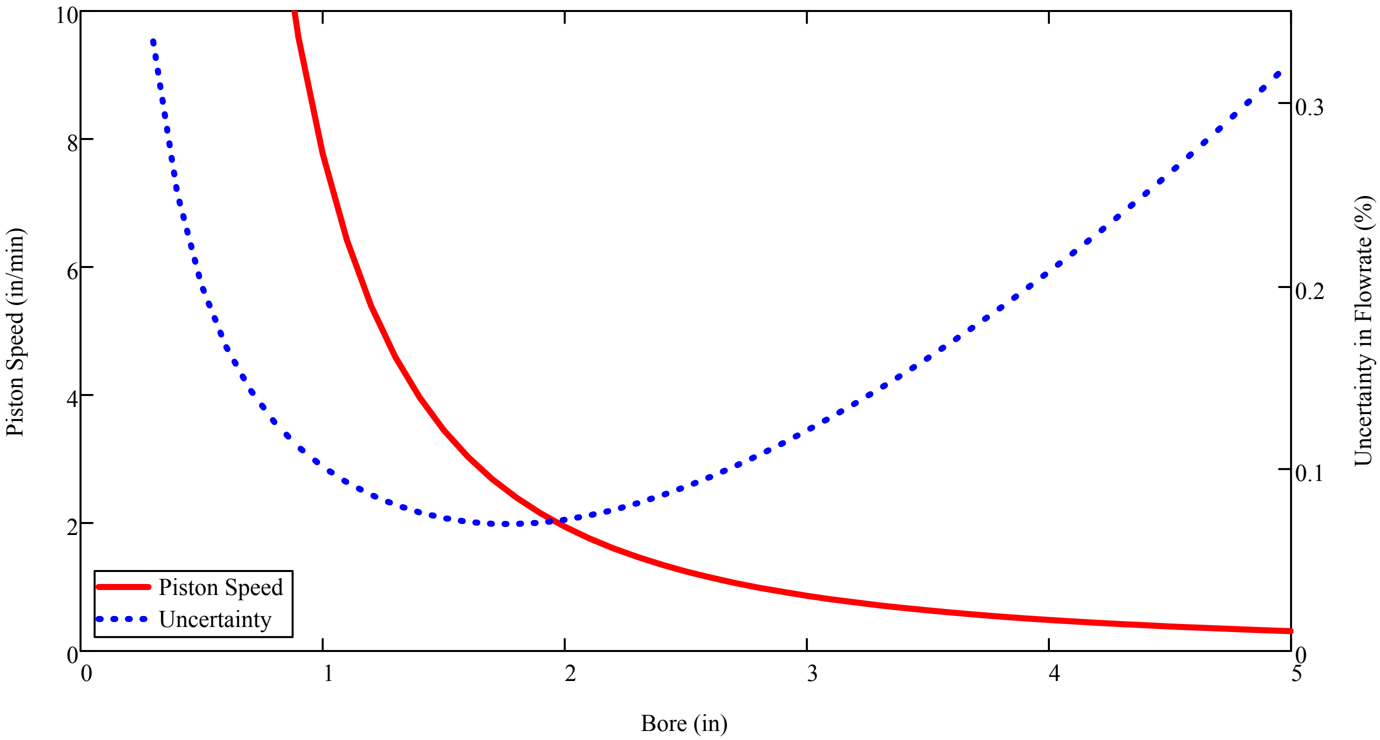
Substituting expressions in terms of bore we get:

$$D_{opt} = \left[\frac{32}{\pi^2} \cdot \left(\frac{\delta D \cdot Q}{\delta V}\right)^2\right]^{\frac{1}{6}}$$

Substituting all known values yields $\quad D_{opt} = 1.76 \cdot in$

Plugging this into the expression for the piston speed yields $\quad V_{opt} = 2.50 \cdot \dfrac{in}{min}$ and the uncertainty is $\quad u_{opt} = 0.0694 \cdot \%$

Graphs of the piston speed and the uncertainty in the flowrate as a function of the bore are shown on the following page.



Piston Speed (in/min)
Uncertainty in Flowrate (%)
Bore (in)
Piston Speed
Uncertainty
0
2
4
6
8
10
0
0.1
0.2
0.3
0
1
2
3
4
5

Problem 2.1

[Difficulty: 1]

2.1 For the velocity fields given below, determine:

a. whether the flow field is one-, two-, or three-dimensional, and why.

b. whether the flow is steady or unsteady, and why.

(The quantities a and b are constants.)

(1) $\vec{V} = [(ax+t)e^{by}]\hat{i}$ (2) $\vec{V} = (ax - by)\hat{i}$

(3) $\vec{V} = ax\hat{i} + [e^{bx}]\hat{j}$ (4) $\vec{V} = ax\hat{i} + bx^2\hat{j} + ax\hat{k}$

(5) $\vec{V} = ax\hat{i} + [e^{bt}]\hat{j}$ (6) $\vec{V} = ax\hat{i} + bx^2\hat{j} + ay\hat{k}$

(7) $\vec{V} = ax\hat{i} + [e^{bt}]\hat{j} + ay\hat{k}$ (8) $\vec{V} = ax\hat{i} + [e^{by}]\hat{j} + az\hat{k}$

Given: Velocity fields

Find: Whether flows are 1, 2 or 3D, steady or unsteady.

Solution:

(1)	$\vec{V} = \vec{V}(x,y)$	2D	$\vec{V} = \vec{V}(t)$	Unsteady
(2)	$\vec{V} = \vec{V}(x,y)$	2D	$\vec{V} \neq \vec{V}(t)$	Steady
(3)	$\vec{V} = \vec{V}(x)$	1D	$\vec{V} \neq \vec{V}(t)$	Steady
(4)	$\vec{V} = \vec{V}(x)$	1D	$\vec{V} \neq \vec{V}(t)$	Steady
(5)	$\vec{V} = \vec{V}(x)$	1D	$\vec{V} = \vec{V}(t)$	Unsteady
(6)	$\vec{V} = \vec{V}(x,y)$	2D	$\vec{V} \neq \vec{V}(t)$	Steady
(7)	$\vec{V} = \vec{V}(x,y)$	2D	$\vec{V} = \vec{V}(t)$	Unsteady
(8)	$\vec{V} = \vec{V}(x,y,z)$	3D	$\vec{V} \neq \vec{V}(t)$	Steady

Problem 2.2

[Difficulty: 1]

2.2 For the velocity fields given below, determine:

a. whether the flow field is one-, two-, or three-dimensional, and why.

b. whether the flow is steady or unsteady, and why.

(The quantities a and b are constants.)

(1) $\vec{V} = [ay^2 e^{-bt}]\hat{i}$ (2) $\vec{V} = ax^2\hat{i} + bx\hat{j} + c\hat{k}$

(3) $\vec{V} = axy\hat{i} - byt\hat{j}$ (4) $\vec{V} = ax\hat{i} - by\hat{j} + ct\hat{k}$

(5) $\vec{V} = [ae^{-bx}]\hat{i} + bt^2\hat{j}$ (6) $\vec{V} = a(x^2 + y^2)^{1/2}(1/z^3)\hat{k}$

(7) $\vec{V} = (ax + t)\hat{i} - by^2\hat{j}$ (8) $\vec{V} = ax^2\hat{i} + bxz\hat{j} + cy\hat{k}$

Given: Velocity fields

Find: Whether flows are 1, 2 or 3D, steady or unsteady.

Solution:

(1)	$\vec{V} = \vec{V}(y)$	1D	$\vec{V} = \vec{V}(t)$	Unsteady
(2)	$\vec{V} = \vec{V}(x)$	1D	$\vec{V} \neq \vec{V}(t)$	Steady
(3)	$\vec{V} = \vec{V}(x,y)$	2D	$\vec{V} = \vec{V}(t)$	Unsteady
(4)	$\vec{V} = \vec{V}(x,y)$	2D	$\vec{V} = \vec{V}(t)$	Unsteady
(5)	$\vec{V} = \vec{V}(x)$	1D	$\vec{V} = \vec{V}(t)$	Unsteady
(6)	$\vec{V} = \vec{V}(x,y,z)$	3D	$\vec{V} \neq \vec{V}(t)$	Steady
(7)	$\vec{V} = \vec{V}(x,y)$	2D	$\vec{V} = \vec{V}(t)$	Unsteady
(8)	$\vec{V} = \vec{V}(x,y,z)$	3D	$\vec{V} \neq \vec{V}(t)$	Steady

Problem 2.3

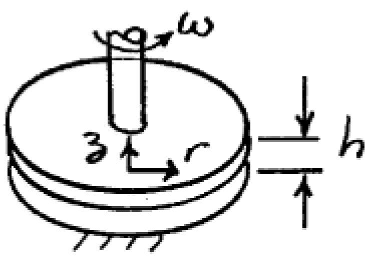
[Difficulty: 2]

2.3 A viscous liquid is sheared between two parallel disks; the upper disk rotates and the lower one is fixed. The velocity field between the disks is given by $\vec{V} = \hat{e}_\theta r\omega z/h$. (The origin of coordinates is located at the center of the lower disk; the upper disk is located at $z = h$.) What are the dimensions of this velocity field? Does this velocity field satisfy appropriate physical boundary conditions? What are they?

Given: Viscous liquid sheared between parallel disks.

Upper disk rotates, lower fixed.

Velocity field is: $\vec{V} = \hat{e}_\theta \dfrac{r\omega z}{h}$



Find:

a. Dimensions of velocity field.

b. Satisfy physical boundary conditions.

Solution: To find dimensions, compare to $\vec{V} = \vec{V}(x, y, z)$ form.

The given field is $\vec{V} = \vec{V}(r, z)$. Two space coordinates are included, so the field is 2-D.

Flow must satisfy the no-slip condition:

1. At lower disk, $\vec{V} = 0$ since stationary.

   $z = 0$, so $\vec{V} = \hat{e}_\theta \dfrac{r\omega 0}{h} = 0$, so satisfied.

2. At upper disk, $\vec{V} = \hat{e}_\theta r\omega$ since it rotates as a solid body.

   $z = h$, so $\vec{V} = \hat{e}_\theta \dfrac{r\omega h}{h} = \hat{e}_\theta r\omega$, so satisfied.

Problem 2.4

[Difficulty: 1]

2.4 For the velocity field $\vec{V} = Ax^2y\hat{i} + Bxy^2\hat{j}$, where $A = 2\ \mathrm{m^{-2}s^{-1}}$ and $B = 1\ \mathrm{m^{-2}s^{-1}}$, and the coordinates are measured in meters, obtain an equation for the flow streamlines. Plot several streamlines in the first quadrant.

Given: Velocity field

Find: Equation for streamlines

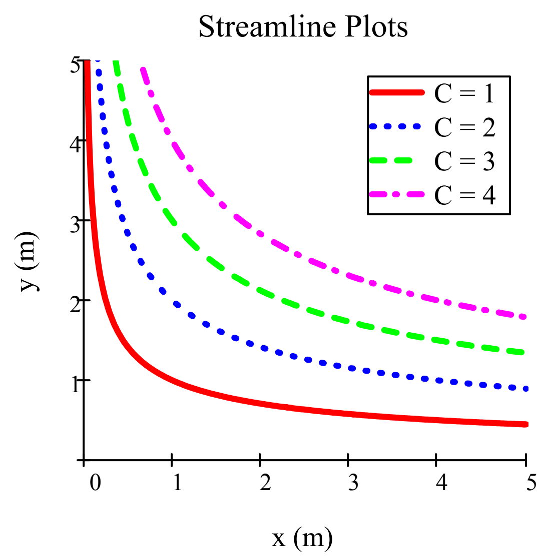
Solution:

For streamlines $\quad \frac{v}{u} = \frac{dy}{dx} = \frac{B \cdot x \cdot y^2}{A \cdot x^2 \cdot y} = \frac{B \cdot y}{A \cdot x}$

So, separating variables $\quad \frac{dy}{y} = \frac{B}{A} \cdot \frac{dx}{x}$

Integrating $\quad \ln(y) = \frac{B}{A} \cdot \ln(x) + c = -\frac{1}{2} \cdot \ln(x) + c$

The solution is $\quad y = \frac{C}{\sqrt{x}}$



The plot can be easily done in *Excel*.

Problem 2.5

[Difficulty: 2]

2.5 The velocity field $\vec{V} = Ax\hat{i} - Ay\hat{j}$, where $A = 2\ s^{-1}$, can be interpreted to represent flow in a corner. Find an equation for the flow streamlines. Explain the relevance of A. Plot several streamlines in the first quadrant, including the one that passes through the point $(x, y) = (0, 0)$.

Given:

Velocity field

Find:

Equation for streamlines; Plot several in the first quadrant, including one that passes through point (0,0)

Solution:

Governing equation: For streamlines $\frac{v}{u} = \frac{dy}{dx}$

Assumption: 2D flow

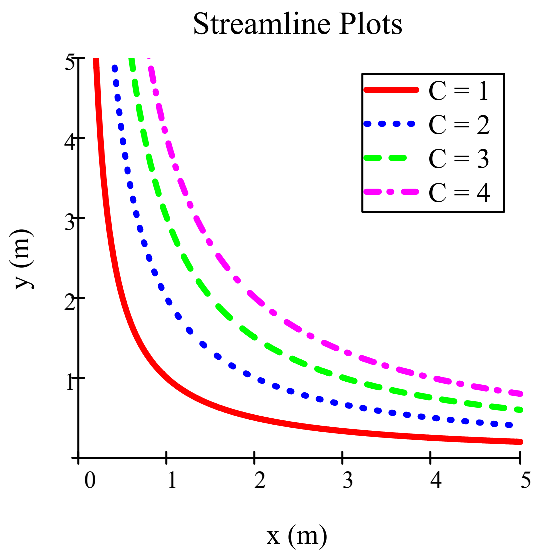
Hence $\frac{v}{u} = \frac{dy}{dx} = -\frac{A \cdot y}{A \cdot x} = -\frac{y}{x}$

So, separating variables $\frac{dy}{y} = -\frac{dx}{x}$

Integrating $\ln(y) = -\ln(x) + c$

The solution is $\ln(x \cdot y) = c$

or $y = \frac{C}{x}$



The plot can be easily done in *Excel*.

The streamline passing through (0,0) is given by the vertical axis, then the horizontal axis.
The value of A is irrelevant to streamline shapes but IS relevant for computing the velocity at each point.

Problem 2.6

[Difficulty: 1]

2.6 A velocity field is specified as $\vec{V} = axy\hat{i} + by^2\hat{j}$, where $a = 2\ \mathrm{m^{-1}s^{-1}}$, $b = -6\ \mathrm{m^{-1}s^{-1}}$, and the coordinates are measured in meters. Is the flow field one-, two-, or three-dimensional? Why? Calculate the velocity components at the point (2, ½). Develop an equation for the streamline passing through this point. Plot several streamlines in the first quadrant including the one that passes through the point (2, ½).

Given:

Velocity field

Find:

Whether field is 1D, 2D or 3D; Velocity components at (2,1/2); Equation for streamlines; Plot

Solution:

The velocity field is a function of *x* and *y*. It is therefore 2D.

At point (2,1/2), the velocity components are

$$u = a\cdot x\cdot y = 2\cdot\frac{1}{m\cdot s}\times 2\cdot m\times\frac{1}{2}\cdot m \qquad u = 2\cdot\frac{m}{s}$$

$$v = b\cdot y^2 = -6\cdot\frac{1}{m\cdot s}\times\left(\frac{1}{2}\cdot m\right)^2 \qquad v = -\frac{3}{2}\cdot\frac{m}{s}$$

For streamlines

$$\frac{v}{u} = \frac{dy}{dx} = \frac{b\cdot y^2}{a\cdot x\cdot y} = \frac{b\cdot y}{a\cdot x}$$

So, separating variables

$$\frac{dy}{y} = \frac{b}{a}\cdot\frac{dx}{x}$$

Integrating

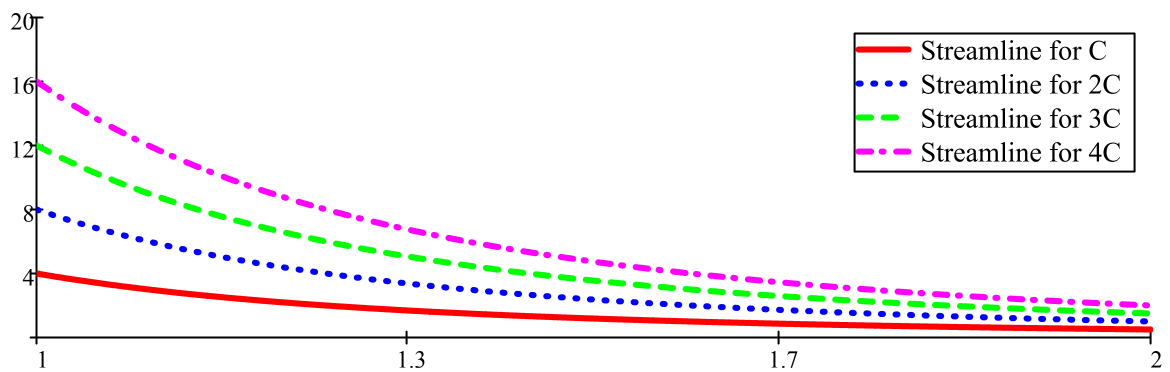
$$\ln(y) = \frac{b}{a}\cdot\ln(x) + c \qquad y = C\cdot x^{\frac{b}{a}}$$

The solution is

$$y = C\cdot x^{-3}$$

The streamline passing through point (2,1/2) is given by

$$\frac{1}{2} = C\cdot 2^{-3} \qquad C = \frac{1}{2}\cdot 2^3 \qquad C = 4 \qquad y = \frac{4}{x^3}$$



This can be plotted in *Excel*.

Problem 2.7

[Difficulty: 2]

2.7 A velocity field is given by $\vec{V} = ax\hat{i} - bty\hat{j}$, where $a = 1\ \mathrm{s}^{-1}$ and $b = 1\ \mathrm{s}^{-2}$. Find the equation of the streamlines at any time t. Plot several streamlines in the first quadrant at $t = 0$ s, $t = 1$ s, and $t = 20$ s.

Given: Velocity field

Find: Equation for streamlines; Plot streamlines

Solution:

For streamlines $$\frac{v}{u} = \frac{dy}{dx} = \frac{-b\cdot t\cdot y}{a\cdot x}$$

So, separating variables $$\frac{dy}{y} = \frac{-b\cdot t}{a}\cdot\frac{dx}{x}$$

Integrating $$\ln(y) = \frac{-b\cdot t}{a}\cdot\ln(x)$$

The solution is $$y = c\cdot x^{\frac{-b}{a}\cdot t}$$

For $t = 0$ s $\quad y = c \qquad$ For $t = 1$ s $\quad y = \frac{c}{x} \qquad$ For $t = 20$ s $\quad y = c\cdot x^{-20}$

t = 0

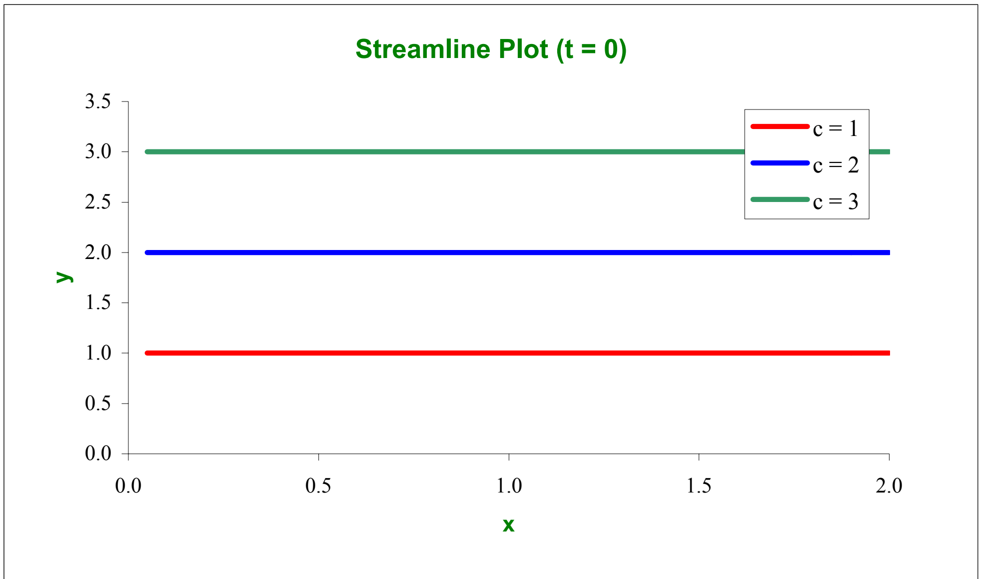
	c = 1	c = 2	c = 3
x	**y**	**y**	**y**
0.05	1.00	2.00	3.00
0.10	1.00	2.00	3.00
0.20	1.00	2.00	3.00
0.30	1.00	2.00	3.00
0.40	1.00	2.00	3.00
0.50	1.00	2.00	3.00
0.60	1.00	2.00	3.00
0.70	1.00	2.00	3.00
0.80	1.00	2.00	3.00
0.90	1.00	2.00	3.00
1.00	1.00	2.00	3.00
1.10	1.00	2.00	3.00
1.20	1.00	2.00	3.00
1.30	1.00	2.00	3.00
1.40	1.00	2.00	3.00
1.50	1.00	2.00	3.00
1.60	1.00	2.00	3.00
1.70	1.00	2.00	3.00
1.80	1.00	2.00	3.00
1.90	1.00	2.00	3.00
2.00	1.00	2.00	3.00

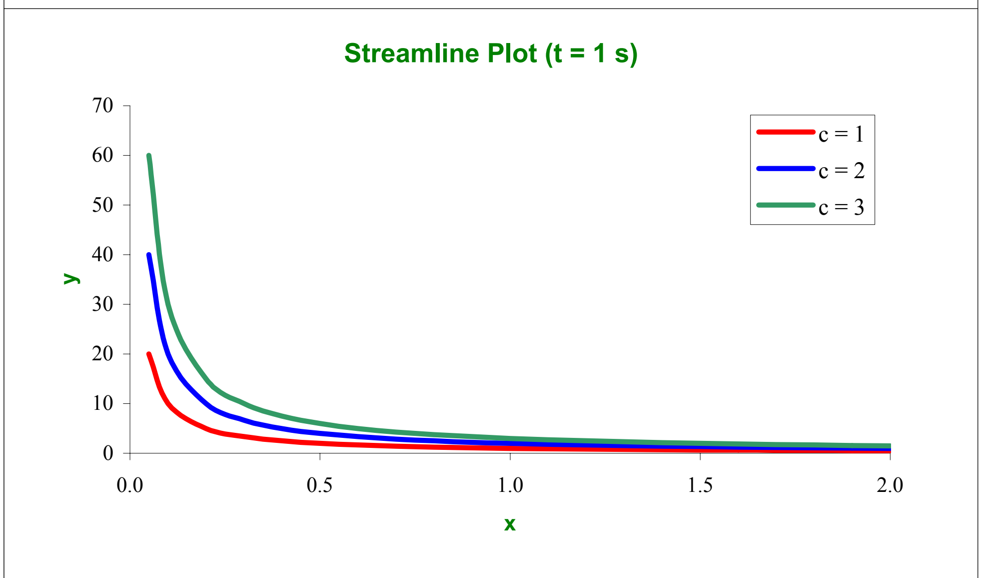
t =1 s
(### means too large to view)

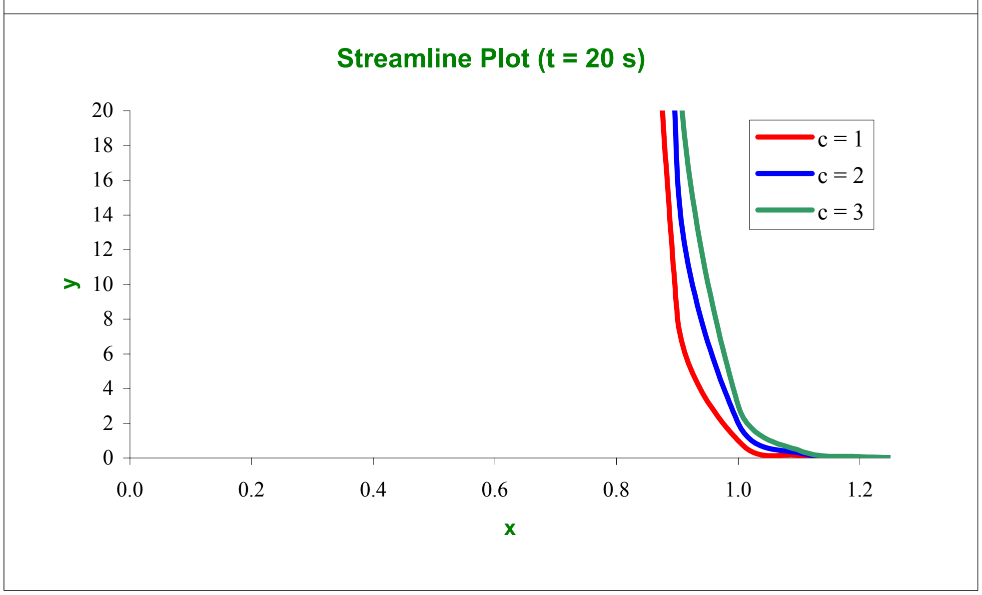
	c = 1	c = 2	c = 3
x	**y**	**y**	**y**
0.05	20.00	40.00	60.00
0.10	10.00	20.00	30.00
0.20	5.00	10.00	15.00
0.30	3.33	6.67	10.00
0.40	2.50	5.00	7.50
0.50	2.00	4.00	6.00
0.60	1.67	3.33	5.00
0.70	1.43	2.86	4.29
0.80	1.25	2.50	3.75
0.90	1.11	2.22	3.33
1.00	1.00	2.00	3.00
1.10	0.91	1.82	2.73
1.20	0.83	1.67	2.50
1.30	0.77	1.54	2.31
1.40	0.71	1.43	2.14
1.50	0.67	1.33	2.00
1.60	0.63	1.25	1.88
1.70	0.59	1.18	1.76
1.80	0.56	1.11	1.67
1.90	0.53	1.05	1.58
2.00	0.50	1.00	1.50

t = 20 s

	c = 1	c = 2	c = 3
x	**y**	**y**	**y**
0.05	######	######	######
0.10	######	######	######
0.20	######	######	######
0.30	######	######	######
0.40	######	######	######
0.50	######	######	######
0.60	######	######	######
0.70	######	######	######
0.80	86.74	173.47	260.21
0.90	8.23	16.45	24.68
1.00	1.00	2.00	3.00
1.10	0.15	0.30	0.45
1.20	0.03	0.05	0.08
1.30	0.01	0.01	0.02
1.40	0.00	0.00	0.00
1.50	0.00	0.00	0.00
1.60	0.00	0.00	0.00
1.70	0.00	0.00	0.00
1.80	0.00	0.00	0.00
1.90	0.00	0.00	0.00
2.00	0.00	0.00	0.00


Streamline Plot (t = 0)
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
y
0.0
0.5
1.0
1.5
2.0
x
c = 1
c = 2
c = 3


Streamline Plot (t = 1 s)
70
60
50
40
30
20
10
0
y
0.0
0.5
1.0
1.5
2.0
x
c = 1
c = 2
c = 3


Streamline Plot (t = 20 s)
20
18
16
14
12
10
8
6
4
2
0
y
0.0
0.2
0.4
0.6
0.8
1.0
1.2
x
c = 1
c = 2
c = 3

Problem 2.8

[Difficulty: 2]

2.8 A velocity field is given by $\vec{V} = ax^3\hat{i} + bxy^3\hat{j}$, where $a = 1\ \text{m}^{-2}\text{s}^{-1}$ and $b = 1\ \text{m}^{-3}\text{s}^{-1}$. Find the equation of the streamlines. Plot several streamlines in the first quadrant.

Given: Velocity field

Find: Equation for streamlines; Plot streamlines

Solution:

Streamlines are given by
$$\frac{v}{u} = \frac{dy}{dx} = \frac{b \cdot x \cdot y^3}{a \cdot x^3}$$

So, separating variables
$$\frac{dy}{y^3} = \frac{b \cdot dx}{a \cdot x^2}$$

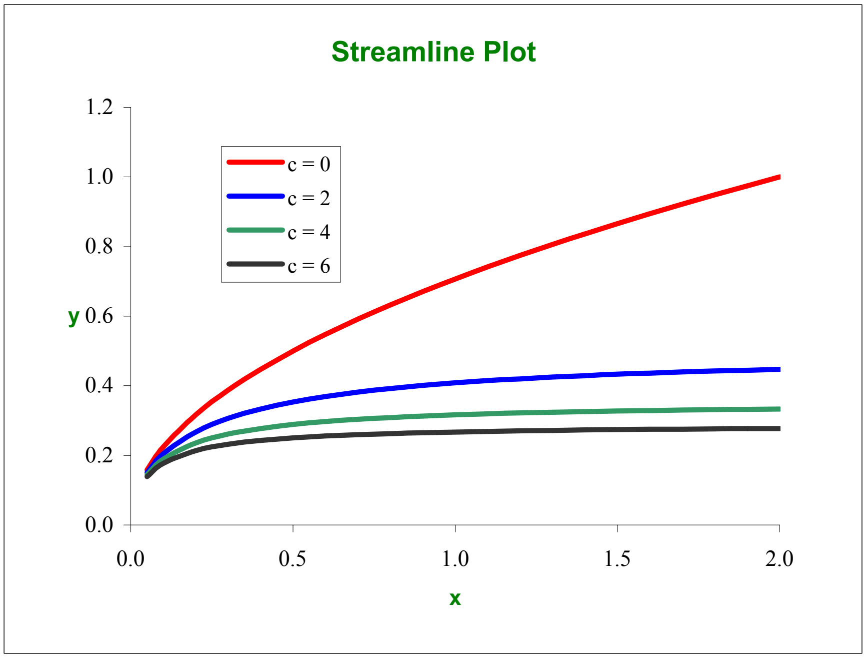
Integrating
$$-\frac{1}{2 \cdot y^2} = \frac{b}{a} \cdot \left(-\frac{1}{x}\right) + C$$

The solution is
$$y = \frac{1}{\sqrt{2 \cdot \left(\frac{b}{a \cdot x} + C\right)}}$$

Note: For convenience the sign of C is changed.

a = 1
b = 1

C =	0	2	4	6
x	**y**	**y**	**y**	**y**
0.05	0.16	0.15	0.14	0.14
0.10	0.22	0.20	0.19	0.18
0.20	0.32	0.27	0.24	0.21
0.30	0.39	0.31	0.26	0.23
0.40	0.45	0.33	0.28	0.24
0.50	0.50	0.35	0.29	0.25
0.60	0.55	0.37	0.30	0.26
0.70	0.59	0.38	0.30	0.26
0.80	0.63	0.39	0.31	0.26
0.90	0.67	0.40	0.31	0.27
1.00	0.71	0.41	0.32	0.27
1.10	0.74	0.41	0.32	0.27
1.20	0.77	0.42	0.32	0.27
1.30	0.81	0.42	0.32	0.27
1.40	0.84	0.43	0.33	0.27
1.50	0.87	0.43	0.33	0.27
1.60	0.89	0.44	0.33	0.27
1.70	0.92	0.44	0.33	0.28
1.80	0.95	0.44	0.33	0.28
1.90	0.97	0.44	0.33	0.28
2.00	1.00	0.45	0.33	0.28



Problem 2.9

[Difficulty: 2]

2.9 A flow is described by the velocity field $\vec{V} = (Ax + B)\hat{i} + (-Ay)\hat{j}$, where $A = 10$ ft/s/ft and $B = 20$ ft/s. Plot a few streamlines in the xy plane, including the one that passes through the point $(x, y) = (1, 2)$.

Given: Velocity field

Find: Plot streamlines

Solution:

Streamlines are given by $\frac{v}{u} = \frac{dy}{dx} = \frac{-A \cdot y}{A \cdot x + B}$

So, separating variables $\frac{dy}{-A \cdot y} = \frac{dx}{A \cdot x + B}$

Integrating $-\frac{1}{A}\ln(y) = \frac{1}{A}\cdot\ln\left(x + \frac{B}{A}\right)$

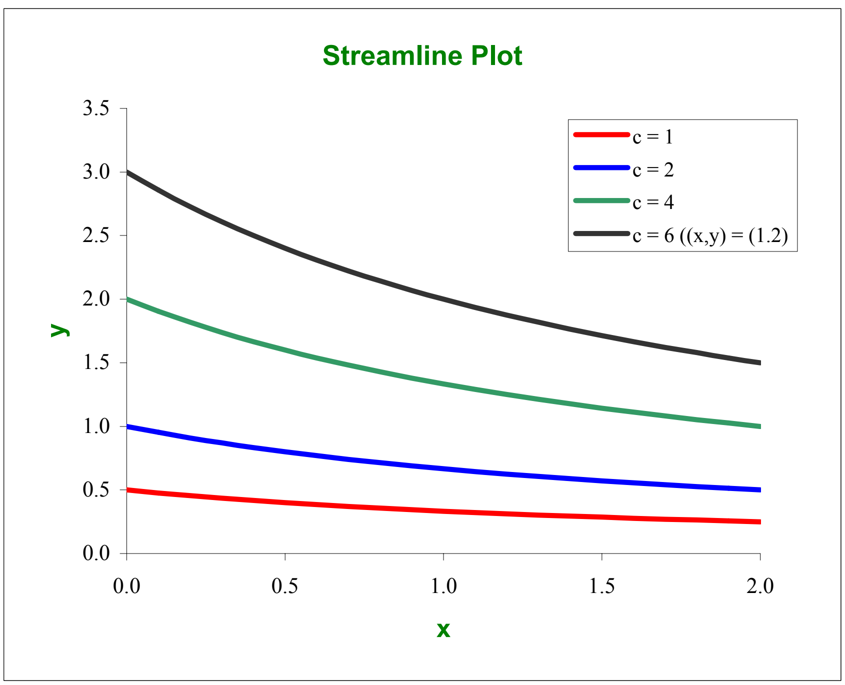
The solution is $y = \frac{C}{x + \frac{B}{A}}$

For the streamline that passes through point (x,y) = (1,2) $C = y \cdot \left(x + \frac{B}{A}\right) = 2 \cdot \left(1 + \frac{20}{10}\right) = 6$ $y = \frac{6}{x + \frac{20}{10}}$

$$y = \frac{6}{x + 2}$$

A = **10**
B = **20**
C =

	1	2	4	6
x	**y**	**y**	**y**	**y**
0.00	0.50	1.00	2.00	3.00
0.10	0.48	0.95	1.90	2.86
0.20	0.45	0.91	1.82	2.73
0.30	0.43	0.87	1.74	2.61
0.40	0.42	0.83	1.67	2.50
0.50	0.40	0.80	1.60	2.40
0.60	0.38	0.77	1.54	2.31
0.70	0.37	0.74	1.48	2.22
0.80	0.36	0.71	1.43	2.14
0.90	0.34	0.69	1.38	2.07
1.00	0.33	0.67	1.33	2.00
1.10	0.32	0.65	1.29	1.94
1.20	0.31	0.63	1.25	1.88
1.30	0.30	0.61	1.21	1.82
1.40	0.29	0.59	1.18	1.76
1.50	0.29	0.57	1.14	1.71
1.60	0.28	0.56	1.11	1.67
1.70	0.27	0.54	1.08	1.62
1.80	0.26	0.53	1.05	1.58
1.90	0.26	0.51	1.03	1.54
2.00	0.25	0.50	1.00	1.50



Problem 2.10

[Difficulty: 2]

2.10 The velocity for a steady, incompressible flow in the xy plane is given by $\vec{V} = \hat{i}A/x + \hat{j}Ay/x^2$, where $A = 2\ m^2/s$, and the coordinates are measured in meters. Obtain an equation for the streamline that passes through the point $(x, y) = (1, 3)$. Calculate the time required for a fluid particle to move from $x = 1$ m to $x = 2$ m in this flow field.

Given: Velocity field

Find: Equation for streamline through (1,3)

Solution:

For streamlines $\quad \dfrac{v}{u} = \dfrac{dy}{dx} = \dfrac{A \cdot \frac{y}{x^2}}{\frac{A}{x}} = \dfrac{y}{x}$

So, separating variables $\quad \dfrac{dy}{y} = \dfrac{dx}{x}$

Integrating $\quad \ln(y) = \ln(x) + c$

The solution is $\quad y = C \cdot x \quad$ which is the equation of a straight line.

For the streamline through point (1,3) $\quad 3 = C \cdot 1 \quad C = 3 \quad$ and $\quad y = 3 \cdot x$

For a particle $\quad u_p = \dfrac{dx}{dt} = \dfrac{A}{x} \quad$ or $\quad x \cdot dx = A \cdot dt \quad x = \sqrt{2 \cdot A \cdot t + c} \quad t = \dfrac{x^2}{2 \cdot A} - \dfrac{c}{2 \cdot A}$

Hence the time for a particle to go from x = 1 to x = 2 m is

$\Delta t = t(x = 2) - t(x = 1) \quad \Delta t = \dfrac{(2 \cdot m)^2 - c}{2 \cdot A} - \dfrac{(1 \cdot m)^2 - c}{2 \cdot A} = \dfrac{4 \cdot m^2 - 1 \cdot m^2}{2 \times 2 \cdot \frac{m^2}{s}} \quad \Delta t = 0.75 \cdot s$

Problem 2.11

[Difficulty: 3]

2.11 The flow field for an atmospheric flow is given by

$$\vec{V} = -\frac{My}{2\pi}\hat{i} + \frac{Mx}{2\pi}\hat{j}$$

where $M = 1\ \mathrm{s}^{-1}$, and the x and y coordinates are the parallel to the local latitude and longitude. Plot the velocity magnitude along the x axis, along the y axis, and along the line $y = x$, and discuss the velocity direction with respect to these three axes. For each plot use a range x or $y = 0$ km to 1 km. Find the equation for the streamlines and sketch several of them. What does this flow field model?

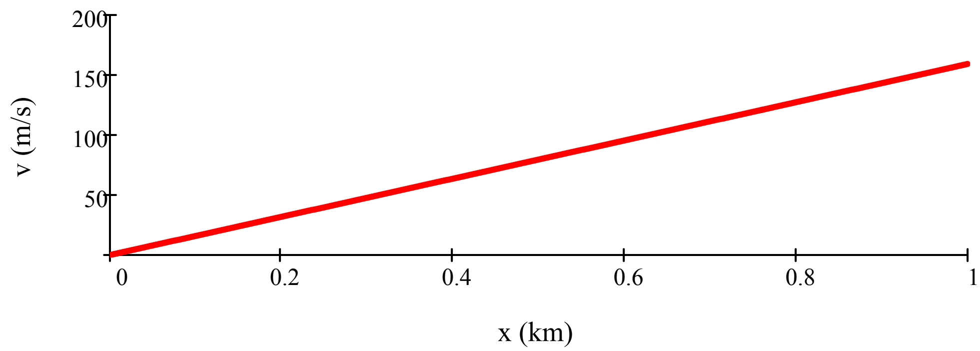
Given: Flow field

Find: Plot of velocity magnitude along axes, and y = x; Equation for streamlines

Solution:

On the x axis, y = 0, so $\quad u = -\frac{M \cdot y}{2 \cdot \pi} = 0 \quad\quad v = \frac{M \cdot x}{2 \cdot \pi}$

Plotting

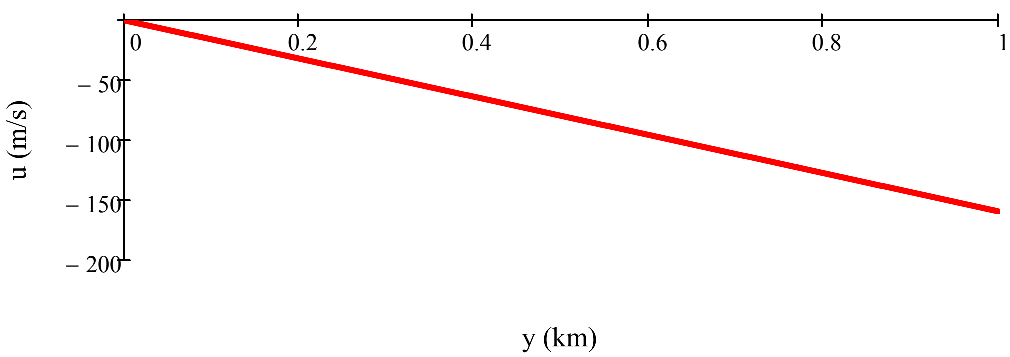


The velocity is perpendicular to the axis and increases linearly with distance x.

This can also be plotted in Excel.

On the y axis, x = 0, so $\quad u = -\frac{M \cdot y}{2 \cdot \pi} \quad\quad v = \frac{M \cdot x}{2 \cdot \pi} = 0$

Plotting



The velocity is perpendicular to the axis and increases linearly with distance y.

This can also be plotted in Excel.

On the y = x axis $\quad u = -\frac{M \cdot y}{2 \cdot \pi} = -\frac{M \cdot x}{2 \cdot \pi} \quad v = \frac{M \cdot x}{2 \cdot \pi}$

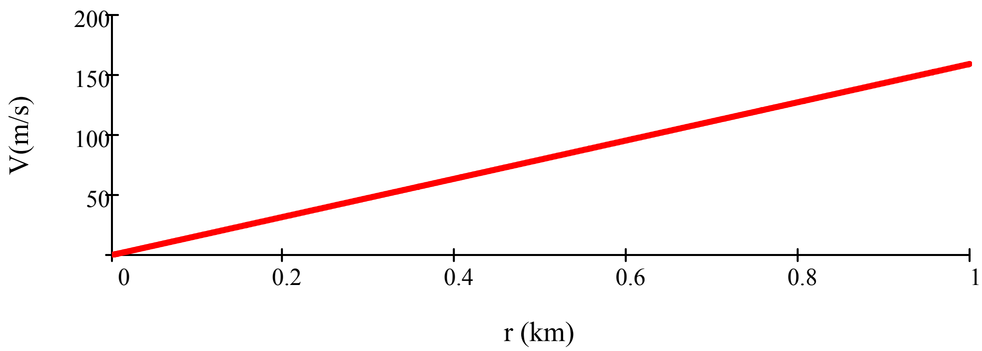
The flow is perpendicular to line y = x: Slope of line y = x: $\quad 1$

Slope of trajectory of motion: $\quad \frac{u}{v} = -1$

If we define the radial position: $\quad r = \sqrt{x^2 + y^2} \quad$ then along y = x $\quad r = \sqrt{x^2 + x^2} = \sqrt{2} \cdot x$

Then the magnitude of the velocity along y = x is $V = \sqrt{u^2 + v^2} = \frac{M}{2 \cdot \pi} \cdot \sqrt{x^2 + x^2} = \frac{M \cdot \sqrt{2} \cdot x}{2 \cdot \pi} = \frac{M \cdot r}{2 \cdot \pi}$

Plotting



This can also be plotted in Excel.

For streamlines $\quad \frac{v}{u} = \frac{dy}{dx} = \frac{\frac{M \cdot x}{2 \cdot \pi}}{-\frac{M \cdot y}{2 \cdot \pi}} = -\frac{x}{y}$

So, separating variables $\quad y \cdot dy = -x \cdot dx$

Integrating $\quad \frac{y^2}{2} = -\frac{x^2}{2} + c$

The solution is $\quad x^2 + y^2 = C \quad$ which is the equation of a circle.

The streamlines form a set of concentric circles.

This flow models a rigid body vortex flow. See Example 5.6 for streamline plots. Streamlines are circular, and the velocity approaches zero as we approach the center. In Problem 2.10, we see that the streamlines are also circular. In a real tornado, at large distances from the center, the velocities behave as in Problem 2.10; close to the center, they behave as in this problem.

Problem 2.12

[Difficulty: 3]

2.12 The flow field for an atmospheric flow is given by

$$\vec{V} = -\frac{Ky}{2\pi(x^2+y^2)}\hat{i} + \frac{Kx}{2\pi(x^2+y^2)}\hat{j}$$

where $K = 10^5\ \mathrm{m^2/s}$, and the x and y coordinates are parallel to the local latitude and longitude. Plot the velocity magnitude along the x axis, along the y axis, and along the line $y = x$, and discuss the velocity direction with respect to these three axes. For each plot use a range x or $y = -1$ km to 1 km, excluding $|x|$ or $|y| < 100$ m. Find the equation for the streamlines and sketch several of them. What does this flow field model?

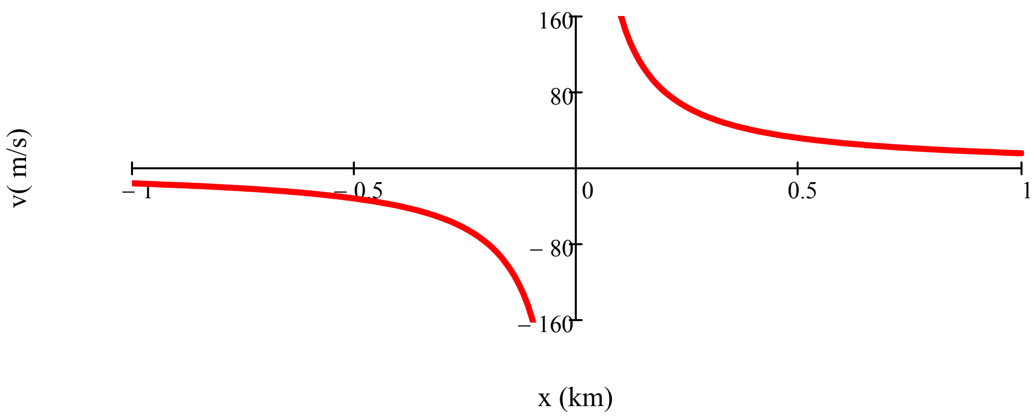
Given: Flow field

Find: Plot of velocity magnitude along axes, and y = x; Equation of streamlines

Solution:

On the x axis, y = 0, so $u = -\frac{K\cdot y}{2\cdot\pi\cdot\left(x^2+y^2\right)} = 0$ $v = \frac{K\cdot x}{2\cdot\pi\cdot\left(x^2+y^2\right)} = \frac{K}{2\cdot\pi\cdot x}$

Plotting

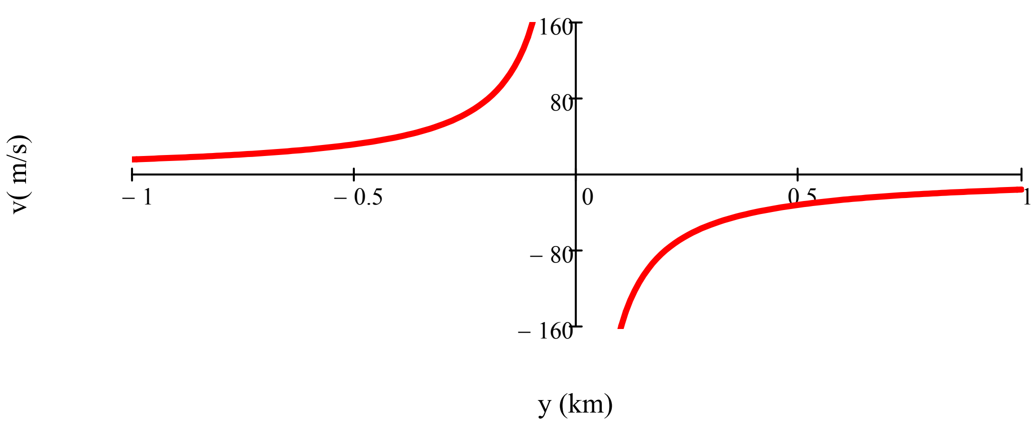


The velocity is perpendicular to the axis, is very high close to the origin, and falls off to zero.

This can also be plotted in Excel.

On the y axis, x = 0, so $u = -\frac{K\cdot y}{2\cdot\pi\cdot\left(x^2+y^2\right)} = -\frac{K}{2\cdot\pi\cdot y}$ $v = \frac{K\cdot x}{2\cdot\pi\cdot\left(x^2+y^2\right)} = 0$

Plotting



The velocity is perpendicular to the axis, is very high close to the origin, and falls off to zero.
This can also be plotted in Excel.

On the y = x axis $\quad u = -\frac{K \cdot x}{2 \cdot \pi \cdot (x^2 + x^2)} = -\frac{K}{4 \cdot \pi \cdot x} \qquad v = \frac{K \cdot x}{2 \cdot \pi \cdot (x^2 + x^2)} = \frac{K}{4 \cdot \pi \cdot x}$

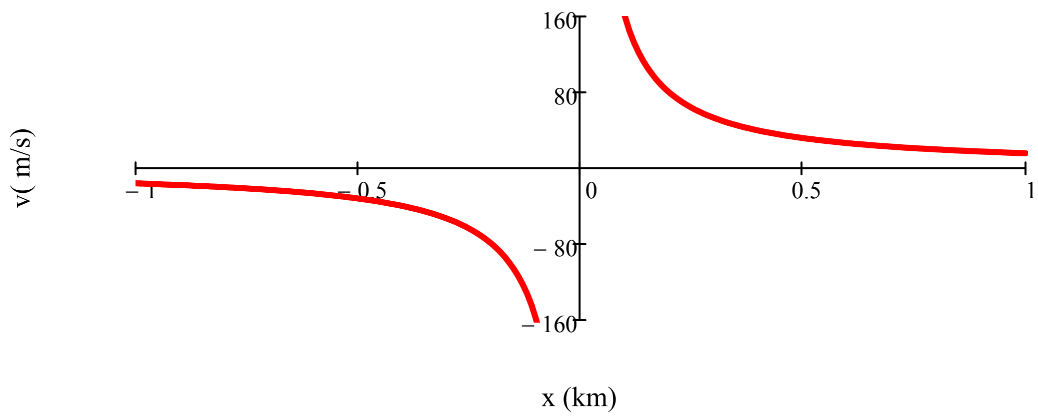
The flow is perpendicular to line y = x: $\quad$ Slope of line y = x: $\quad 1$

Slope of trajectory of motion: $\quad \frac{u}{v} = -1$

If we define the radial position: $\quad r = \sqrt{x^2 + y^2}$ $\quad$ then along y = x $\quad r = \sqrt{x^2 + x^2} = \sqrt{2} \cdot x$

Then the magnitude of the velocity along y = x is $\quad V = \sqrt{u^2 + v^2} = \frac{K}{4 \cdot \pi} \cdot \sqrt{\frac{1}{x^2} + \frac{1}{x^2}} = \frac{K}{2 \cdot \pi \cdot \sqrt{2} \cdot x} = \frac{K}{2 \cdot \pi \cdot r}$

Plotting



This can also be plotted in Excel.

For streamlines $\quad \frac{v}{u} = \frac{dy}{dx} = \frac{\frac{K \cdot x}{2 \cdot \pi \cdot (x^2+y^2)}}{-\frac{K \cdot y}{2 \cdot \pi \cdot (x^2 + y^2)}} = -\frac{x}{y}$

So, separating variables $\quad y \cdot dy = -x \cdot dx$

Integrating $\quad \frac{y^2}{2} = -\frac{x^2}{2} + c$

The solution is $\quad x^2 + y^2 = C$ $\quad$ which is the equation of a circle.

Streamlines form a set of concentric circles.

This flow models a vortex flow. See Example 5.6 for streamline plots. Streamlines are circular, and the velocity approaches infinity as we approach the center. In Problem 2.11, we see that the streamlines are also circular. In a real tornado, at large distances from the center, the velocities behave as in this problem; close to the center, they behave as in Problem 2.11.

Problem 2.13

[Difficulty: 3]

2.13 A flow field is given by

$$\vec{V} = -\frac{qx}{2\pi(x^2+y^2)}\hat{i} - \frac{qy}{2\pi(x^2+y^2)}\hat{j}$$

where $q = 5 \times 10^4$ m^2/s. Plot the velocity magnitude along the x axis, along the y axis, and along the line $y = x$, and discuss the velocity direction with respect to these three axes. For each plot use a range x or $y = -1$ km to 1 km, excluding $|x|$ or $|y| < 100$ m. Find the equation for the streamlines and sketch several of them. What does this flow field model?

Given:
Flow field

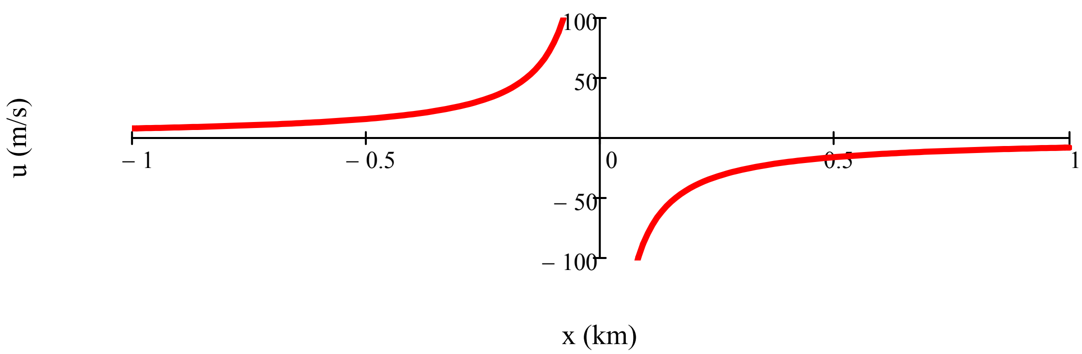
Find:
Plot of velocity magnitude along axes, and y = x; Equations of streamlines

Solution:

On the x axis, y = 0, so

$$u = -\frac{q\cdot x}{2\cdot\pi\cdot\left(x^2+y^2\right)} = -\frac{q}{2\cdot\pi\cdot x} \qquad v = -\frac{q\cdot y}{2\cdot\pi\cdot\left(x^2+y^2\right)} = 0$$

Plotting

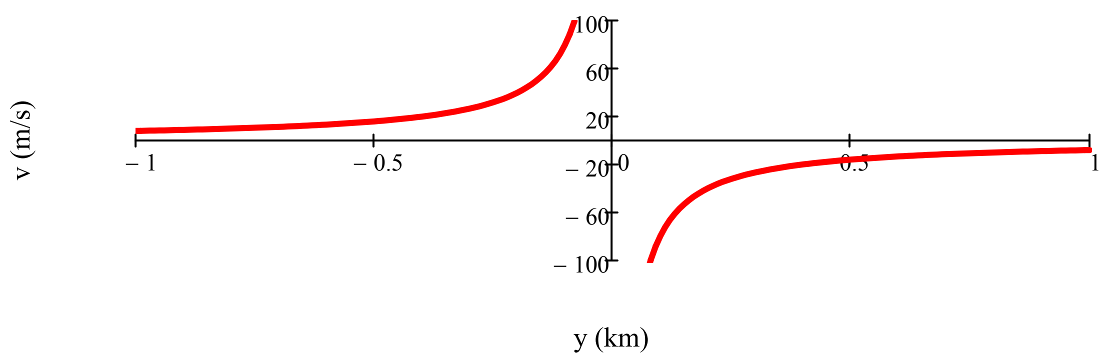


The velocity is very high close to the origin, and falls off to zero. It is also along the axis. This can be plotted in *Excel*.

On the y axis, x = 0, so

$$u = -\frac{q\cdot x}{2\cdot\pi\cdot\left(x^2+y^2\right)} = 0 \qquad v = -\frac{q\cdot y}{2\cdot\pi\cdot\left(x^2+y^2\right)} = -\frac{q}{2\cdot\pi\cdot y}$$

Plotting



The velocity is again very high close to the origin, and falls off to zero. It is also along the axis.

This can also be plotted in *Excel*.

On the y = x axis $\quad u = -\dfrac{q\cdot x}{2\cdot\pi\cdot\left(x^2 + x^2\right)} = -\dfrac{q}{4\cdot\pi\cdot x} \qquad v = -\dfrac{q\cdot x}{2\cdot\pi\cdot\left(x^2 + x^2\right)} = -\dfrac{q}{4\cdot\pi\cdot x}$

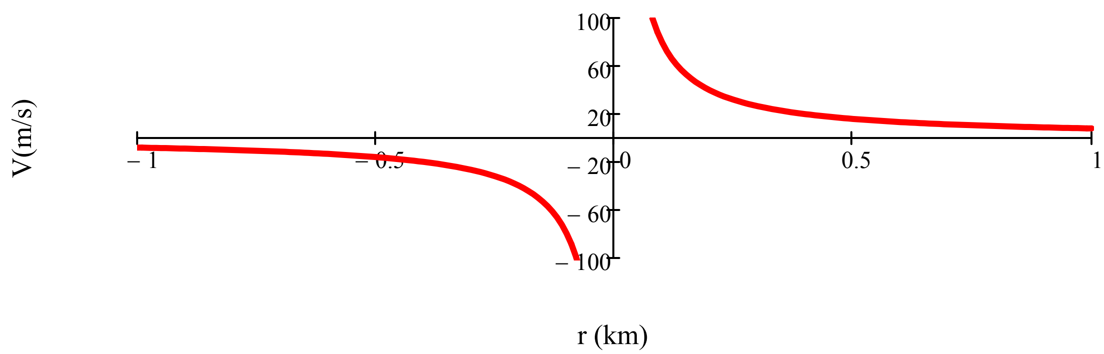
The flow is parallel to line y = x: $\quad$ Slope of line y = x: $\quad 1$

Slope of trajectory of motion: $\quad \dfrac{v}{u} = 1$

If we define the radial position: $\quad r = \sqrt{x^2 + y^2} \quad$ then along y = x $\quad r = \sqrt{x^2 + x^2} = \sqrt{2}\cdot x$

Then the magnitude of the velocity along y = x is $\quad V = \sqrt{u^2 + v^2} = \dfrac{q}{4\cdot\pi}\cdot\sqrt{\dfrac{1}{x^2} + \dfrac{1}{x^2}} = \dfrac{q}{2\cdot\pi\cdot\sqrt{2}\cdot x} = \dfrac{q}{2\cdot\pi\cdot r}$

Plotting



This can also be plotted in Excel.

For streamlines $\quad \dfrac{v}{u} = \dfrac{dy}{dx} = \dfrac{-\dfrac{q\cdot y}{2\cdot\pi\cdot\left(x^2 + y^2\right)}}{-\dfrac{q\cdot x}{2\cdot\pi\cdot\left(x^2 + y^2\right)}} = \dfrac{y}{x}$

So, separating variables $\quad \dfrac{dy}{y} = \dfrac{dx}{x}$

Integrating $\quad \ln(y) = \ln(x) + c$

The solution is $\quad y = C\cdot x \quad$ which is the equation of a straight line.

This flow field corresponds to a sink (discussed in Chapter 6).

Problem 2.14

[Difficulty: 2]

2.14 Beginning with the velocity field of Problem 2.5, show that the parametric equations for particle motion are given by $x_p = c_1 e^{At}$ and $y_p = c_2 e^{-At}$. Obtain the equation for the pathline of the particle located at the point $(x, y) = (2, 2)$ at the instant $t = 0$. Compare this pathline with the streamline through the same point.

Given: Velocity field

Find: Proof that the parametric equations for particle motion are $x_p = c_1 \cdot e^{A \cdot t}$ and $y_p = c_2 \cdot e^{-A \cdot t}$; pathline that was at (2,2) at t = 0; compare to streamline through same point, and explain why they are similar or not.

Solution:

Governing equations: For pathlines $u_p = \frac{dx}{dt}$ $\quad v_p = \frac{dy}{dt}$ $\quad$ For streamlines $\frac{v}{u} = \frac{dy}{dx}$

Assumption: 2D flow

Hence for pathlines $\quad u_p = \frac{dx}{dt} = A \cdot x \qquad v_p = \frac{dy}{dt} = -A \cdot y$

So, separating variables $\quad \frac{dx}{x} = A \cdot dt \qquad \frac{dy}{y} = -A \cdot dt$

Integrating $\quad \ln(x) = A \cdot t + C_1 \qquad \ln(y) = -A \cdot t + C_2$

$$x = e^{A \cdot t + C_1} = e^{C_1} \cdot e^{A \cdot t} = c_1 \cdot e^{A \cdot t} \qquad y = e^{-A \cdot t + C_2} = e^{C_2} \cdot e^{-A \cdot t} = c_2 \cdot e^{-A \cdot t}$$

The pathlines are $\quad x = c_1 \cdot e^{A \cdot t} \qquad y = c_2 \cdot e^{-A \cdot t}$

Eliminating t $\quad t = \frac{1}{A} \cdot \ln\left(\frac{x}{c_1}\right) = -\frac{1}{A} \cdot \ln\left(\frac{y}{c_2}\right) \qquad \ln\left(x^{\frac{1}{A}} \cdot y^{\frac{1}{A}}\right) = \text{const} \quad$ or $\quad \ln\left(x^A \cdot y^A\right) = \text{const}$

so $\quad x^A \cdot y^A = \text{const}$ or $\quad x \cdot y = 4 \quad$ for given data

For streamlines $\quad \frac{v}{u} = \frac{dy}{dx} = -\frac{A \cdot y}{A \cdot x} = \frac{y}{x}$

So, separating variables $\quad \frac{dy}{y} = -\frac{dx}{x}$

Integrating $\quad \ln(y) = -\ln(x) + c$

The solution is $\quad \ln(x \cdot y) = c \quad$ or $\quad x \cdot y = \text{const} \quad$ or $\quad x \cdot y = 4 \quad$ for given data

The streamline passing through (2,2) and the pathline that started at (2,2) coincide because the flow is steady!

Problem 2.15

[Difficulty: 2]

2.15 A flow field is given by $\vec{V} = Ax\hat{i} + 2Ay\hat{j}$, where $A = 2\ \mathrm{s}^{-1}$. Verify that the parametric equations for particle motion are given by $x_p = c_1 e^{At}$ and $y_p = c_2 e^{2At}$. Obtain the equation for the pathline of the particle located at the point $(x, y) = (2,2)$ at the instant $t = 0$. Compare this pathline with the streamline through the same point.

Given: Velocity field

Find: Proof that the parametric equations for particle motion are $x_p = c_1 \cdot e^{A \cdot t}$ and $y_p = c_2 \cdot e^{2 \cdot A \cdot t}$; pathline that was at (2,2) at t = 0; compare to streamline through same point, and explain why they are similar or not.

Solution:

Governing equations: For pathlines $u_p = \dfrac{dx}{dt}$ $\quad v_p = \dfrac{dy}{dt}$ $\quad$ For streamlines $\dfrac{v}{u} = \dfrac{dy}{dx}$

Assumption: 2D flow

Hence for pathlines $\quad u_p = \dfrac{dx}{dt} = A \cdot x \qquad v_p = \dfrac{dy}{dt} = 2 \cdot A \cdot y$

So, separating variables $\quad \dfrac{dx}{x} = A \cdot dt \qquad \dfrac{dy}{y} = 2 \cdot A \cdot dt$

Integrating $\quad \ln(x) = A \cdot t + C_1 \qquad \ln(y) = 2 \cdot A \cdot t + C_2$

$$x = e^{A \cdot t + C_1} = e^{C_1} \cdot e^{A \cdot t} = c_1 \cdot e^{A \cdot t} \qquad y = e^{2 \cdot A \cdot t + C_2} = e^{C_2} \cdot e^{2 \cdot A \cdot t} = c_2 \cdot e^{2 \cdot A \cdot t}$$

The pathlines are $\quad x = c_1 \cdot e^{A \cdot t} \qquad y = c_2 \cdot e^{2 \cdot A \cdot t}$

Eliminating t $\quad y = c_2 \cdot e^{2 \cdot A \cdot t} = c_2 \cdot \left(\dfrac{x}{c_1}\right)^2$ $\quad$ so $\quad y = c \cdot x^2$ $\quad$ or $\quad y = \dfrac{1}{2} \cdot x^2$ $\quad$ for given data

For streamlines $\quad \dfrac{v}{u} = \dfrac{dy}{dx} = \dfrac{2 \cdot A \cdot y}{A \cdot x} = \dfrac{2 \cdot y}{x}$

So, separating variables $\quad \dfrac{dy}{y} = \dfrac{2 \cdot dx}{x}$ $\quad$ Integrating $\quad \ln(y) = 2 \cdot \ln(x) + c$

The solution is $\quad \ln\left(\dfrac{y}{x^2}\right) = c$

or $\quad y = C \cdot x^2$ $\quad$ or $\quad y = \dfrac{1}{2} \cdot x^2$ $\quad$ for given data

The streamline passing through (2,2) and the pathline that started at (2,2) coincide because the flow is steady!

Problem 2.16

[Difficulty: 2]

2.16 A velocity field is given by $\vec{V} = ayt\hat{i} - bx\hat{j}$, where $a = 1\ \text{s}^{-2}$ and $b = 4\ \text{s}^{-1}$. Find the equation of the streamlines at any time t. Plot several streamlines at $t = 0$ s, $t = 1$ s, and $t = 20$ s.

Given: Velocity field

Find: Equation of streamlines; Plot streamlines

Solution:

Streamlines are given by $$\frac{v}{u} = \frac{dy}{dx} = \frac{-b\cdot x}{a\cdot y\cdot t}$$

So, separating variables $$a\cdot t\cdot y\cdot dy = -b\cdot x\cdot dx$$

Integrating $$\frac{1}{2}\cdot a\cdot t\cdot y^2 = -\frac{1}{2}\cdot b\cdot x^2 + C$$

The solution is $$y = \sqrt{C - \frac{b\cdot x^2}{a\cdot t}}$$

For $t = 0$ s $\quad x = c$ $\qquad$ For $t = 1$ s $\quad y = \sqrt{C - 4\cdot x^2}$ $\qquad$ For $t = 20$ s $\quad y = \sqrt{C - \frac{x^2}{5}}$

t = 0

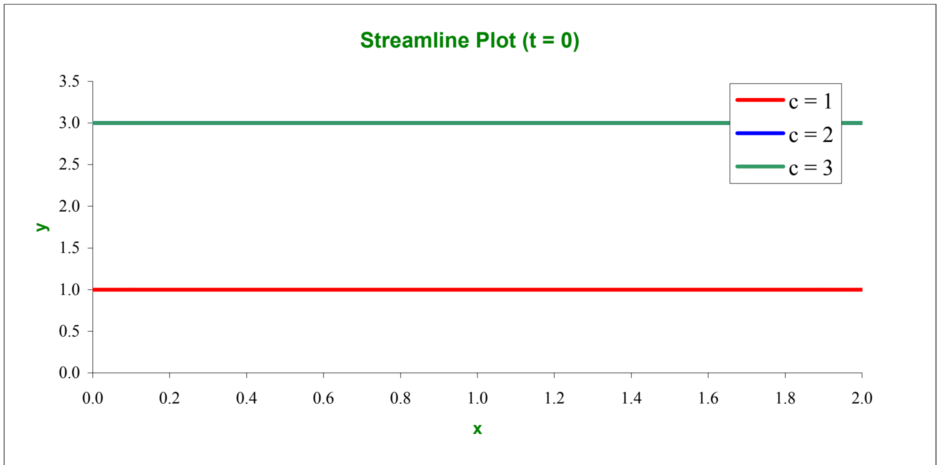
	C = 1	C = 2	C = 3
x	**y**	**y**	**y**
0.00	1.00	2.00	3.00
0.10	1.00	2.00	3.00
0.20	1.00	2.00	3.00
0.30	1.00	2.00	3.00
0.40	1.00	2.00	3.00
0.50	1.00	2.00	3.00
0.60	1.00	2.00	3.00
0.70	1.00	2.00	3.00
0.80	1.00	2.00	3.00
0.90	1.00	2.00	3.00
1.00	1.00	2.00	3.00
1.10	1.00	2.00	3.00
1.20	1.00	2.00	3.00
1.30	1.00	2.00	3.00
1.40	1.00	2.00	3.00
1.50	1.00	2.00	3.00
1.60	1.00	2.00	3.00
1.70	1.00	2.00	3.00
1.80	1.00	2.00	3.00
1.90	1.00	2.00	3.00
2.00	1.00	2.00	3.00

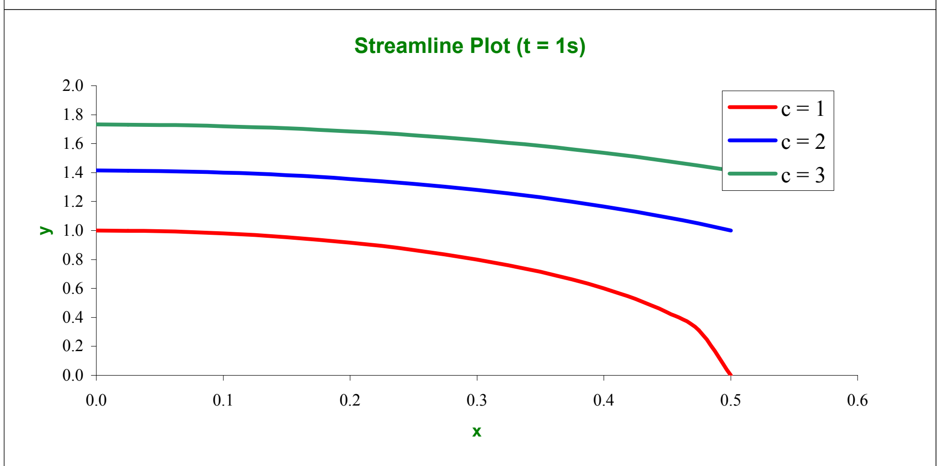
t =1 s

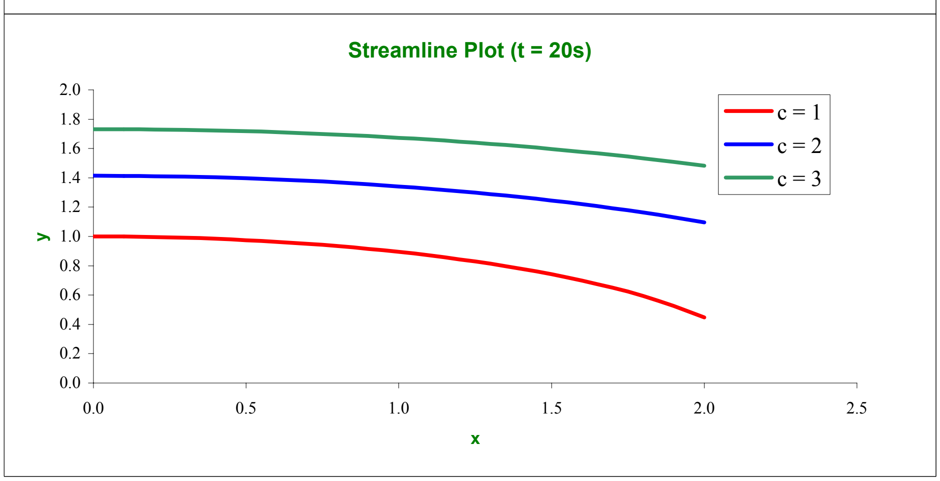
	C = 1	C = 2	C = 3
x	**y**	**y**	**y**
0.000	1.00	1.41	1.73
0.025	1.00	1.41	1.73
0.050	0.99	1.41	1.73
0.075	0.99	1.41	1.73
0.100	0.98	1.40	1.72
0.125	0.97	1.39	1.71
0.150	0.95	1.38	1.71
0.175	0.94	1.37	1.70
0.200	0.92	1.36	1.69
0.225	0.89	1.34	1.67
0.250	0.87	1.32	1.66
0.275	0.84	1.30	1.64
0.300	0.80	1.28	1.62
0.325	0.76	1.26	1.61
0.350	0.71	1.23	1.58
0.375	0.66	1.20	1.56
0.400	0.60	1.17	1.54
0.425	0.53	1.13	1.51
0.450	0.44	1.09	1.48
0.475	0.31	1.05	1.45
0.500	0.00	1.00	1.41

t = 20 s

	C = 1	C = 2	C = 3
x	**y**	**y**	**y**
0.00	1.00	1.41	1.73
0.10	1.00	1.41	1.73
0.20	1.00	1.41	1.73
0.30	0.99	1.41	1.73
0.40	0.98	1.40	1.72
0.50	0.97	1.40	1.72
0.60	0.96	1.39	1.71
0.70	0.95	1.38	1.70
0.80	0.93	1.37	1.69
0.90	0.92	1.36	1.68
1.00	0.89	1.34	1.67
1.10	0.87	1.33	1.66
1.20	0.84	1.31	1.65
1.30	0.81	1.29	1.63
1.40	0.78	1.27	1.61
1.50	0.74	1.24	1.60
1.60	0.70	1.22	1.58
1.70	0.65	1.19	1.56
1.80	0.59	1.16	1.53
1.90	0.53	1.13	1.51
2.00	0.45	1.10	1.48


Streamline Plot (t = 0)
y
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
x
c = 1
c = 2
c = 3


Streamline Plot (t = 1s)
y
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.1
0.2
0.3
0.4
0.5
0.6
x
c = 1
c = 2
c = 3


Streamline Plot (t = 20s)
y
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.5
1.0
1.5
2.0
2.5
x
c = 1
c = 2
c = 3

Problem 2.17

[Difficulty: 4]

2.17 Verify that $x_p = -a\sin(\omega t)$, $y_p = a\cos(\omega t)$ is the equation for the pathlines of particles for the flow field of Problem 2.12. Find the frequency of motion ω as a function of the amplitude of motion, a, and K. Verify that $x_p = -a\sin(\omega t)$, $y_p = a\cos(\omega t)$ is also the equation for the pathlines of particles for the flow field of Problem 2.11, except that ω is now a function of M. Plot typical pathlines for both flow fields and discuss the difference.

Given: Pathlines of particles

Find: Conditions that make them satisfy Problem 2.10 flow field; Also Problem 2.11 flow field; Plot pathlines

Solution:

The given pathlines are $\quad x_p = -a\cdot\sin(\omega\cdot t) \qquad y_p = a\cdot\cos(\omega\cdot t)$

The velocity field of Problem 2.12 is $\quad u = -\dfrac{K\cdot y}{2\cdot\pi\cdot\left(x^2 + y^2\right)} \qquad v = \dfrac{K\cdot x}{2\cdot\pi\cdot\left(x^2 + y^2\right)}$

If the pathlines are correct we should be able to substitute x_p and y_p into the velocity field to find the velocity as a function of time:

$$u = -\frac{K\cdot y}{2\cdot\pi\cdot\left(x^2 + y^2\right)} = -\frac{K\cdot a\cdot\cos(\omega\cdot t)}{2\cdot\pi\cdot\left(a^2\cdot\sin(\omega\cdot t)^2 + a^2\cdot\cos(\omega\cdot t)^2\right)} = -\frac{K\cdot\cos(\omega\cdot t)}{2\cdot\pi\cdot a} \qquad (1)$$

$$v = \frac{K\cdot x}{2\cdot\pi\cdot\left(x^2 + y^2\right)} = -\frac{K\cdot(-a\cdot\sin(\omega\cdot t))}{2\cdot\pi\cdot\left(a^2\cdot\sin(\omega\cdot t)^2 + a^2\cdot\cos(\omega\cdot t)^2\right)} = -\frac{K\cdot\sin(\omega\cdot t)}{2\cdot\pi\cdot a} \qquad (2)$$

We should also be able to find the velocity field as a function of time from the pathline equations (Eq. 2.9):

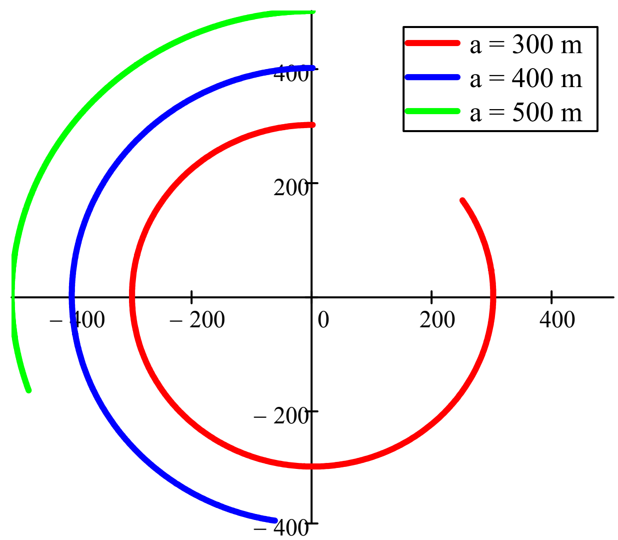
$$\frac{dx_p}{dt} = u \qquad \frac{dx_p}{dt} = v \qquad (2.9)$$

$$u = \frac{dx_p}{dt} = -a\cdot\omega\cdot\cos(\omega\cdot t) \qquad v = \frac{dy_p}{dt} = -a\cdot\omega\cdot\sin(\omega\cdot t) \qquad (3)$$

Comparing Eqs. 1, 2 and 3 $\quad u = -a\cdot\omega\cdot\cos(\omega\cdot t) = -\dfrac{K\cdot\cos(\omega\cdot t)}{2\cdot\pi\cdot a} \qquad v = -a\cdot\omega\cdot\sin(\omega\cdot t) = -\dfrac{K\cdot\sin(\omega\cdot t)}{2\cdot\pi\cdot a}$

Hence we see that $\quad a\cdot\omega = \dfrac{K}{2\cdot\pi\cdot a} \quad$ or $\quad \omega = \dfrac{K}{2\cdot\pi\cdot a^2} \quad$ for the pathlines to be correct.

The pathlines are



To plot this in Excel, compute x_p and y_p for t ranging from 0 to 60 s, with ω given by the above formula. Plot y_p versus x_p. Note that outer particles travel much slower!

This is the free vortex flow discussed in Example 5.6

The velocity field of Problem 2.11 is $u = -\dfrac{M \cdot y}{2 \cdot \pi}$ $\qquad v = \dfrac{M \cdot x}{2 \cdot \pi}$

If the pathlines are correct we should be able to substitute x_p and y_p into the velocity field to find the velocity as a function of time:

$$u = -\frac{M \cdot y}{2 \cdot \pi} = -\frac{M \cdot (a \cdot \cos(\omega \cdot t))}{2 \cdot \pi} = -\frac{M \cdot a \cdot \cos(\omega \cdot t)}{2 \cdot \pi} \qquad (4)$$

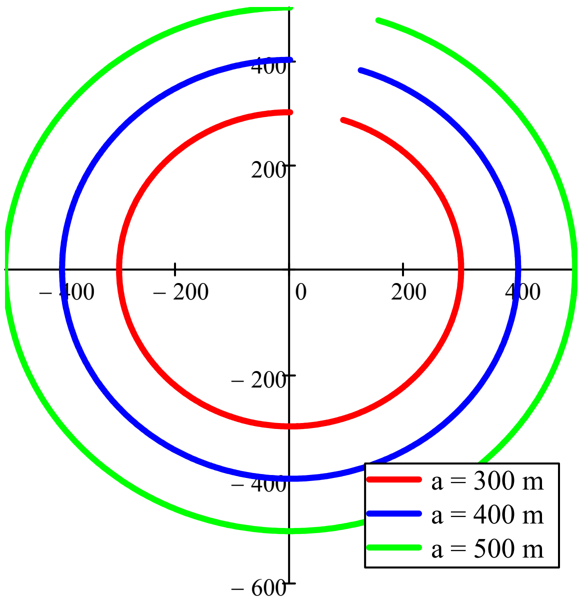
$$v = \frac{M \cdot x}{2 \cdot \pi} = \frac{M \cdot (-a \cdot \sin(\omega \cdot t))}{2 \cdot \pi} = -\frac{M \cdot a \cdot \sin(\omega \cdot t)}{2 \cdot \pi} \qquad (5)$$

Recall that $u = \dfrac{dx_p}{dt} = -a \cdot \omega \cdot \cos(\omega \cdot t)$ $\qquad v = \dfrac{dy_p}{dt} = -a \cdot \omega \cdot \sin(\omega \cdot t) \qquad (3)$

Comparing Eqs. 1, 4 and 5 $\quad u = -a \cdot \omega \cdot \cos(\omega \cdot t) = -\dfrac{M \cdot a \cdot \cos(\omega \cdot t)}{2 \cdot \pi}$ $\qquad v = -a \cdot \omega \cdot \sin(\omega \cdot t) = -\dfrac{M \cdot a \cdot \sin(\omega \cdot t)}{2 \cdot \pi}$

Hence we see that $\quad \omega = \dfrac{M}{2 \cdot \pi}$ $\quad$ for the pathlines to be correct.

The pathlines



To plot this in Excel, compute x_p and y_p for t ranging from 0 to 75 s, with ω given by the above formula. Plot y_p versus x_p. Note that outer particles travel faster!

This is the forced vortex flow discussed in Example 5.6

Note that this is rigid body rotation!

Problem 2.18

[Difficulty: 2]

2.18 Air flows downward toward an infinitely wide horizontal flat plate. The velocity field is given by $\vec{V} = (ax\hat{i} - ay\hat{j})(2 + \cos\omega t)$, where $a = 5\ \text{s}^{-1}$, $\omega = 2\pi\ \text{s}^{-1}$, x and y (measured in meters) are horizontal and vertically upward, respectively, and t is in s. Obtain an algebraic equation for a streamline at $t = 0$. Plot the streamline that passes through point $(x, y) = (3, 3)$ at this instant. Will the streamline change with time? Explain briefly. Show the velocity vector on your plot at the same point and time. Is the velocity vector tangent to the streamline? Explain.

Given: Time-varying velocity field

Find: Streamlines at t = 0 s; Streamline through (3,3); velocity vector; will streamlines change with time

Solution:

For streamlines $$\frac{v}{u} = \frac{dy}{dx} = -\frac{a\cdot y\cdot(2 + \cos(\omega\cdot t))}{a\cdot x\cdot(2 + \cos(\omega\cdot t))} = -\frac{y}{x}$$

At t = 0 (actually all times!) $$\frac{dy}{dx} = -\frac{y}{x}$$

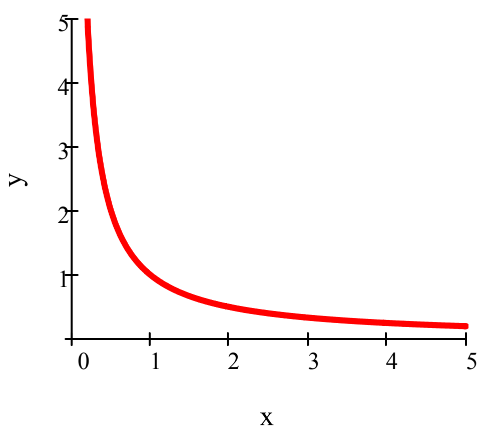
So, separating variables $$\frac{dy}{y} = -\frac{dx}{x}$$

Integrating $$\ln(y) = -\ln(x) + c$$

The solution is $$y = \frac{C}{x}$$ which is the equation of a hyperbola.

For the streamline through point (3,3) $$C = \frac{3}{3} \qquad C = 1 \qquad \text{and} \qquad y = \frac{1}{x}$$

The streamlines will not change with time since dy/dx does not change with time.



This curve can be plotted in Excel.

At t = 0 $$u = a\cdot x\cdot(2 + \cos(\omega\cdot t)) = 5\cdot\frac{1}{s} \times 3\cdot m \times 3$$

$$u = 45\cdot\frac{m}{s}$$

$$v = -a\cdot y\cdot(2 + \cos(\omega\cdot t)) = 5\cdot\frac{1}{s} \times 3\cdot m \times 3$$

$$v = -45\cdot\frac{m}{s}$$

The velocity vector is tangent to the curve;

Tangent of curve at (3,3) is $$\frac{dy}{dx} = -\frac{y}{x} = -1$$

Direction of velocity at (3,3) is $$\frac{v}{u} = -1$$

Problem 2.19

[Difficulty: 3]

2.19 Consider the flow described by the velocity field $\vec{V} = A(1+Bt)\hat{i} + Cty\hat{j}$, with $A = 1$ m/s, $B = 1$ s^{-1}, and $C = 1$ s^{-2}. Coordinates are measured in meters. Plot the pathline traced out by the particle that passes through the point (1, 1) at time $t = 0$. Compare with the streamlines plotted through the same point at the instants $t = 0$, 1, and 2 s.

Given: Velocity field

Find: Plot of pathline traced out by particle that passes through point (1,1) at t = 0; compare to streamlines through same point at the instants t = 0, 1 and 2s

Solution:

Governing equations: For pathlines $u_p = \dfrac{dx}{dt}$ $\quad v_p = \dfrac{dy}{dt}$ $\quad$ For streamlines $\dfrac{v}{u} = \dfrac{dy}{dx}$

Assumption: 2D flow

Hence for pathlines $u_p = \dfrac{dx}{dt} = A\cdot(1 + B\cdot t)$ $\quad A = 1\cdot\dfrac{m}{s}$ $\quad B = 1\cdot\dfrac{1}{s}$ $\quad v_p = \dfrac{dy}{dt} = C\cdot t\cdot y$ $\quad C = 1\cdot\dfrac{1}{s^2}$

So, separating variables $dx = A\cdot(1 + B\cdot t)\cdot dt$ $\quad \dfrac{dy}{y} = C\cdot t\cdot dt$

Integrating $x = A\cdot\left(t + B\cdot\dfrac{t^2}{2}\right) + C_1$ $\quad \ln(y) = \dfrac{1}{2}\cdot C\cdot t^2 + C_2$

$$y = e^{\frac{1}{2}\cdot C\cdot t^2 + C_2} = e^{C_2}\cdot e^{\frac{1}{2}\cdot C\cdot t^2} = c_2\cdot e^{\frac{1}{2}\cdot C\cdot t^2}$$

The pathlines are $x = A\cdot\left(t + B\cdot\dfrac{t^2}{2}\right) + C_1$ $\quad y = c_2\cdot e^{\frac{1}{2}\cdot C\cdot t^2}$

Using given data $x = A\cdot\left(t + B\cdot\dfrac{t^2}{2}\right) + 1$ $\quad y = e^{\frac{1}{2}\cdot C\cdot t^2}$

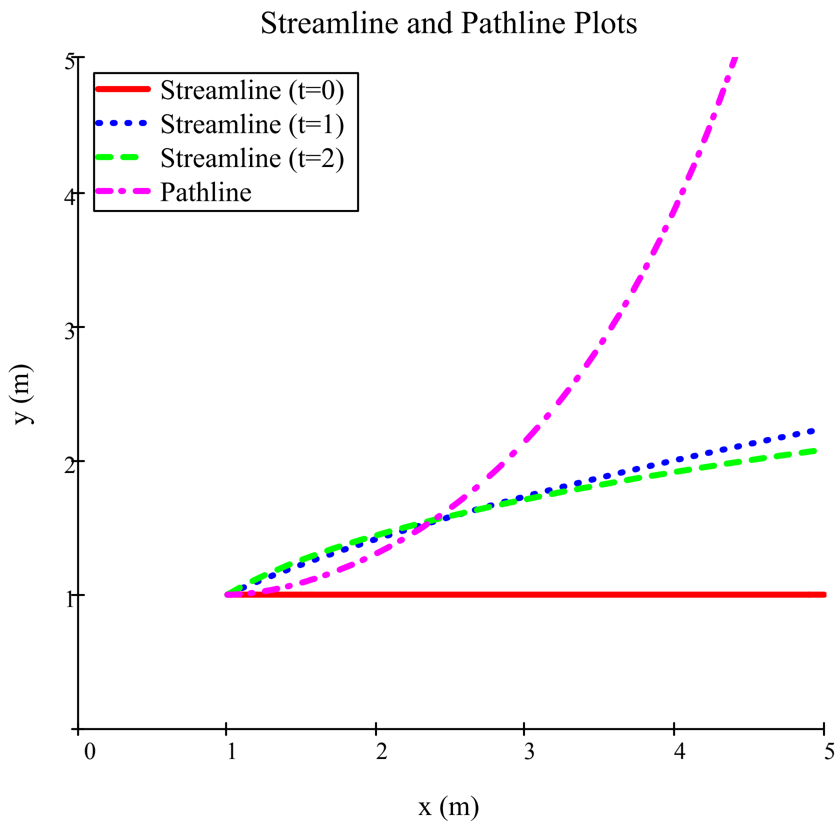
For streamlines $\dfrac{v}{u} = \dfrac{dy}{dx} = \dfrac{C\cdot y\cdot t}{A\cdot(1 + B\cdot t)}$

So, separating variables $(1 + B\cdot t)\cdot\dfrac{dy}{y} = \dfrac{C}{A}\cdot t\cdot dx$ $\quad$ which we can integrate for any given t (t is treated as a constant)

Integrating $(1 + B\cdot t)\cdot\ln(y) = \dfrac{C}{A}\cdot t\cdot x + c$

The solution is $y^{1+B\cdot t} = \dfrac{C}{A}\cdot t\cdot x + const$ $\quad y = \left(\dfrac{C}{A}\cdot t\cdot x + const\right)^{\frac{1}{(1+B\cdot t)}}$

For particles at (1,1) at t = 0, 1, and 2s, using A, B, and C data: $y = 1$ $\quad y = x^{\frac{1}{2}}$ $\quad y = (2\cdot x - 1)^{\frac{1}{3}}$



Problem 2.20

[Difficulty: 3]

2.20 Consider the flow described by the velocity field $\vec{V} = Bx(1 + At)\hat{i} + Cy\hat{j}$, with $A = 0.5\ \text{s}^{-1}$ and $B = C = 1\ \text{s}^{-1}$. Coordinates are measured in meters. Plot the pathline traced out by the particle that passes through the point (1, 1) at time $t = 0$. Compare with the streamlines plotted through the same point at the instants $t = 0$, 1, and 2 s.

Given:
Velocity field

Find:
Plot of pathline traced out by particle that passes through point (1,1) at t = 0; compare to streamlines through same point at the instants t = 0, 1 and 2s

Solution:

Governing equations: For pathlines $u_p = \frac{dx}{dt}$ $\quad v_p = \frac{dy}{dt}$ $\quad$ For streamlines $\frac{v}{u} = \frac{dy}{dx}$

Assumption: 2D flow

Hence for pathlines $u_p = \frac{dx}{dt} = B\cdot x\cdot(1 + A\cdot t)$ $\quad A = 0.5\cdot\frac{1}{s}$ $\quad B = 1\cdot\frac{1}{s}$ $\quad v_p = \frac{dy}{dt} = C\cdot y$ $\quad C = 1\cdot\frac{1}{s}$

So, separating variables $\frac{dx}{x} = B\cdot(1 + A\cdot t)\cdot dt$ $\qquad \frac{dy}{y} = C\cdot dt$

Integrating $\ln(x) = B\cdot\left(t + A\cdot\frac{t^2}{2}\right) + C_1$ $\qquad \ln(y) = C\cdot t + C_2$

$$x = e^{B\cdot\left(t+A\cdot\frac{t^2}{2}\right)+C_1} = e^{C_1}\cdot e^{B\cdot\left(t+A\cdot\frac{t^2}{2}\right)} = c_1\cdot e^{B\cdot\left(t+A\cdot\frac{t^2}{2}\right)} \qquad y = e^{C\cdot t+C_2} = e^{C_2}\cdot e^{C\cdot t} = c_2\cdot e^{C\cdot t}$$

The pathlines are $x = c_1\cdot e^{B\cdot\left(t+A\cdot\frac{t^2}{2}\right)}$ $\qquad y = c_2\cdot e^{C\cdot t}$

Using given data $x = e^{B\cdot\left(t+A\cdot\frac{t^2}{2}\right)}$ $\qquad y = e^{C\cdot t}$

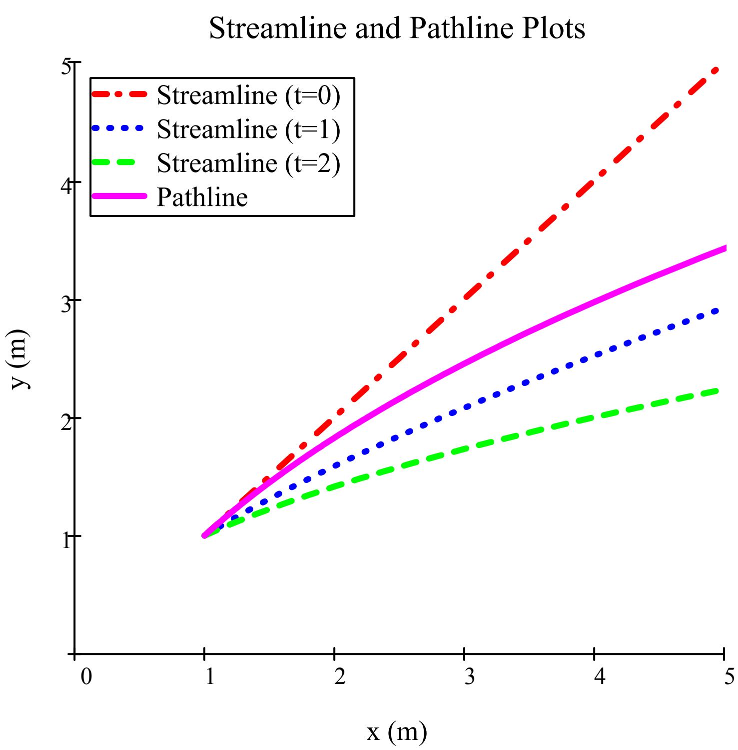
For streamlines $\frac{v}{u} = \frac{dy}{dx} = \frac{C\cdot y}{B\cdot x\cdot(1 + A\cdot t)}$

So, separating variables $(1 + A\cdot t)\cdot\frac{dy}{y} = \frac{C}{B}\cdot\frac{dx}{x}$ $\quad$ which we can integrate for any given t (t is treated as a constant)

Integrating $(1 + A\cdot t)\cdot\ln(y) = \frac{C}{B}\cdot\ln(x) + c$

The solution is $y^{1+A\cdot t} = \text{const}\cdot x^{\frac{C}{B}}$ or $y = \text{const}\cdot x$

For particles at (1,1) at t = 0, 1, and 2s $y = x^{\frac{C}{B}}$ $y = x^{\frac{C}{(1+A)B}}$ $y = x^{\frac{C}{(1+2\cdot A)B}}$



Problem 2.21

[Difficulty: 3]

2.21 Consider the flow field given in Eulerian description by the expression $\vec{V} = A\hat{i} - Bt\hat{j}$, where $A = 2$ m/s, $B = 2$ m/s^2, and the coordinates are measured in meters. Derive the Lagrangian position functions for the fluid particle that was located at the point $(x, y) = (1, 1)$ at the instant $t = 0$. Obtain an algebraic expression for the pathline followed by this particle. Plot the pathline and compare with the streamlines plotted through the same point at the instants $t = 0$, 1, and 2 s.

Given:

Eulerian Velocity field

Find:

Lagrangian position function that was at point (1,1) at t = 0; expression for pathline; plot pathline and compare to streamlines through same point at the instants t = 0, 1 and 2s

Solution:

Governing equations: For pathlines (Lagrangian description) $u_p = \dfrac{dx}{dt}$ $\quad v_p = \dfrac{dy}{dt}$ $\quad$ For streamlines $\dfrac{v}{u} = \dfrac{dy}{dx}$

Assumption: 2D flow

Hence for pathlines $\quad u_p = \dfrac{dx}{dt} = A \quad A = 2\,\dfrac{m}{s} \quad v_p = \dfrac{dy}{dt} = -B \cdot t \quad B = 2\,\dfrac{m}{s^2}$

So, separating variables $\quad dx = A \cdot dt \quad dy = -B \cdot t \cdot dt$

Integrating $\quad x = A \cdot t + x_0 \quad x_0 = 1\ \text{m} \quad y = -B \cdot \dfrac{t^2}{2} + y_0 \quad y_0 = 1\ \text{m}$

The Lagrangian description is $\quad x(t) = A \cdot t + x_0 \quad y(t) = -B \cdot \dfrac{t^2}{2} + y_0$

Using given data $\quad x(t) = 2 \cdot t + 1 \quad y(t) = 1 - t^2$

The pathlines are given by combining the equations $\quad t = \dfrac{x - x_0}{A} \quad y = -B \cdot \dfrac{t^2}{2} + y_0 = -B \cdot \dfrac{\left(x - x_0\right)^2}{2 \cdot A^2} + y_0$

Hence $\quad y(x) = y_0 - B \cdot \dfrac{\left(x - x_0\right)^2}{2 \cdot A^2}$ $\quad$ or, using given data $\quad y(x) = 1 - \dfrac{(x-1)^2}{4}$

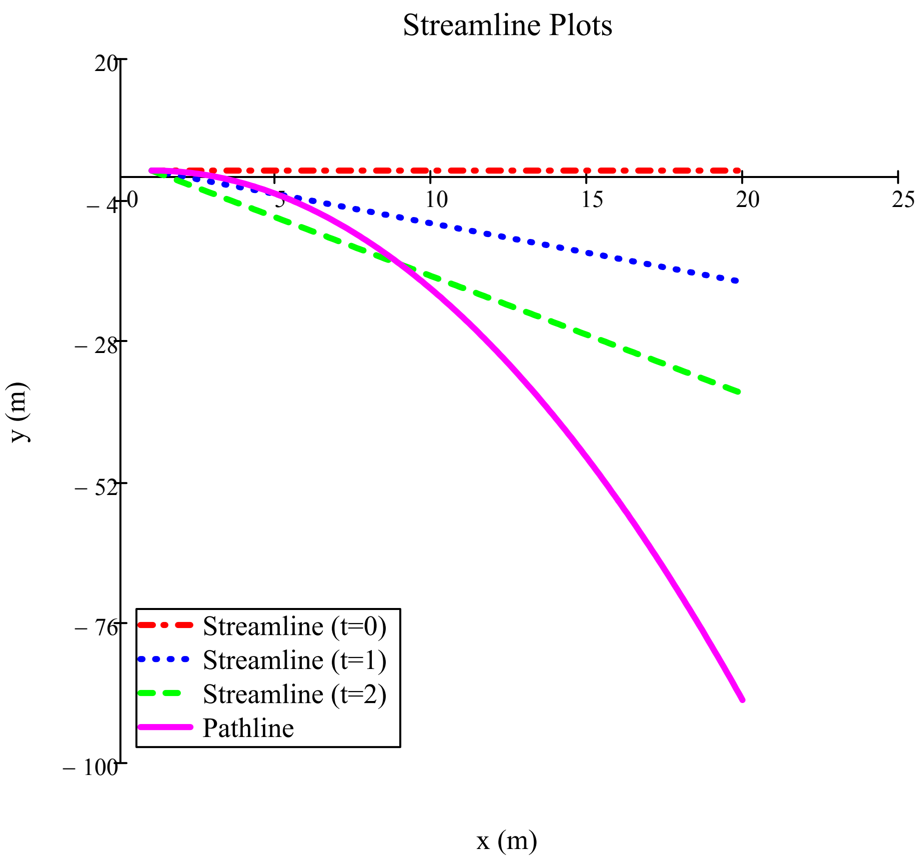
For streamlines $\quad \dfrac{v}{u} = \dfrac{dy}{dx} = \dfrac{-B \cdot t}{A}$

So, separating variables $\quad dy = -\dfrac{B \cdot t}{A} \cdot dx$ $\quad$ which we can integrate for any given t (t is treated as a constant)

The solution is $y = -\frac{B \cdot t}{A} \cdot x + c$ and for the one through (1,1) $1 = -\frac{B \cdot t}{A} \cdot 1 + c$ $c = 1 + \frac{B \cdot t}{A}$

$$y = -\frac{B \cdot t}{A} \cdot (x - 1) + 1 \qquad y = 1 - t \cdot (x - 1)$$

$$x = 1, 1.1 .. 20$$



Problem 2.22

[Difficulty: 3]

2.22 Consider the velocity field $V = ax\hat{i} + by(1 + ct)\hat{j}$, where $a = b = 2\ s^{-1}$ and $c = 0.4\ s^{-1}$. Coordinates are measured in meters. For the particle that passes through the point $(x, y) = (1, 1)$ at the instant $t = 0$, plot the pathline during the interval from $t = 0$ to 1.5 s. Compare this pathline with the streamlines plotted through the same point at the instants $t = 0$, 1, and 1.5 s.

Given: Velocity field

Find: Plot of pathline of particle for t = 0 to 1.5 s that was at point (1,1) at t = 0; compare to streamlines through same point at the instants t = 0, 1 and 1.5 s

Solution:

Governing equations: For pathlines $u_p = \frac{dx}{dt}$ $v_p = \frac{dy}{dt}$ For streamlines $\frac{v}{u} = \frac{dy}{dx}$

Assumption: 2D flow

Hence for pathlines $u_p = \frac{dx}{dt} = ax$ $a = 2\ \frac{1}{s}$ $v_p = \frac{dy}{dt} = b \cdot y \cdot (1 + c \cdot t)$ $b = 2\ \frac{1}{s^2}$ $c = 0.4\ \frac{1}{s}$

So, separating variables $\frac{dx}{x} = a \cdot dt$ $dy = b \cdot y \cdot (1 + c \cdot t) \cdot dt$ $\frac{dy}{y} = b \cdot (1 + c \cdot t) \cdot dt$

Integrating $\ln\left(\frac{x}{x_0}\right) = a \cdot t$ $x_0 = 1\ m$ $\ln\left(\frac{y}{y_0}\right) = b \cdot \left(t + \frac{1}{2} \cdot c \cdot t^2\right)$ $y_0 = 1\ m$

Hence $x(t) = x_0 \cdot e^{a \cdot t}$ $y(t) = e^{b \cdot \left(t + \frac{1}{2} \cdot c \cdot t^2\right)}$

Using given data $x(t) = e^{2 \cdot t}$ $y(t) = e^{2 \cdot t + 0.4 \cdot t^2}$

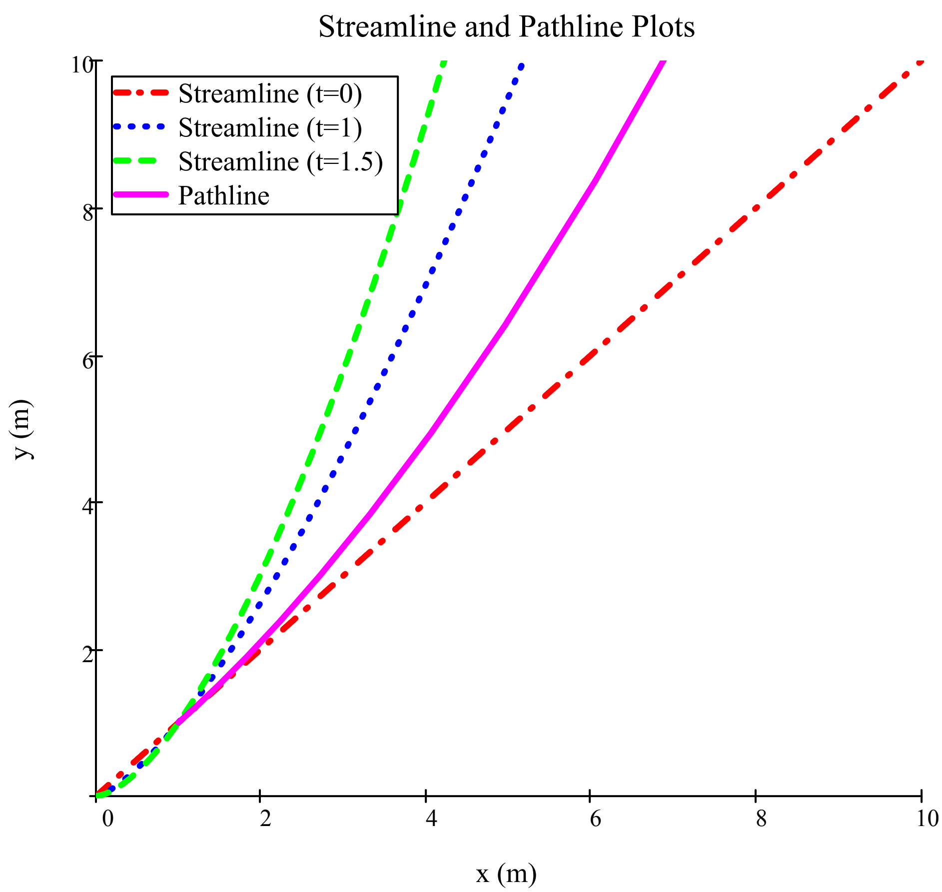
For streamlines $\frac{v}{u} = \frac{dy}{dx} = \frac{b \cdot y \cdot (1 + c \cdot t)}{a \cdot x}$

So, separating variables $\frac{dy}{y} = \frac{b \cdot (1 + c \cdot t)}{a \cdot x} \cdot dx$ which we can integrate for any given t (t is treated as a constant)

Hence $\ln\left(\frac{y}{y_0}\right) = \frac{b}{a} \cdot (1 + c \cdot t) \cdot \ln\left(\frac{x}{x_0}\right)$

The solution is $y = y_0 \cdot \left(\frac{x}{x_0}\right)^{\frac{b}{a} \cdot (1 + c \cdot t)}$

For $t = 0$ $\quad y = y_0 \cdot \left(\frac{x}{x_0}\right)^{\frac{b}{a}\cdot(1+c\cdot t)} = x$ $\quad t = 1$ $\quad y = y_0 \cdot \left(\frac{x}{x_0}\right)^{\frac{b}{a}\cdot(1+c\cdot t)} = x^{1.4}$ $\quad t = 1.5$ $\quad y = y_0 \cdot \left(\frac{x}{x_0}\right)^{\frac{b}{a}\cdot(1+c\cdot t)} = x^{1.6}$



Problem 2.23

[Difficulty: 3]

2.23 Consider the flow field given in Eulerian descriptionby the expression $\vec{V} = ax\hat{i} + byt\hat{j}$, where $a = 0.2\ \mathrm{s}^{-1}$, $b = 0.04\ \mathrm{s}^{-2}$, and the coordinates are measured in meters. Derive the Lagrangian position functions for the fluid particle that was located at the point $(x, y) = (1, 1)$ at the instant $t = 0$. Obtain an algebraic expression for the pathline followed by this particle. Plot the pathline and compare with the streamlines plotted through the same point at the instants $t = 0$, 10, and 20 s.

Given: Velocity field

Find: Plot of pathline of particle for t = 0 to 1.5 s that was at point (1,1) at t = 0; compare to streamlines through same point at the instants t = 0, 1 and 1.5 s

Solution:

Governing equations: For pathlines $u_p = \frac{dx}{dt}$ $\quad v_p = \frac{dy}{dt}$ $\quad$ For streamlines $\frac{v}{u} = \frac{dy}{dx}$

Assumption: 2D flow

Hence for pathlines $u_p = \frac{dx}{dt} = a \cdot x$ $\quad a = \frac{1}{5} \frac{1}{s}$ $\quad v_p = \frac{dy}{dt} = b \cdot y \cdot t$ $\quad b = \frac{1}{25} \frac{1}{s^2}$

So, separating variables $\frac{dx}{x} = a \cdot dt$ $\quad dy = b \cdot y \cdot t \cdot dt$ $\quad \frac{dy}{y} = b \cdot t \cdot dt$

Integrating $\ln\left(\frac{x}{x_0}\right) = a \cdot t$ $\quad x_0 = 1\ \ m$ $\quad \ln\left(\frac{y}{y_0}\right) = b \cdot \frac{1}{2} \cdot t^2$ $\quad y_0 = 1\ \ m$

Hence $x(t) = x_0 \cdot e^{a \cdot t}$ $\quad y(t) = y_0 \cdot e^{\frac{1}{2} \cdot b \cdot t^2}$

Using given data $x(t) = e^{\frac{t}{5}}$ $\quad y(t) = e^{\frac{t^2}{50}}$

For streamlines $\frac{v}{u} = \frac{dy}{dx} = \frac{b \cdot y \cdot t}{a \cdot x}$

So, separating variables $\frac{dy}{y} = \frac{b \cdot t}{a \cdot x} \cdot dx$ $\quad$ which we can integrate for any given t (t is treated as a constant)

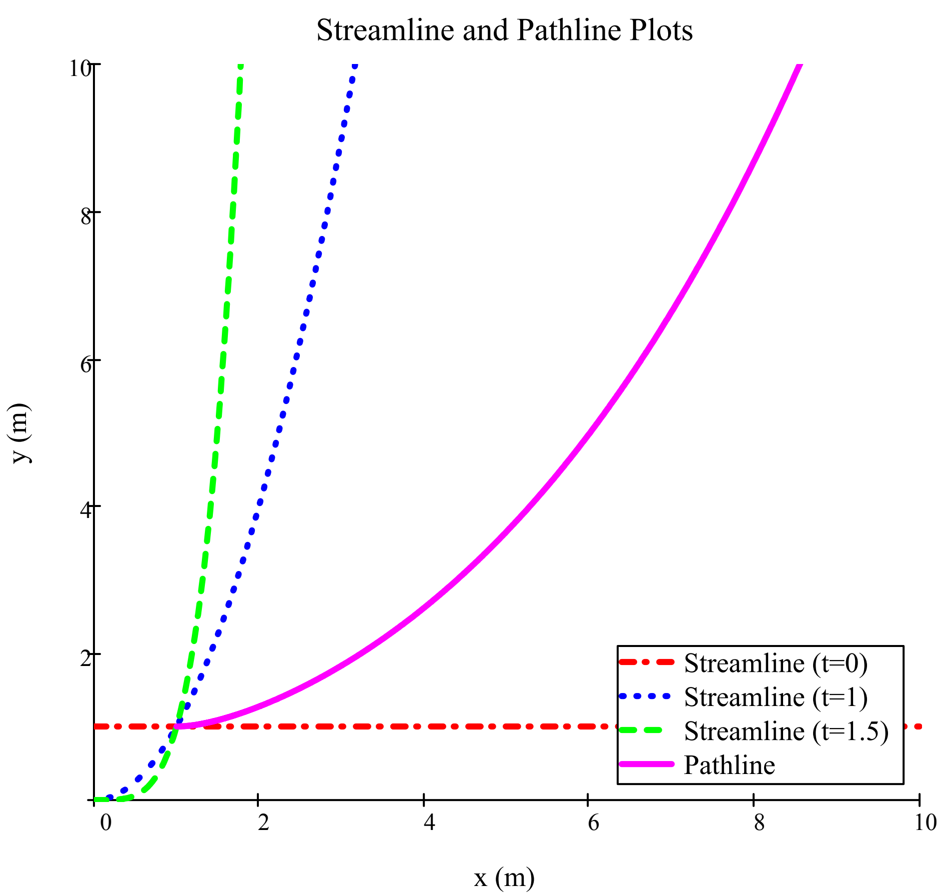
Hence $\ln\left(\frac{y}{y_0}\right) = \frac{b}{a} \cdot t \cdot \ln\left(\frac{x}{x_0}\right)$

The solution is $y = y_0 \cdot \left(\frac{x}{x_0}\right)^{\frac{b}{a} \cdot t}$ $\quad \frac{b}{a} = 0.2$ $\quad x_0 = 1$ $\quad y_0 = 1$

For $\quad t = 0 \qquad y = y_0 \cdot \left(\frac{x}{x_0}\right)^{\frac{b}{a} \cdot t} = 1$

$t = 5 \qquad y = y_0 \cdot \left(\frac{x}{x_0}\right)^{\frac{b}{a} \cdot t} = x \qquad \frac{b}{a} \cdot t = 1$

$t = 10 \qquad y = y_0 \cdot \left(\frac{x}{x_0}\right)^{\frac{b}{a} \cdot t} = x^2 \qquad \frac{b}{a} \cdot t = 2$



Problem 2.24

[Difficulty: 3]

2.24 A velocity field is given by $\vec{V} = axt\hat{i} - by\hat{j}$, where $a = 0.1\ s^{-2}$ and $b = 1\ s^{-1}$. For the particle that passes through the point $(x, y) = (1, 1)$ at instant $t = 0$ s, plot the pathline during the interval from $t = 0$ to $t = 3$ s. Compare with the streamlines plotted through the same point at the instants $t = 0$, 1, and 2 s.

Given: Velocity field

Find: Plot pathlines and streamlines

Solution:

Pathlines are given by $\frac{dx}{dt} = u = a \cdot x \cdot t$ $\qquad$ $\frac{dy}{dt} = v = -b \cdot y$

So, separating variables $\frac{dx}{x} = a \cdot t \cdot dt$ $\qquad$ $\frac{dy}{y} = -b \cdot dt$

Integrating $\ln(x) = \frac{1}{2} \cdot a \cdot t^2 + c_1$ $\qquad$ $\ln(y) = -b \cdot t + c_2$

For initial position (x_0, y_0) $\quad x = x_0 \cdot e^{\frac{a}{2} \cdot t^2}$ $\qquad$ $y = y_0 \cdot e^{-b \cdot t}$

Using the given data, and IC $(x_0, y_0) = (1,1)$ at $t = 0$

$$x = e^{0.05 \cdot t^2} \qquad y = e^{-t}$$

Streamlines are given by $\frac{v}{u} = \frac{dy}{dx} = \frac{-b \cdot y}{a \cdot x \cdot t}$

So, separating variables $\frac{dy}{y} = -\frac{b}{a \cdot t} \cdot \frac{dx}{x}$ $\qquad$ Integrating $\ln(y) = -\frac{b}{a \cdot t} \cdot \ln(x) + C$

The solution is $y = C \cdot x^{-\frac{b}{a \cdot t}}$

For streamline at (1,1) at $t = 0$ s $\quad x = c$

For streamline at (1,1) at $t = 1$ s $\quad y = x^{-10}$

For streamline at (1,1) at $t = 2$ s $\quad y = x^{-5}$

Pathline

t	x	y
0.00	1.00	1.00
0.25	1.00	0.78
0.50	1.01	0.61
0.75	1.03	0.47
1.00	1.05	0.37
1.25	1.08	0.29
1.50	1.12	0.22
1.75	1.17	0.17
2.00	1.22	0.14
2.25	1.29	0.11
2.50	1.37	0.08
2.75	1.46	0.06
3.00	1.57	0.05
3.25	1.70	0.04
3.50	1.85	0.03
3.75	2.02	0.02
4.00	2.23	0.02
4.25	2.47	0.01
4.50	2.75	0.01
4.75	3.09	0.01
5.00	3.49	0.01

Streamlines

t = 0

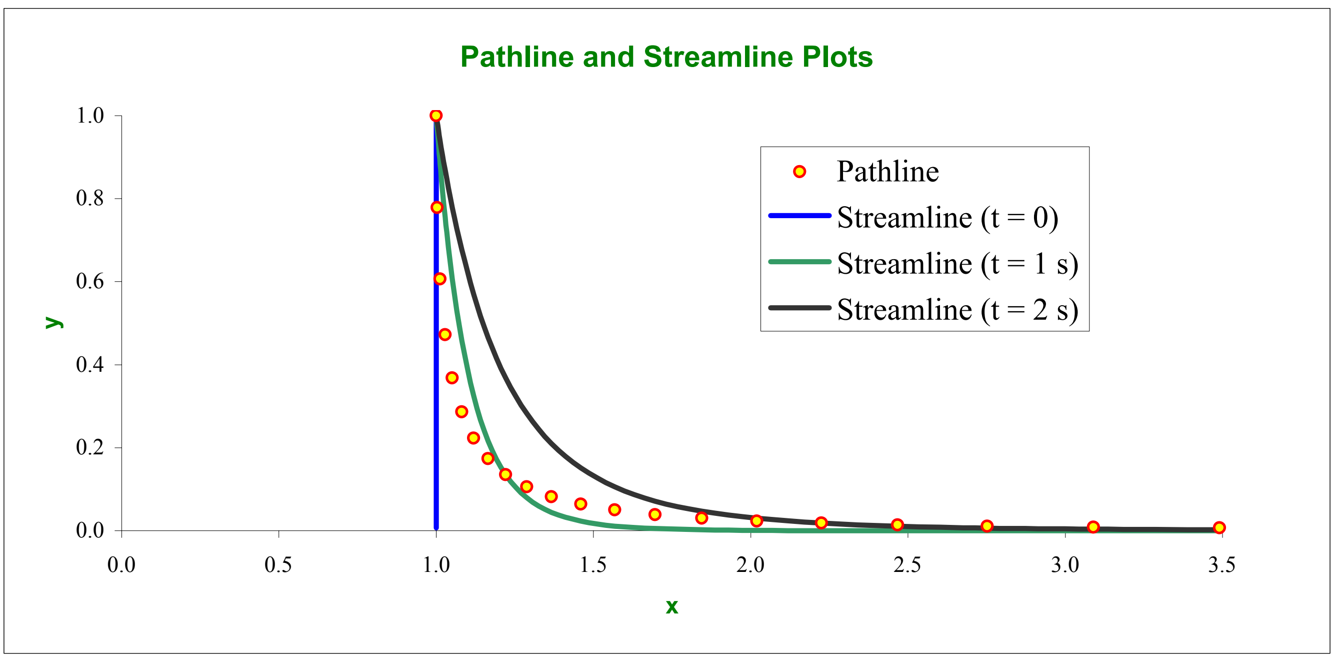
x	y
1.00	1.00
1.00	0.78
1.00	0.61
1.00	0.47
1.00	0.37
1.00	0.29
1.00	0.22
1.00	0.17
1.00	0.14
1.00	0.11
1.00	0.08
1.00	0.06
1.00	0.05
1.00	0.04
1.00	0.03
1.00	0.02
1.00	0.02
1.00	0.01
1.00	0.01
1.00	0.01
1.00	0.01

t = 1 s

x	y
1.00	1.00
1.00	0.97
1.01	0.88
1.03	0.75
1.05	0.61
1.08	0.46
1.12	0.32
1.17	0.22
1.22	0.14
1.29	0.08
1.37	0.04
1.46	0.02
1.57	0.01
1.70	0.01
1.85	0.00
2.02	0.00
2.23	0.00
2.47	0.00
2.75	0.00
3.09	0.00
3.49	0.00

t = 2 s

x	y
1.00	1.00
1.00	0.98
1.01	0.94
1.03	0.87
1.05	0.78
1.08	0.68
1.12	0.57
1.17	0.47
1.22	0.37
1.29	0.28
1.37	0.21
1.46	0.15
1.57	0.11
1.70	0.07
1.85	0.05
2.02	0.03
2.23	0.02
2.47	0.01
2.75	0.01
3.09	0.00
3.49	0.00



Problem 2.25

[Difficulty: 3]

2.25 Consider the flow field $\vec{V} = axt\hat{i} + b\hat{j}$, where $a = 0.1\ \text{s}^{-2}$ and $b = 4$ m/s. Coordinates are measured in meters. For the particle that passes through the point $(x, y) = (3, 1)$ at the instant $t = 0$, plot the pathline during the interval from $t = 0$ to 3 s. Compare this pathline with the streamlines plotted through the same point at the instants $t = 1$, 2, and 3 s.

Given: Flow field

Find: Pathline for particle starting at (3,1); Streamlines through same point at t = 1, 2, and 3 s

Solution:

For particle paths $\quad \frac{dx}{dt} = u = a\cdot x\cdot t \quad$ and $\quad \frac{dy}{dt} = v = b$

Separating variables and integrating $\quad \frac{dx}{x} = a\cdot t\cdot dt \quad$ or $\quad \ln(x) = \frac{1}{2}\cdot a\cdot t^2 + c_1$

$dy = b\cdot dt \quad$ or $\quad y = b\cdot t + c_2$

Using initial condition (x,y) = (3,1) and the given values for a and b

$c_1 = \ln(3\cdot m) \quad$ and $\quad c_2 = 1\cdot m$

The pathline is then $\quad x = 3\cdot e^{0.05\cdot t^2} \quad$ and $\quad y = 4\cdot t + 1$

For streamlines (at any time t) $\quad \frac{v}{u} = \frac{dy}{dx} = \frac{b}{a\cdot x\cdot t}$

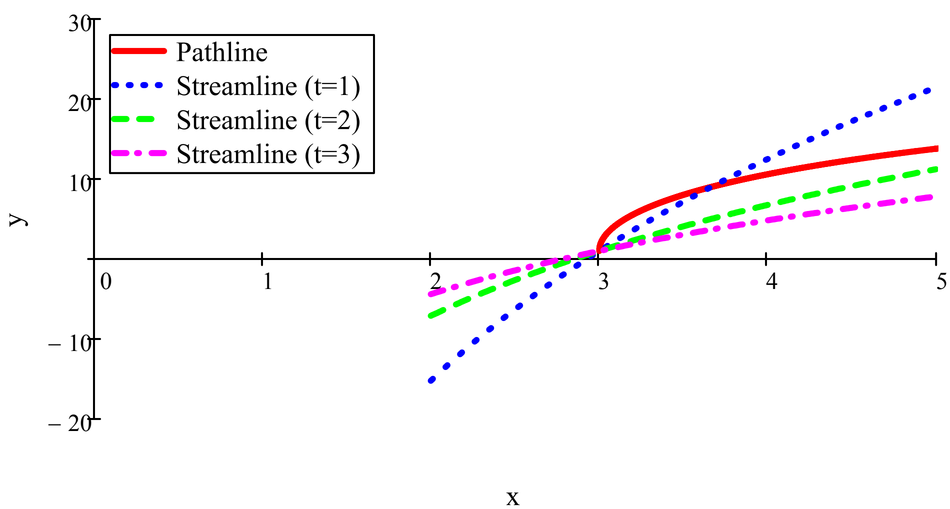
So, separating variables $\quad dy = \frac{b}{a\cdot t}\cdot\frac{dx}{x}$

Integrating $\quad y = \frac{b}{a\cdot t}\cdot\ln(x) + c$

We are interested in instantaneous streamlines at various times that always pass through point (3,1). Using a and b values:

$$c = y - \frac{b}{a\cdot t}\cdot\ln(x) = 1 - \frac{4}{0.1\cdot t}\cdot\ln(3)$$

The streamline equation is $\quad y = 1 + \frac{40}{t}\cdot\ln\left(\frac{x}{3}\right)$



These curves can be plotted in Excel.

Problem 2.26

[Difficulty: 4]

2.26 Consider the garden hose of Fig. 2.5. Suppose the velocity field is given by $\vec{V} = u_0\hat{i} + v_0\sin[\omega(t - x/u_0)]\hat{j}$, where the x direction is horizontal and the origin is at the mean position of the hose, $u_0 = 10$ m/s, $v_0 = 2$ m/s, and $\omega = 5$ cycle/s. Find and plot on one graph the instantaneous streamlines that pass through the origin at $t = 0$ s, 0.05 s, 0.1 s, and 0.15 s. Also find and plot on one graph the pathlines of particles that left the origin at the same four times.

Given: Velocity field

Find: Plot streamlines that are at origin at various times and pathlines that left origin at these times

Solution:

For streamlines
$$\frac{v}{u} = \frac{dy}{dx} = \frac{v_0\cdot\sin\left[\omega\cdot\left(t - \frac{x}{u_0}\right)\right]}{u_0}$$

So, separating variables (t=const)
$$dy = \frac{v_0\cdot\sin\left[\omega\cdot\left(t - \frac{x}{u_0}\right)\right]}{u_0}\cdot dx$$

Integrating
$$y = \frac{v_0\cdot\cos\left[\omega\cdot\left(t - \frac{x}{u_0}\right)\right]}{\omega} + c$$

Using condition y = 0 when x = 0
$$y = \frac{v_0\cdot\left[\cos\left[\omega\cdot\left(t - \frac{x}{u_0}\right)\right] - \cos(\omega\cdot t)\right]}{\omega}$$
This gives streamlines y(x) at each time t

For particle paths, first find x(t)
$$\frac{dx}{dt} = u = u_0$$

Separating variables and integrating
$$dx = u_0\cdot dt \qquad \text{or} \qquad x = u_0\cdot t + c_1$$

Using initial condition x = 0 at t = τ
$$c_1 = -u_0\cdot\tau \qquad x = u_0\cdot(t - \tau)$$

For y(t) we have
$$\frac{dy}{dt} = v = v_0\cdot\sin\left[\omega\cdot\left(t - \frac{x}{u_0}\right)\right] \qquad \text{so} \qquad \frac{dy}{dt} = v = v_0\cdot\sin\left[\omega\cdot\left[t - \frac{u_0\cdot(t-\tau)}{u_0}\right]\right]$$

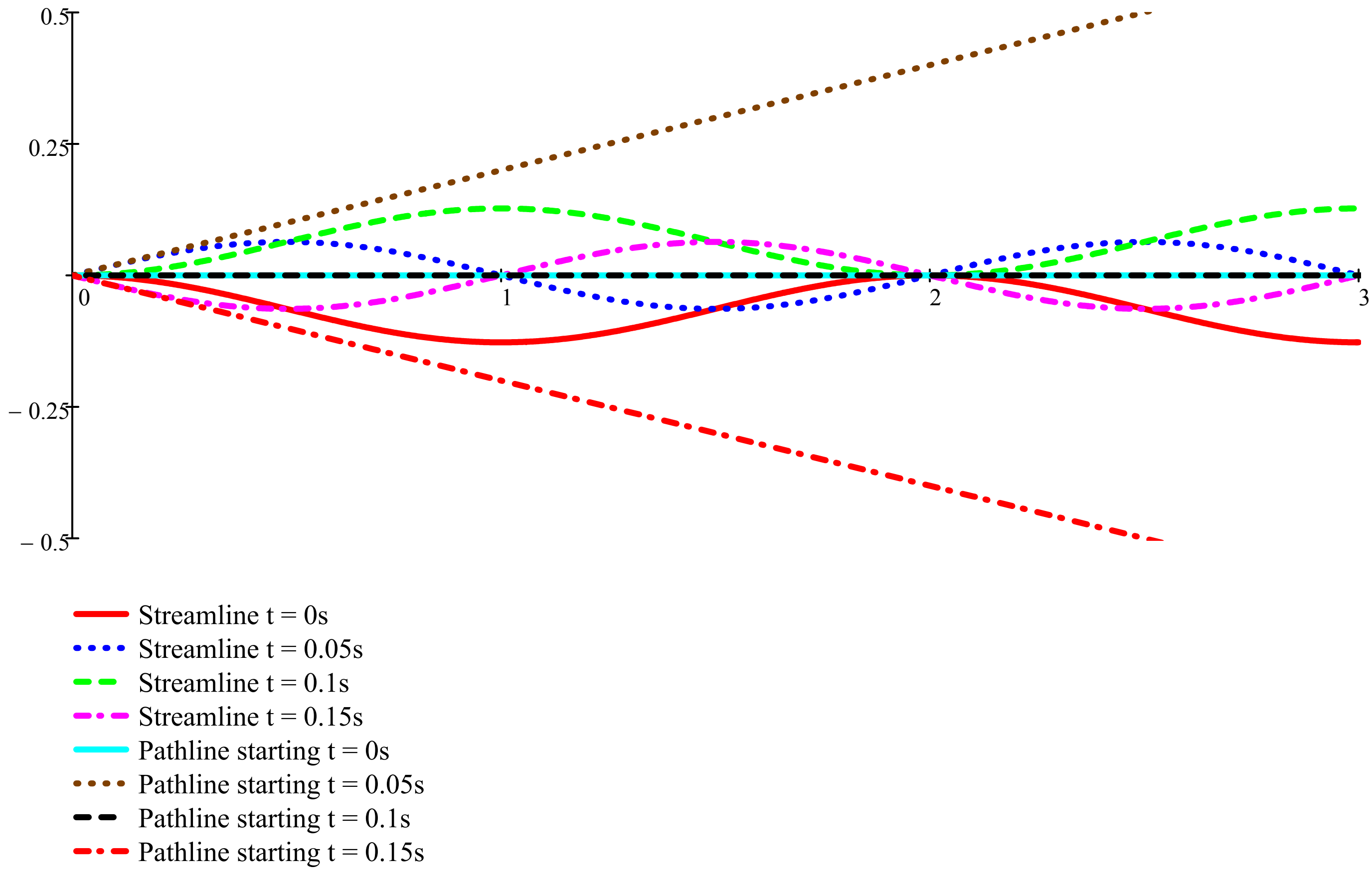
and
$$\frac{dy}{dt} = v = v_0\cdot\sin(\omega\cdot\tau)$$

Separating variables and integrating
$$dy = v_0\cdot\sin(\omega\cdot\tau)\cdot dt \qquad y = v_0\cdot\sin(\omega\cdot\tau)\cdot t + c_2$$

Using initial condition y = 0 at t = τ
$$c_2 = -v_0\cdot\sin(\omega\cdot\tau)\cdot\tau \qquad y = v_0\cdot\sin(\omega\cdot\tau)\cdot(t - \tau)$$

The pathline is then

$$x(t,\tau) = u_0\cdot(t - \tau) \qquad y(t,\tau) = v_0\cdot\sin(\omega\cdot\tau)\cdot(t - \tau)$$
These terms give the path of a particle (x(t),y(t)) that started at t = τ.



The streamlines are sinusoids; the pathlines are straight (once a water particle is fired it travels in a straight line).

These curves can be plotted in *Excel*.

Problem 2.27 [Difficulty: 5]

2.27 Using the data of Problem 2.26, find and plot the streakline shape produced after the first second of flow.

Given: Velocity field

Find: Plot streakline for first second of flow

Solution:

Following the discussion leading up to Eq. 2.10, we first find equations for the pathlines in form

$$x_p(t) = x(t, x_0, y_0, t_0) \qquad \text{and} \qquad y_p(t) = y(t, x_0, y_0, t_0)$$

where x_0, y_0 is the position of the particle at $t = t_0$, and re-interprete the results as streaklines

$$x_{st}(t_0) = x(t, x_0, y_0, t_0) \qquad \text{and} \qquad y_{st}(t_0) = y(t, x_0, y_0, t_0)$$

which gives the streakline at t, where x_0, y_0 is the point at which dye is released (t_0 is varied from 0 to t)

For particle paths, first find x(t) $\quad \dfrac{dx}{dt} = u = u_0$

Separating variables and integrating $\quad dx = u_0 \cdot dt \quad$ or $\quad x = x_0 + u_0 \cdot (t - t_0)$

For y(t) we have $\quad \dfrac{dy}{dt} = v = v_0 \cdot \sin\left[\omega \cdot \left(t - \dfrac{x}{u_0}\right)\right] \quad$ so $\quad \dfrac{dy}{dt} = v = v_0 \cdot \sin\left[\omega \cdot \left[t - \dfrac{x_0 + u_0 \cdot (t - t_0)}{u_0}\right]\right]$

and $\quad \dfrac{dy}{dt} = v = v_0 \cdot \sin\left[\omega \cdot \left(t_0 - \dfrac{x_0}{u_0}\right)\right]$

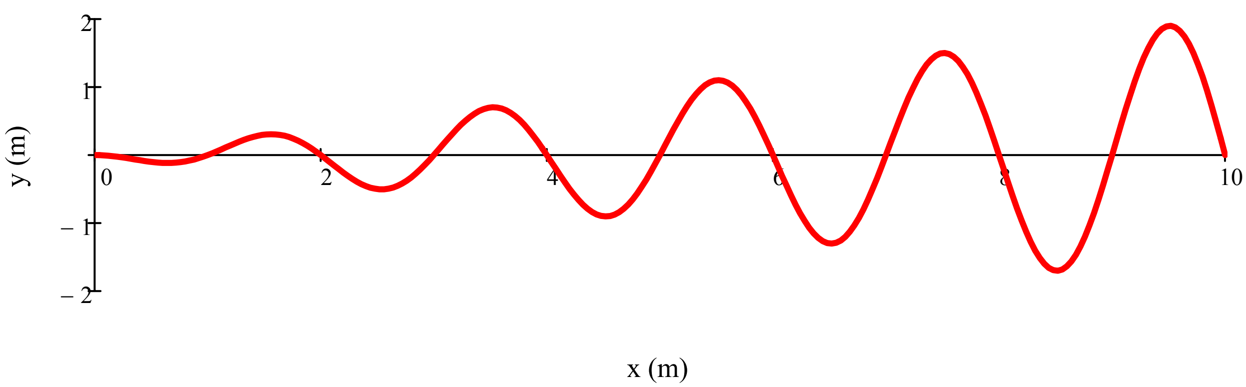
Separating variables and integrating $\quad dy = v_0 \cdot \sin\left[\omega \cdot \left(t_0 - \dfrac{x_0}{u_0}\right)\right] \cdot dt \qquad y = y_0 + v_0 \cdot \sin\left[\omega \cdot \left(t_0 - \dfrac{x_0}{u_0}\right)\right] \cdot (t - t_0)$

The streakline is then $\quad x_{st}(t_0) = x_0 + u_0 (t - t_0) \qquad y_{st}(t_0) = y_0 + v_0 \cdot \sin\left[\omega \cdot \left(t_0 - \dfrac{x_0}{u_0}\right)\right] \cdot (t - t_0)$

With $\quad x_0 = y_0 = 0$

$$x_{st}(t_0) = u_0 \cdot (t - t_0) \qquad y_{st}(t_0) = v_0 \cdot \sin[\omega \cdot (t_0)] \cdot (t - t_0)$$

Streakline for First Second



This curve can be plotted in *Excel*. For t = 1, t_0 ranges from 0 to t.

Problem 2.28

[Difficulty: 4]

2.28 Consider the velocity field of Problem 2.20. Plot the streakline formed by particles that passed through the point (1, 1) during the interval from $t = 0$ to $t = 3$ s. Compare with the streamlines plotted through the same point at the instants $t = 0$, 1, and 2 s.

Given: Velocity field

Find: Plot of streakline for t = 0 to 3 s at point (1,1); compare to streamlines through same point at the instants t = 0, 1 and 2 s

Solution:

Governing equations: For pathlines $u_p = \frac{dx}{dt}$ $\quad v_p = \frac{dy}{dt}$ $\quad$ For streamlines $\frac{v}{u} = \frac{dy}{dx}$

Following the discussion leading up to Eq. 2.10, we first find equations for the pathlines in form

$$x_p(t) = x(t, x_0, y_0, t_0) \quad \text{and} \quad y_p(t) = y(t, x_0, y_0, t_0)$$

$$x_{st}(t_0) = x(t, x_0, y_0, t_0) \quad \text{and} \quad y_{st}(t_0) = y(t, x_0, y_0, t_0)$$

which gives the streakline at t, where x_0, y_0 is the point at which dye is released (t_0 is varied from 0 to t)

Assumption: 2D flow

For pathlines $u_p = \frac{dx}{dt} = B\cdot x\cdot(1 + A\cdot t)$ $\quad A = 0.5\ \frac{1}{s}$ $\quad B = 1\ \frac{1}{s}$ $\quad v_p = \frac{dy}{dt} = C\cdot y$ $\quad C = 1\ \frac{1}{s}$

So, separating variables $\frac{dx}{x} = B\cdot(1 + A\cdot t)\cdot dt$ $\qquad \frac{dy}{y} = C\cdot dt$

Integrating

$$\ln\left(\frac{x}{x_0}\right) = B\cdot\left(t - t_0 + A\cdot\frac{t^2 - t_0^{\,2}}{2}\right) \qquad \ln\left(\frac{y}{y_0}\right) = C\cdot(t - t_0)$$

$$x = x_0\cdot e^{B\cdot\left(t - t_0 + A\cdot\frac{t^2 - t_0^{\,2}}{2}\right)} \qquad y = y_0\cdot e^{C\cdot(t - t_0)}$$

The pathlines are

$$x_p(t) = x_0\cdot e^{B\cdot\left(t - t_0 + A\cdot\frac{t^2 - t_0^{\,2}}{2}\right)} \qquad y_p(t) = y_0\cdot e^{C\cdot(t - t_0)}$$

where x_0, y_0 is the position of the particle at $t = t_0$. Re-interpreting the results as streaklines:

The streaklines are then

$$x_{st}(t_0) = x_0\cdot e^{B\cdot\left(t - t_0 + A\cdot\frac{t^2 - t_0^{\,2}}{2}\right)} \qquad y_{st}(t_0) = y_0\cdot e^{C\cdot(t - t_0)}$$

where x_0, y_0 is the point at which dye is released (t_0 is varied from 0 to t)

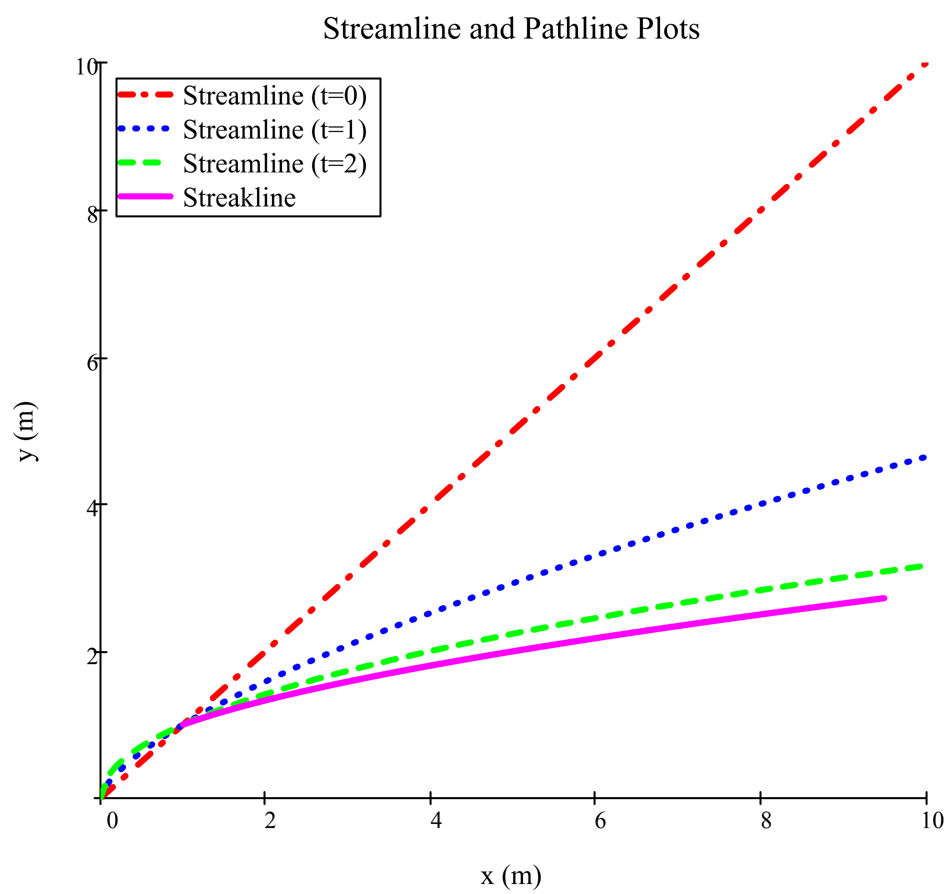
For streamlines $\quad \dfrac{v}{u} = \dfrac{dy}{dx} = \dfrac{C \cdot y}{B \cdot x \cdot (1 + A \cdot t)}$

So, separating variables $\quad (1 + A \cdot t) \cdot \dfrac{dy}{y} = \dfrac{C}{B} \cdot \dfrac{dx}{x}$ $\quad$ which we can integrate for any given t (t is treated as a constant)

Integrating $\quad (1 + A \cdot t) \cdot \ln(y) = \dfrac{C}{B} \cdot \ln(x) + \text{const}$

The solution is $\quad y^{1+A \cdot t} = \text{const} \cdot x^{\frac{C}{B}}$

For particles at (1,1) at t = 0, 1, and 2s $\quad y = x \qquad y = x^{\frac{2}{3}} \qquad y = x^{\frac{1}{2}}$



Problem 2.29

[Difficulty: 4]

2.29 Streaklines are traced out by neutrally buoyant marker fluid injected into a flow field from a fixed point in space. A particle of the marker fluid that is at point (x, y) at time t must have passed through the injection point (x_0, y_0) at some earlier instant $t = \tau$. The time history of a marker particle may be found by solving the pathline equations for the initial conditions that $x = x_0$, $y = y_0$ when $t = \tau$. The present locations of particles on the streakline are obtained by setting τ equal to values in the range $0 \le \tau \le t$. Consider the flow field $\vec{V} = ax(1+bt)\hat{i} + cy\hat{j}$, where $a = c = 1\ \text{s}^{-1}$ and $b = 0.2\ \text{s}^{-1}$. Coordinates are measured in meters. Plot the streakline that passes through the initial point $(x_0, y_0) = (1, 1)$, during the interval from $t = 0$ to $t = 3$ s. Compare with the streamline plotted through the same point at the instants $t = 0$, 1, and 2 s.

Given: Velocity field

Find: Plot of streakline for t = 0 to 3 s at point (1,1); compare to streamlines through same point at the instants t = 0, 1 and 2 s

Solution:

Governing equations: For pathlines $u_p = \dfrac{dx}{dt}$ $\quad v_p = \dfrac{dy}{dt}$ $\quad$ For streamlines $\dfrac{v}{u} = \dfrac{dy}{dx}$

Following the discussion leading up to Eq. 2.10, we first find equations for the pathlines in form

$$x_p(t) = x(t, x_0, y_0, t_0) \quad \text{and} \quad y_p(t) = y(t, x_0, y_0, t_0)$$

$$x_{st}(t_0) = x(t, x_0, y_0, t_0) \quad \text{and} \quad y_{st}(t_0) = y(t, x_0, y_0, t_0)$$

which gives the streakline at t, where x_0, y_0 is the point at which dye is released (t_0 is varied from 0 to t)

Assumption: 2D flow

For pathlines $\quad u_p = \dfrac{dx}{dt} = a \cdot x \cdot (1 + b \cdot t) \quad a = 1\ \dfrac{1}{s} \quad b = \dfrac{1}{5}\ \dfrac{1}{s} \quad v_p = \dfrac{dy}{dt} = c \cdot y \quad c = 1\ \dfrac{1}{s}$

So, separating variables $\quad \dfrac{dx}{x} = a \cdot (1 + b \cdot t) \cdot dt \qquad \dfrac{dy}{y} = c \cdot dt$

Integrating $\quad \ln\left(\dfrac{x}{x_0}\right) = a \cdot \left(t - t_0 + b \cdot \dfrac{t^2 - t_0^2}{2}\right) \qquad \ln\left(\dfrac{y}{y_0}\right) = c \cdot (t - t_0)$

$$x = x_0 \cdot e^{a \cdot \left(t - t_0 + b \cdot \frac{t^2 - t_0^2}{2}\right)} \qquad y = y_0 \cdot e^{c \cdot (t - t_0)}$$

The pathlines are $\quad x_p(t) = x_0 \cdot e^{a \cdot \left(t - t_0 + b \cdot \frac{t^2 - t_0^{\,2}}{2}\right)} \qquad y_p(t) = y_0 \cdot e^{c \cdot \left(t - t_0\right)}$

where x_0, y_0 is the position of the particle at $t = t_0$. Re-interpreting the results as streaklines:

The streaklines are then $\quad x_{st}(t_0) = x_0 \cdot e^{a \cdot \left(t - t_0 + b \cdot \frac{t^2 - t_0^{\,2}}{2}\right)} \qquad y_{st}(t_0) = y_0 \cdot e^{c \cdot \left(t - t_0\right)}$

where x_0, y_0 is the point at which dye is released (t_0 is varied from 0 to t)

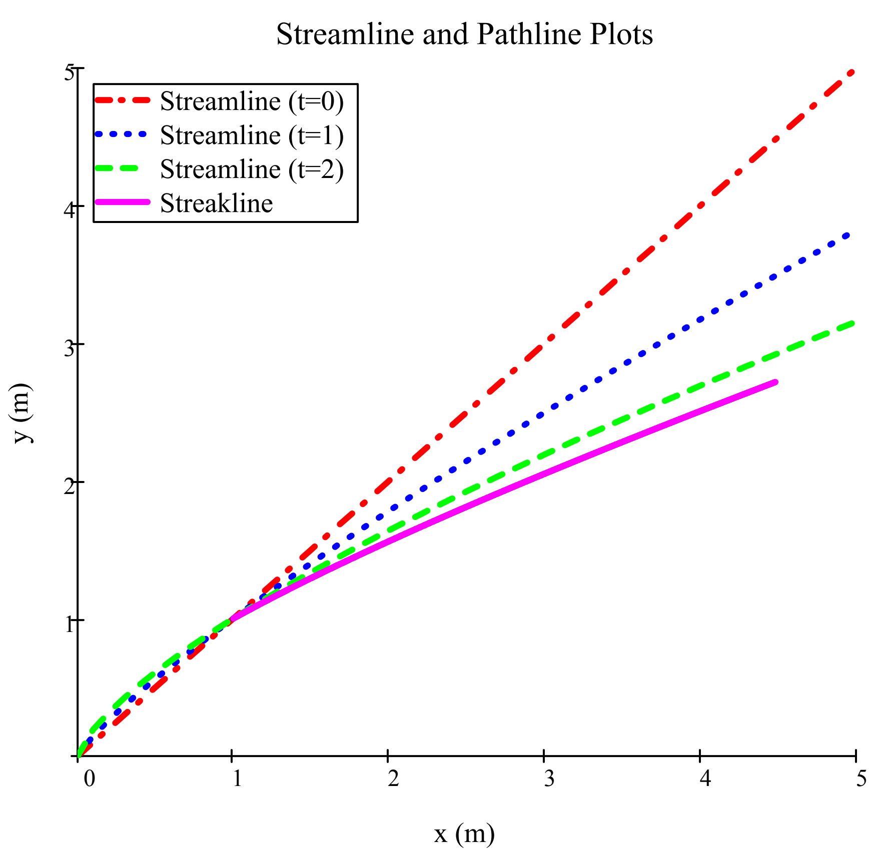
For streamlines $\quad \frac{v}{u} = \frac{dy}{dx} = \frac{c \cdot y}{a \cdot x \cdot (1 + b \cdot t)}$

So, separating variables $\quad (1 + b \cdot t) \cdot \frac{dy}{y} = \frac{c}{a} \cdot \frac{dx}{x} \qquad$ which we can integrate for any given t (t is treated as a constant)

Integrating $\quad (1 + b \cdot t) \cdot \ln(y) = \frac{c}{a} \cdot \ln(x) + const$

The solution is $\quad y^{1+b \cdot t} = const \cdot x^{\frac{c}{a}}$

For particles at (1,1) at t = 0, 1, and 2s $\quad y = x \qquad y = x^{\frac{2}{3}} \qquad y = x^{\frac{1}{2}}$



Problem 2.30

[Difficulty: 4]

2.30 Consider the flow field $\vec{V} = axt\hat{i} + b\hat{j}$, where $a = 1/4\ \mathrm{s}^{-2}$ and $b = 1/3$ m/s. Coordinates are measured in meters. For the particle that passes through the point $(x, y) = (1, 2)$ at the instant $t = 0$, plot the pathline during the time interval from $t = 0$ to 3 s. Compare this pathline with the streakline through the same point at the instant $t = 3$ s.

Given:

Velocity field

Find:

Plot of pathline for t = 0 to 3 s for particle that started at point (1,2) at t = 0; compare to streakline through same point at the instant t = 3

Solution:

Governing equations: For pathlines

$$u_p = \frac{dx}{dt} \qquad v_p = \frac{dy}{dt}$$

Following the discussion leading up to Eq. 2.10, we first find equations for the pathlines in form

$$x_p(t) = x(t, x_0, y_0, t_0) \quad \text{and} \quad y_p(t) = y(t, x_0, y_0, t_0)$$

$$x_{st}(t_0) = x(t, x_0, y_0, t_0) \quad \text{and} \quad y_{st}(t_0) = y(t, x_0, y_0, t_0)$$

which gives the streakline at t, where x_0, y_0 is the point at which dye is released (t_0 is varied from 0 to t)

Assumption: 2D flow

For pathlines

$$u_p = \frac{dx}{dt} = a \cdot x \cdot t \qquad a = \frac{1}{4} \cdot \frac{1}{s^2} \qquad b = \frac{1}{3} \cdot \frac{m}{s} \qquad v_p = \frac{dy}{dt} = b$$

So, separating variables

$$\frac{dx}{x} = a \cdot t \cdot dt \qquad dy = b \cdot dt$$

Integrating

$$\ln\left(\frac{x}{x_0}\right) = \frac{a}{2} \cdot \left(t^2 - t_0^{\,2}\right) \qquad y - y_0 = b \cdot (t - t_0)$$

$$x = x_0 \cdot e^{\frac{a}{2} \cdot \left(t^2 - t_0^{\,2}\right)} \qquad y = y_0 + b \cdot (t - t_0)$$

The pathlines are

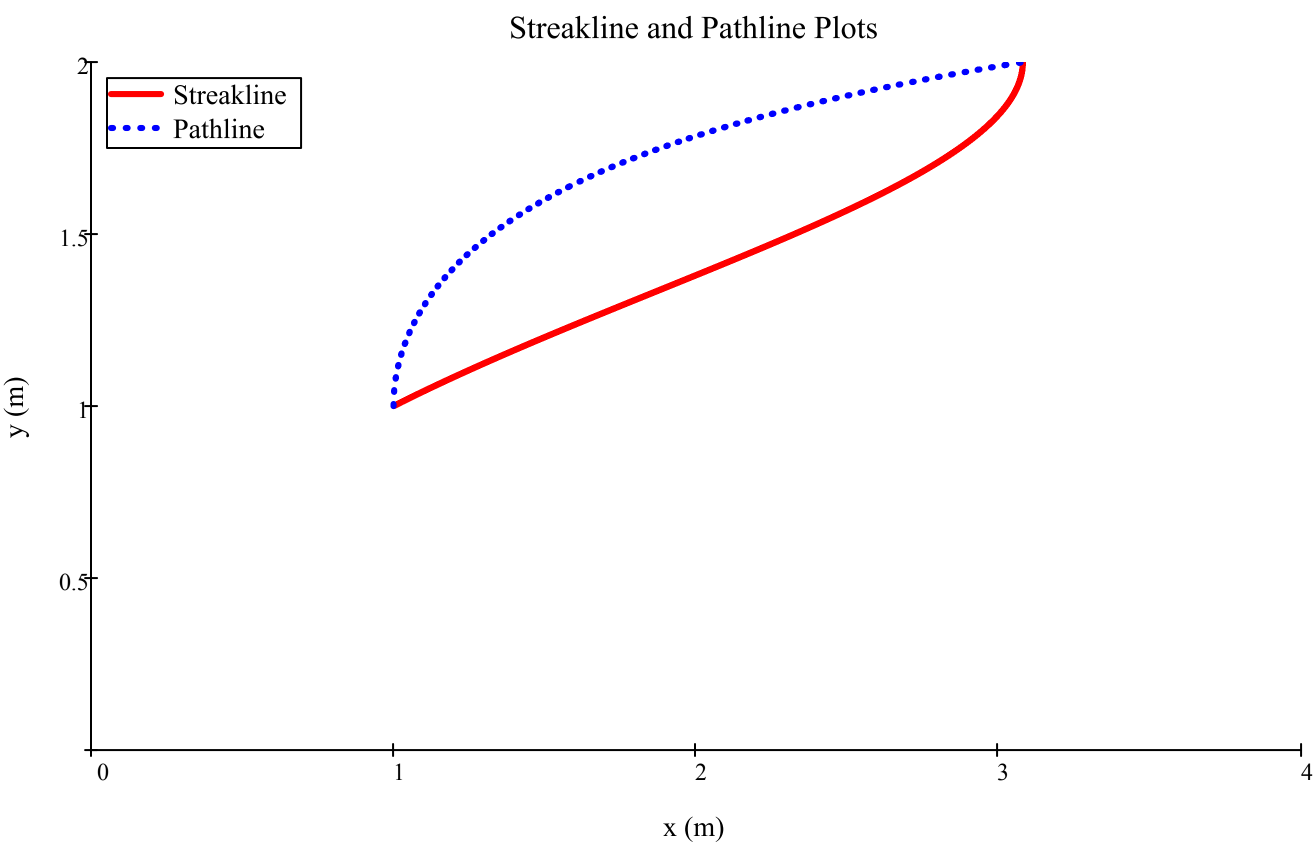
$$x_p(t) = x_0 \cdot e^{\frac{a}{2} \cdot \left(t^2 - t_0^{\,2}\right)} \qquad y_p(t) = y_0 + b \cdot (t - t_0)$$

where x_0, y_0 is the position of the particle at $t = t_0$. Re-interpreting the results as streaklines:

The pathlines are then

$$x_{st}(t_0) = x_0 \cdot e^{\frac{a}{2} \cdot \left(t^2 - t_0^{\,2}\right)} \qquad y_{st}(t_0) = y_0 + b \cdot (t - t_0)$$

where x_0, y_0 is the point at which dye is released (t_0 is varied from 0 to t)


Streakline and Pathline Plots
Streakline
Pathline
2
1.5
1
0.5
0
1
2
3
4
y (m)
x (m)

Problem 2.31

[Difficulty: 4]

2.31 A flow is described by velocity field $\vec{V} = ay^2\hat{i} + b\hat{j}$, where $a = 1\ \text{m}^{-1}\text{s}^{-1}$ and $b = 2$ m/s. Coordinates are measured in meters. Obtain the equation for the streamline passing through point (6, 6). At $t = 1$ s, what are the coordinates of the particle that passed through point (1, 4) at $t = 0$? At $t = 3$ s, what are the coordinates of the particle that passed through point $(-3, 0)$ 2 s earlier? Show that pathlines, streamlines, and streaklines for this flow coincide.

Given: 2D velocity field

Find: Streamlines passing through (6,6); Coordinates of particle starting at (1,4); that pathlines, streamlines and streaklines coincide

Solution:

For streamlines $\quad \dfrac{v}{u} = \dfrac{dy}{dx} = \dfrac{b}{a\cdot y^2} \quad$ or $\quad \displaystyle\int a\cdot y^2\,dy = \int b\,dx$

Integrating $\quad \dfrac{a\cdot y^3}{3} = b\cdot x + c$

For the streamline through point (6,6) $\quad c = 60 \quad$ and $\quad y^3 = 6\cdot x + 180$

For particle that passed through (1,4) at t = 0 $\quad u = \dfrac{dx}{dt} = a\cdot y^2 \quad \displaystyle\int 1\,dx = x - x_0 = \int a\cdot y^2\,dt \quad$ We need y(t)

$v = \dfrac{dy}{dt} = b \quad \displaystyle\int 1\,dy = \int b\,dt \quad y = y_0 + b\cdot t = y_0 + 2\cdot t$

Then $\quad \displaystyle x - x_0 = \int_0^t a\cdot\left(y_0 + b\cdot t\right)^2 dt \quad x = x_0 + a\cdot\left(y_0^2\cdot t + b\cdot y_0\cdot t^2 + \frac{b^2\cdot t^3}{3}\right)$

Hence, with $x_0 = 1 \quad y_0 = 4 \quad x = 1 + 16\cdot t + 8\cdot t^2 + \dfrac{4}{3}\cdot t^3 \quad$ At t = 1 s $\quad x = 26.3\cdot m$

$y = 4 + 2\cdot t \quad y = 6\cdot m$

For particle that passed through (-3,0) at t = 1 $\quad \displaystyle\int 1\,dy = \int b\,dt \quad y = y_0 + b\cdot\left(t - t_0\right)$

$$x - x_0 = \int_{t_0}^{t} a\cdot\left(y_0 + b\cdot t\right)^2 dt \qquad x = x_0 + a\cdot\left[y_0^2\cdot\left(t - t_0\right) + b\cdot y_0\cdot\left(t^2 - t_0^2\right) + \frac{b^2}{3}\cdot\left(t^3 - t_0^3\right)\right]$$

Hence, with $x_0 = -3$, $y_0 = 0$ at $t_0 = 1$ $\quad x = -3 + \dfrac{4}{3}\cdot\left(t^3 - 1\right) = \dfrac{1}{3}\cdot\left(4\cdot t^3 - 13\right) \quad y = 2\cdot(t - 1)$

Evaluating at t = 3 $\quad x = 31.7\cdot m \quad y = 4\cdot m$

This is a steady flow, so pathlines, streamlines and streaklines always coincide

Problem 2.32

[Difficulty: 3]

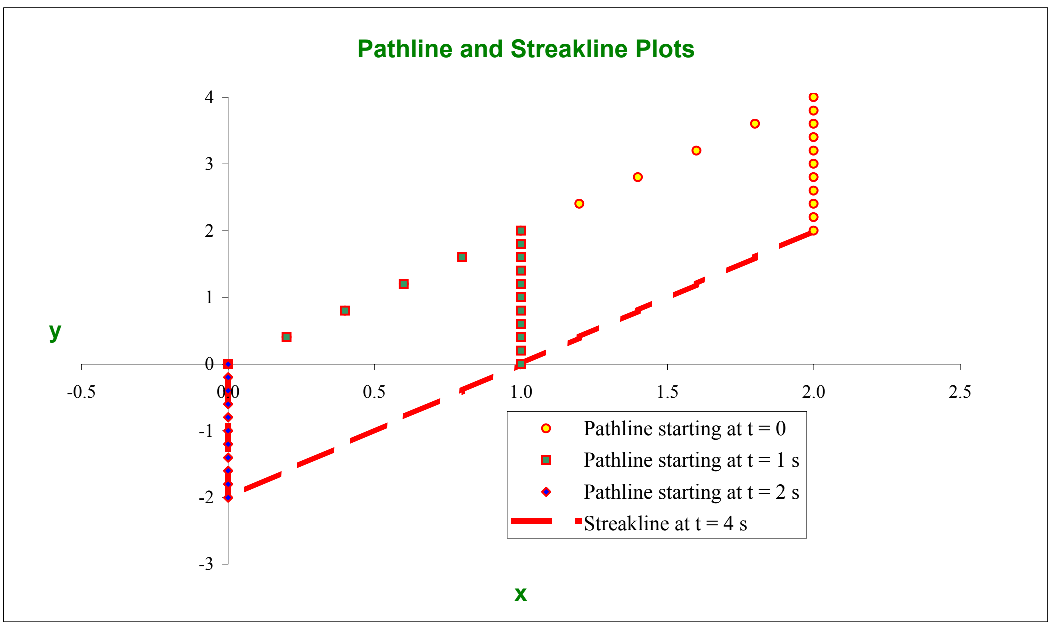
2.32 Tiny hydrogen bubbles are being used as tracers to visualize a flow. All the bubbles are generated at the origin ($x = 0$, $y = 0$). The velocity field is unsteady and obeys the equations:

$$u = 1\ \text{m/s} \qquad v = 2\ \text{m/s} \qquad 0 \le t < 2\ \text{s}$$
$$u = 0 \qquad v = -1\ \text{m/s} \qquad 0 \le t \le 4\ \text{s}$$

Plot the pathlines of bubbles that leave the origin at $t = 0$, 1, 2, 3, and 4 s. Mark the locations of these five bubbles at $t = 4$ s. Use a dashed line to indicate the position of a streakline at $t = 4$ s.

Solution The particle starting at t = 3 s follows the particle starting at t = 2 s;
The particle starting at t = 4 s doesn't move!

Pathlines:	**Starting at t = 0**		**Starting at t = 1 s**		**Starting at t = 2 s**		**Streakline at t = 4 s**	
t	**x**	**y**	**x**	**y**	**x**	**y**	**x**	**y**
0.00	0.00	0.00					2.00	2.00
0.20	0.20	0.40					1.80	1.60
0.40	0.40	0.80					1.60	1.20
0.60	0.60	1.20					1.40	0.80
0.80	0.80	1.60					1.20	0.40
1.00	1.00	2.00	0.00	0.00			1.00	0.00
1.20	1.20	2.40	0.20	0.40			0.80	-0.40
1.40	1.40	2.80	0.40	0.80			0.60	-0.80
1.60	1.60	3.20	0.60	1.20			0.40	-1.20
1.80	1.80	3.60	0.80	1.60			0.20	-1.60
2.00	2.00	4.00	1.00	2.00	0.00	0.00	0.00	-2.00
2.20	2.00	3.80	1.00	1.80	0.00	-0.20	0.00	-1.80
2.40	2.00	3.60	1.00	1.60	0.00	-0.40	0.00	-1.60
2.60	2.00	3.40	1.00	1.40	0.00	-0.60	0.00	-1.40
2.80	2.00	3.20	1.00	1.20	0.00	-0.80	0.00	-1.20
3.00	2.00	3.00	1.00	1.00	0.00	-1.00	0.00	-1.00
3.20	2.00	2.80	1.00	0.80	0.00	-1.20	0.00	-0.80
3.40	2.00	2.60	1.00	0.60	0.00	-1.40	0.00	-0.60
3.60	2.00	2.40	1.00	0.40	0.00	-1.60	0.00	-0.40
3.80	2.00	2.20	1.00	0.20	0.00	-1.80	0.00	-0.20
4.00	2.00	2.00	1.00	0.00	0.00	-2.00	0.00	0.00



Problem 2.33

[Difficulty: 3]

2.33 A flow is described by velocity field $\vec{V} = ax\hat{i} + b\hat{j}$, where $a = 1/5\ \mathrm{s}^{-1}$ and $b = 1$ m/s. Coordinates are measured in meters. Obtain the equation for the streamline passing through point (1, 1). At $t = 5$ s, what are the coordinates of the particle that initially (at $t = 0$) passed through point (1, 1)? What are its coordinates at $t = 10$ s? Plot the streamline and the initial, 5 s, and 10 s positions of the particle. What conclusions can you draw about the pathline, streamline, and streakline for this flow?

Given:
Velocity field

Find:
Equation for streamline through point (1.1); coordinates of particle at t = 5 s and t = 10 s that was at (1,1) at t = 0; compare pathline, streamline, streakline

Solution:

Governing equations: For streamlines $\frac{v}{u} = \frac{dy}{dx}$ For pathlines $u_p = \frac{dx}{dt}$ $v_p = \frac{dy}{dt}$

Assumption: 2D flow

Given data $a = \frac{1}{5} \cdot \frac{1}{s}$ $b = 1 \cdot \frac{m}{s}$ $x_0 = 1$ $y_0 = 1$ $t_0 = 0$

For streamlines $\frac{v}{u} = \frac{dy}{dx} = \frac{b}{a \cdot x}$

So, separating variables $\frac{a}{b} \cdot dy = \frac{dx}{x}$

Integrating $\frac{a}{b} \cdot (y - y_0) = \ln\left(\frac{x}{x_0}\right)$

The solution is then $y = y_0 + \frac{b}{a} \cdot \ln\left(\frac{x}{x_0}\right) = 5 \cdot \ln(x) + 1$

Hence for pathlines $u_p = \frac{dx}{dt} = a \cdot x$ $v_p = \frac{dy}{dt} = b$

Hence $\frac{dx}{x} = a \cdot dt$ $dy = b \cdot dt$

Integrating $\ln\left(\frac{x}{x_0}\right) = a \cdot (t - t_0)$ $y - y_0 = b \cdot (t - t_0)$

The pathlines are $x = x_0 \cdot e^{a \cdot (t - t_0)}$ $y = y_0 + b \cdot (t - t_0)$ or $y = y_0 + \frac{b}{a} \cdot \ln\left(\frac{x}{x_0}\right)$

For a particle that was at $x_0 = 1$ m, $y_0 = 1$ m at $t_0 = 0$ s, at time $t = 1$ s we find the position is

$$x = x_0 \cdot e^{a \cdot (t - t_0)} = e^{\frac{1}{5}} \text{ m} \qquad y = y_0 + b \cdot (t - t_0) = 2 \text{ m}$$

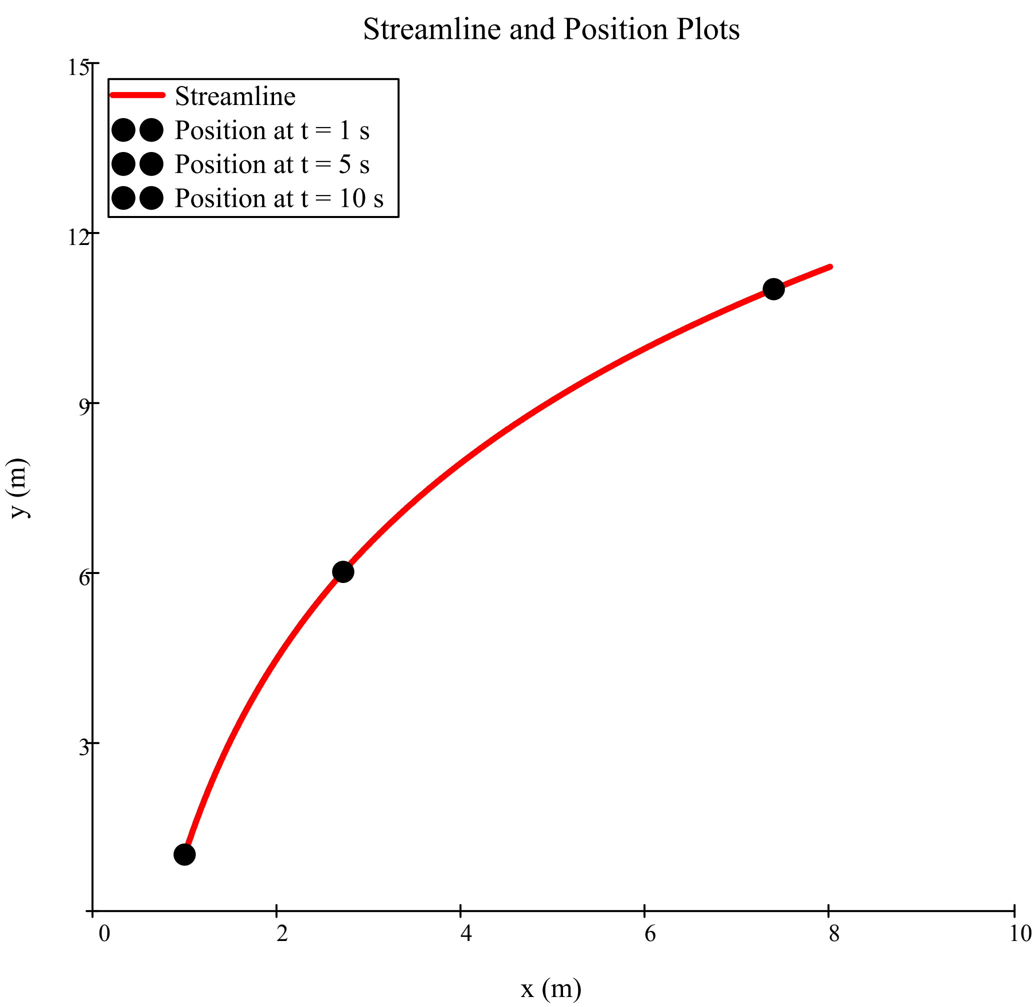
For a particle that was at $x_0 = 1$ m, $y_0 = 1$ m at $t_0 = 0$ s, at time $t = 5$ s we find the position is

$$x = x_0 \cdot e^{a \cdot (t - t_0)} = e \text{ m} \qquad y = y_0 + b \cdot (t - t_0) = 6 \text{ m}$$

For a particle that was at $x_0 = 1$ m, $y_0 = 1$ at $t_0 = 0$ s, at time $t = 10$ s we find the position is

$$x = x_0 \cdot e^{a \cdot (t - t_0)} = e^2 \text{ m} \qquad y = y_0 + b \cdot (t - t_0) = 11 \text{ m}$$

For this steady flow streamlines, streaklines and pathlines coincide



Problem 2.34

[Difficulty: 3]

2.34 A flow is described by velocity field $\vec{V} = a\hat{i} + bx\hat{j}$, where $a = 2$ m/s and $b = 1\ s^{-1}$. Coordinates are measured in meters. Obtain the equation for the streamline passing through point (2, 5). At $t = 2$ s, what are the coordinates of the particle that passed through point (0, 4) at $t = 0$? At $t = 3$ s, what are the coordinates of the particle that passed through point (1, 4.25) 2 s earlier? What conclusions can you draw about the pathline, streamline, and streakline for this flow?

Given: Velocity field

Find: Equation for streamline through point (2.5); coordinates of particle at t = 2 s that was at (0,4) at t = 0; coordinates of particle at t = 3 s that was at (1,4.25) at t = 1 s; compare pathline, streamline, streakline

Solution:

Governing equations: For streamlines $\frac{v}{u} = \frac{dy}{dx}$ For pathlines $u_p = \frac{dx}{dt}$ $v_p = \frac{dy}{dt}$

Assumption: 2D flow

Given data $a = 2\ \frac{m}{s}$ $b = 1\ \frac{1}{s}$ $x_0 = 2$ $y_0 = 5$ $x = 1$ $x = x$

For streamlines $\frac{v}{u} = \frac{dy}{dx} = \frac{b \cdot x}{a}$

So, separating variables $\frac{a}{b} \cdot dy = x \cdot dx$

Integrating $\frac{a}{b} \cdot \left(y - y_0\right) = \frac{1}{2} \cdot \left(x^2 - x_0^{\ 2}\right)$

The solution is then $y = y_0 + \frac{b}{2 \cdot a} \cdot \left(x^2 - x_0^{\ 2}\right) = \frac{x^2}{4} + 4$

Hence for pathlines $u_p = \frac{dx}{dt} = a$ $v_p = \frac{dy}{dt} = b \cdot x$

Hence $dx = a \cdot dt$ $dy = b \cdot x \cdot dt$

Integrating $x - x_0 = a \cdot \left(t - t_0\right)$ $dy = b \cdot \left[x_0 + a \cdot \left(t - t_0\right)\right] \cdot dt$

$$y - y_0 = b \cdot \left[x_0 \cdot \left(t - t_0\right) + \frac{a}{2} \cdot \left(\left(t^2 - t_0^{\ 2}\right)\right) - a \cdot t_0 \cdot \left(t - t_0\right)\right]$$

The pathlines are $x = x_0 + a \cdot \left(t - t_0\right)$ $y = y_0 + b \cdot \left[x_0 \cdot \left(t - t_0\right) + \frac{a}{2} \cdot \left(\left(t^2 - t_0^{\ 2}\right)\right) - a \cdot t_0 \cdot \left(t - t_0\right)\right]$

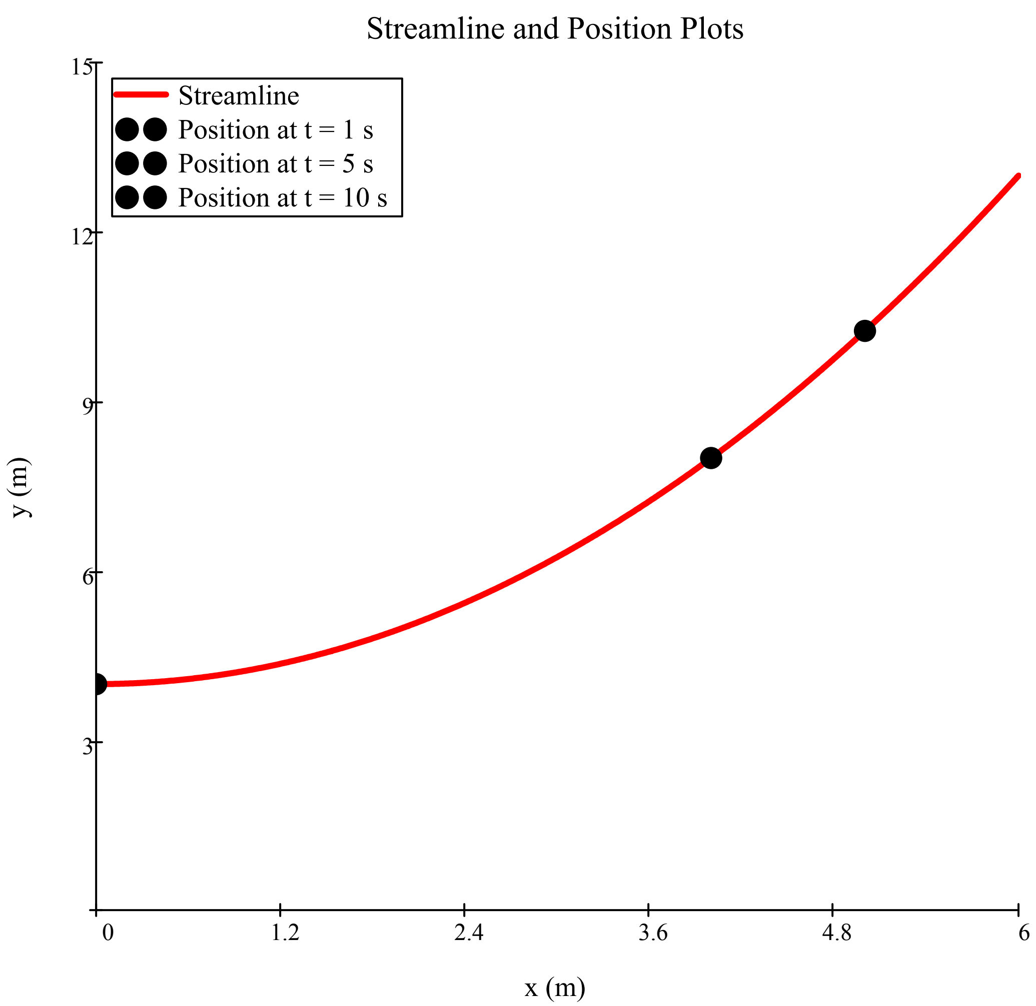
For a particle that was at $x_0 = 0\,m$, $y_0 = 4\,m$ at $t_0 = 0s$, at time $t = 2\,s$ we find the position is

$$x = x_0 + a \cdot (t - t_0) = 4\,m \qquad y = y_0 + b \cdot \left[x_0 \cdot (t - t_0) + \frac{a}{2} \cdot \left(\left(t^2 - t_0^{\,2} \right) \right) - a \cdot t_0 \cdot (t - t_0) \right] = \ m$$

For a particle that was at $x_0 = 1\,m$, $y_0 = 4.25\,m$ at $t_0 = 1\,s$, at time $t = 3\,s$ we find the position is

$$x = x_0 + a \cdot (t - t_0) = 5\,m \qquad y = y_0 + b \cdot \left[x_0 \cdot (t - t_0) + \frac{a}{2} \cdot \left(\left(t^2 - t_0^{\,2} \right) \right) - a \cdot t_0 \cdot (t - t_0) \right] = 10.\,m$$

For this steady flow streamlines, streaklines and pathlines coincide; the particles refered to are the same particle!



Problem 2.35

[Difficulty: 4]

2.35 A flow is described by velocity field $\vec{V} = ay\hat{i} + bt\hat{j}$, where $a = 0.2\ \text{s}^{-1}$ and $b = 0.4\ \text{m/s}^2$. At $t = 2$ s, what are the coordinates of the particle that passed through point (1, 2) at $t = 0$? At $t = 3$ s, what are the coordinates of the particle that passed through point (1, 2) at $t = 2$ s? Plot the pathline and streakline through point (1, 2), and plot the streamlines through the same point at the instants $t = 0$, 1, 2, and 3 s.

Given: Velocity field

Find: Coordinates of particle at t = 2 s that was at (1,2) at t = 0; coordinates of particle at t = 3 s that was at (1,2) at t = 2 s; plot pathline and streakline through point (1,2) and compare with streamlines through same point at t = 0, 1 and 2 s

Solution:

Governing equations: For pathlines $u_p = \frac{dx}{dt}$ $\quad v_p = \frac{dy}{dt}$ $\quad$ For streamlines $\frac{v}{u} = \frac{dy}{dx}$

Following the discussion leading up to Eq. 2.10, we first find equations for the pathlines in form

$$x_p(t) = x(t, x_0, y_0, t_0) \quad \text{and} \quad y_p(t) = y(t, x_0, y_0, t_0)$$

$$x_{st}(t_0) = x(t, x_0, y_0, t_0) \quad \text{and} \quad y_{st}(t_0) = y(t, x_0, y_0, t_0)$$

which gives the streakline at t, where x_0, y_0 is the point at which dye is released (t_0 is varied from 0 to t)

Assumption: 2D flow

Given data $\quad a = 0.2\ \frac{1}{s} \quad b = 0.4\ \frac{m}{s^2}$

Hence for pathlines $\quad u_p = \frac{dx}{dt} = a \cdot y \quad v_p = \frac{dy}{dt} = b \cdot t$

Hence $\quad dx = a \cdot y \cdot dt \quad dy = b \cdot t \cdot dt \quad y - y_0 = \frac{b}{2} \cdot \left(t^2 - t_0^2\right)$

For x $\quad dx = \left[a \cdot y_0 + a \cdot \frac{b}{2} \cdot \left(t^2 - t_0^2\right)\right] \cdot dt$

Integrating $\quad x - x_0 = a \cdot y_0 \cdot (t - t_0) + a \cdot \frac{b}{2} \cdot \left[\frac{t^3}{3} - \frac{t_0^3}{3} - t_0^2 \cdot (t - t_0)\right]$

The pathlines are $\quad x(t) = x_0 + a \cdot y_0 \cdot (t - t_0) + a \cdot \frac{b}{2} \cdot \left[\frac{t^3}{3} - \frac{t_0^3}{3} - t_0^2 \cdot (t - t_0)\right] \quad y(t) = y_0 + \frac{b}{2} \cdot \left(t^2 - t_0^2\right)$

These give the position (x,y) at any time t of a particle that was at (x_0,y_0) at time t_0

Note that streaklines are obtained using the logic of the Governing equations, above

The streaklines are $x(t_0) = x_0 + a \cdot y_0 \cdot (t - t_0) + a \cdot \frac{b}{2} \cdot \left[\frac{t^3}{3} - \frac{t_0^{\ 3}}{3} - t_0^{\ 2} \cdot (t - t_0) \right]$ $y(t_0) = y_0 + \frac{b}{2} \cdot \left(t^2 - t_0^{\ 2} \right)$

These gives the streakline at t, where x_0, y_0 is the point at which dye is released (t_0 is varied from 0 to t)

For a particle that was at $x_0 = 1\ \text{m}$, $y_0 = 2\ \text{m}$ at $t_0 = 0\text{s}$, at time $t = 2\ \text{s}$ we find the position is (from pathline equations)

$$x = x_0 + a \cdot y_0 \cdot (t - t_0) + a \cdot \frac{b}{2} \cdot \left[\frac{t^3}{3} - \frac{t_0^{\ 3}}{3} - t_0^{\ 2} \cdot (t - t_0) \right] = 1.9\ \text{m} \qquad y = y_0 + \frac{b}{2} \cdot \left(t^2 - t_0^{\ 2} \right) = 2.8\ \text{m}$$

For a particle that was at $x_0 = 1\ \text{m}$, $y_0 = 2\ \text{m}$ at $t_0 = 2\ \text{s}$, at time $t = 3\ \text{s}$ we find the position is

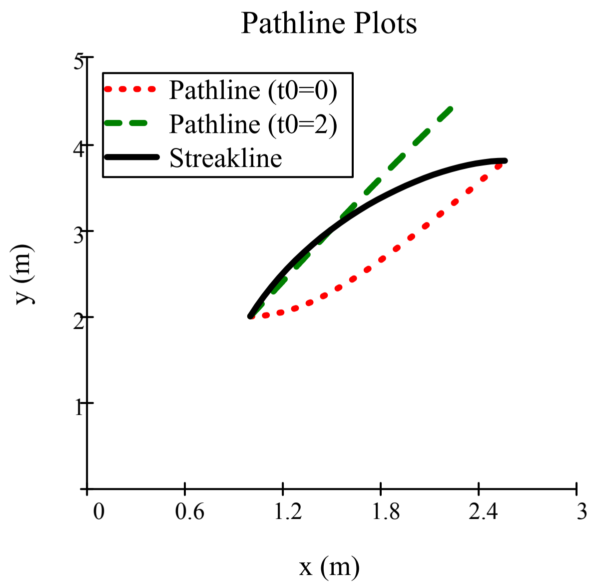
$$x = x_0 + a \cdot y_0 \cdot (t - t_0) + a \cdot \frac{b}{2} \cdot \left[\frac{t^3}{3} - \frac{t_0^{\ 3}}{3} - t_0^{\ 2} \cdot (t - t_0) \right] = 1.4\ \text{m} \qquad y = y_0 + \frac{b}{2} \cdot \left(t^2 - t_0^{\ 2} \right) = 3.0 \quad \text{m}$$

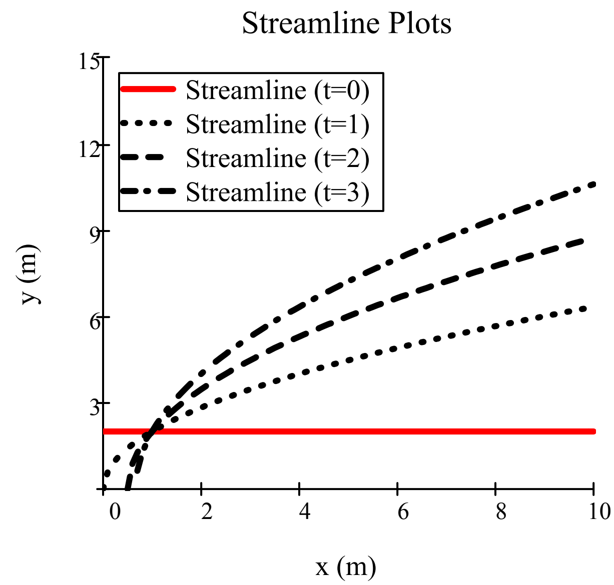
For streamlines $\frac{v}{u} = \frac{dy}{dx} = \frac{b \cdot t}{a \cdot y}$

So, separating variables $y \cdot dy = \frac{b}{a} \cdot t \cdot dx$ where we treat t as a constant

Integrating $\frac{y^2 - y_0^{\ 2}}{2} = \frac{b \cdot t}{a} \cdot (x - x_0)$ and we have $x_0 = 1\ \ \text{m}$ $y_0 = 2\ \ \text{m}$

The streamlines are then $y = \sqrt{y_0^{\ 2} + \frac{2 \cdot b \cdot t}{a} \cdot (x - x_0)} = \sqrt{4 \cdot t \cdot (x - 1) + 4}$


Pathline Plots
Pathline (t0=0)
Pathline (t0=2)
Streakline
y (m)
x (m)
0
0.6
1.2
1.8
2.4
3
1
2
3
4
5


Streamline Plots
Streamline (t=0)
Streamline (t=1)
Streamline (t=2)
Streamline (t=3)
y (m)
x (m)
0
2
4
6
8
10
3
6
9
12
15

Problem 2.36

[Difficulty: 4]

2.36 A flow is described by velocity field $\vec{V} = at\hat{i} + b\hat{j}$, where $a = 0.4$ m/s^2 and $b = 2$ m/s. At $t = 2$ s, what are the coordinates of the particle that passed through point (2, 1) at $t = 0$? At $t = 3$ s, what are the coordinates of the particle that passed through point (2, 1) at $t = 2$ s? Plot the pathline and streakline through point (2, 1) and compare with the streamlines through the same point at the instants $t = 0$, 1, and 2 s.

Given: Velocity field

Find: Coordinates of particle at t = 2 s that was at (2,1) at t = 0; coordinates of particle at t = 3 s that was at (2,1) at t = 2 s; plot pathline and streakline through point (2,1) and compare with streamlines through same point at t = 0, 1 and 2 s

Solution:

Governing equations: For pathlines $u_p = \frac{dx}{dt}$ $\quad v_p = \frac{dy}{dt}$ $\quad$ For streamlines $\frac{v}{u} = \frac{dy}{dx}$

Following the discussion leading up to Eq. 2.10, we first find equations for the pathlines in form

$$x_p(t) = x(t, x_0, y_0, t_0) \qquad \text{and} \qquad y_p(t) = y(t, x_0, y_0, t_0)$$

$$x_{st}(t_0) = x(t, x_0, y_0, t_0) \qquad \text{and} \qquad y_{st}(t_0) = y(t, x_0, y_0, t_0)$$

which gives the streakline at t, where x_0, y_0 is the point at which dye is released (t_0 is varied from 0 to t)

Assumption: 2D flow

Given data $\quad a = 0.4 \frac{m}{s^2} \qquad b = 2 \frac{m}{s^2}$

Hence for pathlines $\quad u_p = \frac{dx}{dt} = a \cdot t \qquad v_p = \frac{dy}{dt} = b$

Hence $\quad dx = a \cdot t \cdot dt \qquad dy = b \cdot dt$

Integrating $\quad x - x_0 = \frac{a}{2} \cdot \left(t^2 - t_0^2\right) \qquad y - y_0 = b \cdot (t - t_0)$

The pathlines are $\quad x(t) = x_0 + \frac{a}{2} \cdot \left(t^2 - t_0^2\right) \qquad y(t) = y_0 + b \cdot (t - t_0)$

These give the position (x,y) at any time t of a particle that was at (x_0,y_0) at time t_0

Note that streaklines are obtained using the logic of the Governing equations, above

The streaklines are $\quad x(t_0) = x_0 + \frac{a}{2} \cdot \left(t^2 - t_0^2\right) \qquad y(t_0) = y_0 + b \cdot (t - t_0)$

These gives the streakline at t, where x_0, y_0 is the point at which dye is released (t_0 is varied from 0 to t)

For a particle that was at $x_0 = 2$ m, $y_0 = 1$ m at $t_0 = 0$s, at time $t = 2$ s we find the position is (from pathline equations)

$$x = x_0 + \frac{a}{2}\cdot\left(t^2 - t_0^{\,2}\right) = 2.8 \text{ m} \qquad\qquad y = y_0 + b\cdot\left(t - t_0\right) = 5 \text{ m}$$

For a particle that was at $x_0 = 2$ m, $y_0 = 1$ m at $t_0 = 2$ s, at time $t = 3$ s we find the position is

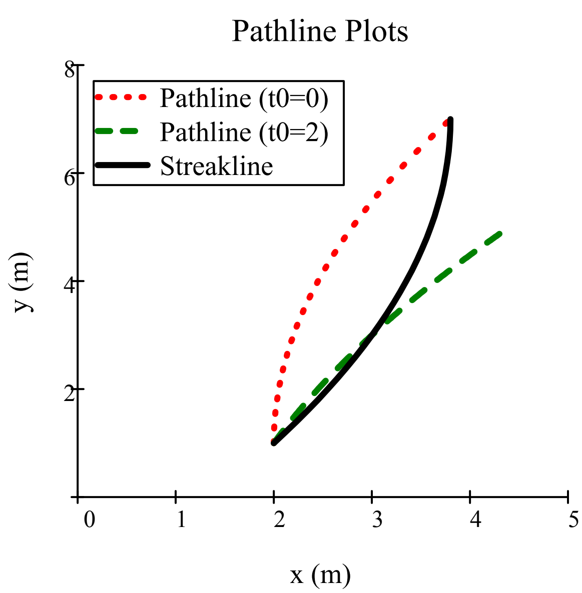
$$x = x_0 + \frac{a}{2}\cdot\left(t^2 - t_0^{\,2}\right) = 3 \text{ m} \qquad\qquad y = y_0 + b\cdot\left(t - t_0\right) = 3 \text{ m}$$

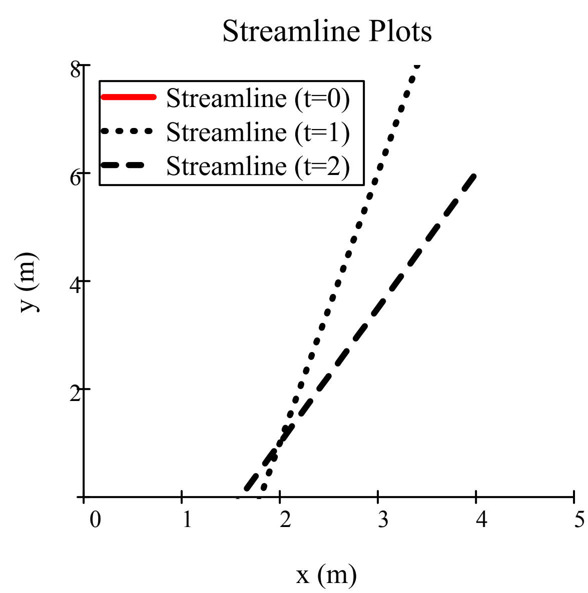
For streamlines $\quad \frac{v}{u} = \frac{dy}{dx} = \frac{b}{a\cdot t}$

So, separating variables $\quad dy = \frac{b}{a\cdot t}\cdot dx \quad$ where we treat t as a constant

Integrating $\quad y - y_0 = \frac{b}{a\cdot t}\cdot\left(x - x_0\right) \quad$ and we have $\quad x_0 = 2$ m $\quad y_0 = 1$ m

The streamlines are then $\quad y = y_0 + \frac{b}{a\cdot t}\cdot\left(x - x_0\right) = \frac{5\cdot(x-2)}{t} + 1$





Problem 2.37

[Difficulty: 2]

2.37 The variation with temperature of the viscosity of air is represented well by the empirical Sutherland correlation

$$\mu = \frac{bT^{1/2}}{1+S/T}$$

Best-fit values of b and S are given in Appendix A. Develop an equation in SI units for kinematic viscosity versus temperature for air at atmospheric pressure. Assume ideal gas behavior. Check by using the equation to compute the kinematic viscosity of air at 0°C and at 100°C and comparing to the data in Appendix 10 (Table A.10); plot the kinematic viscosity for a temperature range of 0°C to 100°C, using the equation and the data in Table A.10.

Given: Sutherland equation

Find: Corresponding equation for kinematic viscosity

Solution:

Governing equation: $\mu = \dfrac{b\cdot T^{\frac{1}{2}}}{1+\dfrac{S}{T}}$ Sutherland equation $\qquad p = \rho\cdot R\cdot T$ Ideal gas equation

Assumptions: Sutherland equation is valid; air is an ideal gas

The given data is $\quad b = 1.458\times 10^{-6}\cdot\dfrac{kg}{m\cdot s\cdot K^{\frac{1}{2}}} \qquad S = 110.4\cdot K \qquad R = 286.9\cdot\dfrac{J}{kg\cdot K} \qquad p = 101.3\cdot kPa$

The kinematic viscosity is
$$\nu = \frac{\mu}{\rho} = \frac{\mu\cdot R\cdot T}{p} = \frac{R\cdot T}{p}\cdot\frac{b\cdot T^{\frac{1}{2}}}{1+\frac{S}{T}} = \frac{R\cdot b}{p}\cdot\frac{T^{\frac{3}{2}}}{1+\frac{S}{T}} = \frac{b'\cdot T^{\frac{3}{2}}}{1+\frac{S}{T}}$$

where $\quad b' = \dfrac{R\cdot b}{p} \qquad b' = 4.129\times 10^{-9}\dfrac{m^2}{K^{1.5}\cdot s}$

$$b' = 286.9\cdot\frac{N\cdot m}{kg\cdot K}\times 1.458\times 10^{-6}\cdot\frac{kg}{m\cdot s\cdot K^{\frac{1}{2}}}\times\frac{m^2}{101.3\times 10^3\cdot N} = 4.129\times 10^{-9}\cdot\frac{m^2}{s\cdot K^{\frac{3}{2}}}$$

Hence $\quad \nu = \dfrac{b'\cdot T^{\frac{3}{2}}}{1+\dfrac{S}{T}} \quad$ with $\quad b' = 4.129\times 10^{-9}\cdot\dfrac{m^2}{s\cdot K^{\frac{3}{2}}} \qquad S = 110.4\,K$

Check with Appendix A, Table A.10. At $T = 0\,°C$ we find $\quad T = 273.1\,K \quad \nu = 1.33 \times 10^{-5}\cdot\dfrac{m^2}{s}$

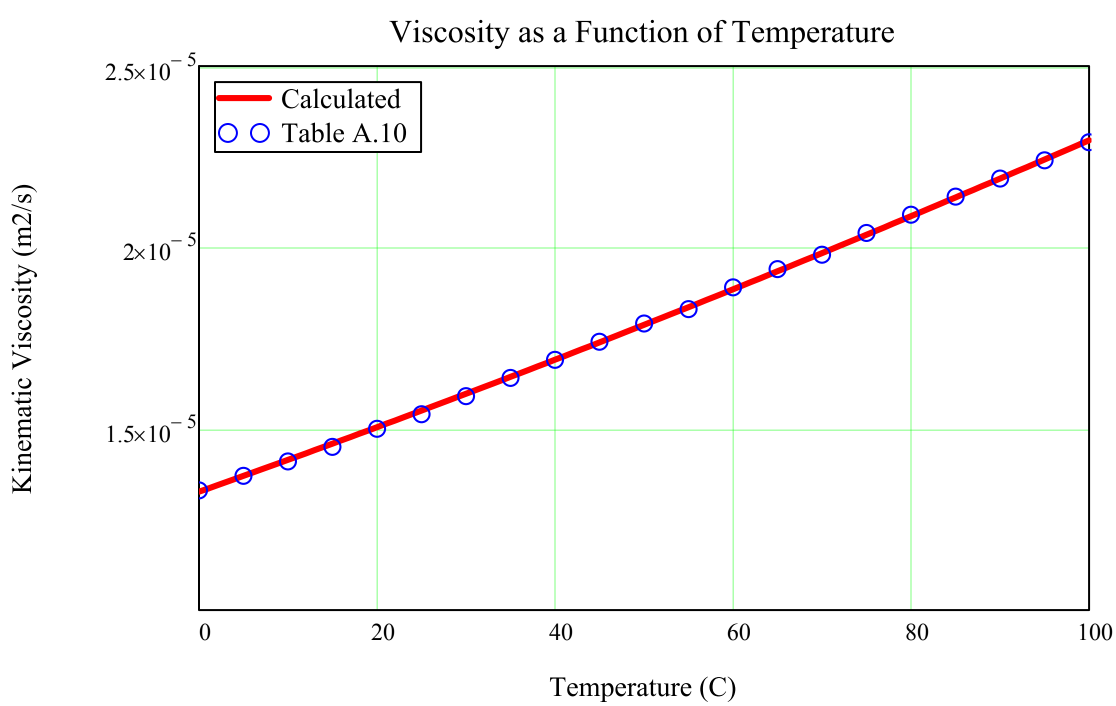
$$\nu = \frac{4.129 \times 10^{-9}\,\dfrac{m^2}{s\cdot K^{\frac{3}{2}}} \times (273.1\cdot K)^{\frac{3}{2}}}{1 + \dfrac{110.4}{273.1}} \qquad \nu = 1.33 \times 10^{-5}\cdot\frac{m^2}{s}$$

Check!

At $T = 100\,°C$ we find $\quad T = 373.1\,K \quad \nu = 2.29 \times 10^{-5}\cdot\dfrac{m^2}{s}$

$$\nu = \frac{4.129 \times 10^{-9}\,\dfrac{m^2}{s\cdot K^{\frac{3}{2}}} \times (373.1\cdot K)^{\frac{3}{2}}}{1 + \dfrac{110.4}{373.1}} \qquad \nu = 2.30 \times 10^{-5}\cdot\frac{m^2}{s}$$

Check!



Problem 2.38

[Difficulty: 2]

2.38 The variation with temperature of the viscosity of air is correlated well by the empirical Sutherland equation

$$\mu = \frac{bT^{1/2}}{1 + S/T}$$

Best-fit values of b and S are given in Appendix A for use with SI units. Use these values to develop an equation for calculating air viscosity in British Gravitational units as a function of absolute temperature in degrees Rankine. Check your result using data from Appendix A.

Given: Sutherland equation with SI units

Find: Corresponding equation in BG units

Solution:

Governing equation: $\mu = \dfrac{b \cdot T^{\frac{1}{2}}}{1 + \dfrac{S}{T}}$ Sutherland equation

Assumption: Sutherland equation is valid

The given data is $b = 1.458 \times 10^{-6} \cdot \dfrac{kg}{m \cdot s \cdot K^{\frac{1}{2}}}$ $S = 110.4 \cdot K$

Converting constants $b = 1.458 \times 10^{-6} \cdot \dfrac{kg}{m \cdot s \cdot K^{\frac{1}{2}}} \times \dfrac{lbm}{0.454 \cdot kg} \times \dfrac{slug}{32.2 \cdot lbm} \times \dfrac{0.3048 \cdot m}{ft} \times \left(\dfrac{5 \cdot K}{9 \cdot R}\right)^{\frac{1}{2}}$ $b = 2.27 \times 10^{-8} \cdot \dfrac{slug}{ft \cdot s \cdot R^{\frac{1}{2}}}$

Alternatively $b = 2.27 \times 10^{-8} \dfrac{slug}{ft \cdot s \cdot R^{\frac{1}{2}}} \times \dfrac{lbf \cdot s^2}{slug \cdot ft}$ $b = 2.27 \times 10^{-8} \cdot \dfrac{lbf \cdot s}{ft^2 \cdot R^{\frac{1}{2}}}$

Also $S = 110.4 \cdot K \times \dfrac{9 \cdot R}{5 \cdot K}$ $S = 198.7 \cdot R$

and $\mu = \dfrac{b \cdot T^{\frac{1}{2}}}{1 + \dfrac{S}{T}}$ with T in Rankine, μ in $\dfrac{lbf \cdot s}{ft^2}$

Check with Appendix A, Table A.9. At $T = 68\,°F$ we find $\quad T = 527.7 \cdot R \quad \mu = 3.79 \times 10^{-7} \cdot \frac{lbf \cdot s}{ft^2}$

$$\mu = \frac{2.27 \times 10^{-8} \dfrac{lbf \cdot s}{ft^2 \cdot R^{\frac{1}{2}}} \times (527.7 \cdot R)^{\frac{1}{2}}}{1 + \dfrac{198.7}{527.7}} \qquad \mu = 3.79 \times 10^{-7} \cdot \frac{lbf \cdot s}{ft^2} \qquad \text{Check!}$$

At $T = 200\,°F$ we find $\quad T = 659.7 \cdot R \quad \mu = 4.48 \times 10^{-7} \cdot \frac{lbf \cdot s}{ft^2}$

$$\mu = \frac{2.27 \times 10^{-8} \dfrac{lbf \cdot s}{ft^2 \cdot R^{\frac{1}{2}}} \times (659.7 \cdot R)^{\frac{1}{2}}}{1 + \dfrac{198.7}{659.7}} \qquad \mu = 4.48 \times 10^{-7} \cdot \frac{lbf \cdot s}{ft^2} \qquad \text{Check!}$$

Problem 2.39

[Difficulty: 3]

2.39 Some experimental data for the viscosity of helium at 1 atm are

T, °C	0	100	200	300	400
μ, N·s/m^2(× 10^5)	1.86	2.31	2.72	3.11	3.46

Using the approach described in Appendix A.3, correlate these data to the empirical Sutherland equation

$$\mu = \frac{bT^{1/2}}{1 + S/T}$$

(where T is in kelvin) and obtain values for constants b and S.

Given: Viscosity data

Find: Obtain values for coefficients in Sutherland equation

Solution:

Data:

T (°C)	T (K)	μ(x10^5)
0	273	1.86E-05
100	373	2.31E-05
200	473	2.72E-05
300	573	3.11E-05
400	673	3.46E-05

Using procedure of Appendix A.3:

T (K)	$T^{3/2}/\mu$
273	2.43E+08
373	3.12E+08
473	3.78E+08
573	4.41E+08
673	5.05E+08

The equation to solve for coefficients S and b is

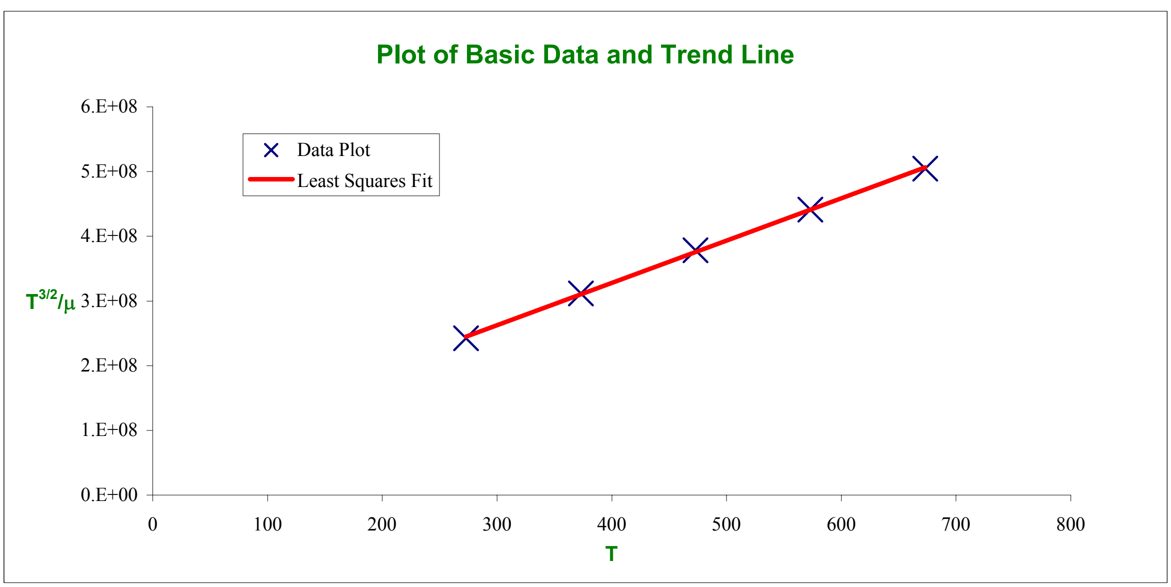
$$\frac{T^{3/2}}{\mu} = \left(\frac{1}{b}\right)T + \frac{S}{b}$$

From the built-in *Excel* *Linear Regression* functions:

Slope = 6.534E+05
Intercept = 6.660E+07
R^2 = 0.9996

Hence:

b = 1.531E-06 kg/m·s·K$^{1/2}$
S = 101.9 K



Problem 2.40

[Difficulty: 2]

2.40 The velocity distribution for laminar flow between parallel plates is given by

$$\frac{u}{u_{max}} = 1 - \left(\frac{2y}{h}\right)^2$$

where h is the distance separating the plates and the origin is placed midway between the plates. Consider a flow of water at 15°C, with $u_{max} = 0.10$ m/s and $h = 0.1$ mm. Calculate the shear stress on the upper plate and give its direction. Sketch the variation of shear stress across the channel.

Given: Velocity distribution between flat plates

Find: Shear stress on upper plate; Sketch stress distribution

Solution:

Basic equation $\qquad \tau_{yx} = \mu \cdot \frac{du}{dy} \qquad \frac{du}{dy} = \frac{d}{dy} u_{max} \cdot \left[1 - \left(\frac{2 \cdot y}{h}\right)^2\right] = u_{max} \cdot \left(-\frac{4}{h^2}\right) \cdot 2 \cdot y = -\frac{8 \cdot u_{max} \cdot y}{h^2}$

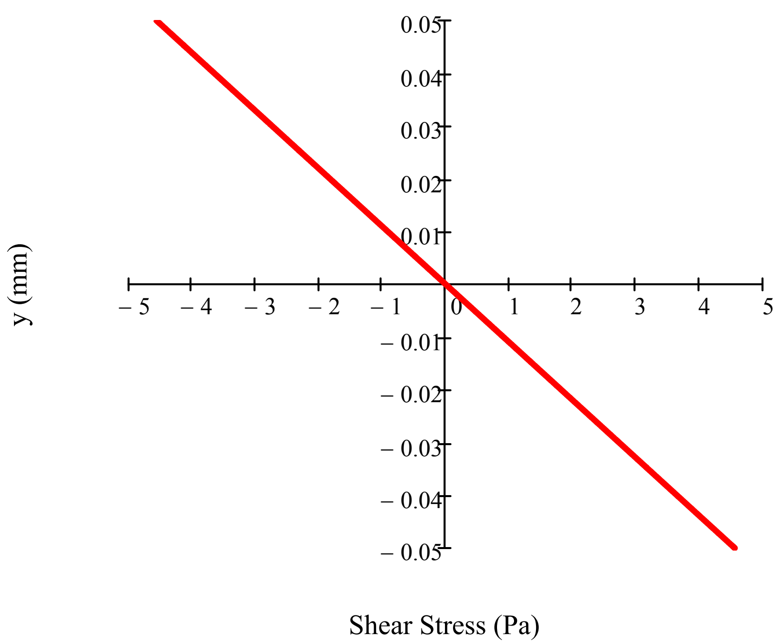
$$\tau_{yx} = -\frac{8 \cdot \mu \cdot u_{max} \cdot y}{h^2}$$

At the upper surface $\qquad y = \frac{h}{2} \quad$ and $\quad h = 0.1 \cdot mm \qquad u_{max} = 0.1 \cdot \frac{m}{s} \qquad \mu = 1.14 \times 10^{-3} \cdot \frac{N \cdot s}{m^2} \quad$ (Table A.8)

Hence $\qquad \tau_{yx} = -8 \times 1.14 \times 10^{-3} \cdot \frac{N \cdot s}{m^2} \times 0.1 \cdot \frac{m}{s} \times \frac{0.1}{2} \cdot mm \times \frac{1 \cdot m}{1000 \cdot mm} \times \left(\frac{1}{0.1 \cdot mm} \times \frac{1000 \cdot mm}{1 \cdot m}\right)^2 \qquad \tau_{yx} = -4.56 \cdot \frac{N}{m^2}$

The upper plate is a minus y surface. Since $\tau_{yx} < 0$, the shear stress on the upper plate must act in the plus x direction.

The shear stress varies linearly with y $\qquad \tau_{yx}(y) = -\left(\frac{8 \cdot \mu \cdot u_{max}}{h^2}\right) \cdot y$



Problem 2.41

[Difficulty: 2]

2.41 The velocity distribution for laminar flow between parallel plates is given by

$$\frac{u}{u_{max}} = 1 - \left(\frac{2y}{h}\right)^2$$

where h is the distance separating the plates and the origin is placed midway between the plates. Consider a flow of water at 15°C with maximum speed of 0.05 m/s and $h = 0.1$ mm. Calculate the force on a 1 m^2 section of the lower plate and give its direction.

Given: Velocity distribution between parallel plates

Find: Force on lower plate

Solution:

Basic equations $\quad F = \tau_{yx} \cdot A \qquad \tau_{yx} = \mu \cdot \frac{du}{dy}$

$$\frac{du}{dy} = \frac{d}{dy} u_{max} \cdot \left[1 - \left(\frac{2 \cdot y}{h}\right)^2\right] = u_{max} \cdot \left(-\frac{4}{h^2}\right) \cdot 2 \cdot y = -\frac{8 \cdot u_{max} \cdot y}{h^2}$$

so $\quad \tau_{yx} = -\frac{8 \cdot \mu \cdot u_{max} \cdot y}{h^2} \quad$ and $\quad F = -\frac{8 \cdot A \cdot \mu \cdot u_{max} \cdot y}{h^2}$

At the lower surface $\quad y = -\frac{h}{2} \quad$ and $\quad h = 0.1 \cdot mm \qquad A = 1 \cdot m^2$

$u_{max} = 0.05 \cdot \frac{m}{s} \qquad \mu = 1.14 \times 10^{-3} \cdot \frac{N \cdot s}{m^2}$ (Table A.8)

Hence $\quad F = -8 \times 1 \cdot m^2 \times 1.14 \times 10^{-3} \cdot \frac{N \cdot s}{m^2} \times 0.05 \cdot \frac{m}{s} \times \frac{-0.1}{2} \cdot mm \times \frac{1 \cdot m}{1000 \cdot mm} \times \left(\frac{1}{0.1} \cdot \frac{1}{mm} \times \frac{1000 \cdot mm}{1 \cdot m}\right)^2$

$F = 2.28 \cdot N \qquad$ (to the right)

Problem 2.42

[Difficulty: 2]

2.42 Explain how an ice skate interacts with the ice surface. What mechanism acts to reduce sliding friction between skate and ice?

Open-Ended Problem Statement: Explain how an ice skate interacts with the ice surface. What mechanism acts to reduce sliding friction between skate and ice?

Discussion: The normal freezing and melting temperature of ice is 0°C (32°F) at atmospheric pressure. The melting temperature of ice decreases as pressure is increased. Therefore ice can be caused to melt at a temperature below the normal melting temperature when the ice is subjected to increased pressure.

A skater is supported by relatively narrow blades with a short contact against the ice. The blade of a typical skate is less than 3 mm wide. The length of blade in contact with the ice may be just ten or so millimeters. With a 3 mm by 10 mm contact patch, a 75 kg skater is supported by a pressure between skate blade and ice on the order of tens of megaPascals (hundreds of atmospheres). Such a pressure is enough to cause ice to melt rapidly.

When pressure is applied to the ice surface by the skater, a thin surface layer of ice melts to become liquid water and the skate glides on this thin liquid film. Viscous friction is quite small, so the effective friction coefficient is much smaller than for sliding friction.

The magnitude of the viscous drag force acting on each skate blade depends on the speed of the skater, the area of contact, and the thickness of the water layer on top of the ice.

The phenomenon of static friction giving way to viscous friction is similar to the hydroplaning of a pneumatic tire caused by a layer of water on the road surface.

Problem 2.43

[Difficulty: 2]

2.43 Crude oil, with specific gravity $SG = 0.85$ and viscosity $\mu = 2.15 \times 10^{-3}$ lbf · s/ft², flows steadily down a surface inclined $\theta = 45$ degrees below the horizontal in a film of thickness $h = 0.1$ in. The velocity profile is given by

$$u = \frac{\rho g}{\mu}\left(hy - \frac{y^2}{2}\right)\sin\theta$$

(Coordinate x is along the surface and y is normal to the surface.) Plot the velocity profile. Determine the magnitude and direction of the shear stress that acts on the surface.

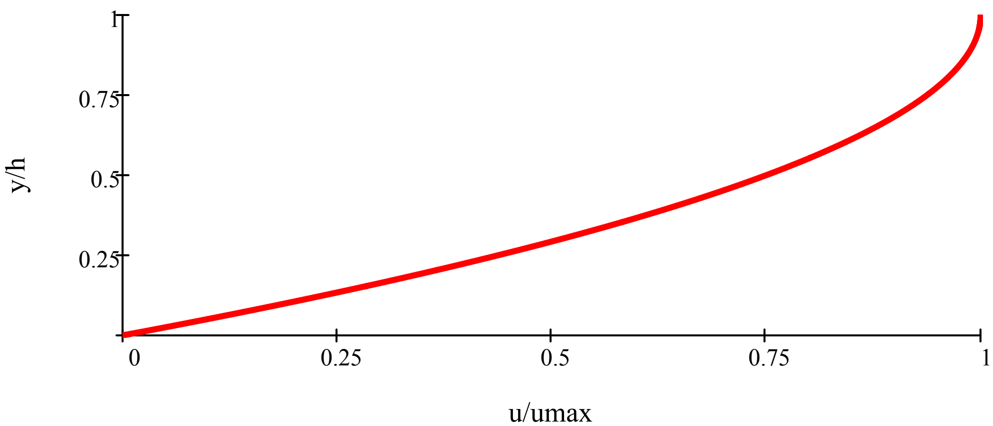
Given: Velocity profile

Find: Plot of velocity profile; shear stress on surface

Solution:

The velocity profile is $u = \frac{\rho \cdot g}{\mu} \cdot \left(h \cdot y - \frac{y^2}{2}\right) \cdot \sin(\theta)$ so the maximum velocity is at y = h $u_{max} = \frac{\rho \cdot g}{\mu} \cdot \frac{h^2}{2} \cdot \sin(\theta)$

Hence we can plot $\frac{u}{u_{max}} = 2 \cdot \left[\frac{y}{h} - \frac{1}{2} \cdot \left(\frac{y}{h}\right)^2\right]$



This graph can be plotted in *Excel*

The given data is $h = 0.1 \cdot in$ $\mu = 2.15 \times 10^{-3} \cdot \frac{lbf \cdot s}{ft^2}$ $\theta = 45 \cdot deg$

Basic equation $\tau_{yx} = \mu \cdot \frac{du}{dy}$ $\tau_{yx} = \mu \cdot \frac{du}{dy} = \mu \cdot \frac{d}{dy} \frac{\rho \cdot g}{\mu} \cdot \left(h \cdot y - \frac{y^2}{2}\right) \cdot \sin(\theta) = \rho \cdot g \cdot (h - y) \cdot \sin(\theta)$

At the surface y = 0 $\tau_{yx} = \rho \cdot g \cdot h \cdot \sin(\theta)$

Hence $\tau_{yx} = 0.85 \times 1.94 \cdot \frac{slug}{ft^3} \times 32.2 \cdot \frac{ft}{s^2} \times 0.1 \cdot in \times \frac{1 \cdot ft}{12 \cdot in} \times \sin(45 \cdot deg) \times \frac{lbf \cdot s^2}{slug \cdot ft}$ $\tau_{yx} = 0.313 \cdot \frac{lbf}{ft^2}$

The surface is a positive y surface. Since $\tau_{yx} > 0$, the shear stress on the surface must act in the plus x direction.

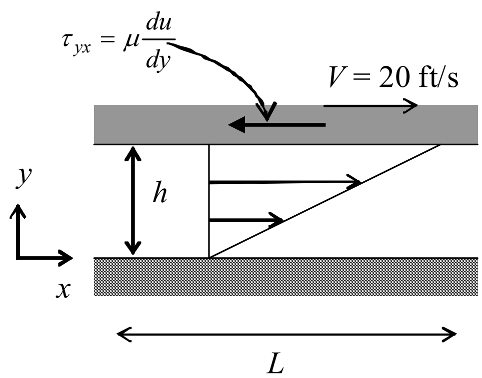
Problem 2.44

[Difficulty: 2]

2.44 A female freestyle ice skater, weighing 100 lbf, glides on one skate at speed $V = 20$ ft/s. Her weight is supported by a thin film of liquid water melted from the ice by the pressure of the skate blade. Assume the blade is $L = 11.5$ in. long and $w = 0.125$ in. wide, and that the water film is $h = 0.0000575$ in. thick. Estimate the deceleration of the skater that results from viscous shear in the water film, if end effects are neglected.

Given: Ice skater and skate geometry

Find: Deceleration of skater



Solution:

Governing equation: $\tau_{yx} = \mu \cdot \dfrac{du}{dy}$ $\qquad \Sigma F_x = M \cdot a_x$

Assumptions: Laminar flow

The given data is $\quad W = 100 \cdot \text{lbf} \quad V = 20 \cdot \dfrac{\text{ft}}{\text{s}} \quad L = 11.5 \cdot \text{in} \quad w = 0.125 \cdot \text{in} \quad h = 0.0000575 \cdot \text{in}$

$$\mu = 3.68 \times 10^{-5} \cdot \frac{\text{lbf} \cdot \text{s}}{\text{ft}^2} \qquad \text{Table A.7 @32°F}$$

Then

$$\tau_{yx} = \mu \cdot \frac{du}{dy} = \mu \cdot \frac{V}{h} = 3.68 \times 10^{-5} \cdot \frac{\text{lbf} \cdot \text{s}}{\text{ft}^2} \times 20 \cdot \frac{\text{ft}}{\text{s}} \times \frac{1}{0.0000575 \cdot \text{in}} \times \frac{12 \cdot \text{in}}{\text{ft}}$$

$$\tau_{yx} = 154 \cdot \frac{\text{lbf}}{\text{ft}^2}$$

Equation of motion $\quad \Sigma F_x = M \cdot a_x \quad$ or $\quad \tau_{yx} \cdot A = \dfrac{-W}{g} \cdot a_x$

$$a_x = -\frac{\tau_{yx} \cdot A \cdot g}{W} = -\frac{\tau_{yx} \cdot L \cdot w \cdot g}{W}$$

$$a_x = -154 \frac{\text{lbf}}{\text{ft}^2} \times 11.5 \cdot \text{in} \times 0.125 \cdot \text{in} \times 32.2 \cdot \frac{\text{ft}}{\text{s}^2} \times \frac{1}{100 \cdot \text{lbf}} \times \frac{\text{ft}^2}{(12 \cdot \text{in})^2}$$

$$a_x = -0.495 \cdot \frac{\text{ft}}{\text{s}^2}$$

Problem 2.45

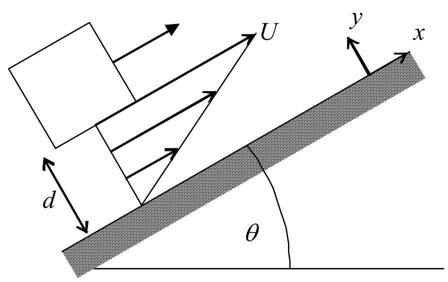
[Difficulty: 2]

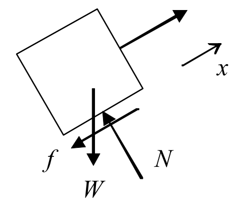
2.45 A block weighing 10 lbf and having dimensions 10 in. on each edge is pulled up an inclined surface on which there is a film of SAE 10W oil at 100°F. If the speed of the block is 2 ft/s and the oil film is 0.001 in. thick, find the force required to pull the block. Assume the velocity distribution in the oil film is linear. The surface is inclined at an angle of 25° from the horizontal.

Given: Block pulled up incline on oil layer

Find: Force required to pull the block

Solution:





Governing equations:

$$\tau_{yx} = \mu \cdot \frac{du}{dy}$$

$$\Sigma F_x = M \cdot a_x$$

Assumptions: Laminar flow

The given data is $W = 10 \cdot lbf$ $\quad U = 2 \cdot \frac{ft}{s}$ $\quad w = 10 \cdot in$ $\quad d = 0.001 \cdot in$ $\quad \theta = 25 \cdot deg$

$$\mu = 3.7 \times 10^{-2} \cdot \frac{N \cdot s}{m^2}$$ Fig. A.2 @100°F (38°C)

Equation of motion $\quad \Sigma F_x = M \cdot a_x = 0 \quad$ so $\quad F - f - W \cdot \sin(\theta) = 0$

The friction force is

$$f = \tau_{yx} \cdot A = \mu \cdot \frac{du}{dy} \cdot A = \mu \cdot \frac{U}{d} \cdot w^2$$

Hence

$$F = f + W \cdot \sin(\theta) = \mu \cdot \frac{U}{d} \cdot w^2 + W \cdot \sin(\theta)$$

$$F = 3.7 \times 10^{-2} \cdot \frac{N \cdot s}{m^2} \times 0.0209 \cdot \frac{lbf \cdot s}{ft^2} \cdot \frac{m^2}{N \cdot s} \times 2 \cdot \frac{ft}{s} \times \frac{1}{0.001 \cdot in} \times (10 \cdot in)^2 \times \frac{ft}{12 \cdot in} + 10 \cdot lbf \cdot \sin(25 \cdot deg)$$

$$F = 17.1 \cdot lbf$$

Problem 2.46

[Difficulty: 2]

2.46 A block of mass 10 kg and measuring 250 mm on each edge is pulled up an inclined surface on which there is a film of SAE 10W-30 oil at 30°F (the oil film is 0.025 mm thick). Find the steady speed of the block if it is released. If a force of 75 N is applied to pull the block up the incline, find the steady speed of the block. If the force is now applied to push the block down the incline, find the steady speed of the block. Assume the velocity distribution in the oil film is linear. The surface is inclined at an angle of 30° from the horizontal.

Given:

Block moving on incline on oil layer

Find:

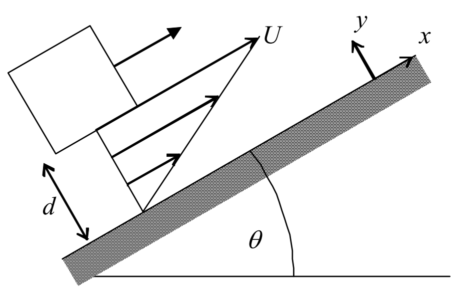
Speed of block when free, pulled, and pushed

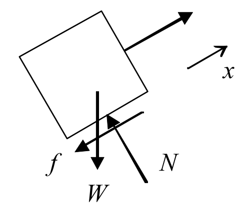
Solution:

Governing equations:

$$\tau_{yx} = \mu \cdot \frac{du}{dy}$$

$$\Sigma F_x = M \cdot a_x$$





Assumptions: Laminar flow

The given data is $M = 10 \cdot kg$ $\quad W = M \cdot g$ $\quad W = 98.066\,N$ $\quad w = 250 \cdot mm$

$d = 0.025 \cdot mm$ $\quad \theta = 30 \cdot deg$ $\quad F = 75 \cdot N$

$$\mu = 10^{-1} \cdot \frac{N \cdot s}{m^2}$$

Fig. A.2 SAE 10-39 @30°C

Equation of motion $\quad \Sigma F_x = M \cdot a_x = 0 \quad$ so $\quad F - f - W \cdot \sin(\theta) = 0$

The friction force is

$$f = \tau_{yx} \cdot A = \mu \cdot \frac{du}{dy} \cdot A = \mu \cdot \frac{U}{d} \cdot w^2$$

Hence for uphill motion

$$F = f + W \cdot \sin(\theta) = \mu \cdot \frac{U}{d} \cdot w^2 + W \cdot \sin(\theta) \qquad U = \frac{d \cdot (F - W \cdot \sin(\theta))}{\mu \cdot w^2}$$

(For downpush change sign of W)

For no force: $\quad U = \dfrac{d \cdot W \cdot \sin(\theta)}{\mu \cdot w^2} \qquad U = 0.196 \dfrac{m}{s}$

Pushing up: $\quad U = \dfrac{d \cdot (F - W \cdot \sin(\theta))}{\mu \cdot w^2} \qquad U = 0.104 \dfrac{m}{s}$

Pushing down: $\quad U = \dfrac{d \cdot (F + W \cdot \sin(\theta))}{\mu \cdot w^2} \qquad U = 0.496 \dfrac{m}{s}$

Problem 2.47

[Difficulty: 2]

2.47 Tape is to be coated on both sides with glue by drawing it through a narrow gap. The tape is 0.015 in. thick and 1.00 in. wide. It is centered in the gap with a clearance of 0.012 in. on each side. The glue, of viscosity $\mu = 0.02$ slug/(ft·s), completely fills the space between the tape and gap. If the tape can withstand a maximum tensile force of 25 lbf, determine the maximum gap region through which it can be pulled at a speed of 3 ft/s.

Given: Data on tape mechanism

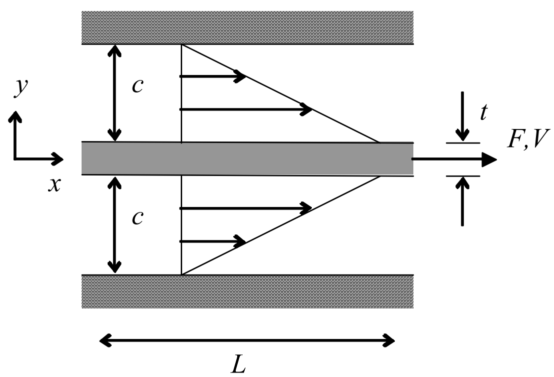
Find: Maximum gap region that can be pulled without breaking tape

Solution:

Basic equation $\quad \tau_{yx} = \mu \cdot \frac{du}{dy} \quad$ and $\quad F = \tau_{yx} \cdot A$

Here F is the force on each side of the tape; the total force is then $\quad F_T = 2 \cdot F = 2 \cdot \tau_{yx} \cdot A$

The velocity gradient is linear as shown $\quad \frac{du}{dy} = \frac{V - 0}{c} = \frac{V}{c}$



The area of contact is $\quad A = w \cdot L$

Combining these results $\quad F_T = 2 \cdot \mu \cdot \frac{V}{c} \cdot w \cdot L$

Solving for L $\quad L = \frac{F_T \cdot c}{2 \cdot \mu \cdot V \cdot w}$

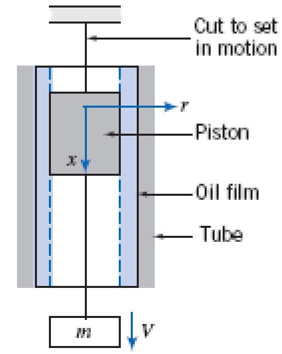
The given data is $\quad F_T = 25 \cdot \text{lbf} \quad c = 0.012 \cdot \text{in} \quad \mu = 0.02 \cdot \frac{\text{slug}}{\text{ft} \cdot \text{s}} \quad V = 3 \cdot \frac{\text{ft}}{\text{s}} \quad w = 1 \cdot \text{in}$

Hence $$L = 25 \cdot \text{lbf} \times 0.012 \cdot \text{in} \times \frac{1 \cdot \text{ft}}{12 \cdot \text{in}} \times \frac{1}{2} \times \frac{1}{0.02} \cdot \frac{\text{ft} \cdot \text{s}}{\text{slug}} \times \frac{1}{3} \cdot \frac{\text{s}}{\text{ft}} \times \frac{1}{1} \cdot \frac{1}{\text{in}} \times \frac{12 \cdot \text{in}}{1 \cdot \text{ft}} \times \frac{\text{slug} \cdot \text{ft}}{\text{s}^2 \cdot \text{lbf}} \qquad L = 2.5 \cdot \text{ft}$$

Problem 2.48

[Difficulty: 2]

2.48 A 73-mm-diameter aluminum (SG = 2.64) piston of 100-mm length resides in a stationary 75-mm-inner-diameter steel tube lined with SAE 10W-30 oil at 25°C. A mass $m = 2$ kg is suspended from the free end of the piston. The piston is set into motion by cutting a support cord. What is the terminal velocity of mass m? Assume a linear velocity profile within the oil.



Given: Flow data on apparatus

Find: The terminal velocity of mass *m*

Solution:

Given data: $D_{piston} = 73 \cdot mm$ $\quad D_{tube} = 75 \cdot mm$ $\quad Mass = 2 \cdot kg$ $\quad L = 100 \cdot mm$ $\quad SG_{Al} = 2.64$

Reference data: $\rho_{water} = 1000 \cdot \frac{kg}{m^3}$ (maximum density of water)

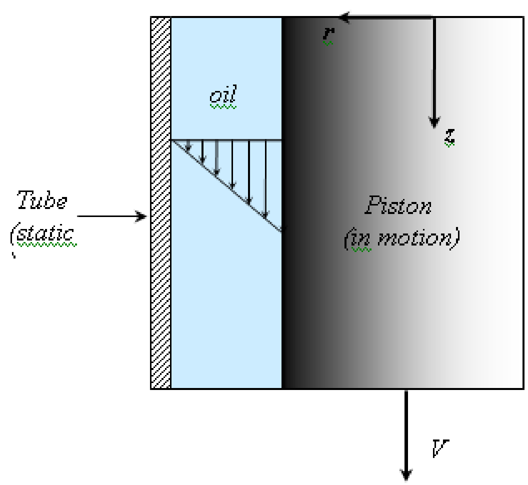
From Fig. A.2:, the dynamic viscosity of SAE 10W-30 oil at 25°C is: $\mu = 0.13 \cdot \frac{N \cdot s}{m^2}$

The terminal velocity of the mass *m* is equivalent to the terminal velocity of the piston. At that terminal speed, the acceleration of the piston is zero. Therefore, all forces acting on the piston must be balanced. This means that the force driving the motion (i.e. the weight of mass *m* and the piston) balances the viscous forces acting on the surface of the piston. Thus, at r = R_{piston}:

$$\left[Mass + SG_{Al} \cdot \rho_{water} \cdot \left(\frac{\pi \cdot D_{piston}^{2} \cdot L}{4}\right)\right] \cdot g = \tau_{rz} \cdot A = \left(\mu \cdot \frac{d}{dr} V_z\right) \cdot \left(\pi \cdot D_{piston} \cdot L\right)$$

The velocity profile within the oil film is linear ...

Therefore $\quad \frac{d}{dr} V_z = \frac{V}{\left(\frac{D_{tube} - D_{piston}}{2}\right)}$



Thus, the terminal velocity of the piston, *V*, is:

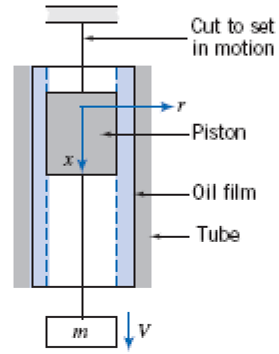
$$V = \frac{g \cdot \left(SG_{Al} \cdot \rho_{water} \cdot \pi \cdot D_{piston}^{2} \cdot L + 4 \cdot Mass\right) \cdot \left(D_{tube} - D_{piston}\right)}{8 \cdot \mu \cdot \pi \cdot D_{piston} \cdot L}$$

or $\quad V = 10.2 \frac{m}{s}$

Problem 2.49

[Difficulty: 3]

2.49 The piston in Problem 2.48 is traveling at terminal speed. The mass m now disconnects from the piston. Plot the piston speed vs. time. How long does it take the piston to come within 1 percent of its new terminal speed?



Given: Flow data on apparatus

Find: Sketch of piston speed vs time; the time needed for the piston to reach 99% of its new terminal speed.

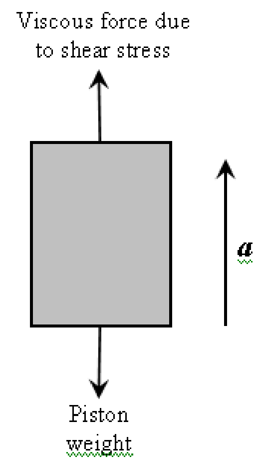
Solution:

Given data: $D_{piston} = 73\cdot mm$ $\quad D_{tube} = 75\cdot mm$ $\quad L = 100\cdot mm$ $\quad SG_{Al} = 2.64$ $\quad V_0 = 10.2\cdot\frac{m}{s}$

Reference data: $\rho_{water} = 1000\cdot\frac{kg}{m^3}$ (maximum density of water) (From Problem 2.48)

From Fig. A.2, the dynamic viscosity of SAE 10W-30 oil at 25ºC is: $\mu = 0.13\cdot\frac{N\cdot s}{m^2}$

The free body diagram of the piston after the cord is cut is:



Piston weight: $W_{piston} = SG_{Al}\cdot\rho_{water}\cdot g\cdot\left(\frac{\pi\cdot D_{piston}^{\ 2}}{4}\right)\cdot L$

Viscous force: $F_{viscous}(V) = \tau_{rz}\cdot A$

or $F_{viscous}(V) = \mu\cdot\left[\frac{V}{\frac{1}{2}\cdot\left(D_{tube} - D_{piston}\right)}\right]\cdot\left(\pi\cdot D_{piston}\cdot L\right)$

Applying Newton's second law: $m_{piston}\cdot\frac{dV}{dt} = W_{piston} - F_{viscous}(V)$

Therefore $\frac{dV}{dt} = g - a\cdot V$ where $a = \frac{8\cdot\mu}{SG_{Al}\cdot\rho_{water}\cdot D_{piston}\cdot\left(D_{tube} - D_{piston}\right)}$

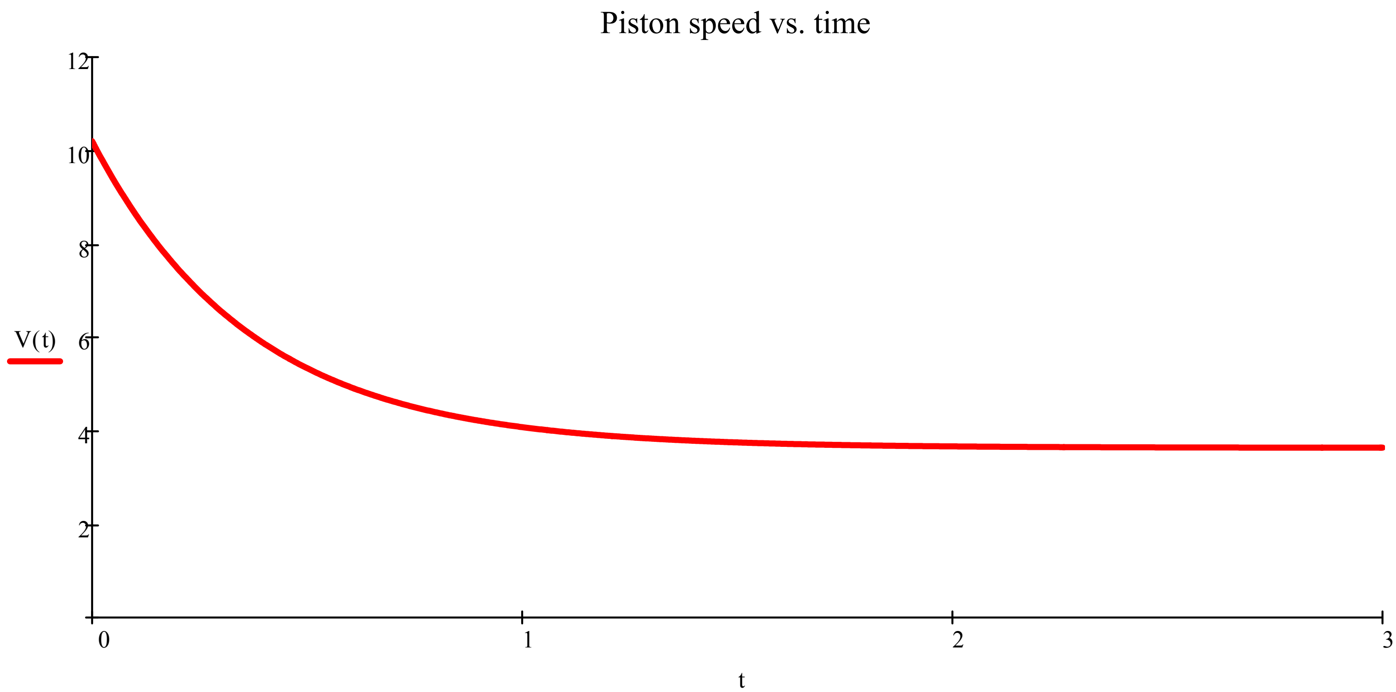
If $V = g - a\cdot V$ then $\frac{dX}{dt} = -a\cdot\frac{dV}{dt}$

The differential equation becomes $\frac{dX}{dt} = -a\cdot X$ where $X(0) = g - a\cdot V_0$

The solution to this differential equation is: $X(t) = X_0\cdot e^{-a\cdot t}$ or $g - a\cdot V(t) = \left(g - a\cdot V_0\right)\cdot e^{-a\cdot t}$

Therefore $$V(t) = \left(V_0 - \frac{g}{a}\right) \cdot e^{(-a \cdot t)} + \frac{g}{a}$$

Plotting piston speed vs. time (which can be done in Excel)



The terminal speed of the piston, V_t, is evaluated as t approaches infinity

$$V_t = \frac{g}{a} \qquad \text{or} \qquad V_t = 3.63\,\frac{m}{s}$$

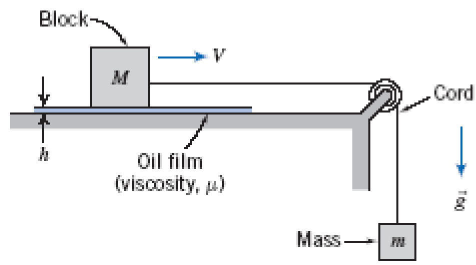
The time needed for the piston to slow down to within 1% of its terminal velocity is:

$$t = \frac{1}{a} \cdot \ln\left(\frac{V_0 - \frac{g}{a}}{1.01 \cdot V_t - \frac{g}{a}}\right) \qquad \text{or} \qquad t = 1.93\,s$$

Problem 2.50

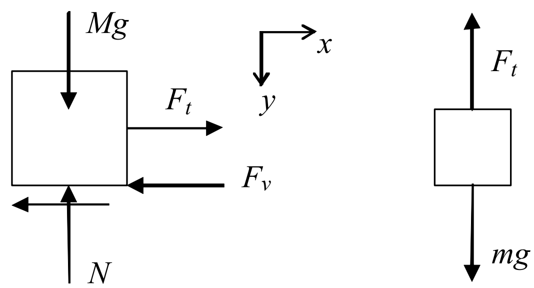
[Difficulty: 3]

2.50 A block of mass M slides on a thin film of oil. The film thickness is h and the area of the block is A. When released, mass m exerts tension on the cord, causing the block to accelerate. Neglect friction in the pulley and air resistance. Develop an algebraic expression for the viscous force that acts on the block when it moves at speed V. Derive a differential equation for the block speed as a function of time. Obtain an expression for the block speed as a function of time. The mass $M = 5$ kg, $m = 1$ kg, $A = 25\ \text{cm}^2$, and $h = 0.5$ mm. If it takes 1 s for the speed to reach 1 m/s, find the oil viscosity μ. Plot the curve for $V(t)$.



Given: Block on oil layer pulled by hanging weight

Find: Expression for viscous force at speed V; differential equation for motion; block speed as function of time; oil viscosity



Solution:

Governing equations: $\tau_{yx} = \mu\cdot\frac{du}{dy}$ $\qquad \Sigma F_x = M\cdot a_x$

Assumptions: Laminar flow; linear velocity profile in oil layer

The given data is $\quad M = 5\cdot kg \qquad W = m\cdot g = 9.81\cdot N \qquad A = 25\cdot cm^2 \qquad h = 0.05\cdot mm$

Equation of motion (block) $\quad \Sigma F_x = M\cdot a_x \quad$ so $\quad F_t - F_v = M\cdot\frac{dV}{dt} \quad$ (1)

Equation of motion (block) $\quad \Sigma F_y = m\cdot a_y \quad$ so $\quad m\cdot g - F_t = m\cdot\frac{dV}{dt} \quad$ (2)

Adding Eqs. (1) and (2) $\quad m\cdot g - F_v = (M + m)\cdot\frac{dV}{dt}$

The friction force is $\quad F_v = \tau_{yx}\cdot A = \mu\cdot\frac{du}{dy}\cdot A = \mu\cdot\frac{V}{h}\cdot A$

Hence $\quad m\cdot g - \frac{\mu\cdot A}{h}\cdot V = (M + m)\cdot\frac{dV}{dt}$

To solve separate variables $\quad dt = \frac{M + m}{m\cdot g - \frac{\mu\cdot A}{h}\cdot V}\cdot dV$

$$t = -\frac{(M + m)\cdot h}{\mu\cdot A}\cdot\left(\ln\left(m\cdot g - \frac{\mu\cdot A}{h}\cdot V\right) - \ln(m\cdot g)\right) = -\frac{(M + m)\cdot h}{\mu\cdot A}\cdot\ln\left(1 - \frac{\mu\cdot A}{m\cdot g\cdot h}\cdot V\right)$$

Hence taking antilogarithms $\quad 1 - \frac{\mu\cdot A}{m\cdot g\cdot h}\cdot V = e^{-\frac{\mu\cdot A}{(M+m)\cdot h}\cdot t}$

Finally

$$V = \frac{m \cdot g \cdot h}{\mu \cdot A} \cdot \left[1 - e^{-\frac{\mu \cdot A}{(M+m) \cdot h} \cdot t} \right]$$

The maximum velocity is $V = \dfrac{m \cdot g \cdot h}{\mu \cdot A}$

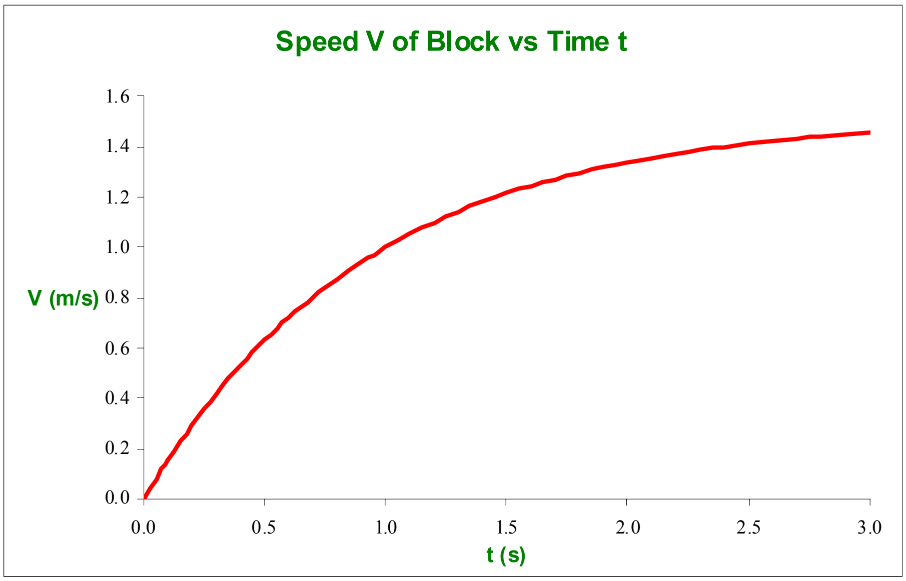
In Excel:

The data is

M =	5.00	kg
m =	1.00	kg
g =	9.81	m/s^2
0 =	1.30	$N.s/m^2$
A =	25	cm^2
h =	0.5	mm

To find the viscosity for which the speed is 1 m/s after 1 s use *Goal Seek* with the velocity targeted to be 1 m/s by varying the viscosity in the set of cell below:

t (s)	V (m/s)
1.00	1.000

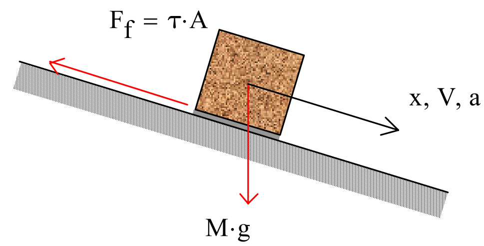


t (s)	V (m/s)
0.00	0.000
0.10	0.155
0.20	0.294
0.30	0.419
0.40	0.531
0.50	0.632
0.60	0.722
0.70	0.803
0.80	0.876
0.90	0.941
1.00	1.00
1.10	1.05
1.20	1.10
1.30	1.14
1.40	1.18
1.50	1.21
1.60	1.25
1.70	1.27
1.80	1.30
1.90	1.32
2.00	1.34
2.10	1.36
2.20	1.37
2.30	1.39
2.40	1.40
2.50	1.41
2.60	1.42
2.70	1.43
2.80	1.44
2.90	1.45
3.00	1.46

Problem 2.51

[Difficulty: 4]

2.51 A block 0.1 m square, with 5 kg mass, slides down a smooth incline, 30° below the horizontal, on a film of SAE 30 oil at 20°C that is 0.20 mm thick. If the block is released from rest at $t = 0$, what is its initial acceleration? Derive an expression for the speed of the block as a function of time. Plot the curve for $V(t)$. Find the speed after 0.1 s. If we want the mass to instead reach a speed of 0.3 m/s at this time, find the viscosity μ of the oil we would have to use.



Given:
Data on the block and incline

Find:
Initial acceleration; formula for speed of block; plot; find speed after 0.1 s. Find oil viscosity if speed is 0.3 m/s after 0.1 s

Solution:

Given data $\quad M = 5\cdot kg \quad A = (0.1\cdot m)^2 \quad d = 0.2\cdot mm \quad \theta = 30\cdot deg$

From Fig. A.2 $\quad \mu = 0.4\cdot\dfrac{N\cdot s}{m^2}$

Applying Newton's 2nd law to initial instant (no friction) $\quad M\cdot a = M\cdot g\cdot \sin(\theta) - F_f = M\cdot g\cdot \sin(\theta)$

so $\quad a_{init} = g\cdot\sin(\theta) = 9.81\cdot\dfrac{m}{s^2}\times\sin(30\cdot deg) \qquad a_{init} = 4.9\dfrac{m}{s^2}$

Applying Newton's 2nd law at any instant $\quad M\cdot a = M\cdot g\cdot\sin(\theta) - F_f \quad$ and $\quad F_f = \tau\cdot A = \mu\cdot\dfrac{du}{dy}\cdot A = \mu\cdot\dfrac{V}{d}\cdot A$

so $\quad M\cdot a = M\cdot\dfrac{dV}{dt} = M\cdot g\cdot\sin(\theta) - \dfrac{\mu\cdot A}{d}\cdot V$

Separating variables

$$\frac{dV}{g\cdot\sin(\theta) - \frac{\mu\cdot A}{M\cdot d}\cdot V} = dt$$

Integrating and using limits

$$-\frac{M\cdot d}{\mu\cdot A}\cdot\ln\left(1 - \frac{\mu\cdot A}{M\cdot g\cdot d\cdot\sin(\theta)}\cdot V\right) = t$$

or

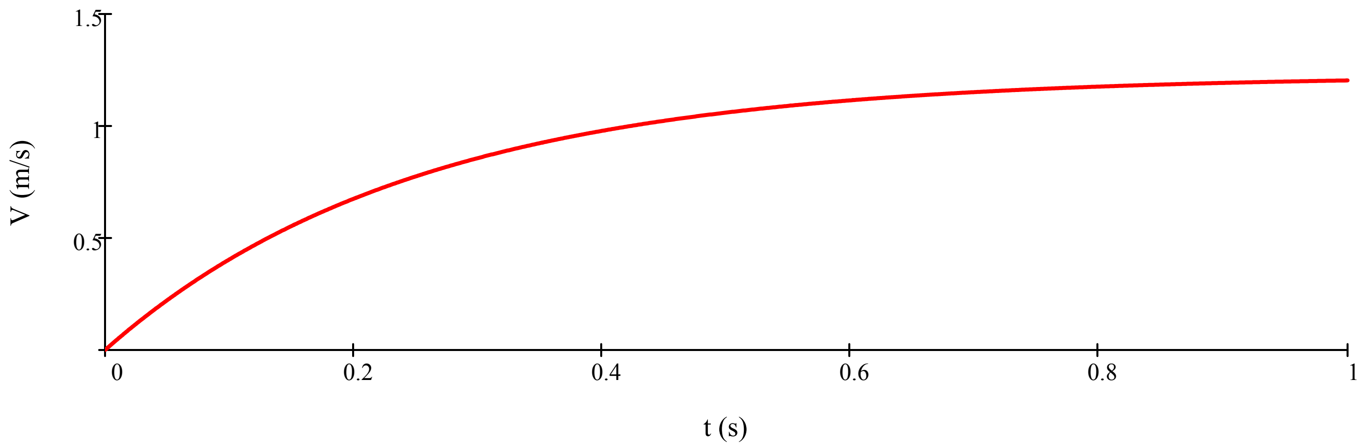
$$V(t) = \frac{M\cdot g\cdot d\cdot\sin(\theta)}{\mu\cdot A}\cdot\left(1 - e^{\frac{-\mu\cdot A}{M\cdot d}\cdot t}\right)$$

At t = 0.1 s

$$V = 5\cdot kg\times 9.81\cdot\frac{m}{s^2}\times 0.0002\cdot m\cdot\sin(30\cdot deg)\times\frac{m^2}{0.4\cdot N\cdot s\cdot(0.1\cdot m)^2}\times\frac{N\cdot s^2}{kg\cdot m}\times\left[1 - e^{-\left(\frac{0.4\cdot 0.01}{5\cdot 0.0002}\cdot 0.1\right)}\right]$$

$$V(0.1\cdot s) = 0.404\cdot\frac{m}{s}$$

The plot looks like



To find the viscosity for which V(0.1 s) = 0.3 m/s, we must solve

$$V(t = 0.1\cdot s) = \frac{M\cdot g\cdot d\cdot \sin(\theta)}{\mu\cdot A}\cdot\left[1 - e^{\frac{-\mu\cdot A}{M\cdot d}\cdot(t=0.1\cdot s)}\right]$$

The viscosity μ is implicit in this equation, so solution must be found by manual iteration, or by any of a number of classic root-finding numerical methods, or by using *Excel*'s *Goal Seek*

Using *Excel*: $\mu = 1.08\cdot\frac{N\cdot s}{m^2}$

Problem 2.52

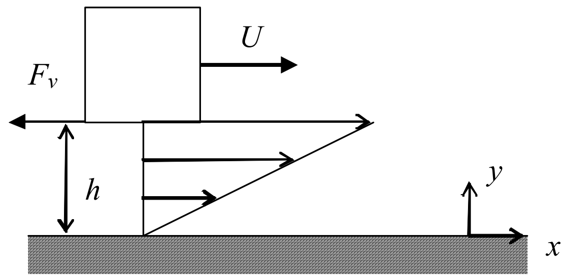
[Difficulty: 3]

2.52 A block that is *a* mm square slides across a flat plate on a thin film of oil. The oil has viscosity μ and the film is *h* mm thick. The block of mass *M* moves at steady speed *U* under the influence of constant force *F*. Indicate the magnitude and direction of the shear stresses on the bottom of the block and the plate. If the force is removed suddenly and the block begins to slow, sketch the resulting speed versus time curve for the block. Obtain an expression for the time required for the block to lose 95 percent of its initial speed.

Given: Block sliding on oil layer

Find: Direction of friction on bottom of block and on plate; expression for speed U versus time; time required to lose 95% of initial speed

Solution:



Governing equations: $\tau_{yx} = \mu\cdot\frac{du}{dy}$ $\qquad$ $\Sigma F_x = M\cdot a_x$

Assumptions: Laminar flow; linear velocity profile in oil layer

The bottom of the block is a -y surface, so τ_{yx} acts to the left; The plate is a +y surface, so τ_{yx} acts to the right

Equation of motion $\quad \Sigma F_x = M\cdot a_x \quad$ so $\quad F_v = M\cdot\frac{dU}{dt}$

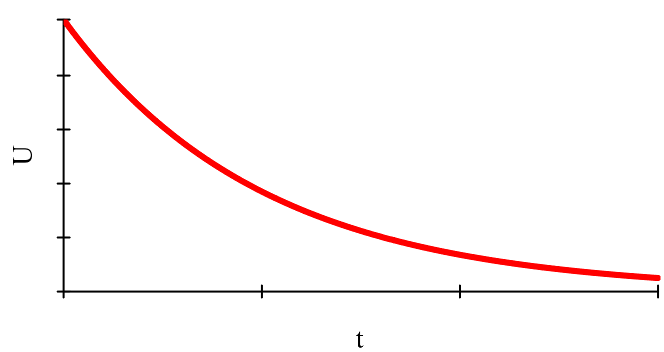
The friction force is $\quad F_v = \tau_{yx}\cdot A = \mu\cdot\frac{du}{dy}\cdot A = \mu\cdot\frac{U}{h}\cdot a^2$

Hence $\quad -\frac{\mu\cdot a^2}{h}\cdot U = M\cdot\frac{dU}{dt}$

To solve separate variables $\quad \frac{1}{U}\cdot dU = -\frac{\mu\cdot a^2}{M\cdot h}\cdot dt$

$$\ln\left(\frac{U}{U_0}\right) = -\frac{\mu\cdot a^2}{M\cdot h}\cdot t$$

Hence taking antilogarithms $\quad U = U_0\cdot e^{-\frac{\mu\cdot a^2}{M\cdot h}\cdot t}$



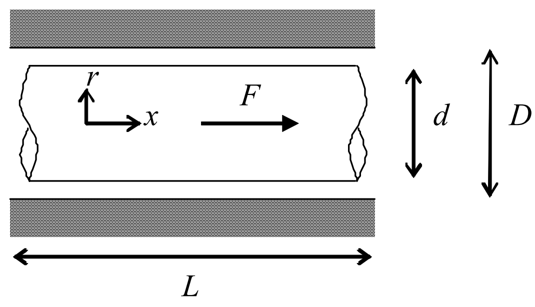
Solving for t $\quad t = -\frac{M\cdot h}{\mu\cdot a^2}\cdot\ln\left(\frac{U}{U_0}\right)$

Hence for $\frac{U}{U_0} = 0.05 \quad t = 3.0\cdot\frac{M\cdot h}{\mu\cdot a^2}$

Problem 2.53

[Difficulty: 2]

2.53 Magnet wire is to be coated with varnish for insulation by drawing it through a circular die of 1.0 mm diameter. The wire diameter is 0.9 mm and it is centered in the die. The varnish ($\mu = 20$ centipoise) completely fills the space between the wire and the die for a length of 50 mm. The wire is drawn through the die at a speed of 50 m/s. Determine the force required to pull the wire.



Given: Varnish-coated wire drawn through die

Find: Force required to pull wire

Solution:

Governing equations: $\tau_{yx} = \mu \cdot \frac{du}{dy}$ $\qquad \Sigma F_x = M \cdot a_x$

Assumptions: Laminar flow; linear velocity profile in varnish layer

The given data is $\quad D = 1 \cdot mm \quad d = 0.9 \cdot mm \quad L = 50 \cdot mm \quad V = 50 \cdot \frac{m}{s} \quad \mu = 20 \times 10^{-2} poise$

Equation of motion $\quad \Sigma F_x = M \cdot a_x \quad$ so $\quad F - F_V = 0 \quad$ for steady speed

The friction force is

$$F_V = \tau_{yx} \cdot A = \mu \cdot \frac{du}{dr} \cdot A = \mu \cdot \frac{V}{\left(\frac{D-d}{2}\right)} \cdot \pi \cdot d \cdot L$$

Hence $\quad F - F_V = 0$

so

$$F = \frac{2 \cdot \pi \cdot \mu \cdot V \cdot d \cdot L}{D - d}$$

$$F = 2 \cdot \pi \times 20 \times 10^{-2} poise \times \frac{0.1 \cdot kg}{m \cdot s \cdot poise} \times 50 \cdot \frac{m}{s} \times 0.9 \cdot mm \times 50 \cdot mm \times \frac{1}{(1 - 0.9) \cdot mm} \times \frac{m}{1000 \cdot mm}$$

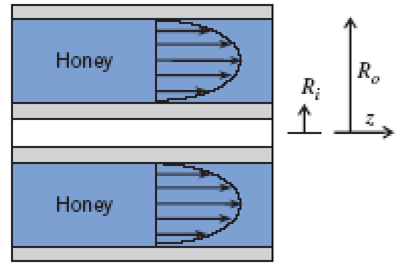
$$F = 2.83\,N$$

Problem 2.54 [Difficulty: 3]

2.54 In a food-processing plant, honey is pumped through an annular tube. The tube is $L = 2$ m long, with inner and outer radii of $R_i = 5$ mm and $R_o = 25$ mm, respectively. The applied pressure difference is $\Delta p = 125$ kPa, and the honey viscosity is $\mu = 5\ \text{N}\cdot\text{s/m}^2$. The theoretical velocity profile for laminar flow through an annulus is:

$$u_z(r) = \frac{1}{4\mu}\left(\frac{\Delta p}{L}\right)\left[R_i^2 - r^2 - \frac{R_o^2 - R_i^2}{\ln\left(\frac{R_i}{R_o}\right)}\cdot \ln\left(\frac{r}{R_i}\right)\right]$$

Show that the no-slip condition is satisfied by this expression. Find the location at which the shear stress is zero. Find the viscous forces acting on the inner and outer surfaces, and compare these to the force $\Delta p \pi(R_o^2 - R_i^2)$. Explain.



Given: Data on annular tube

Find: Whether no-slip is satisfied; location of zeroshear stress; viscous forces

Solution:

The velocity profile is

$$u_z(r) = \frac{1}{4\cdot\mu}\cdot\frac{\Delta p}{L}\cdot\left(R_i^2 - r^2 - \frac{R_o^2 - R_i^2}{\ln\left(\frac{R_i}{R_o}\right)}\cdot\ln\left(\frac{r}{Ri}\right)\right)$$

Check the no-slip condition. When $r = R_o$

$$u_z(R_o) = \frac{1}{4\cdot\mu}\cdot\frac{\Delta p}{L}\cdot\left(R_i^2 - R_o^2 - \frac{R_o^2 - R_i^2}{\ln\left(\frac{R_i}{R_o}\right)}\cdot\ln\left(\frac{R_o}{Ri}\right)\right)$$

$$u_z(R_o) = \frac{1}{4\cdot\mu}\cdot\frac{\Delta p}{L}\cdot\left[R_i^2 - R_o^2 + \left(R_o^2 - R_i^2\right)\right] = 0$$

When $r = R_i$

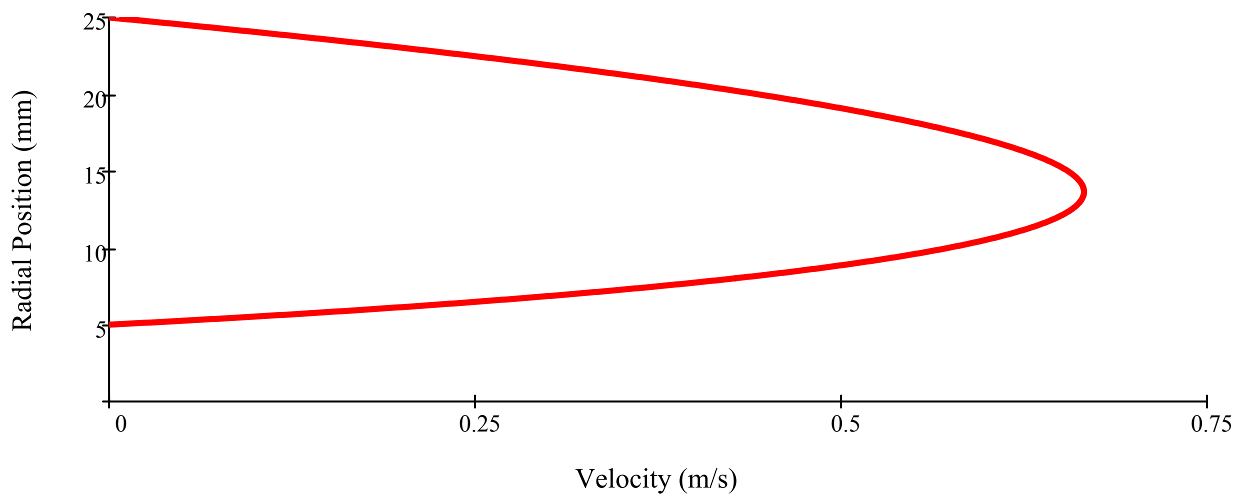
$$u_z(R_i) = \frac{1}{4\cdot\mu}\cdot\frac{\Delta p}{L}\cdot\left(R_i^2 - R_i^2 - \frac{R_o^2 - R_i^2}{\ln\left(\frac{R_i}{R_o}\right)}\cdot\ln\left(\frac{R_i}{R_i}\right)\right) = 0$$

The no-slip condition is satisfied.

The given data is $R_i = 5\cdot\text{mm}$ $R_o = 25\cdot\text{mm}$ $\Delta p = 125\cdot\text{kPa}$ $L = 2\cdot\text{m}$

The viscosity of the honey is $\mu = 5\cdot\frac{\text{N}\cdot\text{s}}{\text{m}^2}$

The plot looks like



For each, shear stress is given by $$\tau_{rx} = \mu \cdot \frac{du}{dr}$$

$$\tau_{rx} = \mu \cdot \frac{du_z(r)}{dr} = \mu \cdot \frac{d}{dr}\left[\frac{1}{4 \cdot \mu} \cdot \frac{\Delta p}{L} \cdot \left(R_i^{\ 2} - r^2 - \frac{R_o^{\ 2} - R_i^{\ 2}}{\ln\left(\frac{R_i}{R_o}\right)} \cdot \ln\left(\frac{r}{Ri}\right)\right)\right]$$

Hence $$\tau_{rx} = \frac{1}{4} \cdot \frac{\Delta p}{L} \cdot \left(-2 \cdot r - \frac{R_o^{\ 2} - R_i^{\ 2}}{\ln\left(\frac{R_i}{R_o}\right) \cdot r}\right)$$

For zero stress $$-2 \cdot r - \frac{R_o^{\ 2} - R_i^{\ 2}}{\ln\left(\frac{R_i}{R_o}\right) \cdot r} = 0$$ or $$r = \sqrt{\frac{R_i^{\ 2} - R_o^{\ 2}}{2 \cdot \ln\left(\frac{R_i}{R_o}\right)}}$$ $$r = 13.7 \cdot mm$$

On the outer surface $$F_o = \tau_{rx} \cdot A = \frac{1}{4} \cdot \frac{\Delta p}{L} \cdot \left(-2 \cdot R_o - \frac{R_o^{\ 2} - R_i^{\ 2}}{\ln\left(\frac{R_i}{R_o}\right) \cdot R_o}\right) \cdot 2 \cdot \pi \cdot R_o \cdot L$$

$$F_o = \Delta p \cdot \pi \cdot \left(-R_o^{\ 2} - \frac{R_o^{\ 2} - R_i^{\ 2}}{2 \cdot \ln\left(\frac{R_i}{R_o}\right)}\right)$$

$$F_o = 125 \times 10^3 \cdot \frac{N}{m^2} \times \pi \times \left[-\left(25 \cdot mm \times \frac{1 \cdot m}{1000 \cdot mm}\right)^2 - \frac{\left[(25 \cdot mm)^2 - (5 \cdot mm)^2\right] \times \left(\frac{1 \cdot m}{1000 \cdot mm}\right)^2}{2 \cdot \ln\left(\frac{5}{25}\right)} \right]$$

$$F_o = -172\,N$$

On the inner surface

$$F_i = \tau_{rx} \cdot A = \frac{1}{4} \cdot \frac{\Delta p}{L} \cdot \left(-2 \cdot R_i - \frac{R_o^2 - R_i^2}{\ln\left(\frac{R_i}{R_o}\right) \cdot R_i} \right) \cdot 2 \cdot \pi \cdot R_i \cdot L$$

$$F_i = \Delta p \cdot \pi \cdot \left(-R_i^2 - \frac{R_o^2 - R_i^2}{2 \cdot \ln\left(\frac{R_i}{R_o}\right)} \right)$$

Hence

$$F_i = 125 \times 10^3 \cdot \frac{N}{m^2} \times \pi \times \left[-\left(5 \cdot mm \times \frac{1 \cdot m}{1000 \cdot mm}\right)^2 - \frac{\left[(25 \cdot mm)^2 - (5 \cdot mm)^2\right] \times \left(\frac{1 \cdot m}{1000 \cdot mm}\right)^2}{2 \cdot \ln\left(\frac{5}{25}\right)} \right]$$

$$F_i = 63.4\,N$$

Note that $F_o - F_i = -236\,N$ and $\Delta p \cdot \pi \cdot \left(R_o^2 - R_i^2\right) = 236\,N$

The net pressure force just balances the net viscous force!

Problem 2.55

[Difficulty: 3]

2.55 SAE 10W-30 oil at 100°C is pumped through a tube L = 10 m long, diameter D = 20 mm. The applied pressure difference is Δp = 5 kPa. On the centerline of the tube is a metal filament of diameter d = 1 μm. The theoretical velocity profile for laminar flow through the tube is:

$$V(r) = \frac{1}{16\mu}\left(\frac{\Delta p}{L}\right)\left[d^2 - 4r^2 - \frac{D^2 - d^2}{\ln\left(\frac{d}{D}\right)}\cdot\ln\left(\frac{2r}{d}\right)\right]$$

Show that the no-slip condition is satisfied by this expression. Find the location at which the shear stress is zero, and the stress on the tube and on the filament. Plot the velocity distribution and the stress distribution. (For the stress curve, set an upper limit on stress of 5 Pa.) Discuss the results.

Given: Data on flow through a tube with a filament

Find: Whether no-slip is satisfied; location of zero stress;stress on tube and filament

Solution:

The velocity profile is
$$V(r) = \frac{1}{16\cdot\mu}\cdot\frac{\Delta p}{L}\cdot\left(d^2 - 4\cdot r^2 - \frac{D^2 - d^2}{\ln\left(\frac{d}{D}\right)}\cdot\ln\left(\frac{2\cdot r}{d}\right)\right)$$

Check the no-slip condition.
When $r = \frac{D}{2}$
$$V\left(\frac{D}{2}\right) = \frac{1}{16\cdot\mu}\cdot\frac{\Delta p}{L}\cdot\left(d^2 - D^2 - \frac{D^2 - d^2}{\ln\left(\frac{d}{D}\right)}\cdot\ln\left(\frac{D}{d}\right)\right)$$

$$V(D) = \frac{1}{16\cdot\mu}\cdot\frac{\Delta p}{L}\cdot\left[d^2 - D^2 + \left(D^2 - d^2\right)\right] = 0$$

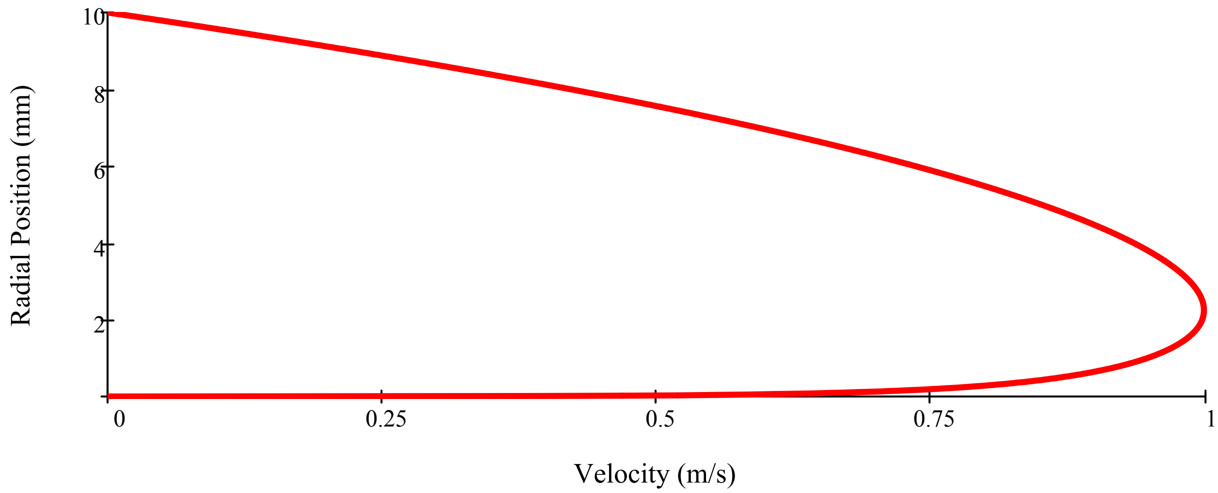
When $r = \frac{d}{2}$
$$V(d) = \frac{1}{16\cdot\mu}\cdot\frac{\Delta p}{L}\cdot\left(d^2 - d^2 - \frac{D^2 - d^2}{\ln\left(\frac{d}{D}\right)}\cdot\ln\left(\frac{d}{d}\right)\right) = 0$$

The no-slip condition is satisfied.

The given data is $d = 1\cdot\mu m$ $\quad D = 20\cdot mm$ $\quad \Delta p = 5\cdot kPa$ $\quad L = 10\cdot m$

The viscosity of SAE 10-30 oil at 100°C is (Fig. A.2) $\mu = 1\times 10^{-2}\cdot\frac{N\cdot s}{m^2}$

The plot looks like

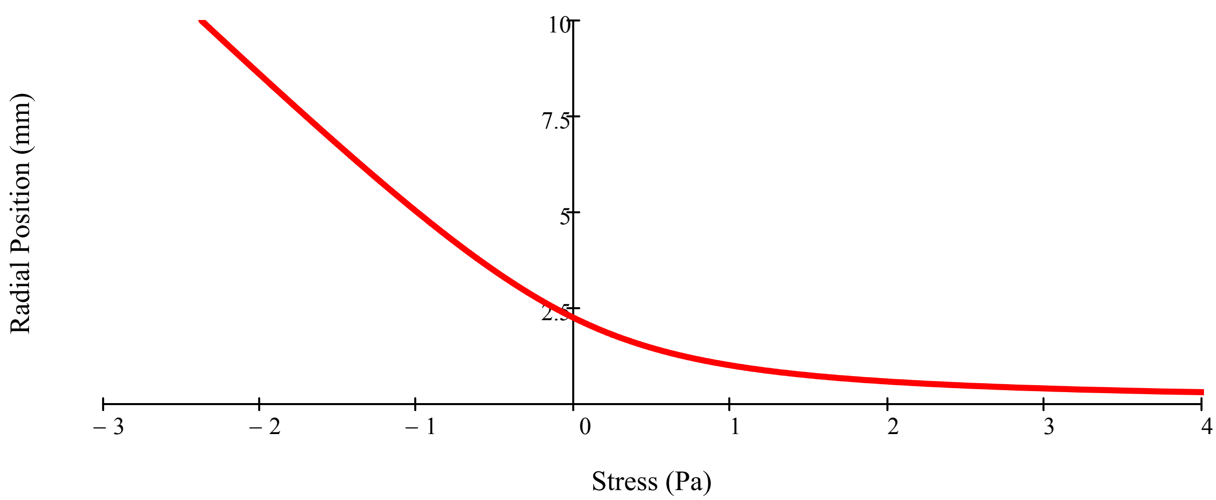


For each, shear stress is given by $\tau_{rx} = \mu \cdot \frac{du}{dr}$

$$\tau_{rx} = \mu \cdot \frac{dV(r)}{dr} = \mu \cdot \frac{d}{dr}\left[\frac{1}{16 \cdot \mu} \cdot \frac{\Delta p}{L} \cdot \left(d^2 - 4 \cdot r^2 - \frac{D^2 - d^2}{\ln\left(\frac{d}{D}\right)} \cdot \ln\left(\frac{2 \cdot r}{D_i}\right)\right)\right]$$

$$\tau_{rx}(r) = \frac{1}{16} \cdot \frac{\Delta p}{L} \cdot \left(-8 \cdot r - \frac{D^2 - d^2}{\ln\left(\frac{d}{D}\right) \cdot r}\right)$$

For the zero-stress point $-8 \cdot r - \frac{D^2 - d^2}{\ln\left(\frac{d}{D}\right) \cdot r} = 0$ or $r = \sqrt{\frac{d^2 - D^2}{8 \cdot \ln\left(\frac{d}{D}\right)}}$ $r = 2.25 \cdot mm$

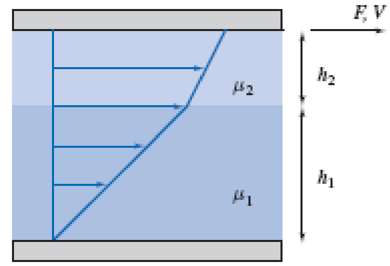


Using the stress formula $\tau_{rx}\left(\frac{D}{2}\right) = -2.374\ Pa$ $\tau_{rx}\left(\frac{d}{2}\right) = 2.524 \cdot kPa$

Problem 2.56

[Difficulty: 2]

2.56 Fluids of viscosities $\mu_1 = 0.1\ \mathrm{N \cdot s/m^2}$ and $\mu_2 = 0.15\ \mathrm{N \cdot s/m^2}$ are contained between two plates (each plate is 1 m² in area). The thicknesses are $h_1 = 0.5$ mm and $h_2 = 0.3$ mm, respectively. Find the force F to make the upper plate move at a speed of 1 m/s. What is the fluid velocity at the interface between the two fluids?



Given: Flow between two plates

Find: Force to move upper plate; Interface velocity

Solution:

The shear stress is the same throughout (the velocity gradients are linear, and the stresses in the fluid at the interface must be equal and opposite).

Hence $\quad \tau = \mu_1 \cdot \dfrac{du_1}{dy} = \mu_2 \cdot \dfrac{du_2}{dy} \quad$ or $\quad \mu_1 \cdot \dfrac{V_i}{h_1} = \mu_2 \cdot \dfrac{(V - V_i)}{h_2} \quad$ where V_i is the interface velocity

Solving for the interface velocity V_i

$$V_i = \frac{V}{1 + \frac{\mu_1}{\mu_2} \cdot \frac{h_2}{h_1}} = \frac{1 \cdot \frac{m}{s}}{1 + \frac{0.1}{0.15} \cdot \frac{0.3}{0.5}} \qquad V_i = 0.714 \frac{m}{s}$$

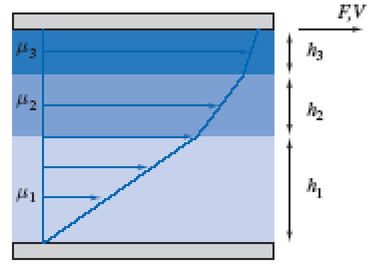
Then the force required is

$$F = \tau \cdot A = \mu_1 \cdot \frac{V_i}{h_1} \cdot A = 0.1 \cdot \frac{N \cdot s}{m^2} \times 0.714 \cdot \frac{m}{s} \times \frac{1}{0.5 \cdot mm} \times \frac{1000 \cdot mm}{1 \cdot m} \times 1 \cdot m^2 \qquad F = 143\ N$$

Problem 2.57

[Difficulty: 2]

2.57 Fluids of viscosities $\mu_1 = 0.15\ \mathrm{N \cdot s/m^2}$, $\mu_2 = 0.5\ \mathrm{N \cdot s/m^2}$, and $\mu_3 = 0.2\ \mathrm{N \cdot s/m^2}$ are contained between two plates (each plate is $1\ \mathrm{m^2}$ in area). The thicknesses are $h_1 = 0.5$ mm, $h_2 = 0.25$ mm, and $h_3 = 0.2$ mm, respectively. Find the steady speed V of the upper plate and the velocities at the two interfaces due to a force $F = 100$ N. Plot the velocity distribution.



Given:

Flow of three fluids between two plates

Find:

Upper plate velocity; Interface velocities; plot velocity distribution

Solution:

The shear stress is the same throughout (the velocity gradients are linear, and the stresses in the fluids at the interfaces must be equal and opposite).

Given data $F = 100 \cdot N$ $\quad h_1 = 0.5 \cdot mm$ $\quad h_2 = 0.25 \cdot mm$ $\quad h_3 = 0.2 \cdot mm$

$A = 1 \cdot m^2$ $\quad \mu_1 = 0.15 \cdot \frac{N \cdot s}{m^2}$ $\quad \mu_2 = 0.5 \cdot \frac{N \cdot s}{m^2}$ $\quad \mu_3 = 0.2 \cdot \frac{N \cdot s}{m^2}$

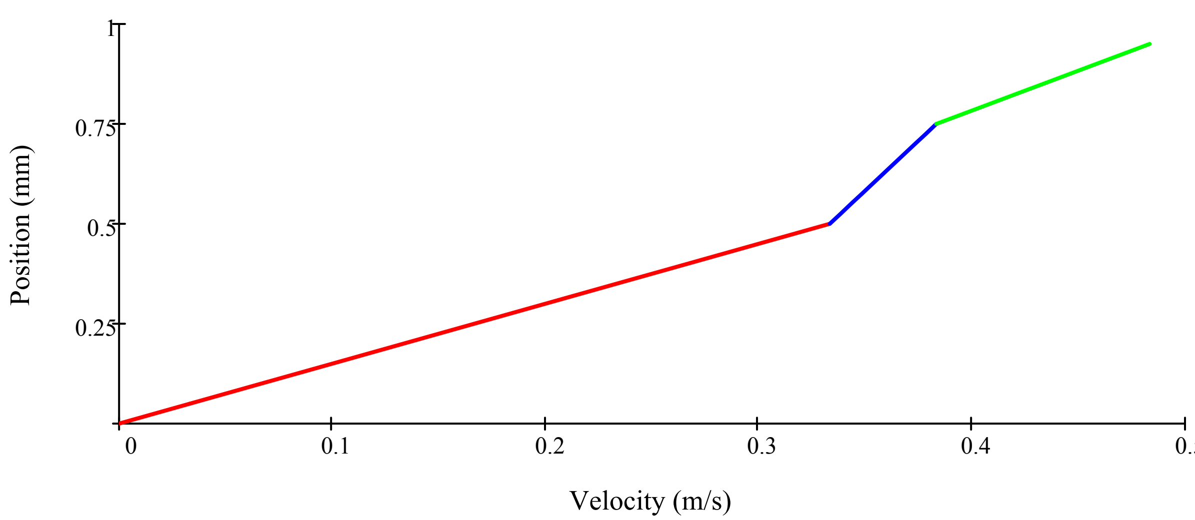
The (constant) stress is $\tau = \frac{F}{A}$ $\quad \tau = 100\ \mathrm{Pa}$

For each fluid $\tau = \mu \cdot \frac{\Delta V}{\Delta y}$ or $\Delta V = \frac{\tau \cdot \Delta y}{\mu}$ where ΔV is the overall change in velocity over distance Δy

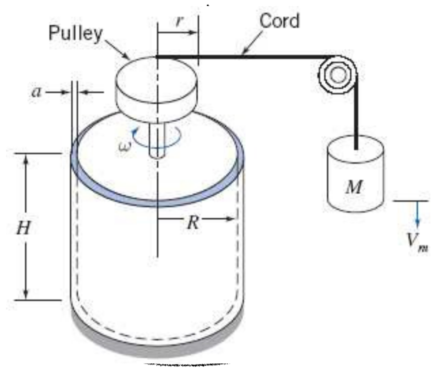
Hence $V_{12} = \frac{\tau \cdot h_1}{\mu_1}$ $\quad V_{12} = 0.333 \frac{m}{s}$ where V_{12} is the velocity at the 1 - 2 interface

Hence $V_{23} = \frac{\tau \cdot h_2}{\mu_2} + V_{12}$ $\quad V_{23} = 0.383 \frac{m}{s}$ where V_{23} is the velocity at the 2 - 3 interface

Hence $V = \frac{\tau \cdot h_3}{\mu_3} + V_{23}$ $\quad V = 0.483 \frac{m}{s}$ where V is the velocity at the upper plate



2.58 A concentric cylinder viscometer may be formed by rotating the inner member of a pair of closely fitting cylinders. The annular gap is small so that a linear velocity profile will exist in the liquid sample. Consider a viscometer with an inner cylinder of 4 in. diameter and 8 in. height, and a clearance gap width of 0.001 in., filled with castor oil at 90°F. Determine the torque required to turn the inner cylinder at 400 rpm.



Solution:

The required torque must balance the resisting torque of the shear force.

The shear force is given by $F = \tau A$ where $A = 2\pi R h$

For a Newtonian fluid $\tau = \mu \frac{du}{dy}$

For small gap (linear profile) $\tau = \mu \frac{V}{d}$

where V = tangential velocity of inner cylinder $= R\omega$

Hence $F = \tau A = \mu \frac{R\omega}{d} 2\pi R h = \frac{2\pi \mu R^2 \omega h}{d}$

and the torque $T = RF = \frac{2\pi\mu R^3 \omega h}{d}$

From Fig A.2, for castor oil at 90°F (32°C), $\mu = 3.80 \times 10^{-1}\ N.s/m^2$

Substituting numerical values.

$$T = \frac{2\pi\mu R^3 \omega h}{d} = 2\pi \times 3.80\times10^{-1} \frac{N.s}{m^2} \times 2.09\times10^{-2} \frac{lbf.s.m^2}{ft^2.N.s} \times (2.0)^3 in^3 \times 400 \frac{rev}{min} \times 8 in \times \frac{1}{10^{-3} in}$$

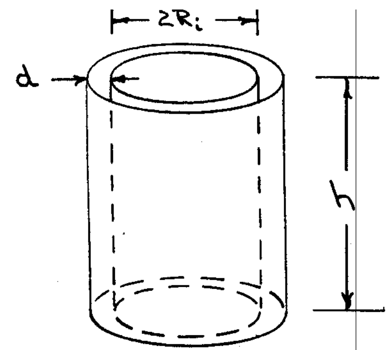
$$\times\ 2\pi \frac{rad}{rev} \times \frac{min}{60\,s} \times \frac{ft^3}{1728\,in^3}$$

$T = 77.4\ ft\cdot lbf$ ← Torque

Problem 2.59

[Difficulty: 2]

2.59 A concentric cylinder viscometer may be formed by rotating the inner member of a pair of closely fitting cylinders. For small clearances, a linear velocity profile may be assumed in the liquid filling the annular clearance gap. A viscometer has an inner cylinder of 75 mm diameter and 150 mm height, with a clearance gap width of 0.02 mm. A torque of 0.021 N·m is required to turn the inner cylinder at 100 rpm. Determine the viscosity of the liquid in the clearance gap of the viscometer.



Solution

The imposed torque must balance the resisting torque of the shear force.

The shear force is given by $F = \tau A$ where $A = 2\pi R_i h$

For a Newtonian fluid $\tau = \mu \dfrac{du}{dy}$

Since the velocity profile is assumed to be linear, $\tau = \mu \dfrac{V}{d}$

where V is the tangential velocity of the inner cylinder, $V = R_i \omega$

Thus,

$$F = \tau A = \mu \frac{V}{d} 2\pi R_i h = \frac{2\pi \mu R_i^2 \omega h}{d}$$

and the torque $T = R_i F = \dfrac{2\pi \mu R_i^3 \omega h}{d}$

Solving for μ,

$$\mu = \frac{T d}{2\pi R_i^3 \omega h} = 0.021\ \text{N}\cdot\text{m} \times 0.02\ \text{mm} \times \frac{1}{2\pi} \times \frac{1}{(37.5)^3\ \text{mm}^3} \times \frac{\text{min}}{100\ \text{rev}} \times \frac{1}{150\ \text{mm}}$$

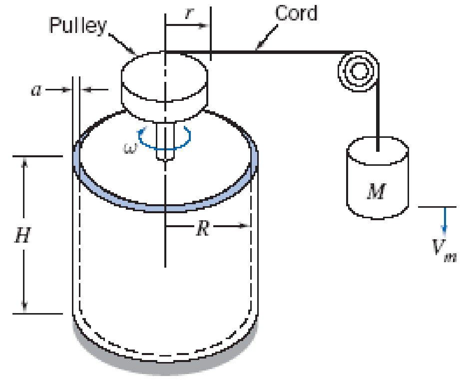
$$\times \frac{\text{rev}}{2\pi\ \text{rad}} \times \frac{60\ \text{s}}{\text{min}} \times (1000)^3 \frac{\text{mm}^3}{\text{m}^3}$$

$$\mu = 8.07 \times 10^{-4}\ \text{N}\cdot\text{s/m}^2 \longleftarrow \mu$$

Problem 2.60

[Difficulty: 2]

2.60 A concentric cylinder viscometer is driven by a falling mass M connected by a cord and pulley to the inner cylinder, as shown. The liquid to be tested fills the annular gap of width a and height H. After a brief starting transient, the mass falls at constant speed V_m. Develop an algebraic expression for the viscosity of the liquid in the device in terms of M, g, V_m, r, R, a, and H. Evaluate the viscosity of the liquid using:



$M = 0.10$ kg	$r = 25$ mm
$R = 50$ mm	$a = 0.20$ mm
$H = 80$ mm	$V_m = 30$ mm/s

Solution: Apply Newton's law of viscosity.

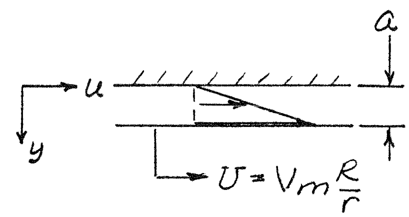
Basic equations: $\tau = \mu \dfrac{du}{dy}$ $\quad \Sigma M = 0 \quad T = \tau A R$

Assumptions: (1) Newtonian liquid
(2) Narrow gap, so linear velocity profile
(3) Steady angular speed

Summing torques on the rotor

$$\Sigma M = Mgr - \tau A R = I\overset{=0(3)}{\cancel{\alpha}} = 0 \; ; \; A = 2\pi R H$$

Because $a \ll R$, treat the gap as plane. Then



$$\tau = \mu \frac{du}{dy} \approx \mu \frac{\Delta u}{\Delta y} \approx \mu \frac{U-0}{a-0} \approx \mu \frac{U}{a} = \frac{\mu V_m R}{a r}$$

Substituting,

$$Mgr - \frac{\mu V_m R}{ar} 2\pi R H \, R = Mgr - \frac{2\pi \mu V_m R^3 H}{ar} = 0$$

so

$$\mu = \frac{M g r^2 a}{2\pi V_m R^3 H}$$ μ

Evaluating for the given data

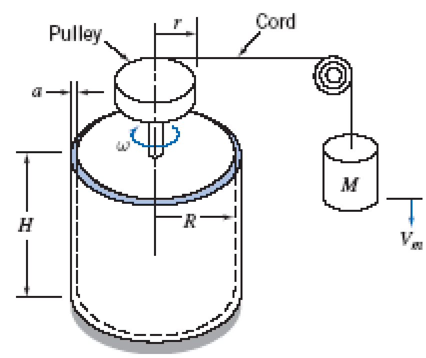
$$\mu = \frac{1}{2\pi} \times 0.10\,kg \times 9.81\,\frac{m}{s^2} \times (0.025)^2 m^2 \times 0.0002\,m \times \frac{s}{0.030\,m} \times \frac{1}{(0.050)^3 m^3} \times \frac{1}{0.080\,m} \times \frac{N \cdot s^2}{kg \cdot m}$$

$$\mu = 0.0651 \; N \cdot s/m^2 \quad (65.1 \; mPa \cdot s)$$ μ

Problem 2.61

[Difficulty: 3]

2.61 The viscometer of Problem 2.60 is being used to verify that the viscosity of a particular fluid is $\mu = 0.1\ \mathrm{N \cdot s/m^2}$. Unfortunately the cord snaps during the experiment. How long will it take the cylinder to lose 99% of its speed? The moment of inertia of the cylinder/pulley system is $0.0273\ \mathrm{kg \cdot m^2}$.



Given: Data on the viscometer

Find: Time for viscometer to lose 99% of speed

Solution:

The given data is $R = 50 \cdot mm$ $\quad H = 80 \cdot mm$ $\quad a = 0.20 \cdot mm$ $\quad I = 0.0273 \cdot kg \cdot m^2$ $\quad \mu = 0.1 \cdot \frac{N \cdot s}{m^2}$

The equation of motion for the slowing viscometer is $\quad I \cdot \alpha = Torque = -\tau \cdot A \cdot R$

where α is the angular acceleration and τ is the viscous stress, and A is the surface area of the viscometer

The stress is given by
$$\tau = \mu \cdot \frac{du}{dy} = \mu \cdot \frac{V - 0}{a} = \frac{\mu \cdot V}{a} = \frac{\mu \cdot R \cdot \omega}{a}$$

where V and ω are the instantaneous linear and angular velocities.

Hence
$$I \cdot \alpha = I \cdot \frac{d\omega}{dt} = -\frac{\mu \cdot R \cdot \omega}{a} \cdot A \cdot R = \frac{\mu \cdot R^2 \cdot A}{a} \cdot \omega$$

Separating variables
$$\frac{d\omega}{\omega} = -\frac{\mu \cdot R^2 \cdot A}{a \cdot I} \cdot dt$$

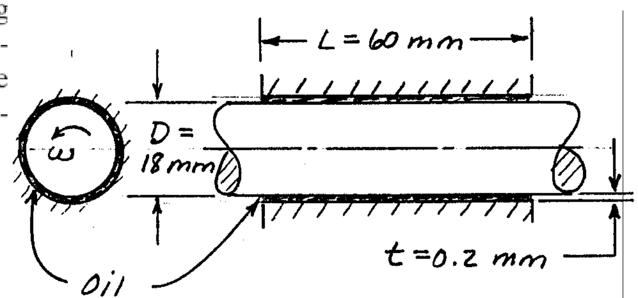
Integrating and using IC $\omega = \omega_0$
$$\omega(t) = \omega_0 \cdot e^{-\frac{\mu \cdot R^2 \cdot A}{a \cdot I} \cdot t}$$

The time to slow down by 99% is obtained from solving
$$0.01 \cdot \omega_0 = \omega_0 \cdot e^{-\frac{\mu \cdot R^2 \cdot A}{a \cdot I} \cdot t} \quad \text{so} \quad t = -\frac{a \cdot I}{\mu \cdot R^2 \cdot A} \cdot \ln(0.01)$$

Note that
$$A = 2 \cdot \pi \cdot R \cdot H \quad \text{so} \quad t = -\frac{a \cdot I}{2 \cdot \pi \cdot \mu \cdot R^3 \cdot H} \cdot \ln(0.01)$$

$$t = -\frac{0.0002 \cdot m \cdot 0.0273 \cdot kg \cdot m^2}{2 \cdot \pi} \cdot \frac{m^2}{0.1 \cdot N \cdot s} \cdot \frac{1}{(0.05 \cdot m)^3} \cdot \frac{1}{0.08 \cdot m} \cdot \frac{N \cdot s^2}{kg \cdot m} \cdot \ln(0.01) \qquad t = 4.00\,s$$

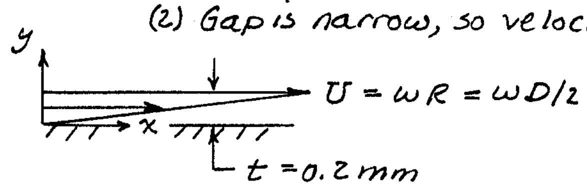
2.62 A shaft with outside diameter of 18 mm turns at 20 revolutions per second inside a stationary journal bearing 60 mm long. A thin film of oil 0.2 mm thick fills the concentric annulus between the shaft and journal. The torque needed to turn the shaft is 0.0036 N·m. Estimate the viscosity of the oil that fills the gap.



Solution: Basic equation $\tau_{yx} = \mu \frac{du}{dy}$

Assumptions: (1) Newtonian fluid
(2) Gap is narrow, so velocity profile is linear, $\frac{du}{dy} \approx \frac{\Delta u}{\Delta y}$

Then



Shear stress is

$$\tau_{yx} \approx \mu \frac{\Delta u}{\Delta y} = \mu \frac{U}{t} = \frac{\mu \omega D}{2t}$$

Neglecting end effects, torque is

$$T = FR = \tau_{yx} A R = \tau_{yx} (\pi D L) \frac{D}{2} = \frac{\mu \pi \omega D^3 L}{4t}$$

Solving for viscosity

$$\mu = \frac{4tT}{\pi \omega D^3 L}$$

$$= \frac{4}{\pi} \times 0.2\,mm \times 0.0036\,N{\cdot}m \times \frac{s}{20\,rev} \times \frac{1}{(18)^3\,mm^3} \times \frac{1}{60\,mm} \times \frac{rev}{2\pi\,rad} \times (1000)^3 \frac{mm^3}{m^3}$$

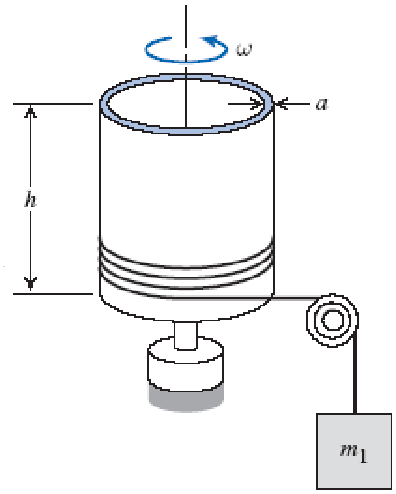
$\mu = 0.0208\ N{\cdot}s / m^2$ ← μ

{ From Fig. A.2, this oil appears somewhat less viscous than SAE 10W, assuming the oil is at room temperature. }

Problem 2.63

[Difficulty: 4]

2.63 The thin outer cylinder (mass m_2 and radius R) of a small portable concentric cylinder viscometer is driven by a falling mass, m_1, attached to a cord. The inner cylinder is stationary. The clearance between the cylinders is a. Neglect bearing friction, air resistance, and the mass of liquid in the viscometer. Obtain an algebraic expression for the torque due to viscous shear that acts on the cylinder at angular speed ω. Derive and solve a differential equation for the angular speed of the outer cylinder as a function of time. Obtain an expression for the maximum angular speed of the cylinder.



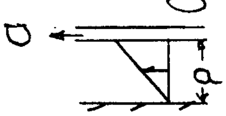
Solution:

Basic equations: $\tau = \mu \dfrac{du}{dy}$

$\Sigma F = ma$, $\Sigma M = I\alpha$

Assume: (1) Newtonian fluid
(2) linear velocity profile

In the gap,

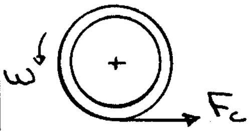


$$\tau = \mu \frac{du}{dy} = \mu \frac{U}{a} = \frac{\mu R \omega}{a}$$

$$T = \tau A R = \frac{\mu R \omega}{a}(2\pi R h) R$$

$$T = \frac{2\pi R^3 \mu h}{a}\,\omega \qquad T$$

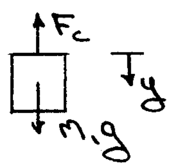
During acceleration, let the tension in the cord be F_c



For the cylinder $\Sigma M = F_c R - T = I\alpha = m_2 R^2 \dfrac{d\omega}{dt}$ ... (1)

For the mass $\Sigma F_y = m_1 g - F_c = m a = m_1 \dfrac{dV}{dt} = m_1 R \dfrac{d\omega}{dt}$ ... (2)

$\therefore \; F_c = m_1 g - m_1 R \dfrac{d\omega}{dt}$



Substituting into eq. (1)

$$m_1 g R - \frac{2\pi R^3 \mu h}{a}\,\omega = (m_1 + m_2) R^2 \frac{d\omega}{dt}$$

Let $m_1 g R = b$, $-2\pi R^3 \mu h / a = c$, $(m_1+m_2)R^2 = f$

Then, $b + c\omega = f \dfrac{d\omega}{dt}$ or $\displaystyle\int_0^t \frac{1}{f}\,dt = \int_0^\omega \frac{d\omega}{(b+c\omega)}$

Integrating, $\dfrac{1}{f} t = \dfrac{1}{c} \ln(b + c\omega)\Big]_0^\omega = \dfrac{1}{c} \ln \dfrac{(b+c\omega)}{b} = \dfrac{1}{c} \ln\left(1 + \dfrac{c}{b}\omega\right)$.

$$\frac{c}{f} t = \ln\left(1 + \frac{c}{b}\omega\right) \Rightarrow e^{\frac{c}{f}t} = \left(1 + \frac{c}{b}\omega\right) \Rightarrow \omega = \frac{b}{c}\left(e^{\frac{c}{f}t} - 1\right)$$

Substituting for b, c, and f

$$\omega = \frac{m_1 g \cancel{R} a}{2\pi R^{\cancel{3}} \mu h}\left(1 - e^{-\frac{2\pi R^3 \mu h}{a(m_1+m_2)R^2} t}\right) = \frac{m g a}{2\pi R^2 \mu h}\left[1 - e^{-\frac{2\pi R \mu h}{a(m_1+m_2)} t}\right] \qquad \omega(t)$$

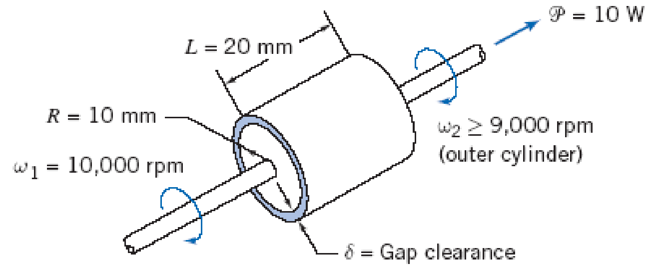
Maximum ω occurs at $t \to \infty$

$$\omega_{max} = \frac{m g a}{2\pi R^2 \mu h} \qquad \omega_{max}$$

Problem 2.64

[Difficulty: 3]

2.64 A shock-free coupling for a low-power mechanical drive is to be made from a pair of concentric cylinders. The annular space between the cylinders is to be filled with oil. The drive must transmit power, $\mathcal{P} = 10$ W. Other dimensions and properties are as shown. Neglect any bearing friction and end effects. Assume the minimum practical gap clearance δ for the device is $\delta = 0.25$ mm. Dow manufactures silicone fluids with viscosities as high as 10^6 centipoise. Determine the viscosity that should be specified to satisfy the requirement for this device.



Given: Shock-free coupling assembly

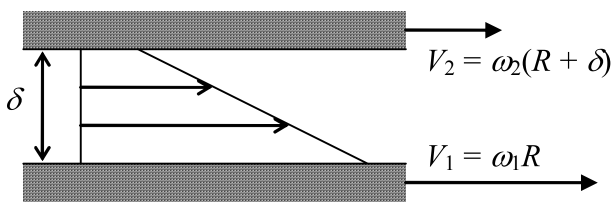
Find: Required viscosity

Solution:

Basic equation $\tau_{r\theta} = \mu \cdot \dfrac{du}{dr}$ Shear force $F = \tau \cdot A$ Torque $T = F \cdot R$ Power $P = T \cdot \omega$

Assumptions: Newtonian fluid, linear velocity profile

$$\tau_{r\theta} = \mu \cdot \frac{du}{dr} = \mu \cdot \frac{\Delta V}{\Delta r} = \mu \cdot \frac{\left[\omega_1 \cdot R - \omega_2 \cdot (R + \delta)\right]}{\delta}$$



$$\tau_{r\theta} = \mu \cdot \frac{(\omega_1 - \omega_2) \cdot R}{\delta}$$ Because $\delta << R$

Then $$P = T \cdot \omega_2 = F \cdot R \cdot \omega_2 = \tau \cdot A_2 \cdot R \cdot \omega_2 = \frac{\mu \cdot (\omega_1 - \omega_2) \cdot R}{\delta} \cdot 2 \cdot \pi \cdot R \cdot L \cdot R \cdot \omega_2$$

$$P = \frac{2 \cdot \pi \cdot \mu \cdot \omega_2 \cdot (\omega_1 - \omega_2) \cdot R^3 \cdot L}{\delta}$$

Hence $$\mu = \frac{P \cdot \delta}{2 \cdot \pi \cdot \omega_2 \cdot (\omega_1 - \omega_2) \cdot R^3 \cdot L}$$

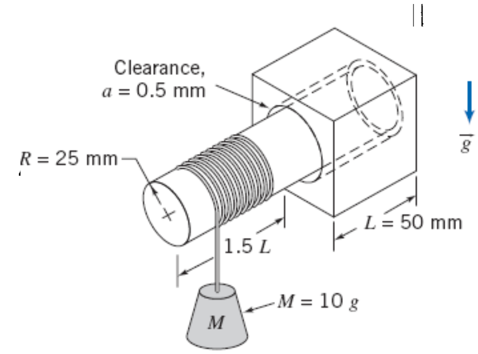
$$\mu = \frac{10 \cdot W \times 2.5 \times 10^{-4} \cdot m}{2 \cdot \pi} \times \frac{1}{9000} \cdot \frac{min}{rev} \times \frac{1}{1000} \cdot \frac{min}{rev} \times \frac{1}{(.01 \cdot m)^3} \times \frac{1}{0.02 \cdot m} \times \frac{N \cdot m}{s \cdot W} \times \left(\frac{rev}{2 \cdot \pi \cdot rad}\right)^2 \times \left(\frac{60 \cdot s}{min}\right)^2$$

$\mu = 0.202 \cdot \dfrac{N \cdot s}{m^2}$ $\mu = 2.02 \cdot poise$ which corresponds to SAE 30 oil at 30°C.

Problem 2.65

[Difficulty: 4] Part 1/2

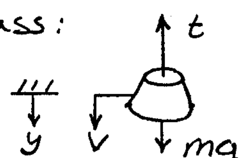
2.65 A circular aluminum shaft mounted in a journal is shown. The symmetric clearance gap between the shaft and journal is filled with SAE 10W-30 oil at $T = 30°C$. The shaft is caused to turn by the attached mass and cord. Develop and solve a differential equation for the angular speed of the shaft as a function of time. Calculate the maximum angular speed of the shaft and the time required to reach 95 percent of this speed.



Solution: Apply summation of torques and Newton's second law.

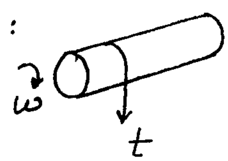
Basic equations: $\sum T = I\frac{d\omega}{dt}$ $\quad \sum F = m\frac{dV}{dt}$ $\quad V = R\omega$

For the mass:



$$\sum F_y = mg - t = m\frac{dV}{dt} = mR\frac{d\omega}{dt} \qquad (1)$$

For the shaft:



$$\sum T = tR - T_{viscous} = I\frac{d\omega}{dt} \qquad (2)$$

$$T_{viscous} = \tau R A = \mu\frac{V}{a}R\,2\pi R L = \frac{2\pi\mu\omega R^3 L}{a}$$

Assume: (1) Newtonian liquid, (2) Small gap, (3) Linear profile

Then Eq. 2 becomes $\quad tR - \frac{2\pi\mu R^3 L}{a}\omega = I\frac{d\omega}{dt}$; $I = \frac{1}{2}MR^2 \qquad (3)$

Multiplying Eq. 1 by R and combining with Eq. 3 gives

$$mgR - mR^2\frac{d\omega}{dt} - \frac{2\pi\mu R^3 L}{a}\omega = I\frac{d\omega}{dt} \quad \text{or} \quad mgR - \frac{2\pi\mu R^3 L}{a}\omega = (I + mR^2)\frac{d\omega}{dt} \qquad (4)$$

This may be written $A - B\omega = C\frac{d\omega}{dt}$ where $A = mgR$, $B = \frac{2\pi\mu R^3 L}{a}$, $C = I + mR^2$

Separating variables $\quad \frac{d\omega}{A - B\omega} = \frac{dt}{C}$

Integrating $\quad \int_0^\omega \frac{d\omega}{A - B\omega} = -\frac{1}{B}\ln(A - B\omega)\Big]_0^\omega = -\frac{1}{B}\ln\left(1 - \frac{B\omega}{A}\right) = \int_0^t \frac{dt}{C} = \frac{t}{C}$

Simplifying $\quad 1 - \frac{B\omega}{A} = e^{-Bt/C}$ or $\omega = \frac{A}{B}\left[1 - e^{-Bt/C}\right] \qquad (5) \qquad \omega(t)$

The maximum angular speed $(t \to \infty)$ is $\omega = A/B$.

$$A = mgR = 0.010\,kg \times 9.81\frac{m}{s^2} \times 0.025\,m \times \frac{N \cdot s^2}{kg \cdot m} = 2.45 \times 10^{-3}\,N \cdot m$$

$$B = \frac{2\pi\mu R^3 L}{a} = 2\pi \times 0.095\frac{kg}{m \cdot s} \times (0.025)^3 m^3 \times 0.050\,m \times \frac{1}{0.0005\,m} \times \frac{N \cdot s^2}{kg \cdot m} = 9.33 \times 10^{-4}\,N \cdot m \cdot s$$

Evaluating, $\omega_{max} = \dfrac{A}{B} = 2.45\times10^{-5}\,N\cdot m \times \dfrac{1}{9.33\times10^{-4}\,N\cdot m\cdot sec} = 2.63\ rad/s$

Thus

$$\omega_{max} = 2.63\,\frac{rad}{s}\times\frac{rev}{2\pi\,rad}\times\frac{60\,s}{min} = 25.1\ rpm$$ ω_{max}

From Eq. 5, $\omega = 0.95\,\omega_{max}$ when $e^{-Bt/C} = 0.05$, or $Bt/C \simeq 3;\ t \simeq \dfrac{3C}{B}$

$$C = I + mR^2 = \tfrac{1}{2}MR^2 + mR^2 = \left(\tfrac{1}{2}M + m\right)R^2$$

$$M = \pi R^2(1.5L + L)\rho = 2.5\pi R^2 L\, SG\, \rho_w$$

$$M = 2.5\pi\times(0.025)^2 m^2\times 0.050\,m\times(2.64)1000\,\frac{kg}{m^3} = 0.648\ kg$$

$$C = \left(\tfrac{1}{2}\times 0.648\,kg + 0.010\,kg\right)(0.025)^2 m^2 = 2.09\times10^{-4}\ kg\cdot m^2$$

Thus

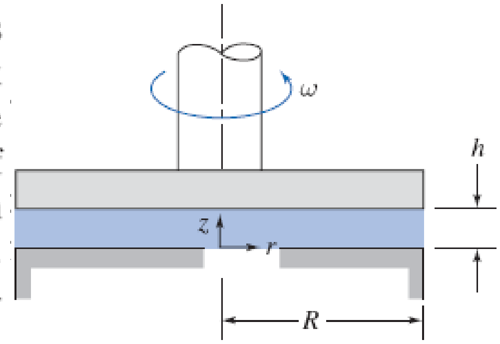
$$t_{95} = 3\times 2.09\times10^{-4}\,kg\cdot m^2\times\frac{1}{9.33\times10^{-4}\,N\cdot m\cdot s}\times\frac{N\cdot s^2}{kg\cdot m} = 0.671\ s$$ t_{95}

{The terminal speed could have been computed from Eq. 4 by setting $d\omega/dt \to 0$, without solving the differential equation.}

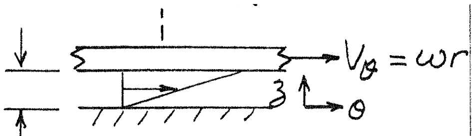
Problem 2.66

[Difficulty: 4]

2.66 A proposal has been made to use a pair of parallel disks to measure the viscosity of a liquid sample. The upper disk rotates at height h above the lower disk. The viscosity of the liquid in the gap is to be calculated from measurements of the torque needed to turn the upper disk steadily. Obtain an algebraic expression for the torque needed to turn the disk. Could we use this device to measure the viscosity of a non-Newtonian fluid? Explain.



Solution: Use r, θ, z coordinates at right:



Basic equations: $\tau_{z\theta} = \mu \frac{dV_\theta}{dz}$

$$dT = r\,dF = r\,\tau_{z\theta}\,dA$$

Assumptions: (1) Newtonian fluid
(2) No-slip condition
(3) Linear velocity profile (in narrow gap)

The velocity at any radial location on the rotating disk is $V_\theta = \omega r$.

Since the velocity profile is linear, then

$$\tau_{z\theta} = \mu \frac{dV_\theta}{dz} = \mu \frac{\Delta V}{\Delta z} = \mu \frac{(\omega r - 0)}{(h - 0)} = \frac{\mu \omega r}{h}$$ $\tau_{z\theta}$

and

$$dT = r\,\tau_{z\theta}\,dA = r\,\mu \frac{\omega r}{h}\, 2\pi r\,dr = \frac{2\pi\mu\omega r^3}{h}\,dr$$

Integrating

$$T = \int_A dT = \int_0^R \frac{2\pi\mu\omega r^3}{h}\,dr = \left.\frac{\pi\mu\omega r^4}{2h}\right]_0^R$$

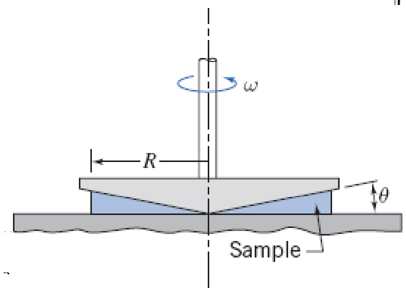
$$T = \frac{\pi\mu\omega R^4}{2h}$$ T

The device could not be used to measure the viscosity of a non-Newtonian fluid because the applied shear stress is not uniform. It varies from zero at the center of the disks to $\mu\omega R/h$ at the edge

Problem 2.67

[Difficulty: 4]

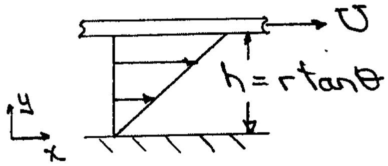
2.67 The cone and plate viscometer shown is an instrument used frequently to characterize non-Newtonian fluids. It consists of a flat plate and a rotating cone with a very obtuse angle (typically θ is less than 0.5 degrees). The apex of the cone just touches the plate surface and the liquid to be tested fills the narrow gap formed by the cone and plate. Derive an expression for the shear rate in the liquid that fills the gap in terms of the geometry of the system. Evaluate the torque on the driven cone in terms of the shear stress and geometry of the system.



Solution:

Since the angle θ is very small, the average gap width is also very small.

It is reasonable to assume a linear velocity profile across the gap and to neglect end effects.



The shear (deformation) rate is given by

$$\dot{\gamma} = \frac{du}{dy} = \frac{\Delta u}{\Delta y}$$

At any radius, r,

the velocity $U = \omega r$ and

the gap width $h = r\tan\theta$

$$\therefore \dot{\gamma} = \frac{\omega r}{r\tan\theta} = \frac{\omega}{\tan\theta}$$

Since θ is very small, $\tan\theta \simeq \theta$ and

$$\dot{\gamma} = \frac{\omega}{\theta}$$ $\dot{\gamma}$

Note: The shear rate is independent of r. The entire sample is subjected to the same shear rate.

The torque on the driven cone is given by

$T = \int r\,dF$ where $dF = \tau_{yx}\,dA$

Since $\dot{\gamma}$ is a constant (for a given ω) then τ_{yx} = constant

and

$$T = \int r\,dF = \int_A r\,\tau_{yx}\,dA = \tau_{yx}\int_0^R r\,2\pi r\,dr$$

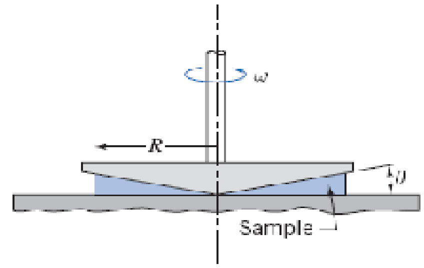
$$T = \frac{2\pi}{3}R^3\tau_{yx}$$ T

Problem 2.68

[Difficulty: 4]

2.68 The viscometer of Problem 2.67 is used to measure the apparent viscosity of a fluid. The data below are obtained. What kind of non-Newtonian fluid is this? Find the values of k and n used in Eqs. 2.16 and 2.17 in defining the apparent viscosity of a fluid. (Assume θ is 0.5 degrees.) Predict the viscosity at 90 and 100 rpm.

Speed (rpm)	10	20	30	40	50	60	70	80
μ (N·s/m²)	0.121	0.139	0.153	0.159	0.172	0.172	0.183	0.185



Given: Data on the viscometer

Find: The values of coefficients k and n; determine the kind of non-Newtonial fluid it is; estimate viscosity at 90 and 100 rpm

Solution:

The velocity gradient at any radius r is $\quad \dfrac{du}{dy} = \dfrac{r \cdot \omega}{r \cdot \tan(\theta)}$

where ω (rad/s) is the angular velocity $\quad \omega = \dfrac{2 \cdot \pi \cdot N}{60} \quad$ where N is the speed in rpm

For small θ, tan(θ) can be replace with θ, so $\quad \dfrac{du}{dy} = \dfrac{\omega}{\theta}$

From Eq 2.11. $\quad k \cdot \left(\left|\dfrac{du}{dy}\right|\right)^{n-1} \dfrac{du}{dy} = \eta \cdot \dfrac{du}{dy}$

where η is the apparent viscosity. Hence $\quad \eta = k \cdot \left(\dfrac{du}{dy}\right)^{n-1} = k \cdot \left(\dfrac{\omega}{\theta}\right)^{n-1}$

The data is

N (rpm)	μ (N s/m²)
10	0.121
20	0.139
30	0.153
40	0.159
50	0.172
60	0.172
70	0.183
80	0.185

The computed data is

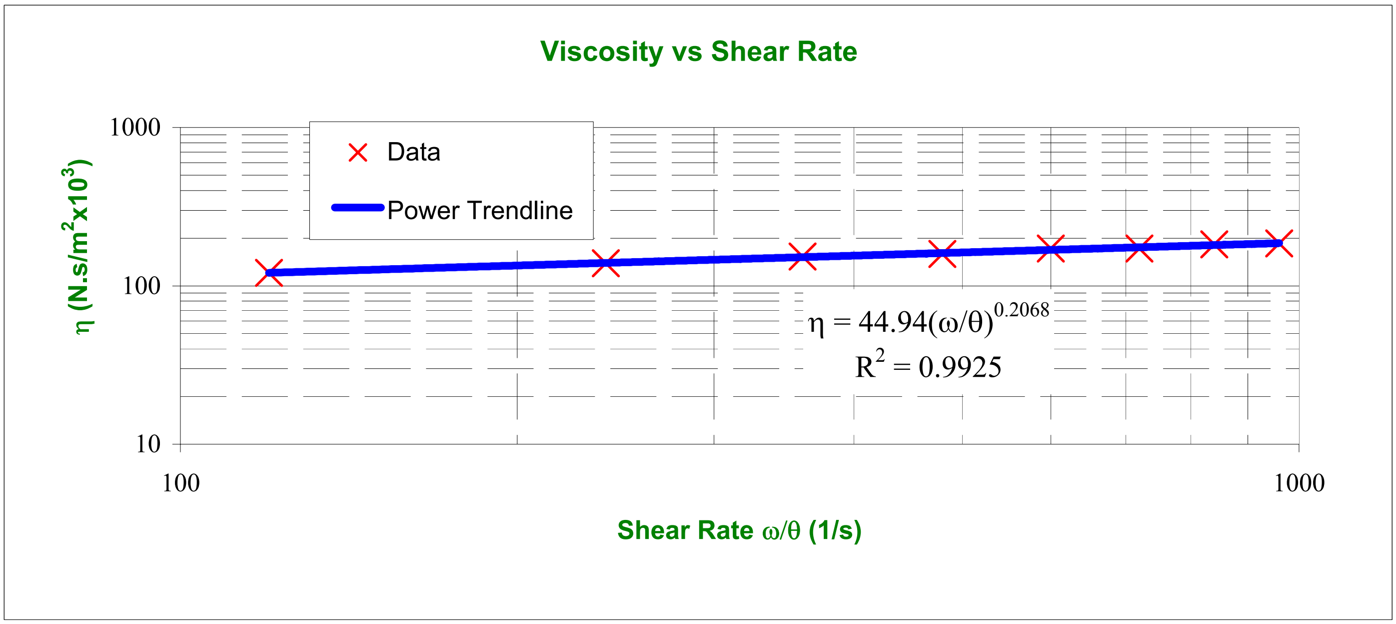
ω (rad/s)	ω/θ (1/s)	η (N s/m^{2}x10^3)
1.047	120	121
2.094	240	139
3.142	360	153
4.189	480	159
5.236	600	172
6.283	720	172
7.330	840	183
8.378	960	185

From the *Trendline* analysis

k = 0.0449
n - 1 = 0.2068
n = 1.21 The fluid is dilatant

The apparent viscosities at 90 and 100 rpm can now be computed

N (rpm)	ω (rad/s)	ω/θ (1/s)	η (N s/m^{2}x10^3)
90	9.42	1080	191
100	10.47	1200	195



2.69 An insulation company is examining a new material for extruding into cavities. The experimental data is given below for the speed U of the upper plate, which is separated from a fixed lower plate by a 1-mm-thick sample of the material, when a given shear stress is applied. Determine the type of material. If a replacement material with a minimum yield stress of 250 Pa is needed, what viscosity will the material need to have the same behavior as the current material at a shear stress of 450 Pa?

τ (Pa)	50	100	150	163	171	170	202	246	349	444
U (m/s)	0	0	0	0.005	0.01	0.025	0.05	0.1	0.2	0.3

Given: Data on insulation material

Find: Type of material; replacement material

Solution:

The velocity gradient is

$du/dy = U/\delta$ where $\delta =$ 0.001 m

Data and computations

τ (Pa)	U (m/s)	du/dy (s^{-1})
50	0.000	0
100	0.000	0
150	0.000	0
163	0.005	5
171	0.01	10
170	0.03	25
202	0.05	50
246	0.1	100
349	0.2	200
444	0.3	300

Hence we have a Bingham plastic, with $\tau_y =$ 154 Pa

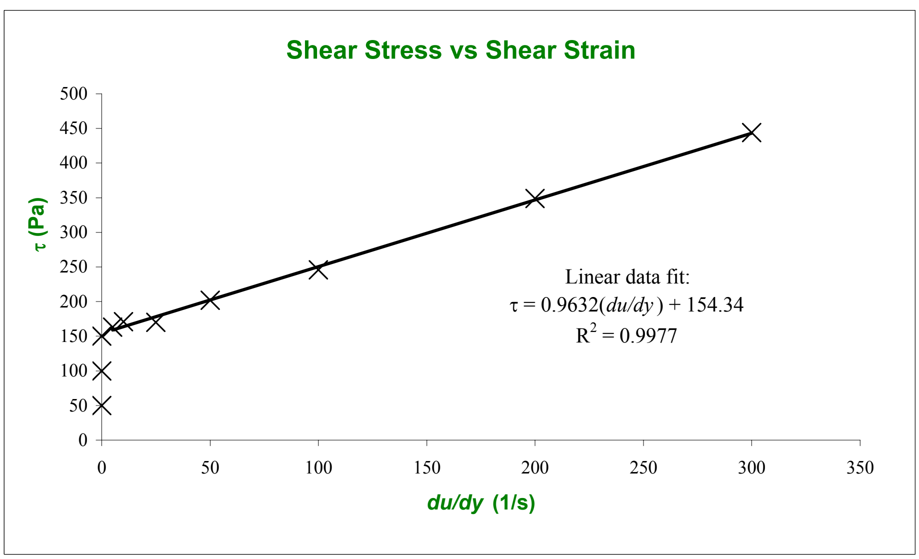
$\mu_p =$ 0.963 N s/m^2

At τ = 450 Pa, based on the linear fit $du/dy =$ 307 s^{-1}

For a fluid with $\tau_y =$ 250 Pa

we can use the Bingham plastic formula to solve for μ_p given τ, τ_y and du/dy from above

$\mu_p =$ 0.652 N s/m^2

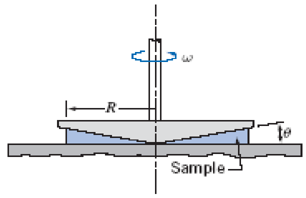


Problem 2.70

[Difficulty: 3]

2.70 A viscometer is used to measure the viscosity of a patient's blood. The deformation rate (shear rate)–shear stress data is shown below. Plot the apparent viscosity versus deformation rate. Find the value of k and n in Eq. 2.17, and from this examine the aphorism "Blood is thicker than water."

du/dy (s^{-1})	5	10	25	50	100	200	300	400
τ (Pa)	0.0457	0.119	0.241	0.375	0.634	1.06	1.46	1.78



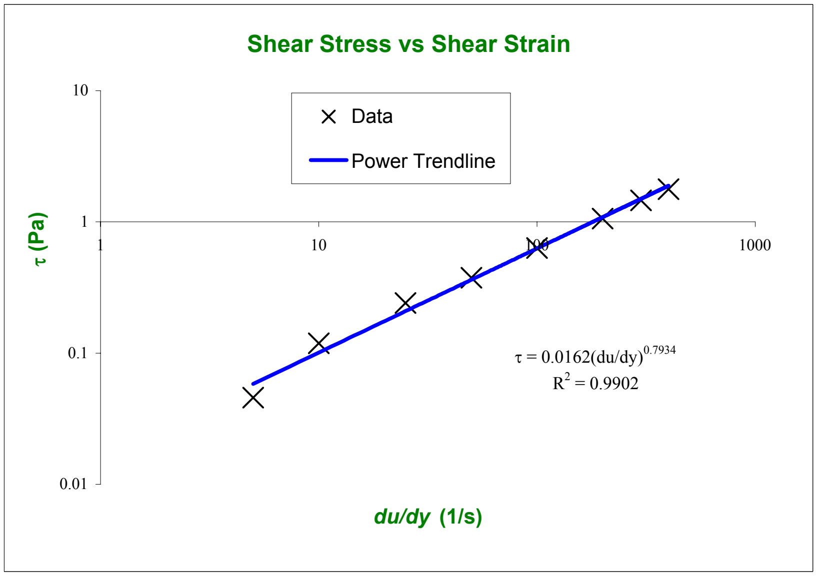
Given: Viscometer data

Find: Value of k and n in Eq. 2.17

Solution:

The data is

τ (Pa)	du/dy (s^{-1})
0.0457	5
0.119	10
0.241	25
0.375	50
0.634	100
1.06	200
1.46	300
1.78	400



Hence we have $k = 0.0162$

$n = 0.7934$ Blood is pseudoplastic (shear thinning)

The apparent viscosity from $\eta = k\,(du/dy)^{n-1}$

du/dy (s^{-1})	η ($N \cdot s/m^2$)
5	0.0116
10	0.0101
25	0.0083
50	0.0072
100	0.0063
200	0.0054
300	0.0050
400	0.0047

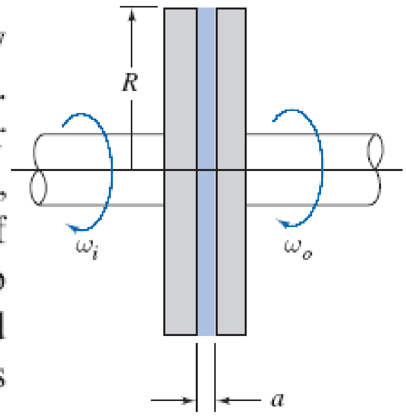
$\mu_{water} = 0.001\ N \cdot s/m^2$ at 20°C

Hence, blood is "thicker" than water!

Problem 2.71

[Difficulty: 5]

2.71 A viscous clutch is to be made from a pair of closely spaced parallel disks enclosing a thin layer of viscous liquid. Develop algebraic expressions for the torque and the power transmitted by the disk pair, in terms of liquid viscosity, μ, disk radius, R, disk spacing, a, and the angular speeds: ω_i of the input disk and ω_o of the output disk. Also develop expressions for the slip ratio, $s = \Delta\omega/\omega_i$, in terms of ω_i and the torque transmitted. Determine the efficiency, η, in terms of the slip ratio.



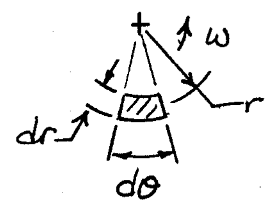
Solution: Apply Newton's law of viscosity

Basic equations: $\tau = \mu \dfrac{du}{dy}$ $\quad dF = \tau dA \quad dT = r dF$

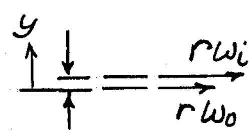
Assumptions: (1) Newtonian liquid
(2) Narrow gap so velocity profile is linear

Consider a segment of plates:

$$\tau = \mu \frac{du}{dy} = \mu \frac{\Delta u}{\Delta y} = \mu \frac{r(\omega_i - \omega_o)}{a}$$



End View



Bottom View

$dA = r\,dr\,d\theta$

$$dF = \tau dA = \frac{\mu r \Delta\omega}{a} r\,dr\,d\theta = \frac{\mu\Delta\omega}{a} r^2 dr\,d\theta \; ; \; dT = r dF = \frac{\mu\Delta\omega}{a} r^3 dr\,d\theta$$

Integrating

$$T = \int_0^{2\pi}\int_0^R dT = \frac{\mu\Delta\omega}{a}\int_0^{2\pi}\int_0^R r^3 dr\,d\theta = \frac{2\pi\mu\Delta\omega}{a}\int_0^R r^3 dr = \frac{\pi\mu\Delta\omega R^4}{2a}$$ T

$$P_o = T\omega_o = \frac{\pi\mu\omega_o\Delta\omega R^4}{2a} \quad \text{(power transmitted)}$$ P

$$s = \frac{\Delta\omega}{\omega_i} = \frac{2aT}{\pi\mu R^4 \omega_i}$$ s

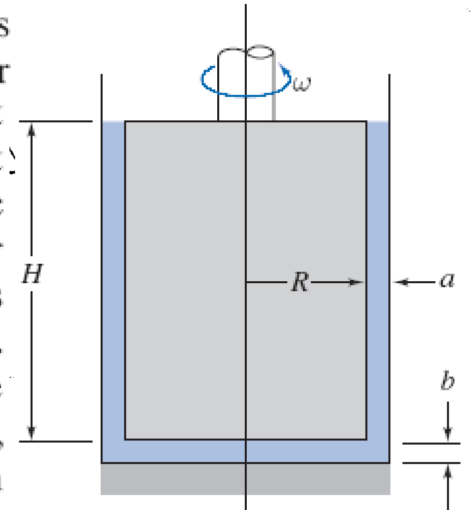
Efficiency is $\eta = \dfrac{\text{Power out}}{\text{Power in}} = \dfrac{T\omega_o}{T\omega_i} = \dfrac{\omega_o}{\omega_i}$. But $\omega_o = \omega_i - \Delta\omega$, so

$$\eta = \frac{\omega_i - \Delta\omega}{\omega_i} = 1 - \frac{\Delta\omega}{\omega_i} = 1 - s$$ η

Problem 2.72

[Difficulty: 5]

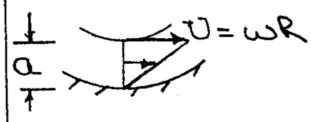
2.72 A concentric-cylinder viscometer is shown. Viscous torque is produced by the annular gap around the inner cylinder. Additional viscous torque is produced by the flat bottom of the inner cylinder as it rotates above the flat bottom of the stationary outer cylinder. Obtain an algebraic expression for the viscous torque due to flow in the annular gap of width a. Obtain an algebraic expression for the viscous torque due to flow in the bottom clearance gap of height b. Prepare a plot showing the ratio, b/a, required to hold the bottom torque to 1 percent or less of the annulus torque, versus the other geometric variables. What are the design implications? What modifications to the design can you recommend?



Solution: Basic equation $\tau_{yx} = \mu \frac{du}{dy}$

Assumptions: (1) linear velocity profile (2) Newtonian liquid

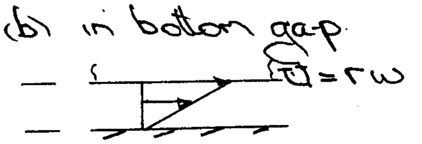
(a) in annular gap



$$\tau = \mu \frac{du}{dr} = \mu \frac{\Delta u}{\Delta r} = \mu \frac{U}{a} = \mu \frac{\omega R}{a}$$

$$\text{Torque} = RF_t = R\tau A = R\mu \frac{\omega R}{a}(2\pi R H) = \frac{2\pi\mu\omega R^3 H}{a} \quad (a)$$

(b) in bottom gap



$$\tau = \mu \frac{du}{dz} = \mu \frac{\Delta u}{\Delta z} = \mu \frac{U}{a} = \mu \frac{\omega r}{b}$$
(varies with r)

$$\text{Torque} = \int dT = \int r\,dF = \int r\tau\,dA = \int_0^R r\mu\frac{\omega r}{b}2\pi r\,dr$$

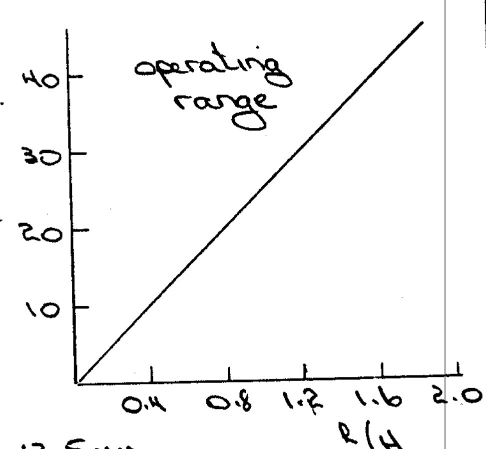
$$\text{Torque} = \frac{2\pi\mu\omega}{b}\int_0^R r^3\,dr = \frac{2\pi\mu\omega}{b}\left[\frac{r^4}{4}\right]_0^R = \frac{\pi\mu\omega}{2b}R^4 \quad (b)$$

(c) For $T_{bottom}/T_{annulus} \le \frac{1}{100}$, then

$$\frac{T_{bot}}{T_{an}} = \frac{\pi\mu\omega}{2b}R^4 \times \frac{a}{2\pi\mu\omega R^3 H} \le \frac{1}{100}$$

$$\frac{aR}{4bH} \le \frac{1}{100}$$

or $\frac{b}{a} \ge 25\frac{R}{H}$



(d) The plot shows the operating range. Specific design would depend on other constraints.

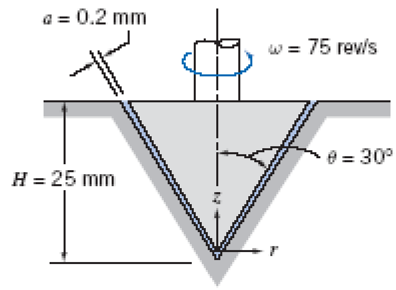
For $a = 1\text{mm}$ with $R/H = 1/2$ gives $b = 12.5\text{mm}$

(e) For a given value of R/H, the dimension b could be effectively increased by "hollowing out" the inner cylinder as shown by the dashed lines in the diagram above.

Problem 2.73

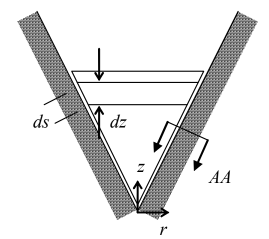
[Difficulty: 4]

2.73 A viscometer is built from a conical pointed shaft that turns in a conical bearing, as shown. The gap between shaft and bearing is filled with a sample of the test oil. Obtain an algebraic expression for the viscosity μ of the oil as a function of viscometer geometry (H, a, and θ), turning speed ω, and applied torque T. For the data given, find by referring to Figure A.2 in Appendix A, the type of oil for which the applied torque is 0.325 N·m. The oil is at 20°C. *Hint:* First obtain an expression for the shear stress on the surface of the conical shaft as a function of z.



Given: Conical bearing geometry

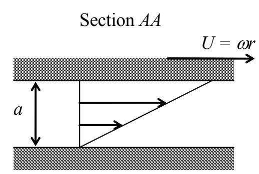
Find: Expression for shear stress; Viscous torque on shaft



Solution:

Basic equation $\qquad \tau = \mu \cdot \frac{du}{dy} \qquad dT = r \cdot \tau \cdot dA \qquad$ Infinitesimal shear torque

Assumptions: Newtonian fluid, linear velocity profile (in narrow clearance gap), no slip condition



$$\tan(\theta) = \frac{r}{z} \qquad \text{so} \qquad r = z \cdot \tan(\theta)$$

Then $\qquad \tau = \mu \cdot \frac{du}{dy} = \mu \cdot \frac{\Delta u}{\Delta y} = \mu \cdot \frac{(\omega \cdot r - 0)}{(a - 0)} = \frac{\mu \cdot \omega \cdot z \cdot \tan(\theta)}{a}$

As we move up the device, shear stress increases linearly (because rate of shear strain does)

But from the sketch $\qquad dz = ds \cdot \cos(\theta) \qquad dA = 2 \cdot \pi \cdot r \cdot ds = 2 \cdot \pi \cdot r \cdot \frac{dz}{\cos(\theta)}$

The viscous torque on the element of area is $\qquad dT = r \cdot \tau \cdot dA = r \cdot \frac{\mu \cdot \omega \cdot z \cdot \tan(\theta)}{a} \cdot 2 \cdot \pi \cdot r \cdot \frac{dz}{\cos(\theta)} \qquad dT = \frac{2 \cdot \pi \cdot \mu \cdot \omega \cdot z^3 \cdot \tan(\theta)^3}{a \cdot \cos(\theta)} \cdot dz$

Integrating and using limits z = H and z = 0 $\qquad T = \frac{\pi \cdot \mu \cdot \omega \cdot \tan(\theta)^3 \cdot H^4}{2 \cdot a \cdot \cos(\theta)}$

Solving for μ $\qquad \mu = \frac{2 \cdot a \cdot \cos(\theta) \cdot T}{\pi \cdot \omega \cdot \tan(\theta)^3 \cdot H^4}$

Using given data $\qquad H = 25 \cdot mm \qquad \theta = 30 \cdot deg \qquad a = 0.2 \cdot mm \qquad \omega = 75 \cdot \frac{rev}{s} \qquad T = 0.325 \cdot N \cdot m$

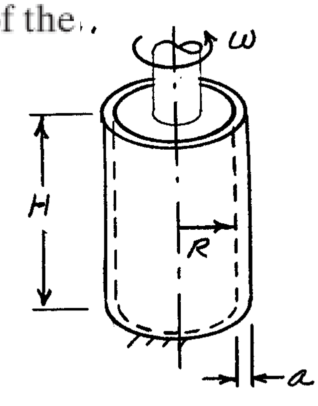
$$\mu = \frac{2 \cdot a \cdot \cos(\theta) \cdot T}{\pi \cdot \omega \cdot \tan(\theta)^3 \cdot H^4} \qquad \mu = 1.012 \cdot \frac{N \cdot s}{m^2}$$

From Fig. A.2, at 20°C, CASTOR OIL has this viscosity!

Problem 2.74

[Difficulty: 5]

2.74 Design a concentric-cylinder viscometer to measure the viscosity of a liquid similar to water. The goal is to achieve a measurement accuracy of ±1 percent. Specify the configuration and dimensions of the viscometer. Indicate what measured parameter will be used to infer the viscosity of the liquid sample.

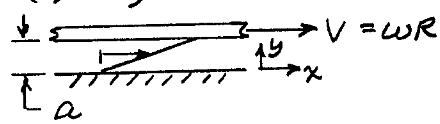


Solution: Apply definition of Newtonian fluid

Computing equation: $\tau = \mu \frac{du}{dy}$

Assumptions: (1) Steady
(2) Newtonian liquid
(3) Narrow gap, so "unroll" it
(4) Linear velocity profile in gap
(5) Neglect end effects

Flow model:



$u = V\frac{y}{a} = \omega R \frac{y}{a}$; $\frac{du}{dy} = \frac{\omega R}{a}$

Thus $\tau = \mu \frac{du}{dy} = \mu \frac{\omega R}{a}$ and torque on rotor is $T = R\tau A$, where $A = 2\pi R H$

Consequently $T = R\mu \frac{\omega R}{a} 2\pi R H = \frac{2\pi \mu \omega R^3 H}{a}$, or

$$\mu = \frac{Ta}{2\pi \omega R^3 H}$$ μ

From this equation the uncertainty in μ is (see Appendix F),

$$u_\mu = \pm\left[u_T^2 + u_a^2 + u_\omega^2 + (3u_R)^2 + u_H^2\right]^{\frac{1}{2}} = \pm\left[13\, u^2\right]^{\frac{1}{2}} = \pm 3.61\, u$$

if the uncertainty of each parameter equals u. Thus

$$u = \pm\frac{u_\mu}{3.61} = \pm\frac{1 \text{ percent}}{3.61} = \pm 0.277 \text{ percent}$$ u

Typical dimensions for a bench-top unit might be

$H = 200$ mm, $R = 75$ mm, $a = 0.02$ mm, and $\omega = 10.5$ rad/s (100 rpm)

From Appendix A, Table A.8, water has $\mu = 1.00 \times 10^{-3}$ N·s/m² at $T = 20°C$.

The corresponding torque would be

$$T = 2\pi \times 1.00\times10^{-3}\frac{N\cdot s}{m^2} \times \frac{10.5}{s} \times (0.075)^3 m^3 \times 0.2\,m \times \frac{1}{0.00002\,m} = 0.278 \text{ N·m}$$ T

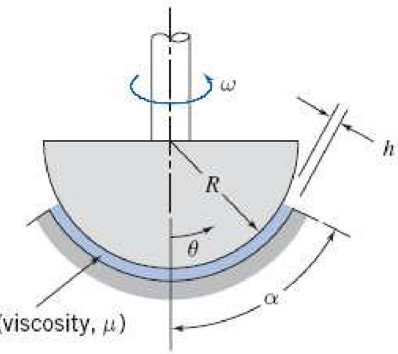
It should be possible to measure this torque quite accurately.

{Many details would need to be considered (e.g. bearings, temperature rise, etc.) to produce a workable device.}

Problem 2.75

[Difficulty: 5]

2.75 A spherical thrust bearing is shown. The gap between the spherical member and the housing is of constant width h. Obtain and plot an algebraic expression for the nondimensional torque on the spherical member, as a function of angle α.



Solution: Apply definitions

Computing equations: $\tau = \mu \frac{du}{dy}$ $\quad T = \int_A r\tau\, dA$

Assumptions: (1) Newtonian fluid, (2) Narrow gap, (3) Laminar flow

From the figure, $r = R\sin\theta$ $\quad u = \omega r = \omega R \sin\theta$

$$\tau = \mu\frac{du}{dy} = \mu\left(\frac{u-0}{h}\right) = \mu\frac{u}{h} = \frac{\mu\omega R\sin\theta}{h}$$

$$dA = 2\pi r R\, d\theta = 2\pi R^2 \sin\theta\, d\theta$$

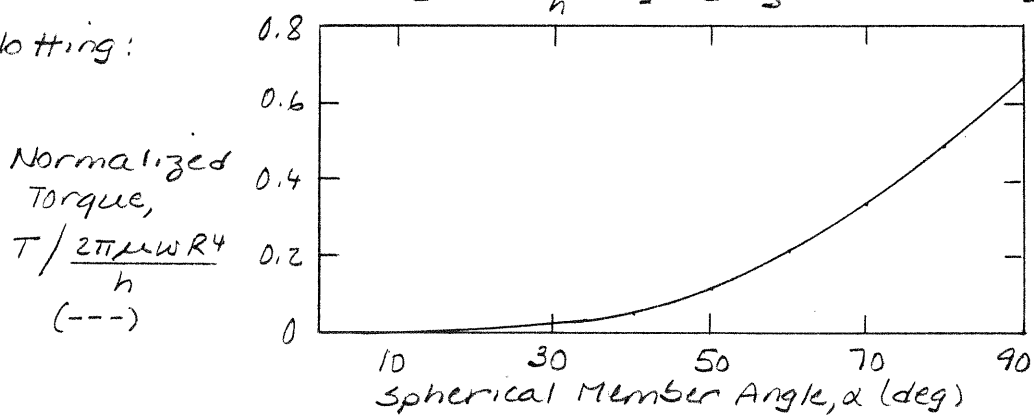
Thus

$$T = \int_0^\alpha R\sin\theta\left(\frac{\mu\omega R\sin\theta}{h}\right) 2\pi R^2 \sin\theta\, d\theta = \frac{2\pi\mu\omega R^4}{h}\int_0^\alpha \sin^3\theta\, d\theta$$

$$T = \frac{2\pi\mu\omega R^4}{h}\left[\frac{\cos^3\theta}{3} - \cos\theta\right]_0^\alpha = \frac{2\pi\mu\omega R^4}{h}\left[\frac{\cos^3\alpha}{3} - \cos\alpha + \frac{2}{3}\right]$$ T

To plot, normalize to $\left[T\Big/\frac{2\pi\mu\omega R^4}{h}\right] = \left[\frac{\cos^3\alpha}{3} - \cos\alpha + \frac{2}{3}\right]$

Plotting:

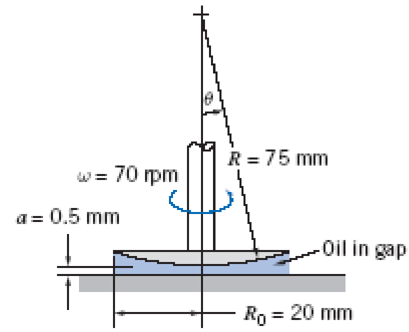


$\left\{\text{Check dimensions: } \left[\frac{\mu\omega R^4}{h}\right] = \frac{Ft}{L^2}\times\frac{1}{t}\times L^4\times\frac{1}{L} = FL \quad \checkmark\checkmark\right\}$

Problem 2.76

[Difficulty: 5]

2.76 A cross section of a rotating bearing is shown. The spherical member rotates with angular speed ω, a small distance, a, above the plane surface. The narrow gap is filled with viscous oil, having $\mu = 1250$ cp. Obtain an algebraic expression for the shear stress acting on the spherical member. Evaluate the maximum shear stress that acts on the spherical member for the conditions shown. (Is the maximum necessarily located at the maximum radius?) Develop an algebraic expression (in the form of an integral) for the total viscous shear torque that acts on the spherical member. Calculate the torque using the dimensions shown.



Given: Geometry of rotating bearing

Find: Expression for shear stress; Maximum shear stress; Expression for total torque; Total torque

Solution:

Basic equation $\quad \tau = \mu \cdot \dfrac{du}{dy} \qquad dT = r \cdot \tau \cdot dA$

Assumptions: Newtonian fluid, narrow clearance gap, laminar motion

From the figure $\quad r = R \cdot \sin(\theta) \qquad u = \omega \cdot r = \omega \cdot R \cdot \sin(\theta) \qquad \dfrac{du}{dy} = \dfrac{u - 0}{h} = \dfrac{u}{h}$

$h = a + R \cdot (1 - \cos(\theta)) \qquad dA = 2 \cdot \pi \cdot r \cdot dr = 2 \cdot \pi\, R \cdot \sin(\theta) \cdot R \cdot \cos(\theta) \cdot d\theta$

Then $\quad \tau = \mu \cdot \dfrac{du}{dy} = \dfrac{\mu \cdot \omega \cdot R \cdot \sin(\theta)}{a + R \cdot (1 - \cos(\theta))}$

To find the maximum τ set $\quad \dfrac{d}{d\theta}\left[\dfrac{\mu \cdot \omega \cdot R \cdot \sin(\theta)}{a + R \cdot (1 - \cos(\theta))}\right] = 0 \qquad$ so $\qquad \dfrac{R \cdot \mu \cdot \omega \cdot (R \cdot \cos(\theta) - R + a \cdot \cos(\theta))}{(R + a - R \cdot \cos(\theta))^2} = 0$

$R \cdot \cos(\theta) - R + a \cdot \cos(\theta) = 0 \qquad \theta = \operatorname{acos}\left(\dfrac{R}{R + a}\right) = \operatorname{acos}\left(\dfrac{75}{75 + 0.5}\right) \qquad \theta = 6.6 \cdot \text{deg}$

$$\tau = 12.5 \cdot \text{poise} \times 0.1 \cdot \frac{\frac{\text{kg}}{\text{m} \cdot \text{s}}}{\text{poise}} \times 2 \cdot \pi \cdot \frac{70}{60} \cdot \frac{\text{rad}}{\text{s}} \times 0.075 \cdot \text{m} \times \sin(6.6 \cdot \text{deg}) \times \frac{1}{[0.0005 + 0.075 \cdot (1 - \cos(6.6 \cdot \text{deg}))] \cdot \text{m}} \times \frac{\text{N} \cdot \text{s}^2}{\text{m} \cdot \text{kg}}$$

$$\tau = 79.2 \cdot \frac{\text{N}}{\text{m}^2}$$

The torque is $\quad T = \displaystyle\int r \cdot \tau \cdot A\, d\theta = \int_0^{\theta_{max}} \frac{\mu \cdot \omega \cdot R^4 \cdot \sin(\theta)^2 \cdot \cos(\theta)}{a + R \cdot (1 - \cos(\theta))}\, d\theta \qquad$ wher e $\qquad \theta_{max} = \operatorname{asin}\left(\dfrac{R_0}{R}\right) \qquad \theta_{max} = 15.5 \cdot \text{deg}$

This integral is best evaluated numerically using Excel, Mathcad, or a good calculator $\quad T = 1.02 \times 10^{-3} \cdot \text{N} \cdot \text{m}$

Problem 2.77

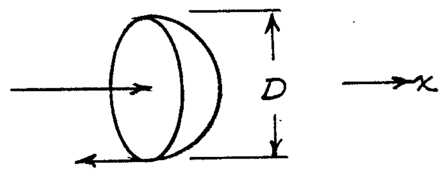
[Difficulty: 2]

2.77 Small gas bubbles form in soda when a bottle or can is opened. The average bubble diameter is about 0.1 mm. Estimate the pressure difference between the inside and outside of such a bubble.

Solution: Consider a free-body diagram of half a bubble:

Two forces act:

Pressure: $F_p = \Delta p \dfrac{\pi D^2}{4}$



Surface tension: $F_\sigma = \sigma \pi D$

Summing forces for equilibrium

$$\Sigma F_x = F_p - F_\sigma = \Delta p \frac{\pi D^2}{4} - \sigma \pi D = 0$$

so $\dfrac{\Delta p D}{4} - \sigma = 0$ or $\Delta p = \dfrac{4\sigma}{D}$

Assuming soda-gas interface is similar to water-air, then $\sigma = 72.8 \text{ mN/m}$, and

$$\Delta p = 4 \times 72.8 \times 10^{-3} \frac{N}{m} \times \frac{1}{0.1 \times 10^{-3} m} = 2.91 \times 10^3 \frac{N}{m^2} = 2.91 \text{ kPa}$$

Δp

Problem 2.78

[Difficulty: 2]

2.78 You intend to gently place several steel needles on the free surface of the water in a large tank. The needles come in two lengths: Some are 5 cm long, and some are 10 cm long. Needles of each length are available with diameters of 1 mm, 2.5 mm, and 5 mm. Make a prediction as to which needles, if any, will float.

Given: Data on size of various needles

Find: Which needles, if any, will float

Solution:

For a steel needle of length *L*, diameter *D*, density ρ_s, to float in water with surface tension σ and contact angle θ, the vertical force due to surface tension must equal or exceed the weight

$$2\cdot L\cdot\sigma\cdot\cos(\theta) \geq W = m\cdot g = \frac{\pi\cdot D^2}{4}\cdot\rho_s\cdot L\cdot g \qquad \text{or} \qquad D \leq \sqrt{\frac{8\cdot\sigma\cdot\cos(\theta)}{\pi\cdot\rho_s\cdot g}}$$

From Table A.4 $\quad \sigma = 72.8\times10^{-3}\cdot\frac{N}{m} \quad \theta = 0\cdot deg \quad$ and for water $\quad \rho = 1000\cdot\frac{kg}{m^3}$

From Table A.1, for steel $\quad SG = 7.83$

Hence

$$\sqrt{\frac{8\cdot\sigma\cdot\cos(\theta)}{\pi\cdot SG\cdot\rho\cdot g}} = \sqrt{\frac{8}{\pi\cdot7.83}\times72.8\times10^{-3}\cdot\frac{N}{m}\times\frac{m^3}{999\cdot kg}\times\frac{s^2}{9.81\cdot m}\times\frac{kg\cdot m}{N\cdot s^2}} = 1.55\times10^{-3}\cdot m = 1.55\cdot mm$$

Hence $D < 1.55$ mm. Only the 1 mm needles float (needle length is irrelevant)

Problem 2.79

[Difficulty: 3]

2.79 According to Folsom [6], the capillary rise Δh (in.) of a water-air interface in a tube is correlated by the following empirical expression:

$$\Delta h = Ae^{-b \cdot D}$$

where D (in.) is the tube diameter, $A = 0.400$, and $b = 4.37$. You do an experiment to measure Δh versus D and obtain:

D (in.)	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	1.1
Δh (in.)	0.232	0.183	0.09	0.059	0.052	0.033	0.017	0.01	0.006	0.004	0.003

What are the values of A and b that best fit this data using *Excel's Trendline* feature? Do they agree with Folsom's values? How good is the data?

Given: Caplillary rise data

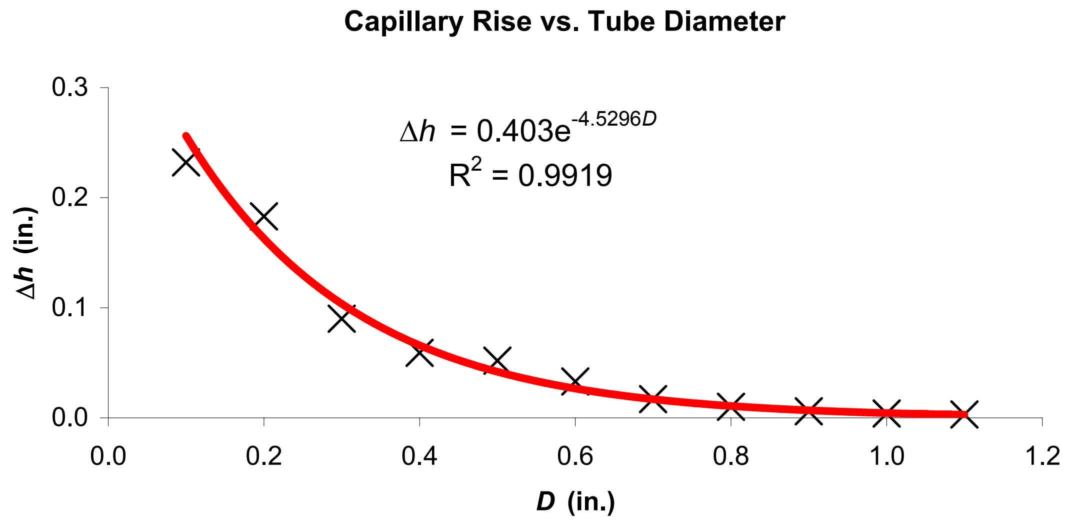
Find: Values of A and b

Solution:

D (in.)	Δh (in.)
0.1	0.232
0.2	0.183
0.3	0.090
0.4	0.059
0.5	0.052
0.6	0.033
0.7	0.017
0.8	0.010
0.9	0.006
1.0	0.004
1.1	0.003

$$\boxed{\begin{aligned} A &= 0.403 \\ b &= 4.530 \end{aligned}}$$

The fit is a good one ($R^2 = 0.9919$)



Problem 2.80

[Difficulty: 2]

2.80 Slowly fill a glass with water to the maximum possible level. Observe the water level closely. Explain how it can be higher than the rim of the glass.

Open-Ended Problem Statement: Slowly fill a glass with water to the maximum possible level before it overflows. Observe the water level closely. Explain how it can be higher than the rim of the glass.

Discussion: Surface tension can cause the maximum water level in a glass to be higher than the rim of the glass. The same phenomenon causes an isolated drop of water to "bead up" on a smooth surface.

Surface tension between the water/air interface and the glass acts as an invisible membrane that allows trapped water to rise above the level of the rim of the glass. The mechanism can be envisioned as forces that act in the surface of the liquid above the rim of the glass. Thus the water appears to defy gravity by attaining a level higher than the rim of the glass.

To experimentally demonstrate that this phenomenon is the result of surface tension, set the liquid level nearly as far above the glass rim as you can get it, using plain water. Add a drop of liquid detergent (the detergent contains additives that reduce the surface tension of water). Watch as the excess water runs over the side of the glass.

Problem 2.81

[Difficulty: 5]

2.81 Plan an experiment to measure the surface tension of a liquid similar to water. If necessary, review the NCFMF video *Surface Tension* for ideas. Which method would be most suitable for use in an undergraduate laboratory? What experimental precision could be expected?

Open-Ended Problem Statement: Plan an experiment to measure the surface tension of a liquid similar to water. If necessary, review the NCFMF video *Surface Tension* for ideas. Which method would be most suitable for use in an undergraduate laboratory? What experimental precision could be expected?

Discussion: Two basic kinds of experiment are possible for an undergraduate laboratory:

1. Using a clear small-diameter tube, compare the capillary rise of the unknown liquid with that of a known liquid (compare with water, because it is similar to the unknown liquid).

   This method would be simple to set up and should give fairly accurate results. A vertical traversing optical microscope could be used to increase the precision of measuring the liquid height in each tube.

   A drawback to this method is that the specific gravity and co ntact angle of the two liquids must be the same to allow the capillary rises to be compared.

   The capillary rise would be largest and therefore easiest to measure accurately in a tube with the smallest practical diameter. Tubes of several diameters could be used if desired.

2. Dip an object into a pool of test liquid and measure the vertical force required to pull the object from the liquid surface.

   The object might be made rectangular (e.g., a sheet of plastic material) or circular (e.g., a metal ring). The net force needed to pull the same object from each liquid should be proportional to the surface tension of each liquid.

   This method would be simple to set up. However, the force magnitudes to be measured would be quite small.

   A drawback to this method is that the contact angles of the two liquids must be the same.

The first method is probably best for undergraduate laboratory use. A quantitative estimate of experimental measurement uncertainty is impossible without knowing details of the test setup. It might be reasonable to expect results accurate to within ± 10% of the true surface tension.

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*Net force is the total vertical force minus the weight of the object. A buoyancy correction would be necessary if part of the object were submerged in the test liquid.

Problem 2.82

[Difficulty: 2]

2.82 Water usually is assumed to be incompressible when evaluating static pressure variations. Actually it is 100 times more compressible than steel. Assuming the bulk modulus of water is constant, compute the percentage change in density for water raised to a gage pressure of 100 atm. Plot the percentage change in water density as a function of p/p_{atm} up to a pressure of 50,000 psi, which is the approximate pressure used for high-speed cutting jets of water to cut concrete and other composite materials. Would constant density be a reasonable assumption for engineering calculations for cutting jets?

Solution: By definition, $E_v = \dfrac{dp}{d\rho/\rho}$. Assume E_v = constant. Then

$$\frac{d\rho}{\rho} = \frac{dp}{E_v}$$

Integrating, from ρ_0 to ρ gives $\ln\dfrac{\rho}{\rho_0} = \dfrac{p-p_0}{E_v} = \dfrac{\Delta p}{E_v}$, so $\dfrac{\rho}{\rho_0} = e^{\Delta p/E_v}$

The relative change in density is

$$\frac{\Delta\rho}{\rho_0} = \frac{\rho-\rho_0}{\rho_0} = \frac{\rho}{\rho_0} - 1 = e^{\Delta p/E_v} - 1$$

From Table A.2, $E_v = 2.24$ GPa for water at 20°C.

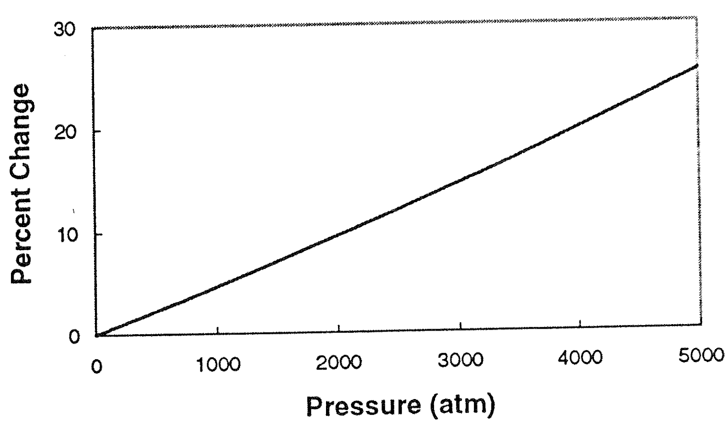
For $p = 100$ atm (gage), $\Delta p = 100$ atm, so

$$\frac{\Delta\rho}{\rho_0} = \exp\left(100\,\text{atm} \times \frac{1}{2.24\times10^9\,\text{Pa}} \times 101.325\times10^3\,\frac{\text{Pa}}{\text{atm}}\right) - 1 = 0.00453, \text{ or } 0.453\%$$

For $\Delta p = 50{,}000$ psi,

$$\frac{\Delta\rho}{\rho_0} = \exp\left(50{,}000\,\text{psi} \times \frac{1}{2.24\times10^9\,\text{Pa}} \times \frac{101.325\times10^3\,\text{Pa}}{14.696\,\text{psi}}\right) - 1 = 0.166 \text{ or } 16.6\%$$

Thus constant density is not a reasonable assumption for a cutting jet operating at 50,000 psi. Constant density (5% change) would be reasonable up to $\Delta p \approx 16{,}000$ psi.



Problem 2.83

[Difficulty: 2]

2.83 The viscous boundary layer velocity profile shown in Fig. 2.15 can be approximated by a parabolic equation,

$$u(y) = a + b\left(\frac{y}{\delta}\right) + c\left(\frac{y}{\delta}\right)^2$$

The boundary condition is $u = U$ (the free stream velocity) at the boundary edge δ (where the viscous friction becomes zero). Find the values of a, b, and c.

Given: Boundary layer velocity profile in terms of constants a, b and c

Find: Constants a, b and c

Solution:

Basic equation $$u = a + b\cdot\left(\frac{y}{\delta}\right) + c\cdot\left(\frac{y}{\delta}\right)^2$$

Assumptions: No slip, at outer edge u = U and τ = 0

At y = 0 $$0 = a \qquad\qquad a = 0$$

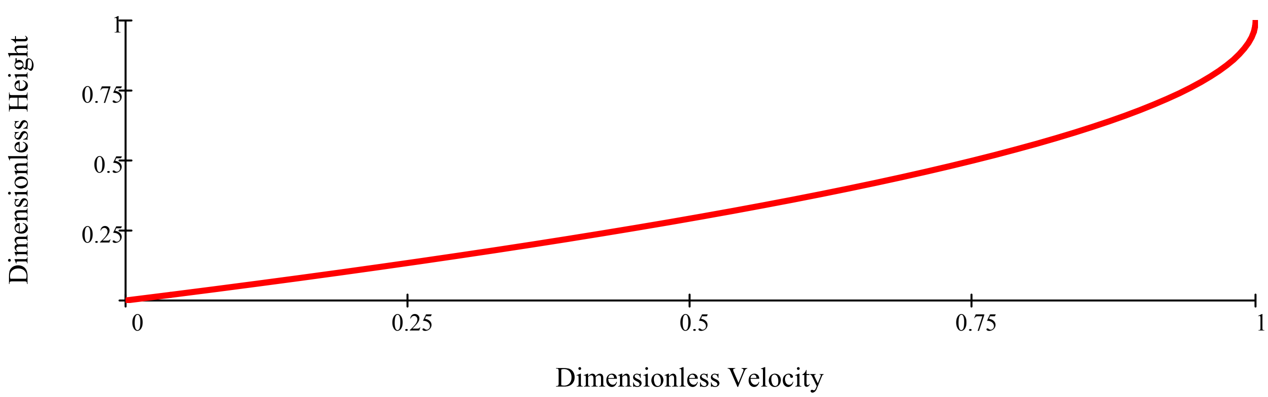
At y = δ $$U = a + b + c \qquad\qquad b + c = U \qquad (1)$$

At y = δ $$\tau = \mu\cdot\frac{du}{dy} = 0$$

$$0 = \frac{d}{dy}\left[a + b\cdot\left(\frac{y}{\delta}\right) + c\cdot\left(\frac{y}{\delta}\right)^2\right] = \frac{b}{\delta} + 2\cdot c\cdot\frac{y}{\delta^2} = \frac{b}{\delta} + 2\cdot\frac{c}{\delta} \qquad\qquad b + 2\cdot c = 0 \qquad (2)$$

From 1 and 2 $$c = -U \qquad\qquad b = 2\cdot U$$

Hence $$u = 2\cdot U\cdot\left(\frac{y}{\delta}\right) - U\cdot\left(\frac{y}{\delta}\right)^2 \qquad\qquad \frac{u}{U} = 2\cdot\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$$



Problem 2.84

[Difficulty: 2]

2.84 The viscous boundary layer velocity profile shown in Fig. 2.15 can be approximated by a cubic equation,

$$u(y) = a + b\left(\frac{y}{\delta}\right) + c\left(\frac{y}{\delta}\right)^3$$

The boundary condition is $u = U$ (the free stream velocity) at the boundary edge δ (where the viscous friction becomes zero). Find the values of a, b, and c.

Given: Boundary layer velocity profile in terms of constants a, b and c

Find: Constants a, b and c

Solution:

Basic equation $\quad u = a + b\cdot\left(\frac{y}{\delta}\right) + c\cdot\left(\frac{y}{\delta}\right)^3$

Assumptions: No slip, at outer edge u = U and τ = 0

At y = 0 $\quad 0 = a \qquad a = 0$

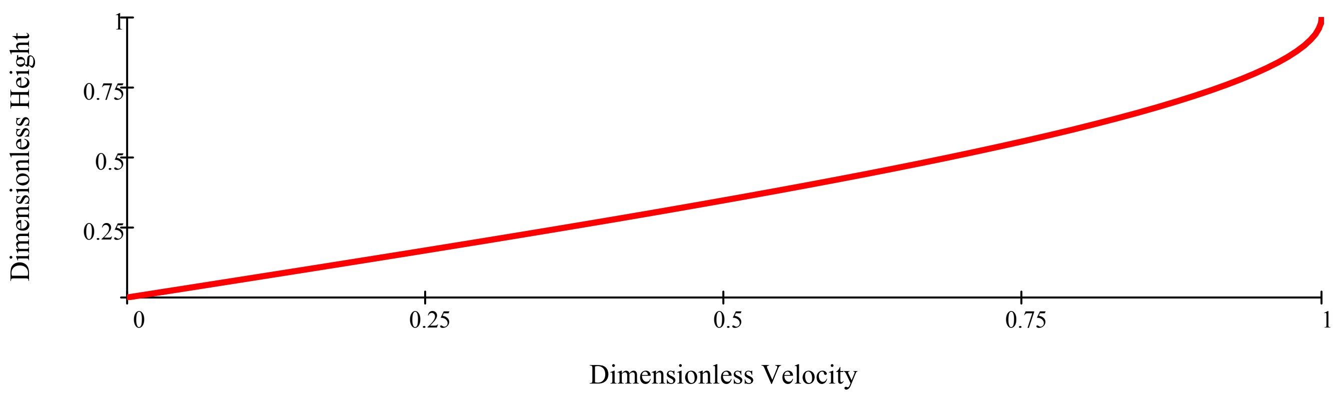
At y = δ $\quad U = a + b + c \qquad b + c = U \qquad (1)$

At y = δ $\quad \tau = \mu\cdot\frac{du}{dy} = 0$

$$0 = \frac{d}{dy}\left[a + b\cdot\left(\frac{y}{\delta}\right) + c\cdot\left(\frac{y}{\delta}\right)^3\right] = \frac{b}{\delta} + 3\cdot c\cdot\frac{y^2}{\delta^3} = \frac{b}{\delta} + 3\cdot\frac{c}{\delta} \qquad b + 3\cdot c = 0 \qquad (2)$$

From 1 and 2 $\quad c = -\frac{U}{2} \qquad b = \frac{3}{2}\cdot U$

Hence $\quad u = \frac{3\cdot U}{2}\cdot\left(\frac{y}{\delta}\right) - \frac{U}{2}\cdot\left(\frac{y}{\delta}\right)^3 \qquad \frac{u}{U} = \frac{3}{2}\cdot\left(\frac{y}{\delta}\right) - \frac{1}{2}\cdot\left(\frac{y}{\delta}\right)^3$



Problem 2.85

[Difficulty: 1]

2.85 At what minimum speed (in mph) would an automobile have to travel for compressibility effects to be important? Assume the local air temperature is 60°F.

Given: Local temperature

Find: Minimum speed for compressibility effects

Solution:

Basic equation $V = M \cdot c$ and $M = 0.3$ for compressibility effects

$c = \sqrt{k \cdot R \cdot T}$ For air at STP, k = 1.40 and R = 286.9J/kg.K (53.33 ft.lbf/lbm°R).

Hence $V = M \cdot c = M \cdot \sqrt{k \cdot R \cdot T}$

$$V = 0.3 \times \left[1.4 \times 53.33 \cdot \frac{ft \cdot lbf}{lbm \cdot R} \times \frac{32.2 \cdot lbm \cdot ft}{lbf \cdot s^2} \times (60 + 460) \cdot R\right]^{\frac{1}{2}} \cdot \frac{60 \cdot mph}{88 \cdot \frac{ft}{s}} \qquad V = 229 \cdot mph$$

Problem 2.86

[Difficulty: 3]

2.86 In a food industry process, carbon tetrachloride at 20°C flows through a tapered nozzle from an inlet diameter $D_{in} = 50$ mm to an outlet diameter of D_{out}. The area varies linearly with distance along the nozzle, and the exit area is one-fifth of the inlet area; the nozzle length is 250 mm. The flow rate is $Q = 2$ L/min. It is important for the process that the flow exits the nozzle as a turbulent flow. Does it? If so, at what point along the nozzle does the flow become turbulent?

Given: Geometry of and flow rate through tapered nozzle

Find: At which point becomes turbulent

Solution:

Basic equation — For pipe flow (Section 2-6) $\quad Re = \dfrac{\rho \cdot V \cdot D}{\mu} = 2300$ for transition to turbulence

Also flow rate Q is given by $\quad Q = \dfrac{\pi \cdot D^2}{4} \cdot V$

We can combine these equations and eliminate V to obtain an expression for Re in terms of D and Q

$$Re = \frac{\rho \cdot V \cdot D}{\mu} = \frac{\rho \cdot D}{\mu} \cdot \frac{4 \cdot Q}{\pi \cdot D^2} = \frac{4 \cdot Q \cdot \rho}{\pi \cdot \mu \cdot D} \qquad Re = \frac{4 \cdot Q \cdot \rho}{\pi \cdot \mu \cdot D}$$

For a given flow rate Q, as the diameter is reduced the Reynolds number increases (due to the velocity increasing with A^{-1} or D^{-2}).

Hence for turbulence (Re = 2300), solving for D $\quad D = \dfrac{4 \cdot Q \cdot \rho}{2300 \cdot \pi \cdot \mu}$

The nozzle is tapered: $\quad D_{in} = 50 \cdot mm \qquad D_{out} = \dfrac{D_{in}}{\sqrt{5}} \qquad D_{out} = 22.4 \cdot mm$

Carbon tetrachloride: $\quad \mu_{CT} = 10^{-3} \cdot \dfrac{N \cdot s}{m^2}$ (Fig A.2) $\qquad$ For water $\quad \rho = 1000 \cdot \dfrac{kg}{m^3}$

$SG = 1.595$ (Table A.2) $\qquad \rho_{CT} = SG \cdot \rho \qquad \rho_{CT} = 1595 \dfrac{kg}{m^3}$

For the given flow rate $\quad Q = 2 \cdot \dfrac{L}{min} \qquad \dfrac{4 \cdot Q \cdot \rho_{CT}}{\pi \cdot \mu_{CT} \cdot D_{in}} = 1354$ LAMINAR $\qquad \dfrac{4 \cdot Q \cdot \rho_{CT}}{\pi \cdot \mu_{CT} \cdot D_{out}} = 3027$ TURBULENT

For the diameter at which we reach turbulence $\quad D = \dfrac{4 \cdot Q \cdot \rho_{CT}}{2300 \cdot \pi \cdot \mu_{CT}} \qquad D = 29.4 \cdot mm$

But $\quad L = 250 \cdot mm$ and linear ratios leads to the distance from D_{in} at which $D = 29.4 \cdot mm$ $\qquad \dfrac{L_{turb}}{L} = \dfrac{D - D_{in}}{D_{out} - D_{in}}$

$$L_{turb} = L \cdot \frac{D - D_{in}}{D_{out} - D_{in}} \qquad L_{turb} = 186 \cdot mm$$

Problem 2.87

[Difficulty: 2]

2.87 What is the Reynolds number of water at 20°C flowing at 0.25 m/s through a 5-mm-diameter tube? If the pipe is now heated, at what mean water temperature will the flow transition to turbulence? Assume the velocity of the flow remains constant.

Given: Data on water tube

Find: Reynolds number of flow; Temperature at which flow becomes turbulent

Solution:

Basic equation For pipe flow (Section 2-6) $\quad Re = \frac{\rho \cdot V \cdot D}{\mu} = \frac{V \cdot D}{\nu}$

At 20°C, from Fig. A.3 $\quad \nu = 9 \times 10^{-7} \cdot \frac{m^2}{s}$ and so $\quad Re = 0.25 \cdot \frac{m}{s} \times 0.005 \cdot m \times \frac{1}{9 \times 10^{-7}} \cdot \frac{s}{m^2} \quad Re = 1389$

For the heated pipe $\quad Re = \frac{V \cdot D}{\nu} = 2300$ for transition to turbulence

Hence $\quad \nu = \frac{V \cdot D}{2300} = \frac{1}{2300} \times 0.25 \cdot \frac{m}{s} \times 0.005 \cdot m \quad \nu = 5.435 \times 10^{-7} \frac{m^2}{s}$

From Fig. A.3, the temperature of water at this viscosity is approximately $\quad T = 52 \cdot C$

Problem 2.88

[Difficulty: 3]

2.88 A supersonic aircraft travels at 2700 km/hr at an altitude of 27 km. What is the Mach number of the aircraft? At what approximate distance measured from the leading edge of the aircraft's wing does the boundary layer change from laminar to turbulent?

Given: Data on supersonic aircraft

Find: Mach number; Point at which boundary layer becomes turbulent

Solution:

Basic equation $V = M \cdot c$ and $c = \sqrt{k \cdot R \cdot T}$ For air at STP, k = 1.40 and R = 286.9J/kg.K (53.33 ft.lbf/lbm°R).

Hence $M = \frac{V}{c} = \frac{V}{\sqrt{k \cdot R \cdot T}}$

At 27 km the temperature is approximately (from Table A.3) $T = 223.5 \cdot K$

$$M = \left(2700 \times 10^3 \cdot \frac{m}{hr} \times \frac{1 \cdot hr}{3600 \cdot s}\right) \cdot \left(\frac{1}{1.4} \times \frac{1}{286.9} \cdot \frac{kg \cdot K}{N \cdot m} \times \frac{1 \cdot N \cdot s^2}{kg \cdot m} \times \frac{1}{223.5} \cdot \frac{1}{K}\right)^{\frac{1}{2}} \qquad M = 2.5$$

For boundary layer transition, from Section 2-6 $Re_{trans} = 500000$

Then $Re_{trans} = \frac{\rho \cdot V \cdot x_{trans}}{\mu}$ so $x_{trans} = \frac{\mu \cdot Re_{trans}}{\rho \cdot V}$

We need to find the viscosity and density at this altitude and pressure. The viscosity depends on temperature only, but at 223.5 K = - 50°C, it is off scale of Fig. A.3. Instead we need to use formulas as in Appendix A

At this altitude the density is (Table A.3) $\rho = 0.02422 \times 1.225 \frac{kg}{m^3}$ $\qquad \rho = 0.0297 \frac{kg}{m^3}$

For μ $\qquad \mu = \frac{b \cdot T^{\frac{1}{2}}}{1 + \frac{S}{T}}$ where $b = 1.458 \times 10^{-6} \cdot \frac{kg}{m \cdot s \cdot K^{\frac{1}{2}}}$ $\qquad S = 110.4 \cdot K$

$$\mu = 1.459 \times 10^{-5} \frac{kg}{m \cdot s} \qquad \mu = 1.459 \times 10^{-5} \cdot \frac{N \cdot s}{m^2}$$

Hence $x_{trans} = 1.459 \times 10^{-5} \cdot \frac{kg}{m \cdot s} \times 500000 \times \frac{1}{0.0297} \cdot \frac{m^3}{kg} \times \frac{1}{2700} \times \frac{1}{10^3} \cdot \frac{hr}{m} \times \frac{3600 \cdot s}{1 \cdot hr}$ $\qquad x_{trans} = 0.327m$

Problem 2.89

[Difficulty: 2]

2.89 SAE 30 oil at 100°C flows through a 12-mm-diameter stainless-steel tube. What is the specific gravity and specific weight of the oil? If the oil discharged from the tube fills a 100-mL graduated cylinder in 9 seconds, is the flow laminar or turbulent?

Given: Type of oil, flow rate, and tube geometry

Find: Whether flow is laminar or turbulent

Solution:

Data on SAE 30 oil SG or density is limited in the Appendix. We can Google it or use the following $\nu = \frac{\mu}{\rho}$ so $\rho = \frac{\mu}{\nu}$

At 100°C, from Figs. A.2 and A.3 $\mu = 9 \times 10^{-3} \cdot \frac{N \cdot s}{m^2}$ $\nu = 1 \times 10^{-5} \cdot \frac{m^2}{s}$

$$\rho = 9 \times 10^{-3} \cdot \frac{N \cdot s}{m^2} \times \frac{1}{1 \times 10^{-5}} \cdot \frac{s}{m^2} \times \frac{kg \cdot m}{s^2 \cdot N} \qquad \rho = 900 \frac{kg}{m^3}$$

Hence $SG = \frac{\rho}{\rho_{water}}$ $\rho_{water} = 1000 \cdot \frac{kg}{m^3}$ $SG = 0.9$

The specific weight is $\gamma = \rho \cdot g$ $\gamma = 900 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times \frac{N \cdot s^2}{kg \cdot m}$ $\gamma = 8.829 \times 10^3 \cdot \frac{N}{m^3}$

For pipe flow (Section 2-6) $Q = \frac{\pi \cdot D^2}{4} \cdot V$ so $V = \frac{4 \cdot Q}{\pi \cdot D^2}$

$$Q = 100 \cdot mL \times \frac{10^{-6} \cdot m^3}{1 \cdot mL} \times \frac{1}{9} \cdot \frac{1}{s} \qquad Q = 1.111 \times 10^{-5} \frac{m^3}{s}$$

Then

$$V = \frac{4}{\pi} \times 1.11 \times 10^{-5} \cdot \frac{m^3}{s} \times \left(\frac{1}{12} \cdot \frac{1}{mm} \times \frac{1000 \cdot mm}{1 \cdot m} \right)^2 \qquad V = 0.0981 \frac{m}{s}$$

Hence $Re = \frac{\rho \cdot V \cdot D}{\mu}$

$$Re = 900 \cdot \frac{kg}{m^3} \times 0.0981 \cdot \frac{m}{s} \times 0.012 \cdot m \times \frac{1}{9 \times 10^{-3}} \cdot \frac{m^2}{N \cdot s} \times \frac{N \cdot s^2}{kg \cdot m} \qquad Re = 118$$

Flow is laminar

Problem 2.90

[Difficulty: 2]

2.90 A seaplane is flying at 100 mph through air at 45°F. At what distance from the leading edge of the underside of the fuselage does the boundary layer transition to turbulence? How does this boundary layer transition change as the underside of the fuselage touches the water during landing? Assume the water temperature is also 45°F.

Given: Data on seaplane

Find: Transition point of boundary layer

Solution:

For boundary layer transition, from Section 2-6 $\quad Re_{trans} = 500000$

Then $\quad Re_{trans} = \dfrac{\rho \cdot V \cdot x_{trans}}{\mu} = \dfrac{V \cdot x_{trans}}{\nu} \quad$ so $\quad x_{trans} = \dfrac{\nu \cdot Re_{trans}}{V}$

At 45°F = 7.2°C (Fig A.3) $\quad \nu = 0.8 \times 10^{-5} \cdot \dfrac{m^2}{s} \times \dfrac{10.8 \cdot \dfrac{ft^2}{s}}{1 \cdot \dfrac{m^2}{s}} \qquad \nu = 8.64 \times 10^{-5} \cdot \dfrac{ft^2}{s}$

$$x_{trans} = 8.64 \times 10^{-5} \cdot \frac{ft^2}{s} \cdot 500000 \times \frac{1}{100 \cdot mph} \times \frac{60 \cdot mph}{88 \cdot \frac{ft}{s}} \qquad x_{trans} = 0.295 \cdot ft$$

As the seaplane touches down:

At 45°F = 7.2°C (Fig A.3) $\quad \nu = 1.5 \times 10^{-5} \cdot \dfrac{m^2}{s} \times \dfrac{10.8 \cdot \dfrac{ft^2}{s}}{1 \cdot \dfrac{m^2}{s}} \qquad \nu = 1.62 \times 10^{-4} \cdot \dfrac{ft^2}{s}$

$$x_{trans} = 1.62 \times 10^{-4} \cdot \frac{ft^2}{s} \cdot 500000 \times \frac{1}{100 \cdot mph} \times \frac{60 \cdot mph}{88 \cdot \frac{ft}{s}} \qquad x_{trans} = 0.552 \cdot ft$$

Problem 2.91

[Difficulty: 3]

2.91 An airliner is cruising at an altitude of 5.5 km with a speed of 700 km/hr. As the airliner increases its altitude, it adjusts its speed so that the Mach number remains constant. Provide a sketch of speed vs. altitude. What is the speed of the airliner at an altitude of 8 km?

Given: Data on airliner

Find: Sketch of speed versus altitude (M = const)

Solution:

Data on temperature versus height can be obtained from Table A.3

At 5.5 km the temperature is approximately 252 K

The speed of sound is obtained from $c = \sqrt{k \cdot R \cdot T}$

where $k = 1.4$

$R = 286.9$ J/kg·K (Table A.6)

$c = 318$ m/s

We also have

$V = 700$ km/hr

or $V = 194$ m/s

Hence $M = V/c$ or

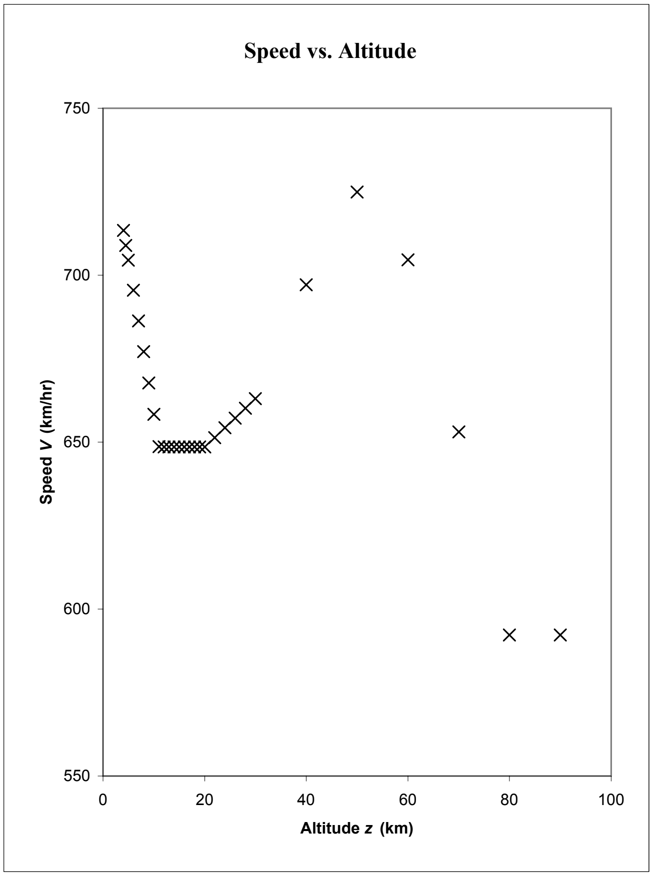
$M = 0.611$

To compute V for constant M, we use $V = M \cdot c = 0.611\ c$

At a height of 8 km: $V = 677$ km/hr

NOTE: Realistically, the aiplane will fly to a maximum height of about 10 km!

z (km)	T (K)	c (m/s)	V (km/hr)
4	262	325	713
5	259	322	709
5	256	320	704
6	249	316	695
7	243	312	686
8	236	308	677
9	230	304	668
10	223	299	658
11	217	295	649
12	217	295	649
13	217	295	649
14	217	295	649
15	217	295	649
16	217	295	649
17	217	295	649
18	217	295	649
19	217	295	649
20	217	295	649
22	219	296	651
24	221	298	654
26	223	299	657
28	225	300	660
30	227	302	663
40	250	317	697
50	271	330	725
60	256	321	705
70	220	297	653
80	181	269	592
90	181	269	592



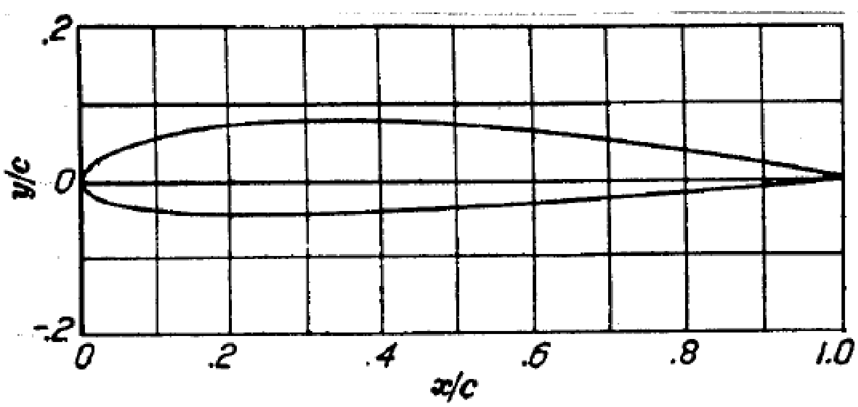
Problem 2.92

[Difficulty: 4]

2.92 How does an airplane wing develop lift?

Open-Ended Problem Statement: How does an airplane wing develop lift?

Discussion: The sketch shows the cross-section of a typical airplane wing. The airfoil section is rounded at the front, curved across the top, reaches maximum thickness about a third of the way back, and then tapers slowly to a fine trailing edge. The bottom of the airfoil section is relatively flat. (The discussion below also applies to a symmetric airfoil at an angle of incidence that produces lift.)



NACA 2412 Wing Section

It is both a popular expectation and an experimental fact that air flows more rapidly over the curved top surface of the airfoil section than along the relatively flat bottom. In the NCFMF video *Flow Visualization*, timelines placed in front of the airfoil indicate that fluid flows more rapidly along the top of the section than along the bottom.

In the absence of viscous effects (this is a valid assumption outside the boundary layers on the airfoil) pressure falls when flow speed increases. Thus the pressures on the top surface of the airfoil where flow speed is higher are lower than the pressures on the bottom surface where flow speed does not increase. (Actual pressure profiles measured for a lifting section are shown in the NCFMF video *Boundary Layer Control*.) The unbalanced pressures on the top and bottom surfaces of the airfoil section create a net force that tends to develop lift on the profile.

Problem 3.1

[Difficulty: 2]

3.1 Compressed nitrogen (140 lbm) is stored in a spherical tank of diameter $D = 2.5$ ft at a temperature of 77°F. What is the pressure inside the tank? If the maximum allowable stress in the tank is 30 ksi, find the minimum theoretical wall thickness of the tank.

Given:
Data on nitrogen tank

Find:
Pressure of nitrogen; minimum required wall thickness

Assumption:
Ideal gas behavior

Solution:

Ideal gas equation of state:
$$p \cdot V = M \cdot R \cdot T$$

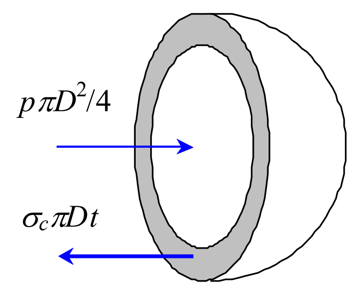
where, from Table A.6, for nitrogen
$$R = 55.16 \cdot \frac{\text{ft} \cdot \text{lbf}}{\text{lbm} \cdot \text{R}}$$

Then the pressure of nitrogen is
$$p = \frac{M \cdot R \cdot T}{V} = M \cdot R \cdot T \cdot \left(\frac{6}{\pi \cdot D^3}\right)$$

$$p = 140 \cdot \text{lbm} \times 55.16 \cdot \frac{\text{ft} \cdot \text{lbf}}{\text{lbm} \cdot \text{R}} \times (77 + 460) \cdot \text{R} \times \left[\frac{6}{\pi \times (2.5 \cdot \text{ft})^3}\right] \times \left(\frac{\text{ft}}{12 \cdot \text{in}}\right)^2$$

$$p = 3520 \cdot \frac{\text{lbf}}{\text{in}^2}$$

To determine wall thickness, consider a free body diagram for one hemisphere:



$$\Sigma F = 0 = p \cdot \frac{\pi \cdot D^2}{4} - \sigma_c \cdot \pi \cdot D \cdot t$$

where σ_c is the circumferential stress in the container

Then
$$t = \frac{p \cdot \pi \cdot D^2}{4 \cdot \pi \cdot D \cdot \sigma_c} = \frac{p \cdot D}{4 \cdot \sigma_c}$$

$$t = 3520 \cdot \frac{\text{lbf}}{\text{in}^2} \times \frac{2.5 \cdot \text{ft}}{4} \times \frac{\text{in}^2}{30 \times 10^3 \cdot \text{lbf}}$$

$$t = 0.0733 \cdot \text{ft} \qquad t = 0.880 \cdot \text{in}$$

Problem 3.2

[Difficulty: 2]

3.2 Because the pressure falls, water boils at a lower temperature with increasing altitude. Consequently, cake mixes and boiled eggs, among other foods, must be cooked different lengths of time. Determine the boiling temperature of water at 1000 and 2000 m elevation on a standard day, and compare with the sea-level value.

Given: Pure water on a standard day

Find: Boiling temperature at (a) 1000 m and (b) 2000 m, and compare with sea level value.

Solution:

We can determine the atmospheric pressure at the given altitudes from table A.3, Appendix A

The data are

Elevation (m)	p/p_o	p (kPa)
0	1.000	101.3
1000	0.887	89.9
2000	0.785	79.5

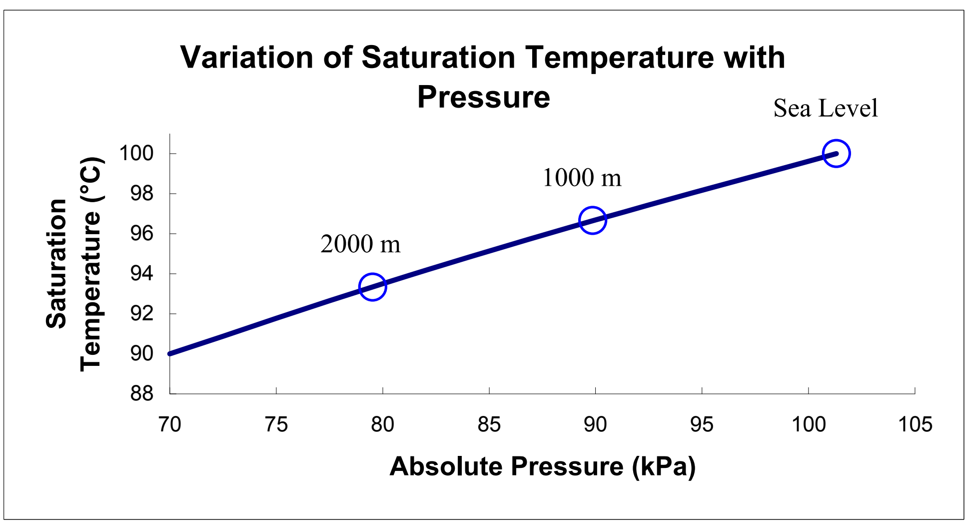
We can also consult steam tables for the variation of saturation temperature with pressure:

p (kPa)	T_{sat} (°C)
70	90.0
80	93.5
90	96.7
101.3	100.0

We can interpolate the data from the steam tables to correlate saturation temperature with altitude:

Elevation (m)	p/p_o	p (kPa)	T_{sat} (°C)
0	1.000	101.3	100.0
1000	0.887	89.9	96.7
2000	0.785	79.5	93.3

The data are plotted here. They show that the saturation temperature drops approximately 3.4°C/1000 m.



Problem 3.3

[Difficulty: 2]

3.3 Ear "popping" is an unpleasant phenomenon sometimes experienced when a change in pressure occurs, for example in a fast-moving elevator or in an airplane. If you are in a two-seater airplane at 3000 m and a descent of 100 m causes your ears to "pop," what is the pressure change that your ears "pop" at, in millimeters of mercury? If the airplane now rises to 8000 m and again begins descending, how far will the airplane descend before your ears "pop" again? Assume a U.S. Standard Atmosphere.

Given:

Data on flight of airplane

Find:

Pressure change in mm Hg for ears to "pop"; descent distance from 8000 m to cause ears to "pop."

Solution:

Assume the air density is approximately constant constant from 3000 m to 2900 m.
From table A.3

$$\rho_{SL} = 1.225 \cdot \frac{kg}{m^3} \qquad \rho_{air} = 0.7423 \cdot \rho_{SL} \qquad \rho_{air} = 0.909 \frac{kg}{m^3}$$

We also have from the manometer equation, Eq. 3.7

$$\Delta p = -\rho_{air} \cdot g \cdot \Delta z \qquad \text{and also} \qquad \Delta p = -\rho_{Hg} \cdot g \cdot \Delta h_{Hg}$$

Combining

$$\Delta h_{Hg} = \frac{\rho_{air}}{\rho_{Hg}} \cdot \Delta z = \frac{\rho_{air}}{SG_{Hg} \cdot \rho_{H2O}} \cdot \Delta z \qquad SG_{Hg} = 13.55 \text{ from Table A.2}$$

$$\Delta h_{Hg} = \frac{0.909}{13.55 \times 999} \times 100 \cdot m \qquad \Delta h_{Hg} = 6.72 \cdot mm$$

For the ear popping descending from 8000 m, again assume the air density is approximately constant constant, this time at 8000 m.
From table A.3

$$\rho_{air} = 0.4292 \cdot \rho_{SL} \qquad \rho_{air} = 0.526 \frac{kg}{m^3}$$

We also have from the manometer equation

$$\rho_{air8000} \cdot g \cdot \Delta z_{8000} = \rho_{air3000} \cdot g \cdot \Delta z_{3000}$$

where the numerical subscripts refer to conditions at 3000m and 8000m.
Hence

$$\Delta z_{8000} = \frac{\rho_{air3000} \cdot g}{\rho_{air8000} \cdot g} \cdot \Delta z_{3000} = \frac{\rho_{air3000}}{\rho_{air8000}} \cdot \Delta z_{3000} \qquad \Delta z_{8000} = \frac{0.909}{0.526} \times 100 \cdot m \qquad \Delta z_{8000} = 173\,m$$

Problem 3.4

[Difficulty: 3]

3.4 When you are on a mountain face and boil water, you notice that the water temperature is 195°F. What is your approximate altitude? The next day, you are at a location where it boils at 185°F. How high did you climb between the two days? Assume a U.S. Standard Atmosphere.

Given: Boiling points of water at different elevations

Find: Change in elevation

Solution:

From the steam tables, we have the following data for the boiling point (saturation temperature) of water

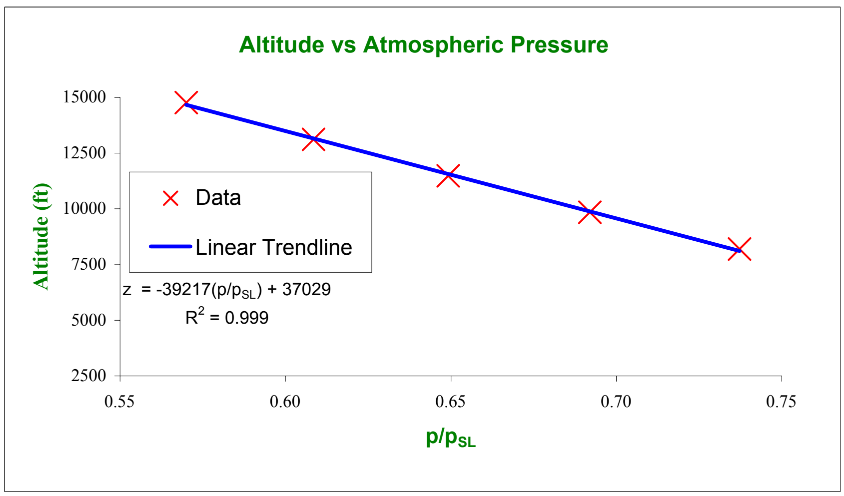
T_{sat} (°F)	p (psia)
195	10.39
185	8.39

The sea level pressure, from Table A.3, is

p_{SL} = 14.696 psia

Hence

T_{sat} (°F)	p/p_{SL}
195	0.707
185	0.571



From Table A.3

p/p_{SL}	Altitude (m)	Altitude (ft)
0.7372	2500	8203
0.6920	3000	9843
0.6492	3500	11484
0.6085	4000	13124
0.5700	4500	14765

Then, any one of a number of *Excel* functions can be used to interpolate
(Here we use *Excel*'s *Trendline* analysis)

p/p_{SL}	Altitude (ft)
0.707	9303
0.571	14640

Current altitude is approximately 9303 ft

The change in altitude is then 5337 ft

Alternatively, we can interpolate for each altitude by using a linear regression between adjacent data points

For

p/p_{SL}	Altitude (m)	Altitude (ft)
0.7372	2500	8203
0.6920	3000	9843

p/p_{SL}	Altitude (m)	Altitude (ft)
0.6085	4000	13124
0.5700	4500	14765

Then

0.7070	2834	9299

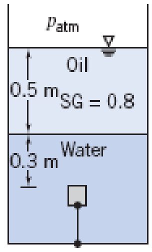
0.5730	4461	14637

The change in altitude is then 5338 ft

Problem 3.5

[Difficulty: 2]

3.5 A 125-mL cube of solid oak is held submerged by a tether as shown. Calculate the actual force of the water on the bottom surface of the cube and the tension in the tether.



Given:

Data on system

Find:

Force on bottom of cube; tension in tether

Solution:

Basic equation $\quad \dfrac{dp}{dy} = -\rho \cdot g \quad$ or, for constant ρ $\quad \Delta p = \rho \cdot g \cdot h \quad$ where h is measured downwards

The absolute pressure at the interface is $\quad p_{interface} = p_{atm} + SG_{oil} \cdot \rho \cdot g \cdot h_{oil}$

Then the pressure on the lower surface is $\quad p_L = p_{interface} + \rho \cdot g \cdot h_L = p_{atm} + \rho \cdot g \cdot \left(SG_{oil} \cdot h_{oil} + h_L\right)$

For the cube $\quad V = 125 \cdot mL \qquad V = 1.25 \times 10^{-4} \cdot m^3$

Then the size of the cube is $\quad d = V^{\frac{1}{3}} \qquad d = 0.05\,m \quad$ and the depth in water to the upper surface is $h_U = 0.3 \cdot m$

Hence $\quad h_L = h_U + d \qquad h_L = 0.35\,m \quad$ where h_L is the depth in water to the lower surface

The force on the lower surface is $\quad F_L = p_L \cdot A \quad$ where $\quad A = d^2 \qquad A = 0.0025\,m^2$

$$F_L = \left[p_{atm} + \rho \cdot g \cdot \left(SG_{oil} \cdot h_{oil} + h_L\right)\right] \cdot A$$

$$F_L = \left[101 \times 10^3 \cdot \frac{N}{m^2} + 1000 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times (0.8 \times 0.5 \cdot m + 0.35 \cdot m) \times \frac{N \cdot s^2}{kg \cdot m}\right] \times 0.0025 \cdot m^2$$

$$F_L = 270.894\,N$$

Note: Extra decimals needed for computing T later!

For the tension in the tether, an FBD gives $\quad \Sigma F_y = 0 \qquad F_L - F_U - W - T = 0 \quad$ or $\quad T = F_L - F_U - W$

where $\quad F_U = \left[p_{atm} + \rho \cdot g \cdot \left(SG_{oil} \cdot h_{oil} + h_U\right)\right] \cdot A$

Note that we could instead compute $\Delta F = F_L - F_U = \rho \cdot g \cdot SG_{oil} \cdot \left(h_L - h_U\right) \cdot A$ and $T = \Delta F - W$

Using F_U

$$F_U = \left[101 \times 10^3 \cdot \frac{N}{m^2} + 1000 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times (0.8 \times 0.5 \cdot m + 0.3 \cdot m) \times \frac{N \cdot s^2}{kg \cdot m}\right] \times 0.0025 \cdot m^2$$

$F_U = 269.668\,N$ Note: Extra decimals needed for computing T later!

For the oak block (Table A.1) $SG_{oak} = 0.77$ so $W = SG_{oak} \cdot \rho \cdot g \cdot V$

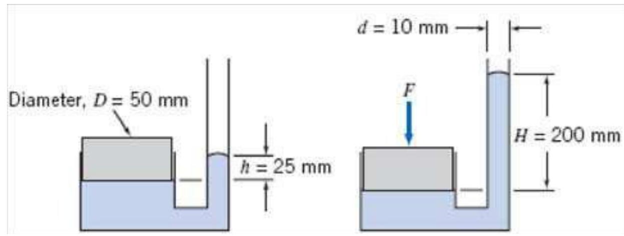
$$W = 0.77 \times 1000 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times 1.25 \times 10^{-4} \cdot m^3 \times \frac{N \cdot s^2}{kg \cdot m} \qquad W = 0.944\,N$$

$T = F_L - F_U - W$ $T = 0.282\,N$

Problem 3.6

[Difficulty: 2]

3.6 The tube shown is filled with mercury at 20°C. Calculate the force applied to the piston.



Given: Data on system before and after applied force

Find: Applied force

Solution:

Basic equation $\quad \dfrac{dp}{dy} = -\rho \cdot g \quad$ or, for constant $\rho \quad p = p_{atm} - \rho \cdot g \cdot (y - y_0) \quad$ with $\quad p(y_0) = p_{atm}$

For initial state $\quad p_1 = p_{atm} + \rho \cdot g \cdot h \quad$ and $\quad F_1 = p_1 \cdot A = \rho \cdot g \cdot h \cdot A \quad$ (Gage; F_1 is hydrostatic upwards force)

For the initial FBD $\quad \Sigma F_y = 0 \quad F_1 - W = 0 \quad W = F_1 = \rho \cdot g \cdot h \cdot A$

For final state $\quad p_2 = p_{atm} + \rho \cdot g \cdot H \quad$ and $\quad F_2 = p_2 \cdot A = \rho \cdot g \cdot H \cdot A \quad$ (Gage; F_2 is hydrostatic upwards force)

For the final FBD $\quad \Sigma F_y = 0 \quad F_2 - W - F = 0 \quad F = F_2 - W = \rho \cdot g \cdot H \cdot A - \rho \cdot g \cdot h \cdot A = \rho \cdot g \cdot A \cdot (H - h)$

$$F = \rho_{H2O} \cdot SG \cdot g \cdot \frac{\pi \cdot D^2}{4} \cdot (H - h)$$

From Fig. A.1 $\quad SG = 13.54$

$$F = 1000 \cdot \frac{kg}{m^3} \times 13.54 \times 9.81 \cdot \frac{m}{s^2} \times \frac{\pi}{4} \times (0.05 \cdot m)^2 \times (0.2 - 0.025) \cdot m \times \frac{N \cdot s^2}{kg \cdot m}$$

$$F = 45.6\,N$$

Problem 3.7 [Difficulty: 1]

3.7 The following pressure and temperature measurements were taken by a meteorological balloon rising through the lower atmosphere:

p (psia)	14.71	14.62	14.53	14.45	14.36	14.27	14.18	14.1	14.01	13.92	13.84
T (°F)	53.6	52	50.9	50.4	50.2	50	50.5	51.4	52.9	54	53.8

The initial values (top of table) correspond to ground level. Using the ideal gas law ($p = \rho RT$ with $R = 53.3$ ft·lbf/lbm·°R), compute and plot the variation of air density (in lbm/ft^3) with height.

Given: Pressure and temperature data from balloon

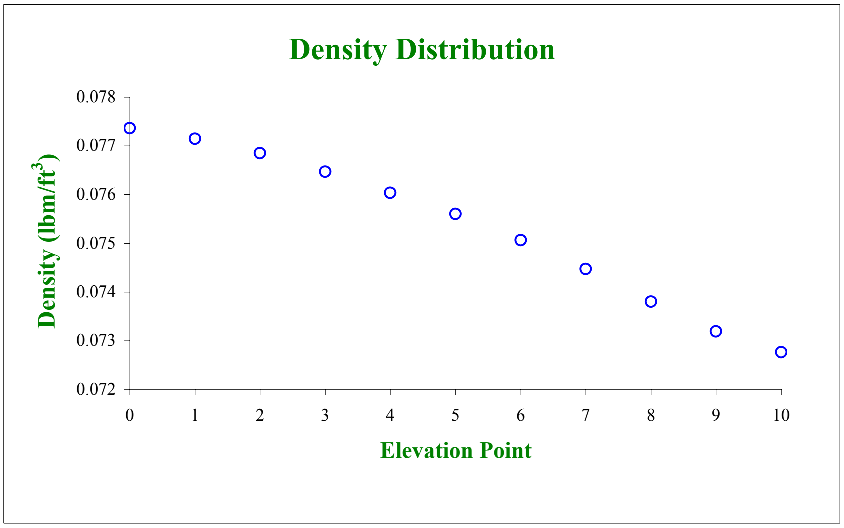
Find: Plot density change as a function of elevation

Assumption: Ideal gas behavior

Solution:

Using the ideal gas equation, $\rho = p/RT$

p (psia)	T (°F)	ρ (lbm/ft^3)
14.71	53.6	0.07736
14.62	52.0	0.07715
14.53	50.9	0.07685
14.45	50.4	0.07647
14.36	50.2	0.07604
14.27	50.0	0.07560
14.18	50.5	0.07506
14.10	51.4	0.07447
14.01	52.9	0.07380
13.92	54.0	0.07319
13.84	53.8	0.07276



Problem 3.8

[Difficulty: 2]

3.8 A hollow metal cube with sides 100 mm floats at the interface between a layer of water and a layer of SAE 10W oil such that 10% of the cube is exposed to the oil. What is the pressure difference between the upper and lower horizontal surfaces? What is the average density of the cube?

Given: Properties of a cube floating at an interface

Find: The pressures difference between the upper and lower surfaces; average cube density

Solution:

The pressure difference is obtained from two applications of Eq. 3.7

$$p_U = p_0 + \rho_{SAE10} \cdot g \cdot (H - 0.1 \cdot d) \qquad p_L = p_0 + \rho_{SAE10} \cdot g \cdot H + \rho_{H2O} \cdot g \cdot 0.9 \cdot d$$

where p_U and p_L are the upper and lower pressures, p_0 is the oil free surface pressure, H is the depth of the interface, and d is the cube size

Hence the pressure difference is

$$\Delta p = p_L - p_U = \rho_{H2O} \cdot g \cdot 0.9 \cdot d + \rho_{SAE10} \cdot g \cdot 0.1 \cdot d \qquad \Delta p = \rho_{H2O} \cdot g \cdot d \cdot \left(0.9 + SG_{SAE10} \cdot 0.1\right)$$

From Table A.2 $\quad SG_{SAE10} = 0.92$

$$\Delta p = 999 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times 0.1 \cdot m \times (0.9 + 0.92 \times 0.1) \times \frac{N \cdot s^2}{kg \cdot m} \qquad \Delta p = 972\,Pa$$

For the cube density, set up a free body force balance for the cube

$$\Sigma F = 0 = \Delta p \cdot A - W$$

Hence $\quad W = \Delta p \cdot A = \Delta p \cdot d^2$

$$\rho_{cube} = \frac{m}{d^3} = \frac{W}{d^3 \cdot g} = \frac{\Delta p \cdot d^2}{d^3 \cdot g} = \frac{\Delta p}{d \cdot g}$$

$$\rho_{cube} = 972 \cdot \frac{N}{m^2} \times \frac{1}{0.1 \cdot m} \times \frac{s^2}{9.81 \cdot m} \times \frac{kg \cdot m}{N \cdot s^2} \qquad \rho_{cube} = 991 \frac{kg}{m^3}$$

Problem 3.9

[Difficulty: 2]

3.9 Your pressure gage indicates that the pressure in your cold tires is 0.25 MPa (gage) on a mountain at an elevation of 3500 m. What is the absolute pressure? After you drive down to sea level, your tires have warmed to 25°C. What pressure does your gage now indicate? Assume a U.S. Standard Atmosphere.

Given: Data on tire at 3500 m and at sea level

Find: Absolute pressure at 3500 m; pressure at sea level

Solution:

At an elevation of 3500 m, from Table A.3:

$$p_{SL} = 101 \cdot kPa \qquad p_{atm} = 0.6492 \cdot p_{SL} \qquad p_{atm} = 65.6 \cdot kPa$$

and we have $\quad p_g = 0.25 \cdot MPa \qquad p_g = 250 \cdot kPa \qquad p = p_g + p_{atm} \qquad p = 316 \cdot kPa$

At sea level $\quad p_{atm} = 101 \cdot kPa$

Meanwhile, the tire has warmed up, from the ambient temperature at 3500 m, to 25°C.

At an elevation of 3500 m, from Table A.3 $\quad T_{cold} = 265.4 \cdot K \quad$ and $\quad T_{hot} = (25 + 273) \cdot K \qquad T_{hot} = 298\,K$

Hence, assuming ideal gas behavior, $pV = mRT$, and that the tire is approximately a rigid container, the absolute pressure of the hot tire is

$$p_{hot} = \frac{T_{hot}}{T_{cold}} \cdot p \qquad p_{hot} = 354 \cdot kPa$$

Then the gage pressure is

$$p_g = p_{hot} - p_{atm} \qquad p_g = 253 \cdot kPa$$

Problem 3.10

[Difficulty: 2]

3.10 An air bubble, 0.3 in. in diameter, is released from the regulator of a scuba diver swimming 100 ft below the sea surface. (The water temperature is 86°F.) Estimate the diameter of the bubble just before it reaches the water surface.

Given: Data on air bubble

Find: Bubble diameter as it reaches surface

Solution:

Basic equation $\frac{dp}{dy} = -\rho_{sea} \cdot g$ and the ideal gas equation $p = \rho \cdot R \cdot T = \frac{M}{V} \cdot R \cdot T$

We assume the temperature is constant, and the density of sea water is constant

For constant sea water density $p = p_{atm} + SG_{sea} \cdot \rho \cdot g \cdot h$ where p is the pressure at any depth h

Then the pressure at the initial depth is $p_1 = p_{atm} + SG_{sea} \cdot \rho \cdot g \cdot h_1$

The pressure as it reaches the surface is $p_2 = p_{atm}$

For the bubble $p = \frac{M \cdot R \cdot T}{V}$ but M and T are constant $M \cdot R \cdot T = const = p \cdot V$

Hence $p_1 \cdot V_1 = p_2 \cdot V_2$ or $V_2 = V_1 \cdot \frac{P_1}{p_2}$ or $D_2^{\ 3} = D_1^{\ 3} \cdot \frac{p_1}{p_2}$

Then the size of the bubble at the surface is $D_2 = D_1 \cdot \left(\frac{p_1}{p_2}\right)^{\frac{1}{3}} = D_1 \cdot \left[\frac{\left(p_{atm} + \rho_{sea} \cdot g \cdot h_1\right)}{p_{atm}}\right]^{\frac{1}{3}} = D_1 \cdot \left(1 + \frac{\rho_{sea} \cdot g \cdot h_1}{p_{atm}}\right)^{\frac{1}{3}}$

From Table A.2 $SG_{sea} = 1.025$ (This is at 68°F)

$$D_2 = 0.3 \cdot in \times \left[1 + 1.025 \times 1.94 \cdot \frac{slug}{ft^3} \times 32.2 \times \frac{ft}{s^2} \times 100 \cdot ft \times \frac{in^2}{14.7 \cdot lbf} \times \left(\frac{1 \cdot ft}{12 \cdot in}\right)^2 \times \frac{lbf \cdot s^2}{slug\,ft}\right]^{\frac{1}{3}}$$

$$D_2 = 0.477 \cdot in$$

Problem 3.11

[Difficulty: 2]

3.11 A cube with 6 in. sides is suspended in a fluid by a wire. The top of the cube is horizontal and 8 in. below the free surface. If the cube has a mass of 2 slugs and the tension in the wire is $T = 50.7$ lbf, compute the fluid specific gravity, and from this determine the fluid. What are the gage pressures on the upper and lower surfaces?

Given: Properties of a cube suspended by a wire in a fluid

Find: The fluid specific gravity; the gage pressures on the upper and lower surfaces

Solution:

From a free body analysis of the cube: $\Sigma F = 0 = T + (p_L - p_U)\cdot d^2 - M\cdot g$

where M and d are the cube mass and size and p_L and p_U are the pressures on the lower and upper surfaces

For each pressure we can use Eq. 3.7 $\quad p = p_0 + \rho\cdot g\cdot h$

Hence $\quad p_L - p_U = \left[p_0 + \rho\cdot g\cdot(H + d)\right] - (p_0 + \rho\cdot g\cdot H) = \rho\cdot g\cdot d = SG\cdot\rho_{H2O}\cdot d$

where H is the depth of the upper surface

Hence the force balance gives $\quad SG = \dfrac{M\cdot g - T}{\rho_{H2O}\cdot g\cdot d^3}$

$$SG = \frac{2\cdot slug\times 32.2\cdot\frac{ft}{s^2}\times\frac{lbf\cdot s^2}{slug\cdot ft} - 50.7\cdot lbf}{1.94\cdot\frac{slug}{ft^3}\times 32.2\cdot\frac{ft}{s^2}\times\frac{lbf\cdot s^2}{slug\cdot ft}\times(0.5\cdot ft)^3} \qquad SG = 1.75$$

From Table A.1, the fluid is Meriam blue.

The individual pressures are computed from Eq 3.7

$$p = p_0 + \rho\cdot g\cdot h \quad \text{or} \quad p_g = \rho\cdot g\cdot h = SG\cdot\rho_{H2O}\cdot h$$

For the upper surface

$$p_g = 1.754\times 1.94\cdot\frac{slug}{ft^3}\times 32.2\cdot\frac{ft}{s^2}\times\frac{2}{3}\cdot ft\times\frac{lbf\cdot s^2}{slug\cdot ft}\times\left(\frac{1\cdot ft}{12\cdot in}\right)^2 \qquad p_g = 0.507\cdot psi$$

For the lower surface

$$p_g = 1.754\times 1.94\cdot\frac{slug}{ft^3}\times 32.2\cdot\frac{ft}{s^2}\times\left(\frac{2}{3}+\frac{1}{2}\right)\cdot ft\times\frac{lbf\cdot s^2}{slug\cdot ft}\times\left(\frac{1\cdot ft}{12\cdot in}\right)^2 \qquad p_g = 0.888\cdot psi$$

Note that the SG calculation can also be performed using a buoyancy approach (discussed later in the chapter):

Consider a free body diagram of the cube: $\quad \Sigma F = 0 = T + F_B - M\cdot g$

where M is the cube mass and F_B is the buoyancy force $\quad F_B = SG\cdot\rho_{H2O}\cdot L^3\cdot g$

Hence $\quad T + SG\cdot\rho_{H2O}\cdot L^3\cdot g - M\cdot g = 0 \quad$ or $\quad SG = \dfrac{M\cdot g - T}{\rho_{H2O}\cdot g\cdot L^3}\quad$ as before $\qquad SG = 1.75$

Problem 3.12

[Difficulty: 4]

3.12 Assuming the bulk modulus is constant for seawater, derive an expression for the density variation with depth, h, below the surface. Show that the result may be written

$$\rho \approx \rho_0 + bh$$

where ρ_0 is the density at the surface. Evaluate the constant b. Then, using the approximation, obtain an equation for the variation of pressure with depth below the surface. Determine the depth in feet at which the error in pressure predicted by the approximate solution is 0.01 percent.

Given: Model behavior of seawater by assuming constant bulk modulus

Find: (a) Expression for density as a function of depth h.
(b) Show that result may be written as

$$\rho = \rho_o + bh$$

(c) Evaluate the constant b
(d) Use results of (b) to obtain equation for p(h)
(e) Determine depth at which error in predicted pressure is 0.01%

Solution: From Table A.2, App. A: $SG_o = 1.025 \quad E_v = 2.42 \cdot GPa = 3.51 \times 10^5 \cdot psi$

Governing Equations:

$$\frac{dp}{dh} = \rho \cdot g$$ (Hydrostatic Pressure - h is positive downwards)

$$E_v = \frac{dp}{\frac{d\rho}{\rho}}$$ (Definition of Bulk Modulus)

Then $dp = \rho \cdot g \cdot dh = E_v \cdot \frac{d\rho}{\rho}$ or $\frac{d\rho}{\rho^2} = \frac{g}{E_v} dh$ Now if we integrate: $\int_{\rho_o}^{\rho} \frac{1}{\rho^2} d\rho = \int_0^h \frac{g}{E_v} dh$

After integrating: $\frac{\rho - \rho_o}{\rho \cdot \rho_o} = \frac{g \cdot h}{E_v}$ Therefore: $\rho = \frac{E_v \cdot \rho_o}{E_v - g \cdot h \cdot \rho_o}$ and $\frac{\rho}{\rho_o} = \frac{1}{1 - \frac{\rho_o \cdot g \cdot h}{E_v}}$

Now for $\frac{\rho_o \cdot g \cdot h}{E_v}$ <<1, the binomial expansion may be used to approximate the density: $\frac{\rho}{\rho_o} = 1 + \frac{\rho_o \cdot g \cdot h}{E_v}$ (Binomial expansion may be found in a host of sources, e.g. *CRC Handbook of Mathematics*)

In other words, $\rho = \rho_o + b \cdot h$ where $b = \frac{\rho_o^2 \cdot g}{E_v}$

Since $dp = \rho \cdot g \cdot dh$ then an approximate expression for the pressure as a function of depth is:

$$p_{approx} - p_{atm} = \int_0^h (\rho_o + b \cdot h) \cdot g\, dh \rightarrow p_{approx} - p_{atm} = \frac{g \cdot h \cdot (2 \cdot \rho_o + b \cdot h)}{2}$$

Solving for p_{approx} we get:

$$p_{approx} = p_{atm} + \frac{g \cdot h \cdot (2 \cdot \rho_o + b \cdot h)}{2} = p_{atm} + \rho_o \cdot g \cdot h + \frac{b \cdot g \cdot h^2}{2} = p_{atm} + \left(\rho_o \cdot h + \frac{b \cdot h^2}{2}\right) \cdot g$$

Now if we subsitiute in the expression for b and simplify, we get:

$$p_{approx} = p_{atm} + \left(\rho_o \cdot h + \frac{\rho_o{}^2 \cdot g}{E_v} \cdot \frac{h^2}{2}\right) \cdot g = p_{atm} + \rho_o \cdot g \cdot h \cdot \left(1 + \frac{\rho_o \cdot g \cdot h}{2 \cdot E_v}\right) \qquad p_{approx} = p_{atm} + \rho_o \cdot g \cdot h \cdot \left(1 + \frac{\rho_o \cdot g \cdot h}{2E_v}\right)$$

The exact soution for p(h) is obtained by utilizing the exact solution for ρ(h). Thus:

$$p_{exact} - p_{atm} = \int_{\rho_o}^{\rho} \frac{E_v}{\rho} \, d\rho = E_v \cdot \ln\left(\frac{\rho}{\rho_o}\right)$$ Subsitiuting for $\frac{\rho}{\rho_o}$ we get: $$p_{exact} = p_{atm} + E_v \cdot \ln\left(1 - \frac{\rho_o \cdot g \cdot h}{E_v}\right)^{-1}$$

If we let $x = \frac{\rho_o \cdot g \cdot h}{E_v}$ For the error to be 0.01%: $$\frac{\Delta p_{exact} - \Delta p_{approx}}{\Delta p_{exact}} = 1 - \frac{\rho_o \cdot g \cdot h \cdot \left(1 + \frac{x}{2}\right)}{E_v \cdot \ln\left[(1-x)^{-1}\right]} = 1 - \frac{x \cdot \left(1 + \frac{x}{2}\right)}{\ln\left[(1-x)^{-1}\right]} = 0.0001$$

This equation requires an iterative solution, e.g. Excel's Goal Seek. The result is: $x = 0.01728$ Solving x for h:

$$h = \frac{x \cdot E_v}{\rho_o \cdot g} \qquad h = 0.01728 \times 3.51 \times 10^5 \cdot \frac{lbf}{in^2} \times \frac{ft^3}{1.025 \times 1.94 \cdot slug} \times \frac{s^2}{32.2 \cdot ft} \times \left(\frac{12 \cdot in}{ft}\right)^2 \times \frac{slug \cdot ft}{lbf \cdot s^2} \qquad h = 1.364 \times 10^4 \cdot ft$$

This depth is over 2.5 miles, so the incompressible fluid approximation is a reasonable one at all but the lowest depths of the ocean.

Problem 3.13

[Difficulty: 3]

3.13 Oceanographic research vessels have descended to 6.5 mi below sea level. At these extreme depths, the compressibility of seawater can be significant. One may model the behavior of seawater by assuming that its bulk modulus remains constant. Using this assumption, evaluate the deviations in density and pressure compared with values computed using the incompressible assumption at a depth, h, of 6.5 mi in seawater. Express your answers as a percentage. Plot the results over the range $0 \leq h \leq 7$ mi.

Given: Model behavior of seawater by assuming constant bulk modulus

Find: The percent deviations in (a) density and (b) pressure at depth h = 6.5 mi, as compared to values assuming constant density.
Plot results over the range of 0 mi - 7 mi.

Solution: From Table A.2, App. A: $SG_o = 1.025$ $\quad E_v = 2.42 \cdot \text{GPa} = 3.51 \times 10^5 \cdot \text{psi}$ $\quad h = 6.5 \cdot \text{mi}$

Governing Equations:

$$\frac{dp}{dh} = \rho \cdot g$$

(Hydrostatic Pressure - h is positive downwards)

$$E_v = \frac{dp}{\frac{d\rho}{\rho}}$$

(Definition of Bulk Modulus)

Then $dp = \rho \cdot g \cdot dh = E_v \cdot \frac{d\rho}{\rho}$ or $\frac{d\rho}{\rho^2} = \frac{g}{E_v} dh$ Now if we integrate: $\int_{\rho_o}^{\rho} \frac{1}{\rho^2} d\rho = \int_0^h \frac{g}{E_v} dh$

After integrating: $\frac{\rho - \rho_o}{\rho \cdot \rho_o} = \frac{g \cdot h}{E_v}$ Therefore: $\rho = \frac{E_v \cdot \rho_o}{E_v - \rho_o \cdot g \cdot h}$ and $\frac{\rho}{\rho_o} = \frac{1}{1 - \frac{\rho_o \cdot g \cdot h}{E_v}}$

$$\frac{\Delta\rho}{\rho_o} = \frac{\rho - \rho_o}{\rho_o} = \frac{\rho}{\rho_o} - 1 = \frac{1}{1 - \frac{\rho_o \cdot g \cdot h}{E_v}} - 1 = \frac{1 - \left(1 - \frac{\rho_o \cdot g \cdot h}{E_v}\right)}{1 - \frac{\rho_o \cdot g \cdot h}{E_v}} = \frac{\frac{\rho_o \cdot g \cdot h}{E_v}}{1 - \frac{\rho_o \cdot g \cdot h}{E_v}}$$

$$\frac{\Delta\rho}{\rho_o} = \frac{\frac{\rho_o \cdot g \cdot h}{E_v}}{1 - \frac{\rho_o \cdot g \cdot h}{E_v}}$$

To determine an expression for the percent deviation in pressure, we find $p - p_{atm}$ for variable ρ, and then for constant ρ.

For variable density and constant bulk modulus: $p - p_{atm} = \int_{\rho_o}^{\rho} \frac{E_v}{\rho} d\rho = E_v \cdot \ln\left(\frac{\rho}{\rho_o}\right)$

For constant density: $p_{const\rho} - p_{atm} = \int_0^h \rho_o g\, dh = \rho_o \cdot g \cdot h$

$$\frac{\Delta p}{p_{const\rho}} = \frac{p - p_{const\rho}}{p_{const\rho}} = \frac{E_v \cdot \ln\left(\frac{\rho}{\rho_o}\right) - \rho_o \cdot g \cdot h}{\rho_o \cdot g \cdot h} = \frac{E_v}{\rho_o \cdot g \cdot h} \cdot \ln\left(\frac{\rho}{\rho_o}\right) - 1 \qquad \frac{\Delta p}{p_{const\rho}} = \frac{E_v}{\rho_o \cdot g \cdot h} \cdot \ln\left[\left(1 - \frac{\rho_o \cdot g \cdot h}{E_v}\right)^{-1}\right] - 1$$

If we let $x = \frac{E_v}{\rho_o \cdot g}$ $\qquad x = 3.51 \times 10^5 \frac{lbf}{in^2} \times \frac{1}{1.025} \times \frac{1}{1.94} \frac{ft^3}{slug} \times \frac{1}{32.2} \frac{s^2}{ft} \times \left(\frac{12 \cdot in}{ft}\right)^2 \times \frac{slug \cdot ft}{lbf \cdot s^2} \times \frac{mi}{5280 \cdot ft}$ $\qquad x = 149.5 \cdot mi$

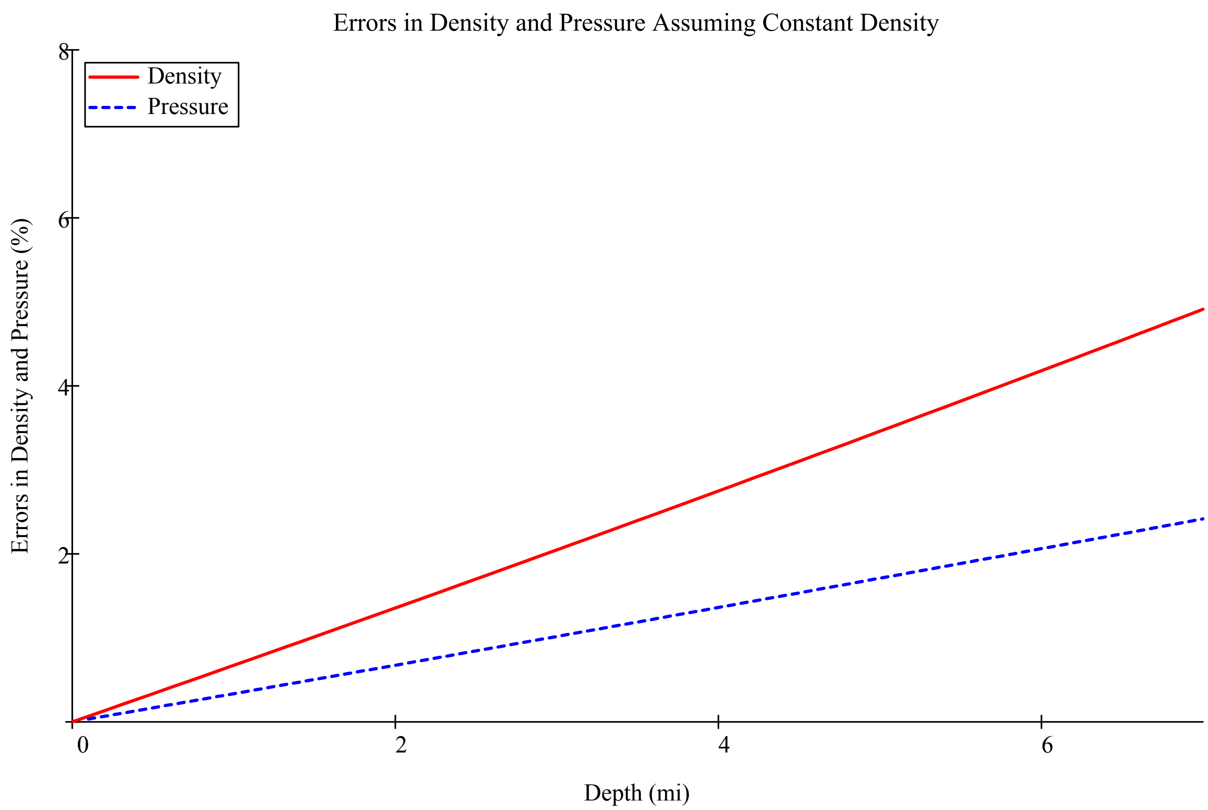
Substituting into the expressions for the deviations we get:

$$dev_\rho = \frac{\Delta\rho}{\rho_o} = \frac{\frac{h}{x}}{1 - \frac{h}{x}} = \frac{h}{x - h} = \frac{h}{149.5 \cdot mi - h}$$

$$dev_p = \frac{\Delta p}{p_{const\rho}} = \frac{x}{h} \cdot \ln\left[\left(1 - \frac{h}{x}\right)^{-1}\right] - 1 = \frac{149.5 \cdot mi}{h} \cdot \ln\left[\left(1 - \frac{h}{149.5 \cdot mi}\right)^{-1}\right] - 1$$

For h = 6.5 mi we get: $\qquad dev_\rho = 4.55 \cdot \%$ $\qquad dev_p = 2.24 \cdot \%$

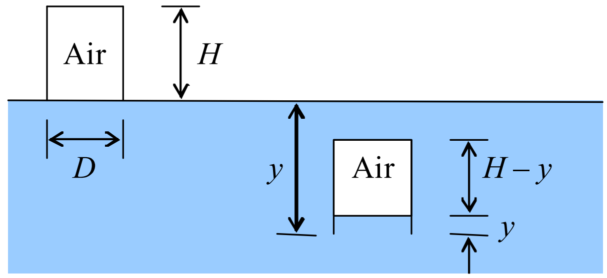
The plot below shows the deviations in density and pressure as a function of depth from 0 mi to 7 mi:



Problem 3.14

[Difficulty: 3]

3.14 An inverted cylindrical container is lowered slowly beneath the surface of a pool of water. Air trapped in the container is compressed isothermally as the hydrostatic pressure increases. Develop an expression for the water height, *y*, inside the container in terms of the container height, *H*, and depth of submersion, *h*. Plot *y*/*H* versus *h*/*H*.



Given: Cylindrical cup lowered slowly beneath pool surface

Find: Expression for y in terms of h and H.
Plot y/H vs. h/H.

Solution:

Governing Equations: $\frac{dp}{dh} = \rho \cdot g$ (Hydrostatic Pressure - h is positive downwards)

$p \cdot V = M \cdot R \cdot T$ (Ideal Gas Equation)

Assumptions:
(1) Constant temperature compression of air inside cup
(2) Static liquid
(3) Incompressible liquid

First we apply the ideal gas equation (at constant temperature) for the pressure of the air in the cup: $p \cdot V = \text{constant}$

Therefore: $p \cdot V = p_a \cdot \frac{\pi}{4} \cdot D^2 \cdot H = p \cdot \frac{\pi}{4} \cdot D^2 \cdot (H - y)$ and upon simplification: $p_a \cdot H = p \cdot (H - y)$

Now we look at the hydrostatic pressure equation for the pressure exerted by the water. Since ρ is constant, we integrate:

$$p - p_a = \rho \cdot g \cdot (h - y) \quad \text{at the water-air interface in the cup.}$$

Since the cup is submerged to a depth of h, these pressures must be equal:

$$p_a \cdot H = \left[p_a + \rho \cdot g \cdot (h - y)\right] \cdot (H - y) = p_a \cdot H - p_a \cdot y + \rho \cdot g \cdot (h - y) \cdot (H - y)$$

Explanding out the right hand side of this expression:

$$0 = -p_a \cdot y + \rho \cdot g \cdot (h - y) \cdot (H - y) = \rho \cdot g \cdot h \cdot H - \rho \cdot g \cdot h \cdot y - \rho \cdot g \cdot H \cdot y + \rho \cdot g \cdot y^2 - p_a \cdot y$$

$$\rho \cdot g \cdot y^2 - \left[p_a + \rho \cdot g \cdot (h + H)\right] \cdot y + \rho \cdot g \cdot h \cdot H = 0 \qquad y^2 - \left[\frac{p_a}{\rho \cdot g} + (h + H)\right] \cdot y + h \cdot H = 0$$

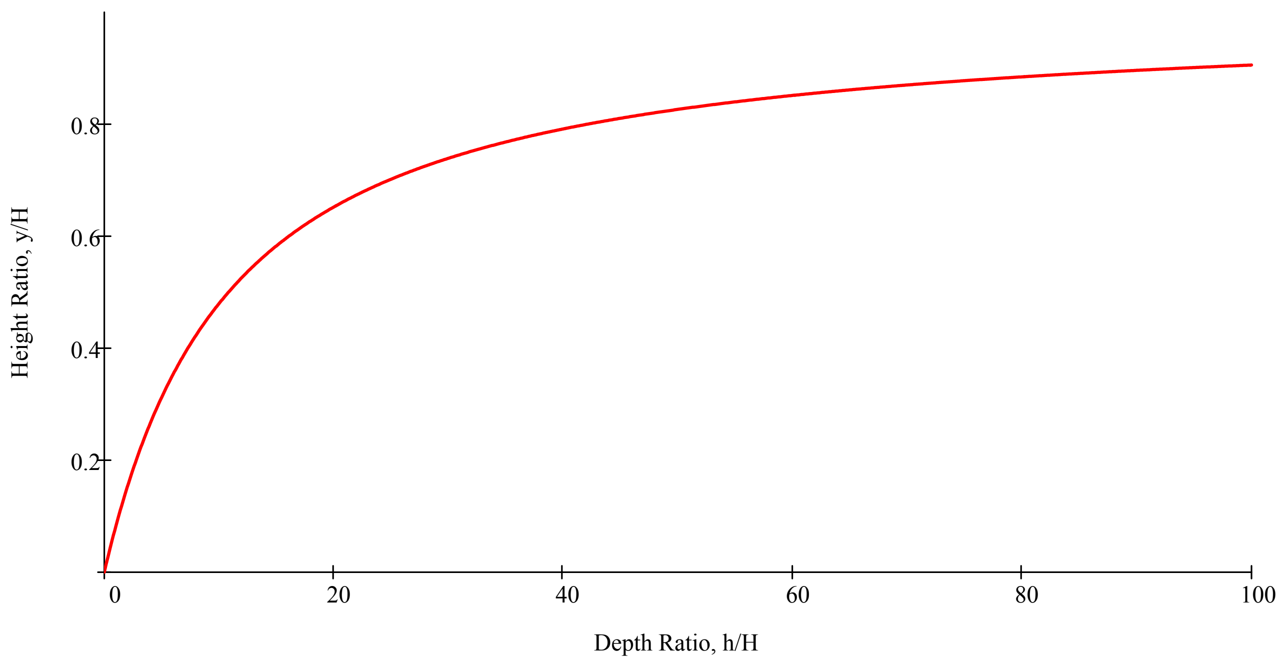
We now use the quadratic equation: $$y = \frac{\left[\frac{p_a}{\rho \cdot g} + (h + H)\right] - \sqrt{\left[\frac{p_a}{\rho \cdot g} + (h + H)\right]^2 - 4 \cdot h \cdot H}}{2}$$ we only use the minus sign because y can never be larger than H.

Now if we divide both sides by H, we get an expression for y/H:

$$\frac{y}{H} = \frac{\left(\frac{p_a}{\rho \cdot g \cdot H} + \frac{h}{H} + 1\right) - \sqrt{\left(\frac{p_a}{\rho \cdot g \cdot H} + \frac{h}{H} + 1\right)^2 - 4 \cdot \frac{h}{H}}}{2}$$

The exact shape of this curve will depend upon the height of the cup. The plot below was generated assuming:

$$p_a = 101.3 \cdot kPa$$
$$H = 1 \cdot m$$



Problem 3.15

[Difficulty: 1]

3.15 You close the top of your straw with your thumb and lift the straw out of your glass containing Coke. Holding it vertically, the total length of the straw is 45 cm, but the Coke held in the straw is in the bottom 15 cm. What is the pressure in the straw just below your thumb? Ignore any surface tension effects.

Given: Geometry of straw

Find: Pressure just below the thumb

Assumptions:
(1) Coke is incompressible
(2) Pressure variation within the air column is negligible
(3) Coke has density of water

Solution:

Basic equation $\frac{dp}{dy} = -\rho \cdot g$ or, for constant ρ $\Delta p = \rho \cdot g \cdot h$ where h is measured downwards

This equation only applies in the 15 cm of coke in the straw - in the other 30 cm of air the pressure is essentially constant.

The gage pressure at the coke surface is $p_{coke} = \rho \cdot g \cdot h_{coke}$

Hence, with $h_{coke} = -15 \cdot cm$ because h is measured downwards

$$p_{coke} = -1000 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times 15 \cdot cm \times \frac{m}{100 \cdot cm} \times \frac{N \cdot s^2}{kg \cdot m} \times \frac{kPa \cdot m^2}{1000 \cdot N}$$

$p_{coke} = -1.471 \cdot kPa$ gage

$p_{coke} = 99.9 \cdot kPa$

Problem 3.16

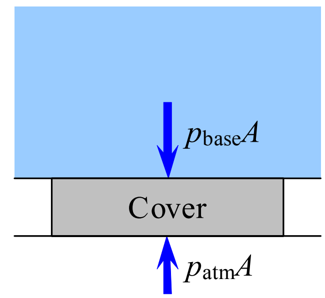
[Difficulty: 2]

3.16 A water tank filled with water to a depth of 16 ft has in inspection cover (1 in. × 1 in.) at its base, held in place by a plastic bracket. The bracket can hold a load of 9 lbf. Is the bracket strong enough? If it is, what would the water depth have to be to cause the bracket to break?

Given: Data on water tank and inspection cover

Find: If the support bracket is strong enough; at what water depth would it fail

Assumptions: Water is incompressible and static



Solution:

Basic equation $\quad \dfrac{dp}{dy} = -\rho \cdot g \quad$ or, for constant ρ $\quad \Delta p = \rho \cdot g \cdot h \quad$ where h is measured downwards

The absolute pressure at the base is $\quad p_{base} = p_{atm} + \rho \cdot g \cdot h \quad$ where $\quad h = 16 \cdot ft$

The gage pressure at the base is $\quad p_{base} = \rho \cdot g \cdot h \quad$ This is the pressure to use as we have p_{atm} on the outside of the cover.

The force on the inspection cover is $\quad F = p_{base} \cdot A \quad$ where $\quad A = 1 \cdot in \times 1 \cdot in \quad$ $A = 1 \cdot in^2$

$$F = \rho \cdot g \cdot h \cdot A$$

$$F = 1.94 \cdot \frac{slug}{ft^3} \times 32.2 \cdot \frac{ft}{s^2} \times 16 \cdot ft \times 1 \cdot in^2 \times \left(\frac{ft}{12 \cdot in}\right)^2 \times \frac{lbf \cdot s^2}{slug \cdot ft}$$

$F = 6.94 \cdot lbf$ $\quad$ The bracket is strong enough (it can take 9 lbf).

To find the maximum depth we start with $\quad F = 9.00 \cdot lbf$

$$h = \frac{F}{\rho \cdot g \cdot A}$$

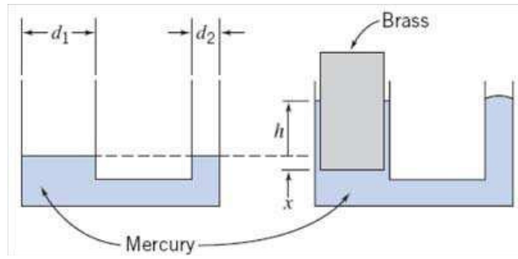
$$h = 9 \cdot lbf \times \frac{1}{1.94} \cdot \frac{ft^3}{slug} \times \frac{1}{32.2} \cdot \frac{s^2}{ft} \times \frac{1}{in^2} \times \left(\frac{12 \cdot in}{ft}\right)^2 \times \frac{slug \cdot ft}{lbf \cdot s^2}$$

$$h = 20.7 \cdot ft$$

Problem 3.17

[Difficulty: 4]

3.17 A container with two circular vertical tubes of diameters $d_1 = 39.5$ mm and $d_2 = 12.7$ mm is partially filled with mercury. The equilibrium level of the liquid is shown in the left diagram. A cylindrical object made from solid brass is placed in the larger tube so that it floats, as shown in the right diagram. The object is $D = 37.5$ mm in diameter and $H = 76.2$ mm high. Calculate the pressure at the lower surface needed to float the object. Determine the new equilibrium level, h, of the mercury with the brass cylinder in place.



Given: Container of mercury with vertical tubes of known diameter, brass cylinder of known dimensions introduced into larger tube, where it floats.

$d_1 = 39.5 \cdot mm \quad d_2 = 12.7 \cdot mm \quad D = 37.5 \cdot mm \quad H = 76.2 \cdot mm \quad SG_{Hg} = 13.55 \quad SG_b = 8.55$

Find: (a) Pressureon the bottom of the cylinder
(b) New equlibrium level, h, of the mercury

Solution: We will analyze a free body diagram of the cylinder, and apply the hydrostatics equation.

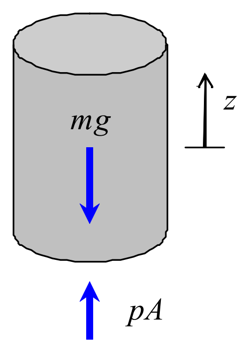
Governing equations:

$\Sigma F_z = 0$ (Vertical Equilibrium)

$\frac{dp}{dz} = -\rho \cdot g$ (Hydrostatic Pressure - z is positive upwards)

$\rho = SG \cdot \rho_{water}$ (Definition of Specific Gravity)

Assumptions: (1) Static liquid
(2) Incompressible liquid



If we take a free body diagram of the cylinder:

$$\Sigma F_z = p \cdot \frac{\pi}{4} \cdot D^2 - \rho_b \cdot g \cdot \frac{\pi}{4} \cdot D^2 \cdot H = 0 \qquad \text{thus:} \quad p = \rho_b \cdot g \cdot H = SG_b \cdot \rho_{water} \cdot g \cdot H$$

$$p = 8.55 \times 1000 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times 76.2 \cdot mm \times \frac{m}{10^3 \cdot mm} \qquad p = 6.39 \cdot kPa \quad (gage)$$

This pressure must be generated by a column of mercury h+x in height. Thus:

$$p = \rho_{Hg} \cdot g \cdot (h + x) = SG_{Hg} \cdot \rho_{water} \cdot g \cdot (h + x) = SG_b \cdot \rho_{water} \cdot g \cdot H \quad \text{Thus:} \quad h + x = \frac{SG_b}{SG_{Hg}} \cdot H$$

The value of x can be found by realizing that the volume of mercury in the system remains constant. Therefore:

$$\frac{\pi}{4} \cdot D^2 \cdot x = \frac{\pi}{4} \cdot \left(d_1^{\,2} - D^2\right) \cdot h + \frac{\pi}{4} \cdot d_2^{\,2} \cdot h \qquad \text{Now if we solve for x:} \quad x = \left[\left(\frac{d_1}{D}\right)^2 - 1 + \left(\frac{d_2}{D}\right)^2\right] \cdot h$$

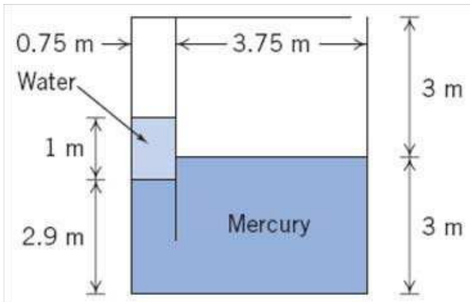
These expressions now allow us to solve for h: $h = \frac{SG_b}{SG_{Hg}} \cdot \frac{D^2}{d_1^{\,2} + d_2^{\,2}} \cdot H$ Substituting in values:

$$h = \frac{8.55}{13.55} \times \frac{(37.5 \cdot mm)^2}{(39.5 \cdot mm)^2 + (12.7 \cdot mm)^2} \times 76.2 \cdot mm \qquad h = 39.3 \cdot mm$$

Problem 3.18

[Difficulty: 2]

3.18 A partitioned tank as shown contains water and mercury. What is the gage pressure in the air trapped in the left chamber? What pressure would the air on the left need to be pumped to in order to bring the water and mercury free surfaces level?



Given:

Data on partitioned tank

Find:

Gage pressure of trapped air; pressure to make water and mercury levels equal

Solution:

The pressure difference is obtained from repeated application of Eq. 3.7, or in other words, from Eq. 3.8. Starting from the right air chamber

$$p_{gage} = SG_{Hg} \times \rho_{H2O} \times g \times (3 \cdot m - 2.9 \cdot m) - \rho_{H2O} \times g \times 1 \cdot m$$

$$p_{gage} = \rho_{H2O} \times g \times (SG_{Hg} \times 0.1 \cdot m - 1.0 \cdot m)$$

$$p_{gage} = 999 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times (13.55 \times 0.1 \cdot m - 1.0 \cdot m) \times \frac{N \cdot s^2}{kg \cdot m}$$

$$p_{gage} = 3.48 \cdot kPa$$

If the left air pressure is now increased until the water and mercury levels are now equal, Eq. 3.8 leads to

$$p_{gage} = SG_{Hg} \times \rho_{H2O} \times g \times 1.0 \cdot m - \rho_{H2O} \times g \times 1.0 \cdot m$$

$$p_{gage} = \rho_{H2O} \times g \times (SG_{Hg} \times 1 \cdot m - 1.0 \cdot m)$$

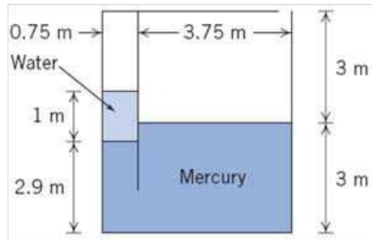
$$p_{gage} = 999 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times (13.55 \times 1 \cdot m - 1.0 \cdot m) \times \frac{N \cdot s^2}{kg \cdot m}$$

$$p_{gage} = 123 \cdot kPa$$

Problem 3.19

[Difficulty: 2]

3.19 In the tank of Problem 3.18, if the opening to atmosphere on the right chamber is first sealed, what pressure would the air on the left now need to be pumped to in order to bring the water and mercury free surfaces level? (Assume the air trapped in the right chamber behaves isothermally.)



Given:

Data on partitioned tank

Find:

Pressure of trapped air required to bring water and mercury levels equal if right air opening is sealed

Solution:

First we need to determine how far each free surface moves.

In the tank of Problem 3.18, the ratio of cross section areas of the partitions is 0.75/3.75 or 1:5. Suppose the water surface (and therefore the mercury on the left) must move down distance x to bring the water and mercury levels equal. Then by mercury volume conservation, the mercury free surface (on the right) moves up $(0.75/3.75)x = x/5$. These two changes in level must cancel the original discrepancy in free surface levels, of (1m + 2.9m) - 3 m = 0.9 m. Hence $x + x/5 = 0.9$ m, or $x = 0.75$ m. The mercury level thus moves up $x/5 = 0.15$ m.

Assuming the air (an ideal gas, $pV=RT$) in the right behaves isothermally, the new pressure there will be

$$p_{right} = \frac{V_{rightold}}{V_{rightnew}} \cdot p_{atm} = \frac{A_{right} \cdot L_{rightold}}{A_{right} \cdot L_{rightnew}} \cdot p_{atm} = \frac{L_{rightold}}{L_{rightnew}} \cdot p_{atm}$$

where V, A and L represent volume, cross-section area, and vertical length
Hence

$$p_{right} = \frac{3}{3 - 0.15} \times 101 \cdot \text{kPa} \qquad p_{right} = 106 \cdot \text{kPa}$$

When the water and mercury levels are equal application of Eq. 3.8 gives:

$$p_{left} = p_{right} + SG_{Hg} \times \rho_{H2O} \times g \times 1.0 \cdot \text{m} - \rho_{H2O} \times g \times 1.0 \cdot \text{m}$$

$$p_{left} = p_{right} + \rho_{H2O} \times g \times \left(SG_{Hg} \times 1.0 \cdot \text{m} - 1.0 \cdot \text{m}\right)$$

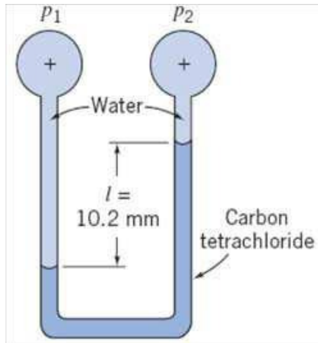
$$p_{left} = 106 \cdot \text{kPa} + 999 \cdot \frac{\text{kg}}{\text{m}^3} \times 9.81 \cdot \frac{\text{m}}{\text{s}^2} \times (13.55 \cdot 1.0 \cdot \text{m} - 1.0 \cdot \text{m}) \times \frac{\text{N} \cdot \text{s}^2}{\text{kg} \cdot \text{m}} \qquad p_{left} = 229 \cdot \text{kPa}$$

$$p_{gage} = p_{left} - p_{atm} \qquad p_{gage} = 229 \cdot \text{kPa} - 101 \cdot \text{kPa} \qquad p_{gage} = 128 \cdot \text{kPa}$$

Problem 3.20

[Difficulty: 2]

3.20 Consider the two-fluid manometer shown. Calculate the applied pressure difference.



Given: Two-fluid manometer as shown

$l = 10.2 \cdot mm \quad SG_{ct} = 1.595$ (From Table A.1, App. A)

Find: Pressure difference

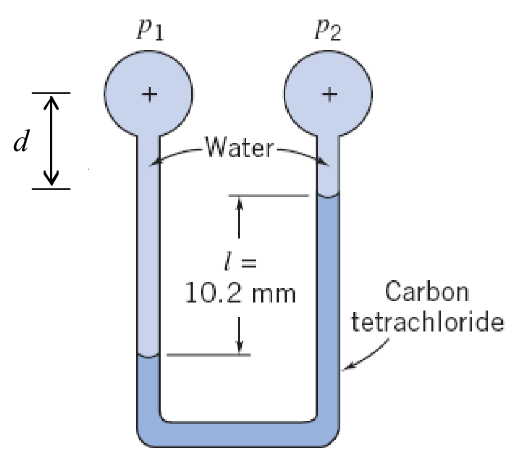
Solution: We will apply the hydrostatics equation.

Governing equations:

$$\frac{dp}{dh} = \rho \cdot g$$ (Hydrostatic Pressure - h is positive downwards)

$$\rho = SG \cdot \rho_{water}$$ (Definition of Specific Gravity)

Assumptions: (1) Static liquid
(2) Incompressible liquid



Starting at point 1 and progressing to point 2 we have:

$$p_1 + \rho_{water} \cdot g \cdot (d + l) - \rho_{ct} \cdot g \cdot l - \rho_{water} \cdot g \cdot d = p_2$$

Simplifying and solving for $p_2 - p_1$ we have:

$$\Delta p = p_2 - p_1 = \rho_{ct} \cdot g \cdot l - \rho_{water} \cdot g \cdot l = \left(SG_{ct} - 1\right) \cdot \rho_{water} \cdot g \cdot l$$

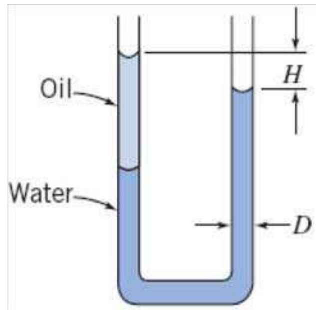
Substituting the known data:

$$\Delta p = (1.591 - 1) \times 1000 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times 10.2 \cdot mm \times \frac{m}{10^3 \cdot mm} \qquad \Delta p = 59.1\, Pa$$

Problem 3.21

[Difficulty: 2]

3.21 A manometer is formed from glass tubing with uniform inside diameter, $D = 6.35$ mm, as shown. The U-tube is partially filled with water. Then $V = 3.25\ \text{cm}^3$ of Meriam red oil is added to the left side. Calculate the equilibrium height, H, when both legs of the U-tube are open to the atmosphere.



Given: U-tube manometer, partiall filled with water, then a given volume of Meriam red oil is added to the left side

$D = 6.35 \cdot \text{mm}$ $\quad V_{oil} = 3.25 \cdot \text{cm}^3$ $\quad SG_{oil} = 0.827$ $\quad$ (From Table A.1, App. A)

Find: Equilibrium height, H, when both legs are open to atmosphere.

Solution: We will apply the basic pressure-height relation.

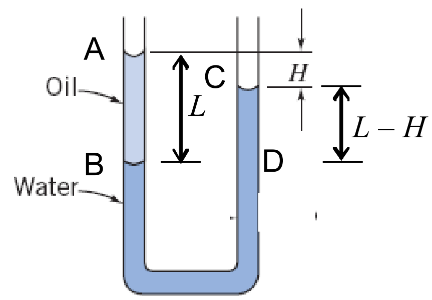
Governing Equations:

$$\frac{dp}{dh} = \rho \cdot g$$ (Hydrostatic Pressure - h is positive downwards)

$$\rho = SG \cdot \rho_{water}$$ (Definition of Specific Gravity)

Assumptions:
(1) Static liquid
(2) Incompressible liquid

Integration of the pressure equation gives: $\quad p_2 - p_1 = \rho \cdot g \cdot (h_2 - h_1)$



Thus: $\quad p_B - p_A = \rho_{oil} \cdot g \cdot L$ and $\quad p_D - p_C = \rho_{water} \cdot g \cdot (L - H)$

Since $\quad p_A = p_C = p_{atm}$ and $\quad p_B = p_D$ since they are at the same height:

$$\rho_{oil} \cdot g \cdot L = \rho_{water} \cdot g \cdot (L - H) \quad \text{or} \quad SG_{oil} \cdot L = L - H$$

Solving for H: $\quad H = L \cdot (1 - SG_{oil})$

The value of L comes from the volume of the oil: $\quad V_{oil} = \frac{\pi}{4} \cdot D^2 \cdot L$

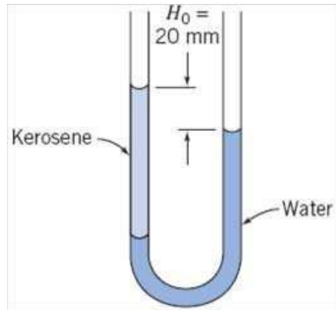
Solving for L: $\quad L = \frac{4 \cdot V_{oil}}{\pi \cdot D^2} \qquad L = \frac{4 \times 3.25 \cdot \text{cm}^3}{\pi \times (6.35 \cdot \text{mm})^2} \times \left(\frac{10 \cdot \text{mm}}{\text{cm}}\right)^3 \qquad L = 102.62 \cdot \text{mm}$

We can now calculate H: $\quad H = 102.62 \cdot \text{mm} \cdot (1 - 0.827) \qquad H = 17.75 \cdot \text{mm}$

Problem 3.22

[Difficulty: 2]

3.22 The manometer shown contains water and kerosene. With both tubes open to the atmosphere, the free-surface elevations differ by $H_0 = 20.0$ mm. Determine the elevation difference when a pressure of 98.0 Pa (gage) is applied to the right tube.



Given:

Two fluid manometer contains water and kerosene. With both tubes open to atmosphere, the difference in free surface elevations is known

$H_o = 20 \cdot mm \quad SG_k = 0.82$ (From Table A.1, App. A)

Find:

The elevation difference, H, between the free surfaces of the fluids when a gage pressure of 98.0 Pa is applied to the right tube.

Solution:

We will apply the hydrostatics equation.

Governing Equations:

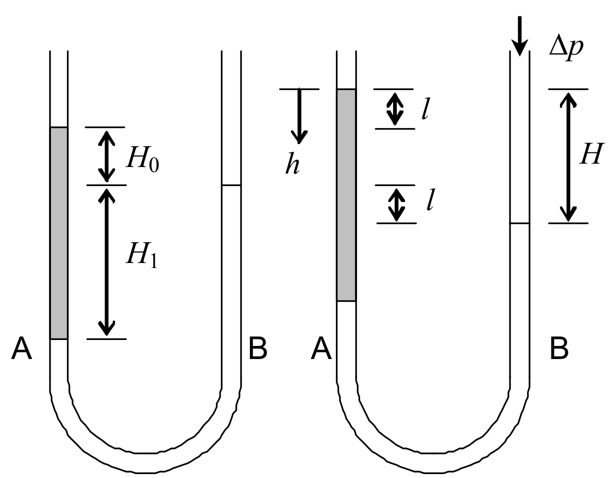
$$\frac{dp}{dh} = \rho \cdot g$$ (Hydrostatic Pressure - h is positive downwards)

$$\rho = SG \cdot \rho_{water}$$ (Definition of Specific Gravity)

Assumptions:

(1) Static liquid
(2) Incompressible liquid

When the gage pressure Δp is applied to the right tube, the water in the right tube is displaced downward by a distance, l. The kerosene in the left tube is displaced upward by the same distance, l.



Under the applied gage pressure Δp, the elevation difference, H, is:

$$H = H_o + 2 \cdot l$$

Since points A and B are at the same elevation in the same fluid, their pressures are the same. Initially:

$$p_A = \rho_k \cdot g \cdot (H_o + H_1) \qquad p_B = \rho_{water} \cdot g \cdot H_1$$

Setting these pressures equal:

$$\rho_k \cdot g \cdot (H_o + H_1) = \rho_{water} \cdot g \cdot H_1$$

Solving for H_1

$$H_1 = \frac{\rho_k \cdot H_o}{\rho_{water} - \rho_k} = \frac{SG_k \cdot H_o}{1 - SG_k} \qquad H_1 = \frac{0.82 \times 20 \cdot mm}{1 - 0.82} \qquad H_1 = 91.11 \cdot mm$$

Now under the applied gage pressure:

$$p_A = \rho_k \cdot g \cdot (H_o + H_1) + \rho_{water} \cdot g \cdot l \qquad p_B = \Delta p + \rho_{water} \cdot g \cdot (H_1 - l)$$

Setting these pressures equal:

$$SG_k \cdot \left(H_o + H_1\right) + l = \frac{\Delta p}{\rho_{water} \cdot g} + \left(H_1 - l\right) \qquad l = \frac{1}{2}\left[\frac{\Delta p}{\rho_{water} \cdot g} + H_1 - SG_k \cdot \left(H_o + H_1\right)\right]$$

Substituting in known values we get:

$$l = \frac{1}{2} \times \left[98.0 \cdot \frac{N}{m^2} \times \frac{1}{999}\frac{m^3}{kg} \times \frac{1}{9.81} \cdot \frac{s^2}{m} \times \frac{kg \cdot m}{N \cdot s^2} + [91.11 \cdot mm - 0.82 \times (20 \cdot mm + 91.11 \cdot mm)] \times \frac{m}{10^3 \cdot mm}\right] \qquad l = 5.000 \cdot mm$$

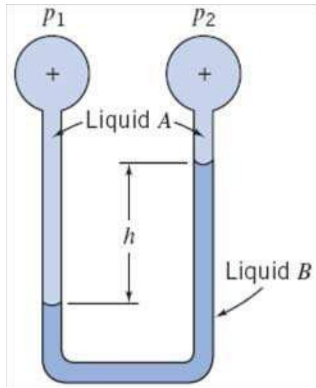
Now we solve for H:

$$H = 20 \cdot mm + 2 \times 5.000 \cdot mm \qquad H = 30.0 \cdot mm$$

Problem 3.23

[Difficulty: 2]

3.23 The manometer shown contains two liquids. Liquid A has SG = 0.88 and liquid B has SG = 2.95. Calculate the deflection, h, when the applied pressure difference is $p_1 - p_2 = 18$ lbf/ft^2.



Given:
Data on manometer

Find:
Deflection due to pressure difference

Solution:

Basic equation $\quad \dfrac{dp}{dy} = -\rho \cdot g \quad$ or, for constant ρ $\quad \Delta p = \rho \cdot g \cdot \Delta h \quad$ where h is measured downwards

Starting at p_1 $\quad p_A = p_1 + SG_A \cdot \rho \cdot g \cdot (h + l) \quad$ where l is the (unknown) distance from the level of the right interface

Next, from A to B $\quad p_B = p_A - SG_B \cdot \rho \cdot g \cdot h$

Finally, from A to the location of p_2 $\quad p_2 = p_B - SG_A \cdot \rho \cdot g \cdot l$

Combining the three equations $\quad p_2 = \left(p_A - SG_B \cdot \rho \cdot g \cdot h\right) - SG_A \cdot \rho \cdot g \cdot l = \left[p_1 + SG_A \cdot \rho \cdot g \cdot (h + l) - SG_B \cdot \rho \cdot g \cdot h\right] - SG_A \cdot \rho \cdot g \cdot l$

$$p_2 - p_1 = \left(SG_A - SG_B\right) \cdot \rho \cdot g \cdot h$$

$$h = \frac{p_1 - p_2}{\left(SG_B - SG_A\right) \cdot \rho \cdot g}$$

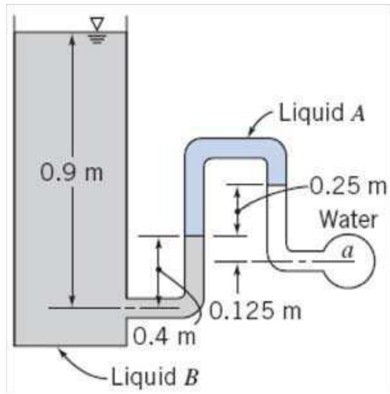
$$h = 18 \cdot \frac{lbf}{ft^2} \times \frac{1}{(2.95 - 0.88)} \times \frac{1}{1.94} \cdot \frac{ft^3}{slug} \times \frac{1}{32.2} \cdot \frac{s^2}{ft} \times \frac{slug \cdot ft}{s^2 \cdot lbf}$$

$h = 0.139 \cdot ft \qquad h = 1.67 \cdot in$

Problem 3.24

[Difficulty: 2]

3.24 Determine the gage pressure in kPa at point a, if liquid A has SG = 1.20 and liquid B has SG = 0.75. The liquid surrounding point a is water, and the tank on the left is open to the atmosphere.

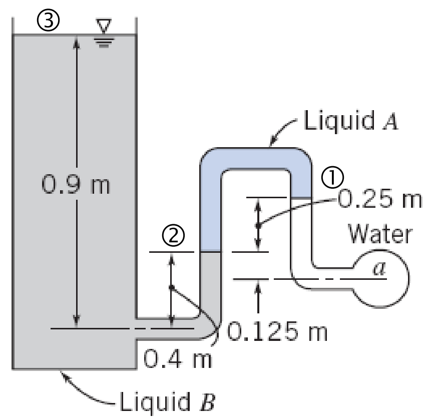


Given: Data on manometer

Find: Gage pressure at point a

Assumption: Water, liquids A and B are static and incompressible

Solution:



Basic equation $\frac{dp}{dy} = -\rho \cdot g$ or, for constant ρ $\Delta p = \rho \cdot g \cdot \Delta h$

where Δh is height difference

Starting at point a $\quad p_1 = p_a - \rho_{H2O} \cdot g \cdot h_1 \quad$ where $\quad h_1 = 0.125 \cdot m + 0.25 \cdot m \quad h_1 = 0.375\,m$

Next, in liquid A $\quad p_2 = p_1 + SG_A \cdot \rho_{H2O} \cdot g \cdot h_2 \quad$ where $\quad h_2 = 0.25 \cdot m$

Finally, in liquid B $\quad p_{atm} = p_2 - SG_B \cdot \rho_{H2O} \cdot g \cdot h_3 \quad$ where $\quad h_3 = 0.9 \cdot m - 0.4 \cdot m \quad h_3 = 0.5\,m$

Combining the three equations

$$p_{atm} = \left(p_1 + SG_A \cdot \rho_{H2O} \cdot g \cdot h_2\right) - SG_B \cdot \rho_{H2O} \cdot g \cdot h_3 = p_a - \rho_{H2O} \cdot g \cdot h_1 + SG_A \cdot \rho_{H2O} \cdot g \cdot h_2 - SG_B \cdot \rho_{H2O} \cdot g \cdot h_3$$

$$p_a = p_{atm} + \rho_{H2O} \cdot g \cdot \left(h_1 - SG_A \cdot h_2 + SG_B \cdot h_3\right)$$

or in gage pressures $\quad p_a = \rho_{H2O} \cdot g \cdot \left(h_1 - SG_A \cdot h_2 + SG_B \cdot h_3\right)$

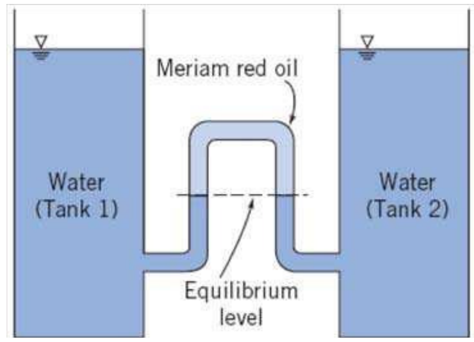
$$p_a = 1000 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times [0.375 - (1.20 \times 0.25) + (0.75 \times 0.5)] \cdot m \times \frac{N \cdot s^2}{kg \cdot m}$$

$p_a = 4.41 \times 10^3\,Pa \qquad p_a = 4.41 \cdot kPa \quad$ (gage)

Problem 3.25

[Difficulty: 2]

3.25 An engineering research company is evaluating using a sophisticated \$80,000 laser system between two large water storage tanks. You suggest that the job can be done with a \$200 manometer arrangement. Oil less dense than water can be used to give a significant amplification of meniscus movement; a small difference in level between the tanks will cause a much larger deflection in the oil levels in the manometer. If you set up a rig using Meriam red oil as the manometer fluid, determine the amplification factor that will be seen in the rig.



Given: Two fluid manometer, Meriam red oil is the second fluid $SG_{oil} = 0.827$ from Table A.1

Find: The amplification factor which will be seen in this demonstrator

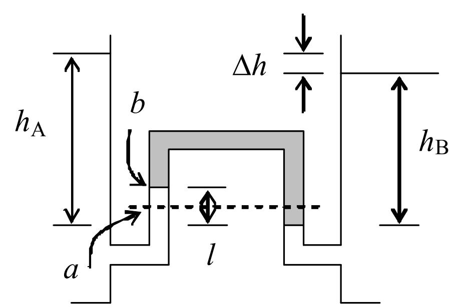
Solution: We will apply the hydrostatics equations to this system.

Governing Equations:

$$\frac{dp}{dh} = \rho \cdot g$$ (Hydrostatic Pressure - h is positive downwards)

$$\rho = SG \cdot \rho_{water}$$ (Definition of Specific Gravity)

Assumptions: (1) Static liquid
(2) Incompressible liquid



Integrating the hydrostatic pressure equation we get:

$$p = p_o + \rho \cdot g \cdot h$$

For the left leg of the manometer: $p_a = p_{atm} + \rho_{water} \cdot g \cdot h_A$

$$p_b = p_a - \rho_{water} \cdot g \cdot l = p_{atm} + \rho_{water} \cdot g \cdot \left(h_A - l\right)$$

For the right leg: $p_a = p_{atm} + \rho_{water} \cdot g \cdot h_B$

$$p_b = p_a - \rho_{oil} \cdot g \cdot l = p_{atm} + \rho_{water} \cdot g \cdot \left(h_B - SG_{oil} \cdot l\right)$$

Combining the right hand sides of these two equations: $p_{atm} + \rho_{water} \cdot g \cdot \left(h_A - l\right) = p_{atm} + \rho_{water} \cdot g \cdot \left(h_B - SG_{oil} \cdot l\right)$

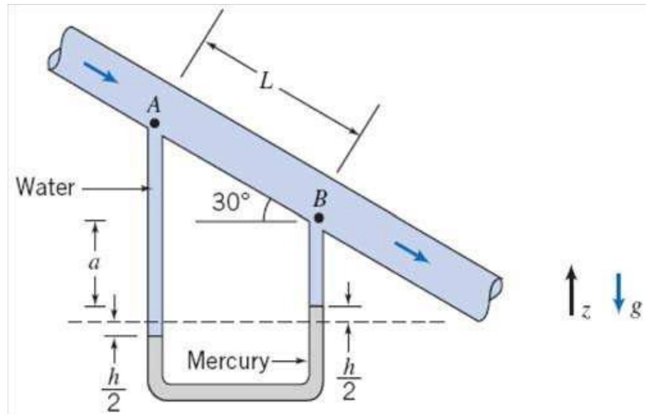
Upon simplification: $h_A - l = h_B - SG_{oil} \cdot l$ $\quad \Delta h = h_A - h_B = l \cdot \left(1 - SG_{oil}\right)$ so the amplification factor would be:

$$AF = \frac{l}{\Delta h} = \frac{1}{1 - SG_{oil}}$$ For Meriam red $$AF = \frac{1}{1 - 0.827} = 5.78$$ $AF = 5.78$

Problem 3.26

[Difficulty: 2]

3.26 Water flows downward along a pipe that is inclined at 30° below the horizontal, as shown. Pressure difference $p_A - p_B$ is due partly to gravity and partly to friction. Derive an algebraic expression for the pressure difference. Evaluate the pressure difference if $L = 5$ ft and $h = 6$ in.



Given: Water flow in an inclined pipe as shown. The pressure difference is measured with a two-fluid manometer

$L = 5\cdot ft \qquad h = 6\cdot in \qquad SG_{Hg} = 13.55$ (From Table A.1, App. A)

Find: Pressure difference between A and B

Solution: We will apply the hydrostatics equations to this system.

Governing Equations:

$$\frac{dp}{dh} = \rho\cdot g$$ (Hydrostatic Pressure - h is positive downwards)

$$\rho = SG\cdot\rho_{water}$$ (Definition of Specific Gravity)

Assumptions:

(1) Static liquid
(2) Incompressible liquid
(3) Gravity is constant

Integrating the hydrostatic pressure equation we get:

$$\Delta p = \rho\cdot g\cdot\Delta h$$

Progressing through the manometer from A to B:

$$p_A + \rho_{water}\cdot g\cdot L\cdot\sin(30\cdot deg) + \rho_{water}\cdot g\cdot a + \rho_{water}\cdot g\cdot h - \rho_{Hg}\cdot g\cdot h - \rho_{water}\cdot g\cdot a = p_B$$

Simplifying terms and solving for the pressure difference:

$$\Delta p = p_A - p_B = \rho_{water}\cdot g\left[h\cdot\left(SG_{Hg} - 1\right) - L\cdot\sin(30\cdot deg)\right]$$

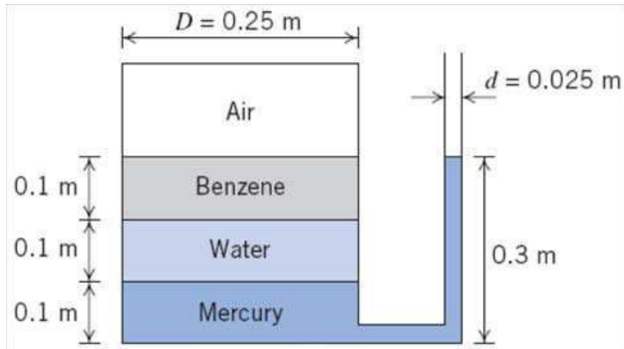
Substituting in values:

$$\Delta p = 1.94\cdot\frac{slug}{ft^3}\times 32.2\frac{ft}{s^2}\times\left[6\cdot in\times\frac{ft}{12\cdot in}\times(13.55 - 1) - 5\cdot ft\times\sin(30\cdot deg)\right]\times\frac{lbf\cdot s^2}{slug\, ft}\times\left(\frac{ft}{12\cdot in}\right)^2 \qquad \Delta p = 1.638\cdot psi$$

Problem 3.27

[Difficulty: 2]

3.27 Consider a tank containing mercury, water, benzene, and air as shown. Find the air pressure (gage). If an opening is made in the top of the tank, find the equilibrium level of the mercury in the manometer.



Given: Data on fluid levels in a tank

Find: Air pressure; new equilibrium level if opening appears

Solution:

Using Eq. 3.8, starting from the open side and working in gage pressure

$$p_{air} = \rho_{H2O} \times g \times \left[SG_{Hg} \times (0.3 - 0.1) \cdot m - 0.1 \cdot m - SG_{Benzene} \times 0.1 \cdot m\right]$$

Using data from Table A.2 $\quad p_{air} = 999 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times (13.55 \times 0.2 \cdot m - 0.1 \cdot m - 0.879 \times 0.1 \cdot m) \times \frac{N \cdot s^2}{kg \cdot m} \qquad p_{air} = 24.7 \cdot kPa$

To compute the new level of mercury in the manometer, assume the change in level from 0.3 m is an increase of x. Then, because the volume of mercury is constant, the tank mercury level will fall by distance $(0.025/0.25)^2 x$. Hence, the gage pressure at the bottom of the ta can be computed from the left and the right, providing a formula for x

$$SG_{Hg} \times \rho_{H2O} \times g \times (0.3 \cdot m + x) = SG_{Hg} \times \rho_{H2O} \times g \times \left[0.1 \cdot m - x \cdot \left(\frac{0.025}{0.25}\right)^2\right] \cdot m \ldots$$
$$+ \rho_{H2O} \times g \times 0.1 \cdot m + SG_{Benzene} \times \rho_{H2O} \times g \times 0.1 \cdot m$$

Hence $\quad x = \dfrac{[0.1 \cdot m + 0.879 \times 0.1 \cdot m + 13.55 \times (0.1 - 0.3) \cdot m]}{\left[1 + \left(\frac{0.025}{0.25}\right)^2\right] \times 13.55} \qquad x = -0.184\,m$

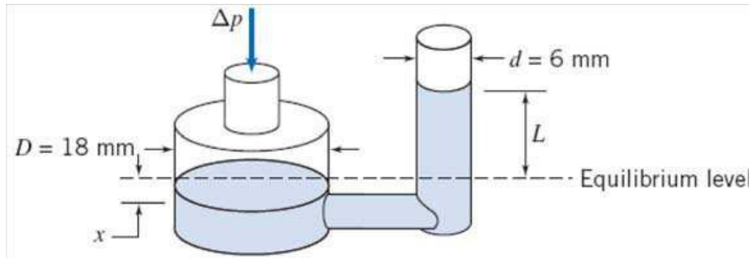
(The negative sign indicates the manometer level actually fell)

The new manometer height is $\quad h = 0.3 \cdot m + x \qquad h = 0.116\,m$

Problem 3.28

[Difficulty: 2]

3.28 A reservoir manometer has vertical tubes of diameter $D = 18$ mm and $d = 6$ mm. The manometer liquid is Meriam red oil. Develop an algebraic expression for liquid deflection L in the small tube when gage pressure Δp is applied to the reservoir. Evaluate the liquid deflection when the applied pressure is equivalent to 25 mm of water (gage).



Given: Reservoir manometer with vertical tubes of knowm diameter. Gage liquid is Meriam red oil

$D = 18\cdot\text{mm} \quad d = 6\cdot\text{mm} \qquad SG_{oil} = 0.827$ (From Table A.1, App. A)

Find: The manometer deflection, L when a gage pressure equal to 25 mm of water is applied to the reservoir.

Solution: We will apply the hydrostatics equations to this system.

Governing Equations:

$$\frac{dp}{dh} = \rho\cdot g$$ (Hydrostatic Pressure - h is positive downwards)

$$\rho = SG\cdot\rho_{water}$$ (Definition of Specific Gravity)

Assumptions: (1) Static liquid
(2) Incompressible liquid

Integrating the hydrostatic pressure equation we get:

$$\Delta p = \rho\cdot g\cdot\Delta h$$

Beginning at the free surface of the reservoir, and accounting for the changes in pressure with elevation:

$$p_{atm} + \Delta p + \rho_{oil}\cdot g\cdot(x + L) = p_{atm}$$

Upon simplification: $x + L = \dfrac{\Delta p}{\rho_{oil}\cdot g}$ The gage pressure is defined as: $\Delta p = \rho_{water}\cdot g\cdot\Delta h$ where $\Delta h = 25\cdot\text{mm}$

Combining these two expressions: $x + L = \dfrac{\rho_{water}\cdot g\cdot h}{\rho_{oil}\cdot g} = \dfrac{\Delta h}{SG_{oil}}$

x and L are related through the manometer dimensions: $\dfrac{\pi}{4}\cdot D^2\cdot x = \dfrac{\pi}{4}\cdot d^2\cdot L \qquad x = \left(\dfrac{d}{D}\right)^2 L$

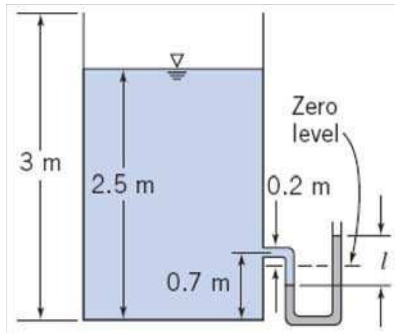
Therefore: $L = \dfrac{\Delta h}{SG_{oil}\cdot\left[1 + \left(\dfrac{d}{D}\right)^2\right]}$ Substituting values into the expression: $L = \dfrac{25\cdot\text{mm}}{0.827\cdot\left[1 + \left(\dfrac{6\cdot\text{mm}}{18\cdot\text{mm}}\right)^2\right]}$

(Note: $s = \dfrac{L}{\Delta h}$ which yields $s = 1.088$ for this manometer.) $L = 27.2\cdot\text{mm}$

Problem 3.29

[Difficulty: 2]

3.29 A rectangular tank, open to the atmosphere, is filled with water to a depth of 2.5 m as shown. A U-tube manometer is connected to the tank at a location 0.7 m above the tank bottom. If the zero level of the Meriam blue manometer fluid is 0.2 m below the connection, determine the deflection l after the manometer is connected and all air has been removed from the connecting leg.



Given: A U-tube manometer is connected to the open tank filled with water as shown (manometer fluid is Meriam blue)

$D_1 = 2.5 \cdot m \quad D_2 = 0.7 \cdot m \quad d = 0.2 \cdot m \quad SG_{oil} = 1.75$ (From Table A.1, App. A)

Find: The manometer deflection, l

Solution: We will apply the hydrostatics equations to this system.

Governing Equations:

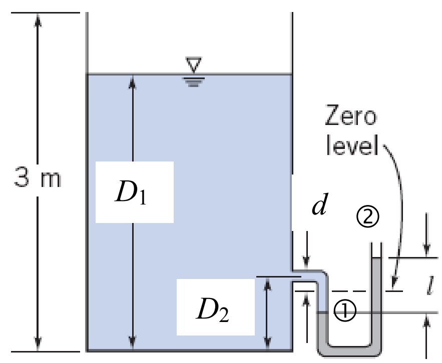
$$\frac{dp}{dh} = \rho \cdot g$$ (Hydrostatic Pressure - h is positive downwards)

$$\rho = SG \cdot \rho_{water}$$ (Definition of Specific Gravity)

Assumptions:
(1) Static liquid
(2) Incompressible liquid

Integrating the hydrostatic pressure equation we get:

$$\Delta p = \rho \cdot g \cdot \Delta h$$



When the tank is filled with water, the oil in the left leg of the manometer is displaced downward by l/2. The oil in the right leg is displaced upward by the same distance, l/2.

Beginning at the free surface of the tank, and accounting for the changes in pressure with elevation:

$$p_{atm} + \rho_{water} \cdot g \cdot \left(D_1 - D_2 + d + \frac{l}{2}\right) - \rho_{oil} \cdot g \cdot l = p_{atm}$$

Upon simplification:

$$\rho_{water} \cdot g \cdot \left(D_1 - D_2 + d + \frac{l}{2}\right) = \rho_{oil} \cdot g \cdot l \qquad D_1 - D_2 + d + \frac{l}{2} = SG_{oil} \cdot l \qquad l = \frac{D_1 - D_2 + d}{SG_{oil} - \frac{1}{2}}$$

$$l = \frac{2.5 \cdot m - 0.7 \cdot m + 0.2 \cdot m}{1.75 - \frac{1}{2}} \qquad l = 1.600\, m$$

Problem 3.30

[Difficulty: 2]

3.30 A reservoir manometer is calibrated for use with a liquid of specific gravity 0.827. The reservoir diameter is 5/8 in. and the (vertical) tube diameter is 3/16 in. Calculate the required distance between marks on the vertical scale for 1 in. of water pressure difference.

Given:

Reservoir manometer with dimensions shown. The manometer fluid specific gravity is given.

$$D = \frac{5}{8}\cdot in \qquad d = \frac{3}{16}\cdot in \quad SG_{oil} = 0.827$$

Find:

The required distance between vertical marks on the scale corresponding to Δp of 1 in water.

Solution:

We will apply the hydrostatics equations to this system.

Governing Equations:

$$\frac{dp}{dz} = -\rho\cdot g \qquad \text{(Hydrostatic Pressure - z is positive upwards)}$$

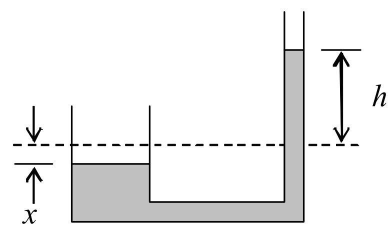
$$\rho = SG\cdot\rho_{water} \qquad \text{(Definition of Specific Gravity)}$$

Assumptions:

(1) Static liquid
(2) Incompressible liquid

Integrating the hydrostatic pressure equation we get:

$$\Delta p = -\rho\cdot g\cdot\Delta z$$



Beginning at the free surface of the tank, and accounting for the changes in pressure with elevation:

$$p_{atm} + \Delta p - \rho_{oil}\cdot g\cdot(x + h) = p_{atm}$$

Upon simplification: $\Delta p = \rho_{oil}\cdot g\cdot(x+h)$ The applied pressure is defined as: $\Delta p = \rho_{water}\cdot g\cdot l$ where $l = 1\cdot in$

Therefore: $\rho_{water}\cdot g\cdot l = \rho_{oil}\cdot g\cdot(x+h)$ $\qquad x + h = \frac{l}{SG_{oil}}$

x and h are related through the manometer dimensions: $\frac{\pi}{4}\cdot D^2\cdot x = \frac{\pi}{4}\cdot d^2\cdot h$ $\qquad x = \left(\frac{d}{D}\right)^2 h$

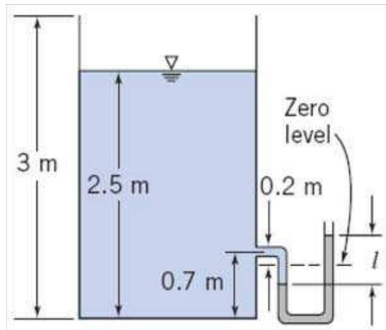
Solving for h: $h = \dfrac{l}{SG_{oil}\cdot\left[1 + \left(\frac{d}{D}\right)^2\right]}$ Substituting values into the expression: $h = \dfrac{1\cdot in}{0.827\cdot\left[1 + \left(\frac{0.1875\cdot in}{0.625\cdot in}\right)^2\right]}$

$$h = 1.109\cdot in$$

Problem 3.31

[Difficulty: 2]

3.31 The manometer fluid of Problem 3.29 is replaced with mercury (same zero level). The tank is sealed and the air pressure is increased to a gage pressure of 0.5 atm. Determine the deflection l.



Given: A U-tube manometer is connected to the open tank filled with water as shown (manometer fluid is mercury). The tank is sealed and pressurized.

$$D_1 = 2.5 \cdot m \quad D_2 = 0.7 \cdot m \quad d = 0.2 \cdot m \quad p_o = 0.5 \cdot atm \qquad SG_{Hg} = 13.55 \text{ (From Table A.1, App. A)}$$

Find: The manometer deflection, l

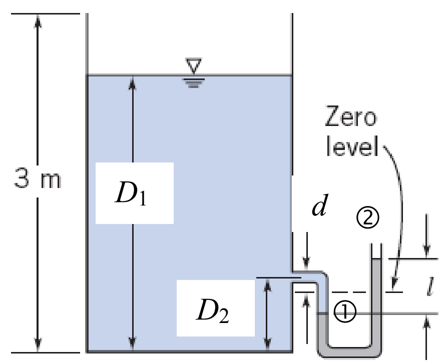
Solution: We will apply the hydrostatics equations to this system.

Governing Equations:

$$\frac{dp}{dh} = \rho \cdot g \qquad \text{(Hydrostatic Pressure - h is positive downwards)}$$

$$\rho = SG \cdot \rho_{water} \qquad \text{(Definition of Specific Gravity)}$$

Assumptions:
(1) Static liquid
(2) Incompressible liquid



Integrating the hydrostatic pressure equation we get:

$$\Delta p = \rho \cdot g \cdot \Delta h$$

When the tank is filled with water and pressurized, the mercury in the left leg of the manometer is displaced downward by l/2. The mercury in the right leg is displaced upward by the same distance, l/2.

Beginning at the free surface of the tank, and accounting for the changes in pressure with elevation:

$$p_{atm} + p_o + \rho_{water} \cdot g \cdot \left(D_1 - D_2 + d + \frac{l}{2} \right) - \rho_{Hg} \cdot g \cdot l = p_{atm}$$

Upon simplification:

$$p_o + \rho_{water} \cdot g \cdot \left(D_1 - D_2 + d + \frac{l}{2} \right) = \rho_{Hg} \cdot g \cdot l \qquad l = \frac{\dfrac{p_o}{\rho_{water} \cdot g} + D_1 - D_2 + d}{SG_{Hg} - \dfrac{1}{2}}$$

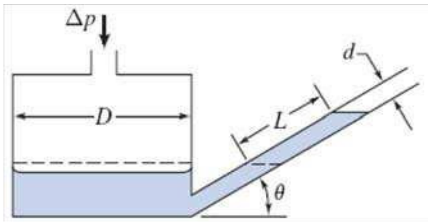
Substituting values into the expression:

$$l = \frac{\left(0.5 \cdot atm \times \dfrac{1.013 \times 10^5 \cdot N}{m^2 \cdot atm} \times \dfrac{1}{1000} \cdot \dfrac{m^3}{kg} \times \dfrac{1}{9.8} \cdot \dfrac{s^2}{m} \right) + 2.5 \cdot m - 0.7 \cdot m + 0.2 \cdot m}{13.55 - \dfrac{1}{2}} \qquad l = 0.549 \, m$$

Problem 3.32

[Difficulty: 3]

3.32 The inclined-tube manometer shown has $D = 96$ mm and $d = 8$ mm. Determine the angle, θ, required to provide a 5 : 1 increase in liquid deflection, L, compared with the total deflection in a regular U-tube manometer. Evaluate the sensitivity of this inclined-tube manometer.



Given: Inclined manometer as shown.

$D = 96\cdot\text{mm} \quad d = 8\cdot\text{mm}$

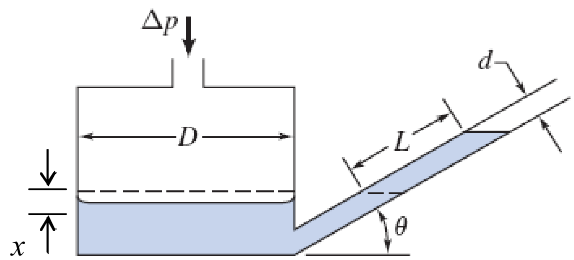
Angle θ is such that the liquid deflection L is five times that of a regular U-tube manometer.

Find: Angle θ and manometer sensitivity.

Solution: We will apply the hydrostatics equations to this system.

Governing Equation: $\dfrac{dp}{dz} = -\rho\cdot g$ (Hydrostatic Pressure - z is positive upwards)

Assumptions: (1) Static liquid
(2) Incompressible liquid



Integrating the hydrostatic pressure equation we get:

$$\Delta p = -\rho\cdot g\cdot\Delta z$$

Applying this equation from point 1 to point 2:

$$p_1 - \rho\cdot g\cdot(x + L\cdot\sin(\theta)) = p_2$$

Upon simplification: $$p_1 - p_2 = \rho\cdot g\cdot(x + L\cdot\sin(\theta))$$

Since the volume of the fluid must remain constant: $\dfrac{\pi}{4}\cdot D^2\cdot x = \dfrac{\pi}{4}\cdot d^2\cdot L \qquad x = \left(\dfrac{d}{D}\right)^2\cdot L$

Therefore: $p_1 - p_2 = \rho\cdot g\cdot L\cdot\left[\left(\dfrac{d}{D}\right)^2 + \sin(\theta)\right]$

Now for a U-tube manometer: $p_1 - p_2 = \rho\cdot g\cdot h$ Hence: $\dfrac{p_{1incl} - p_{2incl}}{p_{1U} - p_{2U}} = \dfrac{\rho\cdot g\cdot L\cdot\left[\left(\dfrac{d}{D}\right)^2 + \sin(\theta)\right]}{\rho\cdot g\cdot h}$

For equal applied pressures: $L\cdot\left[\left(\dfrac{d}{D}\right)^2 + \sin(\theta)\right] = h$ Since L/h = 5: $\sin(\theta) = \dfrac{h}{L} - \left(\dfrac{d}{D}\right)^2 = \dfrac{1}{5} - \left(\dfrac{8\cdot\text{mm}}{96\cdot\text{mm}}\right)^2$

$$\theta = 11.13\cdot\text{deg}$$

The sensitivity of the manometer: $s = \dfrac{L}{\Delta h_e} = \dfrac{L}{SG\cdot h}$ $\qquad s = \dfrac{5}{SG}$

Problem 3.33

[Difficulty: 3]

3.33 The inclined-tube manometer shown has $D = 76$ mm and $d = 8$ mm, and is filled with Meriam red oil. Compute the angle, θ, that will give a 15-cm oil deflection along the inclined tube for an applied pressure of 25 mm of water (gage). Determine the sensitivity of this manometer.

Given: Data on inclined manometer

Find: Angle θ for given data; find sensitivity

Solution:

Basic equation $\frac{dp}{dy} = -\rho \cdot g$ or, for constant ρ $\Delta p = \rho \cdot g \cdot \Delta h$ where Δh is height difference

Under applied pressure $$\Delta p = SG_{Mer} \cdot \rho \cdot g \cdot (L \cdot \sin(\theta) + x) \qquad (1)$$

From Table A.1 $SG_{Mer} = 0.827$

and Δp = 1 in. of water, or $\Delta p = \rho \cdot g \cdot h$ where $h = 25 \cdot mm$ $h = 0.025\,m$

$$\Delta p = 1000 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times 0.025 \cdot m \times \frac{N \cdot s^2}{kg \cdot m} \qquad \Delta p = 245\,Pa$$

The volume of liquid must remain constant, so $x \cdot A_{res} = L \cdot A_{tube}$ $$x = L \cdot \frac{A_{tube}}{A_{res}} = L \cdot \left(\frac{d}{D}\right)^2 \qquad (2)$$

Combining Eqs 1 and 2 $$\Delta p = SG_{Mer} \cdot \rho \cdot g \cdot \left[L \cdot \sin(\theta) + L \cdot \left(\frac{d}{D}\right)^2\right]$$

Solving for θ $$\sin(\theta) = \frac{\Delta p}{SG_{Mer} \cdot \rho \cdot g \cdot L} - \left(\frac{d}{D}\right)^2$$

$$\sin(\theta) = 245 \cdot \frac{N}{m^2} \times \frac{1}{0.827} \times \frac{1}{1000} \cdot \frac{m^3}{kg} \times \frac{1}{9.81} \cdot \frac{s^2}{m} \times \frac{1}{0.15} \cdot \frac{1}{m} \times \frac{kg \cdot m}{s^2 \cdot N} - \left(\frac{8}{76}\right)^2 = 0.186$$

$$\theta = 11 \cdot deg$$

The sensitivity is the ratio of manometer deflection to a vertical water manometer

$$s = \frac{L}{h} = \frac{0.15 \cdot m}{0.025 \cdot m} \qquad s = 6$$

Problem 3.34

[Difficulty: 4]

3.34 A barometer accidentally contains 6.5 inches of water on top of the mercury column (so there is also water vapor instead of a vacuum at the top of the barometer). On a day when the temperature is 70°F, the mercury column height is 28.35 inches (corrected for thermal expansion). Determine the barometric pressure in psia. If the ambient temperature increased to 85°F and the barometric pressure did not change, would the mercury column be longer, be shorter, or remain the same length? Justify your answer.

Given: Barometer with water on top of the mercury column, Temperature is known:

$h_2 = 6.5 \cdot in$ $\quad h_1 = 28.35 \cdot in$ $\quad SG_{Hg} = 13.55$ (From Table A.2, App. A) $\quad T = 70\ °F$

$p_v = 0.363 \cdot psi$ (From Table A.7, App. A)

Find: (a) Barometric pressure in psia
(b) Effect of increase in ambient temperature on length of mercury column for the same barometric pressure: $T_f = 85\ °F$

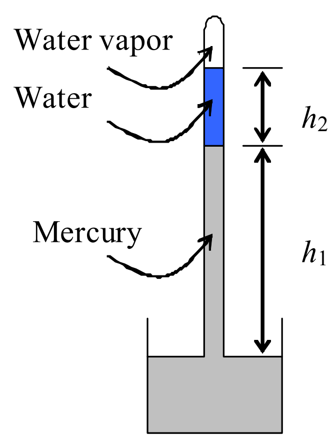
Solution: We will apply the hydrostatics equations to this system.

Governing Equations:

$$\frac{dp}{dh} = -\rho \cdot g \qquad \text{(Hydrostatic Pressure - h is positive downwards)}$$

$$\rho = SG \cdot \rho_{water} \qquad \text{(Definition of Specific Gravity)}$$

Assumptions: (1) Static liquid
(2) Incompressible liquid



Integrating the hydrostatic pressure equation we get:

$$\Delta p = \rho \cdot g \cdot \Delta h$$

Start at the free surface of the mercury and progress through the barometer to the vapor pressure of the water:

$$p_{atm} - \rho_{Hg} \cdot g \cdot h_1 - \rho_{water} \cdot g \cdot h_2 = p_v$$

$$p_{atm} = p_v + \rho_{water} \cdot g \cdot \left(SG_{Hg} \cdot h_1 + h_2\right)$$

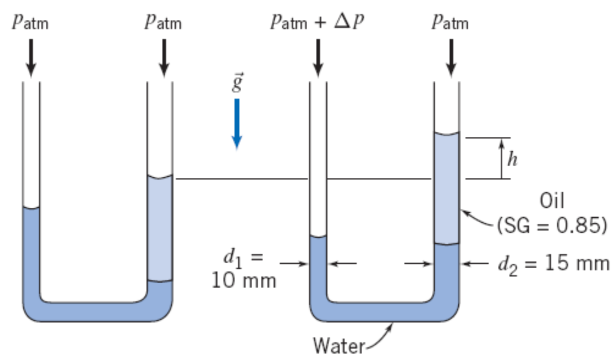
$$p_{atm} = 0.363 \cdot \frac{lbf}{in^2} + 1.93 \cdot \frac{slug}{ft^3} \times 32.2 \cdot \frac{ft}{s^2} \times \frac{lbf \cdot s^2}{slug \cdot ft} \times (13.55 \times 28.35 \cdot in + 6.5 \cdot in) \times \left(\frac{ft}{12 \cdot in}\right)^3 \qquad p_{atm} = 14.41 \cdot \frac{lbf}{in^2}$$

At the higher temperature, the vapor pressure of water increases to 0.60 psi. Therefore, if the atmospheric pressure were to remain constant, the length of the mercury column would have to decrease - the increased water vapor would push the mercury out of the tube!

Problem 3.35

[Difficulty: 3]

3.35 A student wishes to design a manometer with better sensitivity than a water-filled U-tube of constant diameter. The student's concept involves using tubes with different diameters and two liquids, as shown. Evaluate the deflection h of this manometer, if the applied pressure difference is $\Delta p = 250\ \text{N/m}^2$. Determine the sensitivity of this manometer. Plot the manometer sensitivity as a function of the diameter ratio d_2/d_1.



Given: U-tube manometer with tubes of different diameter and two liquids, as shown.

$$d_1 = 10\cdot\text{mm} \quad d_2 = 15\cdot\text{mm} \quad SG_{oil} = 0.85$$

Find: (a) the deflection, h, corresponding to $\quad \Delta p = 250\cdot\frac{\text{N}}{\text{m}^2}$
(b) the sensitivity of the manometer

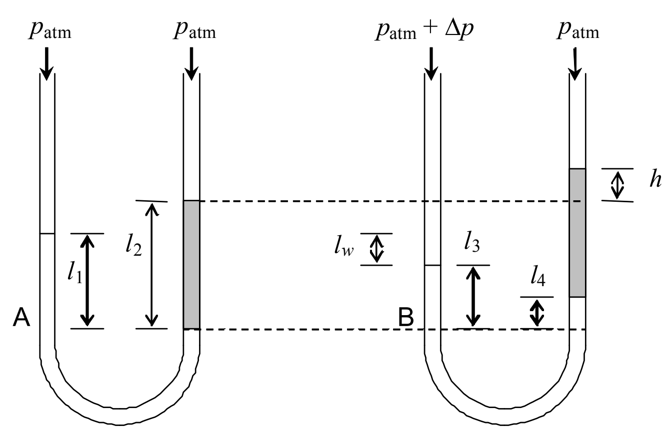
Solution: We will apply the hydrostatics equations to this system.

Governing Equations:

$$\frac{dp}{dz} = -\rho\cdot g \qquad \text{(Hydrostatic Pressure - z is positive upwards)}$$

$$\rho = SG\cdot\rho_{water} \qquad \text{(Definition of Specific Gravity)}$$

Assumptions: (1) Static liquid
(2) Incompressible liquid



Integrating the hydrostatic pressure equation we get:

$$p - p_o = -\rho\cdot g\cdot\left(z - z_o\right) = \rho\cdot g\cdot\left(z_o - z\right)$$

From the left diagram:

$$p_A - p_{atm} = \rho_{water}\cdot g\cdot l_1 = \rho_{oil}\cdot g\cdot l_2 \qquad (1)$$

From the right diagram:

$$p_B - \left(p_{atm} + \Delta p\right) = \rho_{water}\cdot g\cdot l_3 \qquad (2)$$

$$p_B - p_{atm} = \rho_{water}\cdot g\cdot l_4 + \rho_{oil}\cdot g\cdot l_2 \qquad (3)$$

Combining these three equations: $\Delta p = \rho_{water}\cdot g\cdot\left(l_4 - l_3\right) + \rho_{oil}\cdot g\cdot l_2 = \rho_{water}\cdot g\cdot\left(l_4 + l_1 - l_3\right)$

From the diagram we can see $\quad l_w = l_1 - l_3 \quad$ and $\quad h = l_4 \quad$ Therefore:

$$\Delta p = \rho_{water}\cdot g\cdot\left(h + l_w\right) \qquad (4)$$

We can relate l_w to h since the volume of water in the manometer is constant: $\quad \frac{\pi}{4}\cdot d_1^{\,2}\cdot l_w = \frac{\pi}{4}\cdot d_2^{\,2}\cdot h \qquad l_w = \left(\frac{d_2}{d_1}\right)^2\cdot h$

Substituting this into (4) yields: $\Delta p = \rho_{water} \cdot g \cdot h \cdot \left[1 + \left(\frac{d_2}{d_1}\right)^2\right]$ Solving for h: $h = \frac{\Delta p}{\rho_{water} \cdot g \cdot \left[1 + \left(\frac{d_2}{d_1}\right)^2\right]}$

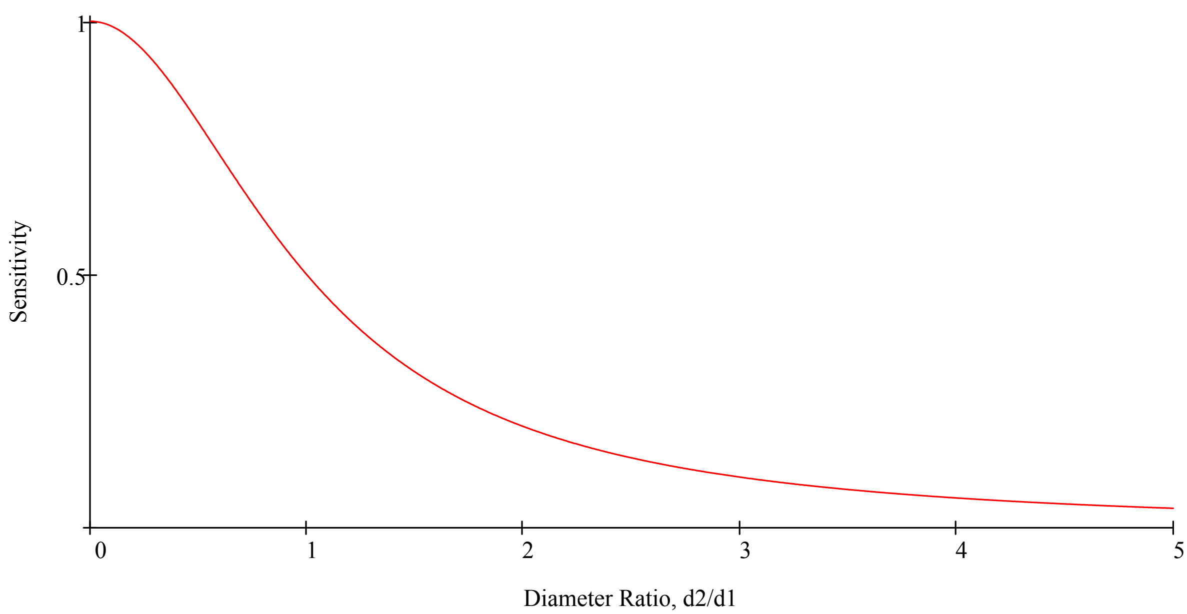
Substituting values into the equation: $h = 250 \cdot \frac{N}{m^2} \times \frac{1}{999} \cdot \frac{m^3}{kg} \times \frac{1}{9.81} \frac{s^2}{m} \times \frac{1}{1 + \left(\frac{15 \cdot mm}{10 \cdot mm}\right)^2} \times \frac{kg \cdot m}{N \cdot s^2} \times \frac{10^3 \cdot mm}{m}$ $h = 7.85 \cdot mm$

The sensitivity for the manometer is defined as: $s = \frac{h}{\Delta h_e}$ where $\Delta p = \rho_{water} \cdot g \cdot \Delta h_e$

Therefore: $s = \frac{1}{1 + \left(\frac{d_2}{d_1}\right)^2}$ $s = \frac{1}{1 + \left(\frac{15 \cdot mm}{10 \cdot mm}\right)^2}$ $s = 0.308$

The design is a poor one. The sensitivity could be improved by interchanging d_2 and d_1, i.e., having d_2 smaller than d_1

A plot of the manometer sensitivity is shown below:



Problem 3.36

[Difficulty: 3]

3.36 A water column stands 50 mm high in a 2.5-mm diameter glass tube. What would be the column height if the surface tension were zero? What would be the column height in a 1.0-mm diameter tube?

Given: Water column standin in glass tube

$$\Delta h = 50\cdot\text{mm} \quad D = 2.5\cdot\text{mm} \quad \sigma = 72.8\times10^{-3}\frac{\text{N}}{\text{m}} \quad \text{(From Table A.4, App. A)}$$

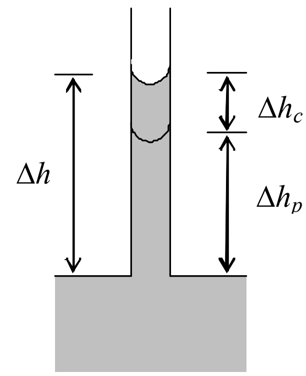
Find: (a) Column height if surface tension were zero.
(b) Column height in 1 mm diameter tube

Solution: We will apply the hydrostatics equations to this system.

Governing Equations:

$$\frac{dp}{dh} = \rho\cdot g \qquad \text{(Hydrostatic Pressure - h is positive downwards)}$$

$$\Sigma F_z = 0 \qquad \text{(Static Equilibrium)}$$

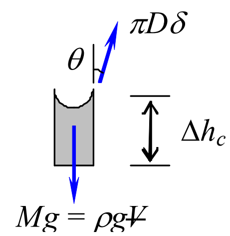


Assumptions:
(1) Static, incompressible liquid
(2) Neglect volume under meniscus
(3) Applied pressure remains constant
(4) Column height is sum of capillary rise and pressure difference

Assumption #4 can be written as: $\Delta h = \Delta h_c + \Delta h_p$

Choose a free-body diagram of the capillary rise portion of the column for analysis:

$$\Sigma F_z = \pi\cdot D\cdot\sigma\cdot\cos(\theta) - \frac{\pi}{4}\cdot D^2\cdot\rho\cdot g\cdot\Delta h_c = 0 \qquad \text{Therefore:} \quad \Delta h_c = \frac{4\cdot\sigma}{\rho\cdot g\cdot D}\cdot\cos(\theta)$$



Substituting values:

$$\Delta h_c = 4\times72.8\times10^{-3}\cdot\frac{\text{N}}{\text{m}}\times\frac{1}{999}\cdot\frac{\text{m}^3}{\text{kg}}\times\frac{1}{9.81}\cdot\frac{\text{s}^2}{\text{m}}\times\frac{1}{2.5}\cdot\frac{1}{\text{mm}}\times\frac{\text{kg}\cdot\text{m}}{\text{N}\cdot\text{s}^2}\times\left(\frac{10^3\cdot\text{mm}}{\text{m}}\right)^2$$

$$\Delta h_c = 11.89\cdot\text{mm}$$

Therefore: $\Delta h_p = \Delta h - \Delta h_c \quad \Delta h_p = 50\cdot\text{mm} - 11.89\cdot\text{mm}$ $\qquad \Delta h_p = 38.1\cdot\text{mm}$ (result for $\sigma = 0$)

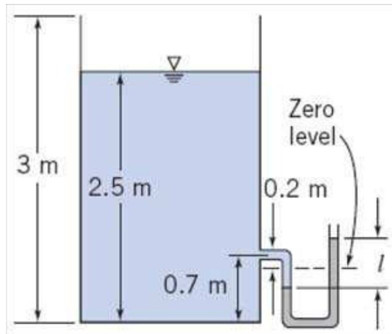
For the 1 mm diameter tube:

$$\Delta h_c = 4\times72.8\times10^{-3}\cdot\frac{\text{N}}{\text{m}}\times\frac{1}{999}\cdot\frac{\text{m}^3}{\text{kg}}\times\frac{1}{9.81}\cdot\frac{\text{s}^2}{\text{m}}\times\frac{1}{1}\cdot\frac{1}{\text{mm}}\times\frac{\text{kg}\cdot\text{m}}{\text{N}\cdot\text{s}^2}\times\left(\frac{10^3\cdot\text{mm}}{\text{m}}\right)^2 \qquad \Delta h_c = 29.71\cdot\text{mm}$$

$$\Delta h = 29.7\cdot\text{mm} + 38.1\cdot\text{mm} \qquad \Delta h = 67.8\cdot\text{mm}$$

Problem 3.37 [Difficulty: 4]

3.37 If the tank of Problem 3.29 is sealed tightly and water drains slowly from the bottom of the tank, determine the deflection, l, after the system has attained equilibrium.



Given: Sealed tank is partially filled with water. Water drains slowly from the tank until the system attains equilibrium. U-tube manometer is connected to the tank as shown. (Meriam blue in manometer)

$L = 3 \cdot m \quad D_1 = 2.5 \cdot m \quad D_2 = 0.7 \cdot m \quad d = 0.2 \cdot m \quad SG_{oil} = 1.75$ (From Table A.2, App. A)

Find: The manometer deflection, l, under equilibrium conditions

Solution: We will apply the hydrostatics equations to this system.

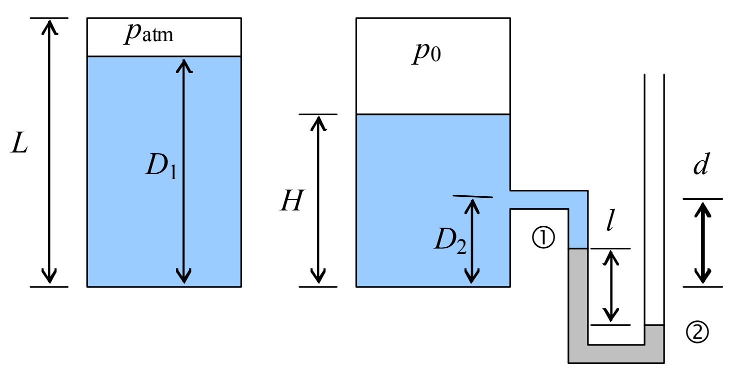
Governing Equations:

$$\frac{dp}{dh} = \rho \cdot g$$ (Hydrostatic Pressure - h is positive downwards)

$$p \cdot V = M \cdot R \cdot T$$ (Ideal gas equation of state)

$$\rho = SG \cdot \rho_{water}$$ (Definition of Specific Gravity)

Assumptions:
(1) Static liquid
(2) Incompressible liquid
(3) Air in tank behaves ideally



Integrating the hydrostatic pressure equation we get:

$$\Delta p = \rho \cdot g \cdot \Delta h$$

To determine the surface pressure p_o under equilibrium conditions we assume that the air expands at constant temperature:

$$\frac{p_a \cdot V_a}{p_o \cdot V_o} = \frac{M \cdot R \cdot T_a}{M \cdot R \cdot T_o} \qquad \text{Thus,} \quad p_o = \frac{V_a}{V_o} \cdot p_a = \frac{(L - D_1) \cdot A}{(L - H) \cdot A} \cdot p_a$$

Simplifying: $p_o = \dfrac{(L - D_1)}{(L - H)} \cdot p_a$ Now under equilibrium conditions: $p_o + \rho_{water} \cdot g \cdot H = p_a$ Combining these expressions:

$$\frac{(L - D_1)}{(L - H)} \cdot p_a + \rho_{water} \cdot g \cdot H = p_a$$

Upon rearranging: $\rho_{water} \cdot g \cdot H^2 - \left(p_a + \rho_{water} \cdot g \cdot L\right) \cdot H + D_1 \cdot p_a = 0$

Now we apply the quadratic formula to solve for H:

$$a = \rho_{water} \cdot g \qquad a = 999 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \qquad a = 9.8 \times 10^3 \cdot \frac{Pa}{m}$$

$$b = -\left(p_a + \rho_{water} \cdot g \cdot L\right) \qquad b = -\left(1.013 \times 10^5 \cdot \frac{N}{m^2} + 999 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times 3 \cdot m\right) \qquad b = -1.307 \times 10^5 \, Pa$$

$$c = D_1 \cdot p_a \qquad c = 2.5 \cdot m \times 1.013 \times 10^5 \cdot \frac{N}{m^2} \qquad c = 2.532 \times 10^5 \cdot Pa \cdot m$$

$$H_{upper} = \frac{-b + \sqrt{b^2 - 4 \cdot a \cdot c}}{2 \cdot a} \qquad H_{upper} = \frac{-\left(-1.307 \times 10^5 \cdot Pa\right) + \sqrt{\left(-1.307 \times 10^5 \cdot Pa\right)^2 - 4 \times 9.8 \times 10^3 \cdot \frac{Pa}{m} \times 2.532 \times 10^5 \cdot Pa \cdot m}}{2 \times 9.8 \times 10^3 \cdot \frac{Pa}{m}}$$

$$H_{upper} = 10.985 \, m$$

$$H_{lower} = \frac{-b - \sqrt{b^2 - 4 \cdot a \cdot c}}{2 \cdot a} \qquad H_{lower} = \frac{-\left(-1.307 \times 10^5 \cdot Pa\right) - \sqrt{\left(-1.307 \times 10^5 \cdot Pa\right)^2 - 4 \times 9.8 \times 10^3 \cdot \frac{Pa}{m} \times 2.532 \times 10^5 \cdot Pa \cdot m}}{2 \times 9.8 \times 10^3 \cdot \frac{Pa}{m}}$$

$$H_{lower} = 2.352 \, m$$

Since H can not be greater than 3 m (otherwise the tank would overflow!), we must select the lower value for H: $H = 2.352 \, m$

Solving for the pressure inside the tank: $p_o = \frac{(3 \cdot m - 2.5 \cdot m)}{(3 \cdot m - 2.352 \cdot m)} \times 1.013 \times 10^5 \cdot Pa \qquad p_o = 7.816 \times 10^4 \, Pa$

Applying the hydrostatic pressure equation to the manometer: $p_o + \rho_{water} \cdot g \cdot \left(H - D_2 + d - \frac{l}{2}\right) + \rho_{oil} \cdot g \cdot l = p_a$

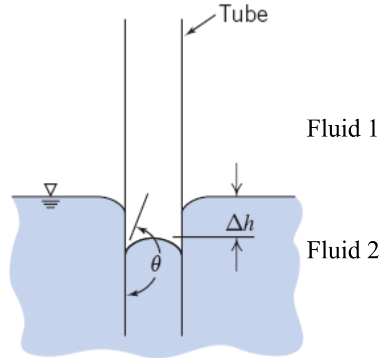
Solving for the manometer deflection: $l = \left(\frac{p_a - p_o}{\rho_{water} \cdot g} - H + D_2 - d\right) \cdot \frac{1}{SG_{oil} - \frac{1}{2}}$

$$l = \left[\left(1.013 \times 10^5 \cdot \frac{N}{m^2} - 7.816 \times 10^4 \cdot \frac{N}{m^2}\right) \times \frac{1}{999} \cdot \frac{m^3}{kg} \times \frac{1}{9.81} \cdot \frac{s^2}{m} \times \frac{kg \cdot m}{N \cdot s^2} - 2.352 \cdot m + 0.7 \cdot m - 0.2 \cdot m\right] \cdot \frac{1}{1.75 - \frac{1}{2}} \qquad l = 0.407 \, m$$

Problem 3.38

[Difficulty :2]

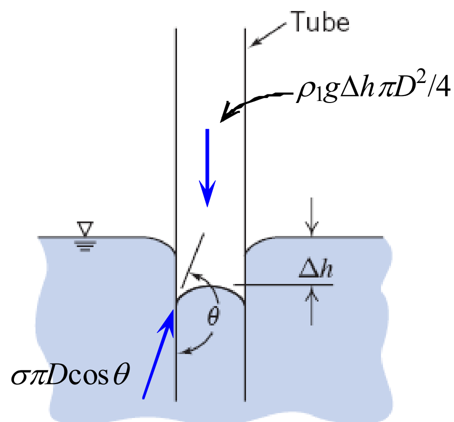
3.38 Consider a small-diameter open-ended tube inserted at the interface between two immiscible fluids of different densities. Derive an expression for the height difference Δh between the interface level inside and outside the tube in terms of tube diameter D, the two fluid densities ρ_1 and ρ_2, and the surface tension σ and angle θ for the two fluids' interface. If the two fluids are water and mercury, find the height difference if the tube diameter is 40 mils (1 mil = 0.001 in.).



Given: Two fluids inside and outside a tube

Find: (a) An expression for height Δh
(b) Height difference when D =0.040 in for water/mercury

Assumptions: (1) Static, incompressible fluids
(2) Neglect meniscus curvature for column height and volume calculations



Solution:

A free-body vertical force analysis for the section of fluid 1 height Δh in the tube below the "free surface" of fluid 2 leads to

$$\sum F = 0 = \Delta p\cdot\frac{\pi\cdot D^2}{4} - \rho_1\cdot g\cdot\Delta h\cdot\frac{\pi\cdot D^2}{4} + \pi\cdot D\cdot\sigma\cdot\cos(\theta)$$

where Δp is the pressure difference generated by fluid 2 over height Δh, $\quad \Delta p = \rho_2\cdot g\cdot\Delta h$

Hence $\quad \Delta p\cdot\frac{\pi\cdot D^2}{4} - \rho_1\cdot g\cdot\Delta h\cdot\frac{\pi\cdot D^2}{4} = \rho_2\cdot g\cdot\Delta h\cdot\frac{\pi\cdot D^2}{4} - \rho_1\cdot g\cdot\Delta h\cdot\frac{\pi\cdot D^2}{4} = -\pi\cdot D\cdot\sigma\cdot\cos(\theta)$

Solving for Δh $\quad \Delta h = -\frac{4\cdot\sigma\cdot\cos(\theta)}{g\cdot D\cdot(\rho_2 - \rho_1)}$

For fluids 1 and 2 being water and mercury (for mercury σ = 375 mN/m and θ = 140º, from Table A.4), solving for Δh when D = 0.040 in

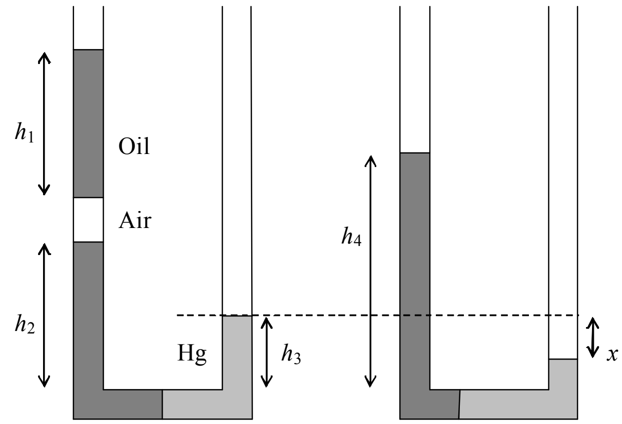
$$\Delta h = -4\times 0.375\cdot\frac{N}{m}\times\frac{lbf}{4.448\cdot N}\times\frac{0.0254m}{in}\times\cos(140\cdot deg)\times\frac{s^2}{32.2\cdot ft}\times\frac{1}{0.040\cdot in}\times\frac{ft^3}{1.94\cdot slug}\times\left(\frac{12\cdot in}{ft}\right)^3\times\frac{1}{(13.6-1)}\times\frac{slug\,ft}{lbf\cdot s^2}$$

$$\Delta h = 0.360\cdot in$$

Problem 3.39

[Difficulty: 2]

3.39 You have a manometer consisting of a tube that is 0.5 in. inner diameter (ID). On one side, the manometer leg contains mercury, 0.6 in.^{3} of an oil (SG = 1.4), and 0.2 in.^{3} of air as a bubble in the oil. The other leg contains only mercury. Both legs are open to the atmosphere and are in a static condition. An accident occurs in which 0.2 in.^{3} of the oil and the air bubble are removed from one leg. How much do the mercury height levels change?



Given:

Data on manometer before and after an "accident"

Find:

Change in mercury level

Assumptions:

(1) Liquids are incompressible and static
(2) Pressure change across air in bubble is negligible
(3) Any curvature of air bubble surface can be neglected in volume calculations

Solution:

Basic equation $\quad \dfrac{dp}{dy} = -\rho \cdot g \quad$ or, for constant $\rho \quad \Delta p = \rho \cdot g \cdot \Delta h \quad$ where Δh is height difference

For the initial state, working from right to left $\quad p_{atm} = p_{atm} + SG_{Hg} \cdot \rho \cdot g \cdot h_3 - SG_{oil} \cdot \rho \cdot g \cdot \left(h_1 + h_2\right)$

$$SG_{Hg} \cdot \rho \cdot g \cdot h_3 = SG_{oil} \cdot \rho \cdot g \cdot \left(h_1 + h_2\right) \qquad (1)$$

Note that the air pocket has no effect!

For the final state, working from right to left $\quad p_{atm} = p_{atm} + SG_{Hg} \cdot \rho \cdot g \cdot \left(h_3 - x\right) - SG_{oil} \cdot \rho \cdot g \cdot h_4$

$$SG_{Hg} \cdot \rho \cdot g \cdot \left(h_3 - x\right) = SG_{oil} \cdot \rho \cdot g \cdot h_4 \qquad (2)$$

The two unknowns here are the mercury levels before and after (i.e., h_3 and x)

Combining Eqs. 1 and 2 $\quad SG_{Hg} \cdot \rho \cdot g \cdot x = SG_{oil} \cdot \rho \cdot g \cdot \left(h_1 + h_2 - h_4\right) \qquad x = \dfrac{SG_{oil}}{SG_{Hg}} \cdot \left(h_1 + h_2 - h_4\right) \qquad (3)$

From Table A.1 $\quad SG_{Hg} = 13.55$

The term $\quad h_1 + h_2 - h_4 \quad$ is the difference between the total height of oil before and after the accident

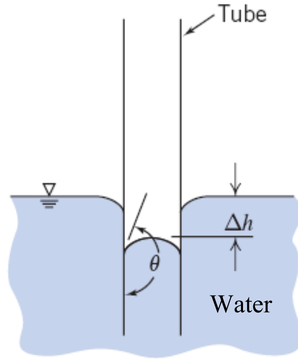
$$h_1 + h_2 - h_4 = \frac{\Delta V}{\left(\frac{\pi \cdot d^2}{4}\right)} = \frac{4}{\pi} \times \left(\frac{1}{0.5 \cdot in}\right)^2 \times 0.2 \cdot in^3 = 1.019 \cdot in$$

Then from Eq. 3 $\quad x = \dfrac{1.4}{13.55} \times 1.019 \cdot in \qquad x = 0.1053 \cdot in$

Problem 3.40

[Difficulty: 2]

3.40 Compare the height due to capillary action of water exposed to air in a circular tube of diameter $D = 0.5$ mm, and between two infinite vertical parallel plates of gap $a = 0.5$ mm.



Given: Water in a tube or between parallel plates

Find: Height Δh for each system

Solution:

a) Tube: A free-body vertical force analysis for the section of water height Δh above the "free surface" in the tube, as shown in the figure, leads to

$$\sum F = 0 = \pi \cdot D \cdot \sigma \cdot \cos(\theta) - \rho \cdot g \cdot \Delta h \cdot \frac{\pi \cdot D^2}{4}$$

Assumption: Neglect meniscus curvature for column height and volume calculations

Solving for Δh $$\Delta h = \frac{4 \cdot \sigma \cdot \cos(\theta)}{\rho \cdot g \cdot D}$$

b) Parallel Plates: A free-body vertical force analysis for the section of water height Δh above the "free surface" between plates arbitrary width w (similar to the figure above), leads to

$$\sum F = 0 = 2 \cdot w \cdot \sigma \cdot \cos(\theta) - \rho \cdot g \cdot \Delta h \cdot w \cdot a$$

Solving for Δh $$\Delta h = \frac{2 \cdot \sigma \cdot \cos(\theta)}{\rho \cdot g \cdot a}$$

For water σ = 72.8 mN/m and $\theta = 0^o$ (Table A.4), so

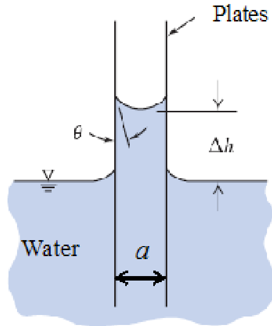
a) Tube $$\Delta h = \frac{4 \times 0.0728 \cdot \frac{N}{m}}{999 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times 0.005 \cdot m} \times \frac{kg \cdot m}{N \cdot s^2}$$ $\Delta h = 5.94 \times 10^{-3} m$ $\Delta h = 5.94 \cdot mm$

b) Parallel Plates $$\Delta h = \frac{2 \times 0.0728 \cdot \frac{N}{m}}{999 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times 0.005 \cdot m} \times \frac{kg \cdot m}{N \cdot s^2}$$ $\Delta h = 2.97 \times 10^{-3} m$ $\Delta h = 2.97 \cdot mm$

Problem 3.41

[Difficulty: 3]

3.41 Two vertical glass plates 12 in. × 12 in. are placed in an open tank containing water. At one end the gap between the plates is 0.004 in., and at the other it is 0.080 in. Plot the curve of water height between the plates from one end of the pair to the other.



Given: Geometry of vertical plates

Find: Curve of water height due to capillary action

Assumption: Water is static and incompressible

Solution:

Parallel Plates: A free-body vertical force analysis for the section of water height Δh above the "free surface" between plates arbitrary width w (similar to the figure above), leads to

$$\sum F = 0 = 2 \cdot w \cdot \sigma \cdot \cos(\theta) - \rho \cdot g \cdot \Delta h \cdot w \cdot a$$

Solving for Δh

$$\Delta h = \frac{2 \cdot \sigma \cdot \cos(\theta)}{\rho \cdot g \cdot a}$$

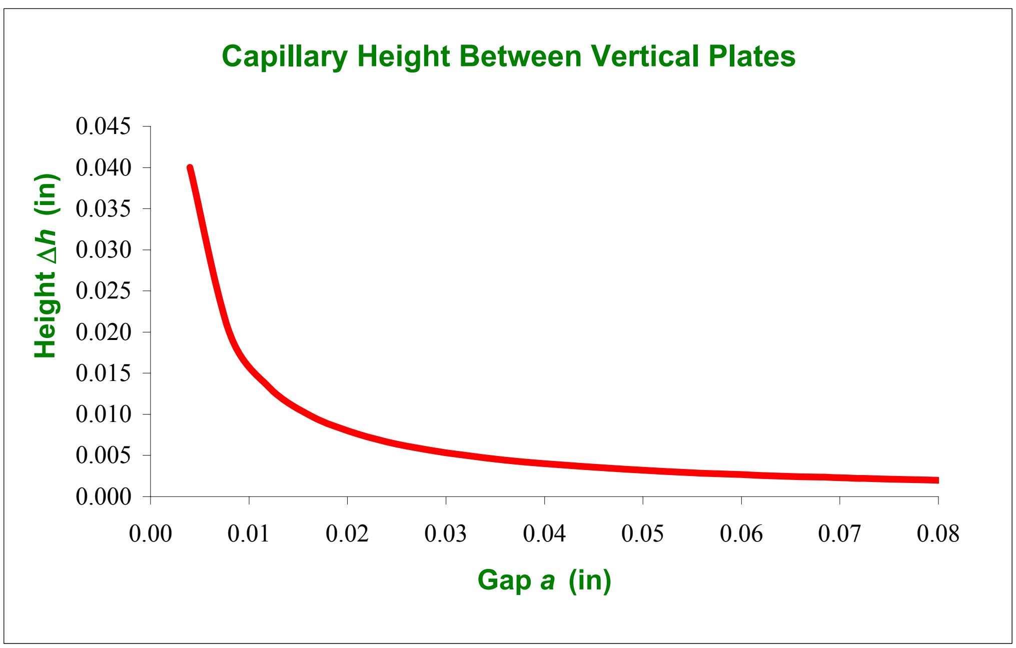
For water σ = 72.8 mN/m = 0.00537 lbf/ft and θ = 0º (Table A.4), so

σ = 0.005 lbf/ft

ρ = 1.94 slug/ft^3

Using the formula above

a (in)	Δh (in)
0.004	0.0400
0.008	0.0200
0.012	0.0133
0.016	0.0100
0.020	0.0080
0.024	0.0067
0.028	0.0057
0.032	0.0050
0.036	0.0044
0.040	0.0040
0.044	0.0036
0.048	0.0033
0.052	0.0031
0.056	0.0029
0.060	0.0027
0.064	0.0025
0.068	0.0024
0.072	0.0022
0.080	0.0020



Problem 3.42

[Difficulty: 3]

3.42 Based on the atmospheric temperature data of the U.S. Standard Atmosphere of Fig. 3.3, compute and plot the pressure variation with altitude, and compare with the pressure data of Table A.3.

Given: Atmospheric temperature data

Find: Pressure variation; compare to Table A.3

Solution:

From Section 3-3: $$\frac{dp}{dz} = -\rho \cdot z$$

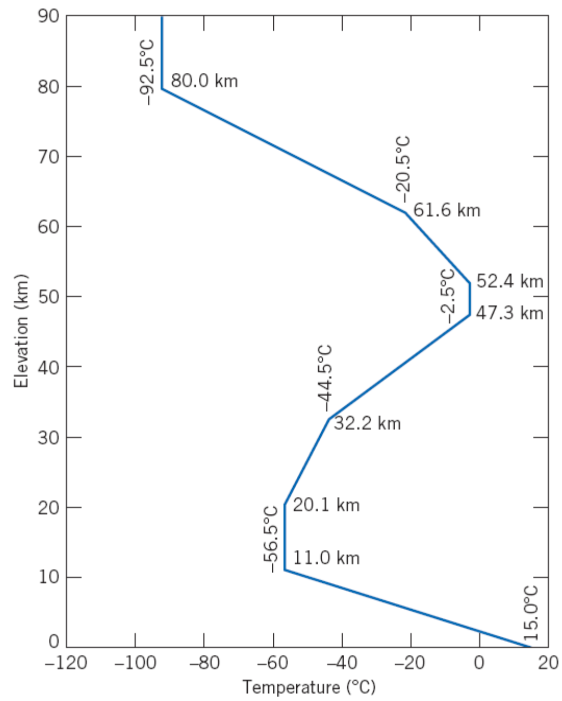
For a linear temperature variation $$m = -\frac{dT}{dz} = \text{const} \qquad p = p_0 \cdot \left(\frac{T}{T_0}\right)^{\frac{g}{m \cdot R}}$$

For isothermal conditions (Example 3.4) $$p = p_0 \cdot e^{-\frac{g \cdot (z - z_0)}{R \cdot T}}$$

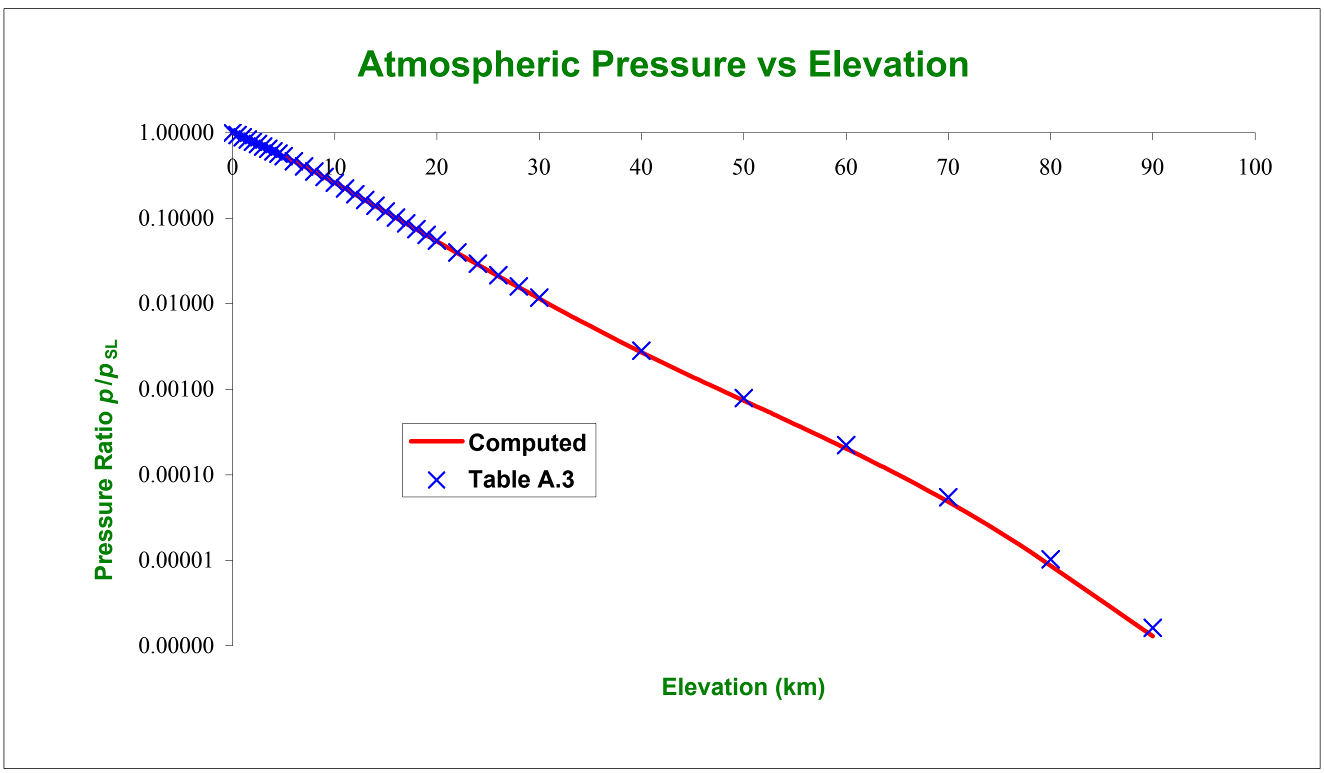
In these equations p_0, T_0, and z_0 are reference conditions

p_{SL} =	101	kPa
R =	286.9	J/kg.K
ρ =	999	kg/m^3

The temperature can be computed from the data in the figure.
The pressures are then computed from the appropriate equation.



From Table A.3



Agreement between calculated and tabulated data is very good (as it should be, considering the table data are also computed!)

z (km)	T (°C)	T (K)		p/p_{SL}
0.0	15.0	288.0	m = 0.0065 (K/m)	1.000
2.0	2.0	275.00		0.784
4.0	-11.0	262.0		0.608
6.0	-24.0	249.0		0.465
8.0	-37.0	236.0		0.351
11.0	-56.5	216.5		0.223
12.0	-56.5	216.5	T = const	0.190
14.0	-56.5	216.5		0.139
16.0	-56.5	216.5		0.101
18.0	-56.5	216.5		0.0738
20.1	-56.5	216.5		0.0530
22.0	-54.6	218.4	m = -0.000991736 (K/m)	0.0393
24.0	-52.6	220.4		0.0288
26.0	-50.6	222.4		0.0211
28.0	-48.7	224.3		0.0155
30.0	-46.7	226.3		0.0115
32.2	-44.5	228.5		0.00824
34.0	-39.5	233.5	m = -0.002781457 (K/m)	0.00632
36.0	-33.9	239.1		0.00473
38.0	-28.4	244.6		0.00356
40.0	-22.8	250.2		0.00270
42.0	-17.2	255.8		0.00206
44.0	-11.7	261.3		0.00158
46.0	-6.1	266.9		0.00122
47.3	-2.5	270.5		0.00104
50.0	-2.5	270.5	T = const	0.000736
52.4	-2.5	270.5		0.000544
54.0	-5.6	267.4	m = 0.001956522 (K/m)	0.000444
56.0	-9.5	263.5		0.000343
58.0	-13.5	259.5		0.000264
60.0	-17.4	255.6		0.000202
61.6	-20.5	252.5		0.000163
64.0	-29.9	243.1	m = 0.003913043 (K/m)	0.000117
66.0	-37.7	235.3		0.0000880
68.0	-45.5	227.5		0.0000655
70.0	-53.4	219.6		0.0000482
72.0	-61.2	211.8		0.0000351
74.0	-69.0	204.0		0.0000253
76.0	-76.8	196.2		0.0000180
78.0	-84.7	188.3		0.0000126
80.0	-92.5	180.5	T = const	0.00000861
82.0	-92.5	180.5		0.00000590
84.0	-92.5	180.5		0.00000404
86.0	-92.5	180.5		0.00000276
88.0	-92.5	180.5		0.00000189
90.0	-92.5	180.5		0.00000130

z (km)	p/p_{SL}
0.0	1.000
0.5	0.942
1.0	0.887
1.5	0.835
2.0	0.785
2.5	0.737
3.0	0.692
3.5	0.649
4.0	0.609
4.5	0.570
5.0	0.533
6.0	0.466
7.0	0.406
8.0	0.352
9.0	0.304
10.0	0.262
11.0	0.224
12.0	0.192
13.0	0.164
14.0	0.140
15.0	0.120
16.0	0.102
17.0	0.0873
18.0	0.0747
19.0	0.0638
20.0	0.0546
22.0	0.0400
24.0	0.0293
26.0	0.0216
28.0	0.0160
30.0	0.0118
40.0	0.00283
50.0	0.000787
60.0	0.000222
70.0	0.0000545
80.0	0.0000102
90.0	0.00000162

Problem 3.43

[Difficulty: 3]

3.43 On a certain calm day, a mild inversion causes the atmospheric temperature to remain constant at 30°C between sea level and 5000-m altitude. Under these conditions, (a) calculate the elevation change for which a 3 percent reduction in air pressure occurs, (b) determine the change of elevation necessary to effect a 5 percent reduction in density, and (c) plot p_2/p_1 and ρ_2/ρ_1 as a function of Δz.

Given: Data on isothermal atmosphere

Find: Elevation changes for 3% pressure change and 5% density change; plot of pressure and density versus elevation

Solution:

Assumptions: Static, isothermal fluid,; g = constant; ideal gas behavior

Basic equations

$$\frac{dp}{dz} = -\rho \cdot g \qquad \text{and} \qquad p = \rho \cdot R \cdot T$$

Then

$$\frac{dp}{dz} = -\rho \cdot g = -\frac{p \cdot g}{R_{air} \cdot T} \qquad \text{and} \qquad \frac{dp}{p} = -\frac{g}{R_{air} \cdot T} \cdot dz$$

Integrating

$$\Delta z = -\frac{R_{air} \cdot T_0}{g} \cdot \ln\left(\frac{p_2}{p_1}\right) \qquad \text{where} \qquad T = T_0$$

For an ideal gas with T constant

$$\frac{p_2}{p_1} = \frac{\rho_2 \cdot R_{air} \cdot T}{\rho_1 \cdot R_{air} \cdot T} = \frac{\rho_2}{\rho_1} \qquad \text{so} \qquad \Delta z = -\frac{R_{air} \cdot T_0}{g} \cdot \ln\left(\frac{\rho_2}{\rho_1}\right) = -C \cdot \ln\left(\frac{\rho_2}{\rho_1}\right) \qquad (1)$$

From Table A.6

$$R_{air} = 287 \cdot \frac{N \cdot m}{kg \cdot K}$$

Evaluating

$$C = \frac{R_{air} \cdot T_0}{g} = 287 \cdot \frac{N \cdot m}{kg \cdot K} \times (30 + 273) \cdot K \times \frac{1}{9.81} \cdot \frac{s^2}{m} \times \frac{kg \cdot m}{N \cdot s^2} \qquad C = 8865 \cdot m$$

For a 3% reduction in pressure

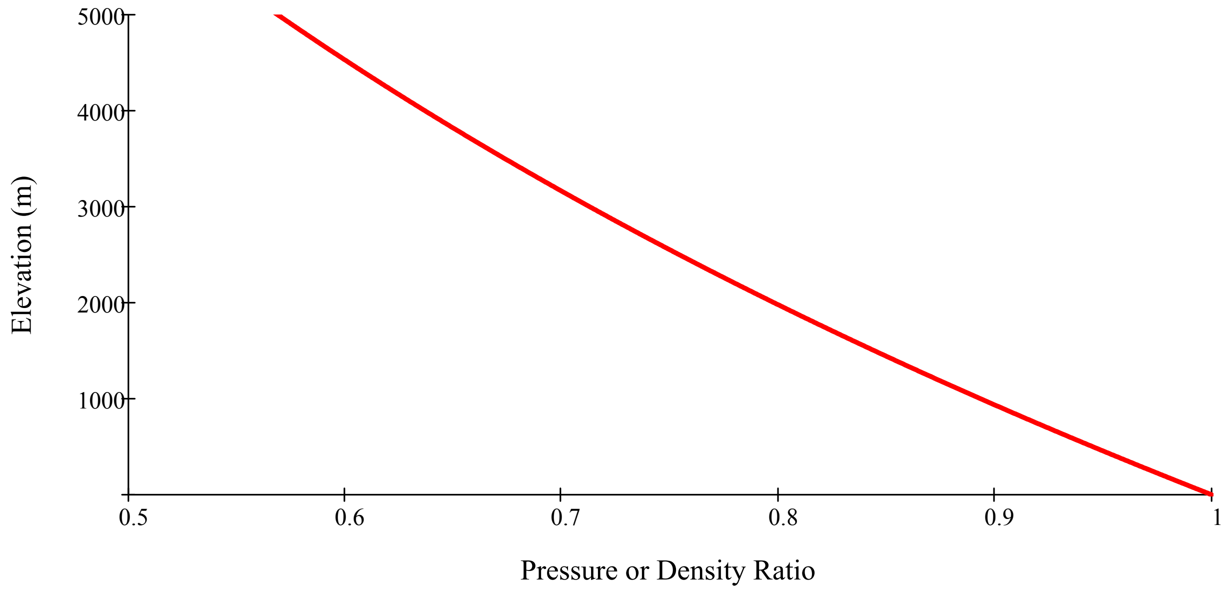
$$\frac{p_2}{p_1} = 0.97 \qquad \text{so from Eq. 1} \qquad \Delta z = -8865 \cdot m \cdot \ln(0.97) \qquad \Delta z = 270 \cdot m$$

For a 5% reduction in density

$$\frac{\rho_2}{\rho_1} = 0.95 \qquad \text{so from Eq. 1} \qquad \Delta z = -8865 \cdot m \cdot \ln(0.95) \qquad \Delta z = 455 \cdot m$$

To plot $\frac{p_2}{p_1}$ and $\frac{\rho_2}{\rho_1}$ we rearrange Eq. 1

$$\frac{\rho_2}{\rho_1} = \frac{p_2}{p_1} = e^{-\frac{\Delta z}{C}}$$



This plot can be plotted in Excel

Problem 3.44

[Difficulty: 3]

3.44 At ground level in Denver, Colorado, the atmospheric pressure and temperature are 83.2 kPa and 25°C. Calculate the pressure on Pike's Peak at an elevation of 2690 m above the city assuming (a) an incompressible and (b) an adiabatic atmosphere. Plot the ratio of pressure to ground level pressure in Denver as a function of elevation for both cases.

Given:

Atmospheric conditions at ground level ($z = 0$) in Denver, Colorado are $p_0 = 83.2$ kPa, $T_0 = 25°$C.
Pike's peak is at elevation z = 2690 m.

Find:

p/p_0 vs z for both cases.

Solution:

Governing Equations:

$$\frac{dp}{dz} = -\rho\cdot g \qquad p = \rho\cdot R\cdot T$$

Assumptions:

(1) Static fluid
(2) Ideal gas behavior

(a) For an incompressible atmosphere:

$$\frac{dp}{dz} = -\rho\cdot g \quad \text{becomes} \quad p - p_0 = -\int_0^z \rho\cdot g\,dz \quad \text{or} \quad p = p_0 - \rho_0\cdot g\cdot z = p_0\cdot\left(1 - \frac{g\cdot z}{R\cdot T_0}\right) \qquad (1)$$

At $z = 2690\cdot m$

$$p = 83.2\cdot kPa \times \left(1 - 9.81\cdot\frac{m}{s^2}\times 2690\cdot m\times\frac{kg\cdot K}{287\cdot N\cdot m}\times\frac{1}{298\cdot K}\times\frac{N\cdot s^2}{kg\cdot m}\right) \qquad p = 57.5\cdot kPa$$

(b) For an adiabatic atmosphere:

$$\frac{p}{\rho^k} = \text{const} \qquad \rho = \rho_0\cdot\left(\frac{p}{p_0}\right)^{\frac{1}{k}}$$

$$\frac{dp}{dz} = -\rho\cdot g \quad \text{becomes} \quad dp = -\rho_0\cdot\left(\frac{p}{p_0}\right)^{\frac{1}{k}}\cdot g\cdot dz \quad \text{or} \quad \frac{1}{p^{\frac{1}{k}}}dp = -\frac{\rho_0\cdot g}{p_0^{\frac{1}{k}}}\cdot dz$$

But

$$\int_{p_0}^{p}\frac{1}{p^{\frac{1}{k}}}\,dp = \frac{k}{k-1}\cdot\left(p - p_0\right)^{\frac{k-1}{k}} \quad \text{hence} \quad \frac{k}{k-1}\cdot\left(p^{\frac{k-1}{k}} - p_0^{\frac{k-1}{k}}\right) = -\frac{\rho_0\cdot g}{p_0^{\frac{1}{k}}}\cdot g\cdot z$$

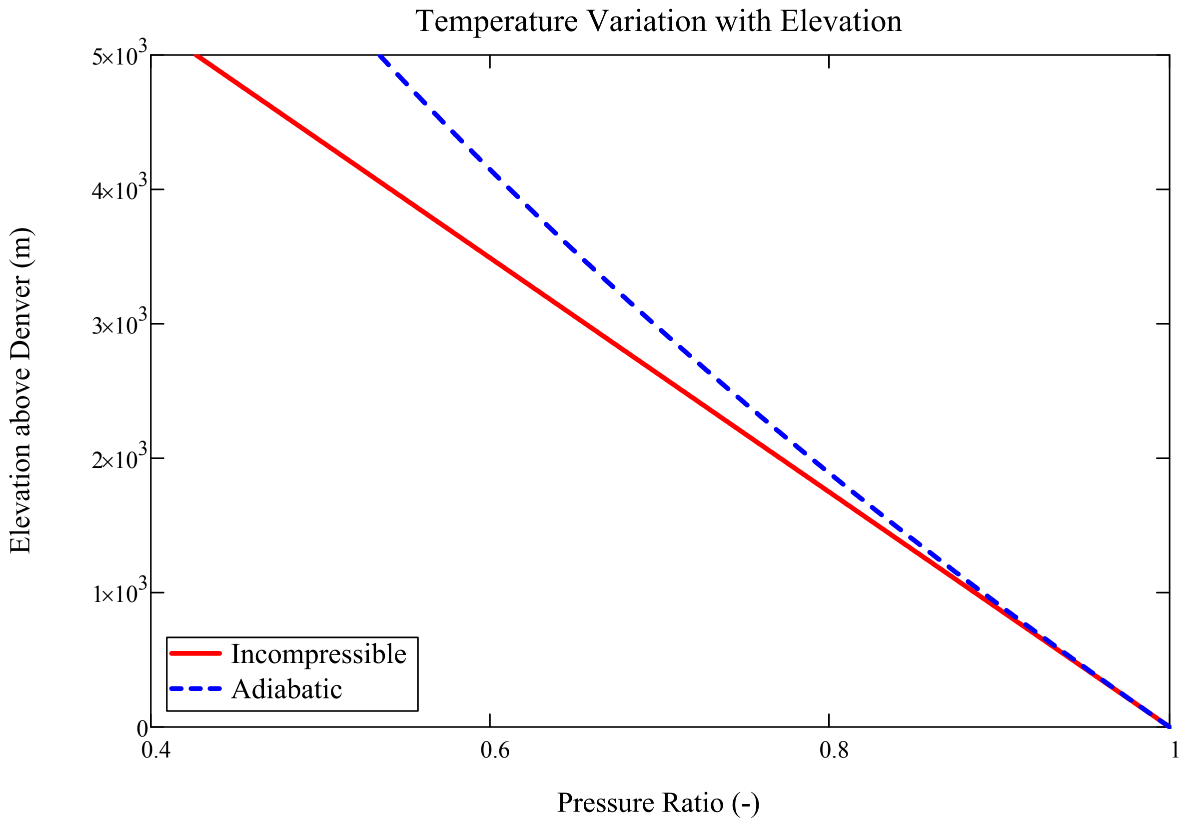
Solving for the pressure ratio

$$\frac{p}{p_0} = \left(1 - \frac{k-1}{k}\cdot\frac{\rho_0}{p_0}\cdot g\cdot z\right)^{\frac{k}{k-1}} \quad \text{or} \quad \frac{p}{p_0} = \left(1 - \frac{k-1}{k}\cdot\frac{g\cdot z}{R\cdot T_0}\right)^{\frac{k}{k-1}} \qquad (2)$$

At $z = 2690\cdot m$

$$p = 83.2\cdot kPa\times\left(1 - \frac{1.4-1}{1.4}\times 9.81\cdot\frac{m}{s^2}\times 2690\cdot m\times\frac{kg\cdot K}{287\cdot N\cdot m}\times\frac{1}{298\cdot K}\times\frac{N\cdot s^2}{kg\cdot m}\right)^{\frac{1.4}{1.4-1}} \qquad p = 60.2\cdot kPa$$

Equations 1 and 2 can be plotted:



Problem 3.45

[Difficulty: 3]

3.45 The Martian atmosphere behaves as an ideal gas with mean molecular mass of 32.0 and constant temperature of 200 K. The atmospheric density at the planet surface is $\rho = 0.015\ \mathrm{kg/m^3}$ and Martian gravity is $3.92\ \mathrm{m/s^2}$. Calculate the density of the Martian atmosphere at height $z = 20$ km above the surface. Plot the ratio of density to surface density as a function of elevation. Compare with that for data on the Earth's atmosphere.

Given: Martian atmosphere behaves as an idel gas, constant temperature

$$M_m = 32.0 \quad T = 200\cdot K \quad g = 3.92\cdot\frac{m}{s^2} \quad \rho_o = 0.015\cdot\frac{kg}{m^3}$$

Find: Density at z = 20 km
Plot the ratio of density to sea level density versus altitude, compare to that of earth.

Solution: We will apply the hydrostatics equations to this system.

Governing Equations:

$$\frac{dp}{dz} = -\rho\cdot g \qquad \text{(Hydrostatic Pressure - z is positive upwards)}$$

$$p = \rho\cdot R\cdot T \qquad \text{(Ideal Gas Equation of State)}$$

$$R = \frac{R_u}{M_m} \qquad \text{(Definition of Gas Constant)}$$

Assumptions:
(1) Static fluid
(2) Constant gravitational acceleration
(3) Ideal gas behavior

Taking the differential of the equation of state (constant temperature): $dp = R\cdot T\cdot d\rho$

Substituting into the hydrostatic pressure equation: $R\cdot T\cdot\frac{d\rho}{dz} = -\rho\cdot g$ Therefore: $\frac{d\rho}{\rho} = -\frac{g}{R\cdot T}\cdot dz$

Integrating this expression: $\int_{\rho_o}^{\rho}\frac{1}{\rho}\,d\rho = -\int_0^z \frac{g}{R\cdot T}\,dz$ $\quad \ln\left(\frac{\rho}{\rho_o}\right) = -\frac{g\cdot z}{R\cdot T}$ or

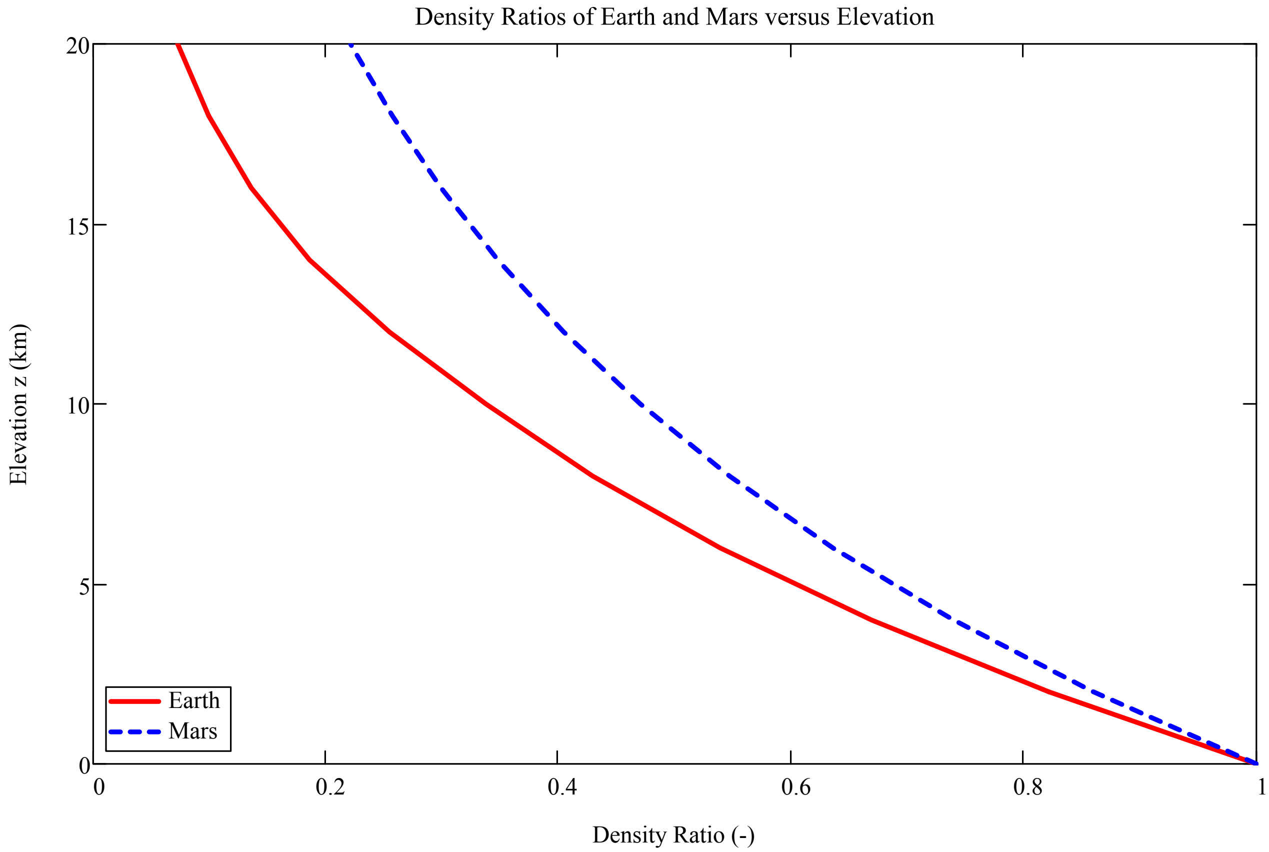
$$\frac{\rho}{\rho_o} = e^{-\frac{g\cdot z}{R\cdot T}} \qquad (1)$$

Evaluating: $R = 8314.3\cdot\frac{N\cdot m}{kg\cdot mol\cdot K}\times\frac{1}{32.0}\cdot\frac{kg\cdot mol}{kg}$ $\qquad R = 259.822\cdot\frac{N\cdot m}{kg\cdot K}$

$$\rho = 0.015\cdot\frac{kg}{m^3}\times e^{-\left(3.92\cdot\frac{m}{s^2}\times 20\times 10^3\cdot m\times\frac{1}{259.822}\cdot\frac{kg\cdot K}{N\cdot m}\times\frac{1}{200}\cdot\frac{1}{K}\times\frac{N\cdot s^2}{kg\cdot m}\right)} \qquad \rho = 3.32\times 10^{-3}\,\frac{kg}{m^3}$$

For the Martian atmosphere, let $x = \frac{g}{R\cdot T}$ $\quad x = 3.92\cdot\frac{m}{s^2}\times\frac{1}{259.822}\cdot\frac{kg\cdot K}{N\cdot m}\times\frac{1}{200}\cdot\frac{1}{K}\times\frac{N\cdot s^2}{kg\cdot m}$ $\quad x = 0.07544\cdot\frac{1}{km}$

Therefore: $\frac{\rho}{\rho_o} = e^{-x \cdot z}$ These data are plotted along with the data for Earth's atmosphere from Table A.3.



Problem 3.46

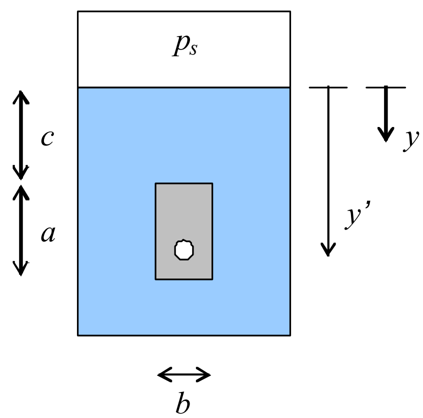
[Difficulty: 3]

3.46 A door 1 m wide and 1.5 m high is located in a plane vertical wall of a water tank. The door is hinged along its upper edge, which is 1 m below the water surface. Atmospheric pressure acts on the outer surface of the door and at the water surface. (a) Determine the magnitude and line of action of the total resultant force from all fluids acting on the door. (b) If the water surface gage pressure is raised to 0.3 atm, what is the resultant force and where is its line of action? (c) Plot the ratios F/F_0 and y'/y_c for different values of the surface pressure ratio p_s/p_{atm}. (F_0 is the resultant force when $p_s = p_{atm}$.)

Given: Door located in plane vertical wall of water tank as shown

$a = 1.5 \cdot m \quad b = 1 \cdot m \quad c = 1 \cdot m$

Atmospheric pressure acts on outer surface of door.



Find: Resultant force and line of action:

(a) for $p_s = p_{atm}$

(b) for $p_{sg} = 0.3 \cdot atm$

Plot F/Fo and y'/yc over range of ps/patm (Fo is force determined in (a), yc is y-ccordinate of door centroid).

Solution: We will apply the hydrostatics equations to this system.

Governing Equations:

$$\frac{dp}{dy} = \rho \cdot g \qquad \text{(Hydrostatic Pressure - y is positive downwards)}$$

$$F_R = \int p\, dA \qquad \text{(Hydrostatic Force on door)}$$

$$y' \cdot F_R = \int y \cdot p\, dA \qquad \text{(First moment of force)}$$

Assumptions: (1) Static fluid
(2) Incompressible fluid

We will obtain a general expression for the force and line of action, and then simplify for parts (a) and (b).

Since $dp = \rho \cdot g \cdot dh$ it follows that $p = p_s + \rho \cdot g \cdot y$

Now because p_{atm} acts on the outside of the door, p_{sg} is the surface gage pressure: $p = p_{sg} + \rho \cdot g \cdot y$

$$F_R = \int p\, dA = \int_c^{c+a} p \cdot b\, dy = \int_c^{c+a} \left(p_{sg} + \rho \cdot g \cdot y\right) \cdot b\, dy = b \cdot \left[p_{sg} \cdot a + \frac{\rho \cdot g}{2} \cdot \left(a^2 + 2 \cdot a \cdot c\right)\right] \qquad (1)$$

$$y' \cdot F_R = \int y \cdot p\, dA \qquad \text{Therefore:} \qquad y' = \frac{1}{F_R} \int y \cdot p\, dA = \frac{1}{F_R} \cdot \int_c^{c+a} y \cdot \left(p_{sg} + \rho \cdot g \cdot y\right) \cdot b\, dy$$

Evaluating the integral: $$y' = \frac{b}{F_R} \left[\frac{p_{sg}}{2} \left[(c+a)^2 - c^2\right] + \frac{\rho \cdot g}{3} \cdot \left[(c+a)^3 - c^3\right]\right]$$

Simplifying: $y' = \frac{b}{F_R}\cdot\left[\frac{p_{sg}}{2}\left(a^2 + 2\cdot a\cdot c\right) + \frac{\rho\cdot g}{3}\cdot\left[a^3 + 3\cdot a\cdot c\cdot(a + c)\right]\right]$ (2)

For part (a) we know $p_{sg} = 0$ so substituting into (1) we get: $F_o = \frac{\rho\cdot g\cdot b}{2}\cdot\left(a^2 + 2\cdot a\cdot c\right)$

$$F_o = \frac{1}{2}\times 999\cdot\frac{kg}{m^3}\times 9.81\cdot\frac{m}{s^2}\times 1\cdot m\times\left[(1.5\cdot m)^2 + 2\times 1.5\cdot m\times 1\cdot m\right]\times\frac{N\cdot s^2}{kg\cdot m}$$

$F_o = 25.7\cdot kN$

Substituting into (2) for the line of action we get: $y' = \frac{\rho\cdot g\cdot b}{3\cdot F_o}\cdot\left[a^3 + 3\cdot a\cdot c\cdot(a + c)\right]$

$$y' = \frac{1}{3}\times 999\cdot\frac{kg}{m^3}\times 9.81\cdot\frac{m}{s^2}\times 1\cdot m\cdot\frac{1}{25.7\times 10^3}\cdot\frac{1}{N}\times\left[(1.5\cdot m)^3 + 3\times 1.5\cdot m\times 1\cdot m\times(1.5\cdot m + 1\cdot m)\right]\times\frac{N\cdot s^2}{kg\cdot m}$$

$y' = 1.9\,m$

For part (b) we know $p_{sg} = 0.3\cdot atm$. Substituting into (1) we get:

$$F_R = 1\cdot m\times\left[0.3\cdot atm\times\frac{1.013\times 10^5\cdot N}{m^2\cdot atm}\times 1.5\cdot m + \frac{1}{2}\times 999\cdot\frac{kg}{m^3}\times 9.81\cdot\frac{m}{s^2}\times\left[(1.5\cdot m)^2 + 2\times 1.5\cdot m\times 1\cdot m\right]\times\frac{N\cdot s^2}{kg\cdot m}\right]$$

$F_R = 71.3\cdot kN$

$$y' = \frac{1\cdot m\times\left[\frac{0.3\cdot atm}{2}\times\frac{1.013\times 10^5\cdot N}{m^2\cdot atm}\times\left[(1.5)^2 + 2\cdot 1.5\cdot 1\right]\cdot m^2 + \frac{999\cdot\frac{kg}{m^3}\times 9.81\cdot\frac{m}{s^2}}{3}\times\left[(1.5)^3 + 3\cdot 1.5\cdot 1\cdot(1.5 + 1)\right]\cdot m^3\times\frac{N\cdot s^2}{kg\cdot m}\right]}{71.3\times 10^3\cdot N}$$

$y' = 1.789\,m$

The value of F/Fo is obtained from Eq. (1) and our result from part (a):

$$\frac{F}{F_o} = \frac{b\cdot\left[p_{sg}\cdot a + \frac{\rho\cdot g}{2}\cdot\left(a^2 + 2\cdot a\cdot c\right)\right]}{\frac{\rho\cdot g\cdot b}{2}\cdot\left(a^2 + 2\cdot a\cdot c\right)} = 1 + \frac{2\cdot p_{sg}}{\rho\cdot g\cdot(a + 2\cdot c)}$$

For the gate $y_c = c + \frac{a}{2}$ Therefore, the value of y'/yc is obtained from Eqs. (1) and (2):

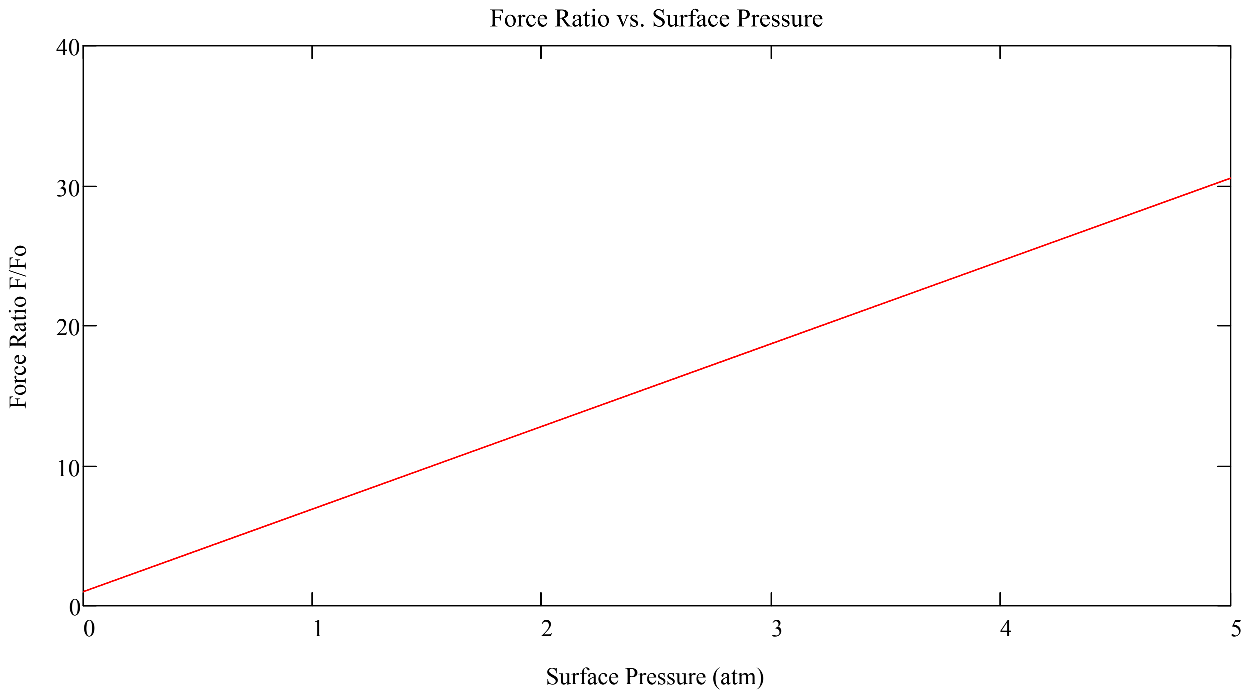
$$\frac{y'}{y_c} = \frac{2\cdot b}{F_R\cdot(2\cdot c + a)}\cdot\left[\frac{p_{sg}}{2}\left(a^2 + 2\cdot a\cdot c\right) + \frac{\rho\cdot g}{3}\cdot\left[a^3 + 3\cdot a\cdot c\cdot(a + c)\right]\right] = \frac{2\cdot b}{(2\cdot c + a)}\cdot\frac{\left[\frac{p_{sg}}{2}\left(a^2 + 2\cdot a\cdot c\right) + \frac{\rho\cdot g}{3}\cdot\left[a^3 + 3\cdot a\cdot c\cdot(a + c)\right]\right]}{\left[b\cdot\left[p_{sg}\cdot a + \frac{\rho\cdot g}{2}\cdot\left(a^2 + 2\cdot a\cdot c\right)\right]\right]}$$

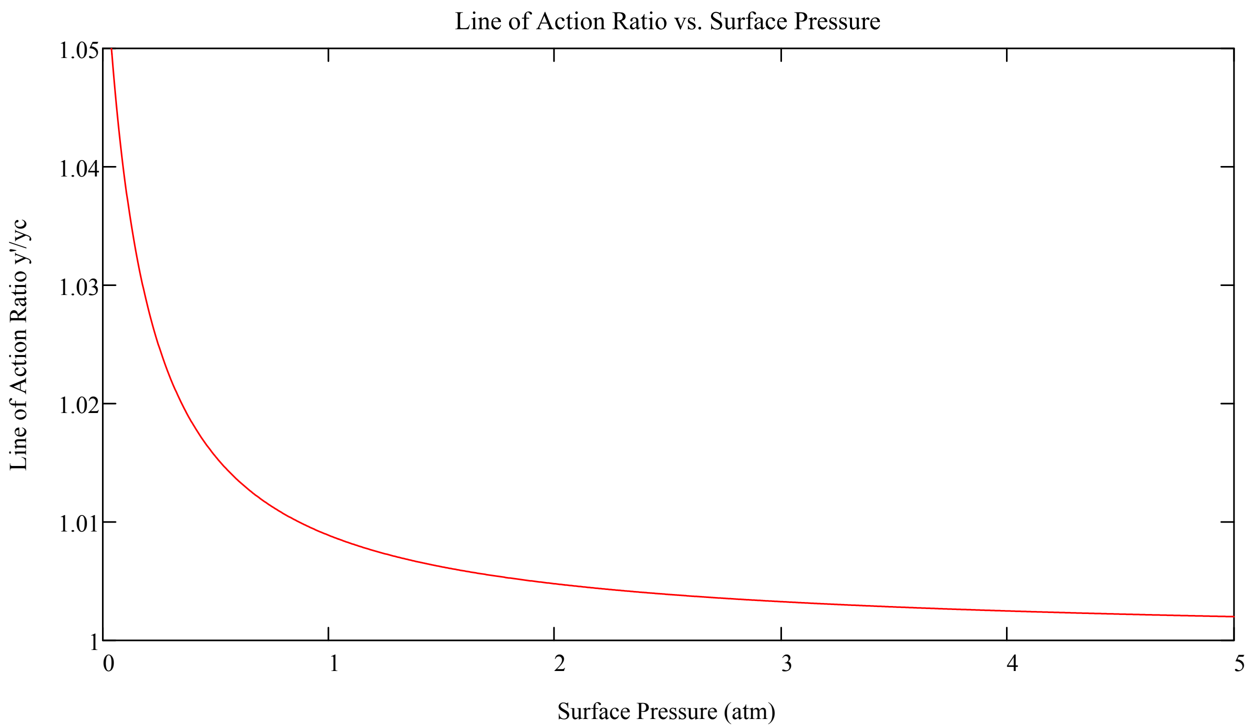
Simplifying this expression we get:

$$\frac{y'}{y_c} = \frac{2}{(2\cdot c + a)}\cdot\frac{\frac{p_{sg}}{2}\left(a^2 + 2\cdot a\cdot c\right) + \frac{\rho\cdot g}{3}\cdot\left[a^3 + 3\cdot a\cdot c\cdot(a + c)\right]}{p_{sg}\cdot a + \frac{\rho\cdot g}{2}\cdot\left(a^2 + 2\cdot a\cdot c\right)}$$

Based on these expressions we see that the force on the gate varies linearly with the increase in surface pressure, and that the line of action of the resultant is always below the centroid of the gate. As the pressure increases, however, the line of action moves closer to the centroid.

Plots of both ratios are shown below:





Problem 3.47

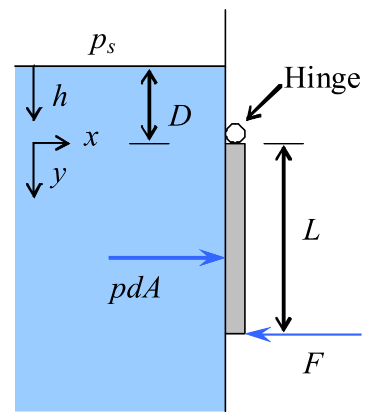
[Difficulty: 2]

3.47 A door 1 m wide and 1.5 m high is located in a plane vertical wall of a water tank. The door is hinged along its upper edge, which is 1 m below the water surface. Atmospheric pressure acts on the outer surface of the door. (a) If the pressure at the water surface is atmospheric, what force must be applied at the lower edge of the door in order to keep the door from opening? (b) If the water surface gage pressure is raised to 0.5 atm, what force must be applied at the lower edge of the door to keep the door from opening? (c) Find the ratio F/F_0 as a function of the surface pressure ratio p_s/p_{atm}. (F_0 is the force required when $p_s = p_{atm}$.)

Given: Door of constant width, located in plane vertical wall of water tank is hinged along upper edge.

$b = 1 \cdot m \quad D = 1 \cdot m \quad L = 1.5 \cdot m$

Atmospheric pressure acts on outer surface of door; force F is applied at lower edge to keep door closed.



Find: (a) Force F, if $p_s = p_{atm}$

(b) Force F, if $p_{sg} = 0.5 \cdot atm$

Plot F/Fo over tange of ps/patm (Fo is force determined in (a)).

Solution: We will apply the hydrostatics equations to this system.

Governing Equations:

$$\frac{dp}{dh} = \rho \cdot g$$ (Hydrostatic Pressure - h is positive downwards)

$$F_R = \int p\, dA$$ (Hydrostatic Force on door)

$$\Sigma M_z = 0$$ (Rotational Equilibrium)

Assumptions:
(1) Static fluid
(2) Constant density
(3) Door is in equilibrium

Taking moments about the hinge: $-F \cdot L + \int y \cdot p\, dA = 0 \qquad dA = b \cdot dy$

Solving for the force: $F = \frac{1}{L} \cdot \int_0^L b \cdot y \cdot p\, dy \quad (1)$ We will obtain a general expression for F and then simplify for parts (a) and (b).

Since $dp = \rho \cdot g \cdot dh$ it follows that $p = p_s + \rho \cdot g \cdot h$ where $h = D + y$

and hence $p = p_s + \rho \cdot g \cdot (D + y)$ Now because p_{atm} acts on the outside of the door, p_{sg} is the surface gage pressure.

From Equation (1): $F = \frac{1}{L} \cdot \int_0^L b \cdot y \cdot \left[p_{sg} + \rho \cdot g \cdot (D + y)\right] dy \qquad F = \frac{b}{L} \cdot \int_0^L \left[\left(p_{sg} + \rho \cdot g \cdot D\right) \cdot y + \rho \cdot g \cdot y^2\right] dy$

After integrating: $F = \frac{b}{L}\cdot\left[\left(p_{sg} + \rho\cdot g\cdot D\right)\cdot\frac{L^2}{2} + \rho\cdot g\cdot\frac{L^3}{3}\right]$ or $F = b\cdot\left[p_{sg}\cdot\frac{L}{2} + \rho\cdot g\cdot L\cdot\left(\frac{D}{2} + \frac{L}{3}\right)\right]$ (2)

(a) For $p_s = p_{atm}$ it follows that $p_{sg} = 0$ Therefore: $F_o = \rho\cdot g\cdot b\cdot L\cdot\left(\frac{D}{2} + \frac{L}{3}\right)$ (3)

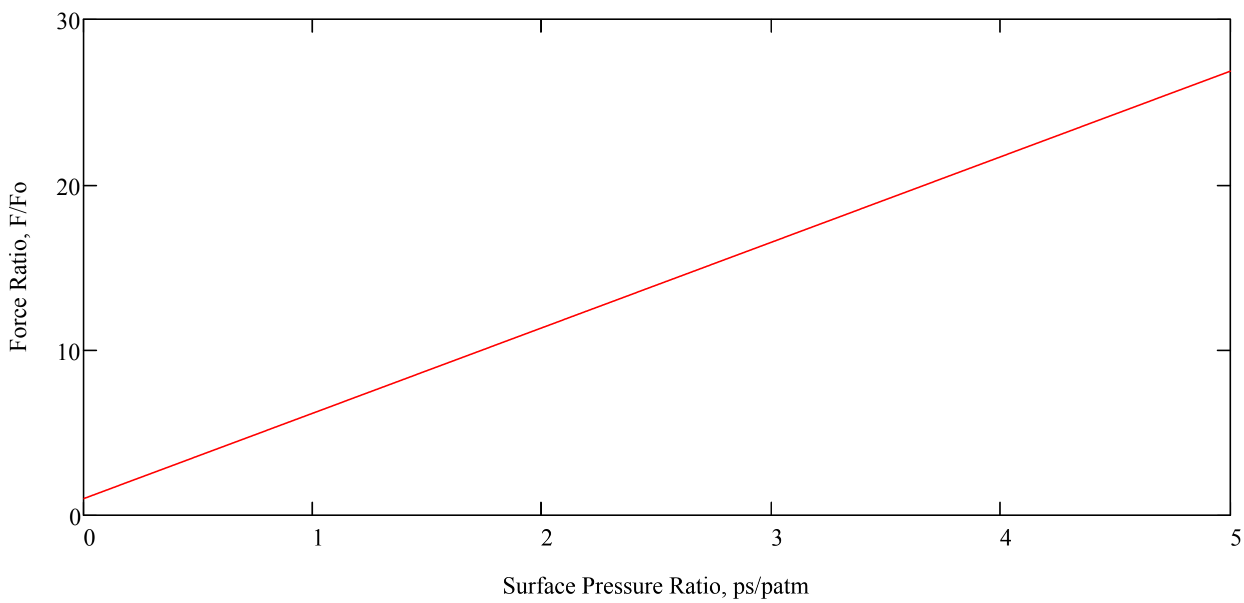
$$F_o = 999\cdot\frac{kg}{m^3} \times 9.81\cdot\frac{m}{s^2} \times 1\cdot m \times 1.5\cdot m \times \left(\frac{1\cdot m}{2} + \frac{1.5\cdot m}{3}\right) \times \frac{N\cdot s^2}{kg\cdot m} \qquad F_o = 14.7\cdot kN$$

(b) For $p_{sg} = 0.5\cdot atm$ we substitute variables:

$$F = 1\cdot m \times \left[0.5\cdot atm \times \frac{101\cdot kPa}{atm} \times \frac{1.5\cdot m}{2} + 999\cdot\frac{kg}{m^3} \times 9.81\cdot\frac{m}{s^2} \times 1.5\cdot m \times \left(\frac{1\cdot m}{2} + \frac{1.5\cdot m}{3}\right) \times \frac{N\cdot s^2}{kg\cdot m}\right] \qquad F = 52.6\cdot kN$$

From Equations (2) and (3) we have: $\frac{F}{F_o} = \frac{b\cdot\left[p_{sg}\cdot\frac{L}{2} + \rho\cdot g\cdot L\cdot\left(\frac{D}{2} + \frac{L}{3}\right)\right]}{\rho\cdot g\cdot b\cdot L\cdot\left(\frac{D}{2} + \frac{L}{3}\right)} = 1 + \frac{p_{sg}}{2\cdot\rho\cdot g\cdot\left(\frac{D}{2} + \frac{L}{3}\right)}$

Here is a plot of the force ratio as a function of the surface pressure:



Problem 3.48

[Difficulty: 5]

3.48 A hydropneumatic elevator consists of a piston-cylinder assembly to lift the elevator cab. Hydraulic oil, stored in an accumulator tank pressurized by air, is valved to the piston as needed to lift the elevator. When the elevator descends, oil is returned to the accumulator. Design the least expensive accumulator that can satisfy the system requirements. Assume the lift is 3 floors, the maximum load is 10 passengers, and the maximum system pressure is 800 kPa (gage). For column bending strength, the piston diameter must be at least 150 mm. The elevator cab and piston have a combined mass of 3000 kg, and are to be purchased. Perform the analysis needed to define, as a function of system operating pressure, the piston diameter, the accumulator volume and diameter, and the wall thickness. Discuss safety features that your company should specify for the complete elevator system. Would it be preferable to use a completely pneumatic design or a completely hydraulic design? Why?

Discussion: The design requirements are specified except that a typical floor height is about 12 ft, making the total required lift about 36 ft. A spreadsheet was used to calculate the system properties for various pressures. Results are presented on the next page, followed by a sample calculation. Total cost dropped quickly as system pressure was increased. A shallow minimum was reached in the 100-110 psig range. The lowest-cost solution was obtained at a system pressure of about 100 psig. At this pressure, the reservoir of 140 gal required a 3.30 ft diameter pressure sphere with a 0.250 in wall thickness. The welding cost was \$155 and the material cost \$433, for a total cost of \$588. Accumulator wall thickness was constrained at 0.250 in for pressures below 100 psi; it increased for higher pressures (this caused the discontinuity in slope of the curve at 100 psig). The mass of steel became constant above 110 psig. No allowance was made for the extra volume needed to pressurize the accumulator. Fail-safe design is essential for an elevator to be used by the public. The control circuitry should be redundant. Failures must be easy to spot. For this reason, hydraulic actuation is good: leaks will be readily apparent. The final design must be reviewed, approved, and stamped by a professional engineer since the design involves public safety. The terminology used in the solution is defined in the following table:

Symbol	Definition	Units
p	System pressure	psig
A_p	Area of lift piston	in^2
$\forall_{oil}$	Volume of oil	gal
D_s	Diameter of spherical accumulator	ft
t	Wall thickness of accumulator	in
A_w	Area of weld	in^2
C_w	Cost of weld	\$
M_s	Mass of steel accumulator	lbm
C_s	Cost of steel	\$
C_t	Total Cost	\$

A sample calculation and the results of the system simulation in Excel are presented below.

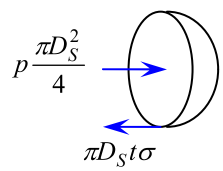
Sample calculation for a pressure of 20 psig:

$$W_t = p \cdot A_p \qquad A_p = \frac{W_t}{p} \qquad A_p = 7500 \cdot lbf \times \frac{1}{20} \cdot \frac{in^2}{lbf} \qquad A_p = 375 \cdot in^2$$

$$V_{oil} = A_p \cdot L \qquad V_{oil} = 375 \cdot in^2 \times 36 \cdot ft \times \left(\frac{ft}{12 \cdot in}\right)^2 \times \frac{7.48 \cdot gal}{ft^3} \qquad V_{oil} = 701 \cdot gal$$

$$V_{oil} = V_s = \frac{4}{3} \cdot \pi \cdot R_s^3 = \frac{\pi}{6} \cdot D_s^3 \qquad D_s = \left(\frac{6 \cdot V_{oil}}{\pi}\right)^{\frac{1}{3}} \qquad D_s = \left(\frac{6}{\pi} \times 701 \cdot gal \times \frac{ft^3}{7.48 \cdot gal}\right)^{\frac{1}{3}} \qquad D_s = 5.64 \cdot ft$$

From a force balance on the sphere:



Thus: $$p \cdot \pi \cdot \frac{D_s^2}{4} = \pi \cdot D_s \cdot t \cdot \sigma \text{ , so} \qquad t = \frac{p}{\sigma} \cdot \frac{D_s}{4} \qquad t = 20 \cdot \frac{lbf}{in^2} \times \frac{1}{4000} \cdot \frac{in^2}{lbf} \times \frac{5.64 \cdot ft}{4} \times \frac{12 \cdot in}{ft} \qquad t = 0.085 \cdot in$$

Since the minimum wall thickness is 0.250 in: $t = 0.250 \cdot in$

$$A_w = \pi \cdot D_s \cdot t \qquad A_w = \pi \cdot 5.64 \cdot ft \cdot 0.250 \cdot in \cdot \frac{12 \cdot in}{ft} \qquad A_w = 53.2 \cdot in^2$$

$$C_w = 5.00 \cdot \frac{1}{in^2} \times 53.2 \cdot in^2 \qquad \text{(cost in \$)} \qquad C_w = 266$$

$$M_s = 4 \cdot \pi \cdot R_s^2 \cdot t \cdot \rho_s = \pi \cdot D_s^2 \cdot t \cdot SG_s \cdot \rho_{water}$$

$$M_s = \pi \times (5.64 \cdot ft)^2 \times 0.250 \cdot in \times \frac{ft}{12in} \times 7.8 \times 62.4 \cdot \frac{lbm}{ft^3} \qquad M_s = 1013 \cdot lbm$$

$$C_s = 1.25 \cdot \frac{1}{lbm} \times 1013 \cdot lbm \qquad \text{(cost in \$)} \qquad C_s = 1266$$

Therefore the total cost is:

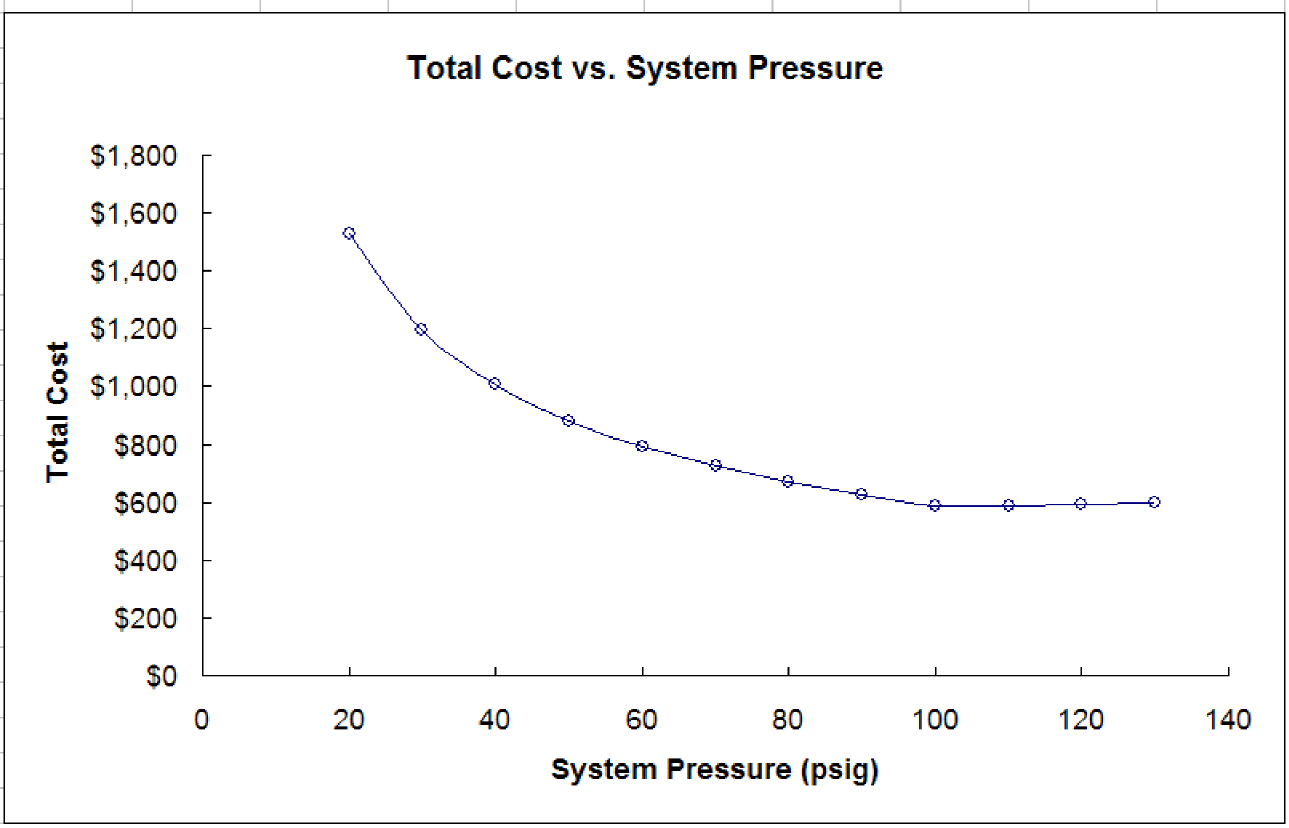
$$C_t = 266 + 1266 \qquad \text{(cost in \$)} \qquad C_t = 1532$$

Results of system simulation:

Input Data:	Cab and piston weight:	W_{cab} =	6000 lbf
	Passenger weight:	W_{pax} =	1500 lbf
	Total weight:	W_{tot} =	7500 lbf
	Allowable stress:	σ =	4000 psi
	Minimum wall thickness:	t =	0.250 in
	Welding cost factor:	cf_w =	5.00 $/in^2
	Steel cost factor:	cf_s =	1.25 $/pound

Results:

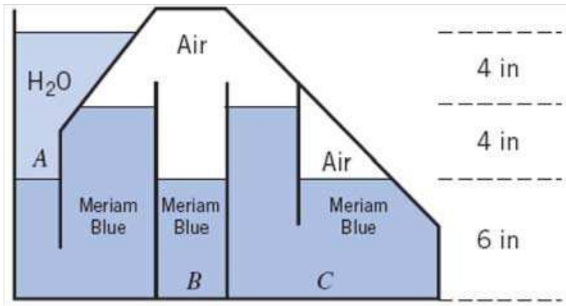
p (psig)	A_p (in^2)	V_{oil} (gal)	D_s (ft)	t (in)	A_w (in^2)	C_w	M_s (lbm)	C_s	C_t
20	375	701	5.64	0.250	53.1	$266	1012	$1,265	$1,531
30	250	468	4.92	0.250	46.4	$232	772	$965	$1,197
40	187.5	351	4.47	0.250	42.2	$211	638	$797	$1,008
50	150.0	281	4.15	0.250	39.1	$196	549	$687	$882
60	125.0	234	3.91	0.250	36.8	$184	487	$608	$792
70	107.1	200	3.71	0.250	35.0	$175	439	$549	$724
80	93.8	175.3	3.55	0.250	33.5	$167	402	$502	$669
90	83.3	155.8	3.41	0.250	32.2	$161	371	$464	$625
100	75.0	140.3	3.30	0.250	31.1	$155	346	$433	$588
110	68.2	127.5	3.19	0.263	31.7	$159	342	$428	$586
120	62.5	116.9	3.10	0.279	32.6	$163	342	$428	$591
130	57.7	107.9	3.02	0.294	33.5	$168	342	$428	$595



Problem 3.49

[Difficulty: 2]

3.49 Find the pressures at points A, B, and C, as shown in the figure, and in the two air cavities.



Given: Geometry of chamber system

Find: Pressure at various locations

Assumptions:
(1) Water and Meriam Blue are static and incompressible
(2) Pressure gradients across air cavities are negligible

Solution:

Basic equation $\frac{dp}{dy} = -\rho \cdot g$ or, for constant ρ $\Delta p = \rho \cdot g \cdot \Delta h$ where Δh is height difference

For point A $p_A = p_{atm} + \rho_{H2O} \cdot g \cdot h_1$ or in gage pressure $p_A = \rho_{H2O} \cdot g \cdot h_1$

Here we have $h_1 = 8 \cdot in$ $h_1 = 0.667 \cdot ft$

$$p_A = 1.94 \cdot \frac{slug}{ft^3} \times 32.2 \cdot \frac{ft}{s^2} \times 0.667 \cdot ft \times \frac{lbf \cdot s^2}{slug\,ft} \times \left(\frac{ft}{12 \cdot in}\right)^2 \qquad p_A = 0.289 \cdot psi \quad \text{(gage)}$$

For the first air cavity $p_{air1} = p_A - SG_{MB} \cdot \rho_{H2O} \cdot g \cdot h_2$ where $h_2 = 4 \cdot in$ $h_2 = 0.333 \cdot ft$

From Table A.1 $SG_{MB} = 1.75$

$$p_{air1} = 0.289 \cdot \frac{lbf}{in^2} - 1.75 \times 1.94 \cdot \frac{slug}{ft^3} \times 32.2 \cdot \frac{ft}{s^2} \times 0.333 \cdot ft \times \frac{lbf \cdot s^2}{slug \cdot ft} \times \left(\frac{ft}{12 \cdot in}\right)^2 \qquad p_{air1} = 0.036 \cdot psi \quad \text{(gage)}$$

Note that p = constant throughout the air pocket

For point B $p_B = p_{air1} + SG_{Hg} \cdot \rho_{H2O} \cdot g \cdot h_3$ where $h_3 = 6 \cdot in$ $h_3 = 0.5 \cdot ft$

$$p_B = 0.036 \cdot \frac{lbf}{in^2} + 1.75 \times 1.94 \cdot \frac{slug}{ft^3} \times 32.2 \cdot \frac{ft}{s^2} \times 0.5 \cdot ft \times \frac{lbf \cdot s^2}{slug \cdot ft} \times \left(\frac{ft}{12 \cdot in}\right)^2 \qquad p_B = 0.416 \cdot psi \quad \text{(gage)}$$

For point C $p_C = p_{air2} + SG_{Hg} \cdot \rho_{H2O} \cdot g \cdot h_4$ where $h_4 = 10 \cdot in$ $h_4 = 0.833 \cdot ft$

$$p_C = 0.416 \cdot \frac{lbf}{in^2} + 1.75 \times 1.94 \cdot \frac{slug}{ft^3} \times 32.2 \cdot \frac{ft}{s^2} \times 0.833 \cdot ft \times \frac{lbf \cdot s^2}{slug \cdot ft} \times \left(\frac{ft}{12 \cdot in}\right)^2 \qquad p_C = 1.048 \cdot psi \quad \text{(gage)}$$

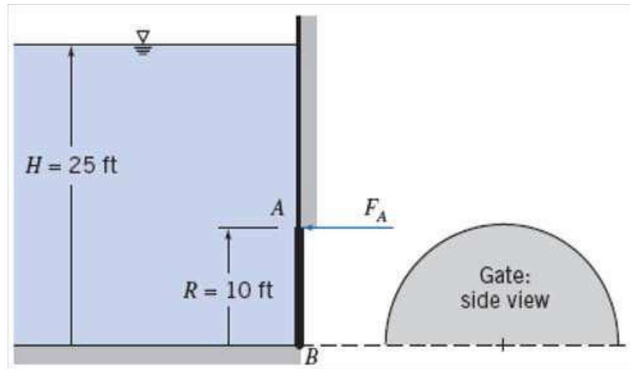
For the second air cavity $p_{air2} = p_C - SG_{Hg} \cdot \rho_{H2O} \cdot h_5$ where $h_5 = 6 \cdot in$ $h_5 = 0.5 \cdot ft$

$$p_{air2} = 1.048 \cdot \frac{lbf}{in^2} - 1.75 \times 1.94 \cdot \frac{slug}{ft^3} \times 32.2 \cdot \frac{ft}{s^2} \times 0.5 \cdot ft \times \frac{lbf \cdot s^2}{slug \cdot ft} \times \left(\frac{ft}{12 \cdot in}\right)^2 \qquad p_{air2} = 0.668 \cdot psi \quad \text{(gage)}$$

Problem 3.50

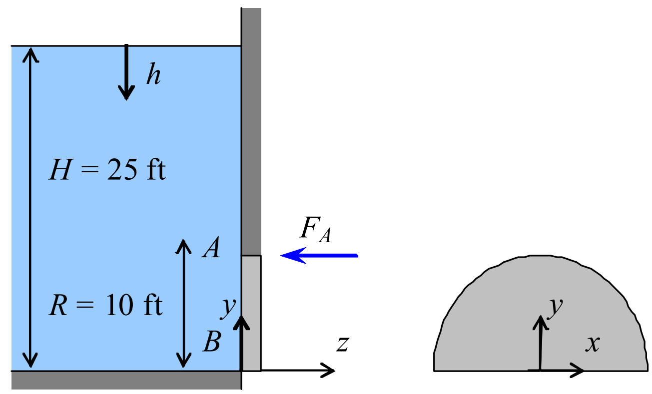
[Difficulty: 3]

3.50 Semicircular plane gate AB is hinged along B and held by horizontal force F_A applied at A. The liquid to the left of the gate is water. Calculate the force F_A required for equilibrium.



Given: Geometry of gate

Find: Force F_A for equilibrium



Solution:

Basic equation $\quad F_R = \int p\,dA \qquad \dfrac{dp}{dh} = \rho \cdot g \qquad \Sigma M_z = 0$

or, use computing equations $\quad F_R = p_c \cdot A \qquad y' = y_c + \dfrac{I_{xx}}{A \cdot y_c}$ where y would be measured from the free surface

Assumptions: static fluid; ρ = constant; p_{atm} on other side; door is in equilibrium

Instead of using either of these approaches, we note the following, using y as in the sketch

$$\Sigma M_z = 0 \qquad F_A \cdot R = \int y \cdot p\,dA \quad \text{with} \quad p = \rho \cdot g \cdot h$$

(Gage pressure, since p = p_{atm} on other side)

$$F_A = \frac{1}{R} \cdot \int y \cdot \rho \cdot g \cdot h\,dA \qquad \text{with} \quad dA = r \cdot dr \cdot d\theta \quad \text{and} \quad y = r \cdot \sin(\theta) \quad h = H - y$$

Hence

$$F_A = \frac{1}{R} \cdot \int_0^{\pi} \int_0^{R} \rho \cdot g \cdot r \cdot \sin(\theta) \cdot (H - r \cdot \sin(\theta)) \cdot r\,dr\,d\theta = \frac{\rho \cdot g}{R} \cdot \int_0^{\pi} \left(\frac{H \cdot R^3}{3} \cdot \sin(\theta) - \frac{R^4}{4} \cdot \sin(\theta)^2 \right) d\theta$$

$$F_R = \frac{\rho \cdot g}{R} \cdot \left(\frac{2 \cdot H \cdot R^3}{3} - \frac{\pi \cdot R^4}{8} \right) = \rho \cdot g \cdot \left(\frac{2 \cdot H \cdot R^2}{3} - \frac{\pi \cdot R^3}{8} \right)$$

Using given data

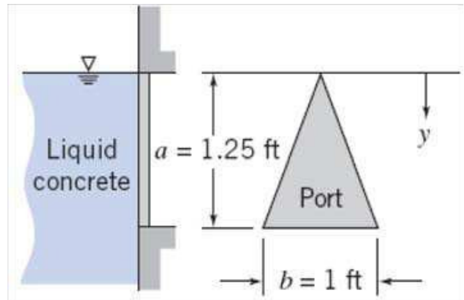
$$F_R = 1.94 \cdot \frac{\text{slug}}{\text{ft}^3} \times 32.2 \cdot \frac{\text{ft}}{\text{s}^2} \times \left[\frac{2}{3} \times 25 \cdot \text{ft} \times (10 \cdot \text{ft})^2 - \frac{\pi}{8} \times (10 \cdot \text{ft})^3 \right] \times \frac{\text{lbf} \cdot \text{s}^2}{\text{slug} \cdot \text{ft}}$$

$$F_R = 7.96 \times 10^4 \cdot \text{lbf}$$

Problem 3.51

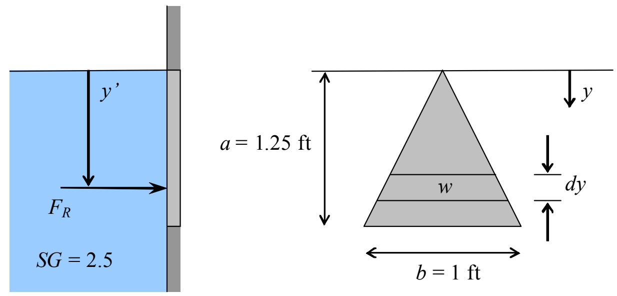
[Difficulty: 2]

3.51 A triangular access port must be provided in the side of a form containing liquid concrete. Using the coordinates and dimensions shown, determine the resultant force that acts on the port and its point of application.



Given: Geometry of access port

Find: Resultant force and location



Solution:

Basic equation $$F_R = \int p\,dA \qquad \frac{dp}{dy} = \rho\cdot g \qquad \Sigma M_s = y'\cdot F_R = \int y\,dF_R = \int y\cdot p\,dA$$

or, use computing equations $$F_R = p_c\cdot A \qquad y' = y_c + \frac{I_{xx}}{A\cdot y_c}$$

We will show both methods

Assumptions: Static fluid; ρ = constant; p_{atm} on other side

$$F_R = \int p\,dA = \int SG\cdot\rho\cdot g\cdot y\,dA \qquad \text{but} \qquad dA = w\cdot dy \quad \text{and} \quad \frac{w}{b} = \frac{y}{a} \qquad w = \frac{b}{a}\cdot y$$

Hence $$F_R = \int_0^a SG\cdot\rho\cdot g\cdot y\cdot\frac{b}{a}\cdot y\,dy = \int_0^a SG\cdot\rho\cdot g\cdot\frac{b}{a}\cdot y^2\,dy = \frac{SG\cdot\rho\cdot g\cdot b\cdot a^2}{3}$$

Alternatively $$F_R = p_c\cdot A \qquad \text{and} \qquad p_c = SG\cdot\rho\cdot g\cdot y_c = SG\cdot\rho\cdot g\cdot\frac{2}{3}\cdot a \qquad \text{with} \qquad A = \frac{1}{2}\cdot a\cdot b$$

Hence $$F_R = \frac{SG\cdot\rho\cdot g\cdot b\cdot a^2}{3}$$

For y' $$y'\cdot F_R = \int y\cdot p\,dA = \int_0^a SG\cdot\rho\cdot g\cdot\frac{b}{a}\cdot y^3\,dy = \frac{SG\cdot\rho\cdot g\cdot b\cdot a^3}{4} \qquad y' = \frac{SG\cdot\rho\cdot g\cdot b\cdot a^3}{4\cdot F_R} = \frac{3}{4}\cdot a$$

Alternatively $$y' = y_c + \frac{I_{xx}}{A\cdot y_c} \qquad \text{and} \qquad I_{xx} = \frac{b\cdot a^3}{36} \qquad \text{(Google it!)} \qquad y' = \frac{2}{3}\cdot a + \frac{b\cdot a^3}{36}\cdot\frac{2}{a\cdot b}\cdot\frac{3}{2\cdot a} = \frac{3}{4}\cdot a$$

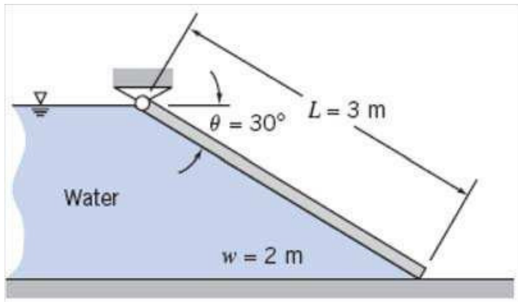
Using given data, and SG = 2.5 (Table A.1) $$F_R = \frac{2.5}{3}\cdot 1.94\cdot\frac{slug}{ft^3}\times 32.2\cdot\frac{ft}{s^2}\times 1\cdot ft\times(1.25\cdot ft)^2\times\frac{lbf\cdot s^2}{slug\cdot ft} \qquad F_R = 81.3\cdot lbf$$

and $$y' = \frac{3}{4}\cdot a \qquad y' = 0.938\cdot ft$$

Problem 3.52

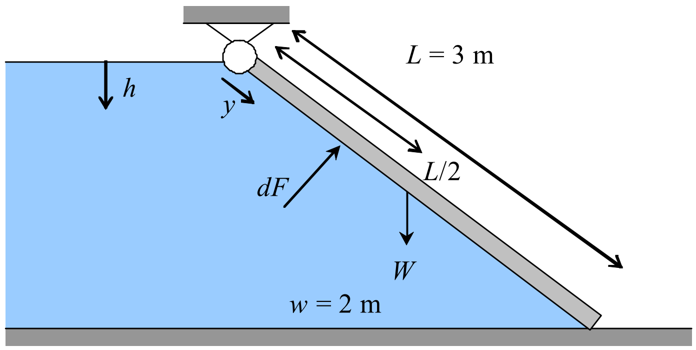
[Difficulty: 3]

3.52 A plane gate of uniform thickness holds back a depth of water as shown. Find the minimum weight needed to keep the gate closed.



Given: Geometry of plane gate

Find: Minimum weight to keep it closed



Solution:

Basic equation $\quad F_R = \int p\,dA \qquad \dfrac{dp}{dh} = \rho \cdot g \qquad \Sigma M_O = 0$

or, use computing equations $\quad F_R = p_c \cdot A \qquad y' = y_c + \dfrac{I_{xx}}{A \cdot y_c}$

Assumptions: static fluid; ρ = constant; p_{atm} on other side; door is in equilibrium

Instead of using either of these approaches, we note the following, using y as in the sketch

$$\Sigma M_O = 0 \qquad W \cdot \frac{L}{2} \cdot \cos(\theta) = \int y\,dF$$

We also have $\quad dF = p \cdot dA \quad$ with $\quad p = \rho \cdot g \cdot h = \rho \cdot g \cdot y \cdot \sin(\theta) \quad$ (Gage pressure, since $p = p_{atm}$ on other side)

Hence

$$W = \frac{2}{L \cdot \cos(\theta)} \cdot \int y \cdot p\,dA = \frac{2}{L \cdot \cos(\theta)} \cdot \int y \cdot \rho \cdot g \cdot y \cdot \sin(\theta) \cdot w\,dy$$

$$W = \frac{2}{L \cdot \cos(\theta)} \cdot \int y \cdot p\,dA = \frac{2 \cdot \rho \cdot g \cdot w \cdot \tan(\theta)}{L} \cdot \int_0^L y^2\,dy = \frac{2}{3} \cdot \rho \cdot g \cdot w \cdot L^2 \cdot \tan(\theta)$$

Using given data

$$W = \frac{2}{3} \cdot 1000 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times 2 \cdot m \times (3 \cdot m)^2 \times \tan(30 \cdot deg) \times \frac{N \cdot s^2}{kg \cdot m} \qquad W = 68 \cdot kN$$

Problem 3.53

[Difficulty: 4]

3.53 Consider a semicylindrical trough of radius R and length L. Develop general expressions for the magnitude and line of action of the hydrostatic force on one end, if the trough is partially filled with water and open to atmosphere. Plot the results (in nondimensional form) over the range of water depth $0 \le d/R \le 1$.

Given: Semicylindrical trough, partly filled with water to depth d.

Find: (a) General expressions for F_R and y' on end of trough, if open to the atmosphere.

(b) Plots of results vs. d/R between 0 and 1.

Solution: We will apply the hydrostatics equations to this system.

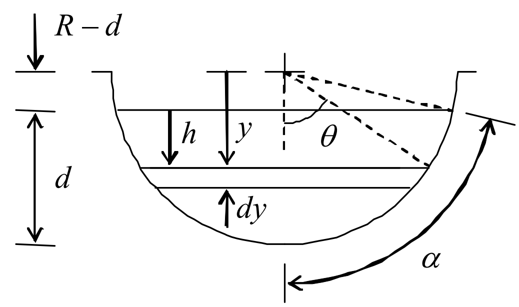
Governing Equations:

$$\frac{dp}{dy} = \rho \cdot g \qquad \text{(Hydrostatic Pressure - y is positive downwards)}$$

$$F_R = \int p \, dA \qquad \text{(Hydrostatic Force on door)}$$

$$y' \cdot F_R = \int y \cdot p \, dA \qquad \text{(First moment of force)}$$

Assumptions:
(1) Static fluid
(2) Incompressible fluid



Integrating the pressure equation: $p = \rho \cdot g \cdot h$ where $h = y - (R - d)$

Therefore: $p = \rho \cdot g \cdot [y - (R - d)] = \rho \cdot g \cdot R \cdot \left[\frac{y}{R} - \left(1 - \frac{d}{R}\right)\right]$

Expressing this in terms of θ and α in the figure: $p = \rho \cdot g \cdot R \cdot (\cos(\theta) - \cos(\alpha))$

For the walls at the end of the trough: $dA = w \cdot dy = 2 \cdot R \cdot \sin(\theta) \cdot dy$ Now since $y = R \cdot \cos(\theta)$ it follows that $dy = -R \cdot \sin(\theta) \cdot d\theta$

Substituting this into the hydrostatic force equation:

$$F_R = \int_{R-d}^{R} p \cdot w \, dy = \int_{\alpha}^{0} \rho \cdot g \cdot R \cdot (\cos(\theta) - \cos(\alpha)) \cdot 2 \cdot R \cdot \sin(\theta) \cdot (-R \cdot \sin(\theta)) \, d\theta$$

Upon simplification:

$$F_R = 2 \cdot \rho \cdot g \cdot R^3 \int_0^{\alpha} \left[\sin(\theta)^2 \cdot \cos(\theta) - (\sin(\theta))^2 \cdot \cos(\alpha)\right] d\theta = 2 \cdot \rho \cdot g \cdot R^3 \cdot \left[\frac{(\sin(\alpha))^3}{3} - \cos(\alpha) \cdot \left(\frac{\alpha}{2} - \frac{\sin(\alpha) \cdot \cos(\alpha)}{2}\right)\right]$$

$$F_R = 2 \cdot \rho \cdot g \cdot R^3 \cdot \left[\frac{(\sin(\alpha))^3}{3} - \cos(\alpha) \cdot \left(\frac{\alpha}{2} - \frac{\sin(\alpha) \cdot \cos(\alpha)}{2}\right)\right]$$

Non-dimensionalizing the force: $$\frac{F_R}{\rho \cdot g \cdot R^3} = 2 \cdot \left[\frac{(\sin(\alpha))^3}{3} - \cos(\alpha) \cdot \left(\frac{\alpha}{2} - \frac{\sin(\alpha) \cdot \cos(\alpha)}{2}\right)\right]$$

To find the line of action of the force:

$$y'\cdot F_R = \int_{R-d}^{R} y\cdot p\cdot w\, dy = \int_{\alpha}^{0} R\cdot\cos(\theta)\cdot\rho\cdot g\cdot R\cdot(\cos(\theta)-\cos(\alpha))\cdot 2\cdot R\cdot\sin(\theta)\cdot(-R\cdot\sin(\theta))\, d\theta$$

Upon simplification:

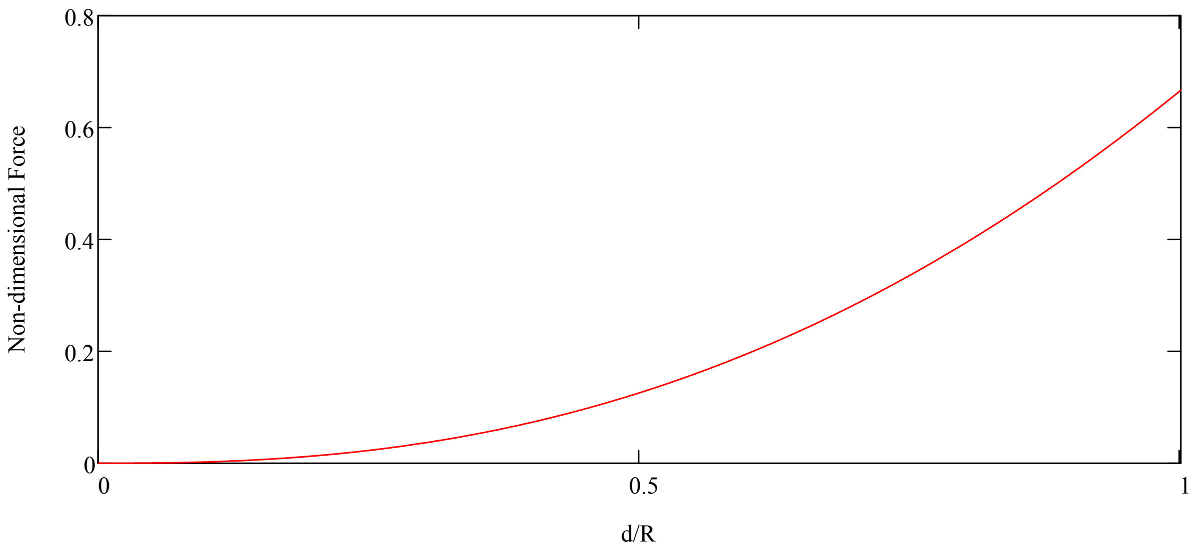
$$y'\cdot F_R = 2\cdot\rho\cdot g\cdot R^4\cdot\int_0^{\alpha}\left[(\sin(\theta))^2\cdot(\cos(\theta))^2-\cos(\alpha)\cdot(\sin(\theta))^2\cdot\cos(\theta)\right]d\theta = 2\cdot\rho\cdot g\cdot R^4\cdot\left[\frac{1}{8}\cdot\left(\alpha-\frac{\sin(4\cdot\alpha)}{4}\right)-\cos(\alpha)\cdot\frac{(\sin(\alpha))^3}{3}\right]$$

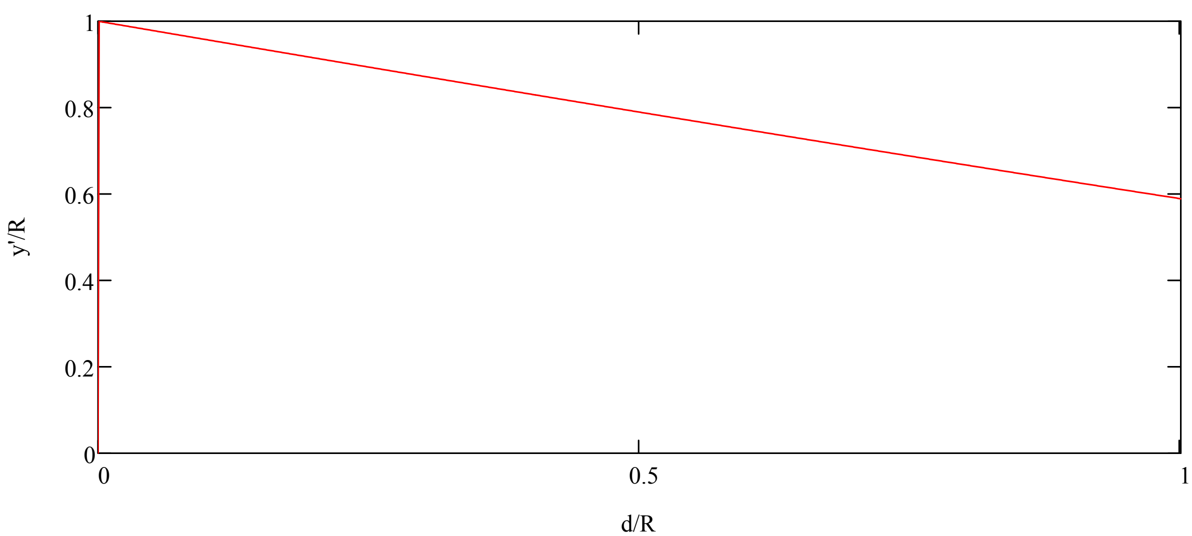
$$y'\cdot F_R = 2\cdot\rho\cdot g\cdot R^4\cdot\left[\frac{1}{8}\cdot\left(\alpha-\frac{\sin(4\cdot\alpha)}{4}\right)-\cos(\alpha)\cdot\frac{(\sin(\alpha))^3}{3}\right]$$ and therefore $y' = \frac{y'\cdot F_R}{F_R}$ or $\frac{y'}{R} = \frac{y'\cdot F_R}{R\cdot F_R}$

Simplifying the expression:

$$\frac{y'}{R} = \frac{\frac{1}{8}\cdot\left(\alpha-\frac{\sin(4\cdot\alpha)}{4}\right)-\cos(\alpha)\cdot\frac{(\sin(\alpha))^3}{3}}{\frac{(\sin(\alpha))^3}{3}-\cos(\alpha)\cdot\left(\frac{\alpha}{2}-\frac{\sin(\alpha)\cdot\cos(\alpha)}{2}\right)}$$

Plots of the non-dimensionalized force and the line of action of the force are shown in the plots below:

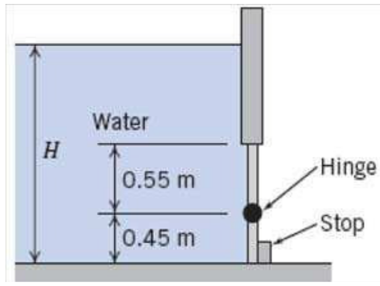




Problem 3.54

[Difficulty: 3]

3.54 A rectangular gate (width $w = 2$ m) is hinged as shown, with a stop on the lower edge. At what depth H will the gate tip?



Given:
Gate geometry

Find:
Depth H at which gate tips

Solution:

This is a problem with atmospheric pressure on both sides of the plate, so we can first determine the location of the center of pressure with respect to the free surface, using Eq.3.11c (assuming depth H)

$$y' = y_c + \frac{I_{xx}}{A \cdot y_c} \qquad \text{and} \qquad I_{xx} = \frac{w \cdot L^3}{12} \qquad \text{with} \qquad y_c = H - \frac{L}{2}$$

where $L = 1$ m is the plate height and w is the plate width

Hence
$$y' = \left(H - \frac{L}{2}\right) + \frac{w \cdot L^3}{12 \cdot w \cdot L \cdot \left(H - \frac{L}{2}\right)} = \left(H - \frac{L}{2}\right) + \frac{L^2}{12 \cdot \left(H - \frac{L}{2}\right)}$$

But for equilibrium, the center of force must always be at or below the level of the hinge so that the stop can hold the gate in place. Hence we must have

$$y' > H - 0.45 \cdot m$$

Combining the two equations
$$\left(H - \frac{L}{2}\right) + \frac{L^2}{12 \cdot \left(H - \frac{L}{2}\right)} \geq H - 0.45 \cdot m$$

Solving for H
$$H \leq \frac{L}{2} + \frac{L^2}{12 \cdot \left(\frac{L}{2} - 0.45 \cdot m\right)} \qquad H \leq \frac{1 \cdot m}{2} + \frac{(1 \cdot m)^2}{12 \times \left(\frac{1 \cdot m}{2} - 0.45 \cdot m\right)} \qquad H \leq 2.17 \cdot m$$

Problem 3.55

[Difficulty: 1]

3.55 For a mug of tea (65 mm diameter), imagine it cut symmetrically in half by a vertical plane. Find the force that each half experiences due to an 80-mm depth of tea.

Given: Geometry of cup

Find: Force on each half of cup

Assumptions: (1) Tea is static and incompressible
(2) Atmospheric pressure on outside of cup

Solution:

Basic equation $\quad F_R = \int p\,dA \qquad \dfrac{dp}{dh} = \rho\cdot g$

or, use computing equation $\quad F_R = p_c\cdot A$

The force on the half-cup is the same as that on a rectangle of size $h = 8\cdot cm$ and $w = 6.5\cdot cm$

$$F_R = \int p\,dA = \int \rho\cdot g\cdot y\,dA \qquad \text{but} \qquad dA = w\cdot dy$$

Hence $\quad F_R = \int_0^h \rho\cdot g\cdot y\cdot w\,dy = \dfrac{\rho\cdot g\cdot w\cdot h^2}{2}$

Alternatively $\quad F_R = p_c\cdot A \quad$ and $\quad F_R = p_c\cdot A = \rho\cdot g\cdot y_c\cdot A = \rho\cdot g\cdot\dfrac{h}{2}\cdot h\cdot w = \dfrac{\rho\cdot g\cdot w\cdot h^2}{2}$

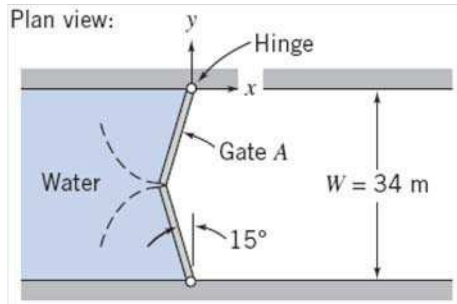
Using given data $\quad F_R = \dfrac{1}{2}\times 999\cdot\dfrac{kg}{m^3}\times 9.81\cdot\dfrac{m}{s^2}\times 6.5\cdot cm\times(8\cdot cm)^2\times\left(\dfrac{m}{100\cdot cm}\right)^3\times\dfrac{N\cdot s^2}{kg\cdot m} \qquad F_R = 2.04\cdot N$

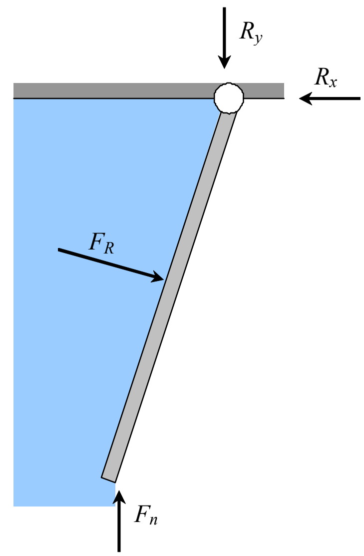
Hence a teacup is being forced apart by about 2 N: not much of a force, so a paper cup works!

Problem 3.56

[Difficulty: 3]

3.56 Gates in the Poe Lock at Sault Ste. Marie, Michigan, close a channel $W = 34$ m wide, $L = 360$ m long, and $D = 10$ m deep. The geometry of one pair of gates is shown; each gate is hinged at the channel wall. When closed, the gate edges are forced together at the center of the channel by water pressure. Evaluate the force exerted by the water on gate A. Determine the magnitude and direction of the force components exerted by the gate on the hinge. (Neglect the weight of the gate.)





Given: Geometry of lock system

Find: Force on gate; reactions at hinge

Solution:

Basic equation $\quad F_R = \int p\,dA \qquad \frac{dp}{dh} = \rho\cdot g$

or, use computing equation $\quad F_R = p_c\cdot A$

Assumptions: static fluid; ρ = constant; p_{atm} on other side

The force on each gate is the same as that on a rectangle of size

$$h = D = 10\cdot m \quad \text{and} \quad w = \frac{W}{2\cdot\cos(15\cdot deg)}$$

$$F_R = \int p\,dA = \int \rho\cdot g\cdot y\,dA \qquad \text{but} \qquad dA = w\cdot dy$$

Hence $\quad F_R = \int_0^h \rho\cdot g\cdot y\cdot w\,dy = \frac{\rho\cdot g\cdot w\cdot h^2}{2}$

Alternatively $\quad F_R = p_c\cdot A \quad$ and $\quad F_R = p_c\cdot A = \rho\cdot g\cdot y_c\cdot A = \rho\cdot g\cdot\frac{h}{2}\cdot h\cdot w = \frac{\rho\cdot g\cdot w\cdot h^2}{2}$

Using given data $\quad F_R = \frac{1}{2}\cdot 1000\cdot\frac{kg}{m^3}\times 9.81\cdot\frac{m}{s^2}\times\frac{34\cdot m}{2\cdot\cos(15\cdot deg)}\times(10\cdot m)^2\times\frac{N\cdot s^2}{kg\cdot m} \qquad F_R = 8.63\cdot MN$

For the force components R_x and R_y we do the following

$$\Sigma M_{hinge} = 0 = F_R\cdot\frac{w}{2} - F_n\cdot w\cdot\sin(15\cdot deg) \qquad F_n = \frac{F_R}{2\cdot\sin(15\cdot deg)} \qquad F_n = 16.7\cdot MN$$

$$\Sigma F_x = 0 = F_R\cdot\cos(15\cdot deg) - R_x = 0 \qquad R_x = F_R\cdot\cos(15\cdot deg) \qquad R_x = 8.34\cdot MN$$

$$\Sigma F_y = 0 = -R_y - F_R\cdot\sin(15\cdot deg) + F_n = 0 \qquad R_y = F_n - F_R\cdot\sin(15\cdot deg) \qquad R_y = 14.4\cdot MN$$

$$R = (8.34\cdot MN, 14.4\cdot MN) \qquad R = 16.7\cdot MN$$

Problem 3.57

[Difficulty: 2]

3.57 A section of vertical wall is to be constructed from ready-mix concrete poured between forms. The wall is to be 3 m high, 0.25 m thick, and 5 m wide. Calculate the force exerted by the ready-mix concrete on each form. Determine the line of application of the force.

Given: Liquid concrete poured between vertical forms as shown

$t = 0.25 \cdot m \quad H = 3 \cdot m \quad W = 5 \cdot m \quad SG_c = 2.5$ (From Table A.1, App. A)

Find: (a) Resultant force on form

(b) Line of application

Solution: We will apply the hydrostatics equations to this system.

Governing Equations:

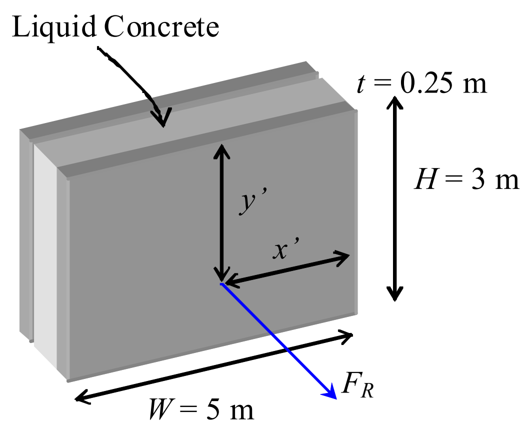
$$\frac{dp}{dy} = \rho \cdot g$$ (Hydrostatic Pressure - y is positive downwards)

$$F_R = p_c \cdot A$$ (Hydrostatic Force)

$$y' = y_c + \frac{I_{xx}}{A \cdot y_c}$$

(Location of line of action)

$$x' = x_c + \frac{I_{xy}}{A \cdot y_c}$$



Assumptions:

(1) Static fluid

(2) Incompressible fluid

(3) Atmospheric pressure acts at free surface and on the outside of the form.

For a rectangular plate: $I_{xx} = \frac{W \cdot H^3}{12} \quad I_{xy} = 0$

$x_c = 2.5 \cdot m \quad y_c = 1.5 \cdot m$

Integrating the hydrostatic pressure equation: $p = \rho \cdot g \cdot y$

The density of concrete is:

$$\rho = 2.5 \times 1000 \cdot \frac{kg}{m^3} \qquad \rho = 2.5 \times 10^3 \frac{kg}{m^3}$$

Therefore, the force is: $F_R = \rho \cdot g \cdot y_c \cdot H \cdot W$

Substituting in values gives us: $F_R = 2.5 \times 10^3 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times 1.5 \cdot m \times 3 \cdot m \times 5 \cdot m \qquad F_R = 552 \cdot kN$

To find the line of action of the resultant force:

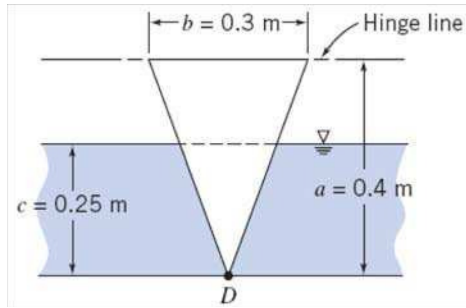
$$y' = y_c + \frac{W \cdot H^3}{12 \cdot W \cdot H \cdot y_c} = y_c + \frac{H^2}{12 \cdot y_c} \qquad y' = 1.5 \cdot m + \frac{(3 \cdot m)^2}{12 \cdot 1.5 \cdot m} \qquad y' = 2.00\,m$$

Since $I_{xy} = 0$ it follows that $x' = x_c \qquad x' = 2.50 \cdot m$

Problem 3.58

[Difficulty: 4]

3.58 A window in the shape of an isosceles triangle and hinged at the top is placed in the vertical wall of a form that contains liquid concrete. Determine the minimum force that must be applied at point D to keep the window closed for the configuration of form and concrete shown. Plot the results over the range of concrete depth $0 \le c \le a$



Given: Window, in shape of isosceles triangle and hinged at the top is located in the vertical wall of a form that contains concrete.

$a = 0.4 \cdot m \quad b = 0.3 \cdot m \quad c = 0.25 \cdot m \quad SG_c = 2.5$ (From Table A.1, App. A)

Find: The minimum force applied at D needed to keep the window closed.
Plot the results over the range of concrete depth between 0 and a.

Solution: We will apply the hydrostatics equations to this system.

Governing Equations:

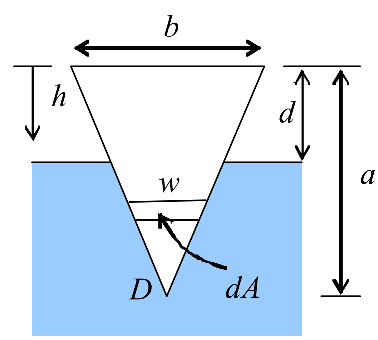
$$\frac{dp}{dh} = \rho \cdot g$$ (Hydrostatic Pressure - h is positive downwards)

$$F_R = \int p\, dA$$ (Hydrostatic Force on door)

$$y' \cdot F_R = \int y \cdot p\, dA$$ (First moment of force)

$$\Sigma M = 0$$ (Rotational equilibrium)

Assumptions:
(1) Static fluid
(2) Incompressible fluid
(3) Atmospheric pressure acts at free surface and on the outside of the window.



Integrating the pressure equation yields: $p = \rho \cdot g \cdot (h - d)$ for $h > d$

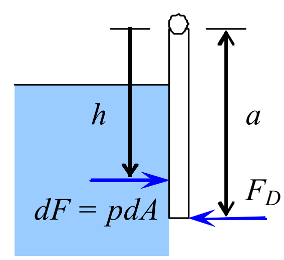
$p = 0$ for $h < d$

where $d = a - c \quad d = 0.15 \cdot m$

Summing moments around the hinge: $-F_D \cdot a + \int h \cdot p\, dA = 0$

$$F_D = \frac{1}{a} \cdot \int h \cdot p\, dA = \frac{1}{a} \cdot \int_d^a h \cdot \rho \cdot g \cdot (h - d) \cdot w\, dh = \frac{\rho \cdot g}{a} \cdot \int_d^a h \cdot (h - d) \cdot w\, dh$$

From the law of similar triangles: $\frac{w}{b} = \frac{a - h}{a}$ Therefore: $w = \frac{b}{a}(a - h)$



Into the expression for the force at D: $F_D = \frac{\rho \cdot g}{a} \cdot \int_d^a \frac{b}{a} \cdot h \cdot (h-d) \cdot (a-h)\, dh = \frac{\rho \cdot g \cdot b}{a^2} \cdot \int_d^a \left[-h^3 + (a+d) \cdot h^2 - a \cdot d \cdot h\right] dh$

Evaluating this integral we get:

$$F_D = \frac{\rho \cdot g \cdot b}{a^2} \cdot \left[-\frac{\left(a^4 - d^4\right)}{4} + \frac{(a+d) \cdot \left(a^3 - d^3\right)}{3} - \frac{a \cdot d \cdot \left(a^2 - d^2\right)}{2}\right]$$

and after collecting terms:

$$F_D = \rho \cdot g \cdot b \cdot a^2 \cdot \left[-\frac{1}{4} \cdot \left[1 - \left(\frac{d}{a}\right)^4\right] + \frac{1}{3} \cdot \left(1 + \frac{d}{a}\right) \cdot \left[1 - \left(\frac{d}{a}\right)^3\right] - \frac{1}{2} \cdot \frac{d}{a} \cdot \left[1 - \left(\frac{d}{a}\right)^2\right]\right] \qquad (1)$$

The density of the concrete is: $\rho = 2.5 \times 1000 \cdot \frac{kg}{m^3}$ $\qquad \rho = 2.5 \times 10^3 \frac{kg}{m^3}$ $\qquad \frac{d}{a} = \frac{0.15}{0.4} = 0.375$

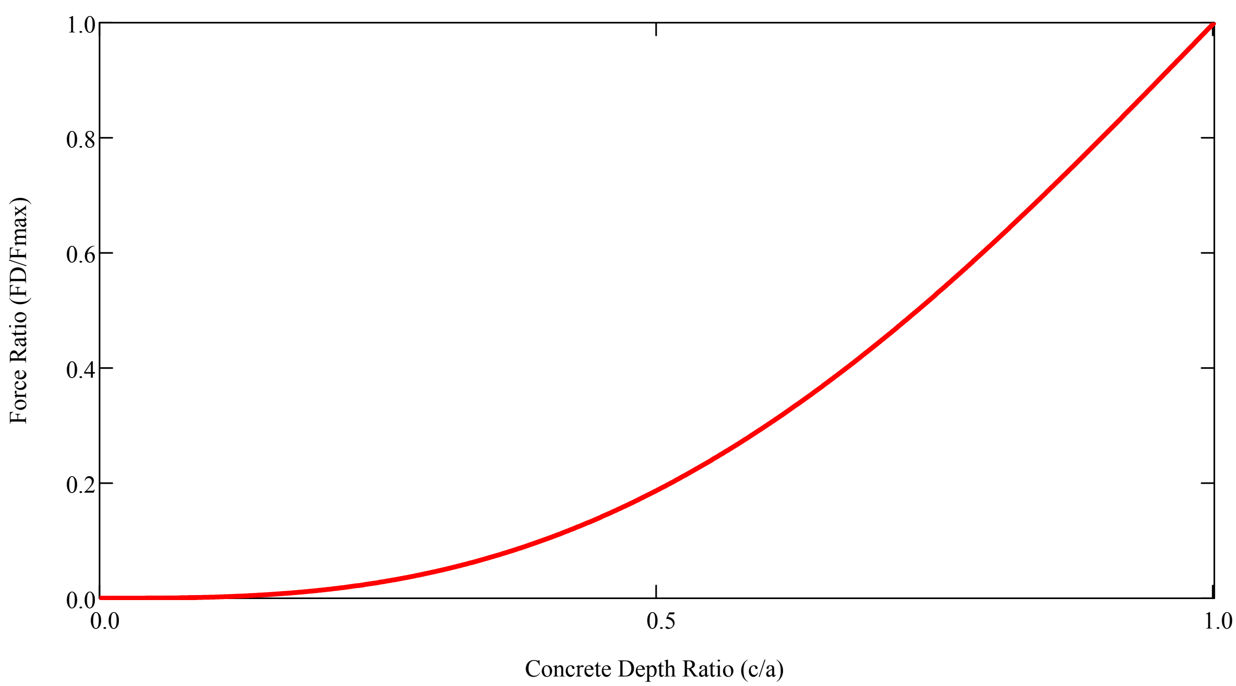
Substituting in values for the force at D:

$$F_D = 2.5 \times 10^3 \cdot \frac{kg}{m^3} \cdot 9.81 \cdot \frac{m}{s^2} \cdot 0.3 \cdot m \cdot (0.4 \cdot m)^2 \cdot \left[-\frac{1}{4} \cdot \left[1 - (0.375)^4\right] + \frac{1}{3} \cdot (1 + 0.375) \cdot \left[1 - (0.375)^3\right] - \frac{0.375}{2} \cdot \left[1 - (0.375)^2\right]\right] \times \frac{N \cdot s^2}{kg \cdot m}$$

To plot the results for different values of c/a, we use Eq. (1) and remember that $d = a - c$ $\qquad F_D = 32.9\,N$

Therefore, it follows that $\frac{d}{a} = 1 - \frac{c}{a}$ In addition, we can maximize the force by the maximum force (when c = a or d = 0):

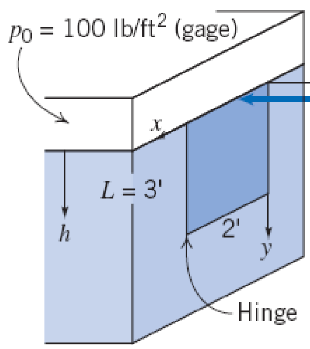
$$F_{max} = \rho \cdot g \cdot b \cdot a^2 \cdot \left(-\frac{1}{4} + \frac{1}{3}\right) = \frac{\rho \cdot g \cdot b \cdot a^2}{12}$$ and so $$\frac{F_D}{F_{max}} = 12 \cdot \left[-\frac{1}{4} \cdot \left[1 - \left(\frac{d}{a}\right)^4\right] + \frac{1}{3} \cdot \left(1 + \frac{d}{a}\right) \cdot \left[1 - \left(\frac{d}{a}\right)^3\right] - \frac{1}{2} \cdot \frac{d}{a} \cdot \left[1 - \left(\frac{d}{a}\right)^2\right]\right]$$

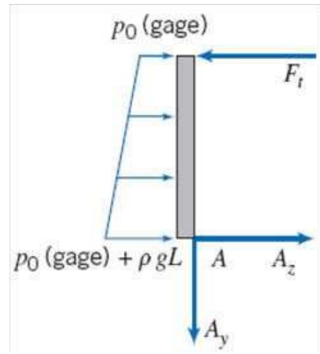


Problem 3.59

[Difficulty: 2]

3.59 Solve Example 3.6 again using the two separate pressures method. Consider the distributed force to be the sum of a force F_1 caused by the uniform gage pressure and a force F_2 caused by the liquid. Solve for these forces and their lines of action. Then sum moments about the hinge axis to calculate F_t.



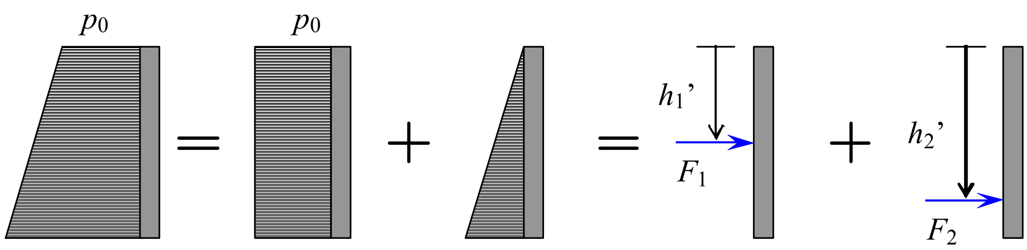


Given: Door as shown; Data from Example 3.6.

Find: Force to keep door shut using the two seperate pressures method.

Solution: We will apply the computing equations to this system.

Governing Equations: $F_R = p_c \cdot A \qquad y' = y_c + \dfrac{I_{xx}}{y_c \cdot A} \qquad I_{xx} = \dfrac{b \cdot L^3}{12}$



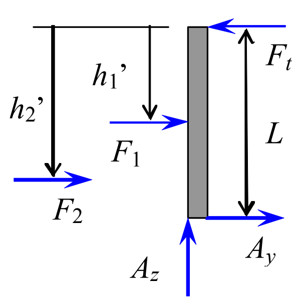
$F_1 = p_0 \cdot A \qquad F_1 = 100 \cdot \dfrac{lbf}{ft^2} \times 3 \cdot ft \times 2 \cdot ft \qquad F_1 = 600\,lbf \qquad x' = 1 \cdot ft \qquad z' = 1.5 \cdot ft$

$F_2 = p_c \cdot A = \rho \cdot g \cdot h_c \cdot L \cdot b = \gamma \cdot h_c \cdot L \cdot b \qquad F_2 = 100 \cdot \dfrac{lbf}{ft^3} \times 1.5 \cdot ft \times 3 \cdot ft \times 2 \cdot ft \qquad F_2 = 900\,lbf$

For the rectangular door $\quad I_{xx} = \dfrac{1}{12} \cdot b \cdot L^3$

$h'_2 = h_c + \dfrac{I_{xx}}{b \cdot L \cdot h_c} = h_c + \dfrac{1}{12} \cdot \dfrac{L^2}{h_c} \qquad h'_2 = 1.5 \cdot m + \dfrac{1}{12} \cdot \dfrac{(3 \cdot m)^2}{1.5 \cdot m} \qquad h'_2 = 2\,m$

The free body diagram of the door is then



$$\Sigma M_{Ax} = 0 = L \cdot F_t - F_1 \cdot (L - h'_1) - F_2 \cdot (L - h'_2)$$

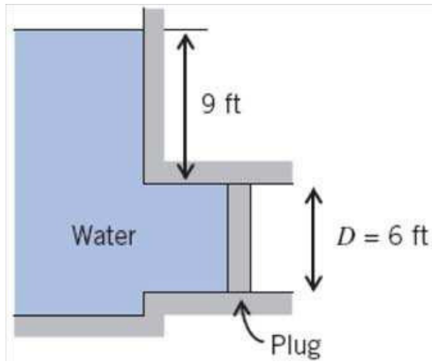
$$F_t = F_1 \cdot \left(1 - \frac{h'_1}{L}\right) + F_2 \cdot \left(1 - \frac{h'_2}{L}\right)$$

$$F_t = 600 \cdot lbf \cdot \left(1 - \frac{1.5}{3}\right) + 900 \cdot lbf \cdot \left(1 - \frac{2}{3}\right) \qquad F_t = 600\,lbf$$

Problem 3.60

[Difficulty: 2]

3.60 A large open tank contains water and is connected to a 6-ft-diameter conduit as shown. A circular plug is used to seal the conduit. Determine the magnitude, direction, and location of the force of the water on the plug.



Given: Plug is used to seal a conduit. $\gamma = 62.4 \cdot \frac{\text{lbf}}{\text{ft}^3}$

Find: Magnitude, direction and location of the force of water on the plug.

Solution: We will apply the hydrostatics equations to this system.

Governing Equations:

$$\frac{dp}{dh} = \gamma$$ (Hydrostatic Pressure - y is positive downwards)

$$F_R = p_c \cdot A$$ (Hydrostatic Force)

$$y' = y_c + \frac{I_{xx}}{A \cdot y_c}$$ (Location of line of action)

Assumptions:
(1) Static fluid
(2) Incompressible fluid
(3) Atmospheric pressure acts on the outside of the plug.

Integrating the hydrostatic pressure equation: $p = \gamma \cdot h$ $\quad F_R = p_c \cdot A = \gamma \cdot h_c \cdot \frac{\pi}{4} \cdot D^2$

$$F_R = 62.4 \cdot \frac{\text{lbf}}{\text{ft}^3} \times 12 \cdot \text{ft} \times \frac{\pi}{4} \times (6 \cdot \text{ft})^2 \qquad F_R = 2.12 \times 10^4 \cdot \text{lbf}$$

For a circular area: $I_{xx} = \frac{\pi}{64} \cdot D^4$ Therefore: $y' = y_c + \dfrac{\frac{\pi}{64} \cdot D^4}{\frac{\pi}{4} \cdot D^2 \cdot y_c} = y_c + \dfrac{D^2}{16 \cdot y_c}$

$$y' = 12 \cdot \text{ft} + \frac{(6 \cdot \text{ft})^2}{16 \times 12 \cdot \text{ft}}$$

$$y' = 12.19 \cdot \text{ft}$$

The force of water is to the right and perpendicular to the plug.

Problem 3.61

[Difficulty: 1]

3.61 What holds up a car on its rubber tires? Most people would tell you that it is the air pressure inside the tires. However, the air pressure is the same all around the hub (inner wheel), and the air pressure inside the tire therefore pushes down from the top as much as it pushes up from below, having no net effect on the hub. Resolve this paradox by explaining where the force is that keeps the car off the ground.

Given: Description of car tire

Find: Explanation of lift effect

Solution:

The explanation is as follows: It is true that the pressure in the entire tire is the same everywhere. However, the tire at the top of the hub will be essentially circular in cross-section, but at the bottom, where the tire meets the ground, the cross section will be approximately a flattened circle, or elliptical. Hence we can explain that the lower cross section has greater upward force than the upper cross section has downward force (providing enough lift to keep the car up) two ways. First, the horizontal projected area of the lower ellipse is larger than that of the upper circular cross section, so that net pressure times area is upwards. Second, any time you have an elliptical cross section that's at high pressure, that pressure will always try to force the ellipse to be circular (thing of a round inflated balloon - if you squeeze it it will resist!). This analysis ignores the stiffness of the tire rubber, which also provides a little lift.

Problem 3.62

[Difficulty: 2]

3.62 The circular access port in the side of a water standpipe has a diameter of 0.6 m and is held in place by eight bolts evenly spaced around the circumference. If the standpipe diameter is 7 m and the center of the port is located 12 m below the free surface of the water, determine (a) the total force on the port and (b) the appropriate bolt diameter.

Given: Circular access port of known diameter in side of water standpipe of known diameter. Port is held in place by eight bolts evenly spaced around the circumference of the port.
Center of the port is located at a know distance below the free surface of the water.

$d = 0.6 \cdot m \quad D = 7 \cdot m \quad L = 12 \cdot m$

Find: (a) Total force on the port
(b) Appropriate bolt diameter

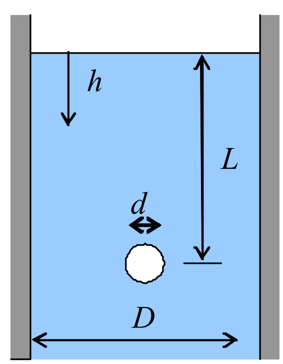
Solution: We will apply the hydrostatics equations to this system.

Governing Equations:

$$\frac{dp}{dh} = \rho \cdot g$$ (Hydrostatic Pressure - y is positive downwards)

$$F_R = p_c \cdot A$$ (Hydrostatic Force)

$$\sigma = \frac{F}{A}$$ (Normal Stress in bolt)



Assumptions:
(1) Static fluid
(2) Incompressible fluid
(3) Force is distributed evenly over all bolts
(4) Appropriate working stress in bolts is 100 MPa
(5) Atmospheric pressure acts at free surface of water and on outside of port.

Integrating the hydrostatic pressure equation: $p = \rho \cdot g \cdot h$

The resultant force on the port is: $F_R = p_c \cdot A = \rho \cdot g \cdot L \cdot \frac{\pi}{4} \cdot d^2 \qquad F_R = 999 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times 12 \cdot m \times \frac{\pi}{4} \times (0.6 \cdot m)^2 \times \frac{N \cdot s^2}{kg \cdot m}$

$$F_R = 33.3 \cdot kN$$

To find the bolt diameter we consider: $\sigma = \frac{F_R}{A}$ where A is the area of all of the bolts: $A = 8 \times \frac{\pi}{4} \cdot d_b^2 = 2 \cdot \pi \cdot d_b^2$

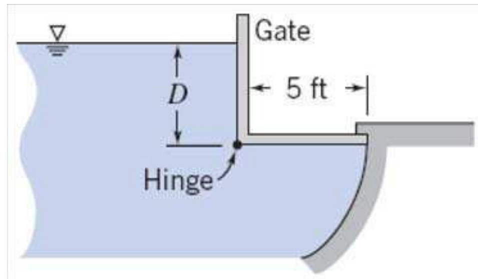
Therefore: $2 \cdot \pi \cdot d_b^2 = \frac{F_R}{\sigma}$ Solving for the bolt diameter we get: $d_b = \left(\frac{F_R}{2 \cdot \pi \cdot \sigma}\right)^{\frac{1}{2}}$

$$d_b = \left(\frac{1}{2 \times \pi} \times 33.3 \times 10^3 \cdot N \times \frac{1}{100 \times 10^6} \cdot \frac{m^2}{N}\right)^{\frac{1}{2}} \times \frac{10^3 \cdot mm}{m} \qquad d_b = 7.28 \cdot mm$$

Problem 3.63

[Difficulty: 3]

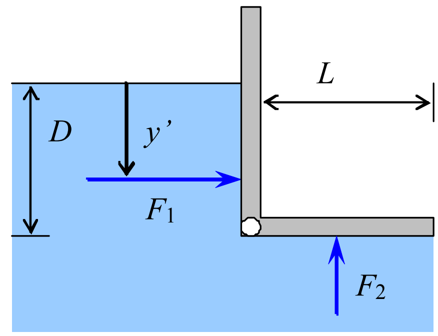
3.63 As water rises on the left side of the rectangular gate, the gate will open automatically. At what depth above the hinge will this occur? Neglect the mass of the gate.



Given: Geometry of rectangular gate

Find: Depth for gate to open

Solution:



Basic equation $\dfrac{dp}{dh} = \rho \cdot g$ $\qquad \Sigma M_z = 0$

Computing equations $F_R = p_c \cdot A$ $\qquad y' = y_c + \dfrac{I_{xx}}{A \cdot y_c}$ $\qquad I_{xx} = \dfrac{b \cdot D^3}{12}$

Assumptions: Static fluid; ρ = constant; p_{atm} on other side; no friction in hinge

For incompressible fluid $\quad p = \rho \cdot g \cdot h \quad$ where p is gage pressure and h is measured downwards

The force on the vertical gate (gate 1) is the same as that on a rectangle of size h = D and width w

Hence
$$F_1 = p_c \cdot A = \rho \cdot g \cdot y_c \cdot A = \rho \cdot g \cdot \frac{D}{2} \cdot D \cdot w = \frac{\rho \cdot g \cdot w \cdot D^2}{2}$$

The location of this force is
$$y' = y_c + \frac{I_{xx}}{A \cdot y_c} = \frac{D}{2} + \frac{w \cdot D^3}{12} \times \frac{1}{w \cdot D} \times \frac{2}{D} = \frac{2}{3} \cdot D$$

The force on the horizontal gate (gate 2) is due to constant pressure, and is at the centroid

$$F_2 = p(y = D) \cdot A = \rho \cdot g \cdot D \cdot w \cdot L$$

Summing moments about the hinge
$$\Sigma M_{hinge} = 0 = -F_1 \cdot (D - y') + F_2 \cdot \frac{L}{2} = -F_1 \cdot \left(D - \frac{2}{3} \cdot D\right) + F_2 \cdot \frac{L}{2}$$

$$F_1 \cdot \frac{D}{3} = \frac{\rho \cdot g \cdot w \cdot D^2}{2} \cdot \frac{D}{3} = F_2 \cdot \frac{L}{2} = \rho \cdot g \cdot D \cdot w \cdot L \cdot \frac{L}{2}$$

$$\frac{\rho \cdot g \cdot w \cdot D^3}{6} = \frac{\rho \cdot g \cdot D \cdot w \cdot L^2}{2}$$

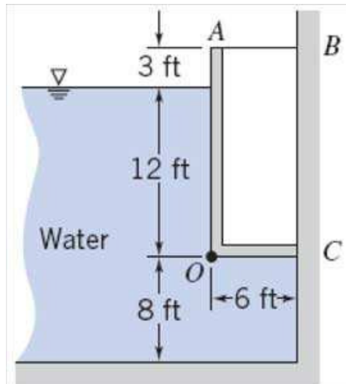
$$D = \sqrt{3} \cdot L = \sqrt{3} \times 5\text{ft}$$

$$D = 8.66 \cdot \text{ft}$$

Problem 3.64

[Difficulty: 3]

3.64 The gate *AOC* shown is 6 ft wide and is hinged along *O*. Neglecting the weight of the gate, determine the force in bar *AB*. The gate is sealed at *C*.



Given: Gate AOC, hinged along O, has known width;
Weight of gate may be neglected. Gate is sealed at C.

$b = 6 \cdot ft$

Find: Force in bar AB

Solution: We will apply the hydrostatics equations to this system.

Governing Equations:

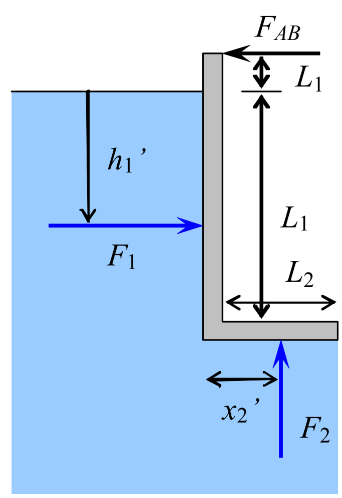
$\frac{dp}{dh} = \rho \cdot g$ (Hydrostatic Pressure - h is positive downwards)

$F_R = p_c \cdot A$ (Hydrostatic Force)

$y' = y_c + \frac{I_{xx}}{A \cdot y_c}$ (Location of line of action)

$\Sigma M_z = 0$ (Rotational equilibrium)

Assumptions:
(1) Static fluid
(2) Incompressible fluid
(3) Atmospheric pressure acts at free surface of water and on outside of gate
(4) No resisting moment in hinge at O
(5) No vertical resisting force at C



Integrating the hydrostatic pressure equation: $p = \rho \cdot g \cdot h$

The free body diagram of the gate is shown here:

F_1 is the resultant of the distributed force on AO

F_2 is the resultant of the distributed force on OC

F_{AB} is the force of the bar

C_x is the sealing force at C

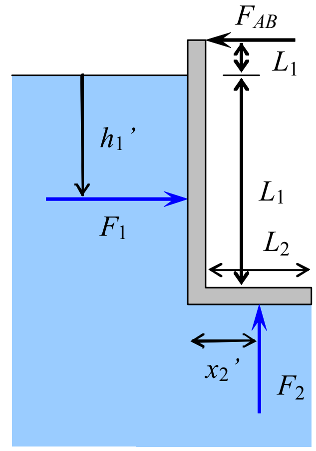
First find the force on AO: $F_1 = p_c \cdot A_1 = \rho \cdot g \cdot h_{c1} \cdot b \cdot L_1$

$$F_1 = 1.94 \cdot \frac{slug}{ft^3} \times 32.2 \cdot \frac{ft}{s^2} \times 6 \cdot ft \times 6 \cdot ft \times 12 \cdot ft \times \frac{lbf \cdot s^2}{slug\,ft} \qquad F_1 = 27.0 \cdot kip$$

$$h'_1 = h_{c1} + \frac{I_{xx}}{A \cdot h_{c1}} = h_{c1} + \frac{b \cdot L_1^{\ 3}}{12 \cdot b \cdot L_1 \cdot h_{c1}} = h_{c1} + \frac{L_1^{\ 2}}{12 \cdot h_{c1}} \qquad h'_1 = 6 \cdot ft + \frac{(12 \cdot ft)^2}{12 \times 6 \cdot ft} \qquad h'_1 = 8 \cdot ft$$

Next find the force on OC: $F_2 = 1.94 \cdot \frac{slug}{ft^3} \times 32.2 \cdot \frac{ft}{s^2} \times 12 \cdot ft \times 6 \cdot ft \times 6 \cdot ft \times \frac{lbf \cdot s^2}{slug \cdot ft}$ $\qquad F_2 = 27.0 \cdot kip$

Since the pressure is uniform over OC, the force acts at the centroid of OC, i.e., $\quad x'_2 = 3 \cdot ft$



Summing moments about the hinge gives: $\quad F_{AB} \cdot (L_1 + L_3) - F_1 \cdot (L_1 - h'_1) + F_2 \cdot x'_2 = 0$

Solving for the force in the bar: $\quad F_{AB} = \frac{F_1 \cdot (L_1 - h'_1) - F_2 \cdot x'_2}{L_1 + L_3}$

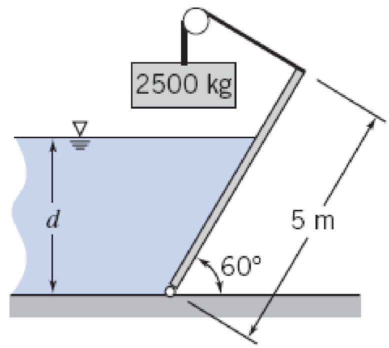
Substituting in values: $\quad F_{AB} = \frac{1}{12 \cdot ft + 3 \cdot ft} \cdot \left[27.0 \times 10^3 \cdot lbf \times (12 \cdot ft - 8 \cdot ft) - 27.0 \times 10^3 \cdot lbf \times 3 \cdot ft\right]$

$F_{AB} = 1800 \cdot lbf$ $\qquad$ Thus bar AB is in compression

Problem 3.65

[Difficulty: 3]

3.65 The gate shown is 3 m wide and for analysis can be considered massless. For what depth of water will this rectangular gate be in equilibrium as shown?



Given: Gate shown with fixed width, bass of gate is negligible.
Gate is in equilibrium.

$b = 3 \cdot m$

Find: Water depth, d

Solution: We will apply the hydrostatics equations to this system.

Governing Equations:

$$\frac{dp}{dh} = \rho \cdot g$$ (Hydrostatic Pressure - h is positive downwards)

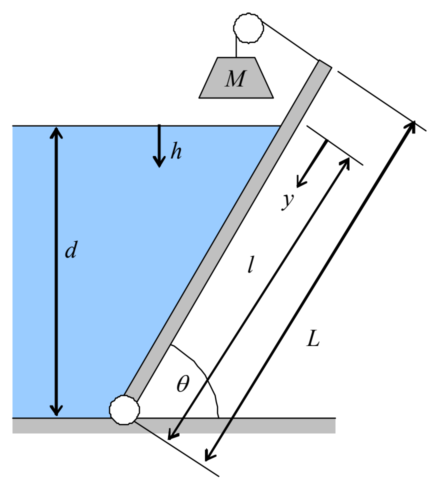
$$F_R = p_c \cdot A$$ (Hydrostatic Force)

$$y' = y_c + \frac{I_{xx}}{A \cdot y_c}$$ (Location of line of action)

$$\Sigma M_z = 0$$ (Rotational equilibrium)

Assumptions:

(1) Static fluid
(2) Incompressible fluid
(3) Atmospheric pressure acts at free surface of water and on outside of gate



Integrating the hydrostatic pressure equation: $p = \rho \cdot g \cdot h$

$$F_R = p_c \cdot A = \rho \cdot g \cdot h_c \cdot A \qquad h_c = \frac{d}{2} \qquad A = b \cdot \frac{d}{\sin(\theta)}$$

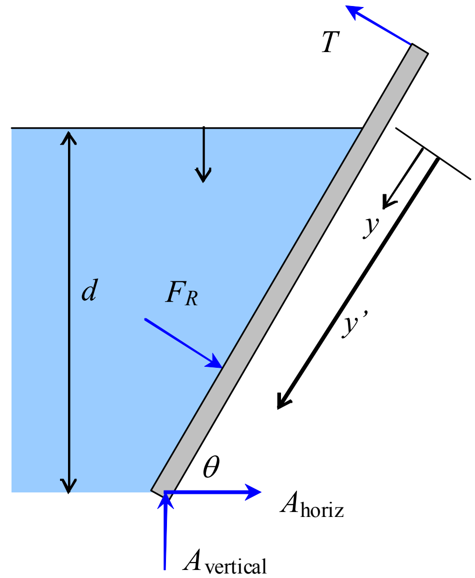
Therefore, $F_R = \dfrac{\rho \cdot g \cdot b \cdot d^2}{2 \cdot \sin(\theta)}$ To find the line of application of this force:

$$y' = y_c + \frac{I_{xx}}{A \cdot y_c}$$ Since $I_{xx} = \dfrac{b \cdot l^3}{12}$ and $A = b \cdot l$ it follows that

$$y' = y_c + \frac{b \cdot l^3}{12 \cdot b \cdot l \cdot y_c} = y_c + \frac{l^2}{12 \cdot y_c}$$ where l is the length of the gate in contact with the water (as seen in diagram)

l and d are related through: $\quad l = \frac{d}{\sin(\theta)}$ Therefore, $\quad y_c = \frac{l}{2} = \frac{d}{2\cdot\sin(\theta)}$ and $\quad y' = \frac{d}{2\cdot\sin(\theta)} + \frac{d^2}{(\sin(\theta))^2}\cdot\frac{2\cdot\sin(\theta)}{12\cdot d} = \frac{2\cdot d}{3\cdot\sin(\theta)}$

The free body diagram of the gate is shown here:



Summing moments about the hinge gives:

$$T\cdot L - (l - y')\cdot F_R = 0 \qquad \text{where} \quad T = M\cdot g$$

Solving for l: $\quad l = \frac{d}{\sin(\theta)} = \frac{M\cdot g\cdot L}{F_R} + y'$ $\qquad$ So upon further substitution we get:

$$d = \left(\frac{2\cdot M\cdot g\cdot L}{\rho\cdot g\cdot b\cdot d^2}\cdot\sin(\theta) + \frac{2\cdot d}{3\cdot\sin(\theta)}\right)\cdot\sin(\theta) \quad \text{or} \quad \frac{d}{3} = \frac{2\cdot M\cdot L\cdot(\sin(\theta))^2}{\rho\cdot b\cdot d^2}$$

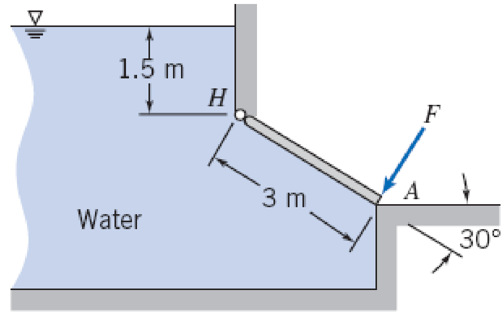
Solving for d: $\quad d = \left[\frac{6\cdot M\cdot L}{\rho\cdot b}\cdot(\sin(\theta))^2\right]^{\frac{1}{3}}$ $\quad$ Substituting in values: $\quad d = \left[6 \times 2500\cdot kg \times 5\cdot m \times \frac{1}{999}\cdot\frac{m^3}{kg} \times \frac{1}{3m} \times (\sin(60\cdot deg))^2\right]^{\frac{1}{3}}$

$$d = 2.66\,m$$

Problem 3.66

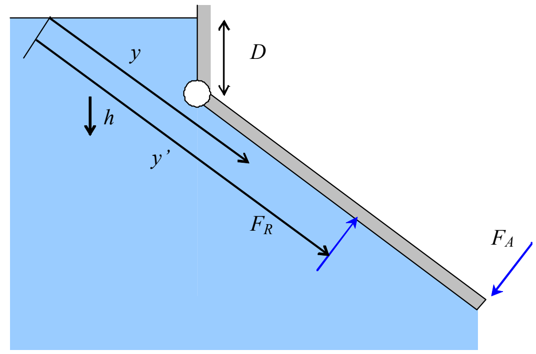
[Difficulty: 3]

3.66 The gate shown is hinged at H. The gate is 3 m wide normal to the plane of the diagram. Calculate the force required at A to hold the gate closed.



Given: Geometry of gate

Find: Force at A to hold gate closed



Solution:

Basic equation $\quad \dfrac{dp}{dh} = \rho \cdot g \qquad \Sigma M_z = 0$

Computing equations $\quad F_R = p_c \cdot A \qquad y' = y_c + \dfrac{I_{xx}}{A \cdot y_c} \qquad I_{xx} = \dfrac{w \cdot L^3}{12}$

Assumptions: Static fluid; ρ = constant; p_{atm} on other side; no friction in hinge

For incompressible fluid $\quad p = \rho \cdot g \cdot h \quad$ where p is gage pressure and h is measured downwards

The hydrostatic force on the gate is that on a rectangle of size L and width w.

Hence
$$F_R = p_c \cdot A = \rho \cdot g \cdot h_c \cdot A = \rho \cdot g \cdot \left(D + \frac{L}{2} \cdot \sin(30 \cdot deg) \right) \cdot L \cdot w$$

$$F_R = 1000 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times \left(1.5 + \frac{3}{2} \sin(30 \cdot deg) \right) \cdot m \times 3 \cdot m \times 3 \cdot m \times \frac{N \cdot s^2}{kg \cdot m} \qquad F_R = 199 \cdot kN$$

The location of this force is given by $y' = y_c + \dfrac{I_{xx}}{A \cdot y_c}$ where y' and y_c are measured along the plane of the gate to the free surface

$$y_c = \frac{D}{\sin(30 \cdot deg)} + \frac{L}{2} \qquad y_c = \frac{1.5 \cdot m}{\sin(30 \cdot deg)} + \frac{3 \cdot m}{2} \qquad y_c = 4.5\,m$$

$$y' = y_c + \frac{I_{xx}}{A \cdot y_c} = y_c + \frac{w \cdot L^3}{12} \cdot \frac{1}{w \cdot L} \cdot \frac{1}{y_c} = y_c + \frac{L^2}{12 \cdot y_c} = 4.5 \cdot m + \frac{(3 \cdot m)^2}{12 \cdot 4.5 \cdot m} \qquad y' = 4.67\,m$$

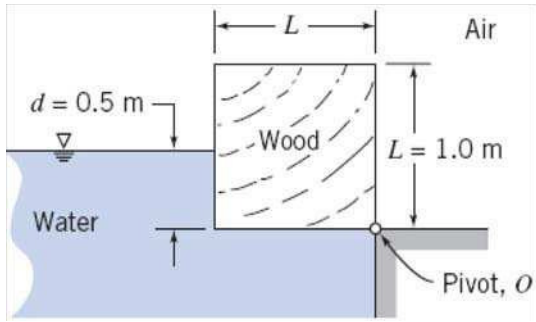
Taking moments about the hinge $\quad \Sigma M_H = 0 = F_R \cdot \left(y' - \dfrac{D}{\sin(30 \cdot deg)} \right) - F_A \cdot L$

$$F_A = F_R \cdot \frac{\left(y' - \dfrac{D}{\sin(30 \cdot deg)} \right)}{L} \qquad F_A = 199 \cdot kN \cdot \frac{\left(4.67 - \dfrac{1.5}{\sin(30 \cdot deg)} \right)}{3} \qquad F_A = 111 \cdot kN$$

Problem 3.67

[Difficulty: 3]

3.67 A long, square wooden block is pivoted along one edge. The block is in equilibrium when immersed in water to the depth shown. Evaluate the specific gravity of the wood, if friction in the pivot is negligible.



Given:
Block hinged and floating

Find:
SG of the wood

Solution:

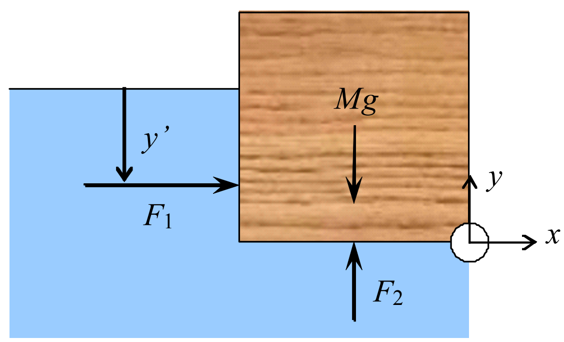
Basic equation $\dfrac{dp}{dh} = \rho \cdot g$ $\qquad \Sigma M_z = 0$

Computing equations $F_R = p_c \cdot A$ $\qquad y' = y_c + \dfrac{I_{xx}}{A \cdot y_c}$

Assumptions: Static fluid; ρ = constant; p_{atm} on other side; no friction in hinge

For incompressible fluid $\quad p = \rho \cdot g \cdot h \quad$ where p is gage pressure and h is measured downwards

The force on the vertical section is the same as that on a rectangle of height d and width L



Hence

$$F_1 = p_c \cdot A = \rho \cdot g \cdot y_c \cdot A = \rho \cdot g \cdot \frac{d}{2} \cdot d \cdot L = \frac{\rho \cdot g \cdot L \cdot d^2}{2}$$

The location of this force is

$$y' = y_c + \frac{I_{xx}}{A \cdot y_c} = \frac{d}{2} + \frac{L \cdot d^3}{12} \times \frac{1}{L \cdot d} \times \frac{2}{d} = \frac{2}{3} \cdot d$$

The force on the horizontal section is due to constant pressure, and is at the centroid

$$F_2 = p(y = d) \cdot A = \rho \cdot g \cdot d \cdot L \cdot L$$

Summing moments about the hinge $\qquad \Sigma M_{hinge} = 0 = -F_1 \cdot (d - y') - F_2 \cdot \frac{L}{2} + M \cdot g \cdot \frac{L}{2}$

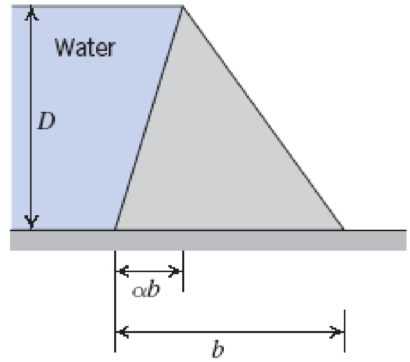
Hence $\qquad F_1 \cdot \left(d - \frac{2}{3} \cdot d\right) + F_2 \cdot \frac{L}{2} = SG \cdot \rho \cdot L^3 \cdot g \cdot \frac{L}{2}$

$$\frac{SG \cdot \rho \cdot g \cdot L^4}{2} = \frac{\rho \cdot g \cdot L \cdot d^2}{2} \cdot \frac{d}{3} + \rho \cdot g \cdot d \cdot L^2 \cdot \frac{L}{2} \qquad SG = \frac{1}{3} \cdot \left(\frac{d}{L}\right)^3 + \frac{d}{L} \qquad SG = \frac{1}{3} \cdot \left(\frac{0.5}{1}\right)^3 + \frac{0.5}{1} \qquad SG = 0.542$$

Problem 3.68

[Difficulty: 4]

3.68 A solid concrete dam is to be built to hold back a depth D of water. For ease of construction the walls of the dam must be planar. Your supervisor asks you to consider the following dam cross-sections: a rectangle, a right triangle with the hypotenuse in contact with the water, and a right triangle with the vertical in contact with the water. She wishes you to determine which of these would require the least amount of concrete. What will your report say? You decide to look at one more possibility: a nonright triangle, as shown. Develop and plot an expression for the cross-section area A as a function of α, and find the minimum cross-sectional area.



Given: Various dam cross-sections

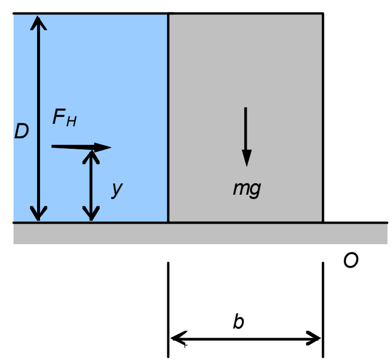
Find: Which requires the least concrete; plot cross-section area A as a function of α

Solution:

For each case, the dam width b has to be large enough so that the weight of the dam exerts enough moment to balance the moment due to fluid hydrostatic force(s). By doing a moment balance this value of b can be found

a) Rectangular dam

Straightforward application of the computing equations of Section 3-5 yields



$$F_H = p_c \cdot A = \rho \cdot g \cdot \frac{D}{2} \cdot w \cdot D = \frac{1}{2} \cdot \rho \cdot g \cdot D^2 \cdot w$$

$$y' = y_c + \frac{I_{xx}}{A \cdot y_c} = \frac{D}{2} + \frac{w \cdot D^3}{12 \cdot w \cdot D \cdot \frac{D}{2}} = \frac{2}{3} \cdot D$$

so $$y = D - y' = \frac{D}{3}$$

Also $$m = \rho_{cement} \cdot g \cdot b \cdot D \cdot w = SG \cdot \rho \cdot g \cdot b \cdot D \cdot w$$

Taking moments about O $$\sum M_{0.} = 0 = -F_H \cdot y + \frac{b}{2} \cdot m \cdot g$$

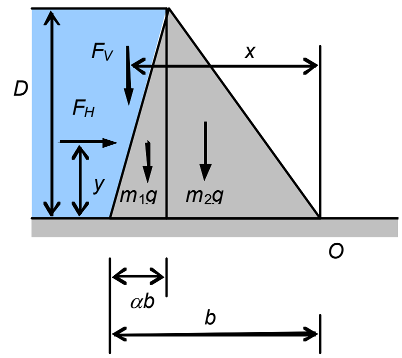
so $$\left(\frac{1}{2} \cdot \rho \cdot g \cdot D^2 \cdot w\right) \cdot \frac{D}{3} = \frac{b}{2} \cdot (SG \cdot \rho \cdot g \cdot b \cdot D \cdot w)$$

Solving for b $$b = \frac{D}{\sqrt{3 \cdot SG}}$$

The minimum rectangular cross-section area is $$A = b \cdot D = \frac{D^2}{\sqrt{3 \cdot SG}}$$

For concrete, from Table A.1, SG = 2.4, so $$A = \frac{D^2}{\sqrt{3 \cdot SG}} = \frac{D^2}{\sqrt{3 \times 2.4}}$$ $$A = 0.373 \cdot D^2$$

b) Triangular dams



Instead of analysing right-triangles, a general analysis is made, at the end of which right triangles are analysed as special cases by setting $\alpha = 0$ or 1.

Straightforward application of the computing equations of Section 3-5 yields

$$F_H = p_c \cdot A = \rho \cdot g \cdot \frac{D}{2} \cdot w \cdot D = \frac{1}{2} \cdot \rho \cdot g \cdot D^2 \cdot w$$

$$y' = y_c + \frac{I_{xx}}{A \cdot y_c} = \frac{D}{2} + \frac{w \cdot D^3}{12 \cdot w \cdot D \cdot \frac{D}{2}} = \frac{2}{3} \cdot D$$

so $$y = D - y' = \frac{D}{3}$$

Also $$F_V = \rho \cdot V \cdot g = \rho \cdot g \cdot \frac{\alpha \cdot b \cdot D}{2} \cdot w = \frac{1}{2} \cdot \rho \cdot g \cdot \alpha \cdot b \cdot D \cdot w \qquad x = (b - \alpha \cdot b) + \frac{2}{3} \cdot \alpha \cdot b = b \cdot \left(1 - \frac{\alpha}{3}\right)$$

For the two triangular masses

$$m_1 = \frac{1}{2} \cdot SG \cdot \rho \cdot g \cdot \alpha \cdot b \cdot D \cdot w \qquad x_1 = (b - \alpha \cdot b) + \frac{1}{3} \cdot \alpha \cdot b = b \cdot \left(1 - \frac{2 \cdot \alpha}{3}\right)$$

$$m_2 = \frac{1}{2} \cdot SG \cdot \rho \cdot g \cdot (1 - \alpha) \cdot b \cdot D \cdot w \qquad x_2 = \frac{2}{3} \cdot b(1 - \alpha)$$

Taking moments about *O*

$$\sum M_{0.} = 0 = -F_H \cdot y + F_V \cdot x + m_1 \cdot g \cdot x_1 + m_2 \cdot g \cdot x_2$$

so $$-\left(\frac{1}{2} \cdot \rho \cdot g \cdot D^2 \cdot w\right) \cdot \frac{D}{3} + \left(\frac{1}{2} \cdot \rho \cdot g \cdot \alpha \cdot b \cdot D \cdot w\right) \cdot b \cdot \left(1 - \frac{\alpha}{3}\right) \ldots + \left(\frac{1}{2} \cdot SG \cdot \rho \cdot g \cdot \alpha \cdot b \cdot D \cdot w\right) \cdot b \cdot \left(1 - \frac{2 \cdot \alpha}{3}\right) + \left[\frac{1}{2} \cdot SG \cdot \rho \cdot g \cdot (1 - \alpha) \cdot b \cdot D \cdot w\right] \cdot \frac{2}{3} \cdot b(1 - \alpha) = 0$$

Solving for *b* $$b = \frac{D}{\sqrt{\left(3 \cdot \alpha - \alpha^2\right) + SG \cdot (2 - \alpha)}}$$

For a right triangle with the hypotenuse in contact with the water, $\alpha = 1$, and

$$b = \frac{D}{\sqrt{3 - 1 + SG}} = \frac{D}{\sqrt{3 - 1 + 2.4}} \qquad b = 0.477 \cdot D$$

The cross-section area is $$A = \frac{b \cdot D}{2} = 0.238 \cdot D^2 \qquad A = 0.238 \cdot D^2$$

For a right triangle with the vertical in contact with the water, $\alpha = 0$, and

$$b = \frac{D}{\sqrt{2 \cdot SG}} = \frac{D}{\sqrt{2 \cdot 2.4}} \qquad b = 0.456 \cdot D$$

The cross-section area is

$$A = \frac{b \cdot D}{2} = 0.228 \cdot D^2 \qquad A = 0.228 \cdot D^2$$

For a general triangle

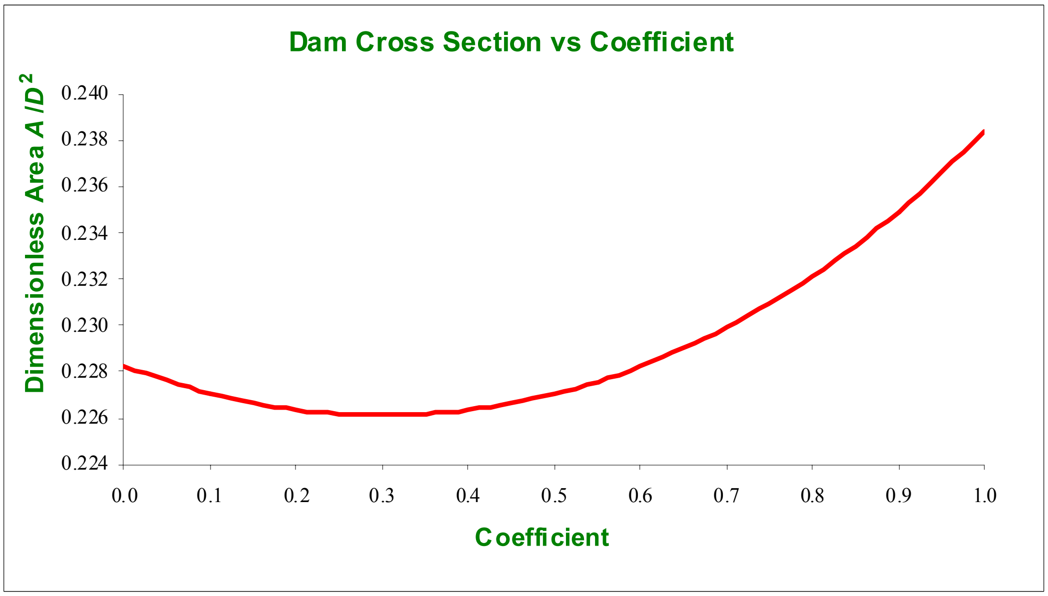
$$A = \frac{b \cdot D}{2} = \frac{D^2}{2 \cdot \sqrt{\left(3 \cdot \alpha - \alpha^2\right) + SG \cdot (2 - \alpha)}} \qquad A = \frac{D^2}{2 \cdot \sqrt{\left(3 \cdot \alpha - \alpha^2\right) + 2.4 \cdot (2 - \alpha)}}$$

The final result is

$$A = \frac{D^2}{2 \cdot \sqrt{4.8 + 0.6 \cdot \alpha - \alpha^2}}$$

The dimensionless area, A/D^2, is plotted

Alpha	A/D^2
0.0	0.2282
0.1	0.2270
0.2	0.2263
0.3	0.2261
0.4	0.2263
0.5	0.2270
0.6	0.2282
0.7	0.2299
0.8	0.2321
0.9	0.2349
1.0	0.2384



Solver can be used to find the minimum area

Alpha	A/D^2
0.300	0.2261

From the Excel workbook, the minimum area occurs at $\alpha = 0.3$

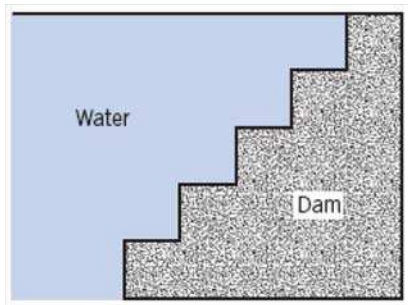
$$A_{min} = \frac{D^2}{2 \cdot \sqrt{4.8 + 0.6 \times 0.3 - 0.3^2}} \qquad A = 0.226 \cdot D^2$$

The final results are that a triangular cross-section with $\alpha = 0.3$ uses the least concrete; the next best is a right triangle with the vertical in contact with the water; next is the right triangle with the hypotenuse in contact with the water; and the cross-section requiring the most concrete is the rectangular cross-section.

Problem 3.69

[Difficulty: 2]

3.69 For the geometry shown, what is the vertical force on the dam? The steps are 0.5 m high, 0.5 m deep, and 3 m wide.



Given:
Geometry of dam

Find:
Vertical force on dam

Assumption:
Water is static and incompressible

Solution:

Basic equation: $$\frac{dp}{dh} = \rho \cdot g$$

For incompressible fluid $p = p_{atm} + \rho \cdot g \cdot h$ where h is measured downwards from the free surface

The force on each horizontal section (depth d = 0.5 m and width w = 3 m) is

$$F = p \cdot A = \left(p_{atm} + \rho \cdot g \cdot h\right) \cdot d \cdot w$$

Hence the total force is

$$F_T = \left[p_{atm} + \left(p_{atm} + \rho \cdot g \cdot h\right) + \left(p_{atm} + \rho \cdot g \cdot 2 \cdot h\right) + \left(p_{atm} + \rho \cdot 3 \cdot g \cdot h\right) + \left(p_{atm} + \rho \cdot g \cdot 4 \cdot h\right)\right] \cdot d \cdot w$$

where we have used h as the height of the steps

$$F_T = d \cdot w \cdot \left(5 \cdot p_{atm} + 10 \cdot \rho \cdot g \cdot h\right)$$

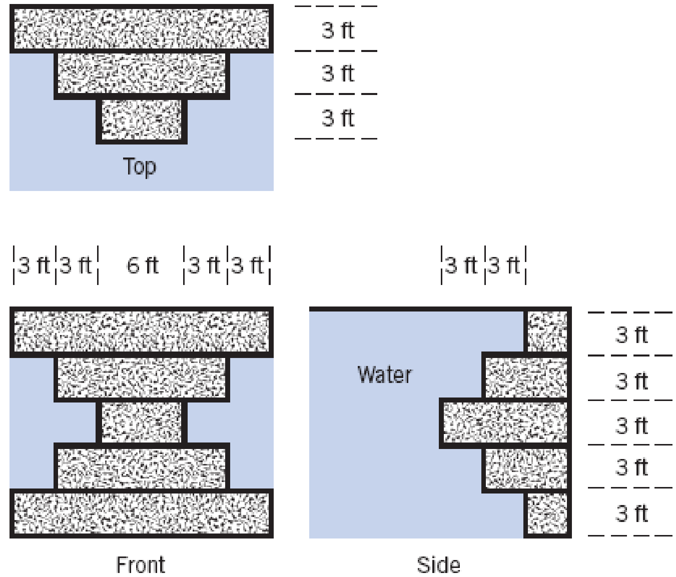
$$F_T = 0.5 \cdot m \times 3 \cdot m \times \left(5 \times 101 \times 10^3 \cdot \frac{N}{m^2} + 10 \times 999 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times 0.5 \cdot m \times \frac{N \cdot s^2}{kg \cdot m}\right)$$

$$F_T = 831 \cdot kN$$

Problem 3.70

[Difficulty: 2]

3.70 For the dam shown, what is the vertical force of the water on the dam?



Given: Geometry of dam

Find: Vertical force on dam

Assumptions:
(1) water is static and incompressible
(2) since we are asked for the force of the water, all pressures will be written as gage

Solution:

Basic equation: $$\frac{dp}{dh} = \rho \cdot g$$

For incompressible fluid $p = \rho \cdot g \cdot h$ where p is gage pressure and h is measured downwards from the free surface

The force on each horizontal section (depth d and width w) is

$$F = p \cdot A = \rho \cdot g \cdot h \cdot d \cdot w$$ (Note that d and w will change in terms of x and y for each section of the dam!)

Hence the total force is (allowing for the fact that some faces experience an upwards (negative) force)

$$F_T = p \cdot A = \Sigma \rho \cdot g \cdot h \cdot d \cdot w = \rho \cdot g \cdot d \cdot \Sigma h \cdot w$$

Starting with the top and working downwards

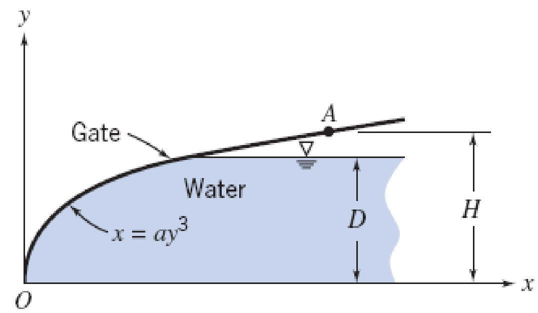
$$F_T = 1.94 \cdot \frac{\text{slug}}{\text{ft}^3} \times 32.2 \cdot \frac{\text{ft}}{\text{s}^2} \times 3 \cdot \text{ft} \times [(3 \cdot \text{ft} \times 12 \cdot \text{ft}) + (3 \cdot \text{ft} \times 6 \cdot \text{ft}) - (9 \cdot \text{ft} \times 6 \cdot \text{ft}) - (12 \cdot \text{ft} \times 12 \cdot \text{ft})] \times \frac{\text{lbf} \cdot \text{s}^2}{\text{slug} \cdot \text{ft}}$$

$F_T = -2.70 \times 10^4 \cdot \text{lbf}$ The negative sign indicates a net upwards force (it's actually a buoyancy effect on the three middle sections)

Problem 3.71

[Difficulty: 3]

3.71 The gate shown is 1.5 m wide and pivoted at O; $a = 1.0\ \text{m}^{-2}$, $D = 1.20$ m, and $H = 1.40$ m. Determine (a) the magnitude and moment of the vertical component of the force about O, and (b) the horizontal force that must be applied at point A to hold the gate in position.



Given: Parabolic gate, hinged at O has a constant width.

$$b = 1.5 \cdot m \quad a = 1.0 \cdot m^{-2} \quad D = 1.2 \cdot m \quad H = 1.4 \cdot m$$

Find: (a) Magnitude and moment of the vertical force on the gate due to water
(b) Horizontal force applied at A required to maintain equilibrium

Solution: We will apply the hydrostatics equations to this system.

Governing Equations:

$$\frac{dp}{dh} = \rho \cdot g \qquad \text{(Hydrostatic Pressure - h is positive downwards)}$$

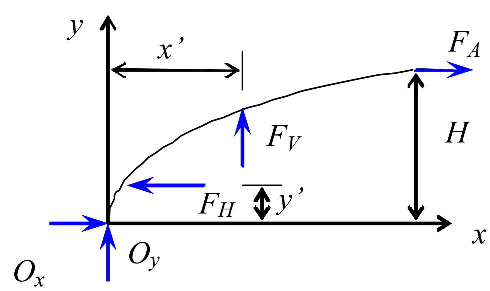
$$\Sigma M_z = 0 \qquad \text{(Rotational equilibrium)}$$

$$F_V = \int p\, dA_y \qquad \text{(Vertical Hydrostatic Force)}$$

$$x' \cdot F_V = \int x\, dF_V \qquad \text{(Moment of vertical force)}$$

$$y' \cdot F_H = \int y\, dF_H \qquad \text{(Moment of Horizontal Hydrostatic Force)}$$

Assumptions: (1) Static fluid
(2) Incompressible fluid
(3) Atmospheric pressure acts at free surface of water and on outside of gate



Integrating the hydrostatic pressure equation: $p = \rho \cdot g \cdot h$

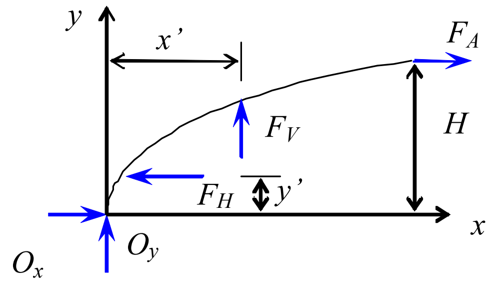
(a) The magnitude and moment of the vertical component of hydrostatic force:

$$F_V = \int p\, dA_y = \int \rho \cdot g \cdot h \cdot b\, dx \quad \text{where} \quad h = D - y \quad x = a \cdot y^3 \quad dx = 3 \cdot a \cdot y^2 \cdot dy$$

Substituting back into the relation for the force: $F_V = \int_0^D \rho \cdot g \cdot (D - y) \cdot b \cdot 3 \cdot a \cdot y^2\, dy = 3 \cdot \rho \cdot g \cdot b \cdot a \cdot \int_0^D \left(D \cdot y^2 - y^3\right) dy$

Evaluating the integral: $F_V = 3 \cdot \rho \cdot g \cdot b \cdot a \cdot \left(\frac{D^4}{3} - \frac{D^4}{4}\right) = \rho \cdot g \cdot b \cdot a \cdot \frac{D^4}{4}$

Substituting values we calculate the force:



$$F_V = 999 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times 1.5 \cdot m \times 1.0 \cdot \frac{1}{m^2} \times \frac{(1.2 \cdot m)^4}{4} \times \frac{N \cdot s^2}{kg \cdot m}$$

$$F_V = 7.62 \cdot kN$$

To find the associated moment: $x' \cdot F_V = \int x\, dF_V = \int x \cdot p\, dA_y$ Using the derivation for the force:

$$x' \cdot F_V = \int_0^D a \cdot y^3 \cdot \rho \cdot g \cdot (D - y) \cdot b \cdot 3 \cdot a \cdot y^2\, dy = 3 \cdot \rho \cdot g \cdot a^2 \cdot b \cdot \int_0^D \left(D \cdot y^5 - y^6\right) dy$$

Evaluating the integral:

$$x' \cdot F_V = 3 \cdot \rho \cdot g \cdot a^2 \cdot b \cdot \left(\frac{D^7}{6} - \frac{D^7}{7}\right) = \frac{3}{42} \cdot \rho \cdot g \cdot a^2 \cdot b \cdot D^7 = \rho \cdot g \cdot a^2 \cdot b \cdot \frac{D^7}{14}$$

Now substituting values into this equation:

$$x'F_V = 999 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times \left(\frac{1.0}{m^2}\right)^2 \times 1.5 \cdot m \times \frac{(1.20 \cdot m)^7}{14} \times \frac{N \cdot s^2}{kg \cdot m}$$

$$x'F_V = 3.76 \cdot kN \cdot m$$ (positive indicates counterclockwise)

(b) Horizontal force at A to maintain equilibrium: we take moments at O:

$$x' \cdot F_V + y' \cdot F_H - H \cdot F_A = 0$$ Solving for the force at A: $$F_A = \frac{1}{H} \cdot \left(x' \cdot F_V + y' \cdot F_H\right)$$

To get the moment of the horizontal hydrostatic force:

$$y' \cdot F_H = \int y\, dF_H = \int y \cdot p\, dA_x = \int y \cdot \rho \cdot g \cdot h \cdot b\, dy = \rho \cdot g \cdot b \cdot \int_0^D y \cdot (D - y)\, dy = \rho \cdot g \cdot b \cdot \int_0^D \left(D \cdot y - y^2\right) dy$$

Evaluating the integral: $y' \cdot F_H = \rho \cdot g \cdot b \cdot \left(\frac{D^3}{2} - \frac{D^3}{3}\right) = \rho \cdot g \cdot b \cdot \frac{D^3}{6}$ Now substituting values into this equation:

$$y'F_H = 999 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times 1.5 \cdot m \times \frac{(1.20 \cdot m)^3}{6} \times \frac{N \cdot s^2}{kg \cdot m}$$

$$y'F_H = 4.23 \cdot kN \cdot m$$ (counterclockwise)

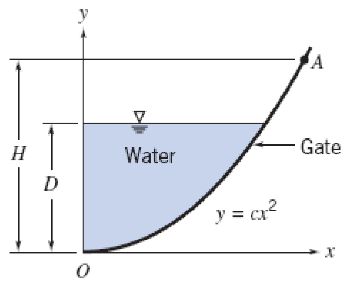
Therefore: $$F_A = \frac{1}{1.4} \cdot \frac{1}{m} \cdot (3.76 \cdot kN \cdot m + 4.23 \cdot kN \cdot m)$$

$$F_A = 5.71 \cdot kN$$

Problem 3.72

[Difficulty: 3]

3.72 The parabolic gate shown is 2 m wide and pivoted at O; $c = 0.25\ \text{m}^{-1}$, $D = 2$ m, and $H = 3$ m. Determine (a) the magnitude and line of action of the vertical force on the gate due to the water, (b) the horizontal force applied at A required to maintain the gate in equilibrium, and (c) the vertical force applied at A required to maintain the gate in equilibrium.



Given: Parabolic gate, hinged at O has a constant width.

$b = 2 \cdot m \quad c = 0.25 \cdot m^{-1} \quad D = 2 \cdot m \quad H = 3 \cdot m$

Find:
(a) Magnitude and line of action of the vertical force on the gate due to water
(b) Horizontal force applied at A required to maintain equilibrium
(c) Vertical force applied at A required to maintain equilibrium

Solution: We will apply the hydrostatics equations to this system.

Governing Equations:

$$\frac{dp}{dh} = \rho \cdot g \qquad \text{(Hydrostatic Pressure - h is positive downwards)}$$

$$\Sigma M_z = 0 \qquad \text{(Rotational equilibrium)}$$

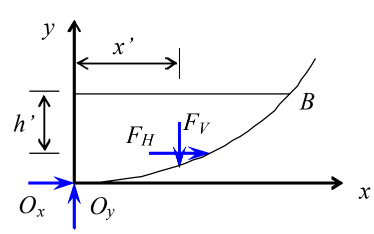
$$F_V = \int p\, dA_y \qquad \text{(Vertical Hydrostatic Force)}$$

$$x' \cdot F_V = \int x\, dF_V \qquad \text{(Location of line of action)}$$

$$F_H = p_c \cdot A \qquad \text{(Horizontal Hydrostatic Force)}$$

$$h' = h_c + \frac{I_{xx}}{A \cdot h_c} \qquad \text{(Location of line of action)}$$

Assumptions:
(1) Static fluid
(2) Incompressible fluid
(3) Atmospheric pressure acts at free surface of water and on outside of gate



Integrating the hydrostatic pressure equation: $p = \rho \cdot g \cdot h$

(a) The magnitude and line of action of the vertical component of hydrostatic force:

$$F_V = \int p\, dA_y = \int_0^{\sqrt{\frac{D}{c}}} \rho \cdot g \cdot h \cdot b\, dx = \int_0^{\sqrt{\frac{D}{c}}} \rho \cdot g \cdot (D - y) b\, dx = \int_0^{\sqrt{\frac{D}{c}}} \rho \cdot g \cdot \left(D - c \cdot x^2\right) b\, dx = \rho \cdot g \cdot b \cdot \int_0^{\sqrt{\frac{D}{c}}} \left(D - c \cdot x^2\right) dx$$

Evaluating the integral: $$F_V = \rho \cdot g \cdot b \cdot \left(\frac{D^{\frac{3}{2}}}{c^{\frac{1}{2}}} - \frac{1}{3} \cdot \frac{D^{\frac{3}{2}}}{c^{\frac{1}{2}}} \right) = \frac{2 \cdot \rho \cdot g \cdot b}{3} \cdot \frac{D^{\frac{3}{2}}}{c^{\frac{1}{2}}} \qquad (1)$$

Substituting values: $$F_V = \frac{2}{3} \times 999 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times 2 \cdot m \times (2 \cdot m)^{\frac{3}{2}} \times \left(\frac{1}{0.25} \cdot m\right)^{\frac{1}{2}} \times \frac{N \cdot s^2}{kg \cdot m} \qquad F_V = 73.9 \cdot kN$$

To find the line of action of this force: $x' \cdot F_V = \int x\, dF_V$ Therefore, $x' = \frac{1}{F_V} \cdot \int x\, dF_V = \frac{1}{F_V} \cdot \int x \cdot p\, dA_y$

Using the derivation for the force: $$x' = \frac{1}{F_V} \cdot \int_0^{\sqrt{\frac{D}{c}}} x \cdot \rho \cdot g \cdot \left(D - c \cdot x^2\right) \cdot b\, dx = \frac{\rho \cdot g \cdot b}{F_V} \cdot \int_0^{\sqrt{\frac{D}{c}}} \left(D \cdot x - c \cdot x^3\right) dx$$

Evaluating the integral: $x' = \frac{\rho \cdot g \cdot b}{F_V} \cdot \left[\frac{D}{2} \cdot \frac{D}{c} - \frac{c}{4} \cdot \left(\frac{D}{c}\right)^2\right] = \frac{\rho \cdot g \cdot b}{F_V} \frac{D^2}{4 \cdot c}$ Now substituting values into this equation:

$$x' = 999 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times 2 \cdot m \times \frac{1}{73.9 \times 10^3} \cdot \frac{1}{N} \times \frac{1}{4} \times (2 \cdot m)^2 \times \frac{1}{0.25} \cdot m \times \frac{N \cdot s^2}{kg \cdot m} \qquad x' = 1.061\, m$$

To find the required force at A for equilibrium, we need to find the horizontal force of the water on the gate and its line of action as well. Once this force is known we take moments about the hinge (point O).

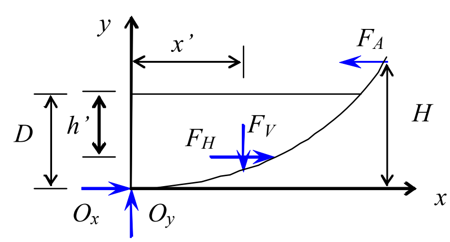
$F_H = p_c \cdot A = \rho \cdot g \cdot h_c \cdot b \cdot D = \rho \cdot g \cdot \frac{D}{2} \cdot b \cdot D = \rho \cdot g \cdot b \cdot \frac{D^2}{2}$ since $h_c = \frac{D}{2}$ Therefore the horizontal force is:

$$F_H = 999 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times 2 \cdot m \times \frac{(2 \cdot m)^2}{2} \times \frac{N \cdot s^2}{kg \cdot m} \qquad F_H = 39.2 \cdot kN$$

To calculate the line of action of this force:

$$h' = h_c + \frac{I_{xx}}{A \cdot h_c} = \frac{D}{2} + \frac{b \cdot D^3}{12} \cdot \frac{1}{b \cdot D} \cdot \frac{2}{D} = \frac{D}{2} + \frac{D}{6} = \frac{2}{3} \cdot D \qquad h' = \frac{2}{3} \cdot 2 \cdot m \qquad h' = 1.333\, m$$

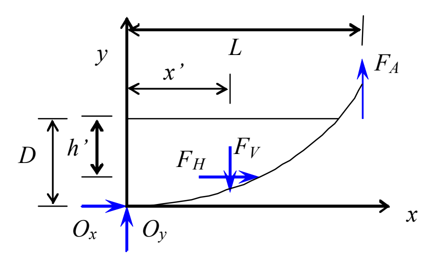
Now we have information to solve parts (b) and (c):
(b) Horizontal force applied at A for equilibrium: take moments about O:



$F_A \cdot H - F_V \cdot x' - F_H \cdot (D - h') = 0$ Solving for F_A $\quad F_A = \frac{F_V \cdot x' + F_H \cdot (D - h')}{H}$

$$F_A = \frac{1}{3} \cdot \frac{1}{m} \times [73.9 \cdot kN \times 1.061 \cdot m + 39.2 \cdot kN \times (2 \cdot m - 1.333 \cdot m)] \qquad F_A = 34.9 \cdot kN$$

(c) Vertical force applied at A for equilibrium: take moments about O:



$F_A \cdot L - F_V \cdot x' - F_H \cdot (D - h') = 0$ Solving for F_A $\quad F_A = \frac{F_V \cdot x' + F_H \cdot (D - h')}{L}$

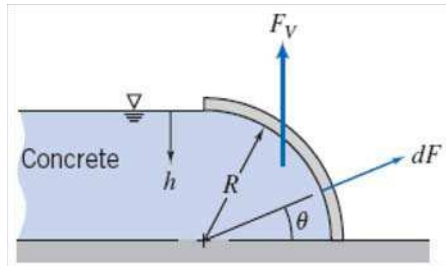
L is the value of x at y = H. Therefore: $L = \sqrt{\frac{H}{c}}$ $\quad L = \sqrt{3 \cdot m \times \frac{1}{0.25} \cdot m}$ $\quad L = 3.464\, m$

$$F_A = \frac{1}{3.464} \cdot \frac{1}{m} \times [73.9 \cdot kN \times 1.061 \cdot m + 39.2 \cdot kN \times (2 \cdot m - 1.333 \cdot m)] \qquad F_A = 30.2 \cdot kN$$

Problem 3.73

[Difficulty: 2]

3.73 Liquid concrete is poured into the form ($R = 2$ ft). The form is $w = 15$ ft wide normal to the diagram. Compute the magnitude of the vertical force exerted on the form by the concrete, and specify its line of action.



Given: Liquid concrete is poured into the form shown

$R = 2 \cdot \text{ft} \qquad w = 15 \cdot \text{ft} \qquad SG_c = 2.5$ (Table A.1, App. A)

Find: Magnitude and line of action of the vertical force on the form

Solution: We will apply the hydrostatics equations to this system.

Governing Equations:

$$\frac{dp}{dh} = \rho \cdot g \qquad \text{(Hydrostatic Pressure - h is positive downwards)}$$

$$F_V = \int p \, dA_y \qquad \text{(Vertical Hydrostatic Force)}$$

$$x' \cdot F_V = \int x \, dF_V \qquad \text{(Moment of vertical force)}$$

Assumptions:
(1) Static fluid
(2) Incompressible fluid
(3) Atmospheric pressure acts at free surface of concrete and on outside of gate

Integrating the hydrostatic pressure equation: $p = \rho \cdot g \cdot h$

$$F_V = \int p \, dA_y = \int \rho \cdot g \cdot h \cdot \sin(\theta) \, dA \quad \text{where} \quad dA = w \cdot R \cdot d\theta \quad \text{and} \quad h = R - y = R - R \cdot \sin(\theta)$$

Therefore,
$$F_V = \int_0^{\frac{\pi}{2}} \rho \cdot g \cdot (R - R \cdot \sin(\theta)) \cdot w \cdot R \cdot \sin(\theta) \, d\theta = \rho \cdot g \cdot w \cdot R^2 \cdot \int_0^{\frac{\pi}{2}} \left[\sin(\theta) - (\sin(\theta))^2\right] d\theta$$

Evaluating the integral:
$$F_V = \rho \cdot g \cdot w \cdot R^2 \cdot \left[-(0-1) - \left(\frac{\pi}{4} - 0\right) + (0-0)\right] = \rho \cdot g \cdot w \cdot R^2 \cdot \left(1 - \frac{\pi}{4}\right)$$

The density of concrete is: $\rho = 2.5 \times 1.94 \cdot \frac{\text{slug}}{\text{ft}^3} \qquad \rho = 4.85 \cdot \frac{\text{slug}}{\text{ft}^3}$

Substituting values we calculate the force:
$$F_V = 4.85 \cdot \frac{\text{slug}}{\text{ft}^3} \times 32.2 \cdot \frac{\text{ft}}{\text{s}^2} \times 15 \cdot \text{ft} \times (2 \cdot \text{ft})^2 \times \left(1 - \frac{\pi}{4}\right) \times \frac{\text{lbf} \cdot \text{s}^2}{\text{slug} \, \text{ft}} \qquad F_V = 2011 \cdot \text{lbf}$$

To find the line of action: $x' \cdot F_V = \int x \, dF_V = \int x \cdot p \, dA_y$ Using the derivation for the force:

$$x' \cdot F_V = \int R \cdot \cos(\theta) \cdot \rho \cdot g \cdot (R - R \cdot \sin(\theta)) \cdot w \cdot R \cdot \sin(\theta)\, d\theta = \rho \cdot g \cdot w \cdot R^3 \cdot \int_0^{\frac{\pi}{2}} \left[\sin(\theta) \cdot \cos(\theta) - (\sin(\theta))^2 \cdot \cos(\theta)\right] d\theta$$

Evaluating the integral: $x' \cdot F_V = \rho \cdot g \cdot w \cdot R^3 \cdot \left(\frac{1}{2} - \frac{1}{3}\right) = \rho \cdot g \cdot w \cdot \frac{R^3}{6}$ Therefore the line of action of the force is:

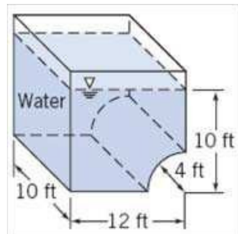
$$x' = \frac{x' \cdot F_V}{F_V} = \frac{\rho \cdot g \cdot w \cdot \frac{R^3}{6}}{\rho \cdot g \cdot w \cdot R^2 \cdot \left(1 - \frac{\pi}{4}\right)} = \frac{R}{6 \cdot \left(1 - \frac{\pi}{4}\right)}$$

Substituting values: $x' = \frac{2 \cdot ft}{6 \cdot \left(1 - \frac{\pi}{4}\right)}$ $x' = 1.553 \cdot ft$

Problem 3.74

[Difficulty: 2]

3.74 An open tank is filled with water to the depth indicated. Atmospheric pressure acts on all outer surfaces of the tank. Determine the magnitude and line of action of the vertical component of the force of the water on the curved part of the tank bottom.



Given: Open tank as shown. Width of curved surface $b = 10 \cdot ft$

Find: (a) Magnitude of the vertical force component on the curved surface
(b) Line of action of the vertical component of the force

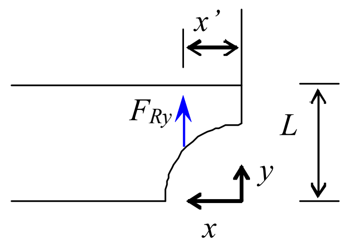
Solution: We will apply the hydrostatics equations to this system.

Governing Equations:

$$\frac{dp}{dh} = \gamma \quad \text{(Hydrostatic Pressure - h is positive downwards)}$$

$$F_V = -\int p\, dA_y \quad \text{(Vertical Hydrostatic Force)}$$

$$x' \cdot F_V = \int x\, dF_V \quad \text{(Moment of vertical force)}$$



Assumptions:
(1) Static fluid
(2) Incompressible fluid
(3) Atmospheric pressure acts at free surface of water and on outside of wall

Integrating the hydrostatic pressure equation: $p = \gamma \cdot h$ We can define along the surface $h = L - \left(R^2 - x^2\right)^{\frac{1}{2}}$

We also define the incremental area on the curved surface as: $dA_y = b \cdot dx$ Substituting these into the force equation we get:

$$F_V = -\int p\, dA_y = -\int_0^R \gamma \cdot \left[L - \left(R^2 - x^2\right)^{\frac{1}{2}}\right] \cdot b\, dx = -\gamma \cdot b \cdot \int_0^R \left(L - \sqrt{R^2 - x^2}\right) dx = -\gamma \cdot b \cdot R \cdot \left(L - R \cdot \frac{\pi}{4}\right)$$

$$F_V = -\left[62.4 \cdot \frac{lbf}{ft^3} \times 10 \cdot ft \times 4 \cdot ft \times \left(10 \cdot ft - 4 \cdot ft \times \frac{\pi}{4}\right)\right] \qquad F_V = -17.12 \times 10^3 \cdot lbf \qquad \text{(negative indicates downward)}$$

To find the line of action of the force: $x' \cdot F_V = \int x\, dF_V$ where $dF_V = -\gamma \cdot b \cdot \left(L - \sqrt{R^2 - x^2}\right) \cdot dx$

Therefore:

$$x' = \frac{x' \cdot F_V}{F_V} = \frac{1}{\gamma \cdot b \cdot R \cdot \left(L - R \cdot \frac{\pi}{4}\right)} \cdot \int_0^R x \cdot \gamma \cdot b \cdot \left(L - \sqrt{R^2 - x^2}\right) dx = \frac{1}{R \cdot \left(L - R \cdot \frac{\pi}{4}\right)} \cdot \int_0^R \left(L \cdot x - x \cdot \sqrt{R^2 - x^2}\right) dx$$

Evaluating the integral:

$$x' = \frac{4}{R \cdot (4 \cdot L - \pi \cdot R)} \cdot \left(\frac{1}{2} \cdot L \cdot R^2 - \frac{1}{3} \cdot R^3\right) = \frac{4 \cdot R^2}{R \cdot (4 \cdot L - \pi \cdot R)} \cdot \left(\frac{L}{2} - \frac{R}{3}\right) = \frac{4 \cdot R}{4 \cdot L - \pi \cdot R} \cdot \left(\frac{L}{2} - \frac{R}{3}\right)$$

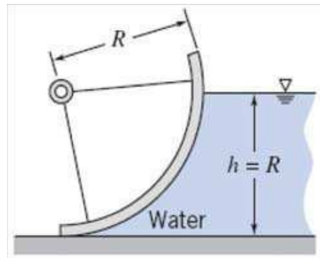
Substituting known values:

$$x' = \frac{4 \cdot 4 \cdot ft}{4 \cdot 10 \cdot ft - \pi \cdot 4 \cdot ft} \cdot \left(\frac{10 \cdot ft}{2} - \frac{4 \cdot ft}{3}\right) \qquad x' = 2.14 \cdot ft$$

Problem 3.75

[Difficulty: 2]

3.75 A spillway gate formed in the shape of a circular arc is w m wide. Find the magnitude and line of action of the vertical component of the force due to all fluids acting on the gate.



Given: Gate formed in the shape of a circular arc has width w. Liquid is water; depth h = R

Find: (a) Magnitude of the net vertical force component due to fluids acting on the gate
(b) Line of action of the vertical component of the force

Solution: We will apply the hydrostatics equations to this system.

Governing Equations:

$$\frac{dp}{dy} = \rho \cdot g$$ (Hydrostatic Pressure - y is positive downwards)

$$F_V = -\int p\, dA_y$$ (Vertical Hydrostatic Force)

$$x' \cdot F_V = \int x\, dF_V$$ (Moment of vertical force)

Assumptions:
(1) Static fluid
(2) Incompressible fluid
(3) Atmospheric pressure acts at free surface of water and on outside of gate

Integrating the hydrostatic pressure equation: $p = \rho \cdot g \cdot y$

Instead of y, we use θ as our variable of integration: $y = R \cdot \sin(\theta)$

Therefore, $dy = R \cdot \cos(\theta) \cdot d\theta$ In addition, $dA_y = w \cdot R \cdot \sin(\theta) \cdot d\theta$

Therefore, $$F_V = -\int_0^{\frac{\pi}{2}} \rho \cdot g \cdot R \cdot \sin(\theta) \cdot w \cdot R \cdot \sin(\theta)\, d\theta = -\rho \cdot g \cdot R^2 \cdot w \cdot \int_0^{\frac{\pi}{2}} (\sin(\theta))^2\, d\theta = -\rho \cdot g \cdot R^2 \cdot w \cdot \frac{\pi}{4} \qquad F_V = -\frac{\pi \cdot \rho \cdot g \cdot R^2 \cdot w}{4}$$

(negative indicates downward)

To find the line of action of the vertical component of the force: $x' \cdot F_V = \int x\, dF_V$ where $x = R \cdot \cos(\theta)$ and the elemental force is

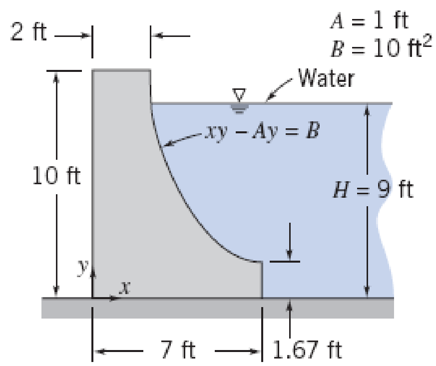
$dF_V = -\rho \cdot g \cdot R^2 \cdot w \cdot (\sin(\theta))^2 \cdot d\theta$ Substituting into the above integral yields:

$$x' = \frac{x' \cdot F_V}{F_V} = -\frac{4}{\pi \cdot \rho \cdot g \cdot R^2 \cdot w} \cdot \int_0^{\frac{\pi}{2}} -(R \cdot \cos(\theta)) \cdot \left[\rho \cdot g \cdot R^2 \cdot w \cdot (\sin(\theta))^2\right] d\theta = \frac{4 \cdot R}{\pi} \cdot \int_0^{\frac{\pi}{2}} (\sin(\theta))^2 \cdot \cos(\theta)\, d\theta = \frac{4 \cdot R}{\pi} \cdot \frac{1}{3} \qquad x' = \frac{4 \cdot R}{3 \cdot \pi}$$

Problem 3.76

[Difficulty: 3]

3.76 A dam is to be constructed using the cross-section shown. Assume the dam width is $w = 160$ ft. For water height $H = 9$ ft, calculate the magnitude and line of action of the vertical force of water on the dam face. Is it possible for water forces to overturn this dam? Under what circumstances will this happen?



Given: Dam with cross-section shown. Width of dam

$b = 160 \cdot ft$

Find: (a) Magnitude and line of action of the vertical force component on the dam
(b) If it is possible for the water to overturn dam

Solution: We will apply the hydrostatics equations to this system.

Governing Equations:

$$\frac{dp}{dh} = \rho \cdot g$$ (Hydrostatic Pressure - h is positive downwards from free surface)

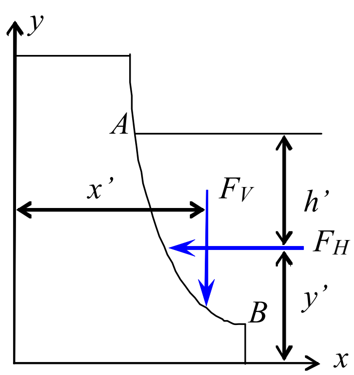
$$F_V = \int p\, dA_y$$ (Vertical Hydrostatic Force)

$$F_H = p_c \cdot A$$ (Horizontal Hydrostatic Force)

$$x' \cdot F_V = \int x\, dF_V$$ (Moment of vertical force)

$$h' = h_c + \frac{I_{xx}}{h_c \cdot A}$$ (Line of action of vertical force)

$$\Sigma M_z = 0$$ (Rotational Equilibrium)



Assumptions: (1) Static fluid
(2) Incompressible fluid
(3) Atmospheric pressure acts at free surface of water and on outside of dam

Integrating the hydrostatic pressure equation: $p = \rho \cdot g \cdot h$

Into the vertical force equation: $F_V = \int p\, dA_y = \int_{x_A}^{x_B} \rho \cdot g \cdot h \cdot b\, dx = \rho \cdot g \cdot b \cdot \int_{x_A}^{x_B} (H - y)\, dx$

From the definition of the dam contour: $x \cdot y - A \cdot y = B$ Therefore: $y = \frac{B}{x - A}$ and $x_A = \frac{10 \cdot ft^2}{9 \cdot ft} + 1 \cdot ft$ $\quad x_A = 2.11 \cdot ft$

Into the force equation: $F_V = \rho \cdot g \cdot b \cdot \int_{x_A}^{x_B} \left(H - \frac{B}{x - A}\right) dx = \rho \cdot g \cdot b \cdot \left[H \cdot \left(x_B - x_A\right) - B \cdot \ln\left(\frac{x_B - A}{x_A - A}\right)\right]$ Substituting known values:

$$F_V = 1.94 \cdot \frac{slug}{ft^3} \times 32.2 \cdot \frac{ft}{s^2} \times 160 \cdot ft \times \left[9 \cdot ft \times (7.0 \cdot ft - 2.11 \cdot ft) - 10 \cdot ft^2 \times \ln\left(\frac{7.0 - 1}{2.11 - 1}\right)\right] \cdot \frac{lbf \cdot s^2}{slug \cdot ft} \qquad F_V = 2.71 \times 10^5 \cdot lbf$$

To find the line of action of the force: $x' \cdot F_V = \int x \, dF_V$ where $dF_V = \rho \cdot g \cdot b \cdot \left(H - \frac{B}{x - A}\right) \cdot dx$ Therefore:

$$x' = \frac{x' \cdot F_V}{F_V} = \frac{1}{F_V} \cdot \int_{x_A}^{x_B} x \cdot \rho \cdot g \cdot b \cdot \left(H - \frac{B}{x - A}\right) dx = \frac{1}{H \cdot \left(x_B - x_A\right) - B \cdot \ln\left(\frac{x_B - A}{x_A - A}\right)} \cdot \int_{x_A}^{x_B} \left(H \cdot x - \frac{B \cdot x}{x - A}\right) dx$$

Evaluating the integral: $x' = \dfrac{\frac{H}{2} \cdot \left(x_B^{\,2} - x_A^{\,2}\right) - B \cdot \left(x_B - x_A\right) - B \cdot A \cdot \ln\left(\frac{x_B - A}{x_A - A}\right)}{H \cdot \left(x_B - x_A\right) - B \cdot \ln\left(\frac{x_B - A}{x_A - A}\right)}$ Substituting known values we get:

$$x' = \frac{\frac{9 \cdot ft}{2} \times \left(7^2 - 2.11^2\right) \cdot ft^2 - 10 \cdot ft^2 \times (7 - 2.11) \cdot ft - 10 \cdot ft^2 \times 1 \cdot ft \times \ln\left(\frac{7 - 1}{2.11 - 1}\right)}{9 \cdot ft \times (7 - 2.11) \cdot ft - 10 \cdot ft^2 \times \ln\left(\frac{7 - 1}{2.11 - 1}\right)} \qquad x' = 4.96 \cdot ft$$

To determine whether or not the water can overturn the dam, we need the horizontal force and its line of action:

$$F_H = p_c \cdot A = \rho \cdot g \cdot \frac{H}{2} \cdot H \cdot b = \frac{\rho \cdot g \cdot b \cdot H^2}{2}$$

Substituting values: $F_H = \frac{1}{2} \times 1.94 \cdot \frac{slug}{ft^3} \times 32.2 \cdot \frac{ft}{s^2} \times 160 \cdot ft \times (9 \cdot ft)^2 \times \frac{lbf \cdot s^2}{slug \cdot ft}$ $\qquad F_H = 4.05 \times 10^5 \cdot lbf$

For the line of action: $h' = h_c + \frac{I_{xx}}{h_c \cdot A}$ where $h_c = \frac{H}{2}$ $\quad A = H \cdot b \quad I_{xx} = \frac{b \cdot H^3}{12}$

Therefore: $h' = \frac{H}{2} + \frac{b \cdot H^3}{12} \cdot \frac{2}{H} \cdot \frac{1}{b \cdot H} = \frac{H}{2} + \frac{H}{6} = \frac{2}{3} \cdot H \qquad h' = \frac{2}{3} \cdot 9 \cdot ft \qquad h' = 6.00 \cdot ft$

Taking moments of the hydrostatic forces about the origin:

$M_w = F_H \cdot (H - h') - F_V \cdot x' \qquad M_w = 4.05 \times 10^5 \cdot lbf \times (9 - 6) \cdot ft - 2.71 \times 10^5 \cdot lbf \times 4.96 \cdot ft \qquad M_w = -1.292 \times 10^5 \cdot lbf \cdot ft$

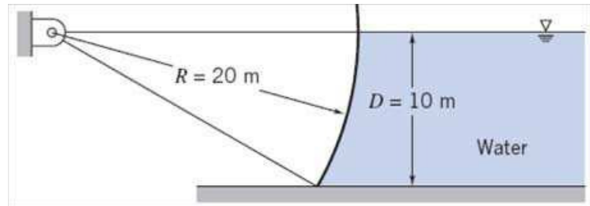
The negative sign indicates that this is a clockwise moment about the origin. Since the weight of the dam will also contribute a clockwise moment about the origin, these two moments should not cause the dam to tip to the left.

Therefore, the water can not overturn the dam.

Problem 3.77

[Difficulty: 3]

3.77 A Tainter gate used to control water flow from the Uniontown Dam on the Ohio River is shown; the gate width is $w = 35$ m. Determine the magnitude, direction, and line of action of the force from the water acting on the gate.



Given:
Tainter gate as shown $\quad w = 35 \cdot m$

Find:
Force of the water acting on the gate

Solution:
We will apply the hydrostatics equations to this system.

Governing Equations:

$$\frac{dp}{dh} = \rho \cdot g$$ (Hydrostatic Pressure - h is positive downwards from free surface)

$$dF = p \cdot dA$$ (Hydrostatic Force)

Assumptions:
(1) Static fluid
(2) Incompressible fluid
(3) Atmospheric pressure acts at free surface of water and on outside of gate

Integrating the hydrostatic pressure equation:

$$p = \rho \cdot g \cdot h = \rho \cdot g \cdot R \cdot \sin(\theta)$$

Resolving the hydrostatic force into horizontal and vertical components:

$$dF_H = dF \cdot \cos(\theta) = p \cdot dA \cdot \cos(\theta) = \rho \cdot g \cdot R \cdot \sin(\theta) \cdot w \cdot R \cdot d\theta \cdot \cos(\theta) \quad \text{since} \quad dA = w \cdot R \cdot d\theta$$

Integrating this expression: $F_H = \int_0^{\theta_1} \rho \cdot g \cdot R^2 \cdot w \cdot \sin(\theta) \cdot \cos(\theta)\, d\theta$ where $\theta_1 = \operatorname{asin}\left(\frac{10 \cdot m}{20 \cdot m}\right) = 30 \cdot deg$

$$F_H = \rho \cdot g \cdot R^2 \cdot w \cdot \int_0^{30 \cdot deg} \sin(\theta) \cdot \cos(\theta)\, d\theta = \rho \cdot g \cdot R^2 \cdot w \cdot \frac{(\sin(30 \cdot deg))^2}{2} = \frac{\rho \cdot g \cdot R^2 \cdot w}{8}$$ Substituting known values:

$$F_H = \frac{1}{8} \times 999 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times (20 \cdot m)^2 \times 35 \cdot m \times \frac{N \cdot s^2}{kg \cdot m}$$

$$F_H = 1.715 \times 10^7 \cdot N$$

Similarly, we can calculate the vertical component of the hydrostatic force: $dF_V = dF \cdot \sin(\theta) = p \cdot dA \cdot \sin(\theta) = \rho \cdot g \cdot R^2 \cdot w \cdot (\sin(\theta))^2 \cdot d\theta$

$$F_V = \rho \cdot g \cdot R^2 \cdot w \cdot \int_0^{30 \cdot deg} (\sin(\theta))^2\, d\theta = \rho \cdot g \cdot R^2 \cdot w \cdot \left(\frac{\pi}{12} - \frac{\sqrt{3}}{8}\right)$$ Substituting known values:

$$F_V = \left(\frac{\pi}{12} - \frac{\sqrt{3}}{8}\right) \times 999 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times (20 \cdot m)^2 \times 35 \cdot m \times \frac{N \cdot s^2}{kg \cdot m}$$

$$F_V = 6.21 \times 10^6 \cdot N$$

Now since the gate surface in contact with the water is a circular arc, all elements dF of the force, and hence the line of action of the resulta must pass through the pivot. Thus:

Magnitude of the resultant force:

$$F_R = \sqrt{F_H^2 + F_V^2} \qquad F_R = \sqrt{\left(1.715 \times 10^7 \cdot N\right)^2 + \left(6.21 \times 10^6 \cdot N\right)^2} \qquad F_R = 1.824 \times 10^7 N$$

The line of action of the force:

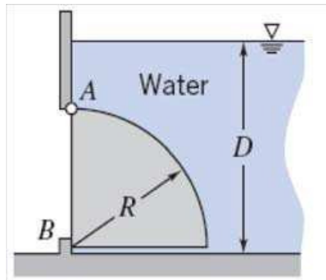
$$\alpha = \operatorname{atan}\left(\frac{F_V}{F_H}\right) \qquad \alpha = \operatorname{atan}\left(\frac{6.21 \times 10^6 \cdot N}{1.715 \times 10^7 \cdot N}\right) \qquad \alpha = 19.9 \cdot deg$$

The force passes through the pivot at an angle α to the horizontal.

Problem 3.78

[Difficulty: 4]

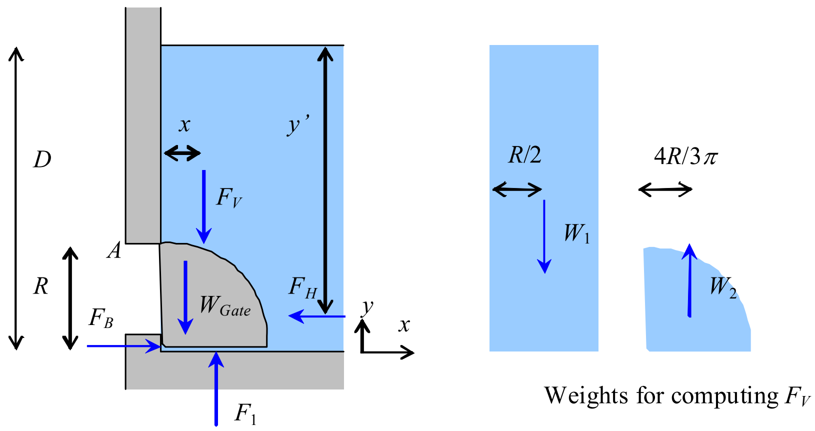
3.78 A gate, in the shape of a quarter-cylinder, hinged at A and sealed at B, is 3 m wide. The bottom of the gate is 4.5 m below the water surface. Determine the force on the stop at B if the gate is made of concrete; $R = 3$ m.



Given: Gate geometry

Find: Force on stop B

Solution:



Weights for computing F_V

Basic equations
$$\frac{dp}{dh} = \rho\cdot g$$
$$\Sigma M_A = 0$$

Assumptions: static fluid; ρ = constant; p_{atm} on other side

For incompressible fluid $\quad p = \rho\cdot g\cdot h \quad$ where p is gage pressure and h is measured downwards

We need to compute force (including location) due to water on curved surface and underneath. For curved surface we could integrate pressure, but here we use the concepts that F_V (see sketch) is equivalent to the weight of fluid above, and F_H is equivalent to the force on a vertical flat plate. Note that the sketch only shows forces that will be used to compute the moment at A

For F_V $\quad F_V = W_1 - W_2$

with
$$W_1 = \rho\cdot g\cdot w\cdot D\cdot R = 1000\cdot\frac{kg}{m^3}\times 9.81\cdot\frac{m}{s^2}\times 3\cdot m\times 4.5\cdot m\times 3\cdot m\times\frac{N\cdot s^2}{kg\cdot m} \qquad W_1 = 397\cdot kN$$

$$W_2 = \rho\cdot g\cdot w\cdot\frac{\pi\cdot R^2}{4} = 1000\cdot\frac{kg}{m^3}\times 9.81\cdot\frac{m}{s^2}\times 3\cdot m\times\frac{\pi}{4}\times(3\cdot m)^2\times\frac{N\cdot s^2}{kg\cdot m} \qquad W_2 = 208\cdot kN$$

$$F_V = W_1 - W_2 \qquad F_V = 189\cdot kN$$

with x given by
$$F_V\cdot x = W_1\cdot\frac{R}{2} - W_2\cdot\frac{4\cdot R}{3\cdot\pi} \qquad \text{or} \qquad x = \frac{W_1}{F_V}\cdot\frac{R}{2} - \frac{W_2}{F_V}\cdot\frac{4\cdot R}{3\cdot\pi}$$

$$x = \frac{397}{189}\times\frac{3\cdot m}{2} - \frac{208}{189}\times\frac{4}{3\cdot\pi}\times 3\cdot m \qquad x = 1.75\,m$$

For F_H $\quad$ Computing equations $\quad F_H = p_c\cdot A \qquad y' = y_c + \frac{I_{xx}}{A\cdot y_c}$

Hence $$F_H = p_c \cdot A = \rho \cdot g \cdot \left(D - \frac{R}{2}\right) \cdot w \cdot R$$

$$F_H = 1000 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times \left(4.5 \cdot m - \frac{3 \cdot m}{2}\right) \times 3 \cdot m \times 3 \cdot m \times \frac{N \cdot s^2}{kg \cdot m} \qquad F_H = 265 \cdot kN$$

The location of this force is

$$y' = y_c + \frac{I_{xx}}{A \cdot y_c} = \left(D - \frac{R}{2}\right) + \frac{w \cdot R^3}{12} \times \frac{1}{w \cdot R \cdot \left(D - \frac{R}{2}\right)} = D - \frac{R}{2} + \frac{R^2}{12 \cdot \left(D - \frac{R}{2}\right)}$$

$$y' = 4.5 \cdot m - \frac{3 \cdot m}{2} + \frac{(3 \cdot m)^2}{12 \times \left(4.5 \cdot m - \frac{3 \cdot m}{2}\right)} \qquad y' = 3.25\,m$$

The force F_1 on the bottom of the gate is $F_1 = p \cdot A = \rho \cdot g \cdot D \cdot w \cdot R$

$$F_1 = 1000 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times 4.5 \cdot m \times 3 \cdot m \times 3 \cdot m \times \frac{N \cdot s^2}{kg \cdot m} \qquad F_1 = 397 \cdot kN$$

For the concrete gate (SG = 2.4 from Table A.2)

$$W_{Gate} = SG \cdot \rho \cdot g \cdot w \cdot \frac{\pi \cdot R^2}{4} = 2.4 \cdot 1000 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times 3 \cdot m \times \frac{\pi}{4} \times (3 \cdot m)^2 \times \frac{N \cdot s^2}{kg \cdot m} \qquad W_{Gate} = 499 \cdot kN$$

Hence, taking moments about A $$F_B \cdot R + F_1 \cdot \frac{R}{2} - W_{Gate} \cdot \frac{4 \cdot R}{3 \cdot \pi} - F_V \cdot x - F_H \cdot [y' - (D - R)] = 0$$

$$F_B = \frac{4}{3 \cdot \pi} \cdot W_{Gate} + \frac{x}{R} \cdot F_V + \frac{[y' - (D - R)]}{R} \cdot F_H - \frac{1}{2} \cdot F_1$$

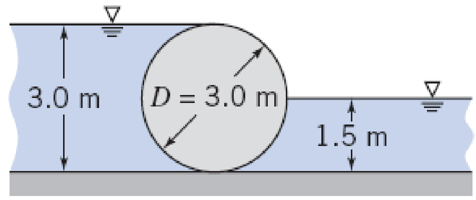
$$F_B = \frac{4}{3 \cdot \pi} \times 499 \cdot kN + \frac{1.75}{3} \times 189 \cdot kN + \frac{[3.25 - (4.5 - 3)]}{3} \times 265 \cdot kN - \frac{1}{2} \times 397 \cdot kN$$

$$F_B = 278 \cdot kN$$

Problem 3.79

[Difficulty: 4]

3.79 Consider the cylindrical weir of diameter 3 m and length 6 m. If the fluid on the left has a specific gravity of 1.6, and on the right has a specific gravity of 0.8, find the magnitude and direction of the resultant force.



Given:
Sphere with different fluids on each side

Find:
Resultant force and direction

Solution:

The horizontal and vertical forces due to each fluid are treated separately. For each, the horizontal force is equivalent to that on a vertical flat plate; the vertical force is equivalent to the weight of fluid "above".

For horizontal forces, the computing equation of Section 3-5 is $F_H = p_c \cdot A$ where A is the area of the equivalent vertical plate.

For vertical forces, the computing equation of Section 3-5 is $F_V = \rho \cdot g \cdot V$ where V is the volume of fluid above the curved surface.

The data is For water $\rho = 999 \cdot \frac{kg}{m^3}$

For the fluids $SG_1 = 1.6$ $SG_2 = 0.8$

For the weir $D = 3 \cdot m$ $L = 6 \cdot m$

(a) Horizontal Forces

For fluid 1 (on the left) $F_{H1} = p_c \cdot A = \left(\rho_1 \cdot g \cdot \frac{D}{2}\right) \cdot D \cdot L = \frac{1}{2} \cdot SG_1 \cdot \rho \cdot g \cdot D^2 \cdot L$

$$F_{H1} = \frac{1}{2} \cdot 1.6 \cdot 999 \cdot \frac{kg}{m^3} \cdot 9.81 \cdot \frac{m}{s^2} \cdot (3 \cdot m)^2 \cdot 6 \cdot m \cdot \frac{N \cdot s^2}{kg \cdot m} \qquad F_{H1} = 423 \cdot kN$$

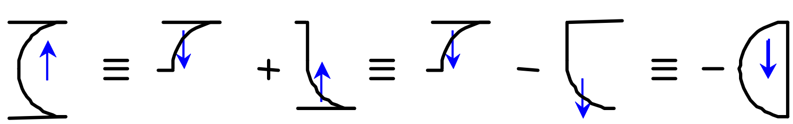
For fluid 2 (on the right) $F_{H2} = p_c \cdot A = \left(\rho_2 \cdot g \cdot \frac{D}{4}\right) \cdot \frac{D}{2} \cdot L = \frac{1}{8} \cdot SG_2 \cdot \rho \cdot g \cdot D^2 \cdot L$

$$F_{H2} = \frac{1}{8} \cdot 0.8 \cdot 999 \cdot \frac{kg}{m^3} \cdot 9.81 \cdot \frac{m}{s^2} \cdot (3 \cdot m)^2 \cdot 6 \cdot m \cdot \frac{N \cdot s^2}{kg \cdot m} \qquad F_{H2} = 52.9 \cdot kN$$

The resultant horizontal force is $F_H = F_{H1} - F_{H2}$ $\qquad F_H = 370 \cdot kN$

(b) Vertical forces

For the left geometry, a "thought experiment" is needed to obtain surfaces with fluid "above"



Hence $$F_{V1} = SG_1 \cdot \rho \cdot g \cdot \frac{\frac{\pi \cdot D^2}{4}}{2} \cdot L$$

$$F_{V1} = 1.6 \times 999 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times \frac{\pi \cdot (3 \cdot m)^2}{8} \times 6 \cdot m \times \frac{N \cdot s^2}{kg \cdot m}$$ $$F_{V1} = 333 \cdot kN$$

(Note: Use of buoyancy leads to the same result!)

For the right side, using a similar logic

$$F_{V2} = SG_2 \cdot \rho \cdot g \cdot \frac{\frac{\pi \cdot D^2}{4}}{4} \cdot L$$

$$F_{V2} = 0.8 \times 999 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times \frac{\pi \cdot (3 \cdot m)^2}{16} \times 6 \cdot m \times \frac{N \cdot s^2}{kg \cdot m}$$ $$F_{V2} = 83.1 \cdot kN$$

The resultant vertical force is $$F_V = F_{V1} + F_{V2}$$ $$F_V = 416 \cdot kN$$

Finally the resultant force and direction can be computed

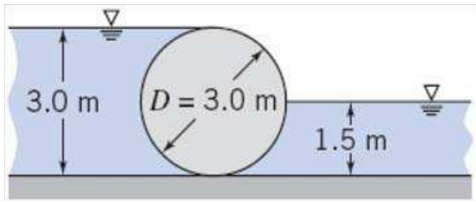
$$F = \sqrt{{F_H}^2 + {F_V}^2}$$ $$F = 557 \cdot kN$$

$$\alpha = \operatorname{atan}\left(\frac{F_V}{F_H}\right)$$ $$\alpha = 48.3 \cdot deg$$

Problem 3.80

[Difficulty: 3]

3.80 A cylindrical weir has a diameter of 3 m and a length of 6 m. Find the magnitude and direction of the resultant force acting on the weir from the water.



Given: Cylindrical weir as shown; liquid is water

Find: Magnitude and direction of the resultant force of the water on the weir

Solution: We will apply the hydrostatics equations to this system.

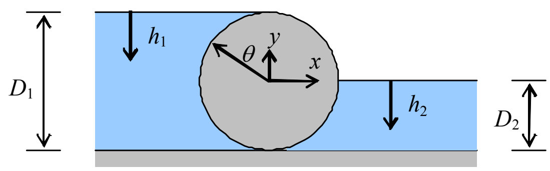
Governing Equations:

$$\frac{dp}{dh} = \rho \cdot g \qquad \text{(Hydrostatic Pressure - h is positive downwards from free surface)}$$

$$\overrightarrow{dF_R} = -p \cdot \overrightarrow{dA} \qquad \text{(Hydrostatic Force)}$$

Assumptions:
(1) Static fluid
(2) Incompressible fluid
(3) Atmospheric pressure acts on free surfaces and on the first quadrant of the cylinder

Using the coordinate system shown in the diagram at the right:



$$F_{Rx} = \overrightarrow{F_R} \cdot \hat{i} = -\int p\, \overrightarrow{dA} \cdot \hat{i} = -\int p \cdot \cos(\theta + 90 \cdot \deg)\, dA = \int p \cdot \sin(\theta)\, dA$$

$$F_{Ry} = \overrightarrow{F_R} \cdot \hat{j} = -\int p\, \overrightarrow{dA} \cdot \hat{j} = -\int p \cdot \cos(\theta)\, dA$$ Now since $dA = L \cdot R \cdot d\theta$ it follows that

$$F_{Rx} = \int_0^{\frac{3 \cdot \pi}{2}} p \cdot L \cdot R \cdot \sin(\theta)\, d\theta \quad \text{and} \quad F_{Ry} = -\int_0^{\frac{3 \cdot \pi}{2}} p \cdot L \cdot R \cdot \cos(\theta)\, d\theta$$

Next, we integrate the hydrostatic pressure equation: $p = \rho \cdot g \cdot h$ Now over the range $0 \le \theta \le \pi$ $\quad h_1 = R(1 - \cos(\theta))$

Over the range $\pi \le \theta \le \frac{3 \cdot \pi}{2}$ $\quad h_2 = -R \cdot \cos(\theta)$

Therefore we can express the pressure in terms of θ and substitute into the force equations:

$$F_{Rx} = \int_0^{\frac{3 \cdot \pi}{2}} p \cdot L \cdot R \cdot \sin(\theta)\, d\theta = \int_0^{\pi} \rho \cdot g \cdot R \cdot (1 - \cos(\theta)) \cdot L \cdot R \cdot \sin(\theta)\, d\theta - \int_{\pi}^{\frac{3 \cdot \pi}{2}} \rho \cdot g \cdot R \cdot \cos(\theta) \cdot L \cdot R \cdot \sin(\theta)\, d\theta$$

$$F_{Rx} = \rho \cdot g \cdot R^2 \cdot L \cdot \int_0^{\pi} (1 - \cos(\theta)) \cdot \sin(\theta)\, d\theta - \rho \cdot g \cdot R^2 \cdot L \cdot \int_{\pi}^{\frac{3 \cdot \pi}{2}} \cos(\theta) \cdot \sin(\theta)\, d\theta$$

$$F_{Rx} = \rho \cdot g \cdot R^2 \cdot L \cdot \left[\int_0^{\pi} (1 - \cos(\theta)) \cdot \sin(\theta)\, d\theta - \int_{\pi}^{\frac{3\cdot\pi}{2}} \cos(\theta) \cdot \sin(\theta)\, d\theta \right] = \rho \cdot g \cdot R^2 \cdot L \cdot \left(2 - \frac{1}{2} \right) = \frac{3}{2} \cdot \rho \cdot g \cdot R^2 \cdot L$$

Substituting known values: $F_{Rx} = \frac{3}{2} \times 999 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times (1.5 \cdot m)^2 \times 6 \cdot m \times \frac{N \cdot s^2}{kg \cdot m}$ $F_{Rx} = 198.5 \cdot kN$

Similarly we can calculate the vertical force component:

$$F_{Ry} = -\int_0^{\frac{3\cdot\pi}{2}} p \cdot L \cdot R \cdot \cos(\theta)\, d\theta = -\left[\int_0^{\pi} \rho \cdot g \cdot R \cdot (1 - \cos(\theta)) \cdot L \cdot R \cdot \cos(\theta)\, d\theta - \int_{\pi}^{\frac{3\cdot\pi}{2}} \rho \cdot g \cdot R \cdot \cos(\theta) \cdot L \cdot R \cdot \cos(\theta)\, d\theta \right]$$

$$F_{Ry} = -\rho \cdot g \cdot R^2 \cdot L \cdot \left[\int_0^{\pi} (1 - \cos(\theta)) \cdot \cos(\theta)\, d\theta - \int_{\pi}^{\frac{3\cdot\pi}{2}} (\cos(\theta))^2\, d\theta \right] = \rho \cdot g \cdot R^2 \cdot L \cdot \left(\frac{\pi}{2} + \frac{3\cdot\pi}{4} - \frac{\pi}{2} \right) = \frac{3\cdot\pi}{4} \cdot \rho \cdot g \cdot R^2 \cdot L$$

Substituting known values: $F_{Ry} = \frac{3\cdot\pi}{4} \times 999 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times (1.5 \cdot m)^2 \times 6 \cdot m \times \frac{N \cdot s^2}{kg \cdot m}$ $F_{Ry} = 312 \cdot kN$

Now since the weir surface in contact with the water is a circular arc, all elements dF of the force, and hence the line of action of the resultant force, must pass through the pivot. Thus:

Magnitude of the resultant force: $F_R = \sqrt{(198.5 \cdot kN)^2 + (312 \cdot kN)^2}$ $F_R = 370 \cdot kN$

The line of action of the force: $\alpha = \operatorname{atan}\left(\frac{312 \cdot kN}{198.5 \cdot kN} \right)$ $\alpha = 57.5 \cdot deg$

Problem 3.81

[Difficulty: 3]

3.81 A cylindrical log of diameter D rests against the top of a dam. The water is level with the top of the log and the center of the log is level with the top of the dam. Obtain expressions for (a) the mass of the log per unit length and (b) the contact force per unit length between the log and dam.

Given: Cylindrical log floating against dam

Find: (a) Mass per unit length of the log (b) Contact force per unit length between log and dam

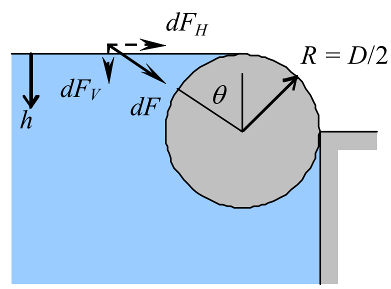
Solution: We will apply the hydrostatics equations to this system.

Governing Equations:

$$\frac{dp}{dh} = \rho \cdot g$$ (Hydrostatic Pressure - h is positive downwards from free surface)

$$\vec{dF} = p \cdot \vec{dA}$$ (Hydrostatic Force)

Assumptions:
(1) Static fluid
(2) Incompressible fluid
(3) Atmospheric pressure acts on free surfaces and on the first quadrant of the log



Integrating the hydrostatic pressure equation: $p = \rho \cdot g \cdot h = \rho \cdot g \cdot R \cdot (1 - \cos(\theta))$

Resolving the incremental force into horizontal and vertical components:

$$dF = p \cdot dA = p \cdot w \cdot R \cdot d\theta = \rho \cdot g \cdot R \cdot (1 - \cos(\theta)) \cdot w \cdot R \cdot d\theta = \rho \cdot g \cdot R^2 \cdot w \cdot (1 - \cos(\theta))$$

$$dF_H = dF \cdot \sin(\theta) = \rho \cdot g \cdot R^2 \cdot w \cdot (1 - \cos(\theta)) \cdot d\theta \cdot \sin(\theta) \qquad dF_V = dF \cdot \cos(\theta) = \rho \cdot g \cdot R^2 \cdot w \cdot (1 - \cos(\theta)) \cdot d\theta \cdot \cos(\theta)$$

Integrating the expression for the horizontal force will provide us with the contact force per unit length:

$$F_H = \int_0^{\frac{3\cdot\pi}{2}} \rho \cdot g \cdot R^2 \cdot w \cdot (1 - \cos(\theta)) \cdot \sin(\theta)\, d\theta = \rho \cdot g \cdot R^2 \cdot w \cdot \int_0^{\frac{3\cdot\pi}{2}} (\sin(\theta) - \sin(\theta) \cdot \cos(\theta))\, d\theta = \rho \cdot g \cdot R^2 \cdot w \cdot \left(-\frac{1}{2} + 1\right) = \frac{\rho \cdot g \cdot R^2 \cdot w}{2}$$

Therefore: $$\frac{F_H}{w} = \frac{\rho \cdot g \cdot R^2}{2}$$

Integrating the expression for the vertical force will provide us with the mass per unit length of the log:

$$F_V = \int_0^{\frac{3\cdot\pi}{2}} \rho \cdot g \cdot R^2 \cdot w \cdot (1 - \cos(\theta)) \cdot \cos(\theta)\, d\theta = \rho \cdot g \cdot R^2 \cdot w \cdot \int_0^{\frac{3\cdot\pi}{2}} (1 - \cos(\theta)) \cdot \cos(\theta)\, d\theta = \rho \cdot g \cdot R^2 \cdot w \cdot \left(-1 - \frac{3\cdot\pi}{4}\right)$$

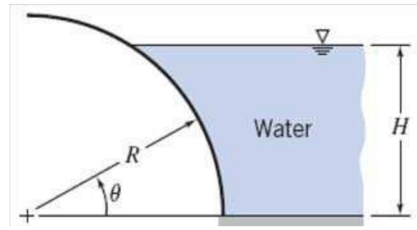
Therefore: $\frac{F_V}{w} = -\rho \cdot g \cdot R^2 \cdot \left(1 + \frac{3\cdot\pi}{4}\right)$ From a free-body diagram for the log: $\Sigma F_y = 0$ $\quad -\frac{m}{w} \cdot g - \frac{F_V}{w} = 0 \quad \frac{m}{w} = -\frac{F_V}{w \cdot g}$

Solving for the mass of the log: $$\frac{m}{w} = \rho \cdot R^2 \cdot \left(1 + \frac{3\cdot\pi}{4}\right)$$

Problem 3.82

[Difficulty: 3]

3.82 A curved surface is formed as a quarter of a circular cylinder with $R = 0.750$ m as shown. The surface is $w = 3.55$ m wide. Water stands to the right of the curved surface to depth $H = 0.650$ m. Calculate the vertical hydrostatic force on the curved surface. Evaluate the line of action of this force. Find the magnitude and line of action of the horizontal force on the surface.



Given: Curved surface, in shape of quarter cylinder, with given radius R and width w; water stands to depth H.

$R = 0.750 \cdot m \quad w = 3.55 \cdot m \quad H = 0.650 \cdot m$

Find: Magnitude and line of action of (a) vertical force and (b) horizontal force on the curved surface

Solution: We will apply the hydrostatics equations to this system.

Governing Equations:

$$\frac{dp}{dh} = \rho \cdot g$$ (Hydrostatic Pressure - h is positive downwards from free surface)

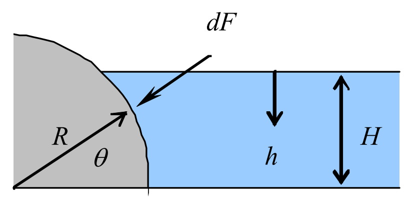
$$F_V = \int p\, dA_y$$ (Vertical Hydrostatic Force)

$$F_H = p_c \cdot A$$ (Horizontal Hydrostatic Force)

$$x' \cdot F_V = \int x\, dF_V$$ (Moment of vertical force)

$$h' = h_c + \frac{I_{xx}}{h_c \cdot A}$$ (Line of action of horizontal force)

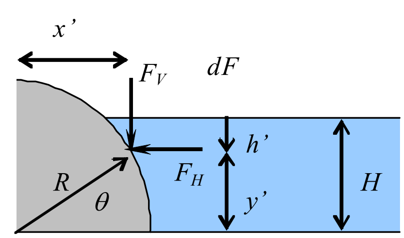
Assumptions:
(1) Static fluid
(2) Incompressible fluid
(3) Atmospheric pressure acts on free surface of the water and on the left side of the curved surface



Integrating the hydrostatic pressure equation: $p = \rho \cdot g \cdot h$

From the geometry: $h = H - R \cdot \sin(\theta) \qquad y = R \cdot \sin(\theta) \qquad x = R \cdot \cos(\theta) \qquad dA = w \cdot R \cdot d\theta$

$$\theta_1 = \operatorname{asin}\left(\frac{H}{R}\right) \qquad \theta_1 = \operatorname{asin}\left(\frac{0.650}{0.750}\right) \qquad \theta_1 = 1.048 \cdot rad$$



Therefore the vertical component of the hydrostatic force is:

$$F_V = \int p\, dA_y = \int \rho \cdot g \cdot h \cdot \sin(\theta)\, dA = \int_0^{\theta_1} \rho \cdot g \cdot (H - R \cdot \sin(\theta)) \cdot \sin(\theta) \cdot w \cdot R\, d\theta$$

$$F_V = \rho \cdot g \cdot w \cdot R \cdot \int_0^{\theta_1} \left[H \cdot \sin(\theta) - R \cdot (\sin(\theta))^2\right] d\theta = \rho \cdot g \cdot w \cdot R \cdot \left[H \cdot \left(1 - \cos\left(\theta_1\right)\right) - R \cdot \left(\frac{\theta_1}{2} - \frac{\sin\left(2 \cdot \theta_1\right)}{4}\right)\right]$$

$$F_V = 999\cdot\frac{kg}{m^3}\times 9.81\cdot\frac{m}{s^2}\times 3.55\cdot m\times 0.750\cdot m\times\left[0.650\cdot m\times(1-\cos(1.048\cdot rad)) - 0.750\cdot m\times\left(\frac{1.048}{2}-\frac{\sin(2\times 1.048\cdot rad)}{4}\right)\right]\times\frac{N\cdot s^2}{kg\cdot m}$$

$$F_V = 2.47\cdot kN$$

To calculate the line of action of this force:

$$x'\cdot F_V = \int R\cdot\cos(\theta)\cdot\rho\cdot g\cdot h\cdot\sin(\theta)\,dA = \rho\cdot g\cdot w\cdot R^2\cdot\int_0^{\theta_1}\left[H\cdot\sin(\theta)\cdot\cos(\theta) - R\cdot(\sin(\theta))^2\cdot\cos(\theta)\right]d\theta$$

Evaluating the integral: $x'\cdot F_V = \rho\cdot g\cdot w\cdot R^2\cdot\left[\frac{H}{2}\cdot\left(\sin\left(\theta_1\right)\right)^2 - \frac{R}{3}\cdot\left(\sin\left(\theta_1\right)\right)^3\right]$ Therefore we may find the line of action:

$$x' = \frac{x'\cdot F_V}{F_V} = \frac{\rho\cdot g\cdot w\cdot R^2}{F_V}\cdot\left[\frac{H}{2}\cdot\left(\sin\left(\theta_1\right)\right)^2 - \frac{R}{3}\cdot\left(\sin\left(\theta_1\right)\right)^3\right]$$ Substituting in known values: $\sin\left(\theta_1\right) = \frac{0.650}{0.750}$

$$x' = 999\cdot\frac{kg}{m^3}\times 9.81\cdot\frac{m}{s^2}\times 3.55\cdot m\times(0.750\cdot m)^2\times\frac{1}{2.47\times 10^3}\cdot\frac{1}{N}\times\left[\frac{0.650\cdot m}{2}\times\left(\frac{0.650}{0.750}\right)^2 - \frac{0.750\cdot m}{3}\times\left(\frac{0.650}{0.750}\right)^3\right]\times\frac{N\cdot s^2}{kg\cdot m}$$

$$x' = 0.645\,m$$

For the horizontal force: $F_H = p_c\cdot A = \rho\cdot g\cdot h_c\cdot H\cdot w = \rho\cdot g\cdot\frac{H}{2}\cdot H\cdot w = \frac{\rho\cdot g\cdot H^2\cdot w}{2}$

$$F_H = \frac{1}{2}\times 999\cdot\frac{kg}{m^3}\times 9.81\cdot\frac{m}{s^2}\times(0.650\cdot m)^2\times 3.55\cdot m\times\frac{N\cdot s^2}{kg\cdot m}$$ $$F_H = 7.35\cdot kN$$

For the line of action of the horizontal force: $h' = h_c + \frac{I_{xx}}{h_c\cdot A}$ where $I_{xx} = \frac{w\cdot H^3}{12}$ $A = w\cdot H$ Therefore:

$$h' = h_c + \frac{I_{xx}}{h_c\cdot A} = \frac{H}{2} + \frac{w\cdot H^3}{12}\cdot\frac{2}{H}\cdot\frac{1}{w\cdot H} = \frac{H}{2}+\frac{H}{6} = \frac{2}{3}\cdot H$$ $$h' = \frac{2}{3}\times 0.650\cdot m$$ $$h' = 0.433\,m$$

Problem 3.83

[Difficulty: 2]

3.83 If you throw an anchor out of your canoe but the rope is too short for the anchor to rest on the bottom of the pond, will your canoe float higher, lower, or stay the same? Prove your answer.

Given: Canoe floating in a pond

Find: What happens when an anchor with too short of a line is thrown from canoe

Solution:

Governing equation:

$$F_B = \rho_w g V_{disp} = W$$

Before the anchor is thrown from the canoe the buoyant force on the canoe balances out the weight of the canoe and anchor:

$$F_{B_1} = W_{canoe} + W_{anchor} = \rho_w g V_{canoe_1}$$

The anchor weight can be expressed as

$$W_{anchor} = \rho_a g V_a$$

so the initial volume displaced by the canoe can be written as

$$V_{canoe_1} = \frac{W_{canoe}}{\rho_w g} + \frac{\rho_a}{\rho_w} V_a$$

After throwing the anchor out of the canoe there will be buoyant forces acting on the canoe and the anchor. Combined, these buoyant forces balance the canoe weight and anchor weight:

$$F_{B_2} = W_{canoe} + W_{anchor} = \rho_w g V_{canoe_2} + \rho_w g V_a$$

$$V_{canoe2} = \frac{W_{canoe}}{\rho_w g} + \frac{W_a}{\rho_w g} - V_a$$

Using the anchor weight,

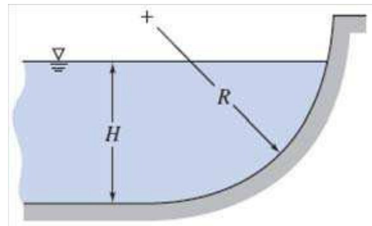
$$V_{canoe2} = \frac{W_{canoe}}{\rho_w g} + \frac{\rho_a}{\rho_w} V_a - V_a$$

Hence the volume displaced by the canoe after throwing the anchor in is less than when the anchor was in the canoe, meaning that the canoe is floating higher.

Problem 3.84

[Difficulty: 3]

3.84 A curved submerged surface, in the shape of a quarter cylinder with radius $R = 1.0$ ft is shown. The form can withstand a maximum vertical load of 350 lbf before breaking. The width is $w = 4$ ft. Find the maximum depth H to which the form may be filled. Find the line of action of the vertical force for this condition. Plot the results over the range of concrete depth $0 \le H \le R$.



Given: Curved surface, in shape of quarter cylinder, with given radius R and width w; liquid concrete stands to depth H.

$R = 1 \cdot ft \qquad w = 4 \cdot ft \qquad F_{vmax} = 350 \cdot lbf \qquad SG = 2.50$ From Table A.1, App A

Find: (a) Maximum depth of concrete to avoid cracking
(b) Line of action on the form.
(c) Plot the vertical force and line of action over H ranging from 0 to R.

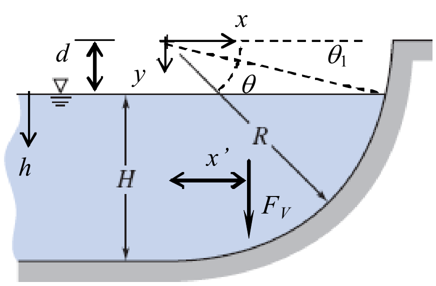
Solution: We will apply the hydrostatics equations to this system.

Governing Equations:

$$\frac{dp}{dh} = \rho \cdot g$$ (Hydrostatic Pressure - h is positive downwards from free surface)

$$F_V = \int p\, dA_y$$ (Vertical Hydrostatic Force)

$$x' \cdot F_V = \int x\, dF_V$$ (Moment of vertical force)



Assumptions: (1) Static fluid
(2) Incompressible fluid
(3) Atmospheric pressure acts on free surface of the concrete

Integrating the hydrostatic pressure equation: $p = \rho \cdot g \cdot h$

From the geometry: $y = R \cdot \sin(\theta) \qquad x = R \cdot \cos(\theta) \qquad h = y - d \qquad d = R - H \qquad dA = w \cdot R \cdot d\theta$

Therefore the vertical component of the hydrostatic force is:

$$F_V = \int p\, dA_y = \int \rho \cdot g \cdot h \cdot \sin(\theta)\, dA = \int_{\theta_1}^{\frac{\pi}{2}} \rho \cdot g \cdot (R \cdot \sin(\theta) - d) \cdot \sin(\theta) \cdot w \cdot R\, d\theta \qquad \text{where} \qquad \theta_1 = \operatorname{asin}\left(\frac{d}{R}\right)$$

$$F_V = \rho \cdot g \cdot w \cdot R \cdot \int_{\theta_1}^{\frac{\pi}{2}} \left[R \cdot (\sin(\theta))^2 - d \cdot (\sin(\theta))\right] d\theta = \rho \cdot g \cdot w \cdot R \cdot \left[R \cdot \left(\frac{\pi}{4} - \frac{\theta_1}{2} + \frac{\sin(2 \cdot \theta_1)}{4}\right) - d \cdot \cos(\theta_1)\right]$$ In terms of H:

$$\sin(\theta_1) = \frac{R - H}{R} \qquad \cos(\theta_1) = \frac{\sqrt{R^2 - (R - H)^2}}{R} = \frac{\sqrt{2 \cdot R \cdot H - H^2}}{R} \qquad \sin(2 \cdot \theta_1) = 2 \cdot \sin(\theta_1) \cdot \cos(\theta_1) = \frac{2 \cdot (R - H) \cdot \sqrt{2 \cdot R \cdot H - H^2}}{R^2}$$

$$F_V = \rho \cdot g \cdot w \cdot R \cdot \left[R \cdot \left[\frac{\pi}{4} - \frac{\operatorname{asin}\left(1 - \frac{H}{R}\right)}{2} + \frac{(R - H)\cdot\sqrt{2\cdot R\cdot H - H^2}}{2R^2} \right] - (R - H)\cdot\frac{\sqrt{2\cdot R\cdot H - H^2}}{R} \right]$$

This equation can be solved iterativ
for H:

$H = 0.773 \cdot ft$

To calculate the line of action of this force:

$$x' \cdot F_V = \int x \cdot \rho \cdot g \cdot h \cdot \sin(\theta)\, dA = \rho \cdot g \cdot R^2 \cdot w \cdot \int_{\theta_1}^{\frac{\pi}{2}} \left[R \cdot (\sin(\theta))^2 \cdot \cos(\theta) - d \cdot \sin(\theta) \cdot \cos(\theta) \right] d\theta$$

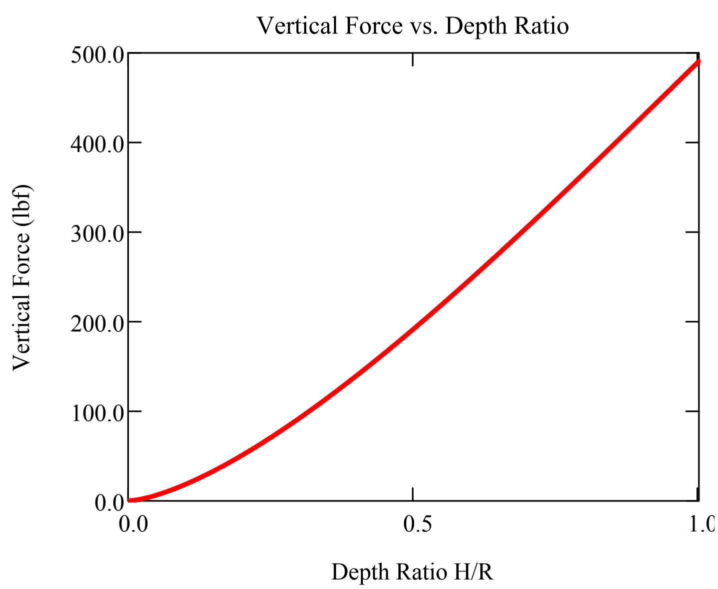
Evaluating the integral: $$x' \cdot F_V = \rho \cdot g \cdot R^2 \cdot w \cdot \left[\frac{R}{3} \cdot \left[1 - \left(\sin\left(\theta_1\right)\right)^3 \right] - \frac{d}{2} \cdot \left(\cos\left(\theta_1\right)\right)^2 \right]$$

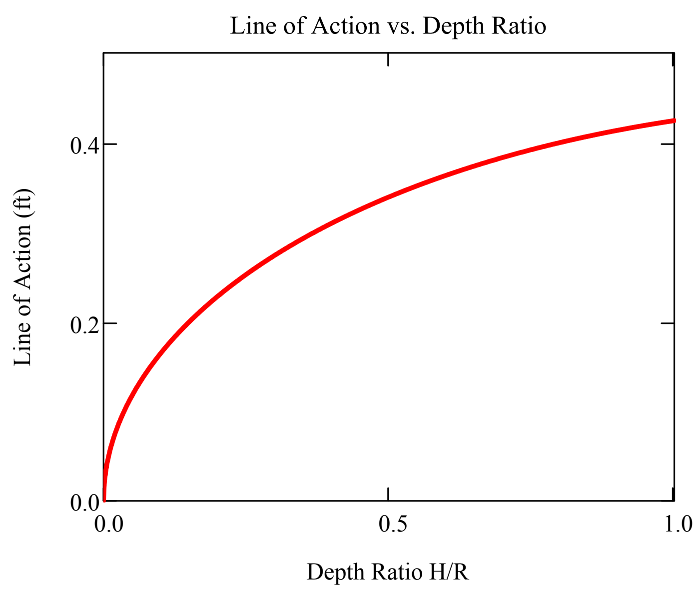
Therefore we may find the line of action: $$x' = \frac{x' \cdot F_V}{F_V} = \frac{\rho \cdot g \cdot R^2 \cdot w}{F_V} \cdot \left[\frac{R}{3} \cdot \left[1 - \left(\sin\left(\theta_1\right)\right)^3 \right] - \frac{d}{2} \cdot \left(\cos\left(\theta_1\right)\right)^2 \right]$$

Substituting in known values: $\sin\left(\theta_1\right) = \frac{1 - 0.773}{1} = 0.227$ $\quad \cos\left(\theta_1\right) = \sqrt{1 - 0.227^2} = 0.9739$

$$x' = \left(2.5 \times 1.94 \cdot \frac{slug}{ft^3}\right) \times 32.2 \cdot \frac{ft}{s^2} \times (1 \cdot ft)^2 \times 4 \cdot ft \times \frac{1}{350} \cdot \frac{1}{lbf} \times \left[\frac{1 \cdot ft}{3} \times \left[1 - (0.227)^3 \right] - \frac{0.227 \cdot ft}{2} \times (0.9739)^2 \right] \times \frac{lbf \cdot s^2}{slug \cdot ft} \qquad x' = 0.396 \cdot ft$$

We may use the equations we developed above to plot the vertical force and line of action as a function of the height of the concrete in the

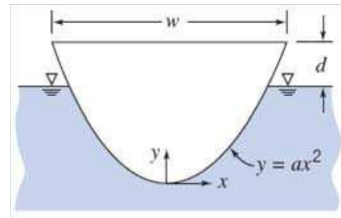




Problem 3.85

[Difficulty: 3]

3.85 The cross-sectional shape of a canoe is modeled by the curve $y = ax^2$, where $a = 1.2\ \text{ft}^{-1}$ and the coordinates are in feet. Assume the width of the canoe is constant at $w = 2$ ft over its entire length $L = 18$ ft. Set up a general algebraic expression relating the total mass of the canoe and its contents to distance d between the water surface and the gunwale of the floating canoe. Calculate the maximum total mass allowable without swamping the canoe.



Given: Model cross section of canoe as a parabola. Assume constant width W over entire length L

$$y = a \cdot x^2 \quad a = 1.2 \cdot \text{ft}^{-1} \quad W = 2 \cdot \text{ft} \quad L = 18 \cdot \text{ft}$$

Find: Expression relating the total mass of canoe and contents to distance d. Determine maximum allowable total mass without swamping the canoe.

Solution: We will apply the hydrostatics equations to this system.

Governing Equations:

$$\frac{dp}{dh} = \rho \cdot g$$ (Hydrostatic Pressure - h is positive downwards from free surface)

$$F_V = \int p\, dA_y$$ (Vertical Hydrostatic Force)

Assumptions:
(1) Static fluid
(2) Incompressible fluid
(3) Atmospheric pressure acts on free surface of the water and inner surface of the canoe.

At any value of d the weight of the canoe and its contents is balanced by the net vertical force of the water on the canoe.

Integrating the hydrostatic pressure equation: $p = \rho \cdot g \cdot h$

$$F_V = \int p\, dA_y = \int \rho \cdot g \cdot h \cdot L\, dx \quad \text{where} \quad h = (H - d) - y$$

To determine the upper limit of integreation we remember that $y = a \cdot x^2$ At the surface

$y = H - d$ Therefore, $x = \sqrt{\frac{H - d}{a}}$ and so the vertical force is:

$$F_V = 2 \cdot \int_0^{\sqrt{\frac{H-d}{a}}} \rho \cdot g \cdot \left[(H - d) - a \cdot x^2\right] \cdot L\, dx = 2 \cdot \rho \cdot g \cdot L \cdot \int_0^{\sqrt{\frac{H-d}{a}}} \left[(H - d) - a \cdot x^2\right] dx = 2 \cdot \rho \cdot g \cdot L \cdot \left[\frac{(H - d)^{\frac{3}{2}}}{\sqrt{a}} - \frac{a}{3} \cdot \left[\frac{(H - d)}{a}\right]^{\frac{3}{2}}\right]$$

Upon simplification: $F_V = 2 \cdot \rho \cdot g \cdot L \cdot \frac{(H - d)^{\frac{3}{2}}}{\sqrt{a}} \cdot \left(1 - \frac{1}{3}\right) = \frac{4 \cdot \rho \cdot g \cdot L}{3\sqrt{a}} \cdot (H - d)^{\frac{3}{2}} = M \cdot g$ or $M = \frac{4 \cdot \rho \cdot L}{3\sqrt{a}} \cdot (H - d)^{\frac{3}{2}}$ where M is the mass of the canoe.

The limit for no swamping is d=0, and so: $M = \frac{4}{3} \times 1.94 \cdot \frac{\text{slug}}{\text{ft}^3} \times 18 \cdot \text{ft} \times \sqrt{\frac{\text{ft}}{1.2}} \times (2.4 \cdot \text{ft})^{\frac{3}{2}} \times \frac{32.174 \cdot \text{lb}}{\text{slug}}$ $\qquad M = 5.08 \times 10^3 \cdot \text{lb}$

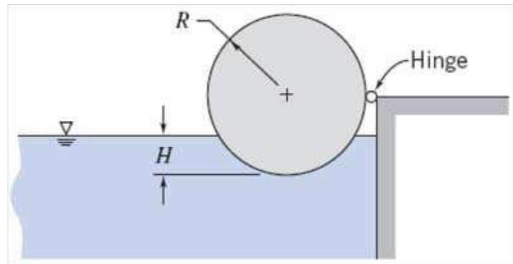
This leaves us no margin, so if we set d=0.2 ft we get $M = \frac{4}{3} \times 1.94 \cdot \frac{\text{slug}}{\text{ft}^3} \times 18 \cdot \text{ft} \times \sqrt{\frac{\text{ft}}{1.2}} \times (2.2 \cdot \text{ft})^{\frac{3}{2}} \cdot \frac{32.174 \cdot \text{lb}}{\text{slug}}$ $\qquad M = 4.46 \times 10^3 \cdot \text{lb}$

Clearly the answer is highly dependent upon the allowed risk of swamping!

Problem 3.86

[Difficulty: 4]

3.86 The cylinder shown is supported by an incompressible liquid of density ρ, and is hinged along its length. The cylinder, of mass M, length L, and radius R, is immersed in liquid to depth H. Obtain a general expression for the cylinder specific gravity versus the ratio of liquid depth to cylinder radius, $\alpha = H/R$, needed to hold the cylinder in equilibrium for $0 \le \alpha < 1$. Plot the results.



Given: Cylinder of mass M, length L, and radius R is hinged along its length and immersed in an incompressilble liquid to deptl

Find: General expression for the cylinder specific gravity as a function of α=H/R needed to hold the cylinder in equilibrium for α ranging from 0 to 1.

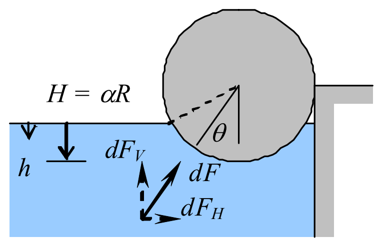
Solution: We will apply the hydrostatics equations to this system.

Governing Equations:

$$\frac{dp}{dh} = \rho \cdot g$$ (Hydrostatic Pressure - h is positive downwards from free surface)

$$F_V = \int p\, dA_y$$ (Vertical Hydrostatic Force)

$$\Sigma M = 0$$ (Rotational Equilibrium)



Assumptions:
(1) Static fluid
(2) Incompressible fluid
(3) Atmospheric pressure acts on free surface of the liquid.

The moments caused by the hydrostatic force and the weight of the cylinder about the hinge need to balance each other.

Integrating the hydrostatic pressure equation: $p = \rho \cdot g \cdot h$

$$dF_V = dF \cdot \cos(\theta) = p \cdot dA \cdot \cos(\theta) = \rho \cdot g \cdot h \cdot w \cdot R \cdot d\theta \cdot \cos(\theta)$$

Now the depth to which the cylinder is submerged is $H = h + R \cdot (1 - \cos(\theta))$

Therefore $h = H - R \cdot (1 - \cos(\theta))$ and into the vertical force equation:

$$dF_V = \rho \cdot g \cdot [H - R \cdot (1 - \cos(\theta))] \cdot w \cdot R \cdot \cos(\theta) \cdot d\theta = \rho \cdot g \cdot w \cdot R^2 \cdot \left[\frac{H}{R} - (1 - \cos(\theta))\right] \cdot \cos(\theta) \cdot d\theta$$

$$dF_V = \rho \cdot g \cdot w \cdot R^2 \cdot \left[(\alpha - 1) \cdot \cos(\theta) + (\cos(\theta))^2\right] \cdot d\theta = \rho \cdot g \cdot w \cdot R^2 \cdot \left[(\alpha - 1) \cdot \cos(\theta) + \frac{1 + \cos(2 \cdot \theta)}{2}\right] \cdot d\theta$$

Now as long as α is not greater than 1, the net horizontal hydrostatic force will be zero due to symmetry, and the vertical force is:

$$F_V = \int_{-\theta_{max}}^{\theta_{max}} 1\, dF_V = \int_0^{\theta_{max}} 2\, dF_V \qquad \text{where} \qquad \cos(\theta_{max}) = \frac{R - H}{R} = 1 - \alpha \qquad \text{or} \qquad \theta_{max} = \mathrm{acos}(1 - \alpha)$$

$$F_V = 2\rho \cdot g \cdot w \cdot R^2 \cdot \int_0^{\theta_{max}} \left[(\alpha - 1) \cdot \cos(\theta) + \frac{1}{2} + \frac{1}{2} \cdot \cos(2 \cdot \theta) \right] d\theta$$

Now upon integration of this expression we have:

$$F_V = \rho \cdot g \cdot w \cdot R^2 \cdot \left[\operatorname{acos}(1 - \alpha) - (1 - \alpha) \cdot \sqrt{\alpha \cdot (2 - \alpha)} \right]$$

The line of action of the vertical force due to the liquid is through the centroid of the displaced liquid, i.e., through the center of the cylinde

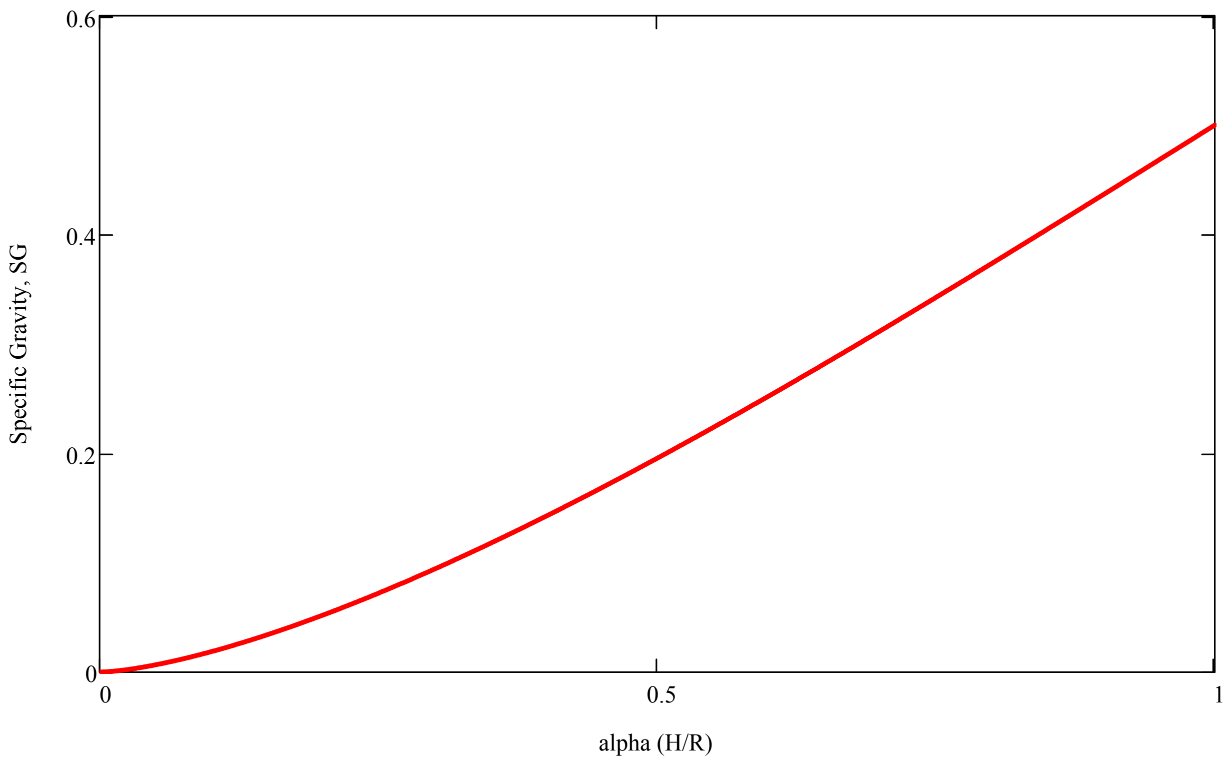
The weight of the cylinder is given by: $W = M \cdot g = \rho_c \cdot V \cdot g = SG \cdot \rho \cdot \pi \cdot R^2 \cdot w \cdot g$ where ρ is the density of the fluid and $SG = \frac{\rho_c}{\rho}$

The line of action of the weight is also throught the center of the cylinder. Taking moment about the hinge we get:

$\Sigma M_O = W \cdot R - F_V \cdot R = 0$ or in other words $W = F_V$ and therefore:

$$SG \cdot \rho \cdot \pi \cdot R^2 \cdot w \cdot g = \rho \cdot g \cdot w \cdot R^2 \cdot \left[\operatorname{acos}(1 - \alpha) - (1 - \alpha) \cdot \sqrt{\alpha \cdot (2 - \alpha)} \right]$$

$$SG = \frac{1}{\pi} \cdot \left[\operatorname{acos}(1 - \alpha) - (1 - \alpha) \cdot \sqrt{\alpha \cdot (2 - \alpha)} \right]$$



Problem 3.87

[Difficulty: 4]

3.87 A canoe is represented by a right semicircular cylinder, with $R = 1.2$ ft and $L = 17$ ft. The canoe floats in water that is $d = 1$ ft deep. Set up a general algebraic expression for the total mass (canoe and contents) that can be floated, as a function of depth. Evaluate for the given conditions. Plot the results over the range of water depth $0 \le d \le R$.

Given: Canoe, modeled as a right semicircular cylindrical shell, floats in water of depth d. The shell has outer radius R and leng

$R = 1.2 \cdot ft \qquad L = 17 \cdot ft \qquad d = 1 \cdot ft$

Find:
(a) General expression for the maximum total mass that can be floated, as a function of depth,
(b) evaluate for the given conditions
(c) plot for range of water depth between 0 and R.

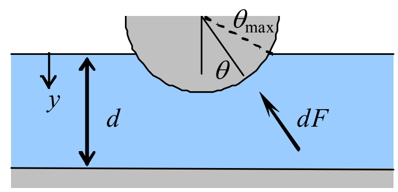
Solution: We will apply the hydrostatics equations to this system.

Governing Equations:

$$\frac{dp}{dy} = \rho \cdot g$$ (Hydrostatic Pressure - y is positive downwards from free surface)

$$F_v = \int p\, dA_y$$ (Vertical Hydrostatic Force)

Assumptions:
(1) Static fluid
(2) Incompressible fluid
(3) Atmospheric pressure acts on free surface of the liquid.



y is a function of θ for a given depth d: $y = d - (R - R\cdot\cos(\theta)) = d - R + R\cdot\cos(\theta)$

The maximum value of θ: $\theta_{max} = \mathrm{acos}\left[\frac{(R-d)}{R}\right]$

A free-body diagram of the canoe gives: $\Sigma F_y = 0 = M\cdot g - F_v$ where F_v is the vertical force of the water on the canoe.

$$F_v = \int p\, dA_y = \int p\cdot\cos(\theta)\, dA = \int_{-\theta_{max}}^{\theta_{max}} \rho\cdot g\cdot y\cdot L\cdot R\cdot\cos(\theta)\, d\theta = 2\cdot\rho\cdot g\cdot L\cdot R\cdot\int_0^{\theta_{max}} (d - R + R\cdot\cos(\theta))\cdot\cos(\theta)\, d\theta$$

$$F_v = 2\cdot\rho\cdot g\cdot L\cdot R\cdot\int_0^{\theta_{max}} \left[(d-R)\cdot\cos(\theta) + R\cdot(\cos(\theta))^2\right] d\theta = 2\cdot\rho\cdot g\cdot L\cdot R\cdot\left[(d-R)\cdot\sin(\theta_{max}) + R\cdot\left(\frac{\theta_{max}}{2} + \frac{\sin(2\cdot\theta_{max})}{4}\right)\right]$$

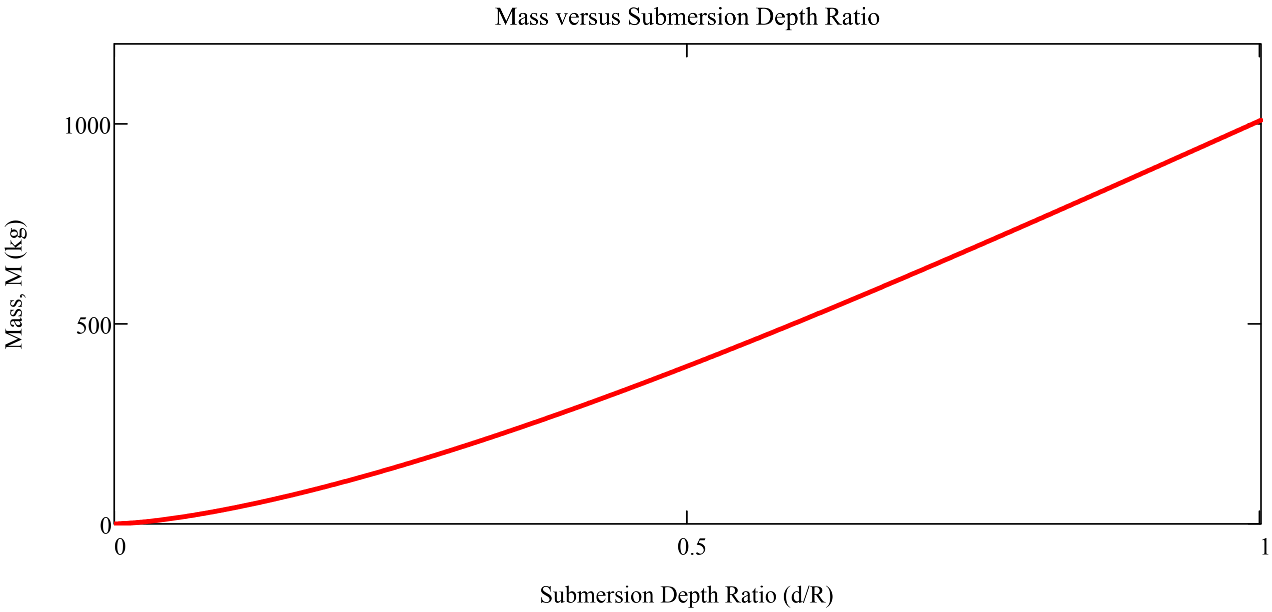
Since $M = \frac{F_v}{g}$ it follows that $M = 2\cdot\rho\cdot L\cdot R\cdot\left[(d-R)\cdot\sin(\theta_{max}) + R\cdot\left(\frac{\theta_{max}}{2} + \frac{\sin(2\cdot\theta_{max})}{4}\right)\right]$

For $R = 1.2\cdot ft$ $L = 17\cdot ft$ and $d = 1\cdot ft$ we can determine the mass: $\theta_{max} = \mathrm{acos}\left[\frac{(1.2-1)}{1.2}\right]$ $\theta_{max} = 1.403\cdot rad$

$$M = 2 \times 1.94 \cdot \frac{\text{slug}}{\text{ft}^3} \times 17 \cdot \text{ft} \times 1.2 \cdot \text{ft} \times \left[(1 \cdot \text{ft} - 1.2 \cdot \text{ft}) \times \sin(1.403 \cdot \text{rad}) + 1.2 \cdot \text{ft} \times \left(\frac{1.403 \cdot \text{rad}}{2} + \frac{\sin(2 \times 1.403 \cdot \text{rad})}{4} \right) \right] \times \frac{32.2 \cdot \text{lbm}}{\text{slug}}$$

$$M = 1895 \cdot \text{lbm}$$

When we enter the values of d/R into the expressions for θ_{max} and M, we get the following graph:



Problem 3.88

[Difficulty: 4]

3.88 A glass observation room is to be installed at the corner of the bottom of an aquarium. The aquarium is filled with seawater to a depth of 35 ft. The glass is a segment of a sphere, radius 5 ft, mounted symmetrically in the corner. Compute the magnitude and direction of the net force on the glass structure.

Given: Geometry of glass observation room

Find: Resultant force and direction

Assumptions: Water in aquarium is static and incompressible

Solution:

The *x*, *y* and *z* components of force due to the fluid are treated separately. For the *x*, *y* components, the horizontal force is equivalent to that on a vertical flat plate; for the *z* component, (vertical force) the force is equivalent to the weight of fluid above.

For horizontal forces, the computing equation of Section 3-5 is $F_H = p_c \cdot A$ where A is the area of the equivalent vertical plate.

For the vertical force, the computing equation of Section 3-5 is $F_V = \rho \cdot g \cdot V$ where V is the volume of fluid above the curved surface.

The data are For water $\rho = 1.94 \cdot \frac{slug}{ft^3}$

For the fluid (Table A.2) $SG = 1.025$

For the aquarium $R = 5 \cdot ft$ $H = 35 \cdot ft$

(a) Horizontal Forces

Consider the *x* component

The center of pressure of the glass is $y_c = H - \frac{4 \cdot R}{3 \cdot \pi}$ $y_c = 32.88 \cdot ft$

Hence $F_{Hx} = p_c \cdot A = (SG \cdot \rho \cdot g \cdot y_c) \cdot \frac{\pi \cdot R^2}{4}$

$$F_{Hx} = 1.025 \times 1.94 \cdot \frac{slug}{ft^3} \times 32.2 \cdot \frac{ft}{s^2} \times 32.88 \cdot ft \times \frac{\pi \cdot (5 \cdot ft)^2}{4} \times \frac{lbf \cdot s^2}{slug \cdot ft} \qquad F_{Hx} = 4.13 \times 10^4 \cdot lbf$$

The *y* component is of the same magnitude as the *x* component

$$F_{Hy} = F_{Hx} \qquad F_{Hy} = 4.13 \times 10^4 \cdot lbf$$

The resultant horizontal force (at 45° to the *x* and *y* axes) is

$$F_H = \sqrt{F_{Hx}^2 + F_{Hy}^2} \qquad F_H = 5.85 \times 10^4 \cdot lbf$$

(b) Vertical forces

The vertical force is equal to the weight of fluid above (a volume defined by a rectangular column minus a segment of a sphere)

The volume is $$V = \frac{\pi \cdot R^2}{4} \cdot H - \frac{\frac{4 \cdot \pi \cdot R^3}{3}}{8} \qquad V = 621.8 \cdot ft^3$$

Then $$F_V = SG \cdot \rho \cdot g \cdot V \qquad F_V = 1.025 \times 1.94 \cdot \frac{slug}{ft^3} \times 32.2 \cdot \frac{ft}{s^2} \times 621.8 \cdot ft^3 \times \frac{lbf \cdot s^2}{slug \cdot ft}$$

$$F_V = 3.98 \times 10^4 \cdot lbf$$

Finally the resultant force and direction can be computed

$$F = \sqrt{F_H^{\ 2} + F_V^{\ 2}} \qquad F = 7.07 \times 10^4 \cdot lbf$$

$$\alpha = \mathrm{atan}\left(\frac{F_V}{F_H}\right) \qquad \alpha = 34.3 \cdot deg$$

Note that α is the angle the resultant force makes with the horizontal

Problem *3.89

[Difficulty: 2]

3.89 A hydrometer is a specific gravity indicator, the value being indicated by the level at which the free surface intersects the stem when floating in a liquid. The 1.0 mark is the level when in distilled water. For the unit shown, the immersed volume in distilled water is 15 cm^3. The stem is 6 mm in diameter. Find the distance, h, from the 1.0 mark to the surface when the hydrometer is placed in a nitric acid solution of specific gravity 1.5.

Given: Hydrometer as shown, submerged in nitric acid. When submerged in water, h = 0 and the immersed volume is 15 cubic cm.

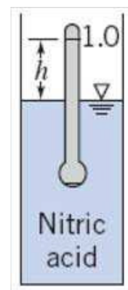
$SG = 1.5 \quad d = 6 \cdot mm$

Find: The distance h when immersed in nitric acid.

Solution: We will apply the hydrostatics equations to this system.

Governing Equations: $F_{buoy} = \rho \cdot g \cdot V_d$ (Buoyant force is equal to weight of displaced fluid)

Assumptions: (1) Static fluid
(2) Incompressible fluid



Taking a free body diagram of the hydrometer: $\Sigma F_z = 0 \quad -M \cdot g + F_{buoy} = 0$

Solving for the mass of the hydrometer: $M = \dfrac{F_{buoy}}{g} = \rho \cdot V_d$

When immersed in water: $M = \rho_w \cdot V_w$ When immersed in nitric acid: $M = \rho_n \cdot V_n$

Since the mass of the hydrometer is the same in both cases: $\rho_w \cdot V_w = \rho_n \cdot V_n$

When the hydrometer is in the nitric acid: $V_n = V_w - \dfrac{\pi}{4} \cdot d^2 \cdot h \quad \rho_n = SG \cdot \rho_w$

Therefore: $\rho_w \cdot V_w = SG \cdot \rho_w \cdot \left(V_w - \dfrac{\pi}{4} \cdot d^2 \cdot h \right)$ Solving for the height h:

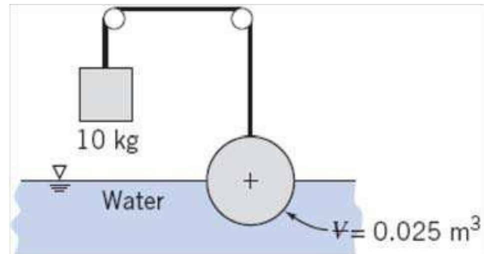
$$V_w = SG \cdot \left(V_w - \frac{\pi}{4} \cdot d^2 \cdot h \right) \qquad V_w \cdot (1 - SG) = -SG \cdot \frac{\pi}{4} \cdot d^2 \cdot h$$

$$h = V_w \cdot \left(\frac{SG - 1}{SG} \right) \cdot \frac{4}{\pi \cdot d^2} \qquad h = 15 \cdot cm^3 \times \left(\frac{1.5 - 1}{1.5} \right) \times \frac{4}{\pi \times (6 \cdot mm)^2} \times \left(\frac{10 \cdot mm}{cm} \right)^3 \qquad h = 177 \cdot mm$$

Problem *3.90

[Difficulty: 3]

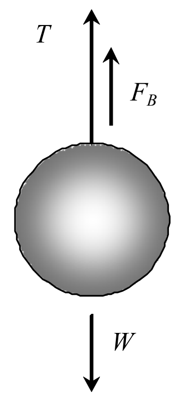
3.90 Find the specific weight of the sphere shown if its volume is 0.025m^3. State all assumptions. What is the equilibrium position of the sphere if the weight is removed?



Given: Data on sphere and weight

Find: SG of sphere; equilibrium position when freely floating

Solution:



Basic equation $\quad F_B = \rho \cdot g \cdot V \quad$ and $\quad \Sigma F_z = 0 \qquad \Sigma F_z = 0 = T + F_B - W$

where $\quad T = M \cdot g \qquad M = 10 \cdot kg \qquad F_B = \rho \cdot g \cdot \frac{V}{2} \qquad W = SG \cdot \rho \cdot g \cdot V$

Hence $\quad M \cdot g + \rho \cdot g \cdot \frac{V}{2} - SG \cdot \rho \cdot g \cdot V = 0 \qquad SG = \frac{M}{\rho \cdot V} + \frac{1}{2}$

$$SG = 10 \cdot kg \times \frac{m^3}{1000 \cdot kg} \times \frac{1}{0.025 \cdot m^3} + \frac{1}{2} \qquad SG = 0.9$$

The specific weight is $\quad \gamma = \frac{\text{Weight}}{\text{Volume}} = \frac{SG \cdot \rho \cdot g \cdot V}{V} = SG \cdot \rho \cdot g \qquad \gamma = 0.9 \times 1000 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times \frac{N \cdot s^2}{kg \cdot m} \qquad \gamma = 8829 \cdot \frac{N}{m^3}$

For the equilibriul position when floating, we repeat the force balance with T = 0

$$F_B - W = 0 \qquad W = F_B \qquad \text{with} \qquad F_B = \rho \cdot g \cdot V_{submerged}$$

From references (trying Googling "partial sphere volume") $\quad V_{submerged} = \frac{\pi \cdot h^2}{3} \cdot (3 \cdot R - h)$

where h is submerged depth and R is the sphere radius $\quad R = \left(\frac{3 \cdot V}{4 \cdot \pi}\right)^{\frac{1}{3}} \qquad R = \left(\frac{3}{4 \cdot \pi} \cdot 0.025 \cdot m^3\right)^{\frac{1}{3}} \qquad R = 0.181\,m$

Hence $\quad W = SG \cdot \rho \cdot g \cdot V = F_B = \rho \cdot g \cdot \frac{\pi \cdot h^2}{3} \cdot (3 \cdot R - h) \qquad h^2 \cdot (3 \cdot R - h) = \frac{3 \cdot SG \cdot V}{\pi}$

$$h^2 \cdot (3 \cdot 0.181 \cdot m - h) = \frac{3 \cdot 0.9 \cdot .025 \cdot m^3}{\pi} \qquad h^2 \cdot (0.544 - h) = 0.0215$$

This is a cubic equation for h. We can keep guessing h values, manually iterate, or use Excel's Goal Seek to find $\quad h = 0.292 \cdot m$

Problem *3.91

[Difficulty: 2]

3.91 The fat-to-muscle ratio of a person may be determined from a specific gravity measurement. The measurement is made by immersing the body in a tank of water and measuring the net weight. Develop an expression for the specific gravity of a person in terms of their weight in air, net weight in water, and SG = *f*(*T*) for water.

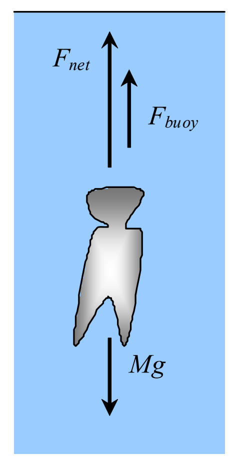
Given: Specific gravity of a person is to be determined from measurements of weight in air and the met weight when totally immersed in water.

Find: Expression for the specific gravity of a person from the measurements.

Solution: We will apply the hydrostatics equations to this system.

Governing Equation: $F_{buoy} = \rho \cdot g \cdot V_d$ (Buoyant force is equal to weight of displaced fluid)

Assumptions: (1) Static fluid
(2) Incompressible fluid



Taking a free body diagram of the body: $\Sigma F_y = 0 \qquad F_{net} - M \cdot g + F_{buoy} = 0$

F_{net} is the weight measurement for the immersed body.

$F_{net} = M \cdot g - F_{buoy} = M \cdot g - \rho_w \cdot g \cdot V_d$ However in air: $F_{air} = M \cdot g$

Therefore the weight measured in water is: $F_{net} = F_{air} - \rho_w \cdot g \cdot V_d$ and $V_d = \dfrac{F_{air} - F_{net}}{\rho_w \cdot g}$

Now in order to find the specific gravity of the person, we need his/her density:

$F_{air} = M \cdot g = \rho \cdot g \cdot V_d = \rho \cdot g \cdot \dfrac{\left(F_{air} - F_{net}\right)}{\rho_w \cdot g}$ Simplifying this expression we get: $F_{air} = \dfrac{\rho}{\rho_w}\left(F_{air} - F_{net}\right)$

Now if we call the density of water at 4 deg C ρ_{w4C} then: $F_{air} = \dfrac{\left(\dfrac{\rho}{\rho_{w4C}}\right)}{\left(\dfrac{\rho_w}{\rho_{w4C}}\right)}\left(F_{air} - F_{net}\right) = \dfrac{SG}{SG_w} \cdot \left(F_{air} - F_{net}\right)$

Solving this expression for the specific gravity of the person SG, we get: $SG = SG_w \cdot \dfrac{F_{air}}{F_{air} - F_{net}}$

Problem *3.92

[Difficulty: 2]

3.92 Quantify the statement, "Only the tip of an iceberg shows (in seawater)."

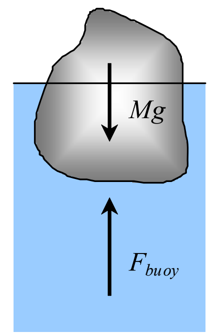
Given: Iceberg floating in seawater

Find: Quantify the statement, "Only the tip of an iceberg shows (in seawater)."

Solution: We will apply the hydrostatics equations to this system.

Governing Equations: $F_{buoy} = \rho \cdot g \cdot V_d$ (Buoyant force is equal to weight of displaced fluid)

Assumptions:
(1) Static fluid
(2) Incompressible fluid



Taking a free body diagram of the iceberg: $\Sigma F_z = 0 \quad -M \cdot g + F_{buoy} = 0$

$M \cdot g = F_{buoy} = \rho_{sw} \cdot g \cdot V_d$ But the mass of the iceberg is also: $M = \rho_{ice} \cdot V_{tot}$

Combining these expressions: $\rho_{ice} \cdot V_{tot} \cdot g = \rho_{sw} \cdot g \cdot V_d \qquad V_d = V_{tot} \cdot \frac{\rho_{ice}}{\rho_{sw}} = V_{tot} \cdot \frac{SG_{ice}}{SG_{sw}}$

The volume of the iceberg above the water is: $V_{show} = V_{tot} - V_d = V_{tot} \cdot \left(1 - \frac{SG_{ice}}{SG_{sw}}\right)$

Therefore we may define a volume fraction: $VF = \frac{V_{show}}{V_{tot}} = 1 - \frac{SG_{ice}}{SG_{sw}}$

Substituting in data from Tables A.1 and A.2 we get: $VF = 1 - \frac{0.917}{1.025} \qquad VF = 0.1054$ Only 10% of the iceberg is above wate

Problem *3.93

[Difficulty: 2]

3.93 An open tank is filled to the top with water. A steel cylindrical container, wall thickness $\delta = 1$ mm, outside diameter $D = 100$ mm, and height $H = 1$ m, with an open top, is gently placed in the water. What is the volume of water that overflows from the tank? How many 1 kg weights must be placed in the container to make it sink? Neglect surface tension effects.

Given: Geometry of steel cylinder

Find: Volume of water displaced; number of 1 kg wts to make it sink

Solution:

The data is For water $\rho = 999 \cdot \frac{kg}{m^3}$

For steel (Table A.1) $SG = 7.83$

For the cylinder $D = 100 \cdot mm$ $\quad H = 1 \cdot m$ $\quad \delta = 1 \cdot mm$

The volume of the cylinder is $V_{steel} = \delta \cdot \left(\frac{\pi \cdot D^2}{4} + \pi \cdot D \cdot H \right)$ $\quad V_{steel} = 3.22 \times 10^{-4} \cdot m^3$

The weight of the cylinder is $W = SG \cdot \rho \cdot g \cdot V_{steel}$

$$W = 7.83 \times 999 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times 3.22 \times 10^{-4} \cdot m^3 \times \frac{N \cdot s^2}{kg \cdot m} \qquad W = 24.7\,N$$

At equilibium, the weight of fluid displaced is equal to the weight of the cylinder

$$W_{displaced} = \rho \cdot g \cdot V_{displaced} = W$$

$$V_{displaced} = \frac{W}{\rho \cdot g} = 24.7 \cdot N \times \frac{m^3}{999 \cdot kg} \times \frac{s^2}{9.81 \cdot m} \times \frac{kg \cdot m}{N \cdot s^2} \qquad V_{displaced} = 2.52\,L$$

To determine how many 1 kg wts will make it sink, we first need to find the extra volume that will need to be dsiplaced

Distance cylinder sank $x_1 = \frac{V_{displaced}}{\left(\frac{\pi \cdot D^2}{4} \right)}$ $\quad x_1 = 0.321\,m$

Hence, the cylinder must be made to sink an additional distance $x_2 = H - x_1$ $\quad x_2 = 0.679\,m$

We deed to add n weights so that $1 \cdot kg \cdot n \cdot g = \rho \cdot g \cdot \frac{\pi \cdot D^2}{4} \cdot x_2$

$$n = \frac{\rho \cdot \pi \cdot D^2 \cdot x_2}{4 \times 1 \cdot kg} = 999 \cdot \frac{kg}{m^3} \times \frac{\pi}{4} \times (0.1 \cdot m)^2 \times 0.679 \cdot m \times \frac{1}{1 \cdot kg} \times \frac{N \cdot s^2}{kg \cdot m} \qquad n = 5.33$$

Hence we need $n = 6$ weights to sink the cylinder

Problem *3.94

[Difficulty: 2]

3.94 Quantify the experiment performed by Archimedes to identify the material content of King Hiero's crown. Assume you can measure the weight of the king's crown in air, W_a, and the weight in water, W_w. Express the specific gravity of the crown as a function of these measured values.

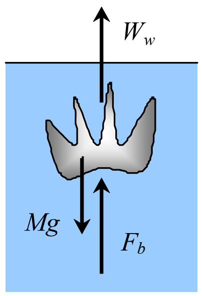
Given: Experiment performed by Archimedes to identify the material conent of King Hiero's crown. The crown was weighed in air and in water.

Find: Expression for the specific gravity of the crown as a function of the weights in water and air.

Solution: We will apply the hydrostatics equations to this system.

Governing Equations: $F_b = \rho \cdot g \cdot V_d$ (Buoyant force is equal to weight of displaced fluid)

Assumptions: (1) Static fluid
(2) Incompressible fluid



Taking a free body diagram of the body: $\Sigma F_z = 0$ $\quad W_w - M \cdot g + F_b = 0$

W_w is the weight of the crown in water.

$W_w = M \cdot g - F_{buoy} = M \cdot g - \rho_w \cdot g \cdot V_d$ However in air: $W_a = M \cdot g$

Therefore the weight measured in water is: $W_w = W_a - \rho_w \cdot g \cdot V_d$

so the volume is: $V_d = \dfrac{W_a - W_w}{\rho_w \cdot g}$ Now the density of the crown is: $\rho_c = \dfrac{M}{V_d} = \dfrac{M \cdot \rho_w \cdot g}{W_a - W_w} = \dfrac{W_a}{W_a - W_w} \cdot \rho_w$

Therefore, the specific gravity of the crown is: $SG = \dfrac{\rho_c}{\rho_w} = \dfrac{W_a}{W_a - W_w}$ $\qquad SG = \dfrac{W_a}{W_a - W_w}$

Note: by definition specific gravity is the density of an object divided by the density of water at 4 degrees Celsius, so the measured temperature of the water in the experiment and the data from tables A.7 or A.8 may be used to correct for the variation in density of the water with temperature.

Problem *3.95

[Difficulty: 2]

3.95 Gas bubbles are released from the regulator of a submerged scuba diver. What happens to the bubbles as they rise through the seawater? Explain.

Open-Ended Problem Statement: Gas bubbles are released from the regulator of a submerged Scuba diver. What happens to the bubbles as they rise through the seawater?

Discussion: Air bubbles released by a submerged diver should be close to ambient pressure at the depth where the diver is swimming. The bubbles are small compared to the depth of submersion, so each bubble is exposed to essentially constant pressure. Therefore the released bubbles are nearly spherical in shape.

The air bubbles are buoyant in water, so they begin to rise toward the surface. The bubbles are quite light, so they reach terminal speed quickly. At low speeds the spherical shape should be maintained. At higher speeds the bubble shape may be distorted.

As the bubbles rise through the water toward the surface, the hydrostatic pressure decreases. Therefore the bubbles expand as they rise. As the bubbles grow larger, one would expect the tendency for distorted bubble shape to be exaggerated.

Problem *3.96

[Difficulty: 2]

3.96 Hot-air ballooning is a popular sport. According to a recent article, "hot-air volumes must be large because air heated to 150°F over ambient lifts only 0.018 lbf/ft^3 compared to 0.066 and 0.071 for helium and hydrogen, respectively." Check these statements for sea-level conditions. Calculate the effect of increasing the hot-air maximum temperature to 250°F above ambient.

Given: Balloons with hot air, helium and hydrogen. Claim lift per cubic foot of 0.018, 0.066, and 0.071 pounds force per cubic f for respective gases, with the air heated to 150 deg. F over ambient.

Find: (a) evaluate the claims of lift per unit volume
(b) determine change in lift when air is heated to 250 deg. F over ambient.

Solution: We will apply the hydrostatics equations to this system.

Governing Equations:

$$L = \rho_a \cdot g \cdot V - \rho_g \cdot g \cdot V$$ (Net lift force is equal to difference in weights of air and gas)

$$p = \rho \cdot R \cdot T$$ (Ideal gas equation of state)

Assumptions:
(1) Static fluid
(2) Incompressible fluid
(3) Ideal gas behavior

The lift per unit volume may be written as: $LV = \frac{L}{V} = g \cdot (\rho_a - \rho_g) = \rho_a \cdot g \cdot \left(1 - \frac{\rho_g}{\rho_a}\right)$ now if we take the ideal gas equation and we take into account that the pressure inside and outside the balloon are equal: $\frac{L}{V} = \rho_a \cdot g \cdot \left(1 - \frac{R_a \cdot T_a}{R_g \cdot T_g}\right) = \gamma_a \cdot \left(1 - \frac{R_a \cdot T_a}{R_g \cdot T_g}\right)$

At standard conditions the specific weight of air is: $\gamma_a = 0.0765 \cdot \frac{lbf}{ft^3}$ the gas constant is: $R_a = 53.33 \cdot \frac{ft \cdot lbf}{lbm \cdot R}$ and $T_a = 519 \cdot R$

For helium: $R_g = 386.1 \cdot \frac{ft \cdot lbf}{lbm \cdot R}$ $T_g = T_a$ and therefore: $LV_{He} = 0.0765 \cdot \frac{lbf}{ft^3} \times \left(1 - \frac{53.33}{386.1}\right)$ $LV_{He} = 0.0659 \cdot \frac{lbf}{ft^3}$

For hydrogen: $R_g = 766.5 \cdot \frac{ft \cdot lbf}{lbm \cdot R}$ $T_g = T_a$ and therefore: $LV_{H2} = 0.0765 \cdot \frac{lbf}{ft^3} \times \left(1 - \frac{53.33}{766.5}\right)$ $LV_{H2} = 0.0712 \cdot \frac{lbf}{ft^3}$

For hot air at 150 degrees above ambient:

$R_g = R_a$ $T_g = T_a + 150 \cdot R$ and therefore: $LV_{air150} = 0.0765 \cdot \frac{lbf}{ft^3} \times \left(1 - \frac{519}{519 + 150}\right)$ $LV_{air150} = 0.0172 \cdot \frac{lbf}{ft^3}$

The agreement with the claims stated above is good.

For hot air at 250 degrees above ambient:

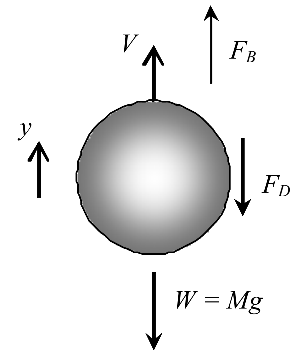
$R_g = R_a$ $T_g = T_a + 250 \cdot R$ and therefore: $LV_{air250} = 0.0765 \cdot \frac{lbf}{ft^3} \times \left(1 - \frac{519}{519 + 250}\right)$ $LV_{air250} = 0.0249 \cdot \frac{lbf}{ft^3}$

$\frac{LV_{air250}}{LV_{air150}} = 1.450$ Air at ΔT of 250 deg. F gives 45% more lift than air at ΔT of 150 deg.F!

Problem *3.97

[Difficulty: 2]

3.97 Hydrogen bubbles are used to visualize water flow streaklines in the video, *Flow Visualization*. A typical hydrogen bubble diameter is $d = 0.001$ in. The bubbles tend to rise slowly in water because of buoyancy; eventually they reach terminal speed relative to the water. The drag force of the water on a bubble is given by $F_D = 3\pi\mu V d$, where μ is the viscosity of water and V is the bubble speed relative to the water. Find the buoyancy force that acts on a hydrogen bubble immersed in water. Estimate the terminal speed of a bubble rising in water.



Given: Data on hydrogen bubbles

Find: Buoyancy force on bubble; terminal speed in water

Solution:

Basic equation $\quad F_B = \rho \cdot g \cdot V = \rho \cdot g \cdot \frac{\pi}{6} \cdot d^3 \quad$ and $\quad \Sigma F_y = M \cdot a_y \quad \Sigma F_y = 0 = F_B - F_D - W \quad$ for terminal speed

$$F_B = 1.94 \cdot \frac{\text{slug}}{\text{ft}^3} \times 32.2 \cdot \frac{\text{ft}}{\text{s}^2} \times \frac{\pi}{6} \times \left(0.001 \cdot \text{in} \times \frac{1 \cdot \text{ft}}{12 \cdot \text{in}}\right)^3 \times \frac{\text{lbf} \cdot \text{s}^2}{\text{slug} \cdot \text{ft}} \qquad F_B = 1.89 \times 10^{-11} \cdot \text{lbf}$$

For terminal speed $\quad F_B - F_D - W = 0 \qquad F_D = 3 \cdot \pi \cdot \mu \cdot V \cdot d = F_B \quad$ where we have ignored W, the weight of the bubble (at STP most gases are about 1/1000 the density of water)

Hence $\quad V = \frac{F_B}{3 \cdot \pi \cdot \mu \cdot d} \quad$ with $\quad \mu = 2.10 \times 10^{-5} \cdot \frac{\text{lbf} \cdot \text{s}}{\text{ft}^2} \quad$ from Table A.7 at 68ºF

$$V = 1.89 \times 10^{-11} \cdot \text{lbf} \times \frac{1}{3 \cdot \pi} \times \frac{1}{2.10 \times 10^{-5}} \cdot \frac{\text{ft}^2}{\text{lbf} \cdot \text{s}} \times \frac{1}{0.001 \cdot \text{in}} \times \frac{12 \cdot \text{in}}{1 \cdot \text{ft}}$$

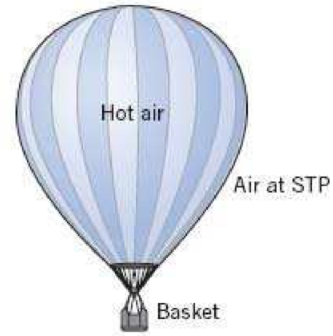
$$V = 1.15 \times 10^{-3} \cdot \frac{\text{ft}}{\text{s}} \qquad V = 0.825 \cdot \frac{\text{in}}{\text{min}}$$

As noted by Professor Kline in the film "Flow Visualization", bubbles rise slowly!

Problem *3.98

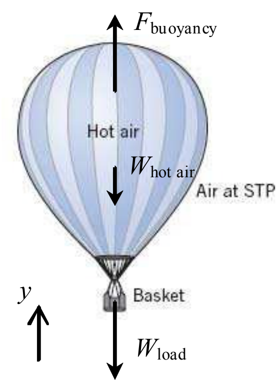
[Difficulty: 3]

3.98 It is desired to use a hot air balloon with a volume of 320,000 ft^3 for rides planned in summer morning hours when the air temperature is about 48°F. The torch will warm the air inside the balloon to a temperature of 160°F. Both inside and outside pressures will be "standard" (14.7 psia). How much mass can be carried by the balloon (basket, fuel, passengers, personal items, and the component of the balloon itself) if neutral buoyancy is to be assured? What mass can be carried by the balloon to ensure vertical takeoff acceleration of 2.5 ft/s^2? For this, consider that both balloon and inside air have to be accelerated, as well as some of the surrounding air (to make way for the balloon). The rule of thumb is that the total mass subject to acceleration is the mass of the balloon, all its appurtenances, and twice its volume of air. Given that the volume of hot air is fixed during the flight, what can the balloonists do when they want to go down?



Given: Data on hot air balloon

Find: Maximum mass of balloon for neutral buoyancy; mass for initial acceleration of 2.5 ft/s^2.



Assumptions: Air is treated as static and incompressible, and an ideal gas

Solution:

Basic equation $\quad F_B = \rho_{atm}\cdot g\cdot V \quad$ and $\quad \Sigma F_y = M\cdot a_y$

Hence $\quad \Sigma F_y = 0 = F_B - W_{hotair} - W_{load} = \rho_{atm}\cdot g\cdot V - \rho_{hotair}\cdot g\cdot V - M\cdot g \quad$ for neutral buoyancy

$$M = V\cdot\left(\rho_{atm} - \rho_{hotair}\right) = \frac{V\cdot p_{atm}}{R}\cdot\left(\frac{1}{T_{atm}} - \frac{1}{T_{hotair}}\right)$$

$$M = 320000\cdot ft^3 \times 14.7\cdot\frac{lbf}{in^2}\times\left(\frac{12\cdot in}{ft}\right)^2\times\frac{lbm\cdot R}{53.33\cdot ft\cdot lbf}\times\left[\frac{1}{(48+460)\cdot R} - \frac{1}{(160+460)\cdot R}\right] \qquad M = 4517\cdot lbm$$

Initial acceleration $\quad \Sigma F_y = F_B - W_{hotair} - W_{load} = \left(\rho_{atm} - \rho_{hotair}\right)\cdot g\cdot V - M_{new}\cdot g = M_{accel}\cdot a = \left(M_{new} + 2\cdot\rho_{hotair}\cdot V\right)\cdot a$

Solving for M_{new} $\quad \left(\rho_{atm} - \rho_{hotair}\right)\cdot g\cdot V - M_{new}\cdot g = \left(M_{new} + 2\cdot\rho_{hotair}\cdot V\right)\cdot a$

$$M_{new} = V\cdot\frac{\left(\rho_{atm} - \rho_{hotair}\right)\cdot g - 2\cdot\rho_{hotair}\cdot a}{a+g} = \frac{V\cdot p_{atm}}{a+g}\cdot\left[g\cdot\left(\frac{1}{T_{atm}} - \frac{1}{T_{hotair}}\right) - \frac{2\cdot a}{T_{hotair}}\right]$$

$$M_{new} = 320000\cdot ft^3\cdot 14.7\cdot\frac{lbf}{in^2}\cdot\left(\frac{12\cdot in}{ft}\right)^2\cdot\frac{lbm\cdot R}{53.33\cdot ft\cdot lbf}\cdot\frac{s^2}{(2.5+32.2)\cdot ft}\cdot\left[32.2\cdot\left[\frac{1}{(48+460)} - \frac{1}{(160+460)}\right] - 2\cdot 2.5\cdot\frac{1}{(160+460)}\right]\cdot\frac{ft}{s^2\cdot R}$$

$$M_{new} = 1239\cdot lbm$$

To make the balloon move up or down during flight, the air needs to be heated to a higher temperature, or let cool (or let in ambient air).

Problem *3.99

[Difficulty: 4]

3.99 Scientific balloons operating at pressure equilibrium with the surroundings have been used to lift instrument packages to extremely high altitudes. One such balloon, filled with helium, constructed of polyester with a skin thickness of 0.013 mm and a diameter of 120 m, lifted a payload of 230 kg. The specific gravity of the skin material is 1.28. Determine the altitude to which the balloon would rise. Assume that the helium used in the balloon is in thermal equilibrium with the ambient air, and that the balloon is a perfect sphere.

Given: Spherical balloon filled with helium lifted a payload of mass M=230 kg. At altitude, helium and air were in thermal equilibrium. Balloon diameter is 120 m and specific gravity of the skin material is 1.28.

Find: The altitude to which the balloon rose.

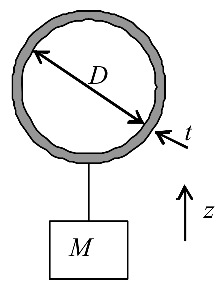
Solution: We will apply the hydrostatics equations to this system.

Governing Equations:

$$F_{buoy} = \rho \cdot g \cdot V_d$$ (Buoyant force is equal to mass of displaced fluid)

$$p = \rho \cdot R \cdot T$$ (Ideal gas equation of state)

Assumptions:
(1) Static, incompressible fluid
(2) Static equilibrium at 49 km altitude
(3) Ideal gas behavior



Taking a free body diagram of the balloon and payload: $\Sigma F_z = F_{buoy} - M_{He} \cdot g - M_s \cdot g - M \cdot g = 0$

Substituting for the buoyant force and knowing that mass is density times volume:

$$\rho_{air} \cdot g \cdot V_b - \rho_{He} \cdot g \cdot V_b - \rho_s \cdot g \cdot V_s - M \cdot g = 0 \qquad \rho_{air} \cdot V_b - \rho_{He} \cdot V_b - \rho_s \cdot V_s - M = 0$$

The volume of the balloon: $V_b = \frac{\pi}{6} \cdot D^3$ The volume of the skin: $V_s = \pi \cdot D^2 \cdot t$ Substituting these into the force equation:

$$\rho_{air} - \rho_{He} = \frac{6}{\pi \cdot D^3} \cdot \left(\pi \cdot \rho_s \cdot t \cdot D^2 + M\right)$$

From the ideal gas equation of state and remembering that pressure and temperature of the air and helium are equal:

$$\frac{p}{T} = \frac{6}{\pi \cdot D^3} \cdot \left(\pi \cdot \rho_s \cdot t \cdot D^2 + M\right) \cdot \frac{1}{\left(\frac{1}{R_{air}} - \frac{1}{R_{He}}\right)}$$

Substituting known values and consulting Appendix A for gas constants:

$$\frac{p}{T} = \frac{6}{\pi} \times \frac{1}{(120 \cdot m)^3} \times \left[\pi \times 1280 \cdot \frac{kg}{m^3} \times 0.013 \cdot 10^{-3} \cdot m \times (120 \cdot m)^2 + 230 \cdot kg\right] \times \frac{1}{\frac{1}{287} - \frac{1}{2080}} \cdot \frac{N \cdot m}{kg \cdot K} \times \frac{Pa \cdot m^2}{N} = 3.616 \times 10^{-4} \cdot \frac{kPa}{K}$$

To determine the altitude, we need to check this ratio against data from Table A.3. We find that the ratio of pressure to temperature matches the result above at:

$$h = 48.3 \cdot km$$

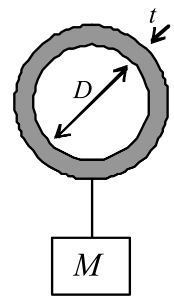
Problem *3.100

[Difficulty: 3]

3.100 A helium balloon is to lift a payload to an altitude of 40 km, where the atmospheric pressure and temperature are 3.0 mbar and −25°C, respectively. The balloon skin is polyester with specific gravity of 1.28 and thickness of 0.015 mm. To maintain a spherical shape, the balloon is pressurized to a gage pressure of 0.45 mbar. Determine the maximum balloon diameter if the allowable tensile stress in the skin is limited to 62 MN/m². What payload can be carried?

Given: A pressurized balloon is to be designed to lift a payload of mass M to an altitude of 40 km, where p = 3.0 mbar and T = -25 deg C. The balloon skin has a specific gravity of 1.28 and thickness 0.015 mm. The gage pressure of the helium is 0.45 mbar. The allowable tensile stress in the balloon is 62 MN/m²

Find: (a) The maximum balloon diameter
(b) The maximum payload mass



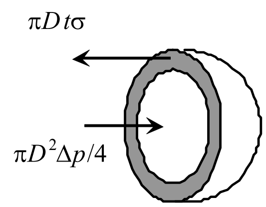
Solution: We will apply the hydrostatics equations to this system.

Governing Equations:

$$F_{buoy} = \rho \cdot g \cdot V_d$$ (Buoyant force is equal to mass of displaced fluid)

$$p = \rho \cdot R \cdot T$$ (Ideal gas equation of state)

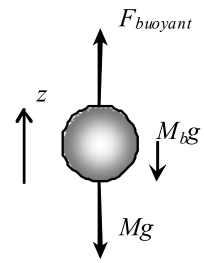
Assumptions:
(1) Static, incompressible fluid
(2) Static equilibrium at 40 km altitude
(3) Ideal gas behavior



The diameter of the balloon is limited by the allowable tensile stress in the skin:

$$\Sigma F = \frac{\pi}{4} \cdot D^2 \cdot \Delta p - \pi \cdot D \cdot t \cdot \sigma = 0$$ Solving this expression for the diameter: $$D_{max} = \frac{4 \cdot t \cdot \sigma}{\Delta p}$$

$$D_{max} = 4 \times 0.015 \times 10^{-3} \cdot m \times 62 \times 10^6 \cdot \frac{N}{m^2} \times \frac{1}{0.45 \cdot 10^{-3} \cdot bar} \times \frac{bar \cdot m^2}{10^5 \cdot N}$$ $$D_{max} = 82.7 m$$



To find the maximum allowable payload we perform a force balance on the system:

$$\Sigma F_z = F_{buoy} - M_{He} \cdot g - M_b \cdot g - M \cdot g = 0 \quad \rho_a \cdot g \cdot V_b - \rho_{He} \cdot g \cdot V_b - \rho_s \cdot g \cdot V_s - M \cdot g = 0$$

Solving for M: $M = (\rho_a - \rho_{He}) \cdot V_b - \rho_s \cdot V_s$ The volume of the balloon is: $V_b = \frac{\pi}{6} \cdot D^3$

The volume of the skin is: $V_s = \pi \cdot D^2 \cdot t$ Therefore, the mass is: $M = \frac{\pi}{6} \cdot (\rho_a - \rho_{He}) \cdot D^3 - \pi \cdot \rho_s \cdot D^2 \cdot t$

The air density: $\rho_a = \frac{p_a}{R_a \cdot T}$ $$\rho_a = 3.0 \times 10^{-3} \cdot bar \times \frac{kg \cdot K}{287 \cdot N \cdot m} \times \frac{1}{(273 - 25) \cdot K} \times \frac{10^5 \cdot N}{bar \cdot m^2}$$ $$\rho_a = 4.215 \times 10^{-3} \frac{kg}{m^3}$$

Repeating for helium: $\rho_{He} = \frac{p}{R \cdot T}$ $$\rho_{He} = 6.688 \times 10^{-4} \frac{kg}{m^3}$$

The payload mass is:

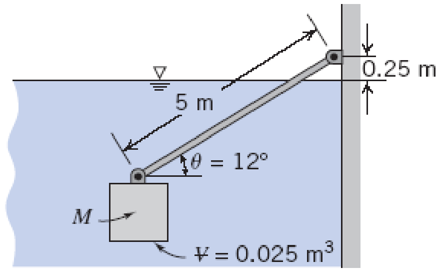
$$M = \frac{\pi}{6} \times (4.215 - 0.6688) \times 10^{-3} \cdot \frac{kg}{m^3} \times (82.7 \cdot m)^3 - \pi \times 1.28 \times 10^3 \cdot \frac{kg}{m^3} \times (82.7 \cdot m)^2 \times 0.015 \times 10^{-3} \cdot m$$

$$M = 638 kg$$

Problem *3.101

[Difficulty: 3]

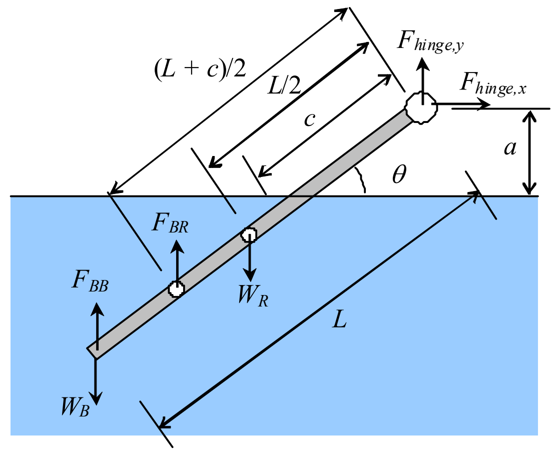
3.101 A block of volume 0.025 m^3 is allowed to sink in water as shown. A circular rod 5 m long and 20 cm^2 in cross-section is attached to the weight and also to the wall. If the rod mass is 1.25 kg and the rod makes an angle of 12 degrees with the horizontal at equilibrium, what is the mass of the block?



Given: Geometry of block and rod

Find: Angle for equilibrium

Assumptions: Water is static and incompressible



Solution:

Basic equations $\qquad \Sigma M_{Hinge} = 0 \qquad F_B = \rho \cdot g \cdot V \quad$ (Buoyancy)

The free body diagram is as shown. F_{BB} and F_{BR} are the buoyancy of the block and rod, respectively; c is the (unknown) exposed length of the rod

Taking moments about the hinge

$$\left(W_B - F_{BB}\right)\cdot L\cdot\cos(\theta) - F_{BR}\cdot\frac{(L+c)}{2}\cdot\cos(\theta) + W_R\cdot\frac{L}{2}\cdot\cos(\theta) = 0$$

with $\quad W_B = M_B\cdot g \qquad F_{BB} = \rho\cdot g\cdot V_B \qquad F_{BR} = \rho\cdot g\cdot(L-c)\cdot A \qquad W_R = M_R\cdot g$

Combining equations $\qquad \left(M_B - \rho\cdot V_B\right)\cdot L - \rho\cdot A\cdot(L-c)\cdot\frac{(L+c)}{2} + M_R\cdot\frac{L}{2} = 0$

We can solve for M_B $\qquad \rho\cdot A\cdot\left(L^2 - c^2\right) = 2\cdot\left(M_B - \rho\cdot V_B + \frac{1}{2}\cdot M_R\right)\cdot L$

$$M_B = \frac{\rho\cdot A}{2\cdot L}\cdot\left(L^2 - c^2\right) + \rho\cdot V_B - \frac{1}{2}\cdot M_R \qquad \text{and since} \qquad c = \frac{a}{\sin(\theta)} \qquad M_B = \frac{\rho\cdot A}{2\cdot L}\cdot\left[L^2 - \left(\frac{a}{\sin(\theta)}\right)^2\right] + \rho\cdot V_B - \frac{1}{2}\cdot M_R$$

$$M_B = \frac{1}{2}\times 1000\cdot\frac{kg}{m^3}\times 20\cdot cm^2\times\left(\frac{m}{100\cdot cm}\right)^2\times\frac{1}{5\cdot m}\cdot\left[(5\cdot m)^2 - \left(\frac{0.25\cdot m}{\sin(12\cdot deg)}\right)^2\right] + 1000\cdot\frac{kg}{m^3}\times 0.025\cdot m^3 - \frac{1}{2}\times 1.25\cdot kg$$

$$M_B = 29.1\ kg$$

Problem *3.102

[Difficulty: 3]

3.102 The stem of a glass hydrometer used to measure specific gravity is 5 mm in diameter. The distance between marks on the stem is 2 mm per 0.1 increment of specific gravity. Calculate the magnitude and direction of the error introduced by surface tension if the hydrometer floats in kerosene. (Assume the contact angle between kerosene and glass is 0°.)

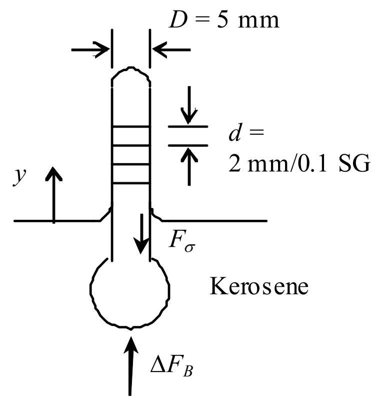
Given: Glass hydrometer used to measure SG of liquids. Stem has diameter D=5 mm, distance between marks on stem is d=2 mm per 0.1 SG. Hydrometer floats in kerosene (Assume zero contact angle between glass and kerosene).

Find: Magnitude of error introduced by surface tension.

Solution: We will apply the hydrostatics equations to this system.

Governing Equations: $F_{buoy} = \rho \cdot g \cdot V_d$ (Buoyant force is equal to weight of displaced fluid)

Assumptions:
(1) Static fluid
(2) Incompressible fluid
(3) Zero contact angle between ethyl alcohol and glass



The surface tension will cause the hydrometer to sink Δh lower into the liquid. Thus for this change:

$$\Sigma F_z = \Delta F_{buoy} - F_\sigma = 0$$

The change in buoyant force is: $\Delta F_{buoy} = \rho \cdot g \cdot \Delta V = \rho \cdot g \cdot \frac{\pi}{4} \cdot D^2 \cdot \Delta h$

The force due to surface tension is: $F_\sigma = \pi \cdot D \cdot \sigma \cdot \cos(\theta) = \pi \cdot D \cdot \sigma$

Thus, $\rho \cdot g \cdot \frac{\pi}{4} \cdot D^2 \cdot \Delta h = \pi \cdot D \cdot \sigma$ Upon simplification: $\frac{\rho \cdot g \cdot D \cdot \Delta h}{4} = \sigma$

Solving for Δh: $\Delta h = \frac{4 \cdot \sigma}{\rho \cdot g \cdot D}$ From Table A.2, SG = 1.43 and from Table A.4, σ = 26.8 mN/m

Therefore, $\Delta h = 4 \times 26.8 \times 10^{-3} \cdot \frac{N}{m} \times \frac{m^3}{1430 \cdot kg} \times \frac{s^2}{9.81 \cdot m} \times \frac{1}{5 \times 10^{-3} \cdot m} \times \frac{kg \cdot m}{s^2 \cdot N}$ $\Delta h = 1.53 \times 10^{-3} \, m$

So the change in specific gravity will be: $\Delta SG = 1.53 \times 10^{-3} \cdot m \times \frac{0.1}{2 \times 10^{-3} \cdot m}$ $\Delta SG = 0.0765$

From the diagram, surface tension acts to cause the hydrometer to float lower in the liquid. Therefore, surface tension results in an indicated specific gravity <u>smaller</u> than the actual specific gravity.

Problem *3.103

[Difficulty:4]

3.103 A sphere, of radius R, is partially immersed, to depth d, in a liquid of specific gravity SG. Obtain an algebraic expression for the buoyancy force acting on the sphere as a function of submersion depth d. Plot the results over the range of water depth $0 \le d \le 2R$.

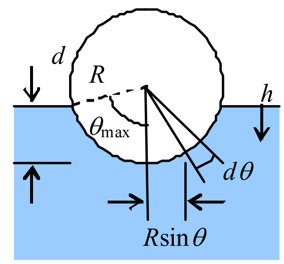
Given: Sphere partially immersed in a liquid of specific gravity SG.

Find: (a) Formula for buoyancy force as a function of the submersion depth d
(b) Plot of results over range of liquid depth

Solution: We will apply the hydrostatics equations to this system.

Governing Equations: $F_{buoy} = \rho \cdot g \cdot V_d$ (Buoyant force is equal to weight of displaced fluid)

Assumptions: (1) Static fluid
(2) Incompressible fluid
(3) Atmospheric pressure acts everywhere



We need an expression for the displaced volume of fluid at an arbitrary depth d. From the diagram we see that:

$d = R(1 - \cos(\theta_{max}))$ at an arbitrary depth h: $h = d - R \cdot (1 - \cos(\theta))$ $r = R \cdot \sin(\theta)$

So if we want to find the volume of the submerged portion of the sphere we calculate:

$$V_d = \int_0^{\theta_{max}} \pi r^2 \, dh = \pi \cdot \int_0^{\theta_{max}} R^2 \cdot (\sin(\theta))^2 \cdot R \cdot \sin(\theta) \, d\theta = \pi \cdot R^3 \cdot \int_0^{\theta_{max}} (\sin(\theta))^3 \, d\theta$$

Evaluating the integral we get:

$$V_d = \pi \cdot R^3 \cdot \left[\frac{(\cos(\theta_{max}))^3}{3} - \cos(\theta_{max}) + \frac{2}{3} \right]$$

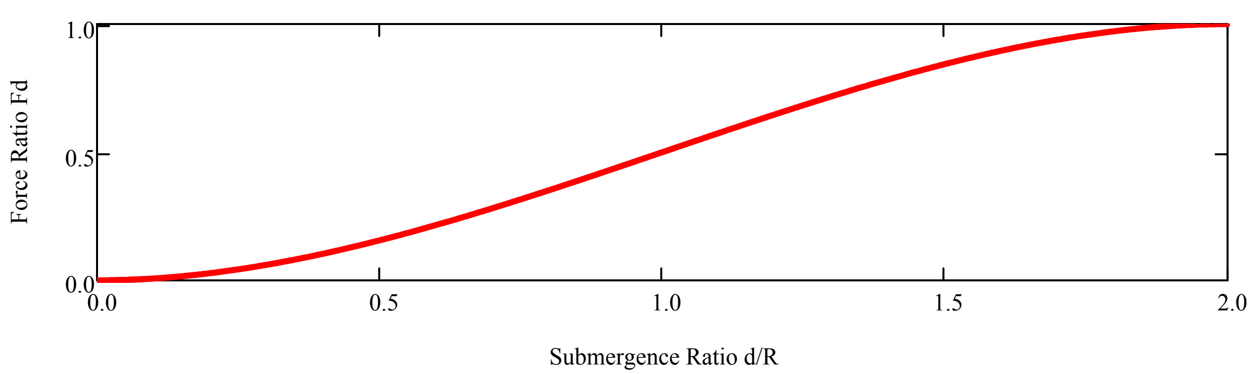
Now since: $\cos(\theta_{max}) = 1 - \frac{d}{R}$ we get: $V_d = \pi \cdot R^3 \cdot \left[\frac{1}{3} \left(1 - \frac{d}{R}\right)^3 - \left(1 - \frac{d}{R}\right) + \frac{2}{3} \right]$

Thus the buoyant force is: $F_{buoy} = \rho_w \cdot SG \cdot g \cdot \pi \cdot R^3 \cdot \left[\frac{1}{3} \left(1 - \frac{d}{R}\right)^3 - \left(1 - \frac{d}{R}\right) + \frac{2}{3} \right]$

If we non-dimensionalize by the force on a fully submerged sphere:

$$F_d = \frac{F_{buoy}}{\rho_w \cdot SG \cdot g \cdot \frac{4}{3} \cdot \pi \cdot R^3} = \frac{3}{4} \left[\frac{1}{3} \cdot \left(1 - \frac{d}{R}\right)^3 - \left(1 - \frac{d}{R}\right) + \frac{2}{3} \right]$$

$$F_d = \frac{3}{4} \left[\frac{1}{3} \cdot \left(1 - \frac{d}{R}\right)^3 - \left(1 - \frac{d}{R}\right) + \frac{2}{3} \right]$$



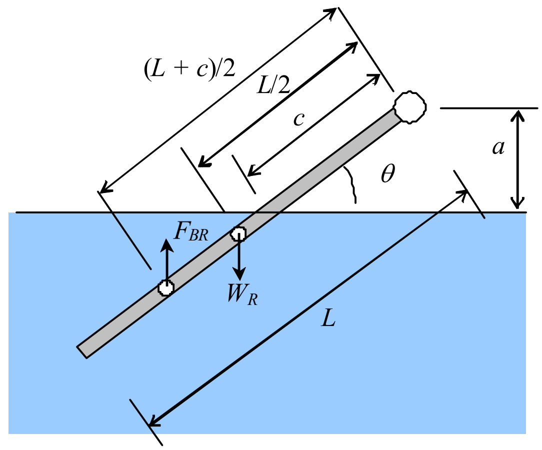
Problem 3.104

[Difficulty: 2]

3.104 If the mass M in Problem 3.101 is released from the rod, at equilibrium how much of the rod will remain submerged? What will be the minimum required upward force at the tip of the rod to just lift it out of the water?

Given: Geometry of rod

Find: How much of rod is submerged; force to lift rod out of water



Solution:

Basic equations $\qquad \Sigma M_{Hinge} = 0 \qquad F_B = \rho \cdot g \cdot V \quad$ (Buoyancy)

The free body diagram is as shown. F_{BR} is the buoyancy of the rod; c is the (unknown) exposed length of the rod

Taking moments about the hinge

$$-F_{BR} \cdot \frac{(L + c)}{2} \cdot \cos(\theta) + W_R \cdot \frac{L}{2} \cdot \cos(\theta) = 0$$

with $\qquad F_{BR} = \rho \cdot g \cdot (L - c) \cdot A \qquad W_R = M_R \cdot g$

Hence $\qquad -\rho \cdot A \cdot (L - c) \cdot \frac{(L + c)}{2} + M_R \cdot \frac{L}{2} = 0$

We can solve for c $\qquad \rho \cdot A \cdot \left(L^2 - c^2\right) = M_R \cdot L$

$$c = \sqrt{L^2 - \frac{L \cdot M_R}{\rho \cdot A}}$$

$$c = \sqrt{(5 \cdot m)^2 - 5 \cdot m \times \frac{m^3}{1000 \cdot kg} \times \frac{1}{20} \cdot \frac{1}{cm^2} \times \left(\frac{100 \cdot cm}{1 \cdot m}\right)^2 \times 1.25 \cdot kg}$$

$$c = 4.68\,m$$

Then the submerged length is $\qquad L - c = 0.323\,m$

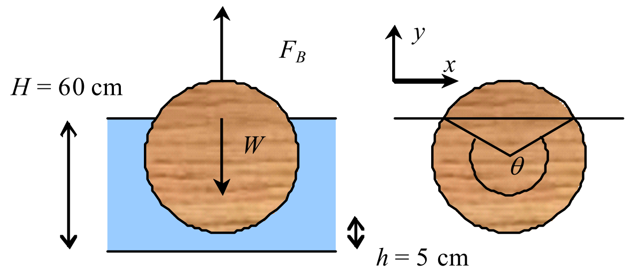
To lift the rod out of the water requires a force equal to half the rod weight (the reaction also takes half the weight)

$$F = \frac{1}{2} \cdot M_R \cdot g = \frac{1}{2} \times 1.25 \cdot kg \times 9.81 \cdot \frac{m}{s^2} \times \frac{N \cdot s^2}{kg \cdot m} \qquad F = 6.1\,N$$

Problem *3.105

[Difficulty: 2]

3.105 In a logging operation, timber floats downstream to a lumber mill. It is a dry year, and the river is running low, as low as 60 cm in some locations. What is the largest diameter log that may be transported in this fashion (leaving a minimum 5 cm clearance between the log and the bottom of the river)? For the wood, SG = 0.8.



Given: Data on river

Find: Largest diameter of log that will be transported

Solution:

Basic equation $\quad F_B = \rho \cdot g \cdot V_{sub} \quad$ and $\quad \Sigma F_y = 0 \quad \Sigma F_y = 0 = F_B - W$

where $\quad F_B = \rho \cdot g \cdot V_{sub} = \rho \cdot g \cdot A_{sub} \cdot L \quad W = SG \cdot \rho \cdot g \cdot V = SG \cdot \rho \cdot g \cdot A \cdot L$

From references (e.g. CRC Mathematics Handbook) $\quad A_{sub} = \frac{R^2}{2} \cdot (\theta - \sin(\theta))$ $\quad$ where R is the radius and θ is the included angle

Hence $\quad \rho \cdot g \cdot \frac{R^2}{2} \cdot (\theta - \sin(\theta)) \cdot L = SG \cdot \rho \cdot g \cdot \pi \cdot R^2 \cdot L$

$$\theta - \sin(\theta) = 2 \cdot SG \cdot \pi = 2 \times 0.8 \times \pi$$

This equation can be solved by manually iterating, or by using a good calculator, or by using Excel's Goal Seek

$$\theta = 239 \cdot deg$$

From geometry the submerged amount of a log is $\quad H - h \quad$ and also $\quad R + R \cdot \cos\left(\pi - \frac{\theta}{2}\right)$

Hence $\quad H - h = R + R \cdot \cos\left(\pi - \frac{\theta}{2}\right)$

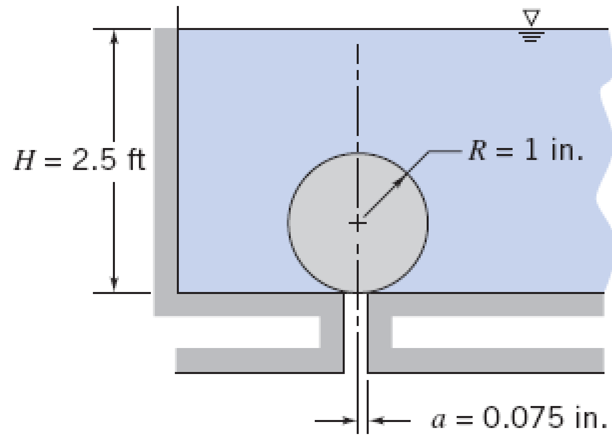
Solving for R $\quad R = \frac{H - h}{1 + \cos\left(180deg - \frac{\theta}{2}\right)} \quad R = \frac{(0.6 - 0.05) \cdot m}{1 + \cos\left[\left(180 - \frac{239}{2}\right) \cdot deg\right]} \quad R = 0.369\,m$

$D = 2 \cdot R \quad D = 0.737\,m$

Problem *3.106

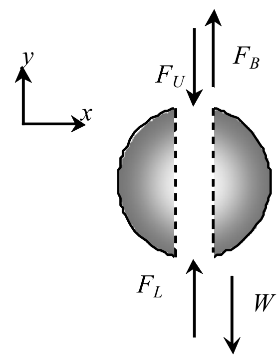
[Difficulty: 4]

3.106 A sphere of radius 1 in., made from material of specific gravity of SG = 0.95, is submerged in a tank of water. The sphere is placed over a hole of radius 0.075 in., in the tank bottom. When the sphere is released, will it stay on the bottom of the tank or float to the surface?



Given: Data on sphere and tank bottom

Find: Expression for SG of sphere at which it will float to surface; minimum SG to remain in position



Assumptions: (1) Water is static and incompressible
(2) Sphere is much larger than the hole at the bottom of the tank

Solution:

Basic equations $\quad F_B = \rho \cdot g \cdot V \quad$ and $\quad \Sigma F_y = F_L - F_U + F_B - W$

where $\quad F_L = p_{atm} \cdot \pi \cdot a^2 \qquad F_U = \left[p_{atm} + \rho \cdot g \cdot (H - 2 \cdot R)\right] \cdot \pi \cdot a^2$

$$F_B = \rho \cdot g \cdot V_{net} \qquad V_{net} = \frac{4}{3} \cdot \pi \cdot R^3 - \pi \cdot a^2 \cdot 2 \cdot R$$

$$W = SG \cdot \rho \cdot g \cdot V \quad \text{with} \quad V = \frac{4}{3} \cdot \pi \cdot R^3$$

Now if the sum of the vertical forces is positive, the sphere will float away, while if the sum is zero or negative the sphere will stay at the bottom of the tank (its weight and the hydrostatic force are greater than the buoyant force).

Hence $$\Sigma F_y = p_{atm} \cdot \pi \cdot a^2 - \left[p_{atm} + \rho \cdot g \cdot (H - 2 \cdot R)\right] \cdot \pi \cdot a^2 + \rho \cdot g \cdot \left(\frac{4}{3} \cdot \pi \cdot R^3 - 2 \cdot \pi \cdot R \cdot a^2\right) - SG \cdot \rho \cdot g \cdot \frac{4}{3} \cdot \pi \cdot R^3$$

This expression simplifies to $$\Sigma F_y = \pi \cdot \rho \cdot g \cdot \left[(1 - SG) \cdot \frac{4}{3} \cdot R^3 - H \cdot a^2\right]$$

$$\Sigma F_y = \pi \times 1.94 \cdot \frac{slug}{ft^3} \times 32.2 \cdot \frac{ft}{s^2} \times \left[\frac{4}{3} \times (1 - 0.95) \times \left(1 \cdot in \times \frac{ft}{12 \cdot in}\right)^3 - 2.5 \cdot ft \times \left(0.075 \cdot in \times \frac{ft}{12 \cdot in}\right)^2\right] \times \frac{lbf \cdot s^2}{slug \cdot ft}$$

$\Sigma F_y = -0.012 \cdot lbf \qquad$ Therefore, the sphere stays at the bottom of the tank.

Problem *3.107

[Difficulty: 4]

3.107 A cylindrical timber, with $D = 1$ ft and $L = 15$ ft, is weighted on its lower end so that it floats vertically with 10 ft submerged in seawater. When displaced vertically from its equilibrium position, the timber oscillates or "heaves" in a vertical direction upon release. Estimate the frequency of oscillation in this heave mode. Neglect viscous effects and water motion.

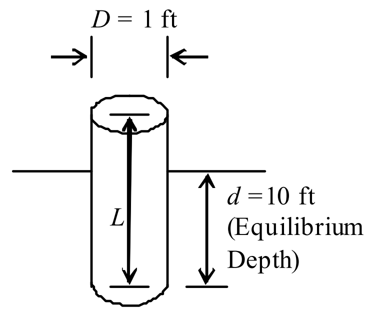
Given: Cylindrical timber, D = 1 ft and L = 15 ft, is weighted on the lower end so that is floats vertically with 10 ft submerged in sea water. When displaced vertically from equilibrium, the timber oscillates in a vertical direction upon release.

Find: Estimate the frequency of the oscillation. Neglect viscous forces or water motion.

Solution: We will apply the hydrostatics equations to this system.

Governing Equations: $F_{buoy} = \rho\cdot g\cdot V_d$ (Buoyant force is equal to weight of displaced fluid)

Assumptions:
(1) Static fluid
(2) Incompressible fluid
(3) Atmospheric pressure acts everywhere
(4) Viscous effects and water motion are negligible.



At equilibrium: $\Sigma F_y = F_{buoy} - M\cdot g = 0 \qquad M = \rho\cdot V_d = \rho\cdot A\cdot d$

Once the timber is displaced: $\Sigma F_y = F_{buoy} - M\cdot g = M\cdot\frac{d^2y}{dt^2}$

$\rho\cdot g\cdot A\cdot(d - y) - M\cdot g = M\cdot\frac{d^2y}{dt^2} \qquad \rho\cdot g\cdot A\cdot d - \rho\cdot g\cdot A\cdot y - \rho\cdot A\cdot d\cdot g = M\cdot\frac{d^2y}{dt^2}$

Thus we have the equation: $M\cdot\frac{d^2y}{dt^2} + \rho\cdot g\cdot A\cdot y = 0$ or: $\frac{d^2y}{dt^2} + \frac{\rho\cdot g\cdot A}{\rho\cdot A\cdot d}\cdot y = 0 \qquad \frac{d^2y}{dt^2} + \frac{g}{d}\cdot y = 0$

This ODE describes simple harmonic motion with the natural frequency ω described by: $\omega^2 = \frac{g}{d}$

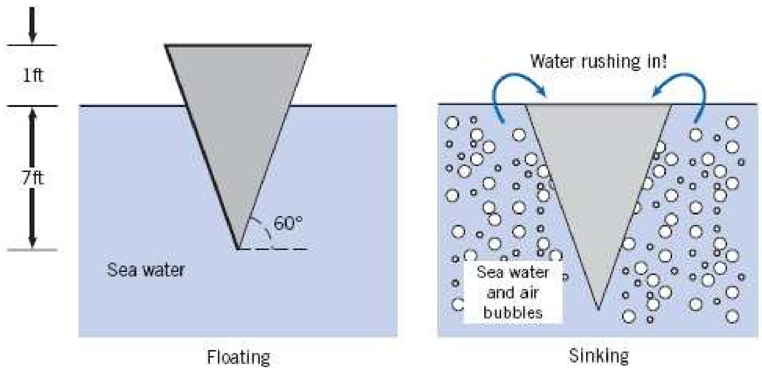
Solving for ω: $\omega = \sqrt{\frac{g}{d}} \qquad \omega = \sqrt{\frac{32.2\cdot ft}{s^2}\times\frac{1}{10\cdot ft}} \qquad \omega = 1.7944\frac{rad}{s}$

To express this as a frequency: $f = \frac{\omega}{2\cdot\pi} \qquad f = \frac{1.7944\frac{1}{s}}{2\cdot\pi} \qquad f = 0.286\,Hz$

Problem *3.108

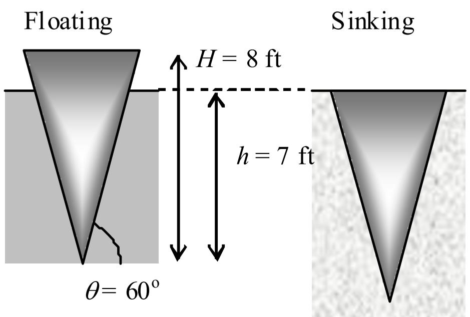
[Difficulty: 3]

3.108 You are in the Bermuda Triangle when you see a bubble plume eruption (a large mass of air bubbles, similar to a foam) off to the side of the boat. Do you want to head toward it and be part of the action? What is the effective density of the water and air bubbles in the drawing on the right that will cause the boat to sink? Your boat is 10 ft long, and weight is the same in both cases.



Given: Data on boat

Find: Effective density of water/air bubble mix if boat sinks



Solution:

Basic equations $\quad F_B = \rho \cdot g \cdot V \quad$ and $\quad \Sigma F_y = 0$

We can apply the sum of forces for the "floating" free body

$$\Sigma F_y = 0 = F_B - W \qquad \text{where} \qquad F_B = SG_{sea} \cdot \rho \cdot g \cdot V_{subfloat}$$

$$V_{subfloat} = \frac{1}{2} \cdot h \cdot \left(\frac{2 \cdot h}{\tan \cdot \theta} \right) \cdot L = \frac{L \cdot h^2}{\tan(\theta)} \qquad SG_{sea} = 1.024 \qquad \text{(Table A.2)}$$

Hence $\quad W = \dfrac{SG_{sea} \cdot \rho \cdot g \cdot L \cdot h^2}{\tan(\theta)} \qquad (1)$

We can apply the sum of forces for the "sinking" free body

$$\Sigma F_y = 0 = F_B - W \qquad \text{where} \qquad F_B = SG_{mix} \cdot \rho \cdot g \cdot V_{sub} \qquad V_{subsink} = \frac{1}{2} \cdot H \cdot \left(\frac{2 \cdot H}{\tan \cdot \theta} \right) \cdot L = \frac{L \cdot H^2}{\tan(\theta)}$$

Hence $\quad W = \dfrac{SG_{mix} \cdot \rho \cdot g \cdot L \cdot H^2}{\tan(\theta)} \qquad (2)$

Comparing Eqs. 1 and 2 $\quad W = \dfrac{SG_{sea} \cdot \rho \cdot g \cdot L \cdot h^2}{\tan(\theta)} = \dfrac{SG_{mix} \cdot \rho \cdot g \cdot L \cdot H^2}{\tan(\theta)}$

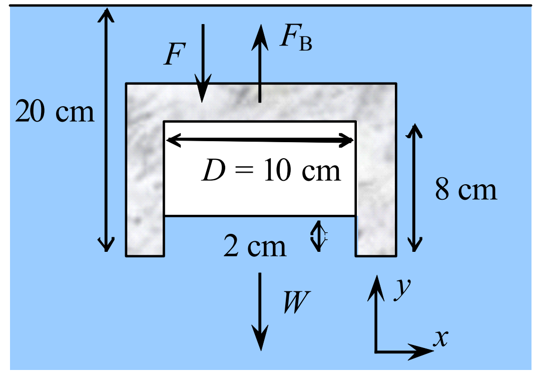
$$SG_{mix} = SG_{sea} \cdot \left(\frac{h}{H} \right)^2 \qquad SG_{mix} = 1.024 \times \left(\frac{7}{8} \right)^2 \qquad SG_{mix} = 0.784$$

The density is $\quad \rho_{mix} = SG_{mix} \cdot \rho \qquad \rho_{mix} = 0.784 \times 1.94 \cdot \dfrac{slug}{ft^3} \qquad \rho_{mix} = 1.52 \cdot \dfrac{slug}{ft^3}$

Problem *3.109

[Difficulty: 2]

3.109 A bowl is inverted symmetrically and held in a dense fluid, SG = 15.6, to a depth of 200 mm measured along the centerline of the bowl from the bowl rim. The bowl height is 80 mm, and the fluid rises 20 mm inside the bowl. The bowl is 100 mm inside diameter, and it is made from an old clay recipe, SG = 6.1. The volume of the bowl itself is about 0.9 L. What is the force required to hold it in place?



Given: Data on inverted bowl and dense fluid

Find: Force to hold in place

Assumption: Fluid is static and incompressible

Solution:

Basic equations $\quad F_B = \rho \cdot g \cdot V \quad$ and $\quad \Sigma F_y = 0 \quad \Sigma F_y = 0 = F_B - F - W$

Hence $\quad F = F_B - W$

For the buoyancy force $\quad F_B = SG_{fluid} \cdot \rho_{H2O} \cdot g \cdot V_{sub} \quad$ with $\quad V_{sub} = V_{bowl} + V_{air}$

For the weight $\quad W = SG_{bowl} \cdot \rho_{H2O} \cdot g \cdot V_{bowl}$

Hence $\quad F = SG_{fluid} \cdot \rho_{H2O} \cdot g \cdot \left(V_{bowl} + V_{air}\right) - SG_{bowl} \cdot \rho_{H2O} \cdot g \cdot V_{bowl}$

$$F = \rho_{H2O} \cdot g \cdot \left[SG_{fluid} \cdot \left(V_{bowl} + V_{air}\right) - SG_{bowl} \cdot V_{bowl}\right]$$

$$F = 999 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times \left[15.6 \times \left[0.9 \cdot L \times \frac{m^3}{1000 \cdot L} + (0.08 - 0.02) \cdot m \cdot \frac{\pi \cdot (0.1 \cdot m)^2}{4}\right] - 5.7 \times \left(0.9 \cdot L \times \frac{m^3}{1000 \cdot L}\right)\right] \times \frac{N \cdot s^2}{kg \cdot m}$$

$$F = 159.4\,N$$

Problem *3.110

[Difficulty: 4]

3.110 In the "Cartesian diver" child's toy, a miniature "diver" is immersed in a column of liquid. When a diaphragm at the top of the column is pushed down, the diver sinks to the bottom. When the diaphragm is released, the diver again rises. Explain how the toy might work.

Open-Ended Problem Statement: In the "Cartesian diver" child's toy, a miniature "diver" is immersed in a column of liquid. When a diaphragm at the top of the column is pushed down, the diver sinks to the bottom. When the diaphragm is released, the diver again rises. Explain how the toy might work.

Discussion: A possible scenario is for the toy to have a flexible bladder that contains air. Pushing down on the diaphragm at the top of the liquid column would increase the pressure at any point in the liquid. The air in the bladder would be compressed slightly as a result. The volume of the bladder, and therefore its buoyancy, would decrease, causing the diver to sink to the bottom of the liquid column.

Releasing the diaphragm would reduce the pressure in the water column. This would allow the bladder to expand again, increasing its volume and therefore the buoyancy of the diver. The increased buoyancy would permit the diver to rise to the top of the liquid column and float in a stable, partially submerged position, on the surface of the liquid.

Problem *3.111

[Difficulty: 4]

3.111 Consider a conical funnel held upside down and submerged slowly in a container of water. Discuss the force needed to submerge the funnel if the spout is open to the atmosphere. Compare with the force needed to submerge the funnel when the spout opening is blocked by a rubber stopper.

Open-Ended Problem Statement: Consider a conical funnel held upside down and submerged slowly in a container of water. Discuss the force needed to submerge the funnel if the spout is open to the atmosphere. Compare with the force needed to submerge the funnel when the spout opening is blocked by a rubber stopper.

Discussion: Let the weight of the funnel in air be W_a. Assume the funnel is held with its spout vertical and the conical section down. Then W_a will also be vertical.

Two possible cases are with the funnel spout open to atmosphere or with the funnel spout sealed.
With the funnel spout open to atmosphere, the pressures inside and outside the funnel are equal, so no net pressure force acts on the funnel. The force needed to support the funnel will remain constant until it first contacts the water. Then a buoyancy force will act vertically upward on every element of volume located beneath the water surface.

The first contact of the funnel with the water will be at the widest part of the conical section. The buoyancy force will be caused by the volume formed by the funnel thickness and diameter as it begins to enter the water. The buoyancy force will reduce the force needed to support the funnel. The buoyancy force will increase as the depth of submergence of the funnel increases until the funnel is fully submerged. At that point the buoyancy force will be constant and equal to the weight of water displaced by the volume of the material from which the funnel is made.

If the funnel material is less dense than water, it would tend to float partially submerged in the water. The force needed to support the funnel would decrease to zero and then become negative (i.e., down) to fully submerge the funnel.

If the funnel material were denser than water it would not tend to float even when fully submerged. The force needed to support the funnel would decrease to a minimum when the funnel became fully submerged, and then would remain constant at deeper submersion depths.
With the funnel spout sealed, air will be trapped inside the funnel. As the funnel is submerged gradually below the water surface, it will displace a volume equal to the volume of the funnel material plus the volume of trapped air. Thus its buoyancy force will be much larger than when the spout is open to atmosphere. Neglecting any change in air volume (pressures caused by submersion should be small compared to atmospheric pressure) the buoyancy force would be from the entire volume encompassed by the outside of the funnel. Finally, when fully submerged, the volume of the rubber stopper (although small) will also contribute to the total buoyancy force acting on the funnel.

Problem *3.112

[Difficulty: 2]

3.112 Three steel balls (each about half an inch in diameter) lie at the bottom of a plastic shell floating on the water surface in a partially filled bucket. Someone removes the steel balls from the shell and carefully lets them fall to the bottom of the bucket, leaving the plastic shell to float empty. What happens to the water level in the bucket? Does it rise, go down, or remain unchanged? Explain.

Given: Steel balls resting in floating plastic shell in a bucket of water

Find: What happens to water level when balls are dropped in water

Solution: Basic equation $F_B = \rho \cdot V_{disp} \cdot g = W$ for a floating body weight W

When the balls are in the plastic shell, the shell and balls displace a volume of water equal to their own weight - a large volume because the balls are dense. When the balls are removed from the shell and dropped in the water, the shell now displaces only a small volume of water, and the balls sink, displacing only their own volume. Hence the difference in displaced water before and after moving the balls is the difference between the volume of water that is equal to the weight of the balls, and the volume of the balls themselves. The amount of water displaced is significantly reduced, so the water level in the bucket drops.

Volume displaced before moving balls: $V_1 = \dfrac{W_{plastic} + W_{balls}}{\rho \cdot g}$

Volume displaced after moving balls: $V_2 = \dfrac{W_{plastic}}{\rho \cdot g} + V_{balls}$

Change in volume displaced $\Delta V = V_2 - V_1 = V_{balls} - \dfrac{W_{balls}}{\rho \cdot g} = V_{balls} - \dfrac{SG_{balls} \cdot \rho \cdot g \cdot V_{balls}}{\rho \cdot g}$

$$\Delta V = V_{balls} \cdot \left(1 - SG_{balls}\right)$$

Hence initially a large volume is displaced; finally a small volume is displaced ($\Delta V < 0$ because $SG_{balls} > 1$)

Problem *3.113

[Difficulty: 4]

3.113 A proposed ocean salvage scheme involves pumping air into "bags" placed within and around a wrecked vessel on the sea bottom. Comment on the practicality of this plan, supporting your conclusions with analyses.

Open-Ended Problem Statement: A proposed ocean salvage scheme involves pumping air into "bags" placed within and around a wrecked vessel on the sea bottom. Comment on the practicality of this plan, supporting your conclusions with analyses.

Discussion: This plan has several problems that render it impractical. First, pressures at the sea bottom are very high. For example, *Titanic* was found in about 12,000 ft of seawater. The corresponding pressure is nearly 6,000 psi. Compressing air to this pressure is possible, but would require a multi-stage compressor and very high power.

Second, it would be necessary to manage the buoyancy force after the bag and object are broken loose from the sea bed and begin to rise toward the surface. Ambient pressure would decrease as the bag and artifact rise toward the surface. The air would tend to expand as the pressure decreases, thereby tending to increase the volume of the bag. The buoyancy force acting on the bag is directly proportional to the bag volume, so it would increase as the assembly rises. The bag and artifact thus would tend to accelerate as they approach the sea surface. The assembly could broach the water surface with the possibility of damaging the artifact or the assembly.

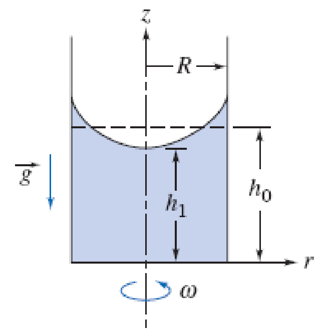
If the bag were of constant volume, the pressure inside the bag would remain essentially constant at the pressure of the sea floor, e.g., 6,000 psi for *Titanic*. As the ambient pressure decreases, the pressure differential from inside the bag to the surroundings would increase. Eventually the difference would equal sea floor pressure. This probably would cause the bag to rupture.

If the bag permitted some expansion, a control scheme would be needed to vent air from the bag during the trip to the surface to maintain a constant buoyancy force just slightly larger than the weight of the artifact in water. Then the trip to the surface could be completed at low speed without danger of broaching the surface or damaging the artifact.

Problem *3.114

[Difficulty: 3]

3.114 A cylindrical container, similar to that analyzed in Example 3.10 (on the Web), is rotated at a constant rate of 2 Hz about its axis. The cylinder is 0.5 m in diameter and initially contains water that is 0.3 m deep. Determine the height of the liquid free surface at the center of the container. Does your answer depend on the density of the liquid? Explain.



Given: Cylindrical container rotating as in Example 3.10

$R = 0.25 \cdot m \quad h_o = 0.3 \cdot m \qquad f = 2 \cdot Hz$

Find: (a) height of free surface at the entrance
(b) if solution depends on ρ

Solution: We will apply the hydrostatics equations to this system.

Governing Equations: $-\nabla p + \rho \vec{g} = \rho \vec{a}$ (Hydrostatic equation)

Assumptions: (1) Incompressible fluid
(2) Atmospheric pressure acts everywhere

In order to obtain the solution we need an expression for the shape of the free surface in terms of ω, r, and h_o. The required expression was derived in Example 3.10. The equation is:

$$z = h_o - \frac{(\omega \cdot R)^2}{2 \cdot g} \cdot \left[\frac{1}{2} - \left(\frac{r}{R}\right)^2\right]$$

The angular velocity ω is related to the frequency of rotation through: $\omega = 2 \cdot \pi \cdot f \qquad \omega = 2 \cdot \pi \times 2 \cdot \frac{rad}{s} = 12.57 \cdot \frac{rad}{s}$

Now since h_1 is the z value which corresponds to r = 0: $h_1 = h_o - \frac{(\omega \cdot R)^2}{4 \cdot g}$

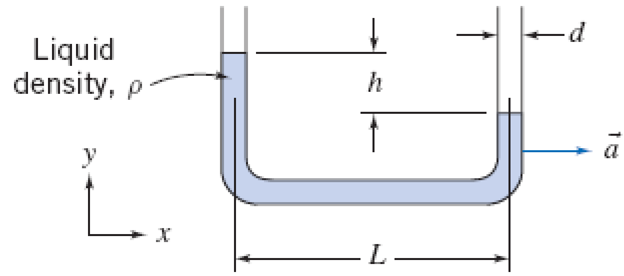
Substituting known values: $h_1 = 0.3 \cdot m - \frac{1}{4} \times \left(12.57 \cdot \frac{rad}{s} \times 0.25 \cdot m\right)^2 \times \frac{s^2}{9.81 \cdot m} \qquad h_1 = 0.05\,m$

The solution is independent of ρ because the equation of the free surface is independent of ρ as well.

Problem *3.115

[Difficulty: 2]

3.115 A crude accelerometer can be made from a liquid-filled U-tube as shown. Derive an expression for the liquid level difference h caused by an acceleration $\vec{a}$, in terms of the tube geometry and fluid properties.



Given: U-tube accelerometer

Find: Acceleration in terms of h and L

Solution: We will apply the hydrostatics equations to this system.

Governing Equations:

$$-\frac{\partial p}{\partial x} + \rho g_x = \rho a_x$$ (Hydrostatic equation in x-direction)

$$-\frac{\partial p}{\partial y} + \rho g_y = \rho a_y$$ (Hydrostatic equation in y-direction)

Assumptions:
(1) Incompressible fluid
(2) Neglect sloshing
(3) Ignore corners
(4) Both ends of U-tube are open to atmosphere

In the coordinate system we are using, we can see that: $a_x = a \quad a_y = 0 \quad g_x = 0 \quad g_y = -g$

Thus, $\frac{\partial p}{\partial x} = -\rho a \qquad \frac{\partial p}{\partial y} = -\rho g$ Now if we evaluate Δp from left to right in the U-tube: $dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy$

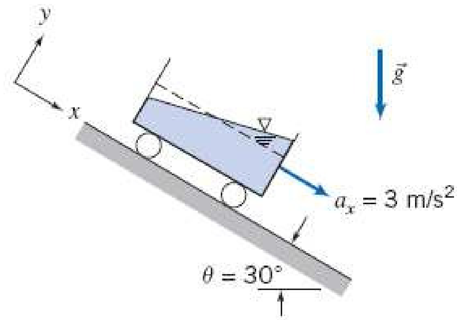
We may also write this expression as: $\Delta p = \frac{\partial p}{\partial x}\Delta x + \frac{\partial p}{\partial y}\Delta y \qquad \Delta p = (-\rho\cdot g)\cdot(-b) + (-\rho\cdot a)\cdot(-L) + (-\rho\cdot g)\cdot(b + h) = 0$

Simplifying this expression: $\Delta p = \rho\cdot a\cdot L - \rho\cdot g\cdot h = 0$ Solving for h: $h = \frac{a\cdot L}{g}$

Problem *3.116

[Difficulty: 2]

3.116 A rectangular container of water undergoes constant acceleration down an incline as shown. Determine the slope of the free surface using the coordinate system shown.



Given: Rectangular container with constant acceleration

Find: Slope of free surface

Solution: Basic equation $-\nabla p + \rho\vec{g} = \rho\vec{a}$

In components $\quad -\frac{\partial}{\partial x}p + \rho\cdot g_x = \rho\cdot a_x \qquad -\frac{\partial}{\partial y}p + \rho\cdot g_y = \rho\cdot a_y \qquad -\frac{\partial}{\partial z}p + \rho\cdot g_z = \rho\cdot a_z$

We have $\quad a_y = a_z = 0 \qquad g_x = g\cdot\sin(\theta) \qquad g_y = -g\cdot\cos(\theta) \qquad g_z = 0$

Hence $\quad -\frac{\partial}{\partial x}p + \rho\cdot g\cdot\sin(\theta) = \rho\cdot a_x \quad (1) \qquad -\frac{\partial}{\partial y}p - \rho\cdot g\cdot\cos(\theta) = 0 \quad (2) \qquad -\frac{\partial}{\partial z}p = 0 \quad (3)$

From Eq. 3 we can simplify from $\quad p = p(x,y,z) \quad$ to $\quad p = p(x,y)$

Hence a change in pressure is given by $\quad dp = \frac{\partial}{\partial x}p\cdot dx + \frac{\partial}{\partial y}p\cdot dy$

At the free surface p = const., so $\quad dp = 0 = \frac{\partial}{\partial x}p\cdot dx + \frac{\partial}{\partial y}p\cdot dy \quad$ or $\quad \frac{dy}{dx} = -\frac{\frac{\partial}{\partial x}p}{\frac{\partial}{\partial y}p} \quad$ at the free surface

Hence at the free surface, using Eqs 1 and 2

$$\frac{dy}{dx} = -\frac{\frac{\partial}{\partial x}p}{\frac{\partial}{\partial y}p} = \frac{\rho\cdot g\cdot\sin(\theta) - \rho\cdot a_x}{\rho\cdot g\cdot\cos(\theta)} = \frac{g\cdot\sin(\theta) - a_x}{g\cdot\cos(\theta)}$$

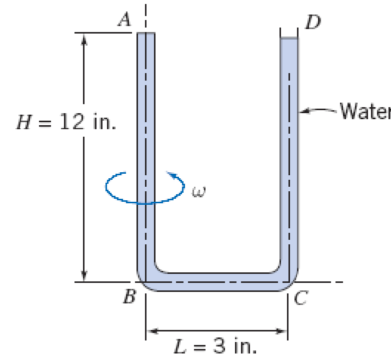
$$\frac{dy}{dx} = \frac{9.81\cdot(0.5)\cdot\frac{m}{s^2} - 3\cdot\frac{m}{s^2}}{9.81\cdot(0.866)\cdot\frac{m}{s^2}}$$

At the free surface, the slope is $\quad \frac{dy}{dx} = 0.224$

Problem *3.117

[Difficulty: 2]

3.117 The U-tube shown is filled with water at $T = 68°F$. It is sealed at A and open to the atmosphere at D. The tube is rotated about vertical axis AB at 1600 rpm. For the dimensions shown, would cavitation occur in the tube?



Given: Spinning U-tube sealed at one end

Find: Maximum angular speed for no cavitation

Assumptions: (1) water is incompressible
(2) constant angular velocity

Solution: Basic equation $-\nabla p + \rho \vec{g} = \rho \vec{a}$

In components $-\left(\frac{\partial}{\partial r}p\right) = \rho \cdot a_r = -\rho \cdot \frac{V^2}{r} = -\rho \cdot \omega^2 \cdot r \qquad \frac{\partial}{\partial z}p = -\rho \cdot g$

Between D and C, r = constant, so $\frac{\partial}{\partial z}p = -\rho \cdot g$ and so $p_D - p_C = -\rho \cdot g \cdot H$ (1)

Between B and A, r = constant, so $\frac{\partial}{\partial z}p = -\rho \cdot g$ and so $p_A - p_B = -\rho \cdot g \cdot H$ (2)

Between B and C, z = constant, so $\frac{\partial}{\partial r}p = \rho \cdot \omega^2 \cdot r$ and so $\int_{p_B}^{p_C} 1\, dp = \int_0^L \rho \cdot \omega^2 \cdot r\, dr$

Integrating $p_C - p_B = \rho \cdot \omega^2 \cdot \frac{L^2}{2}$ (3)

Since $p_D = p_{atm}$, then from Eq 1 $p_C = p_{atm} + \rho \cdot g \cdot H$

From Eq. 3 $p_B = p_C - \rho \cdot \omega^2 \cdot \frac{L^2}{2}$ so $p_B = p_{atm} + \rho \cdot g \cdot H - \rho \cdot \omega^2 \cdot \frac{L^2}{2}$

From Eq. 2 $p_A = p_B - \rho \cdot g \cdot H$ so $p_A = p_{atm} - \rho \cdot \omega^2 \cdot \frac{L^2}{2}$

Thus the minimum pressure occurs at point A (not B). Substituting known data to find the pressure at A:

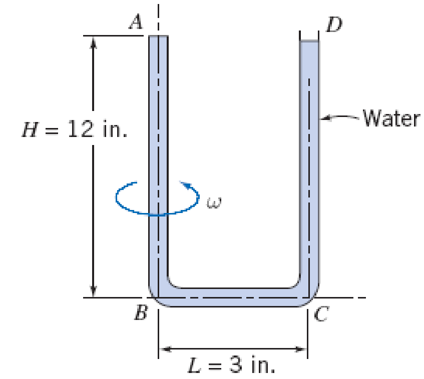
$$p_A = 14.7 \cdot \frac{lbf}{in^2} - 1.94 \cdot \frac{slug}{ft^3} \times \left(1600 \cdot \frac{rev}{min} \times \frac{2 \cdot \pi \cdot rad}{rev} \times \frac{min}{60 \cdot s}\right)^2 \times \frac{1}{2} \times \left(3 \cdot in \times \frac{ft}{12 \cdot in}\right)^2 \times \frac{lbf \cdot s^2}{slug \cdot ft} \times \left(\frac{ft}{12 \cdot in}\right)^2 = 2.881 \cdot \frac{lbf}{in^2}$$

At 68°F from steam tables, the vapor pressure of water is $p_v = 0.339 \cdot psi$ which is less than the pressure at A. Therefore, cavitation does not occur.:

Problem *3.118

[Difficulty: 2]

3.118 If the U-tube of Problem 3.117 is spun at 300 rpm, what will the pressure be at A? If a small leak appears at A, how much water will be lost at D?



Given: Spinning U-tube sealed at one end

Find: Pressure at A; water loss due to leak

Assumption: Water is incompressible; centripetal acceleration is constant

Solution: Basic equation $-\nabla p + \rho\vec{g} = \rho\vec{a}$

From the analysis of Example Problem 3.10, solving the basic equation, the pressure p at any point (r,z) in a continuous rotating fluid is given by

$$p = p_0 + \frac{\rho\cdot\omega^2}{2}\cdot\left(r^2 - r_0^2\right) - \rho\cdot g\cdot\left(z - z_0\right) \qquad (1)$$

where p_0 is a reference pressure at point (r_0,z_0)

In this case $\quad p = p_A \quad p_0 = p_D \quad z = z_A = z_D = z_0 = H \quad r = 0 \quad r_0 = r_D = L$

The speed of rotation is $\quad \omega = 300\cdot\text{rpm} \quad \omega = 31.4\cdot\frac{\text{rad}}{\text{s}}$

The pressure at D is $\quad p_D = 0\cdot\text{kPa} \quad$ (gage)

Hence

$$p_A = \frac{\rho\cdot\omega^2}{2}\cdot\left(-L^2\right) - \rho\cdot g\cdot(0) = -\frac{\rho\cdot\omega^2\cdot L^2}{2} = -\frac{1}{2}\times 1.94\cdot\frac{\text{slug}}{\text{ft}^3}\times\left(31.4\cdot\frac{\text{rad}}{\text{s}}\right)^2\times(3\cdot\text{in})^2\times\left(\frac{1\cdot\text{ft}}{12\cdot\text{in}}\right)^4\times\frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}}$$

$$p_A = -0.42\cdot\text{psi} \qquad \text{(gage)}$$

When the leak appears,the water level at A will fall, forcing water out at point D. Once again, from the analysis of Example Problem 3.10, we can use Eq 1

In this case $\quad p = p_A = 0 \quad p_0 = p_D = 0 \quad z = z_A \quad z_0 = z_D = H \quad r = 0 \quad r_0 = r_D = L$

Hence

$$0 = \frac{\rho\cdot\omega^2}{2}\cdot\left(-L^2\right) - \rho\cdot g\cdot\left(z_A - H\right)$$

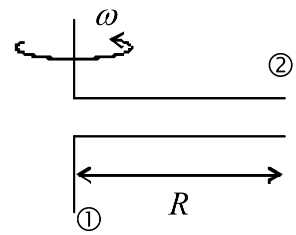
$$z_A = H - \frac{\omega^2\cdot L^2}{2\cdot g} = 12\text{in} - \frac{1}{2}\times\left(31.4\cdot\frac{\text{rad}}{\text{s}}\right)^2\times(3\cdot\text{in})^2\times\frac{\text{s}^2}{32.2\cdot\text{ft}}\times\frac{1\cdot\text{ft}}{12\cdot\text{in}} \qquad z_A = 0.52\cdot\text{in}$$

The amount of water lost is $\quad \Delta h = H - z_A = 12\cdot\text{in} - 0.52\cdot\text{in} \qquad \Delta h = 11.48\cdot\text{in}$

Problem *3.119

[Difficulty: 2]

3.119 A centrifugal micromanometer can be used to create small and accurate differential pressures in air for precise measurement work. The device consists of a pair of parallel disks that rotate to develop a radial pressure difference. There is no flow between the disks. Obtain an expression for pressure difference in terms of rotation speed, radius, and air density. Evaluate the speed of rotation required to develop a differential pressure of 8 μm of water using a device with a 50 mm radius.



Given: Centrifugal manometer consists of pair of parallel disks that rotate to develop a radial pressure difference. There is no flow between the disks.

Find: (a) an expression for the pressure difference, Δp, as a function of ω, R, and ρ.
(b) find ω if Δp = 8 μm H2O and R = 50 mm

Solution: We will apply the hydrostatics equations to this system.

Governing Equations:

$$-\nabla p + \rho\vec{g} = \rho\vec{a} \qquad \text{(Hydrostatic equation)}$$

$$-\frac{\partial p}{\partial r} + \rho g_r = \rho a_r \qquad \text{(Hydrostatic equation in radial direction)}$$

Assumptions:
(1) Incompressible fluid
(2) Standard air between disks
(3) Rigid body motion
(4) Radial direction is horizontal

For rigid body motion: $a_r = -\frac{V^2}{r} = -\frac{(r\cdot\omega)^2}{r} = -r\cdot\omega^2$ In addition, since r is horizontal: $g_r = 0$

Thus, the hydrostatic equation becomes: $\frac{\partial p}{\partial r} = \rho r \omega^2$

We can solve this expression by separating variables and integrating:

$$\Delta p = \rho\cdot\omega^2\cdot\int_0^R r\,dr$$ Evaluating the integral on the right hand side: $$\Delta p = \frac{\rho\cdot\omega^2\cdot R^2}{2}$$

Solving for the rotational frequency: $\omega = \sqrt{\frac{2\cdot\Delta p}{\rho\cdot R^2}}$ The pressure differential can be expressed as: $\Delta p = \rho\cdot g\cdot\Delta h$

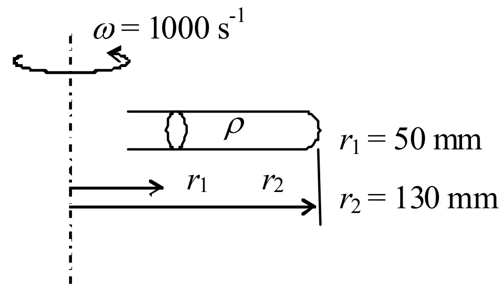
Therefore: $$\omega = \sqrt{2\cdot\frac{\rho_w}{\rho_{air}}\cdot\frac{g\cdot\Delta h}{R^2}}$$

Substituting in values: $$\omega = \sqrt{2\times\frac{999}{1.225}\times 9.81\cdot\frac{m}{s^2}\times 8\times 10^{-6}\cdot m\times\frac{1}{\left(50\times 10^{-3}\cdot m\right)^2}}$$ $$\omega = 7.16\cdot\frac{rad}{s}$$

Problem *3.120

[Difficulty: 2]

3.120 A test tube is spun in a centrifuge. The tube support is mounted on a pivot so that the tube swings outward as rotation speed increases. At high speeds, the tube is nearly horizontal. Find (a) an expression for the radial component of acceleration of a liquid element located at radius r, (b) the radial pressure gradient dp/dr, and (c) the required angular velocity to generate a pressure of 250 MPa in the bottom of a test tube containing water. (The free surface and bottom radii are 50 and 130 mm, respectively.)



Given: Test tube with water

Find:
(a) Radial acceleration
(b) Radial pressure gradient
(c) Rotational speed needed to generate 250 MPa pressure at the bottom of the tube

Solution: We will apply the hydrostatics equations to this system.

Governing Equations:

$$-\nabla p + \rho\vec{g} = \rho\vec{a} \qquad \text{(Hydrostatic equation)}$$

$$-\frac{\partial p}{\partial r} + \rho g_r = \rho a_r \qquad \text{(Hydrostatic equation in radial direction)}$$

Assumptions:
(1) Incompressible fluid
(2) Rigid body motion
(3) Radial direction is horizontal

For rigid body motion: $a_r = -\frac{V^2}{r} = -\frac{(r\cdot\omega)^2}{r} = -r\cdot\omega^2$ $\qquad a_r = -r\cdot\omega^2$

In addition, since r is horizontal: $g_r = 0$ Thus, the hydrostatic equation becomes: $\frac{\partial p}{\partial r} = \rho r\omega^2$

We can solve this expression by separating variables and integrating: $\Delta p = \rho\cdot\omega^2\cdot\int_{r_1}^{r_2} r\,dr$

Evaluating the integral on the right hand side: $\Delta p = \frac{\rho\cdot\omega^2}{2}\cdot\left(r_2^{\,2} - r_1^{\,2}\right)$ Solving for ω: $\omega = \sqrt{\frac{2\cdot\Delta p}{\rho\cdot\left(r_2^{\,2} - r_1^{\,2}\right)}}$

Substituting in values:

$$\omega = \sqrt{2 \times 250 \times 10^6\cdot\frac{N}{m^2} \times \frac{m^3}{999\cdot kg} \times \frac{1}{\left(130 \times 10^{-3}\cdot m\right)^2 - \left(50 \times 10^{-3}\cdot m\right)^2} \times \frac{kg\cdot m}{N\cdot s^2}} \times \frac{rev}{2\cdot\pi\cdot rad}$$

$\omega = 938\cdot Hz$

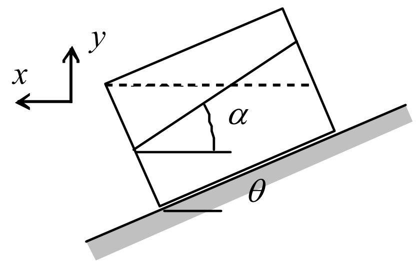
Problem *3.121

[Difficulty: 3]

3.121 A rectangular container, of base dimensions 0.4 m × 0.2 m and height 0.4 m, is filled with water to a depth of 0.2 m; the mass of the empty container is 10 kg. The container is placed on a plane inclined at 30° to the horizontal. If the coefficient of sliding friction between the container and the plane is 0.3, determine the angle of the water surface relative to the horizontal.

Given: Rectangular container of base dimensions 0.4 m x 0.2 m and a height of 0.4 m is filled with water to a depth of d = 0.2 m. Mass of empty container is M_c = 10 kg. The container slides down an incline of θ = 30 deg with respect to the horizontal. The coefficient of sliding friction is 0.30.

Find: The angle of the water surface relative to the horizontal.



Solution: We will apply the hydrostatics equations to this system.

Governing Equations:

$$-\nabla p + \rho\vec{g} = \rho\vec{a} \quad \text{(Hydrostatic equation)}$$

$$\vec{F} = M\vec{a} \quad \text{(Newton's Second Law)}$$

Assumptions:
(1) Incompressible fluid
(2) Rigid body motion

Writing the component relations: $-\frac{\partial p}{\partial x} = \rho a_x \qquad \frac{\partial p}{\partial x} = -\rho a_x \qquad -\frac{\partial p}{\partial y} - \rho g = \rho a_y \qquad \frac{\partial p}{\partial y} = -\rho\left(g + a_y\right)$

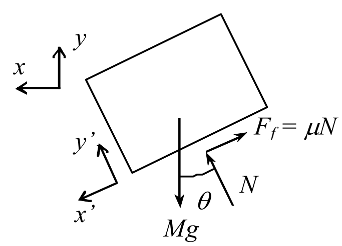
We write the total differential of pressure as: $dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy$ Now along the free surface of the water dp = 0. Thus:

$\frac{dy}{dx} = -\frac{\partial p/\partial x}{\partial p/\partial y} = -\frac{a_x}{g + a_y}$ and $\alpha = \mathrm{atan}\left(-\frac{dy}{dx}\right)$ To determine the acceleration components we analyze a free-body diagram:

$M = M_c + M_w = M_c + \rho_w \cdot V_w \qquad M = 10\cdot kg + 999\cdot\frac{kg}{m^3} \times 0.4\cdot m \times 0.2\cdot m \times 0.2\cdot m \qquad M = 25.98\,kg$

$\Sigma F_{y'} = 0 = N - M\cdot g\cdot\cos(\theta) \qquad N = M\cdot g\cdot\cos(\theta) \qquad N = 25.98\cdot kg \times 9.81\cdot\frac{m}{s^2} \times \cos(30\cdot deg) \times \frac{N\cdot s^2}{kg\cdot m} \qquad N = 220.7\,N$

$\Sigma F_{x'} = M\cdot a_{x'} = M\cdot g\cdot\sin(\theta) - F_f = M\cdot g\cdot\sin(\theta) - \mu\cdot N \qquad a_{x'} = g\cdot\sin(\theta) - \mu\cdot\frac{N}{M}$



$a_{x'} = 9.81\cdot\frac{m}{s^2} \times \sin(30\cdot deg) - 0.30 \times 220.7\cdot N \times \frac{1}{25.98\cdot kg} \times \frac{kg\cdot m}{N\cdot s^2} \qquad a_{x'} = 2.357\,\frac{m}{s^2}$

Now that we have the acceleration in the x'-y' system, we transform it to the x-y system: $a_x = a_{x'}\cdot\cos(\theta) \qquad a_y = -a_{x'}\cdot\sin(\theta)$

$a_x = 2.357\cdot\frac{m}{s^2} \times \cos(30\cdot deg) \qquad a_x = 2.041\,\frac{m}{s^2} \qquad a_y = -2.357\cdot\frac{m}{s^2} \times \sin(30\cdot deg) \qquad a_y = -1.178\,\frac{m}{s^2}$

Thus, $\alpha = \mathrm{atan}\left(\frac{2.041}{9.81 - 1.178}\right) \qquad \alpha = 13.30\cdot deg$

Problem *3.122

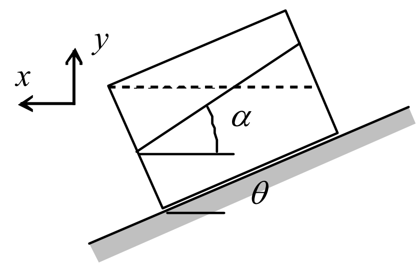
[Difficulty: 3]

3.122 If the container of Problem 3.121 slides without friction, determine the angle of the water surface relative to the horizontal. What is the slope of the free surface for the same acceleration up the plane?

Given: Rectangular container of base dimensions 0.4 m x 0.2 m and a height of 0.4 m is filled with water to a depth of d = 0.2 m. Mass of empty container is M_c = 10 kg. The container slides down an incline of θ = 30 deg with respect to the horizontal without friction.

Find: (a) The angle of the water surface relative to the horizontal.
(b) The slope of the free surface for the same acceleration up the plane.

Solution: We will apply the hydrostatics equations to this system.



Governing Equations:

$$-\nabla p + \rho\vec{g} = \rho\vec{a} \quad \text{(Hydrostatic equation)}$$

$$\vec{F} = M\vec{a} \quad \text{(Newton's Second Law)}$$

Assumptions: (1) Incompressible fluid
(2) Rigid body motion

Writing the component relations: $-\frac{\partial p}{\partial x} = \rho a_x \qquad \frac{\partial p}{\partial x} = -\rho a_x \qquad -\frac{\partial p}{\partial y} - \rho g = \rho a_y \qquad \frac{\partial p}{\partial y} = -\rho\left(g + a_y\right)$

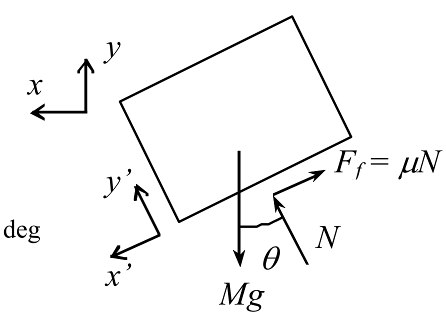
We write the total differential of pressure as: $dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy$ Now along the free surface of the water dp = 0. Thus:

$\frac{dy}{dx} = -\frac{\partial p/\partial x}{\partial p/\partial y} = -\frac{a_x}{g + a_y}$ and $\alpha = \operatorname{atan}\left(-\frac{dy}{dx}\right)$ To determine the acceleration components we analyze a free-body diagram:

$M = M_c + M_w = M_c + \rho_w \cdot V_w \qquad M = 10 \cdot kg + 999 \cdot \frac{kg}{m^3} \times 0.4 \cdot m \times 0.2 \cdot m \times 0.2 \cdot m \qquad M = 25.98\,kg$

$\Sigma F_{x'} = M \cdot a_{x'} = M \cdot g \cdot \sin(\theta) \qquad a_{x'} = g \cdot \sin(\theta) \qquad a_x = a_{x'} \cdot \cos(\theta) = g \cdot \sin(\theta) \cdot \cos(\theta)$

$a_y = -a_{x'} \cdot \sin(\theta) = g \cdot (\sin(\theta))^2$



Thus, $\frac{dy}{dx} = -\frac{g \cdot \sin(\theta) \cdot \cos(\theta)}{g\left[1 - (\sin(\theta))^2\right]} = -\frac{\sin(\theta) \cdot \cos(\theta)}{(\cos(\theta))^2} = -\frac{\sin(\theta)}{\cos(\theta)} = -\tan(\theta) \qquad \alpha = 30 \cdot deg$

For the acceleration up the incline: $a_x = -g \cdot \sin(\theta) \cdot \cos(\theta) \qquad a_y = g \cdot (\sin(\theta))^2$

Thus, $\text{slope} = \frac{g \cdot \sin(\theta) \cdot \cos(\theta)}{g\left[1 + (\sin(\theta))^2\right]} = -\frac{\sin(\theta) \cdot \cos(\theta)}{1 + (\sin(\theta))^2} \qquad \text{slope} = \frac{\sin(30 \cdot deg) \cdot \cos(30 \cdot deg)}{1 + (\sin(30 \cdot deg))^2} \qquad \text{slope} = 0.346$

Problem *3.123

[Difficulty: 3]

3.123 A cubical box, 80 cm on a side, half-filled with oil (SG = 0.80), is given a constant horizontal acceleration of 0.25 g parallel to one edge. Determine the slope of the free surface and the pressure along the horizontal bottom of the box.

Given: Cubical box with constant acceleration

Find: Slope of free surface; pressure along bottom of box

Solution: Basic equation $-\nabla p + \rho\vec{g} = \rho\vec{a}$

In components $\frac{\partial}{\partial x}p + \rho\cdot g_x = \rho\cdot a_x$ $\quad$ $\frac{\partial}{\partial y}p + \rho\cdot g_y = \rho\cdot a_y$ $\quad$ $\frac{\partial}{\partial z}p + \rho\cdot g_z = \rho\cdot a_z$

We have $a_x = a_x \quad g_x = 0$ $\quad$ $a_y = 0 \quad g_y = -g$ $\quad$ $a_z = 0 \quad g_z = 0$

Hence $\frac{\partial}{\partial x}p = -SG\cdot\rho\cdot a_x \quad (1)$ $\quad$ $\frac{\partial}{\partial y}p = -SG\cdot\rho\cdot g \quad (2)$ $\quad$ $\frac{\partial}{\partial z}p = 0 \quad (3)$

From Eq. 3 we can simplify from $p = p(x,y,z)$ to $p = p(x,y)$

Hence a change in pressure is given by $dp = \frac{\partial}{\partial x}p\cdot dx + \frac{\partial}{\partial y}p\cdot dy \quad (4)$

At the free surface p = const., so $dp = 0 = \frac{\partial}{\partial x}p\cdot dx + \frac{\partial}{\partial y}p\cdot dy$ or $\frac{dy}{dx} = -\frac{\frac{\partial}{\partial x}p}{\frac{\partial}{\partial y}p} = -\frac{a_x}{g} = -\frac{0.25\cdot g}{g}$

Hence at the free surface $\frac{dy}{dx} = -0.25$

The equation of the free surface is then $y = -\frac{x}{4} + C$ and through volume conservation the fluid rise in the rear balances the fluid fall in the front, so at the midpoint the free surface has not moved from the rest position

For size $L = 80\cdot cm$ at the midpoint $x = \frac{L}{2}$ $y = \frac{L}{2}$ (box is half filled) $\frac{L}{2} = -\frac{1}{4}\cdot\frac{L}{2} + C$ $\quad C = \frac{5}{8}\cdot L$ $\quad y = \frac{5}{8}\cdot L - \frac{x}{4}$

Combining Eqs 1, 2, and 4 $dp = -SG\cdot\rho\cdot a_x\cdot dx - SG\cdot\rho\cdot g\cdot dy$ or $p = -SG\cdot\rho\cdot a_x\cdot x - SG\cdot\rho\cdot g\cdot y + c$

We have $p = p_{atm}$ when $x = 0$ $y = \frac{5}{8}\cdot L$ so $p_{atm} = -SG\cdot\rho\cdot g\cdot\frac{5}{8}\cdot L + c$ $\quad c = p_{atm} + SG\cdot\rho\cdot g\cdot\frac{5}{8}\cdot L$

$$p(x,y) = p_{atm} + SG\cdot\rho\cdot\left(\frac{5}{8}\cdot g\cdot L - a_x\cdot x - g\cdot y\right) = p_{atm} + SG\cdot\rho\cdot g\cdot\left(\frac{5}{8}\cdot L - \frac{x}{4} - y\right)$$

On the bottom y = 0 so

$$p(x,0) = p_{atm} + SG\cdot\rho\cdot g\cdot\left(\frac{5}{8}\cdot L - \frac{x}{4}\right) = 101 + 0.8\times 1000\cdot\frac{kg}{m^3}\times\frac{N\cdot s^2}{kg\cdot m}\times 9.81\cdot\frac{m}{s^2}\times\left(\frac{5}{8}\times 0.8\cdot m - \frac{x}{4}\right)\times\frac{kPa}{10^3\cdot Pa}$$

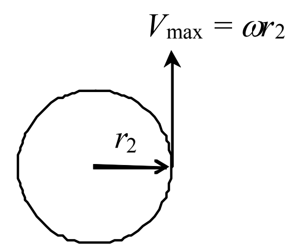
$$p(x,0) = 105 - 1.96\cdot x \quad \text{(p in kPa, x in m)}$$

Problem *3.124

[Difficulty: 3]

3.124 Gas centrifuges are used in one process to produce enriched uranium for nuclear fuel rods. The maximum peripheral speed of a gas centrifuge is limited by stress considerations to about 950 ft/s. Assume a gas centrifuge containing uranium hexafluoride gas, with molecular gas $M_m = 352$, and ideal gas behavior. Develop an expression for the ratio of maximum pressure to pressure at the centrifuge axis. Evaluate the pressure ratio for a gas temperature of 620°F.

Given: Gas centrifuge, with maximum peripheral speed Vmax = 950 ft/s contains uranium hexafluoride gas (M = 352 lb/lbmol) at 620 deg F.



Find: (a) Ratio of maximum pressure to pressure at the centrifuge axis
(b) Evaluate pressure ratio at 620 deg F.

Solution: We will apply the hydrostatics equations to this system.

Governing Equations:

$$-\nabla p + \rho\vec{g} = \rho\vec{a} \qquad \text{(Hydrostatic equation)}$$

$$-\frac{\partial p}{\partial r} + \rho g_r = \rho a_r \qquad \text{(Hydrostatic equation radial component)}$$

Assumptions:
(1) Incompressible fluid
(2) Rigid body motion
(3) Ideal gas behavior, constant temperature

For rigid body motion: $a_r = -\frac{V^2}{r} = -\frac{(r\cdot\omega)^2}{r} = -r\cdot\omega^2$ Thus: $\frac{\partial p}{\partial r} = -\rho a_r = \frac{p}{R_g T} r\omega^2$

Separating variables and integrating:

$$\int_{p_1}^{p_2}\frac{1}{p}\,dp = \frac{\omega^2}{R_g\cdot T}\cdot\int_0^{r_2} r\,dr \qquad \ln\left(\frac{p_2}{p_1}\right) = \frac{\omega^2}{R_g\cdot T}\cdot\frac{r_2^{\,2}}{2}$$

where we define: $V_{max} = \omega\cdot r_2$ therefore: $\ln\left(\frac{p_2}{p_1}\right) = \frac{V_{max}^{\,2}}{2\cdot R_g\cdot T}$

Solving for the pressure ratio: $p_{rat} = \frac{p_2}{p_1} = e^{\left(\frac{V_{max}^{\,2}}{2\cdot R_g\cdot T}\right)}$

The gas constant: $R_g = \frac{1545}{352}\cdot\frac{ft\cdot lbf}{lbm\cdot R}$ $R_g = 4.39\cdot\frac{ft\cdot lbf}{lbm\cdot R}$

Substituting in all known values:

$$p_{rat} = e^{\left[\left(950\cdot\frac{ft}{s}\right)^2\times\frac{1}{2}\times\frac{lbm\cdot R}{4.39\cdot ft\cdot lbf}\times\frac{1}{(620+460)\cdot R}\times\frac{lbf\cdot s^2}{32.2\cdot lbm\cdot ft}\right]}$$

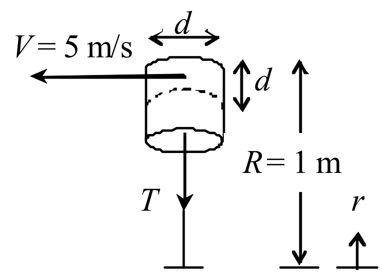
$p_{rat} = 19.2$

Problem *3.125

[Difficulty: 3]

3.125 A pail, 400 mm in diameter and 400 mm deep, weighs 15 N and contains 200 mm of water. The pail is swung in a vertical circle of 1-m radius at a speed of 5 m/s. Assume the water moves as a rigid body. At the instant when the pail is at the top of its trajectory, compute the tension in the string and the pressure on the bottom of the pail from the water.

Given: Pail is swung in a vertical circle. Water moves as a rigid body. Point of interest is the top of the trajectory.



Find: (a) Tension in the string
(b) Pressure on pail bottom from the water.

Solution: We will apply the hydrostatics equations to this system.

Governing Equations:

$$-\nabla p + \rho \vec{g} = \rho \vec{a}$$ (Hydrostatic equation)

$$-\frac{\partial p}{\partial r} + \rho g_r = \rho a_r$$ (Hydrostatic equation radial component)

Assumptions:
(1) Incompressible fluid
(2) Rigid body motion
(3) Center of mass of bucket and water are located at a radius of 1 m where V = rω = 5 m/s

Summing the forces in the radial direction: $-T - (M_b + M_w) \cdot g = (M_b + M_w) a_r$ where $a_r = -\frac{V^2}{r}$

Thus the tension is: $T = (M_b + M_w) \cdot \left(\frac{V^2}{r} - g\right)$ where: $M_b = 15 \cdot N \times \frac{s^2}{9.81 \cdot m} \times \frac{kg \cdot m}{N \cdot s^2}$ $M_b = 1.529 \cdot kg$

and: $M_w = \rho \cdot V = \rho \cdot \frac{\pi}{4} \cdot d^2 \cdot h$ $M_w = 999 \cdot \frac{kg}{m^3} \times \frac{\pi}{4} \times (0.4 \cdot m)^2 \times 0.2 \cdot m$ $M_w = 25.11 \cdot kg$

Now we find T: $T = (1.529 + 25.11) \cdot kg \times \left[\left(5 \cdot \frac{m}{s}\right)^2 \times \frac{1}{1 \cdot m} - 9.81 \cdot \frac{m}{s^2}\right] \times \frac{N \cdot s^2}{kg \cdot m}$ $T = 405 \cdot N$

If we apply this information to the radial hydrostatic equation we get: $-\frac{\partial p}{\partial r} - \rho g = -\rho \frac{V^2}{r}$ Thus: $\frac{\partial p}{\partial r} = \rho\left(\frac{V^2}{r} - g\right)$

If we assume that the radial pressure gradient is constant throughout the water, then the pressure gradient is equal to:

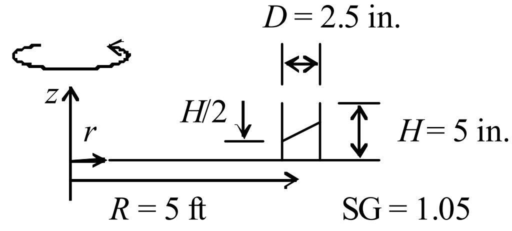
$$p_r = 999 \cdot \frac{kg}{m^3} \times \left[\left(5 \cdot \frac{m}{s}\right)^2 \times \frac{1}{1 \cdot m} - 9.81 \cdot \frac{m}{s^2}\right] \times \frac{N \cdot s^2}{kg \cdot m} \qquad p_r = 15.17 \cdot \frac{kPa}{m}$$

and we may calculate the pressure at the bottom of the bucket: $\Delta p = p_r \cdot \Delta r$ $\Delta p = 15.17 \cdot \frac{kPa}{m} \times 0.2 \cdot m$ $\Delta p = 3.03 \cdot kPa$

Problem *3.126

[Difficulty: 3]

3.126 A partially full can of soda is placed at the outer edge of a child's merry-go-round, located $R = 5$ ft from the axis of rotation. The can diameter and height are 2.5 in. and 5 in., respectively. The can is half full, and the soda has specific gravity SG = 1.05. Evaluate the slope of the liquid surface in the can if the merry-go-round spins at 20 rpm. Calculate the spin rate at which the can would spill, assuming no slippage between the can bottom and the merry-go-round. Would the can most likely spill or slide off the merry-go-round?



Given: Half-filled soft drink can at the outer edge of a merry-go-round

$$\omega = 0.3 \cdot \frac{\text{rev}}{\text{s}}$$

Find:
(a) Slope of free surface
(b) Spin rate necessary for spillage
(c) Likelihood of spilling versus slipping

Solution: We will apply the hydrostatics equations to this system.

Governing Equations:

$$-\nabla p + \rho\vec{g} = \rho\vec{a}$$ (Hydrostatic equation)

$$-\frac{\partial p}{\partial r} + \rho g_r = \rho a_r$$ (Hydrostatic equation radial component)

$$-\frac{\partial p}{\partial z} + \rho g_z = \rho a_z$$ (Hydrostatic equation z component)

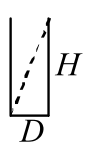
Assumptions:
(1) Incompressible fluid
(2) Rigid body motion
(3) Merry-go-round is horizontal

$$a_r = -\frac{V^2}{r} = -\frac{(r\cdot\omega)^2}{r} = -r\cdot\omega^2 \qquad a_z = 0 \qquad g_r = 0 \qquad g_z = -g$$ Thus: $$\frac{\partial p}{\partial r} = \rho r \omega^2 \qquad \frac{\partial p}{\partial z} = -\rho g$$ So $p = p(r,z)$

$$dp = \frac{\partial p}{\partial r}dr + \frac{\partial p}{\partial z}dz$$ For the free surface the pressure is constant. Therefore: $$\frac{dz}{dr} = -\frac{\partial p/\partial r}{\partial p/\partial z} = -\frac{\rho r \omega^2}{-\rho g} = \frac{r\omega^2}{g}$$

So the slope at the free surface is $$\text{slope} = 5\cdot\text{ft}\times\left(20\cdot\frac{\text{rev}}{\text{min}}\times\frac{\text{min}}{60\cdot\text{s}}\times\frac{2\cdot\pi\cdot\text{rad}}{\text{rev}}\right)^2\times\frac{\text{s}^2}{32.2\cdot\text{ft}} \qquad \text{slope} = 0.681$$

To spill, the slope must be



$$\text{slope}_{sp} = \frac{H}{D} \qquad \text{slope}_{sp} = \frac{5}{2.5} \qquad \text{slope}_{sp} = 2.000$$

Thus, $$\omega_{sp} = \sqrt{\frac{g}{r}\cdot\frac{dz}{dr}} \qquad \omega_{sp} = \sqrt{32.2\cdot\frac{\text{ft}}{\text{s}^2}\times\frac{1}{5\cdot\text{ft}}\times 2} \qquad \omega_{sp} = 3.59\cdot\frac{\text{rad}}{\text{s}}$$

This is nearly double the original speed (2.09 rad/s). Now the coefficient of static friction between the can and the surface of the merry-go-round is probably less than 0.5.Thus the can would not likely spill or tip; it would slide off!

Problem *3.127

[Difficulty: 2]

3.127 When a water polo ball is submerged below the surface in a swimming pool and released from rest, it is observed to pop out of the water. How would you expect the height to which it rises above the water to vary with depth of submersion below the surface? Would you expect the same results for a beach ball? For a table-tennis ball?

Discussion: Separate the problem into two parts: (1) the motion of the ball in water below the pool surface, and (2) the motion of the ball in air above the pool surface.

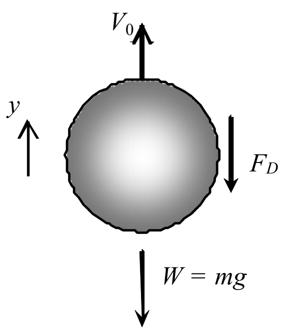
Below the pool water surface the motion of each ball is controlled by buoyancy force and inertia. For small depths of submersion ball speed upon reaching the surface will be small. As depth is increased, ball speed will increase until terminal speed in water is approached. For large depths, the actual depth will be irrelevant because the ball will reach terminal speed before reaching the pool water surface. All three balls are relatively light for their diameters, so terminal speed in water should be reached quickly. The depth of submersion needed to reach terminal speed should be fairly small, perhaps 1 meter or less (The initial water depth required to reach terminal speed may be calculated using the methods of Chapter 9).

Buoyancy is proportional to volume and inertia is proportional to mass. The ball with the largest volume per unit mass should accelerate most quickly to terminal speed. This will probably be the beach ball, followed by the table-tennis ball and the water polo ball.

The ball with the largest diameter has the smallest frontal area per unit volume; the terminal speed should be highest for this ball. Therefore, the beach ball should have the highest terminal speed, followed by the water polo ball and the table-tennis ball.

Above the pool water surface the motion of each ball is controlled by aerodynamic drag force, gravity force, and inertia (see the equation below). Without aerodynamic drag, the height above the pool water surface reached by each ball will depend only on its initial speed (The maximum height reached by a ball in air with aerodynamic drag may be calculated using the methods of Chapter 9). Aerodynamic drag reduces the height reached by the ball.

Aerodynamid drag is proportional to frontal area. The heaviest ball per unit frontal area (probably the water polo ball) should reach the maximum height and the lightest ball per unit area (probably the beach ball) should reach the minimum height.



$$\Sigma F_y = -F_D - m\cdot g = m\cdot a_y = m\cdot\frac{dV}{dt} = -C_D\cdot A\cdot\frac{1}{2}\cdot\rho\cdot V^2 - m\cdot g \quad \text{since} \quad F_D = C_D\cdot A\cdot\frac{1}{2}\cdot\rho\cdot V^2$$

Thus, $$-\frac{C_D\cdot A\cdot\frac{1}{2}\cdot\rho\cdot V^2}{m} - g = \frac{dV}{dt} = V\cdot\frac{dV}{dy} \quad (1)$$

We solve this by separating variables:

$$\frac{V\cdot dV}{1 + \frac{C_D\cdot A\cdot\rho}{m\cdot g}\cdot\frac{V^2}{2}} = -g\cdot dy$$

Integrating this expression over the flight of the ball yields:

$$-\frac{m\cdot g}{\rho\cdot C_D\cdot A}\cdot\ln\left(1 + \frac{\rho\cdot C_D\cdot A}{m\cdot g}\cdot\frac{V_o^2}{2}\right) = -g\cdot y_{max}$$

Solving for the maximum height: $$y_{max} = -\frac{m}{\rho\cdot C_D\cdot A}\cdot\ln\left(1 + \frac{\rho\cdot C_D\cdot A}{m\cdot g}\cdot\frac{V_o^2}{2}\right)$$ Simplifying: $$y_{max} = -\frac{m}{\rho\cdot C_D\cdot A}\cdot\ln\left(1 + \frac{F_{Do}}{m\cdot g}\right) \quad (2)$$

If we neglect drag, equation (1) becomes: $-g\cdot dy = V\cdot dV$ Integrating and solving for the maximum height: $$y_{max} = -\frac{V_o^2}{2\cdot g} \quad (3)$$

Checking the limiting value predicted by Eq (2) as $C_D \rightarrow 0$: we remember that for small x that ln(1+x) = -x. Thus:

$$\lim_{C_D\to 0} y_{max} = \lim_{C_D\to 0}\left(\frac{m}{\rho C_D A}\frac{\rho C_D A}{mg}\frac{V_o^2}{2}\right) = \frac{V_o^2}{2g}$$

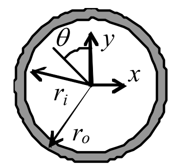
which is the result in Equation (3).

Problem *3.128

[Difficulty: 4]

3.128 Cast iron or steel molds are used in a horizontal-spindle machine to make tubular castings such as liners and tubes. A charge of molten metal is poured into the spinning mold. The radial acceleration permits nearly uniformly thick wall sections to form. A steel liner, of length $L = 6$ ft, outer radius $r_o = 6$ in., and inner radius $r_i = 4$ in., is to be formed by this process. To attain nearly uniform thickness, the angular velocity should be at least 300 rpm. Determine (a) the resulting radial acceleration on the inside surface of the liner and (b) the maximum and minimum pressures on the surface of the mold.

Given: A steel liner is to be formed in a spinning horizontal mold. To insure uniform thickness the minimum angular velocity should be at least 300 rpm. For steel, SG = 7.8



(gravity is downward in this diagram)

Find: (a) The resulting radial acceleration on the inside surface of the liner
(b) the maximum and minimum pressures on the surface of the mold

Solution: We will apply the hydrostatics equations to this system.

Governing Equations:

$$-\nabla p + \rho\vec{g} = \rho\vec{a}$$ (Hydrostatic equation)

$$-\frac{\partial p}{\partial r} + \rho g_r = \rho a_r$$ (Hydrostatic equation radial component)

$$-\frac{1}{r}\frac{\partial p}{\partial \theta} + \rho g_\theta = \rho a_\theta$$ (Hydrostatic equation transeverse component)

$$-\frac{\partial p}{\partial z} + \rho g_z = \rho a_z$$ (Hydrostatic equation z component)

Assumptions:
(1) Incompressible fluid
(2) Rigid body motion

$$a_r = -\frac{V^2}{r} = -\frac{(r\cdot\omega)^2}{r} = -r\cdot\omega^2 \qquad a_\theta = 0 \qquad a_z = 0 \qquad g_r = -g\cdot\cos(\theta) \qquad g_\theta = g\cdot\sin(\theta) \qquad g_z = 0$$

Hence: $$a_r = 4\cdot in\times\left(300\times\frac{rev}{min}\times\frac{2\cdot\pi\cdot rad}{rev}\times\frac{min}{60\cdot s}\right)^2\times\frac{ft}{12\cdot in} \qquad a_r = 329\cdot\frac{ft}{s^2} \qquad a_r = 10.23\cdot g$$

Thus: $$\frac{\partial p}{\partial r} = \rho g_r - \rho a_r = \rho r\omega^2 - \rho g\cos\theta \qquad \frac{\partial p}{\partial \theta} = \rho r g_\theta - \rho r a_\theta = \rho r g\sin\theta \qquad \frac{\partial p}{\partial z} = \rho g_z - \rho a_z = 0$$

$$dp = \frac{\partial p}{\partial r}dr + \frac{\partial p}{\partial \theta}d\theta = \left(\rho r\omega^2 - \rho g\cos\theta\right)dr + \left(\rho r g\sin\theta\right)d\theta$$

We can integrate to find pressure as a function of r and θ. $p(r_i,\theta) = p_{atm}$ $\left.\frac{\partial p}{\partial r}\right)_\theta = \rho r\omega^2 - \rho g\cos\theta$

Therefore, we integrate: $$p - p_{atm} = \int_{r_i}^{r}\left(\rho\cdot r\cdot\omega^2 - \rho\cdot g\cdot\cos(\theta)\right)dr + f(\theta)$$

$$p = p_{atm} + \rho\cdot\omega^2\cdot\frac{\left(r^2 - r_i^2\right)}{2} - \rho\cdot g\cdot\cos(\theta)\cdot\left(r - r_i\right) + f(\theta)$$

Taking the derivative of pressure with respect to θ:

$$\left.\frac{\partial p}{\partial \theta}\right)_r = \rho\left(r - r_i\right)g\sin\theta + \frac{df}{d\theta} = \rho r g \sin\theta$$

Thus, the integration function f(θ) is: $f(\theta) = -\rho\cdot g\cdot r_i\cdot\cos(\theta) + C$

Therefore, the pressure is:

$$p = p_{atm} + \rho\cdot\omega^2\cdot\frac{\left(r^2 - r_i^2\right)}{2} - \rho\cdot g\cdot\left(r - r_i\right)\cdot\cos(\theta) - \rho\cdot g\cdot r_i\cdot\cos(\theta) + C$$

The integration constant is determined from the boundary condition: $p\left(r_i, \theta\right) = p_{atm}$

$$p_{atm} = p_{atm} + \rho\cdot\omega^2\cdot\frac{\left(r_i^2 - r_i^2\right)}{2} - \rho\cdot g\cdot\left(r_i - r_i\right)\cdot\cos(\theta) - \rho\cdot g\cdot r_i\cdot\cos(\theta) + C \qquad -\rho\cdot g\cdot r_i\cdot\cos(\theta) + C = 0 \qquad C = \rho\cdot g\cdot r_i\cdot\cos(\theta)$$

Therefore, the pressure is:

$$p = p_{atm} + \rho\cdot\omega^2\cdot\frac{\left(r^2 - r_i^2\right)}{2} - \rho\cdot g\cdot\left(r - r_i\right)\cdot\cos(\theta)$$

The maximum pressure should occur on the mold surface at θ = π:

$$p_{maxgage} = \left[\left(7.8\cdot1.94\cdot\frac{slug}{ft^3}\right)\times\left(31.42\cdot\frac{rad}{s}\right)^2\times\frac{1}{2}\cdot\left(\frac{6^2 - 4^2}{12^2}\right)\cdot ft^2 - \left(7.8\cdot1.94\cdot\frac{slug}{ft^3}\right)\times 32.2\cdot\frac{ft}{s^2}\times\left(\frac{6-4}{12}\right)\cdot ft\cdot\cos(\pi)\right]\cdot\frac{lbf\cdot s^2}{slug\cdot ft}$$

$p_{maxgage} = 1119\cdot psf$ $\qquad p_{maxgage} = 7.77\cdot psi$

The minimum pressure should occur on the mold surface at θ = 0:

$$p_{mingage} = \left[\left(7.8\cdot1.94\cdot\frac{slug}{ft^3}\right)\times\left(31.42\cdot\frac{rad}{s}\right)^2\times\frac{1}{2}\cdot\left(\frac{6^2 - 4^2}{12^2}\right)\cdot ft^2 - \left(7.8\cdot1.94\cdot\frac{slug}{ft^3}\right)\times 32.2\cdot\frac{ft}{s^2}\times\left(\frac{6-4}{12}\right)\cdot ft\cdot\cos(0)\right]\cdot\frac{lbf\cdot s^2}{slug\cdot ft}$$

$p_{mingage} = 956\cdot psf$ $\qquad p_{mingage} = 6.64\cdot psi$

(In both results we divided by 144 to convert from psf to psi.)

Problem *3.129

[Difficulty: 4]

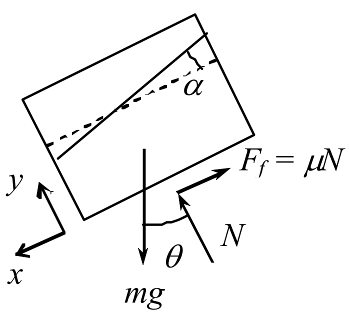
3.129 The analysis of Problem 3.121 suggests that it may be possible to determine the coefficient of sliding friction between two surfaces by measuring the slope of the free surface in a liquid-filled container sliding down an inclined surface. Investigate the feasibility of this idea.

Discussion: A certain minimum angle of inclination would be needed to overcome static friction and start the container into motion down the incline. Once the container is in motion, the retarding force would be provided by sliding (dynamic) friction. the coefficient of dynamic friction usually is smaller than the static friction coefficient. Thus the container would continue to accelerate as it moved down the incline. This acceleration would procide a non-zero slope to the free surface of the liquid in the container.

In principle the slope could be measured and the coefficent of dynamic friction calculated. In practice several problems would arise.

To calculate dynamic friction coefficient one must assume the liquid moves as a solid body, i.e., that there is no sloshing. This condition could only be achieved if there were nminimum initial disturbance and the sliding distance were long.

It would be difficult to measure the slope of the free surface of liquid in the moving container. Images made with a video camera or a digital still camera might be processed to obtain the required slope information.



$\Sigma F_y = 0 = N - M \cdot g \cdot \cos(\theta)$ $\qquad$ $N = M \cdot g \cdot \cos(\theta)$

$\Sigma F_x = M \cdot a_x = M \cdot g \cdot \sin(\theta) - F_f$ $\qquad$ $F_f = \mu_k \cdot N = \mu_k \cdot M \cdot g \cdot \cos \cdot (\theta)$

Thus the acceleration is:

$a_x = g \cdot \sin(\theta) - \mu_k \cdot g \cdot \cos(\theta)$ $\qquad$ Now for a static liquid: $\qquad$ $-\nabla p + \rho \vec{g} = \rho \vec{a}$

$$-\frac{\partial p}{\partial x} + \rho g \sin\theta = \rho a_x = \rho(g \sin\theta - \mu_k g \cos\theta) \qquad \frac{\partial p}{\partial x} = \rho g \mu_k \cos\theta$$

$$-\frac{\partial p}{\partial y} - \rho g \cos\theta = \rho a_x = 0 \qquad \frac{\partial p}{\partial y} = -\rho g \cos\theta$$

$$dp = \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy$$

For the free surface the pressure is constant. Therefore:

$$\frac{dy}{dx} = -\frac{\partial p / \partial x}{\partial p / \partial y} = -\frac{\rho g \mu_k \cos\theta}{-\rho g \cos\theta} = \mu_k$$

So the free surface angle is: $\alpha = \operatorname{atan}(\mu_k)$ Now since it is necessary to make the container slip along the surface,

$$\theta > \operatorname{atan}(\mu_s) > \operatorname{atan}(\mu_k) = \alpha$$

Thus, $\alpha < \theta$, as shown in the sketch above.

Problem 4.1

[Difficulty: 2]

4.1 A mass of 5 lbm is released when it is just in contact with a spring of stiffness 25 lbf/ft that is attached to the ground. What is the maximum spring compression? Compare this to the deflection if the mass was just resting on the compressed spring. What would be the maximum spring compression if the mass was released from a distance of 5 ft above the top of the spring?

Given:

Data on mass and spring

Find:

Maximum spring compression

Solution:

The given data is $M = 5 \cdot lb$ $\quad h = 5 \cdot ft$ $\quad k = 25 \cdot \frac{lbf}{ft}$

Apply the First Law of Thermodynamics: for the system consisting of the mass and the spring (the spring has gravitional potential energy and the spring elastic potential energy)

Total mechanical energy at initial state $\quad E_1 = 0$

Note: The datum for zero potential is the top of the uncompressed spring

Total mechanical energy at instant of maximum compression x $\quad E_2 = M \cdot g \cdot (-x) + \frac{1}{2} \cdot k \cdot x^2$

But $\quad E_1 = E_2$

so $\quad 0 = M \cdot g \cdot (-x) + \frac{1}{2} \cdot k \cdot x^2$

Solving for x $\quad x = \frac{2 \cdot M \cdot g}{k}$ $\quad x = 2 \times 5 \cdot lb \times 32.2 \cdot \frac{ft}{s^2} \times \frac{ft}{25 \cdot lbf} \times \frac{32.2 \cdot lb \cdot ft}{s^2 \cdot lbf}$ $\quad x = 0.401 \cdot ft$

When just resting on the spring $\quad x = \frac{M \cdot g}{k}$ $\quad x = 0.200\ ft$

When dropped from height h:

Total mechanical energy at initial state $\quad E_1 = M \cdot g \cdot h$

Total mechanical energy at instant of maximum compression x $\quad E_2 = M \cdot g \cdot (-x) + \frac{1}{2} \cdot k \cdot x^2$

Note: The datum for zero potential is the top of the uncompressed spring

But $\quad E_1 = E_2$

so $\quad M \cdot g \cdot h = M \cdot g \cdot (-x) + \frac{1}{2} \cdot k \cdot x^2$

Solving for x $$x^2 - \frac{2 \cdot M \cdot g}{k} \cdot x - \frac{2 \cdot M \cdot g \cdot h}{k} = 0$$

$$x = \frac{M \cdot g}{k} + \sqrt{\left(\frac{M \cdot g}{k}\right)^2 + \frac{2 \cdot M \cdot g \cdot h}{k}}$$

$$x = 5 \cdot lb \times 32.2 \cdot \frac{ft}{s^2} \times \frac{ft}{25 \cdot lbf} \times \frac{32.2 \cdot lb \cdot ft}{s^2 \cdot lbf} + \sqrt{\left(5 \cdot lb \times 32.2 \cdot \frac{ft}{s^2} \times \frac{ft}{25 \cdot lbf} \times \frac{32.2 \cdot lb \cdot ft}{s^2 \cdot lbf}\right)^2 + 2 \times 5 \cdot lb \times 32.2 \cdot \frac{ft}{s^2} \times \frac{ft}{25 \cdot lbf} \times \frac{32.2 \cdot lb \cdot ft}{s^2 \cdot lbf} \times 5 \cdot}$$

$$x = 1.63 \cdot ft$$

Note that ignoring the loss of potential of the mass due to spring compression x gives

$$x = \sqrt{\frac{2 \cdot M \cdot g \cdot h}{k}} \qquad x = 1.41 \cdot ft$$

Problem 4.2

[Difficulty: 2]

4.2 An ice-cube tray containing 250 mL of freshwater at 15°C is placed in a freezer at −5°C. Determine the change in internal energy (kJ) and entropy (kJ/K) of the water when it has frozen.

Given: An ice-cube tray with water at 15°C is frozen at –5°C.

Find: Change in internal energy and entropy

Solution: Apply the Tds and internal energy equations

Governing equations: $Tds = du + pdv$ $\qquad du = c_v dT$

Assumption: Neglect volume change
Liquid properties similar to water

The given or available data is:

$$T_1 = (15 + 273)\,\text{K} = 288\ \text{K} \qquad T_2 = (-5 + 273)\,\text{K} = 268\ \text{K}$$

$$c_v = 1\,\frac{\text{kcal}}{\text{kg}\cdot\text{K}} \qquad \rho = 999\,\frac{\text{kg}}{\text{m}^3}$$

Then with the assumption: $Tds = du + pdv = du = c_v dT$

or $$ds = c_v \frac{dT}{T}$$

Integrating $$s_2 - s_1 = c_v \ln\left(\frac{T_2}{T_1}\right) \quad \text{or} \quad \Delta S = m(s_2 - s_1) = \rho V c_v \ln\left(\frac{T_2}{T_1}\right)$$

$$\Delta S = 999\frac{\text{kg}}{\text{m}^3}\times 250\text{mL}\times\frac{10^{-6}\,\text{m}^3}{\text{mL}}\times 1\frac{\text{kcal}}{\text{kg}\cdot\text{K}}\times\ln\left(\frac{268}{288}\right)\times 4190\frac{\text{J}}{\text{kcal}}$$

$$\Delta S = -0.0753\frac{\text{kJ}}{\text{K}}$$

Also $$u_2 - u_1 = c_v(T_2 - T_1) \quad \text{or} \quad \Delta U = mc_v(T_2 - T_1) = \rho V c_v \Delta T$$

$$\Delta U = 999\frac{\text{kg}}{\text{m}^3}\times 250\text{mL}\times\frac{10^{-6}\,\text{m}^3}{\text{mL}}\times 1\frac{\text{kcal}}{\text{kg}\cdot\text{K}}\times(-268-288)\text{K}\times 4190\frac{\text{J}}{\text{kcal}}$$

$$\Delta U = -20.9\,\text{kJ}$$

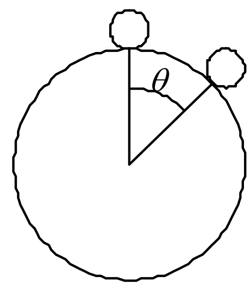
Problem 4.3

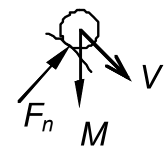
[Difficulty: 2]

4.3 A small steel ball of radius $r = 1$ mm is placed on top of a horizontal pipe of outside radius $R = 50$ mm and begins to roll under the influence of gravity. Rolling resistance and air resistance are negligible. As the speed of the ball increases, it eventually leaves the surface of the pipe and becomes a projectile. Determine the speed and location at which the ball loses contact with the pipe.

Given: Data on ball and pipe

Find: Speed and location at which contact is lost





Solution:

The given data is $\quad r = 1\cdot mm \qquad R = 50\cdot mm$

$$\sum F_n = F_n - m\cdot g\cdot\cos(\theta) = m\cdot a_n$$

$$a_n = -\frac{V^2}{R + r}$$

Contact is lost when $\quad F_n = 0 \quad$ or $\quad -m\cdot g\cdot\cos(\theta) = -m\cdot\frac{V^2}{R + r}$

$$V^2 = g\cdot(R + r)\cdot\cos(\theta) \qquad (1)$$

For no resistance energy is conserved, so $\quad E = m\cdot g\cdot z + m\cdot\frac{V^2}{2} = m\cdot g\cdot(R + r)\cdot\cos(\theta) + m\cdot\frac{V^2}{2} = E_0 = m\cdot g\cdot(R + r)$

Hence from energy considerations $\quad V^2 = 2\cdot g\cdot(R + r)\cdot(1 - \cos(\theta)) \qquad (2)$

Combining 1 and 2, $\quad V^2 = 2\cdot g\cdot(R + r)\cdot(1 - \cos(\theta)) = g\cdot(R + r)\cdot\cos(\theta) \quad$ or $\quad 2\cdot(1 - \cos(\theta)) = \cos(\theta)$

Hence $\quad \theta = \operatorname{acos}\left(\frac{2}{3}\right) \qquad \theta = 48.2\cdot deg$

Then from 1 $\quad V = \sqrt{(R + r)\cdot g\cdot\cos(\theta)} \qquad V = 0.577\,\frac{m}{s}$

Problem 4.4

[Difficulty: 2]

4.4 A fully loaded Boeing 777-200 jet transport aircraft weighs 325,000 kg. The pilot brings the 2 engines to full takeoff thrust of 450 kN each before releasing the brakes. Neglecting aerodynamic and rolling resistance, estimate the minimum runway length and time needed to reach a takeoff speed of 225 kph. Assume engine thrust remains constant during ground roll.

Given: Data on Boeing 777-200 jet

Find: Minimum runway length for takeoff

Solution:

Basic equation $\Sigma F_x = M \cdot \frac{dV}{dt} = M \cdot V \cdot \frac{dV}{dx} = F_t = \text{constant}$ Note that the "weight" is already in mass units!

Separating variables $M \cdot V \cdot dV = F_t \cdot dx$

Integrating $x = \frac{M \cdot V^2}{2 \cdot F_t}$

$$x = \frac{1}{2} \times 325 \times 10^3 \text{kg} \times \left(225 \frac{\text{km}}{\text{hr}} \times \frac{1 \cdot \text{km}}{1000 \cdot \text{m}} \times \frac{1 \cdot \text{hr}}{3600 \cdot \text{s}}\right)^2 \times \frac{1}{2 \times 425 \times 10^3} \cdot \frac{1}{\text{N}} \times \frac{\text{N} \cdot \text{s}^2}{\text{kg} \cdot \text{m}} \qquad x = 747 \text{ m}$$

For time calculation $M \cdot \frac{dV}{dt} = F_t$ $\qquad dV = \frac{F_t}{M} \cdot dt$

Integrating $t = \frac{M \cdot V}{F_t}$

$$t = 325 \times 10^3 \text{kg} \times 225 \frac{\text{km}}{\text{hr}} \times \frac{1 \cdot \text{km}}{1000 \cdot \text{m}} \times \frac{1 \cdot \text{hr}}{3600 \cdot \text{s}} \times \frac{1}{2 \times 425 \times 10^3} \cdot \frac{1}{\text{N}} \times \frac{\text{N} \cdot \text{s}^2}{\text{kg} \cdot \text{m}} \qquad t = 23.9 \text{ s}$$

Aerodynamic and rolling resistances would significantly increase both these results

Problem 4.5

[Difficulty: 2]

4.5 A police investigation of tire marks showed that a car traveling along a straight and level street had skidded to a stop for a total distance of 200 ft after the brakes were applied. The coefficient of friction between tires and pavement is estimated to be $\mu = 0.7$. What was the probable minimum speed (mph) of the car when the brakes were applied? How long did the car skid?

Given:
Car sliding to a stop

Find:
Initial speed; skid time

Solution:

Governing equations: $\Sigma F_x = M \cdot a_x \qquad F_f = \mu \cdot W$

Assumptions: Dry friction; neglect air resistance

Given data $\quad L = 200 \cdot ft \qquad \mu = 0.7$

$$\Sigma F_x = -F_f = -\mu \cdot W = M \cdot a_x = \frac{W}{g} \cdot a_x = \frac{W}{g} \cdot \frac{d^2}{dt^2} x$$

or
$$\frac{d^2}{dt^2} x = -\mu \cdot g$$

Integrating, and using I.C. $V = V_0$ at $t = 0$

Hence
$$\frac{dx}{dt} = -\mu \cdot g \cdot t + c_1 = -\mu \cdot g \cdot t + V_0 \qquad (1)$$

Integrating again
$$x = -\frac{1}{2} \cdot g \cdot t^2 + V_0 \cdot t + c_2 = -\frac{1}{2} \cdot g \cdot t^2 + V_0 \cdot t \qquad \text{since } x = 0 \text{ at } t = 0 \qquad (2)$$

We have the final state, at which $\quad x_f = L$ and $\quad \frac{dx}{dt} = 0 \quad$ at $\quad t = t_f$

From Eq. 1
$$\frac{dx}{dt} = 0 = -\mu \cdot g \cdot t_f + V_0 \qquad \text{so} \qquad t_f = \frac{V_0}{\mu \cdot g}$$

Substituting into Eq. 2
$$x = x_f = L = -\frac{1}{2} \cdot g \cdot t^2 + V_0 \cdot t = -\frac{1}{2} \cdot g \cdot t_f^2 + V_0 \cdot t_f = -\frac{1}{2} \cdot g \cdot \left(\frac{V_0}{\mu \cdot g}\right)^2 + V_0 \cdot \frac{V_0}{\mu \cdot g} = \frac{V_0^2}{2 \cdot \mu \cdot g}$$

Solving
$$L = \frac{V_0}{2 \cdot \mu \cdot g} \qquad \text{or} \qquad V_0 = \sqrt{2 \cdot \mu \cdot g \cdot L}$$

Using the data $\quad V_0 = 64.7 \cdot mph \qquad$ The skid time is $\quad t_f = \frac{V_0}{\mu \cdot g} \qquad t_f = 4.21\ s$

Problem 4.6

[Difficulty: 2]

4.6 A high school experiment consists of a block of mass 2 kg sliding across a surface (coefficient of friction $\mu = 0.6$). If it is given an initial velocity of 5 m/s, how far will it slide, and how long will it take to come to rest? The surface is now roughened a little, so with the same initial speed it travels a distance of 2 m. What is the new coefficient of friction, and how long does it now slide?

Given: Block sliding to a stop

Find: Distance and time traveled; new coeeficient of friction

Solution:

Governing equations: $\Sigma F_x = M \cdot a_x$ $\quad F_f = \mu \cdot W$

Assumptions: Dry friction; neglect air resistance

Given data $\mu = 0.6$ $\quad V_0 = 5 \cdot \frac{m}{s}$ $\quad M = 2 \cdot kg$ $\quad L = 2 \cdot m$

$$\Sigma F_x = -F_f = -\mu \cdot W = M \cdot a_x = \frac{W}{g} \cdot a_x = \frac{W}{g} \cdot \frac{d^2}{dt^2} x \qquad \text{or} \qquad \frac{d^2}{dt^2} x = -\mu \cdot g$$

Integrating, and using I.C. $V = V_0$ at $t = 0$

Hence $\quad \frac{dx}{dt} = -\mu \cdot g \cdot t + c_1 = -\mu \cdot g \cdot t + V_0 \qquad (1)$

Integrating again $\quad x = -\frac{1}{2} \cdot g \cdot t^2 + V_0 \cdot t + c_2 = -\frac{1}{2} \cdot g \cdot t^2 + V_0 \cdot t$ $\quad$ since $x = 0$ at $t = 0$ $\qquad (2)$

We have the final state, at which $\quad x_f = L$ and $\quad \frac{dx}{dt} = 0$ $\quad$ at $\quad t = t_f$

From Eq. 1 $\quad \frac{dx}{dt} = 0 = -\mu \cdot g \cdot t_f + V_0$ $\quad$ so $\quad t_f = \frac{V_0}{\mu \cdot g}$ $\quad$ Using given data $\quad t_f = 0.850\,s$

Substituting into Eq. 2 $\quad x = x_f = L = -\frac{1}{2} \cdot g \cdot t^2 + V_0 \cdot t = -\frac{1}{2} \cdot g \cdot t_f^2 + V_0 \cdot t_f = -\frac{1}{2} \cdot g \cdot \left(\frac{V_0}{\mu \cdot g}\right)^2 + V_0 \cdot \frac{V_0}{\mu \cdot g} = \frac{V_0^2}{2 \cdot \mu \cdot g}$

Solving $\quad x = \frac{V_0^2}{2 \cdot \mu \cdot g} \qquad (3)$ $\qquad$ Using give data $\quad x = 2.12\,m$

For rough surface, using Eq. 3 with $x = L$ $\quad \mu = \frac{V_0^2}{2 \cdot g \cdot L}$ $\quad \mu = 0.637$ $\quad t_f = \frac{V_0}{\mu \cdot g}$ $\quad t_f = 0.800\,s$

Problem 4.7

[Difficulty: 2]

4.7 A car traveling at 30 mph encounters a curve in the road. The radius of the road curve is 100 ft. Find the maximum speeds (mph) before losing traction, if the coefficient of friction on a dry road is $\mu_{dry} = 0.7$ and on a wet road is $\mu_{wet} = 0.3$.

Given: Car entering a curve

Find: Maximum speed

Solution:

Governing equations: $\Sigma F_r = M \cdot a_r$ $\quad F_f = \mu \cdot W$ $\quad a_r = \dfrac{V^2}{r}$

Assumptions: Dry friction; neglect air resistance

Given data $\quad \mu_{dry} = 0.7 \quad \mu_{wet} = 0.3 \quad r = 100 \cdot ft$

$$\Sigma F_r = -F_f = -\mu \cdot W = -\mu \cdot M \cdot g = M \cdot a_x = M \cdot \frac{V^2}{r}$$

or $\quad V = \sqrt{\mu \cdot r \cdot g}$

Hence, using given data $\quad V = \sqrt{\mu_{dry} \cdot r \cdot g} \quad V = 32.4 \cdot mph \quad V = \sqrt{\mu_{wet} \cdot r \cdot g} \quad V = 21.2 \cdot mph$

Problem 4.8

[Difficulty: 2]

4.8 Air at 20°C and an absolute pressure of 1 atm is compressed adiabatically in a piston-cylinder device, without friction, to an absolute pressure of 4 atm in a piston-cylinder device. Find the work done (MJ).

Given: Data on air compression process

Find: Work done

Solution:

Basic equation $\delta w = p \cdot dv$

Assumptions: 1) Adiabatic 2) Frictionless process $\delta w = pdv$

Given data $p_1 = 1 \cdot atm$ $p_2 = 4 \cdot atm$ $T_1 = 20\,°C$ $T_1 = 293\,K$

From Table A.6 $R = 286.9 \cdot \frac{J}{kg \cdot K}$ and $k = 1.4$

Before integrating we need to relate p and v. An adiabatic frictionless (reversible) process is isentropic, which for an ideal gas gives

$$p \cdot v^k = C \qquad \text{where} \qquad k = \frac{c_p}{c_v}$$

$$\delta w = p \cdot dv = C \cdot v^{-k} \cdot dv$$

Integrating
$$w = \frac{C}{k-1} \cdot \left(v_2^{1-k} - v_2^{1-k}\right) = \frac{1}{(k-1)} \cdot \left(p_2 \cdot v_2^k v_2^{1-k} - p_1 \cdot v_1^k \cdot v_2^{1-k}\right)$$

$$w = \frac{R}{(k-1)} \cdot \left(T_2 - T_1\right) = \frac{R \cdot T_1}{(k-1)} \cdot \left(\frac{T_2}{T_1} - 1\right) \qquad (1)$$

But $p \cdot v^k = C$ means $p_1 \cdot v_1^k = p_2 \cdot v_2^k$ or $p_1 \cdot \left(\frac{R \cdot T_1}{p_1}\right)^k = p_2 \cdot \left(\frac{R \cdot T_2}{p_2}\right)^k$

Rearranging
$$\frac{T_2}{T_1} = \left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}}$$

Combining with Eq. 1
$$w = \frac{R \cdot T_1}{k-1} \cdot \left[\left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}} - 1\right]$$

$$w = \frac{1}{0.4} \times 286.9 \cdot \frac{J}{kg \cdot K} \times (20 + 273)K \times \left[\left(\frac{4}{1}\right)^{\frac{1.4-1}{1.4}} - 1\right] \qquad w = 102\,\frac{kJ}{kg}$$

Problem 4.9

[Difficulty: 2]

4.9 In an experiment with a can of soda, it took 2 hr to cool from an initial temperature of 80°F to 45°F in a 35°F refrigerator. If the can is now taken from the refrigerator and placed in a room at 72°F, how long will the can take to reach 60°F? You may assume that for both processes the heat transfer is modeled by $\dot{Q} \approx k(T - T_{amb})$, where T is the can temperature, T_{amb} is the ambient temperature, and k is a heat transfer coefficient.

Given: Data on cooling of a can of soda in a refrigerator

Find: How long it takes to warm up in a room

Solution:

The First Law of Thermodynamics for the can (either warming or cooling) is

$$M \cdot c \cdot \frac{dT}{dt} = -k \cdot (T - T_{amb}) \qquad \text{or} \qquad \frac{dT}{dt} = -k \cdot (T - T_{amb})$$

where M is the can mass, c is the average specific heat of the can and its contents, T is the temperature, and T_{amb} is the ambient temperature

Separating variables $\qquad \dfrac{dT}{T - T_{amb}} = -A \cdot dt$

Integrating $\qquad T(t) = T_{amb} + (T_{init} - T_{amb}) \cdot e^{-A\,t}$

where T_{init} is the initial temperature. The available data from the coolling can now be used to obtain a value for constant A

Given data for cooling $\quad T_{init} = 80\,°F \qquad T_{init} = 540 \cdot R \qquad T_{amb} = 35\,°F \qquad T_{amb} = 495 \cdot R$

$T = 45\,°F \qquad T = 505 \cdot R \qquad$ when $\qquad \tau = 2 \cdot hr$

Hence $\qquad k = \dfrac{1}{\tau} \cdot \ln\left(\dfrac{T_{init} - T_{amb}}{T - T_{amb}}\right) = \dfrac{1}{2 \cdot hr} \times \dfrac{1 \cdot hr}{3600 \cdot s} \times \ln\left(\dfrac{540 - 495}{505 - 495}\right) \qquad k = 2.09 \times 10^{-4}\, s^{-1}$

Then, for the warming up process

$T_{init} = 45\,°F \qquad T_{init} = 505 \cdot R \qquad T_{amb} = 72\,°F \qquad T_{amb} = 532 \cdot R$

$T_{end} = 60\,°F \qquad T_{end} = 520 \cdot R$

with $\qquad T_{end} = T_{amb} + (T_{init} - T_{amb}) \cdot e^{-k\,\tau}$

Hence the time τ is $\qquad \tau = \dfrac{1}{k} \cdot \ln\left(\dfrac{T_{init} - T_{amb}}{T_{end} - T_{amb}}\right) = \dfrac{s}{2.09 \cdot 10^{-4}} \cdot \ln\left(\dfrac{505 - 532}{520 - 532}\right) \qquad \tau = 3.88 \times 10^{3}\, s \qquad \tau = 1.08 \cdot hr$

Problem 4.10

[Difficulty: 2]

4.10 A block of copper of mass 5 kg is heated to 90°C and then plunged into an insulated container containing 4 L of water at 10°C. Find the final temperature of the system. For copper, the specific heat is 385 J/kg·K, and for water the specific heat is 4186 J/kg·K.

Given: Data on heating and cooling a copper block

Find: Final system temperature

Solution:

Basic equation $Q - W = \Delta E$

Assumptions: 1) Stationary system $\Delta E = \Delta U$ 2) No work $W = 0$ 3) Adiabatic $Q = 0$

Then for the system (water and copper)

$$\Delta U = 0 \quad \text{or} \quad M_{copper} \cdot c_{copper} \cdot T_{copper} + M_w \cdot c_w \cdot T_w = \left(M_{copper} \cdot c_{copper} + M_w \cdot c_w\right) \cdot T_f \qquad (1)$$

where T_f is the final temperature of the water ($_w$) and copper ($_{copper}$)

The given data is $M_{copper} = 5 \cdot kg$ $\quad c_{copper} = 385 \cdot \frac{J}{kg \cdot K}$ $\quad c_w = 4186 \cdot \frac{J}{kg \cdot K}$ $\quad V = 4 \cdot L$

$T_{copper} = (90 + 273) \cdot K$ $\quad T_w = (10 + 273) \cdot K$

Also, for the water $\rho = 999 \cdot \frac{kg}{m^3}$ so $M_w = \rho \cdot V$ $\quad M_w = 4.00\,kg$

Solving Eq. 1 for T_f

$$T_f = \frac{M_{copper} \cdot c_{copper} \cdot T_{copper} + M_w \cdot c_w \cdot T_w}{\left(M_{copper} \cdot c_{copper} + M_w \cdot c_w\right)}$$

$T_f = 291\,K$ $\qquad T_f = 18.1 \cdot °C$

Problem 4.11

[Difficulty: 2]

4.11 The average rate of heat loss from a person to the surroundings when not actively working is about 85 W. Suppose that in an auditorium with volume of approximately $3.5 \times 10^5\ m^3$, containing 6000 people, the ventilation system fails. How much does the internal energy of the air in the auditorium increase during the first 15 min after the ventilation system fails? Considering the auditorium and people as a system, and assuming no heat transfer to the surroundings, how much does the internal energy of the system change? How do you account for the fact that the temperature of the air increases? Estimate the rate of temperature rise under these conditions.

Given: Data on heat loss from persons, and people-filled auditorium

Find: Internal energy change of air and of system; air temperature rise

Solution:

Basic equation $\quad Q - W = \Delta E$

Assumptions: 1) Stationary system $\Delta E = \Delta U$ 2) No work $W = 0$

Then for the air $\quad \Delta U = Q = 85 \cdot \frac{W}{person} \times 6000 \cdot people \times 15 \cdot min \times \frac{60 \cdot s}{min} \qquad \Delta U = 459 \cdot MJ$

For the air and people $\quad \Delta U = Q_{surroundings} = 0$

The increase in air energy is equal and opposite to the loss in people energy

For the air $\quad \Delta U = Q \quad$ but for air (an ideal gas) $\quad \Delta U = M \cdot c_V \cdot \Delta T \quad$ with $\quad M = \rho \cdot V = \frac{p \cdot V}{R_{air} \cdot T}$

Hence $\quad \Delta T = \frac{Q}{M \cdot c_V} = \frac{R_{air} \cdot Q \cdot T}{c_V \cdot p \cdot V}$

From Table A.6 $\quad R_{air} = 286.9 \cdot \frac{J}{kg \cdot K} \quad$ and $\quad c_V = 717.4 \cdot \frac{J}{kg \cdot K}$

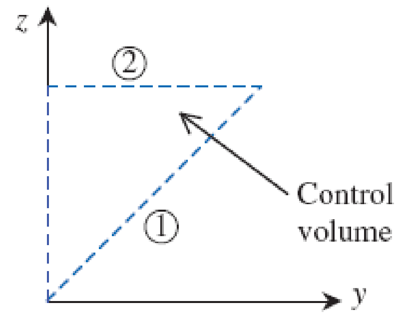
$$\Delta T = \frac{286.9}{717.4} \times 459 \times 10^6 \cdot J \times (20 + 273)K \times \frac{1}{101 \times 10^3} \cdot \frac{m^2}{N} \times \frac{1}{3.5 \times 10^5} \cdot \frac{1}{m^3} \qquad \Delta T = 1.521\ K$$

This is the temperature change in 15 min. The rate of change is then $\quad \frac{\Delta T}{15 \cdot min} = 6.09 \cdot \frac{K}{hr}$

Problem 4.12

[Difficulty: 3]

4.12 The velocity field in the region shown is given by $\vec{V} = (a\hat{j} + by\hat{k})$ where $a = 10$ m/s and $b = 5\ \text{s}^{-1}$. For the 1 m × 1 m triangular control volume (depth $w = 1$ m perpendicular to the diagram), an element of area ① may be represented by $d\vec{A}_1 = wdz\hat{j} - wdy\hat{k}$ and an element of area ② by $d\vec{A}_2 = -wdy\hat{k}$.
(a) Find an expression for $\vec{V} \cdot dA_1$.
(b) Evaluate $\int_{A_1} \vec{V} \cdot dA_1$.
(c) Find an expression for $\vec{V} \cdot dA_2$.
(d) Find an expression for $\vec{V}(\vec{V} \cdot dA_2)$.
(e) Evaluate $\int_{A_2} \vec{V}(\vec{V} \cdot dA_2)$.



Given: Data on velocity field and control volume geometry

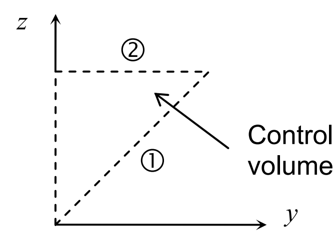
Find: Several surface integrals

Solution:

$$d\vec{A}_1 = wdz\hat{j} - wdy\hat{k} \qquad d\vec{A}_1 = dz\hat{j} - dy\hat{k}$$

$$d\vec{A}_2 = -wdy\hat{k} \qquad d\vec{A}_2 = -dy\hat{k}$$

$$\vec{V} = \left(a\hat{j} + by\hat{k}\right) \qquad \vec{V} = \left(10\,\hat{j} + 5y\hat{k}\right)$$



(a) $$\vec{V} \cdot dA_1 = \left(10\,\hat{j} + 5y\hat{k}\right)\cdot\left(dz\hat{j} - dy\hat{k}\right) = 10dz - 5ydy$$

(b) $$\int_{A_1} \vec{V} \cdot dA_1 = \int_0^1 10dz - \int_0^1 5ydy = 10z\Big|_0^1 - \frac{5}{2}y^2\Big|_0^1 = 7.5$$

(c) $$\vec{V} \cdot dA_2 = \left(10\,\hat{j} + 5y\hat{k}\right)\cdot\left(-dy\hat{k}\right) = -5ydy$$

(d) $$\vec{V}\left(\vec{V} \cdot dA_2\right) = -\left(10\,\hat{j} + 5y\hat{k}\right)5ydy$$

(e) $$\int_{A_2} \vec{V}\left(\vec{V} \cdot dA_2\right) = -\int_0^1 \left(10\,\hat{j} + 5y\hat{k}\right)5ydy = -25y^2\hat{j}\Big|_0^1 - \frac{25}{3}y^3\hat{k}\Big|_0^1 = -25\hat{j} - 8.33\hat{k}$$

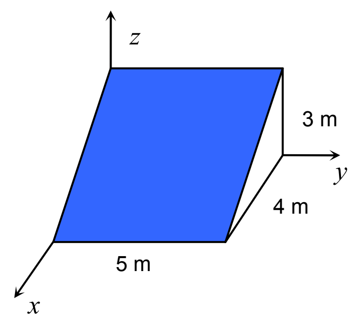
Problem 4.13

[Difficulty: 3]

4.13 The shaded area shown is in a flow where the velocity field is given by $\vec{V} = ax\hat{i} + by\hat{j}$; $a = b = 1\ \text{s}^{-1}$, and the coordinates are measured in meters. Evaluate the volume flow rate and the momentum flux through the shaded area ($\rho = 1\ \text{kg/m}^3$).

Given: Data on velocity field and control volume geometry

Find: Volume flow rate and momentum flux



Solution:

First we define the area and velocity vectors

$$d\vec{A} = dydz\hat{i} + dydx\hat{k}$$

$$\vec{V} = ax\hat{i} + by\hat{j} \qquad \text{or} \qquad \vec{V} = x\hat{i} + y\hat{j}$$

We will need the equation of the surface: $z = 3 - \frac{3}{4}x$ or $x = 4 - \frac{4}{3}z$

Then

a) Volume flow rate

$$Q = \int_A \vec{V} \cdot d\vec{A} = \int_A \left(x\hat{i} + y\hat{j}\right) \cdot \left(dydz\hat{i} + dxdy\hat{k}\right) = \int_0^3 \int_0^5 xdydz = 5\int_0^3 \left(4 - \frac{4}{3}z\right)dz = 5\left(4z - \frac{2}{3}z^2\right)\Bigg|_0^3$$

$$Q = (60 - 30)\frac{\text{m}^3}{\text{s}} = 30\frac{\text{m}^3}{\text{s}}$$

b) Momentum flux

$$\rho\int_A \vec{V}\left(\vec{V} \cdot d\vec{A}\right) = \rho\int_A \left(x\hat{i} + y\hat{j}\right)\left(x\hat{i} + y\hat{j}\right) \cdot \left(dydz\hat{i} + dxdy\hat{k}\right) = \rho\int_A \left(x\hat{i} + y\hat{j}\right)(xdydz)$$

$$= \rho\int_0^3\int_0^5 x^2dydz\hat{i} + \rho\int_0^3\int_0^5 xydydz\hat{j} = 5\int_0^3 \left(4 - \frac{4}{3}z\right)^2 dz\hat{i} + \frac{y^2}{2}\Bigg|_0^5 \int_0^3 \left(4 - \frac{4}{3}z\right)dz\hat{j}$$

$$= 5\int_0^3 \left(16 - \frac{32}{3}z + \frac{16}{9}z^2\right)dz\hat{i} + \frac{25}{2}\left(4z - \frac{2}{3}z^2\right)\Bigg|_0^3 \hat{j} == 5\left(16z - \frac{16}{3}z^2 + \frac{16}{27}z^3\right)\Bigg|_0^3 \hat{i} + \frac{25}{2}(12 - 6)\hat{j}$$

$$= 5(48 - 48 + 16)\hat{i} + 75\hat{j}$$

$$\text{Momentum flux} = 80\hat{i} + 75\hat{j}\ \text{N}$$

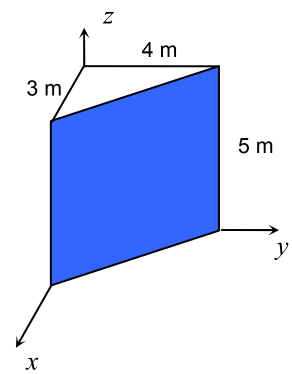
Problem 4.14

[Difficulty: 3]

4.14 The area shown shaded is in a flow where the velocity field is given by $\vec{V} = ax\hat{i} + by\hat{j} + c\hat{k}$; $a = b = 2\ \mathrm{s}^{-1}$ and $c = 1$ m/s. Write a vector expression for an element of the shaded area. Evaluate the integrals $\int_A \vec{V} \cdot dA$ and $\int_A \vec{V}(\vec{V} \cdot d\vec{A})$ over the shaded area.

Given: Data on velocity field and control volume geometry

Find: Surface integrals



Solution:

First we define the area and velocity vectors

$$d\vec{A} = dydz\hat{i} + dxdz\hat{j} \qquad \vec{V} = ax\hat{i} + by\hat{j} + c\hat{k} \quad \text{or} \quad \vec{V} = 2x\hat{i} + 2y\hat{j} + \hat{k}$$

We will need the equation of the surface: $y = \frac{3}{2}x$ or $x = \frac{2}{3}y$

Then

$$\int_A \vec{V} \cdot dA = \int_A \left(-ax\hat{i} + by\hat{j} + c\hat{k}\right) \cdot \left(dydz\hat{i} - dxdz\hat{j}\right)$$

$$= \int_0^2\int_0^3 -axdydz - \int_0^2\int_0^2 bydxdz = -a\int_0^2 dz\int_0^3 \frac{2}{3}ydy - b\int_0^2 dz\int_0^2 \frac{3}{2}xdx = -2a\frac{1}{3}y^2\Big|_0^3 - 2b\frac{3}{4}x^2\Big|_0^2$$

$$Q = (-6a - 6b) = -24\frac{\mathrm{m}^3}{\mathrm{s}}$$

We will again need the equation of the surface: $y = \frac{3}{2}x$ or $x = \frac{2}{3}y$, and also $dy = \frac{3}{2}dx$ and $a = b$

$$\int_A \vec{V}\left(\vec{V} \cdot d\vec{A}\right) = \int_A \left(-ax\hat{i} + by\hat{j} + c\hat{k}\right)\left(-ax\hat{i} + by\hat{j} + c\hat{k}\right) \cdot \left(dydz\hat{i} - dxdz\hat{j}\right)$$

$$= \int_A \left(-ax\hat{i} + by\hat{j} + c\hat{k}\right)\left(-axdydz - bydxdz\right)$$

$$= \int_A \left(-ax\hat{i} + \frac{3}{2}ax\hat{j} + c\hat{k}\right)\left(-ax\frac{3}{2}dxdz - a\frac{3}{2}xdxdz\right)$$

$$= \int_A \left(-ax\hat{i} + \frac{3}{2}ax\hat{j} + c\hat{k}\right)\left(-3axdxdz\right)$$

$$= 3\int_0^2\int_0^2 a^2x^2dxdz\hat{i} - \frac{9}{2}\int_0^2\int_0^2 a^2x^2dxdz\hat{j} - 3\int_0^2\int_0^2 acxdxdz\hat{k}$$

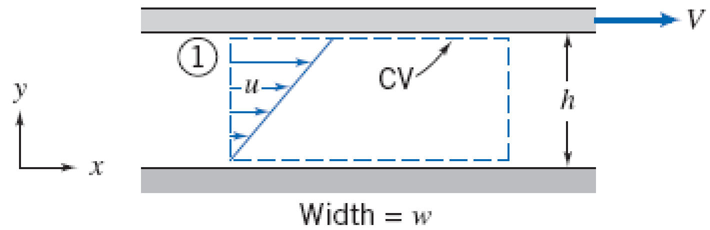
$$= (6)\left(a^2\frac{x^3}{3}\Big|_0^2\right)\hat{i} - (9)\left(a^2\frac{x^3}{3}\Big|_0^2\right)\hat{j} - (6)\left(ac\frac{x^2}{2}\Big|_0^2\right) = 16a^2\hat{i} - 24a^2\hat{j} - 12ac\hat{k}$$

$$= 64\hat{i} - 96\hat{j} - 60\hat{k} \quad \frac{\mathrm{m}^4}{\mathrm{s}^2}$$

Problem 4.15

[Difficulty: 2]

4.15 Obtain expressions for the volume flow rate and the momentum flux through cross section ① of the control volume shown in the diagram.



Given: Control Volume with linear velocity distribution

Find: Volume flow rate and momentum flux

Solution: Apply the expressions for volume and momentum flux

Governing equations: $Q = \int_A \vec{V} \cdot dA$ $\qquad mf = \rho \int_A \vec{V}\left(\vec{V} \cdot d\vec{A}\right)$

Assumption: (1) Incompressible flow

For a linear velocity profile $\vec{V} = \frac{V}{h} y\hat{i}$ and also $d\vec{A} = -wdy\hat{i}$

For the volume flow rate:

$$Q = \int_{y=0}^{h} \frac{V}{h}\hat{i} \cdot \left(-wdy\hat{i}\right) = -\frac{Vw}{h}\int_{y=0}^{h} ydy = -\frac{Vw}{h}\frac{y^2}{2}\Big|_0^h$$

$$Q = -\frac{1}{2}Vhw$$

The momentum flux is

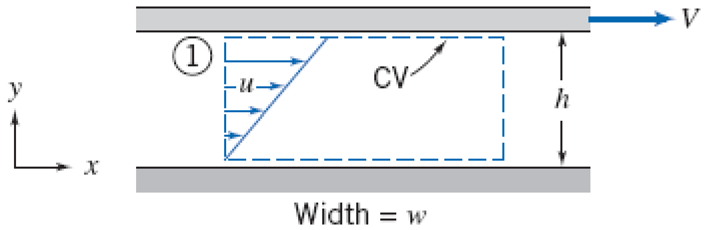
$$mf = \int_{y=0}^{h} \frac{V}{h}\hat{i} \cdot \left(-\rho\frac{Vw}{h}ydy\right) = -\rho\frac{V^2w}{h^2}\hat{i}\int_{y=0}^{h} y^2dy = -\rho\frac{V^2w}{h^2}\hat{i}\frac{y^3}{3}\Big|_0^h$$

$$mf = -\frac{1}{3}\rho V^2wh\hat{i}$$

Problem 4.16

[Difficulty: 2]

4.16 For the flow of Problem 4.15, obtain an expression for the kinetic energy flux, $\int (V^2/2)\rho\vec{V}\cdot d\vec{A}$, through cross section ① of the control volume shown.



Given: Control Volume with linear velocity distribution

Find: Kinetic energy flux

Solution: Apply the expression for kinetic energy flux

Governing equation: $kef = \int_A \frac{V^2}{2}\rho\vec{V}\cdot d\vec{A}$

Assumption: (1) Incompressible flow

For a linear velocity profile $\vec{V} = \frac{V}{h}y\hat{i}$ $\quad V(y) = \frac{V}{h}y$ $\quad$ and also $\quad d\vec{A} = -wdy\hat{i}$

The kinetic energy flux is

$$kef = \int_{y=0}^{h} \frac{1}{2}\left(\frac{V}{h}y\right)^2\left(-\rho\frac{Vw}{h}ydy\right) = -\rho\frac{V^3w}{2h^3}\int_{y=0}^{h} y^3dy = -\rho\frac{V^3w}{2h^3}\left.\frac{y^4}{4}\right|_0^h$$

$$kef = -\frac{1}{8}\rho V^3 wh$$

Problem 4.17

[Difficulty: 2]

4.17 The velocity distribution for laminar flow in a long circular tube of radius R is given by the one-dimensional expression,

$$\vec{V} = u\hat{i} = u_{\max}\left[1 - \left(\frac{r}{R}\right)^2\right]\hat{i}$$

For this profile obtain expressions for the volume flow rate and the momentum flux through a section normal to the pipe axis.

Given: Control Volume with parabolic velocity distribution

Find: Volume flow rate and momentum flux

Solution: Apply the expressions for volume and momentum flux

Governing equations: $Q = \int_A \vec{V} \cdot dA$ $\quad mf = \rho \int_A \vec{V}\left(\vec{V} \cdot d\vec{A}\right)$

Assumption: (1) Incompressible flow

For a linear velocity profile $\vec{V} = u\hat{i} = u_{\max}\left[1 - \left(\frac{r}{R}\right)^2\right]\hat{i}$ and also $d\vec{A} = 2\pi r dr\hat{i}$

For the volume flow rate:

$$Q = \int_{r=0}^{R} u_{\max}\left[1 - \left(\frac{r}{R}\right)^2\right]\hat{i} \cdot \left(2\pi r dr\hat{i}\right) = 2\pi u_{\max}\int_{y=0}^{R}\left[r - \frac{r^3}{R^2}\right]dy = 2\pi u_{\max}\left[\frac{r^2}{2} - \frac{r^4}{4R^2}\right]_0^h = 2\pi u_{\max}\left[\frac{R^2}{2} - \frac{R^2}{4}\right]$$

$$Q = \frac{1}{2}\pi u_{\max} R^2$$

The momentum flux is

$$mf = \int_{r=0}^{R} u_{\max}\left[1 - \left(\frac{r}{R}\right)^2\right]\hat{i}\left\{u_{\max}\left[1 - \left(\frac{r}{R}\right)^2\right]\hat{i} \cdot \left(2\pi r dr\hat{i}\right)\right\} = u_{\max}\left[1 - \left(\frac{r}{R}\right)^2\right]\hat{i}\left\{2\pi u_{\max}\int_{y=0}^{R}\left[r - \frac{r^3}{R^2}\right]dr\right\}$$

$$= 2\pi u_{\max}^2\hat{i}\int_{y=0}^{R}\left[r - \frac{2r^3}{R^2} + \frac{r^5}{R^4}\right]dr$$

$$= 2\pi u_{\max}^2\hat{i}\left[\frac{r^2}{2} - \frac{r^4}{4R^2} + \frac{r^6}{6R^4}\right]_0^h$$

$$= 2\pi u_{\max}^2\hat{i}\left[\frac{R^2}{2} - \frac{R^2}{4} + \frac{R^2}{6}\right]$$

$$mf = \frac{1}{3}\pi u_{\max}^2 R^2\hat{i}$$

Problem 4.18

[Difficulty: 2]

4.18 For the flow of Problem 4.17, obtain an expression for the kinetic energy flux, $\int (V^2/2)\rho\vec{V}\cdot d\vec{A}$, through a section normal to the pipe axis.

Given: Control Volume with parabolic velocity distribution

Find: Kinetic energy flux

Solution: Apply the expressions for kinetic energy flux

Governing equation: $kef = \int_A \frac{V^2}{2}\rho\vec{V}\cdot d\vec{A}$

Assumption: (1) Incompressible flow

For a linear velocity profile $\vec{V} = u\hat{i} = u_{\max}\left[1-\left(\frac{r}{R}\right)^2\right]\hat{i}$ $\quad V = u = u_{\max}\left[1-\left(\frac{r}{R}\right)^2\right]$ and also $d\vec{A} = 2\pi r dr\hat{i}$

For the volume flow rate:

$$kef = \int_{r=0}^{R}\frac{1}{2}\left\{u_{\max}\left[1-\left(\frac{r}{R}\right)^2\right]\right\}^2\rho\left\{u_{\max}\left[1-\left(\frac{r}{R}\right)^2\right]\hat{i}\cdot(2\pi r dr\hat{i})\right\}$$

$$= \int_{r=0}^{R}\frac{1}{2}u_{\max}^2\left[1-2\left(\frac{r}{R}\right)^2+\left(\frac{r}{R}\right)^4\right]\rho\left\{2\pi u_{\max}\left[1-\left(\frac{r}{R}\right)^2\right]rdr\right\}$$

$$= \int_{r=0}^{R}\pi\rho u_{\max}^3\left[1-3\left(\frac{r}{R}\right)^2+3\left(\frac{r}{R}\right)^4-\left(\frac{r}{R}\right)^6\right]rdr$$

$$= \int_{r=0}^{R}\pi\rho u_{\max}^3\left[r-3\frac{r^3}{R^2}+3\frac{r^5}{R^4}-\frac{r^7}{R^6}\right]dr$$

$$= \pi\rho u_{\max}^3\left[\frac{r^2}{2}-\frac{3r^4}{4R^2}+\frac{r^6}{2R^4}-\frac{r^8}{8R^6}\right]_0^h$$

$$kef = \frac{1}{8}\pi\rho u_{\max}^3 R^2$$

Problem 4.19

[Difficulty: 1]

4.19 A shower head fed by a ¾-in. ID water pipe consists of 50 nozzles of ⅓₂-in. ID. Assuming a flow rate of 2.2 gpm, what is the exit velocity (ft/s) of each jet of water? What is the average velocity (ft/s) in the pipe?

Given: Data on flow through nozzles

Find: Exit velocity in each jet; velocity in pipe

Solution:

Basic equation $\sum_{CS} \left(\vec{V}\cdot\vec{A}\right) = 0$

Assumptions: 1) Steady flow 2) Incompressible flow 3) Uniform flow

The given data is $Q = 2.2\cdot gpm$ $\quad d = \frac{1}{32}\cdot in$ $\quad n = 50$ (Number of nozzles) $\quad D = \frac{3}{4}\cdot in$

Area of each nozzle $A = \frac{\pi}{4}\cdot d^2$ $\quad A = 7.67\times 10^{-4}\,in^2$

Area of the pipe $A_{pipe} = \frac{\pi}{4}\cdot D^2$ $\quad A_{pipe} = 0.442 in^2$

Total area of nozzles $A_{total} = n\cdot A$ $\quad A_{total} = 0.0383 in^2$

The jet speeds are then $V = \frac{Q}{A_{total}}$ $\quad V = 18.4\frac{ft}{s}$ (Note that gal = 231 in^3)

Then for the pipe flow $\sum_{CS} \left(\vec{V}\cdot\vec{A}\right) = -V_{pipe}\cdot A_{pipe} + n\cdot V\cdot A = 0$

Hence $V_{pipe} = V\cdot\frac{n\cdot A}{A_{pipe}} = V\cdot n\cdot\left(\frac{d}{D}\right)^2$

$$V_{pipe} = 18.4\cdot\frac{ft}{s}\times 50\times\left(\frac{\frac{1}{32}}{\frac{3}{4}}\right)^2 \qquad V_{pipe} = 1.60\cdot\frac{ft}{s}$$

Problem 4.20

[Difficulty: 1]

4.20 A farmer is spraying a liquid through 10 nozzles, ⅛-in. ID, at an average exit velocity of 10 ft/s. What is the average velocity in the 1-in. ID head feeder? What is the system flow rate, in gpm?

Given: Data on flow through nozzles

Find: Average velocity in head feeder; flow rate

Solution:

Basic equation
$$\sum_{CS} \left(\vec{V} \cdot \vec{A}\right) = 0$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) Uniform flow

Then for the nozzle flow
$$\sum_{CS} \left(\vec{V} \cdot \vec{A}\right) = -V_{feeder} \cdot A_{feeder} + 10 \cdot V_{nozzle} \cdot A_{nozzle} = 0$$

Hence
$$V_{feeder} = V_{nozzle} \cdot \frac{10 \cdot A_{nozzle}}{A_{feeder}} = V_{nozzle} \cdot 10 \cdot \left(\frac{D_{nozzle}}{D_{feeder}}\right)^2$$

$$V_{feeder} = 10 \cdot \frac{ft}{s} \times 10 \times \left(\frac{\frac{1}{8}}{1}\right)^2 \qquad V_{feeder} = 1.56 \cdot \frac{ft}{s}$$

The flow rate is
$$Q = V_{feeder} \cdot A_{feeder} = V_{feeder} \cdot \frac{\pi \cdot D_{feeder}^{\ 2}}{4}$$

$$Q = 1.56 \cdot \frac{ft}{s} \times \frac{\pi}{4} \times \left(1 \cdot in \times \frac{1 \cdot ft}{12 \cdot in}\right)^2 \times \frac{7.48 \cdot gal}{1 \cdot ft^3} \times \frac{60 \cdot s}{1 \cdot min} \qquad Q = 3.82 \cdot gpm$$

Problem 4.21

[Difficulty: 3]

4.21 A cylindrical holding water tank has a 3 m ID and a height of 3 m. There is one inlet of diameter 10 cm, an exit of diameter 8 cm, and a drain. The tank is initially empty when the inlet pump is turned on, producing an average inlet velocity of 5 m/s. When the level in the tank reaches 0.7 m, the exit pump turns on, causing flow out of the exit; the exit average velocity is 3 m/s. When the water level reaches 2 m the drain is opened such that the level remains at 2 m. Find (a) the time at which the exit pump is switched on, (b) the time at which the drain is opened, and (c) the flow rate into the drain (m^3/min).

Given: Data on flow into and out of tank

Find: Time at which exit pump is switched on; time at which drain is opened; flow rate into drain

Solution:

Basic equation $$\frac{\partial}{\partial t}M_{CV} + \sum_{CS}\left(\rho\cdot\vec{V}\cdot\vec{A}\right) = 0$$

Assumptions: 1) Uniform flow 2) Incompressible flow

After inlet pump is on $$\frac{\partial}{\partial t}M_{CV} + \sum_{CS}\left(\rho\cdot\vec{V}\cdot\vec{A}\right) = \frac{\partial}{\partial t}M_{tank} - \rho\cdot V_{in}\cdot A_{in} = 0 \qquad \frac{\partial}{\partial t}M_{tank} = \rho\cdot A_{tank}\cdot\frac{dh}{dt} = \rho\cdot V_{in}\cdot A_{in}$$

$$\frac{dh}{dt} = V_{in}\cdot\frac{A_{in}}{A_{tank}} = V_{in}\cdot\left(\frac{D_{in}}{D_{tank}}\right)^2$$ where h is the level of water in the tank

Hence the time to reach $h_{exit} = 0.7$ m is $$t_{exit} = \frac{h_{exit}}{\frac{dh}{dt}} = \frac{h_{exit}}{V_{in}}\cdot\left(\frac{D_{tank}}{D_{in}}\right)^2 \qquad t_{exit} = 0.7\cdot m\times\frac{1}{5}\cdot\frac{s}{m}\times\left(\frac{3\cdot m}{0.1\cdot m}\right)^2 \qquad t_{exit} = 126\ s$$

After exit pump is on $$\frac{\partial}{\partial t}M_{CV} + \sum_{CS}\left(\rho\cdot\vec{V}\cdot\vec{A}\right) = \frac{\partial}{\partial t}M_{tank} - \rho\cdot V_{in}\cdot A_{in} + \rho\cdot V_{exit}\cdot A_{exit} = 0 \quad A_{tank}\cdot\frac{dh}{dt} = V_{in}\cdot A_{in} - V_{exit}\cdot A_{exit}$$

$$\frac{dh}{dt} = V_{in}\cdot\frac{A_{in}}{A_{tank}} - V_{exit}\cdot\frac{A_{exit}}{A_{tank}} = V_{in}\cdot\left(\frac{D_{in}}{D_{tank}}\right)^2 - V_{exit}\cdot\left(\frac{D_{exit}}{D_{tank}}\right)^2$$

Hence the time to reach $h_{drain} = 2$ m is $$t_{drain} = t_{exit} + \frac{\left(h_{drain} - h_{exit}\right)}{\frac{dh}{dt}} = \frac{\left(h_{drain} - h_{exit}\right)}{V_{in}\cdot\left(\frac{D_{in}}{D_{tank}}\right)^2 - V_{exit}\cdot\left(\frac{D_{exit}}{D_{tank}}\right)^2}$$

$$t_{drain} = 126\cdot s + (2 - 0.7)\cdot m\times\frac{1}{5\cdot\frac{m}{s}\times\left(\frac{0.1\cdot m}{3\cdot m}\right)^2 - 3\cdot\frac{m}{s}\times\left(\frac{0.08\cdot m}{3\cdot m}\right)^2} \qquad t_{drain} = 506\ s$$

The flow rate into the drain is equal to the net inflow (the level in the tank is now constant)

$$Q_{drain} = V_{in}\cdot\frac{\pi\cdot D_{in}^2}{4} - V_{exit}\cdot\frac{\pi\cdot D_{exit}^2}{4} \qquad Q_{drain} = 5\cdot\frac{m}{s}\times\frac{\pi}{4}\times(0.1\cdot m)^2 - 3\cdot\frac{m}{s}\times\frac{\pi}{4}\times(0.08\cdot m)^2 \qquad Q_{drain} = 0.0242\,\frac{m^3}{s}$$

Problem 4.22

[Difficulty: 1]

4.22 A university laboratory that generates 15 m^3/s of air flow at design condition wishes to build a wind tunnel with variable speeds. It is proposed to build the tunnel with a sequence of three circular test sections: section 1 will have a diameter of 1.5 m, section 2 a diameter of 1 m, and section 3 a diameter such that the average speed is 75 m/s.

(a) What will be the speeds in sections 1 and 2?

(b) What must the diameter of section 3 be to attain the desired speed at design condition?

Given: Data on wind tunnel geometry

Find: Average speeds in wind tunnel; diameter of section 3

Solution:

Basic equation $\quad Q = V \cdot A$

Assumptions: 1) Steady flow 2) Incompressible flow 3) Uniform flow

Given data: $\quad Q = 15 \cdot \frac{m^3}{s} \qquad D_1 = 1.5 \cdot m \qquad D_2 = 1 \cdot m \qquad V_3 = 75 \cdot \frac{m}{s}$

Between sections 1 and 2 $\quad Q = V_1 \cdot A_1 = V_1 \cdot \frac{\pi \cdot D_1^2}{4} = V_2 \cdot A_2 = V_2 \cdot \frac{\pi \cdot D_2^2}{4}$

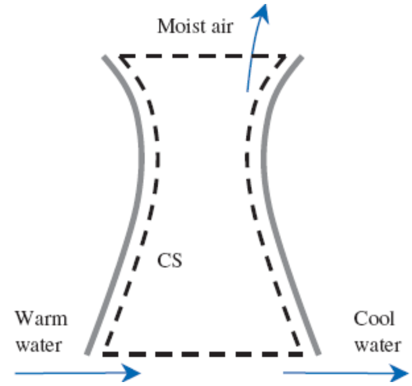
Hence $\quad V_1 = \frac{Q}{\frac{\pi}{4} \cdot D_1^2} \qquad V_1 = 8.49 \frac{m}{s} \qquad V_2 = \frac{Q}{\frac{\pi}{4} \cdot D_2^2} \qquad V_2 = 19.1 \frac{m}{s}$

For section 3 we can use $\quad V_1 \cdot \frac{\pi \cdot D_1^2}{4} = V_3 \cdot \frac{\pi \cdot D_3^2}{4} \quad$ or $\quad D_3 = D_1 \cdot \sqrt{\frac{V_1}{V_3}} \qquad D_3 = 0.505 \, m$

Problem 4.23

Difficulty: 4]

4.23 A wet cooling tower cools warm water by spraying it into a forced dry-air flow. Some of the water evaporates in this air and is carried out of the tower into the atmosphere; the evaporation cools the remaining water droplets, which are collected at the exit pipe (6 in. ID) of the tower. Measurements indicate the warm water mass flow rate is 250,000 lb/hr, and the cool water (70°F) flows at an average speed of 5 ft/s in the exit pipe. The moist air density is 0.065 lb/ft^3. Find (a) the volume flow rate (ft^3/s) and mass flow rate (lb/hr) of the cool water, (b) the mass flow rate (lb/hr) of the moist air, and (c) the mass flow rate (lb/hr) of the dry air. *Hint*: Google "density of moist air" for information on relating moist and dry air densities!



Given: Data on flow into and out of cooling tower

Find: Volume and mass flow rate of cool water; mass flow rate of moist and dry air

Solution:

Basic equation $\quad \sum_{CS} \left(\rho\cdot\vec{V}\cdot\vec{A}\right) = 0 \quad$ and at each inlet/exit $\quad Q = V\cdot A$

Assumptions: 1) Uniform flow 2) Incompressible flow

Given data: $\quad m_{warm} = 2.5\cdot 10^5\cdot\frac{lb}{hr} \quad D = 6\cdot in \quad V = 5\cdot\frac{ft}{s} \quad \rho_{moist} = 0.065\cdot\frac{lb}{ft^3}$

At the cool water exit $\quad Q_{cool} = V\cdot A \quad Q_{cool} = 5\cdot\frac{ft}{s}\times\frac{\pi}{4}\times(0.5\cdot ft)^2 \quad Q_{cool} = 0.982\cdot\frac{ft^3}{s} \quad Q_{cool} = 441\cdot gpm$

The mass flow rate is $\quad m_{cool} = \rho\cdot Q_{cool} \quad m_{cool} = 1.94\cdot\frac{slug}{ft^3}\times 0.982\cdot\frac{ft^3}{s} \quad m_{cool} = 1.91\cdot\frac{slug}{s} \quad m_{cool} = 2.21\times 10^5\cdot\frac{lb}{hr}$

NOTE: Software does not allow dots over terms, so m represents mass flow rate, not mass!

For the water flow we need $\quad \sum_{CS} \left(\rho\cdot\vec{V}\cdot\vec{A}\right) = 0 \quad$ to balance the water flow

We have $\quad -m_{warm} + m_{cool} + m_v = 0 \quad m_v = m_{warm} - m_{cool} \quad m_v = 29341\cdot\frac{lb}{hr}$

This is the mass flow rate of water vapor. To obtain air flow rates, from psychrometrics $\quad x = \frac{m_v}{m_{air}}$

where x is the relative humidity. It is also known (try Googling "density of moist air") that $\quad \frac{\rho_{moist}}{\rho_{dry}} = \frac{1 + x}{1 + x\cdot\frac{R_{H2O}}{R_{air}}}$

We are given $\quad \rho_{moist} = 0.065\cdot\frac{lb}{ft^3}$

For dry air we could use the ideal gas equation $\rho_{dry} = \frac{p}{R \cdot T}$ but here we use atmospheric air density (Table A.3)

$$\rho_{dry} = 0.002377 \cdot \frac{slug}{ft^3} \qquad \rho_{dry} = 0.002377 \cdot \frac{slug}{ft^3} \times 32.2 \cdot \frac{lb}{slug} \qquad \rho_{dry} = 0.0765 \cdot \frac{lb}{ft^3}$$

Note that moist air is less dense than dry air!

Hence $\frac{0.065}{0.0765} = \frac{1 + x}{1 + x \cdot \frac{85.78}{53.33}}$ using data from Table A.6

$$x = \frac{0.0765 - 0.065}{0.065 \cdot \frac{85.78}{53.33} - .0765} \qquad x = 0.410$$

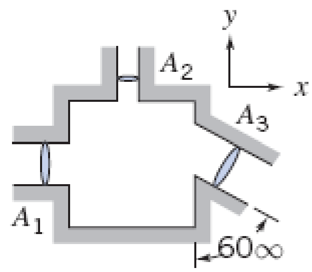
Hence $\frac{m_v}{m_{air}} = x$ leads to $m_{air} = \frac{m_v}{x}$ $\qquad m_{air} = 29341 \cdot \frac{lb}{hr} \times \frac{1}{0.41} \qquad m_{air} = 71563 \cdot \frac{lb}{hr}$

Finally, the mass flow rate of moist air is $m_{moist} = m_v + m_{air}$ $\qquad m_{moist} = 1.01 \times 10^5 \cdot \frac{lb}{hr}$

Problem 4.24

[Difficulty: 1]

4.24 Fluid with 65 lbm/ft^3 density is flowing steadily through the rectangular box shown. Given $A_1 = 0.5$ ft^2, $A_2 = 0.1$ ft^2, $A_3 = 0.6$ ft^2, $\vec{V}_1 = 10\hat{i}$ ft/s, and $\vec{V}_2 = 20\hat{j}$ ft/s, determine velocity $\vec{V}_3$.



Given: Data on flow through box

Find: Velocity at station 3

Solution:

Basic equation
$$\sum_{CS} \left(\vec{V} \cdot \vec{A}\right) = 0$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) Uniform flow

Then for the box
$$\sum_{CS} \left(\vec{V} \cdot \vec{A}\right) = -V_1 \cdot A_1 + V_2 \cdot A_2 + V_3 \cdot A_3 = 0$$

Note that the vectors indicate that flow is in at location 1 and out at location 2; we assume outflow at location 3

Hence
$$V_3 = V_1 \cdot \frac{A_1}{A_3} - V_2 \cdot \frac{A_2}{A_3} \qquad V_3 = 10 \cdot \frac{ft}{s} \times \frac{0.5}{0.6} - 20 \cdot \frac{ft}{s} \times \frac{0.1}{0.6} \qquad V_3 = 5 \cdot \frac{ft}{s}$$

Based on geometry
$$V_x = V_3 \cdot \sin(60 \cdot deg) \qquad V_x = 4.33 \cdot \frac{ft}{s}$$

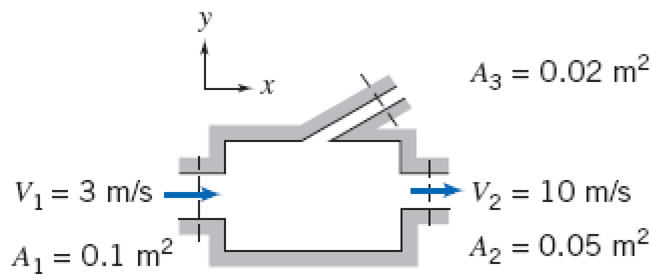
$$V_y = -V_3 \cdot \cos(60 \cdot deg) \qquad V_y = -2.5 \cdot \frac{ft}{s}$$

$$\vec{V_3} = \left(4.33 \cdot \frac{ft}{s}, -2.5 \cdot \frac{ft}{s}\right)$$

Problem 4.25

[Difficulty: 1]

4.25 Consider steady, incompressible flow through the device shown. Determine the magnitude and direction of the volume flow rate through port 3.



Given: Data on flow through device

Find: Volume flow rate at port 3

Solution:

Basic equation $\quad \displaystyle\sum_{CS} \left(\vec{V}\cdot\vec{A}\right) = 0$

Assumptions: 1) Steady flow 2) Incompressible flow 3) Uniform flow

Then for the box $\quad \displaystyle\sum_{CS} \left(\vec{V}\cdot\vec{A}\right) = -V_1 \cdot A_1 + V_2 \cdot A_2 + V_3 \cdot A_3 = -V_1 \cdot A_1 + V_2 \cdot A_2 + Q_3$

Note we assume outflow at port 3

Hence $\quad Q_3 = V_1 \cdot A_1 - V_2 \cdot A_2 \qquad Q_3 = 3 \cdot \frac{m}{s} \times 0.1 \cdot m^2 - 10 \cdot \frac{m}{s} \times 0.05 \cdot m^2 \qquad Q_3 = -0.2 \cdot \frac{m^3}{s}$

The negative sign indicates the flow at port 3 is inwards. $\qquad$ Flow rate at port 3 is 0.2 m^3/s inwards

Problem 4.26

[Difficulty: 1]

4.26 A rice farmer needs to fill her 150 m × 400 m field with water to a depth of 7.5 cm in 1 hr. How many 37.5-cm-diameter supply pipes are needed if the average velocity in each must be less than 2.5 m/s?

Given: Water needs of farmer

Find: Number of supply pipes needed

Solution:

Basic equation $Q = V \cdot A$

Assumptions: 1) Steady flow 2) Incompressible flow 3) Uniform flow

The given data is: $A = 150 \cdot m \cdot 400 \cdot m$ $\quad A = 6 \times 10^4 \, m^2$ $\quad h = 7.5 \cdot cm$ $\quad t = 1 \cdot hr$ $\quad D = 37.5 \cdot cm$ $\quad V = 2.5 \cdot \frac{m}{s}$

Then $Q = \frac{A \cdot h}{t}$ $\quad Q = 1.25 \frac{m^3}{s}$

If n is the number of pipes $Q = V \cdot \frac{\pi}{4} \cdot D^2 \cdot n$ or $n = \frac{4 \cdot Q}{\pi \cdot V \cdot D^2}$ $\quad n = 4.527$

The farmer needs 5 pipes.

Problem 4.27

[Difficulty: 1]

4.27 You are making beer. The first step is filling the glass carboy with the liquid wort. The internal diameter of the carboy is 15 in., and you wish to fill it up to a depth of 2 ft. If your wort is drawn from the kettle using a siphon process that flows at 3 gpm, how long will it take to fill?

Given: Data on filling of glass carboy

Find: Time to fill

Solution:

We can treat this as an unsteady problem if we choose the CS as the entire carboy

Basic equation $$\frac{\partial}{\partial t} M_{CV} + \sum_{CS} \left(\rho \cdot \vec{V} \cdot \vec{A}\right) = 0$$

Assumptions: 1) Incompressible flow 2) Uniform flow

Given data: $Q = 3 \cdot \text{gpm}$ $\quad D = 15 \cdot \text{in}$ $\quad h = 2 \cdot \text{ft}$

Hence $$\frac{\partial}{\partial t} M_{CV} = \rho \cdot A \cdot \frac{dh}{dt} = \rho \cdot A \cdot \frac{h}{\tau} = -\sum_{CS} \left(\rho \cdot \vec{V} \cdot \vec{A}\right) = \rho \cdot Q$$

where Q is the fill rate, A is the carboy cross-section area, dh/dt is the rate of rise in the carboy, and τ is the fill time

Hence $$\tau = \frac{\frac{\pi}{4} \cdot D^2 \cdot h}{Q} \qquad \tau = 6.12 \cdot \text{min}$$

Problem 4.28

[Difficulty: 1]

4.28 In your kitchen, the sink is 2 ft by 18 in. by 12 in. deep. You are filling it with water at the rate of 4 gpm. How long will it take (in min) to half fill the sink? After this you turn off the faucet and open the drain slightly so that the tank starts to drain at 1 gpm. What is the rate (in./min) at which the water level drops?

Given: Data on filling of a sink

Find: Time to half fill; rate at which level drops

Solution:

This is an unsteady problem if we choose the CS as the entire sink

Basic equation $$\frac{\partial}{\partial t}M_{CV} + \sum_{CS}\left(\rho\cdot\vec{V}\cdot\vec{A}\right) = 0$$

Assumptions: 1) Incompressible flow

Given data: $m_{rate} = 4\cdot gpm$ $\quad L = 2\cdot ft$ $\quad w = 18\cdot in$ $\quad d = 12\cdot in$ $\quad Q = 4\cdot gpm$ $\quad Q_{drain} = 1\cdot gpm$

Hence $$\frac{\partial}{\partial t}M_{CV} = -\sum_{CS}\left(\rho\cdot\vec{V}\cdot\vec{A}\right) = \text{Inflow} - \text{Outflow} \qquad (1)$$

To half fill: $V = \frac{1}{2}\cdot L\cdot w\cdot d$ $\quad V = 1.5\,ft^3$ $\quad V = 11.2\,gal$

Then, using Eq 1 $\frac{V}{\tau} = Q$ $\quad \tau = \frac{V}{Q}$ $\quad \tau = 168\,s$ $\quad \tau = 2.81\,min$

After the drain opens, Eq. 1 becomes $\frac{dV}{dt} = L\cdot w\cdot V_{level} = -Q_{drain}$ where V_{level} is the speed of water level drop

$$V_{level} = -\frac{Q_{drain}}{L\cdot w} \qquad V_{level} = -7.43\times 10^{-4}\,\frac{ft}{s} \qquad V_{level} = -0.535\,\frac{in}{min}$$

Problem 4.29

[Difficulty: 1]

4.29 Ventilation air specifications for classrooms require that at least 8.0 L/s of fresh air be supplied for each person in the room (students and instructor). A system needs to be designed that will supply ventilation air to 6 classrooms, each with a capacity of 20 students. Air enters through a central duct, with short branches successively leaving for each classroom. Branch registers are 200 mm high and 500 mm wide. Calculate the volume flow rate and air velocity entering each room. Ventilation noise increases with air velocity. Given a supply duct 500 mm high, find the narrowest supply duct that will limit air velocity to a maximum of 1.75 m/s.

Given: Air flow system

Find: Flow rate and velocity into each room; narrowest supply duct

Solution:

Basic equation $\quad Q = V \cdot A$

Assumptions: 1) Steady flow 2) Incompressible flow 3) Uniform flow

The given data is: $\quad Q_{person} = 8 \cdot \frac{L}{s} \quad n_{rooms} = 6 \quad n_{students} = 20$

$h = 200 \cdot mm \quad w = 500 \cdot mm \quad V_{max} = 1.75 \cdot \frac{m}{s}$

Then for each room $\quad Q_{room} = n_{students} \cdot Q_{person} \quad Q_{room} = 160 \frac{L}{s} \quad Q_{room} = 0.16 \frac{m^3}{s}$

and $\quad V_{room} = \frac{Q_{room}}{w \cdot h} \quad V_{room} = 1.6 \frac{m}{s}$

For the supply duct $\quad Q = n_{rooms} \cdot Q_{room} \quad Q = 960 \frac{L}{s} \quad Q = 0.96 \frac{m^3}{s}$

and $\quad Q = V_{max} \cdot A = V_{max} \cdot w \cdot h$ where w and h are now the supply duct dimensions $\quad h = 500 \cdot mm$

$w = \frac{Q}{V_{max} \cdot h} \quad w = 1.097\,m$

Problem 4.30

[Difficulty: 1]

4.30 You are trying to pump storm water out of your basement during a storm. The pump can extract 27.5 gpm. The water level in the basement is now sinking by about 4 in./hr. What is the flow rate (gpm) from the storm into the basement? The basement is 30 ft × 20 ft.

Given: Data on filling of a basement during a storm

Find: Flow rate of storm into basement

Solution:

This is an unsteady problem if we choose the CS as the entire basement

Basic equation
$$\frac{\partial}{\partial t}M_{CV} + \sum_{CS}\left(\rho\cdot\vec{V}\cdot\vec{A}\right) = 0$$

Assumptions: 1) Incompressible flow

Given data: $Q_{pump} = 27.5\cdot gpm$ $\quad \frac{dh}{dt} = 4\cdot\frac{in}{hr}$ $\quad A = 30\cdot ft\cdot 20\cdot ft$

Hence
$$\frac{\partial}{\partial t}M_{CV} = \rho\cdot A\cdot\frac{dh}{dt} = \sum_{CS}\left(\rho\cdot\vec{V}\cdot\vec{A}\right) = \rho\cdot Q_{storm} - \rho\cdot Q_{pump}$$

where A is the basement area and dh/dt is the rate at which the height of water in the basement changes.

or
$$Q_{storm} = Q_{pump} - A\cdot\frac{dh}{dt}$$

$$Q_{storm} = 27.5\cdot\frac{gal}{min} - 30\cdot ft \times 20\cdot ft \times \left(\frac{4}{12}\cdot\frac{ft}{hr}\right) \times \frac{7.48\cdot gal}{ft^3} \times \frac{1\cdot hr}{60\cdot min}$$

Data on gals from Table G.2

$$Q_{storm} = 2.57\cdot gpm$$

Problem 4.31

[Difficulty: 1]

4.31 In steady-state flow, downstream the density is 1 kg/m^3, the velocity is 1000 m/sec, and the area is 0.1 m^2. Upstream, the velocity is 1500 m/sec, and the area is 0.25 m^2. What is the density upstream?

Given: Data on compressible flow

Find: Downstream density

Solution:

Basic equation
$$\sum_{CS} \left(\rho \cdot \vec{V} \cdot \vec{A}\right) = 0$$

Assumptions: 1) Steady flow 2) Uniform flow

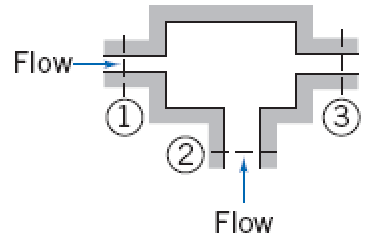
Then for the box
$$\sum_{CS} \left(\rho \cdot \vec{V} \cdot \vec{A}\right) = -\rho_u \cdot V_u \cdot A_u + \rho_d \cdot V_d \cdot A_d = 0$$

Hence
$$\rho_u = \rho_d \cdot \frac{V_d \cdot A_d}{V_u \cdot A_u} \qquad \rho_u = 1 \cdot \frac{kg}{m^3} \cdot \frac{1000 \cdot \frac{m}{s}}{1500 \cdot \frac{m}{s}} \cdot \frac{0.1 \cdot m^2}{0.25 \cdot m^2} \qquad \rho_u = 0.267 \frac{kg}{m^3}$$

Problem 4.32

[Difficulty: 2]

4.32 In the incompressible flow through the device shown, velocities may be considered uniform over the inlet and outlet sections. The following conditions are known: $A_1 = 0.1\ \text{m}^2$, $A_2 = 0.2\ \text{m}^2$, $A_3 = 0.15\ \text{m}^2$, $V_1 = 10e^{-t/2}$ m/s, and $V_2 = 2\cos(2\pi t)$ m/s (t in seconds). Obtain an expression for the velocity at section (3), and plot V_3 as a function of time. At what instant does V_3 first become zero? What is the total mean volumetric flow at section (3)?



Given: Data on flow through device

Find: Velocity V_3; plot V_3 against time; find when V_3 is zero; total mean flow

Solution:

Governing equation: For incompressible flow (Eq. 4.13) and uniform flow

$$\int \vec{V}\,d\vec{A} = \sum \vec{V}\cdot\vec{A} = 0$$

Applying to the device (assuming V_3 is out)

$$-V_1 \cdot A_1 - V_2 \cdot A_2 + V_3 \cdot A_3 = 0$$

$$V_3 = \frac{V_1 \cdot A_1 + V_2 \cdot A_2}{A_3} = \frac{10 \cdot e^{-\frac{t}{2}} \cdot \frac{m}{s} \times 0.1 \cdot m^2 + 2 \cdot \cos(2 \cdot \pi \cdot t) \cdot \frac{m}{s} \times 0.2 \cdot m^2}{0.15 \cdot m^2}$$

The velocity at A_3 is

$$V_3 = 6.67 \cdot e^{-\frac{t}{2}} + 2.67 \cdot \cos(2 \cdot \pi \cdot t)$$

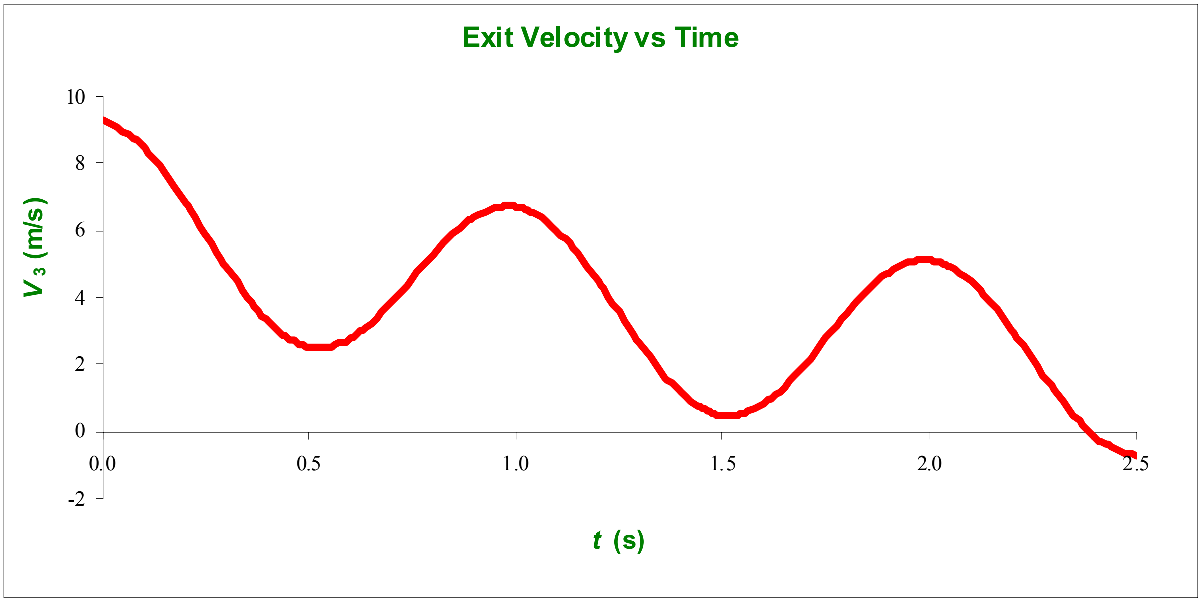
The total mean volumetric flow at A_3 is

$$Q = \int_0^{\infty} V_3 \cdot A_3\, dt = \int_0^{\infty} \left(6.67 \cdot e^{-\frac{t}{2}} + 2.67 \cdot \cos(2 \cdot \pi \cdot t)\right) \cdot 0.15\, dt \cdot \left(\frac{m}{s} \cdot m^2\right)$$

$$Q = \lim_{t \to \infty} \left(-2 \cdot e^{-\frac{t}{2}} + \frac{1}{5 \cdot \pi} \cdot \sin(2 \cdot \pi \cdot t)\right) - (-2) = 2 \cdot m^3 \qquad Q = 2 \cdot m^3$$

The time at which V_3 first is zero, and the plot of V_3 is shown in the corresponding *Excel* workbook $\qquad t = 2.39 \cdot s$

t (s)	V_3 (m/s)
0.00	9.33
0.10	8.50
0.20	6.86
0.30	4.91
0.40	3.30
0.50	2.53
0.60	2.78
0.70	3.87
0.80	5.29
0.90	6.41
1.00	6.71
1.10	6.00
1.20	4.48
1.30	2.66
1.40	1.15
1.50	0.48
1.60	0.84
1.70	2.03
1.80	3.53
1.90	4.74
2.00	5.12
2.10	4.49
2.20	3.04
2.30	1.29
2.40	-0.15
2.50	-0.76



The time at which V_3 first becomes zero can be found using *Goal Seek*

t (s)	V_3 (m/s)
2.39	0.00

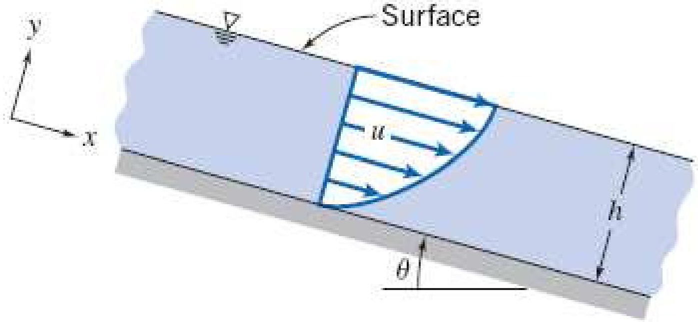
Problem 4.33

[Difficulty: 2]

4.33 Oil flows steadily in a thin layer down an inclined plane. The velocity profile is

$$u = \frac{\rho g \sin\theta}{\mu}\left[hy - \frac{y^2}{2}\right]$$

Express the mass flow rate per unit width in terms of ρ, μ, g, θ, and h.



Given: Data on flow down an inclined plane

Find: Find u_{max}

Solution:

Basic equation
$$m_{flow} = \int \rho u \, dA$$

Assumptions: 1) Steady flow 2) Incompressible flow

Evaluating at 1 and 2
$$m_{flow} = \int_0^h \rho\cdot\frac{\rho\cdot g\cdot\sin(\theta)}{\mu}\cdot\left(h\cdot y - \frac{y^2}{2}\right)\cdot w\,dy = \frac{\rho^2\cdot g\cdot\sin(\theta)\cdot w}{\mu}\cdot\int_0^h \left(h\cdot y - \frac{y^2}{2}\right) dy$$

$$m_{flow} = \frac{\rho^2\cdot g\cdot\sin(\theta)\cdot w}{\mu}\cdot\left(\frac{h^3}{2} - \frac{h^3}{6}\right)$$

Hence
$$m_{flow} = \frac{\rho^2\cdot g\cdot\sin(\theta)\cdot w\cdot h^3}{3\cdot\mu}$$

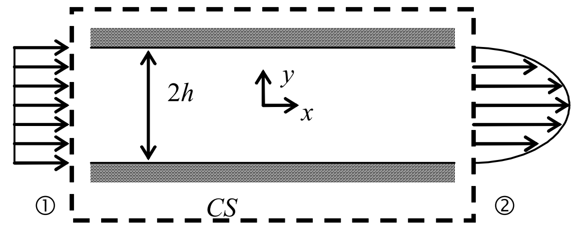
Problem 4.34

[Difficulty: 2]

4.34 Water enters a wide, flat channel of height $2h$ with a uniform velocity of 2.5 m/s. At the channel outlet the velocity distribution is given by

$$\frac{u}{u_{max}} = 1 - \left(\frac{y}{h}\right)^2$$

where y is measured from the centerline of the channel. Determine the exit centerline velocity, u_{max}.



Given: Data on flow at inlet and outlet of channel

Find: Find u_{max}

Solution:

Basic equation $\displaystyle\int_{CS} \rho\vec{V}\cdot d\vec{A} = 0$

Assumptions: 1) Steady flow 2) Incompressible flow

Evaluating at 1 and 2 $\displaystyle -\rho\cdot U\cdot 2\cdot h\cdot w + \int_{-h}^{h} \rho\cdot u(y)\,dy = 0$ $\displaystyle\int_{-h}^{h} u_{max}\cdot\left[1 - \left(\frac{y}{h}\right)^2\right] dy = 2\cdot h\cdot U$

$$u_{max}\cdot\left[[h-(-h)] - \left[\frac{h^3}{3\cdot h^2} - \left(-\frac{h^3}{3\cdot h^2}\right)\right]\right] = 2\cdot h\cdot U \qquad u_{max}\cdot\frac{4}{3}\cdot h = 2\cdot h\cdot U$$

Hence $\displaystyle u_{max} = \frac{3}{2}\cdot U = \frac{3}{2}\times 2.5\cdot\frac{m}{s}$ $\displaystyle u_{max} = 3.75\cdot\frac{m}{s}$

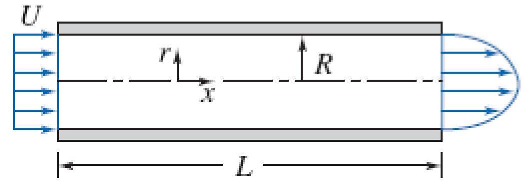
Problem 4.35

[Difficulty: 2]

4.35 Water flows steadily through a pipe of length L and radius $R = 75$ mm. Calculate the uniform inlet velocity, U, if the velocity distribution across the outlet is given by

$$u = u_{max}\left[1 - \frac{r^2}{R^2}\right]$$

and $u_{max} = 3$ m/s.



Given: Data on flow at inlet and outlet of pipe

Find: Find U

Solution:

Basic equation $$\int_{CS} \rho\vec{V}\cdot d\vec{A} = 0$$

Assumptions: 1) Steady flow 2) Incompressible flow

Evaluating at inlet and exit $$-\rho\cdot U\cdot\pi\cdot R^2 + \int_0^R \rho\cdot u(r)\cdot 2\cdot\pi\cdot r\,dr = 0$$ $$\int_0^R u_{max}\cdot\left[1-\left(\frac{r}{R}\right)^2\right]\cdot 2\cdot r\,dr = R^2\cdot U$$

$$u_{max}\cdot\left(R^2 - \frac{1}{2}\cdot R^2\right) = R^2\cdot U$$ $$U = \frac{1}{2}\cdot u_{max}$$

Hence $$U = \frac{1}{2}\times 3\cdot\frac{m}{s}$$ $$U = 1.5\cdot\frac{m}{s}$$

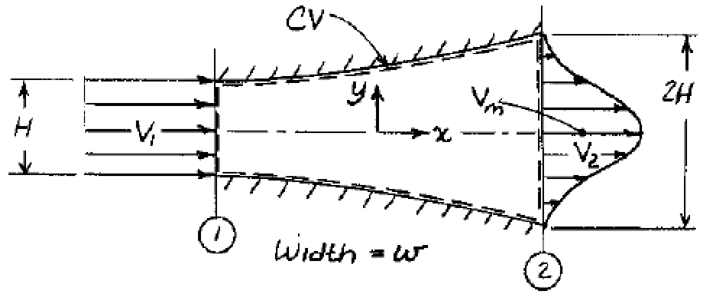
Problem 4.36

[Difficulty: 2]

4.36 Incompressible fluid flows steadily through a plane diverging channel. At the inlet, of height H, the flow is uniform with magnitude V_1. At the outlet, of height $2H$, the velocity profile is

$$V_2 = V_m \cos\left(\frac{\pi y}{2H}\right)$$

where y is measured from the channel centerline. Express V_m in terms of V_1.



Given: Data on flow at inlet and outlet of channel

Find: Find u_{max}

Solution:

Basic equation $$\int_{CS} \rho \vec{V} \cdot d\vec{A} = 0$$

Assumptions: 1) Steady flow 2) Incompressible flow

Evaluating at 1 and 2 $$-\rho \cdot V_1 \cdot H \cdot w + \int_{-H}^{H} \rho \cdot V_2(y) \cdot w \, dy = 0$$

or $$V_1 \cdot H = \int_{-H}^{H} V_m \cdot \cos\left(\frac{\pi \cdot y}{2 \cdot H}\right) dy = 2 \cdot \int_{0}^{H} V_m \cdot \cos\left(\frac{\pi \cdot y}{2 \cdot H}\right) dy = 2 \cdot V_m \cdot \frac{2 \cdot H}{\pi} \cdot \left(\sin\left(\frac{\pi}{2}\right) - \sin(0)\right) = \frac{4 \cdot H \cdot V_m}{\pi}$$

Hence $$V_m = \frac{\pi}{4} \cdot V_1$$

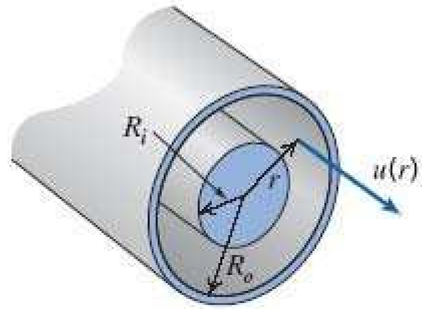
Problem 4.37

[Difficulty: 3]

4.37 The velocity profile for laminar flow in an annulus is given by

$$u(r) = -\frac{\Delta p}{4\mu L}\left[R_o^2 - r^2 + \frac{R_o^2 - R_i^2}{\ln(R_i/R_o)}\ln\frac{R_o}{r}\right]$$

where $\Delta p/L = -10$ kPa/m is the pressure gradient, μ is the viscosity (SAE 10 oil at 20°C), and $R_o = 5$ mm and $R_i = 1$ mm are the outer and inner radii. Find the volume flow rate, the average velocity, and the maximum velocity. Plot the velocity distribution.



Given: Velocity distribution in annulus

Find: Volume flow rate; average velocity; maximum velocity; plot velocity distribution

Solution:

Governing equation: For the flow rate (Eq. 4.14a) and average velocity (Eq. 4.14b)

$$Q = \int \vec{V}\,d\vec{A} \qquad V_{av} = \frac{Q}{A}$$

The given data is $R_o = 5\cdot\text{mm}$ $\quad R_i = 1\cdot\text{mm}$ $\quad \dfrac{\Delta p}{L} = -10\cdot\dfrac{\text{kPa}}{\text{m}}$ $\quad \mu = 0.1\cdot\dfrac{\text{N}\cdot\text{s}}{\text{m}^2}$ (From Fig. A.2)

$$u(r) = \frac{-\Delta p}{4\cdot\mu\cdot L}\cdot\left(R_o^2 - r^2 + \frac{R_o^2 - R_i^2}{\ln\left(\frac{R_i}{R_o}\right)}\cdot\ln\left(\frac{R_o}{r}\right)\right)$$

The flow rate is
$$Q = \int_{R_i}^{R_o} u(r)\cdot 2\cdot\pi\cdot r\,dr$$

Considerable mathematical manipulation leads to
$$Q = \frac{\Delta p\cdot\pi}{8\cdot\mu\cdot L}\cdot\left(R_o^2 - R_i^2\right)\cdot\left[\frac{\left(R_o^2 - R_i^2\right)}{\ln\left(\frac{R_o}{R_i}\right)} - \left(R_i^2 + R_o^2\right)\right]$$

Substituting values
$$Q = \frac{\pi}{8}\cdot\left(-10\cdot 10^3\right)\cdot\frac{\text{N}}{\text{m}^2\cdot\text{m}}\cdot\frac{\text{m}^2}{0.1\cdot\text{N}\cdot\text{s}}\cdot\left(5^2 - 1^2\right)\cdot\left(\frac{\text{m}}{1000}\right)^2\cdot\left[\frac{5^2 - 1^2}{\ln\left(\frac{5}{1}\right)} - \left(5^2 + 1^2\right)\right]\cdot\left(\frac{\text{m}}{1000}\right)^2$$

$$Q = 1.045\times 10^{-5}\,\frac{\text{m}^3}{\text{s}} \qquad Q = 10.45\cdot\frac{\text{mL}}{\text{s}}$$

The average velocity is
$$V_{av} = \frac{Q}{A} = \frac{Q}{\pi\cdot\left(R_o^2 - R_i^2\right)} \qquad V_{av} = \frac{1}{\pi}\times 1.045\times 10^{-5}\cdot\frac{\text{m}^3}{\text{s}}\times\frac{1}{5^2 - 1^2}\cdot\left(\frac{1000}{\text{m}}\right)^2 \qquad V_{av} = 0.139\,\frac{\text{m}}{\text{s}}$$

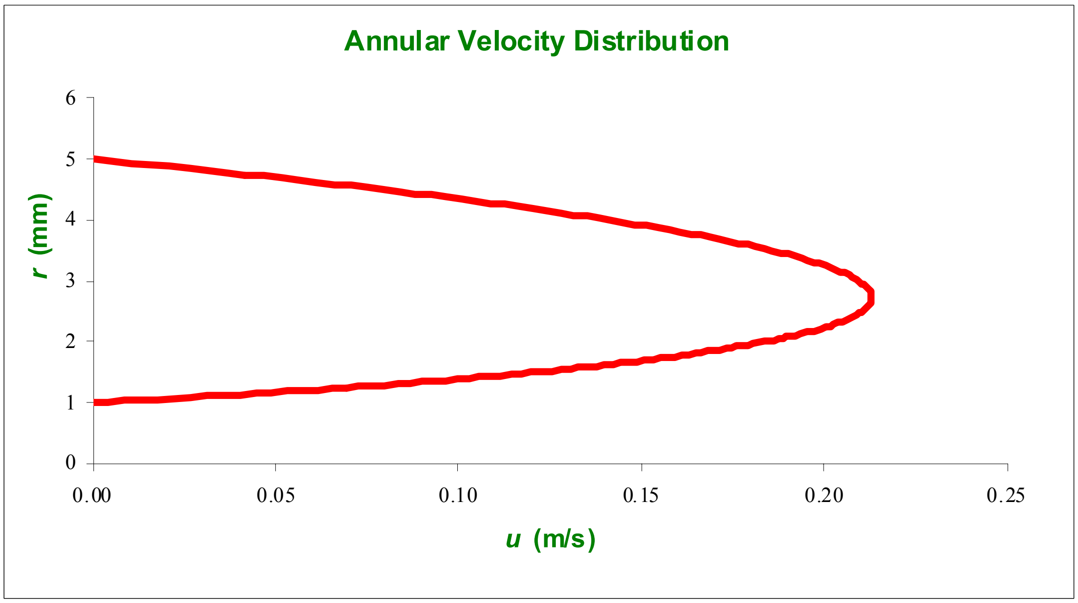
The maximum velocity occurs when
$$\frac{du}{dr} = 0 = \frac{d}{dx}\left[\frac{-\Delta p}{4\cdot\mu\cdot L}\cdot\left(R_o^2 - r^2 + \frac{R_o^2 - R_i^2}{\ln\left(\frac{R_i}{R_o}\right)}\cdot\ln\left(\frac{R_o}{r}\right)\right)\right] = -\frac{\Delta p}{4\cdot\mu\cdot L}\cdot\left[-2\cdot r - \frac{\left(R_o^2 - R_i^2\right)}{\ln\left(\frac{R_i}{R_o}\right)\cdot r}\right]$$

Then $\quad r = \sqrt{\dfrac{R_i^{\,2} - R_o^{\,2}}{2 \cdot \ln\left(\dfrac{R_i}{R_o}\right)}} \qquad r = 2.73 \cdot \text{mm}$ $\quad$ Substituting in u(r) $\qquad u_{max} = u(2.73 \cdot \text{mm}) = 0.213 \cdot \dfrac{m}{s}$

The maximum velocity using Solver instead, and the plot, are also shown in an *Excel* workbook

R_o =	5	mm
R_i =	1	mm
$\Delta p/L$ =	-10	kPa/m
μ =	0.1	N.s/m^2

r (mm)	*u* (m/s)
1.00	0.000
1.25	0.069
1.50	0.120
1.75	0.157
2.00	0.183
2.25	0.201
2.50	0.210
2.75	0.213
3.00	0.210
3.25	0.200
3.50	0.186
3.75	0.166
4.00	0.142
4.25	0.113
4.50	0.079
4.75	0.042
5.00	0.000



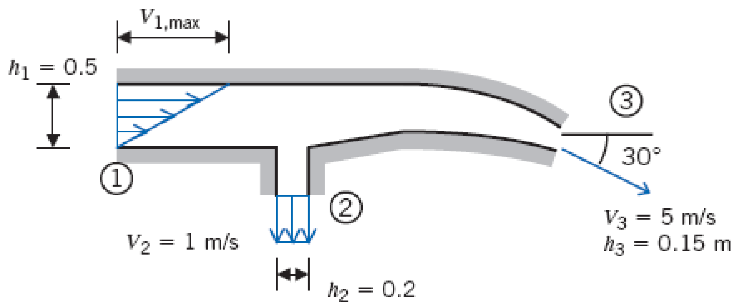
The maximum velocity can be found using *Solver*

r (mm)	*u* (m/s)
2.73	0.213

Problem 4.38

[Difficulty: 2]

4.38 A two-dimensional reducing bend has a linear velocity profile at section ①. The flow is uniform at sections ② and ③. The fluid is incompressible and the flow is steady. Find the maximum velocity, $V_{1,max}$, at section ①.



Given: Data on flow at inlet and outlet of a reducing elbow

Find: Find the maximum velcoity at section 1

Solution:

Basic equation
$$\int_{CS} \rho \vec{V} \cdot d\vec{A} = 0$$

Assumptions: 1) Steady flow 2) Incompressible flow

Evaluating at 1, 2 and 3
$$-\int_0^{h_1} V_1(y)\cdot w\,dy + V_2\cdot w\cdot h_2 + V_3\cdot w\cdot h_3 = 0$$

or
$$\frac{V_{1max}}{h_1}\cdot\int_0^{h_1} y\,dy = \frac{V_{1max}}{h_1}\cdot\frac{h_1^{\ 2}}{2} = V_2\cdot h_2 + V_3\cdot h_3$$

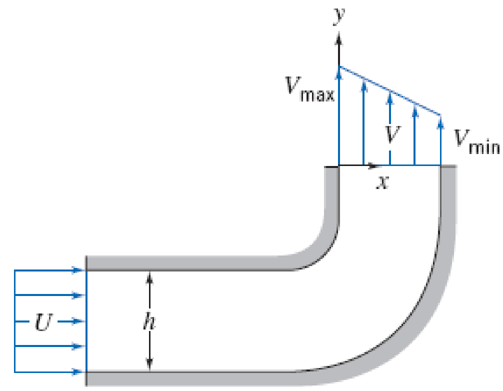
Hence
$$V_{1max} = \frac{2}{h_1}\cdot\left(V_3\cdot h_3 + V_2\cdot h_2\right)$$

$$V_{1max} = \frac{2}{0.5\cdot m}\cdot\left(5\cdot\frac{m}{s}\times 0.15\cdot m + 1\cdot\frac{m}{s}\times 0.2\cdot m\right) \qquad V_{1max} = 3.80\,\frac{m}{s}$$

Problem 4.39

[Difficulty: 2]

4.39 Water enters a two-dimensional, square channel of constant width, $h = 75.5$ mm, with uniform velocity, U. The channel makes a 90° bend that distorts the flow to produce the linear velocity profile shown at the exit, with $v_{max} = 2\,v_{min}$. Evaluate v_{min}, if $U = 7.5$ m/s.



Given: Data on flow at inlet and outlet of channel

Find: Find u_{max}

Solution:

Basic equation $$\int_{CS} \rho \vec{V} \cdot d\vec{A} = 0$$

Assumptions: 1) Steady flow 2) Incompressible flow

Evaluating at inlet and exit $$-U\cdot w\cdot h + \int_0^h V_{exit}(x)\cdot w\,dx = 0$$

Here we have $$V_{exit} = V_{max} - \left(V_{max} - V_{min}\right)\cdot\frac{x}{h}$$ But we also have $$V_{max} = 2\cdot V_{min}$$

Hence $$V_{exit} = 2\cdot V_{min} - V_{min}\cdot\frac{x}{h}$$

$$\int_0^h V_{exit}(x)\cdot w\,dx = \int_0^h \left(2\cdot V_{min} - V_{min}\cdot\frac{x}{h}\right)\cdot w\,dx = \left(2\cdot V_{min}\cdot h - V_{min}\cdot\frac{h^2}{2\cdot h}\right)\cdot w = \frac{3}{2}\cdot V_{min}\cdot h\cdot w$$

Hence $$\frac{3}{2}\cdot V_{min}\cdot h\cdot w = U\cdot w\cdot h \qquad V_{min} = \frac{2}{3}\cdot U$$

$$V_{min} = \frac{2}{3}\times 7.5\cdot\frac{m}{s} \qquad V_{min} = 5.00\cdot\frac{m}{s}$$

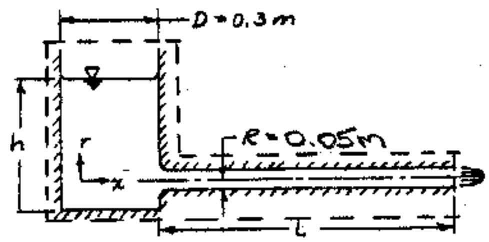
Problem 4.40

[Difficulty: 2]

4.40 Viscous liquid from a circular tank, $D = 300$ mm in diameter, drains through a long circular tube of radius $R = 50$ mm. The velocity profile at the tube discharge is

$$u = u_{max}\left[1 - \left(\frac{r}{R}\right)^2\right]$$

Show that the average speed of flow in the drain tube is $\bar{V} = \frac{1}{2}u_{max}$. Evaluate the rate of change of liquid level in the tank at the instant when $u_{max} = 0.155$ m/s.



Solution:

(a) The average velocity $\bar{V}$ is defined as Q/A.

Since $Q = \int u\,dA$, $dA = 2\pi r\,dr$ and $A = \pi R^2$, then

$$\bar{V} = \frac{Q}{A} = \frac{1}{\pi R^2}\int_0^R u_{max}\left[1 - \left(\frac{r}{R}\right)^2\right]2\pi r\,dr = \frac{2u_{max}}{R^2}\int_0^R\left[1 - \left(\frac{r}{R}\right)^2\right]r\,dr$$

$$\bar{V} = \frac{2u_{max}}{R^2}R^2\int_0^1\left[1 - \left(\frac{r}{R}\right)^2\right]\left(\frac{r}{R}\right)d\left(\frac{r}{R}\right) = 2u_{max}\left[\frac{1}{2}\left(\frac{r}{R}\right)^2 - \frac{1}{4}\left(\frac{r}{R}\right)^4\right]_0^1$$

$$\bar{V} = \frac{1}{2}u_{max}$$

$\bar{V}$

(b) Apply conservation of mass to the CV shown

Basic equation: $\quad 0 = \frac{\partial}{\partial t}\int_{CV}\rho\,d\forall + \int_{CS}\rho\vec{V}\cdot d\vec{A}$

Assumptions: (1) neglect air entering the CV
(2) incompressible flow

Then

$$0 = \rho_L\frac{\partial}{\partial t}\forall_{CV} + \left\{\left|\rho_L\bar{V}A_e\right|\right\} = \rho_L\frac{\partial}{\partial t}\left[\frac{\pi D^2}{4}h + L\pi R^2\right] + \rho\bar{V}\pi R^2$$

$$0 = \frac{\pi D^2}{4}\frac{dh}{dt} + \bar{V}\pi R^2 \qquad \left(\text{note } \frac{dL}{dt} = 0\right)$$

$$\therefore\ \frac{dh}{dt} = -4\bar{V}\left(\frac{R}{D}\right)^2 \qquad \text{But } \bar{V} = \frac{1}{2}u_{max} \text{ and hence}$$

$$\frac{dh}{dt} = -2u_{max}\left(\frac{R}{D}\right)^2 = -2\times 0.155\frac{m}{s}\times\left(\frac{0.05\,m}{0.30\,m}\right)^2\times 1000\frac{mm}{m}$$

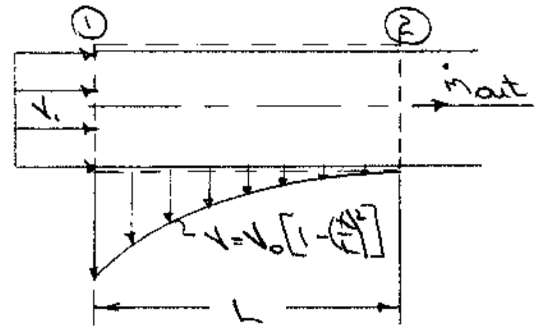
$$\frac{dh}{dt} = -8.61\ mm/s \quad \text{(level is falling)}$$

$\frac{dh}{dt}$

4.41 A porous round tube with $D = 60$ mm carries water. The inlet velocity is uniform with $V_1 = 7.0$ m/s. Water flows radially and axisymmetrically outward through the porous walls with velocity distribution

$$v = V_0\left[1 - \left(\frac{x}{L}\right)^2\right]$$

where $V_0 = 0.03$ m/s and $L = 0.950$ m. Calculate the mass flow rate inside the tube at $x = L$.



Solution:

Basic equation: $0 = \cancel{\frac{\partial}{\partial t}\int_{CV} \rho\, d\forall}^{\,0(1)} + \int_{CS} \rho \vec{V}\cdot d\vec{A}$

Assumptions: (1) steady flow
(2) ρ = constant

Then

$$0 = \int_{A_1} \rho\vec{V}\cdot d\vec{A} + \int_{A_2}\rho\vec{V}\cdot d\vec{A} + \int_{A_{circ}}\rho \vec{V}\cdot d\vec{A}$$

$$= -|\rho V_1 A_1| + \dot m_2 + \int_0^L \rho V_0\left[1-\left(\frac{x}{L}\right)^2\right] 2\pi R\, dx$$

$$\dot m_2 = \rho V_1 A_1 - 2\pi R\rho V_0 \int_0^L \left[1 - \frac{x^2}{L^2}\right] dx$$

$$= \rho V_1 \frac{\pi D_1^2}{4} - 2\pi R\rho V_0 \left[x - \frac{x^3}{3L^2}\right]_0^L$$

$$= \frac{\pi}{4}\rho V_1 D_1^2 - \frac{4}{3}\pi R \rho V_0 L$$

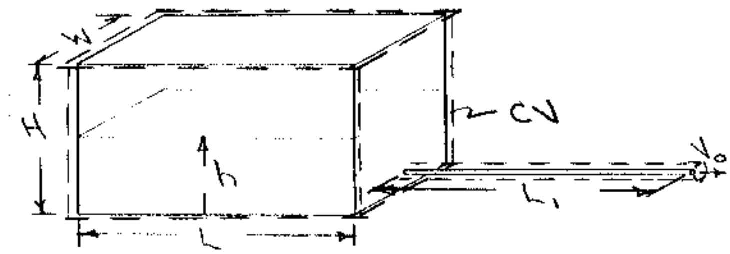
$$\dot m_2 = \frac{\pi}{4} \times 999\,\frac{kg}{m^3} \times 7.0\,\frac{m}{s} \times (0.06)^2 m^2 - \frac{4}{3}\pi \times 0.03\,m \times 999\,\frac{kg}{m^3} \times 0.03\,\frac{m}{s} \times 0.95\,m$$

$$\dot m_2 = 19.8\,\frac{kg}{s} - 3.6\,\frac{kg}{s} = 16.2\,kg/s \longleftarrow \dot m_{out}$$

Problem 4.42

[Difficulty: 2]

4.42 A rectangular tank used to supply water for a Reynolds flow experiment is 230 mm deep. Its width and length are $W = 150$ mm and $L = 230$ mm. Water flows from the outlet tube (inside diameter $D = 6.35$ mm) at Reynolds number $Re = 2000$, when the tank is half full. The supply valve is closed. Find the rate of change of water level in the tank at this instant.



Solution:

Apply conservation of mass to CV which includes tank and tube.

Basic equation: $0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$

Definition: $Re = \frac{\rho D \bar{V}}{\mu} = \frac{D\bar{V}}{\nu}$

Assumptions: (1) uniform flow at exit of tube

(2) incompressible flow

(3) neglect air entering the control volume

Then,

$$0 = \frac{\partial}{\partial t}\left[\cancel{\rho} W L h + \cancel{\rho}\frac{\pi D^2}{4} L_1\right] + \left\{+\left|\cancel{\rho}\bar{V}_o \frac{\pi D^2}{4}\right|\right\}$$

$$0 = WL\frac{dh}{dt} + \bar{V}_o \frac{\pi D^2}{4} \qquad (\text{note } L_1 = \text{constant})$$

$$\therefore \frac{dh}{dt} = -\bar{V}_o \frac{\pi D^2}{4WL}$$

To find $\bar{V}$ use the definition of Re

$$\bar{V}_o = \frac{Re\,\nu}{D}$$

For water at 20C $\nu = 1\times10^{-6}$ m²/sec (Table A.8)

$$V_o = 2000 \times 1\times10^{-6}\,\frac{m^2}{sec} \times \frac{1}{6.35\times10^{-3}\,m} = 0.315\ m/sec$$

$$\frac{dh}{dt} = -\bar{V}_o\frac{\pi D^2}{4WL} = -\frac{0.315}{4}\frac{m}{sec} \times \frac{\pi(6.35)^2\,mm^2}{150\,mm\times230\,mm}\times10^3\frac{mm}{m}$$

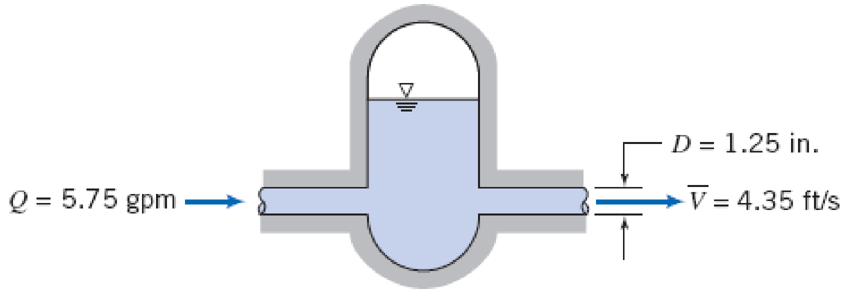
$$\frac{dh}{dt} = -0.289\ mm/sec \quad (\text{falling})$$

$\frac{dh}{dt}$

Problem 4.43

[Difficulty: 2]

4.43 A *hydraulic accumulator* is designed to reduce pressure pulsations in a machine tool hydraulic system. For the instant shown, determine the rate at which the accumulator gains or loses hydraulic oil.



Solution:

Use the control volume shown

Basic equation:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$

Assumptions: (1) uniform flow at section ②
(2) ρ = constant

Then,

$$0 = \frac{\partial}{\partial t}(M_{cv}) + \int_{A_1} \{-|\rho V_1 dA_1|\} + \int_{A_2} \{|\rho V_2 dA_2|\}$$

But $\int_{A_1} \rho V_1 dA_1 = \rho Q_1$ where Q = volume flowrate and $\rho = SG \, \rho_{H_2O}$

So $0 = \frac{\partial}{\partial t} M_{cv} - \rho Q_1 + \rho V_2 A_2$

or

$$\frac{\partial M_{cv}}{\partial t} = \rho (Q_1 - V_2 A_2)$$

$$= SG \, \rho_{H_2O} \left(Q_1 - V_2 \pi \frac{D_2^2}{4}\right) \quad \text{where } SG = 0.88 \text{ (Table A.2)}$$

$$= 0.88 \times 1.94 \frac{slug}{ft^3} \left[5.75 \frac{gal}{min} \times \frac{ft^3}{7.48 gal} \times \frac{min}{60 s} - 4.35 \frac{ft}{s} \times \frac{\pi}{4} \times (1.25)^2 in^2 \times \frac{ft^2}{144 in^2}\right]$$

$$\frac{\partial M_{cv}}{\partial t} = -4.14 \times 10^{-2} \frac{slug}{s} \quad \text{or} \quad -1.33 \ lbm/s \qquad \frac{\partial M_{cv}}{\partial t}$$

(mass is decreasing in the CV)

Since $M_{cv} = \rho_{oil} \forall_{oil}$

$$\frac{\partial M_{cv}}{\partial t} = \frac{\partial}{\partial t}(\rho_{oil} \forall_{oil}) = \rho_{oil} \frac{\partial \forall_{oil}}{\partial t} = SG_{oil} \, \rho_{H_2O} \frac{\partial \forall_{oil}}{\partial t}$$

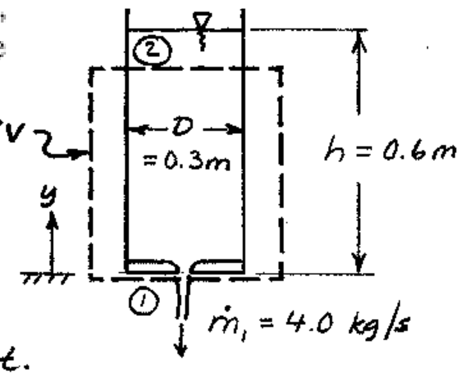
$$\frac{\partial \forall_{oil}}{\partial t} = \frac{1}{SG_{oil} \, \rho_{H_2O}} \frac{\partial M_{cv}}{\partial t} = \frac{1}{0.88} \times \frac{ft^3}{1.94 \, slugs} \times (-4.14) \times 10^{-2} \frac{slug}{s}$$

$$\frac{\partial \forall_{oil}}{\partial t} = -2.43 \times 10^{-2} \frac{ft^3}{s} \quad \text{or} \quad 0.181 \ gal/s \qquad \frac{\partial \forall_{oil}}{\partial t}$$

Problem 4.44

[Difficulty: 2]

4.44 A cylindrical tank, 0.3 m in diameter, drains through a hole in its bottom. At the instant when the water depth is 0.6 m, the flow rate from the tank is observed to be 4 kg/s. Determine the rate of change of water level at this instant.



Solution: Apply conservation of mass to CV shown. Note section 2 cuts below free surface, so $\vec{V}_2$ corresponds to free surface velocity; volume of CV is constant.

Basic equation: $$0 = \cancel{\frac{\partial}{\partial t}\int_{CV} \rho\, d\forall}^{=0(1)} + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$

Assumptions: (1) Incompressible flow, so unsteady term is zero, since volume of CV is fixed
(2) Uniform flow at each section

Then
$$0 = \rho \vec{V}_1 \cdot \vec{A}_1 + \rho \vec{V}_2 \cdot \vec{A}_2 = \dot{m}_1 + \rho \vec{V}_2 \cdot \vec{A}_2$$

and
$$\vec{V}_2 \cdot \vec{A}_2 = -\frac{\dot{m}_1}{\rho} = -\frac{4.0\ \text{kg}}{\text{s}} \times \frac{\text{m}^3}{999\ \text{kg}} = -0.004\ \text{m}^3/\text{s}$$

Since $\vec{V}_2 \cdot \vec{A}_2 < 0$, flow at section 2 is into CV. Therefore

$$V_2 = \frac{|V_2 A_2|}{A_2} = 0.004\ \frac{\text{m}^3}{\text{s}} \times \frac{4}{\pi} \times \frac{1}{(0.3)^2\ \text{m}^2} = 0.0566\ \text{m/s}$$

The water level is falling at 56.6 mm/s.

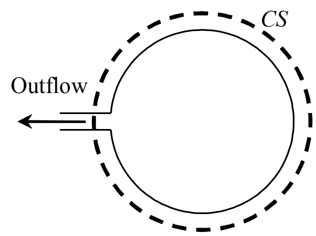
$$\vec{V}_s = -V_2 \hat{j} = -56.6 \hat{j}\ \text{mm/s}$$

$\vec{V}_s$

Problem 4.45

[Difficulty: 2]

4.45 A tank of 0.4 m^3 volume contains compressed air. A valve is opened and air escapes with a velocity of 250 m/s through an opening of 100 mm^2 area. Air temperature passing through the opening is −20°C and the absolute pressure is 300 kPa. Find the rate of change of density of the air in the tank at this moment.



Given: Data on airflow out of tank

Find: Find rate of change of density of air in tank

Solution:

Basic equation
$$\frac{\partial}{\partial t}\int_{CV} \rho\, dV\!\!\!/ + \int_{CS} \rho \vec{V}\cdot d\vec{A} = 0$$

Assumptions: 1) Density in tank is uniform 2) Uniform flow 3) Air is an ideal gas

Hence
$$V_{tank}\cdot\frac{d\rho_{tank}}{dt} + \rho_{exit}\cdot V\cdot A = 0 \qquad \frac{d\rho_{tank}}{dt} = -\frac{\rho_{exit}\cdot V\cdot A}{V_{tank}} = -\frac{p_{exit}\cdot V\cdot A}{R_{air}\cdot T_{exit}\cdot V_{tank}}$$

$$\frac{d\rho_{tank}}{dt} = -300\times 10^3\cdot\frac{N}{m^2}\times 250\cdot\frac{m}{s}\times 100\cdot mm^2\times\left(\frac{1\cdot m}{1000\cdot mm}\right)^2\times\frac{1}{286.9}\cdot\frac{kg\cdot K}{N\cdot m}\times\frac{1}{(-20+273)\cdot K}\times\frac{1}{0.4\cdot m^3}$$

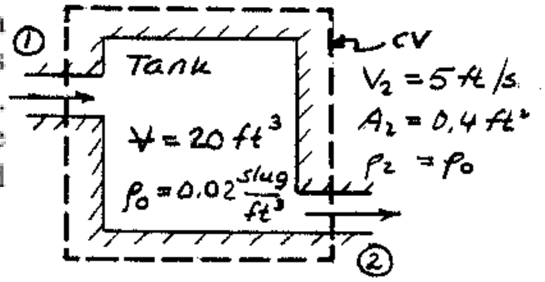
Hence
$$\frac{d\rho_{tank}}{dt} = -0.258\cdot\frac{\frac{kg}{m^3}}{s}$$

The mass in the tank is decreasing, as expected

Problem 4.46

[Difficulty: 2]

4.46 Air enters a tank through an area of 0.2 ft^2 with a velocity of 15 ft/s and a density of 0.03 slug/ft^3. Air leaves with a velocity of 5 ft/s and a density equal to that in the tank. The initial density of the air in the tank is 0.02 slug/ft^3. The total tank volume is 20 ft^3 and the exit area is 0.4 ft^2. Find the initial rate of change of density in the tank.



Solution: Apply conservation of mass, using CV shown.

Basic equation: $0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A}$

Assumptions: (1) Density is uniform in tank, so $\frac{\partial}{\partial t}\int_{CV} \rho\, d\forall = \frac{\partial}{\partial t}(\rho_0 \forall)$

(2) Flow is uniform at inlet and outlet sections.

Then

$$0 = \frac{\partial}{\partial t}(\rho_0 \forall) + \rho_1 \vec{V}_1 \cdot \vec{A}_1 + \rho_0 \vec{V}_2 \cdot \vec{A}_2$$

$$0 = \rho_0 \underbrace{\frac{\partial \forall}{\partial t}}_{=0} + \forall \frac{\partial \rho_0}{\partial t} - |\rho_1 V_1 A_1| + |\rho_0 V_2 A_2|$$

or

$$\frac{\partial \rho_0}{\partial t} = \frac{|\rho_1 V_1 A_1| - |\rho_0 V_2 A_2|}{\forall}$$

Substituting magnitudes

$$\frac{\partial \rho_0}{\partial t} = \frac{1}{20\ ft^3}\left[\frac{0.03\ slug}{ft^3} \times \frac{15\ ft}{s} \times 0.2\ ft^2 - \frac{0.02\ slug}{ft^3} \times \frac{5\ ft}{s} \times 0.4\ ft^2\right]$$

$$\frac{\partial \rho_0}{\partial t} = 2.50 \times 10^{-3}\ slug/(ft^3 \cdot s)$$

$\frac{\partial \rho_0}{\partial t}$

{Note since $\frac{\partial \rho_0}{\partial t} > 0$, mass in tank increases.}

Problem 4.47

[Difficulty: 2]

4.47 A recent TV news story about lowering Lake Shafer near Monticello, Indiana, by increasing the discharge through the dam that impounds the lake, gave the following information for flow through the dam:

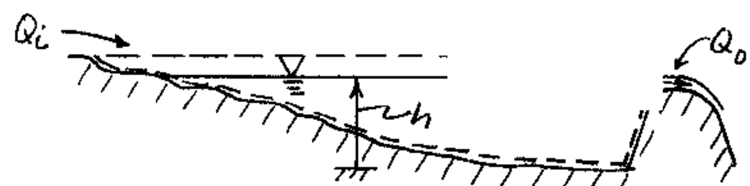
Normal flow rate	290 cfs
Flow rate during draining of lake	2000 cfs

(The flow rate during draining was stated to be equivalent to 16,000 gal/s.) The announcer also said that during draining the lake level was expected to fall at the rate of 1 ft every 8 hr. Calculate the actual flow rate during draining in gal/s. Estimate the surface area of the lake.

Solution: Convert units

$$Q = 2000\,\frac{ft^3}{s} = 2000\,\frac{ft^3}{s} \times 7.48\,\frac{gal}{ft^3} = 1.50 \times 10^4\ gal/s$$ Q

Apply conservation of mass using CV shown:



Basic equation: $0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$

Assumption: (1) ρ = constant

Then $\frac{d\forall}{dt} = A\frac{dh}{dt} = -\int_{CS} \vec{V} \cdot d\vec{A} = -Q_0 + Q_i$

$$A = -\frac{Q_0 - Q_i}{dh/dt} = -\frac{\Delta Q}{dh/dt}\ ; \quad \Delta Q = Q_0 - Q_i$$

But $\Delta Q = 1{,}710\ ft^3/s$ and $dh/dt = -1\ ft/8\ hr$, since decreasing.

Thus

$$A = -1{,}710\,\frac{ft^3}{s} \times \frac{8\,hr}{-1\,ft} \times 3600\,\frac{s}{hr} = 4.92 \times 10^7\ ft^2$$ A

Since 1 acre = 43,600 ft^2,

$$A = 4.92 \times 10^7\ ft^2 \times \frac{acre}{43{,}600\ ft^2} \approx 1{,}130\ acres$$

Since 1 square mile = 640 acres, the lake surface area is slightly less than 2 square miles!

Problem 4.48

[Difficulty: 3]

4.48 A cylindrical tank, of diameter $D = 6$ in., drains through an opening, $d = 0.25$ in., in the bottom of the tank. The speed of the liquid leaving the tank is approximately $V = \sqrt{2gy}$ where y is the height from the tank bottom to the free surface. If the tank is initially filled with water to $y_0 = 3$ ft, determine the water depths at $t = 1$ min, $t = 2$ min, and $t = 3$ min. Plot y (ft) versus t for the first three min.

Given: Data on draining of a tank

Find: Depth at various times; Plot of depth versus time

Solution:

Basic equation
$$\frac{\partial}{\partial t}\int_{CV} \rho\, dV\!\!\!/ + \int_{CS} \rho \vec{V} \cdot d\vec{A} = 0$$

Assumptions: 1) Uniform flow 2) Incompressible flow 3) Neglect air density

Treating the tank as the CV the basic equation becomes

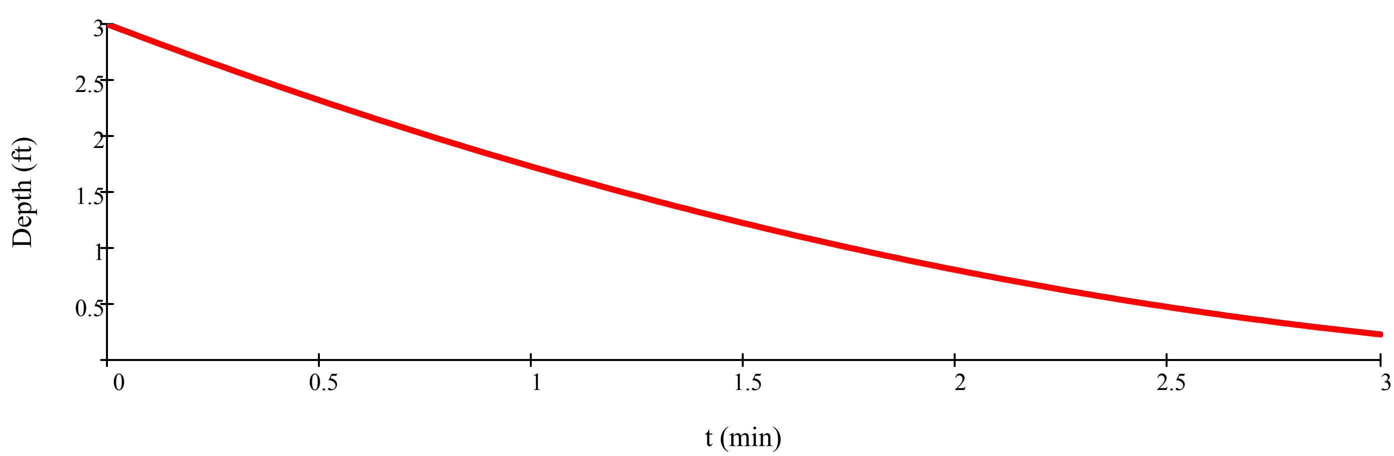
$$\frac{\partial}{\partial t}\int_0^y \rho \cdot A_{tank}\, dy + \rho \cdot V \cdot A_{opening} = 0 \qquad \text{or} \qquad \rho \cdot \frac{\pi}{4} \cdot D^2 \cdot \frac{dy}{dt} + \rho \cdot \frac{\pi}{4} \cdot d^2 \cdot V = 0$$

Using $V = \sqrt{2 \cdot g \cdot y}$ and simplifying $\dfrac{dy}{dt} = -\left(\dfrac{d}{D}\right)^2 \cdot \sqrt{2 \cdot g} \cdot y^{\frac{1}{2}}$

Separating variables $\dfrac{dy}{y^{\frac{1}{2}}} = \left(\dfrac{d}{D}\right)^2 \cdot \sqrt{2 \cdot g} \cdot dt$ and integrating $2 \cdot \left(y^{\frac{1}{2}} - y_0^{\frac{1}{2}}\right) = -\left(\dfrac{d}{D}\right)^2 \cdot \sqrt{2 \cdot g t}$

Solving for y
$$y(t) = y_0 \cdot \left[1 - \sqrt{\frac{g}{2 \cdot y_0}} \cdot \left(\frac{d}{D}\right)^2 \cdot t\right]^2$$

Using the given data $y(1 \cdot min) = 1.73 \cdot ft$ $\quad y(2 \cdot min) = 0.804 \cdot ft$ $\quad y(3 \cdot min) = 0.229 \cdot ft$



Problem 4.49

[Difficulty: 3]

4.49 For the conditions of Problem 4.48, estimate the times required to drain the tank from initial depth to a depth $y = 2$ ft (a change in depth of 1 ft), and from $y = 2$ ft to $y = 1$ ft (also a change in depth of 1 ft). Can you explain the discrepancy in these times? Plot the time to drain to a depth $y = 1$ ft as a function of opening sizes ranging from $d = 0.1$ in. to 0.5 in.

Given: Data on draining of a tank

Find: Times to a depth of 1 foot; Plot of drain timeversus opening size

Solution:

Basic equation $$\frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A} = 0$$

Assumptions: 1) Uniform flow 2) Incompressible flow 3) Neglect air density

Treating the tank as the CV the basic equation becomes

$$\frac{\partial}{\partial t}\int_0^y \rho \cdot A_{tank}\, dy + \rho \cdot V \cdot A_{opening} = 0 \qquad \text{or} \qquad \rho \cdot \frac{\pi}{4} \cdot D^2 \cdot \frac{dy}{dt} + \rho \cdot \frac{\pi}{4} \cdot d^2 \cdot V = 0$$

Using $V = \sqrt{2 \cdot g \cdot y}$ and simplifying $\frac{dy}{dt} = -\left(\frac{d}{D}\right)^2 \cdot \sqrt{2 \cdot g} \cdot y^{\frac{1}{2}}$

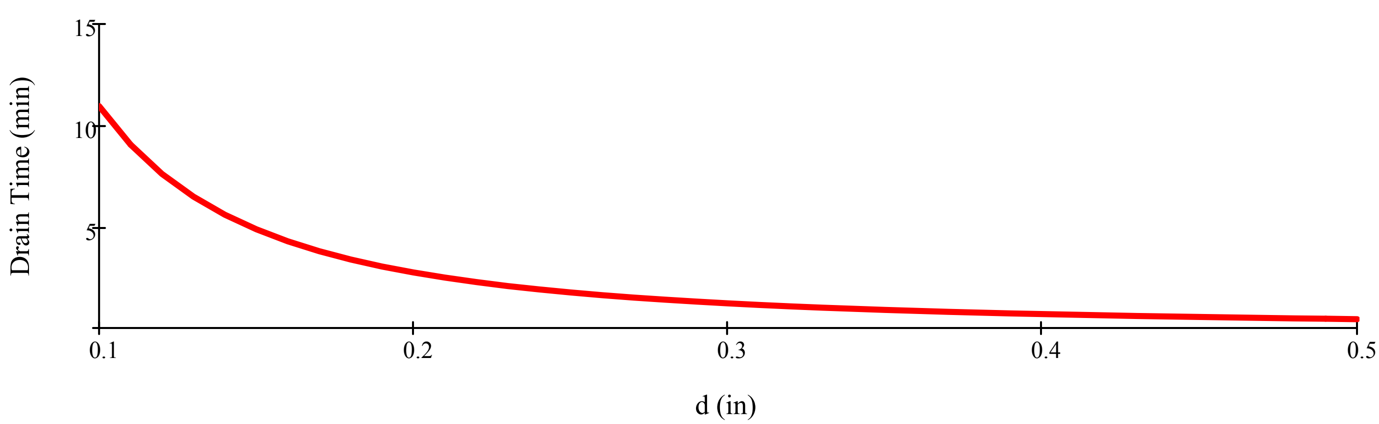
Separating variables $\frac{dy}{y^{\frac{1}{2}}} = \left(\frac{d}{D}\right)^2 \cdot \sqrt{2 \cdot g} \cdot dt$ and integrating $2 \cdot \left(y^{\frac{1}{2}} - y_0^{\frac{1}{2}}\right) = -\left(\frac{d}{D}\right)^2 \cdot \sqrt{2 \cdot g} t$

Solving for t $t = \sqrt{\frac{2 \cdot y_0}{g}} \cdot \left(\frac{D}{d}\right)^2 \cdot \left(1 - \sqrt{\frac{y}{y_0}}\right)$ Using the given data $t(2 \cdot ft) = 45.6\,s$ $t(1 \cdot ft) = 105\,s$

Hence for the first drop of 1 foot $\Delta t = t(2 \cdot ft)$ $\Delta t = 45.6\,s$

For the second drop of 1 foot $\Delta t = t(1 \cdot ft) - t(2 \cdot ft)$ $\Delta t = 59.5\,s$

This is because as the level drops the exit speed, hence drain rate, decreases.

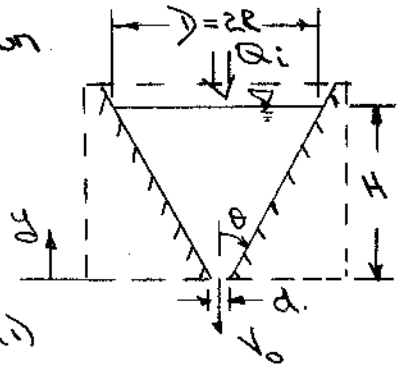


Problem 4.50

[Difficulty: 3]

4.50 A conical flask contains water to height $H = 36.8$ mm, where the flask diameter is $D = 29.4$ mm. Water drains out through a smoothly rounded hole of diameter $d = 7.35$ mm at the apex of the cone. The flow speed at the exit is approximately $V = \sqrt{2gy}$, where y is the height of the liquid free surface above the hole. A stream of water flows into the top of the flask at constant volume flow rate, $Q = 3.75 \times 10^{-7}$ m³/hr. Find the volume flow rate from the bottom of the flask. Evaluate the direction and rate of change of water surface level in the flask at this instant.

Solution: Apply continuity to the CV shown.



Basic eq.: $0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$

Assumptions: (1) uniform flow at each section
(2) neglect mass of air.

Then

$$0 = \rho \left.\frac{d\forall}{dt}\right)_{water} + \rho Q_{out} - \rho Q_{in} \quad \text{.....(1)}$$

$$Q_{out} = V_o A_o = (2gH)^{1/2} \frac{\pi d^2}{4}$$

$$Q_{out} = \left[2 \times 9.81 \frac{m}{s^2} \times 0.0368\, m\right]^{1/2} \frac{\pi}{4} \times (0.00735)^2 \, m^2$$

$$Q_{out} = 3.61 \times 10^{-5} \; m^3/s \quad (0.130 \; m^3/hr)$$ Q_{out}

From eq. (1)

$$\left.\frac{d\forall}{dt}\right)_{water} = Q_{in} - Q_{out}$$

$$\forall = \tfrac{1}{3} \text{ area of base} \times \text{altitude} = \tfrac{1}{3}\pi R^2 y$$

Since $R = y\tan\theta$, $\forall = \frac{1}{3}\pi y^3 \tan^2\theta$

$$\frac{d\forall}{dt} = \frac{1}{3}\pi \tan^2\theta \times 3y^2 \frac{dy}{dt} = \pi y^2 \tan^2\theta \frac{dy}{dt} = \pi R^2 \frac{dy}{dt}$$

$$\therefore \frac{dy}{dt} = \frac{Q_{in} - Q_{out}}{\pi R^2} = \frac{4}{\pi D^2}(Q_{in} - Q_{out})$$

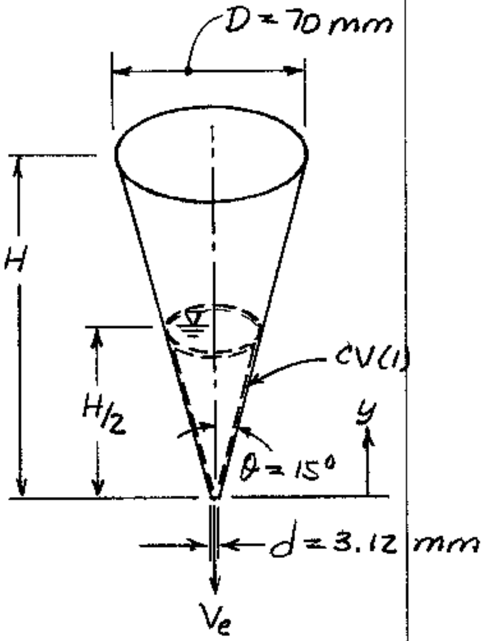
$$= \frac{4}{\pi} \times \frac{1}{(0.0294)^2\, m^2} (3.75 \times 10^{-7} - 0.130) \frac{m^3}{hr} \times \frac{hr}{3600\, s}$$

$$\frac{dy}{dt} = -0.0532 \; m/s \quad \text{(surface moves downward)}$$ $\frac{dy}{dt}$

Problem 4.51

[Difficulty: 3]

4.51 A conical funnel of half-angle $\theta = 15°$, with maximum diameter $D = 70$ mm and height H, drains through a hole (diameter $d = 3.12$ mm) in its bottom. The speed of the liquid leaving the funnel is approximately $V = \sqrt{2gy}$, where y is the height of the liquid free surface above the hole. Find the rate of change of surface level in the funnel at the instant when $y = H/2$.



Solution: Apply conservation of mass.

(1) Choose CV with top just below surface level.

Basic equation: $$0 = \underbrace{\frac{\partial}{\partial t}\int_{CV} \rho \, d\forall}_{=0(1)} + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$

Assumptions: (1) ρ = constant, $\forall$ = const, so $\partial/\partial t = 0$
(2) Uniform flow at each section.

For CV(1): $$0 = \{-|\rho V_s A_s|\}_{in} + \{+|\rho V_e A_e|\}_{out} \quad \text{or} \quad V_s = V_e \frac{A_e}{A_s}$$

Thus
$$V_s = V_e\left(\frac{d}{D/2}\right)^2 = \sqrt{2g\frac{H}{2}}\, 4\left(\frac{d}{D}\right)^2 = 4\sqrt{gH}\left(\frac{d}{D}\right)^2 = -\frac{dy}{dt} \quad \text{(since } y \text{ decreases)}$$

But $$\tan\theta = \frac{D/2}{H} \quad \text{so} \quad H = \frac{D}{2\tan\theta} = \frac{0.070 \text{ m}}{2\tan 15°} = 0.131 \text{ m}$$

Substituting,
$$\frac{dy}{dt} = -4\sqrt{9.81 \frac{\text{m}}{\text{s}^2} \times 0.131 \text{ m}}\left(\frac{0.00312 \text{ m}}{0.070 \text{ m}}\right)^2 1000 \frac{\text{mm}}{\text{m}} = -9.01 \text{ mm/s}$$ $\frac{dy}{dt}$

Alternate solution: Choose CV(2) enclosing entire funnel.

Basic equation: $$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$

Assumptions: (1) ρ = constant, but $\forall$ changes (Note: $\forall = \frac{\pi}{3} r^2 h$ for a cone.)
(2) Neglect air
(3) Uniform flow at outlet section

Then
$$0 = \rho \frac{\partial}{\partial t} \forall_{H_2O} + \{+|\rho V_e A_e|\}_{out} \quad \text{or} \quad \frac{d\forall}{dt} = -V_e A_e$$

The volume of water is $\forall = \frac{\pi}{3} r^2 h = \frac{\pi}{3}(y\tan\theta)^2 y = \frac{\pi y^3 \tan^2\theta}{3}$

so $\frac{d\forall}{dt} = \pi y^2 \tan^2\theta \frac{dy}{dt} = \pi\left(\frac{D}{4}\right)^2 \frac{dy}{dt}$ and $\frac{\pi D^2}{16}\frac{dy}{dt} = -V_e A_e = -\sqrt{2gy}\,\frac{\pi d^2}{4}$

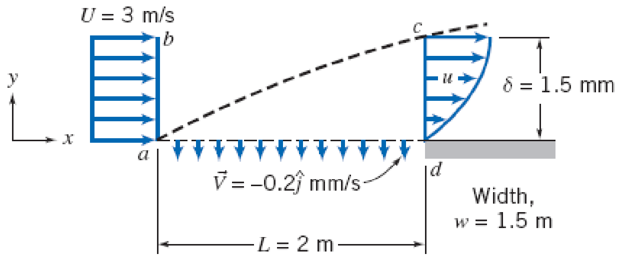
Finally, since $y = H/2$, $\frac{dy}{dt} = -4\sqrt{2gH}\left(\frac{d}{D}\right)^2$ as before.

{Note: Flow is not steady in either CV. The $\partial/\partial t$ term vanishes for CV(1) because there is no change in mass inside the CV.}

4.52 Water flows steadily past a porous flat plate. Constant suction is applied along the porous section. The velocity profile at section *cd* is

$$\frac{u}{U_\infty} = 3\left[\frac{y}{\delta}\right] - 2\left[\frac{y}{\delta}\right]^{3/2}$$

Evaluate the mass flow rate across section *bc*.



Basic equation:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A}$$

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) $\vec{V} = -v_0 \hat{j}$ along da

Then

$$0 = \int_{CS} \rho\vec{V}\cdot d\vec{A} = \int_{ab} \rho\vec{V}\cdot d\vec{A} + \dot{m}_{bc} + \int_{cd} \rho\vec{V}\cdot d\vec{A} + \int_{da} \rho\vec{V}\cdot d\vec{A}$$

or

$$0 = -\rho U_\infty w\delta + \dot{m}_{bc} + \int_0^\delta \rho U_\infty\left[3\left(\tfrac{y}{\delta}\right) - 2\left(\tfrac{y}{\delta}\right)^{1.5}\right] w\,dy + \rho v_0 w L$$

Thus

$$\dot{m}_{bc} = \rho U_\infty w\delta - \rho U_\infty w\delta \int_0^1 \left[3\left(\tfrac{y}{\delta}\right) - 2\left(\tfrac{y}{\delta}\right)^{1.5}\right] d\left(\tfrac{y}{\delta}\right) - \rho v_0 w L$$

$$= \rho w\left\{ U_\infty\delta - U_\infty\delta\left[\tfrac{3}{2}\left(\tfrac{y}{\delta}\right)^2 - \tfrac{2}{2.5}\left(\tfrac{y}{\delta}\right)^{1.5}\right]_0^1 - v_0 L\right\}$$

$$= \rho w\left[U_\infty\delta - U_\infty\delta\left(\tfrac{3}{2} - \tfrac{2}{2.5}\right) - v_0 L\right] = \rho w\left(0.3\,U_\infty\delta - v_0 L\right)$$

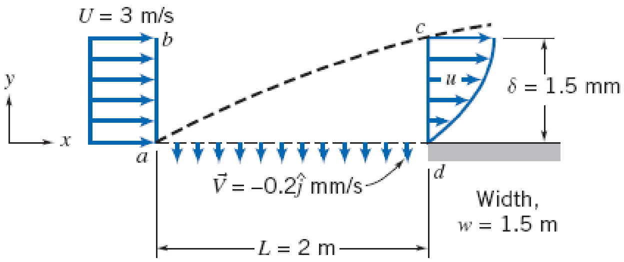
$$= 999\,\frac{kg}{m^3} \times 1.5\,m\left(0.3 \times 3\,\frac{m}{s} \times 0.0015\,m - 0.0002\,\frac{m}{s} \times 2\,m\right)$$

$\dot{m}_{bc} = 1.42\ kg/s$ ($\dot{m} > 0$, so out of CV) $\dot{m}_{bc}$

Problem 4.53

[Difficulty: 3]

4.53 Consider incompressible steady flow of standard air in a boundary layer on the length of porous surface shown. Assume the boundary layer at the downstream end of the surface has an approximately parabolic velocity profile, $u/U_\infty = 2(y/\delta) - (y/\delta)^2$. Uniform suction is applied along the porous surface, as shown. Calculate the volume flow rate across surface cd, through the porous suction surface, and across surface bc.



Solution: Apply conservation of mass to CV shown.

Basic equation:

$$0 = \cancelto{=0(1)}{\frac{\partial}{\partial t}\int_{CV} \rho\, d\forall} + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$

Assumptions: (1) Incompressible flow
(2) Parabolic profile at section cd: $\frac{u}{U_\infty} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$

Then $\quad 0 = \int_{CS} \vec{V} \cdot d\vec{A} = Q_{ab} + Q_{bc} + Q_{cd} + Q_{da} \qquad (1)$

$$Q_{cd} = \int_{cd} \vec{V} \cdot d\vec{A} = \int_0^\delta u\, w\, dy = w U_\infty \delta \int_0^1 \frac{u}{U}\, d\left(\frac{y}{\delta}\right) = w U_\infty \delta \int_0^1 \left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right] d\frac{y}{\delta}$$

$$= w U_\infty \delta \left[\left(\frac{y}{\delta}\right)^2 - \frac{1}{3}\left(\frac{y}{\delta}\right)^3\right]_0^1 = \frac{2}{3} w \delta U_\infty$$

$$Q_{cd} = \frac{2}{3} \times 1.5\, m \times 0.0015\, m \times 3\, \frac{m}{s} = 4.50 \times 10^{-3}\, m^3/s \text{ (out of CV)}$$

Q_{cd}

Flow across ad is uniform, so

$$Q_{ad} = \vec{V} \cdot \vec{A} = v\hat{j} \cdot wL(-\hat{j}) = -v w L$$

$$Q_{ad} = -0.2\, \frac{mm}{s} \times 1.5\, m \times 2\, m \times \frac{m}{1000\, mm} = 6.00 \times 10^{-4}\, m/s \text{ (out of CV)}$$

Q_{ad}

Finally, from Eq. 1,

$$Q_{bc} = -Q_{ab} - Q_{cd} - Q_{da} \qquad (2)$$

But $\quad Q_{ab} = \vec{U}_\infty \cdot \vec{A}_{ab} = U_\infty \hat{i} \cdot w\delta(-\hat{i}) = -w \delta U_\infty$

$$Q_{ab} = -1.5\, m \times 0.0015\, m \times 3\, \frac{m}{s} = -6.75 \times 10^{-3}\, m^3/s \text{ (into CV)}$$

Substituting into Eq. 2,

$$Q_{bc} = \left[-(-6.75 \times 10^{-3}) - 4.50 \times 10^{-3} - 6.00 \times 10^{-4}\right] m^3/s$$

$$Q_{bc} = 1.65 \times 10^{-3}\, m^3/s \text{ (out of CV)}$$

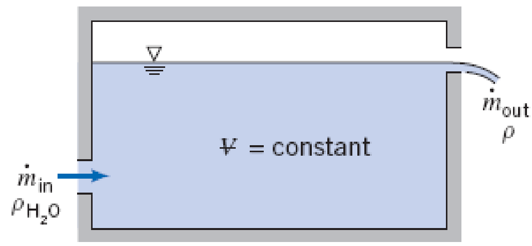
Q_{bc}

Problem 4.54

[Difficulty: 4]

4.54 A tank of fixed volume contains brine with initial density, ρ_i, greater than water. Pure water enters the tank steadily and mixes thoroughly with the brine in the tank. The liquid level in the tank remains constant. Derive expressions for (a) the rate of change of density of the liquid mixture in the tank and (b) the time required for the density to reach the value ρ_f, where $\rho_i > \rho_f > \rho_{H_2O}$.

<u>Solution:</u> Apply conservation of mass using the CV shown.



Basic equation:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$

Assumptions: (1) $\forall_{tank}$ = constant
(2) ρ uniform in tank
(3) Uniform flows at inlet and outlet sections

Then $V_1 A_1 = V_2 A_2$ since tank volume is constant, and

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \rho V A - \rho_{H_2O} V A = \frac{\partial}{\partial t} \rho \forall + (\rho - \rho_{H_2O}) V A = \forall \frac{d\rho}{dt} + (\rho - \rho_{H_2O}) V A$$

So that

$$\frac{d\rho}{dt} = -\frac{(\rho - \rho_{H_2O}) V A}{\forall}$$ $\frac{d\rho}{dt}$

Separating variables,

$$\frac{d\rho}{\rho - \rho_{H_2O}} = -\frac{VA}{\forall} dt$$

Integrating from ρ_i at $t = 0$ to ρ_f at t,

$$\int_{\rho_i}^{\rho_f} \frac{d\rho}{\rho - \rho_{H_2O}} = \ln(\rho - \rho_{H_2O})\Big]_{\rho_i}^{\rho_f} = \ln\left(\frac{\rho_f - \rho_{H_2O}}{\rho_i - \rho_{H_2O}}\right) = \int_0^t -\frac{VA}{\forall} dt = -\frac{VA}{\forall} t$$

Finally,

$$t = -\frac{\forall}{VA} \ln\left(\frac{\rho_f - \rho_{H_2O}}{\rho_i - \rho_{H_2O}}\right)$$ t

{ Note that $\rho_f \to \rho_{H_2O}$ asymptotically as $t \to \infty$. }

Problem 4.55

[Difficulty: 4]

4.55 A conical funnel of half-angle $\theta = 30°$ drains through a small hole of diameter $d = 0.25$ in. at the vertex. The speed of the liquid leaving the funnel is approximately $V = \sqrt{2gy}$, where y is the height of the liquid free surface above the hole. The funnel initially is filled to height $y_0 = 12$ in. Obtain an expression for the time, t, for the funnel to completely drain, and evaluate. Find the time to drain from 12 in. to 6 in. (a change in depth of 6 in.), and from 6 in. to completely empty (also a change in depth of 6 in.). Can you explain the discrepancy in these times? Plot the drain time t as a function diameter d for d ranging from 0.25 in. to 0.5 in.

Given: Data on draining of a funnel

Find: Formula for drain time; time to drain from 12 in to 6 in; plot drain time versus hole diameter

Solution:

Basic equation
$$\frac{\partial}{\partial t}\int_{CV}\rho d\mathit{V\!\!\!-} + \int_{CS}\rho\vec{V}\cdot d\vec{A} = 0$$

Assumptions: 1) Uniform flow 2) Incompressible flow 3) Neglect air density

Treating the funnel as the CV the basic equation becomes

$$\frac{\partial}{\partial t}\int_0^y \rho\cdot A_{funnel}\,dy + \rho\cdot V\cdot A_{opening} = 0$$

For the funnel
$$A_{funnel} = \pi\cdot r^2 = \pi\cdot(y\cdot\tan(\theta))^2$$

Hence
$$\rho\cdot\pi\cdot(\tan(\theta))^2\cdot\frac{\partial}{\partial t}\int_0^y y^2\,dy + \rho\cdot V\cdot\frac{\pi}{4}\cdot d^2 = 0$$
or
$$(\tan(\theta))^2\cdot\frac{d}{dt}\left(\frac{y^3}{3}\right) = -\sqrt{2\cdot g\cdot y}\cdot\frac{d^2}{4}$$

Then
$$(\tan(\theta))^2\cdot y^2\cdot\frac{dy}{dt} = -\sqrt{2\cdot g\cdot y}\cdot\frac{d^2}{4}$$

Separating variables
$$y^{\frac{3}{2}}\cdot dy = -\frac{\sqrt{2\cdot g\cdot d^2}}{4\cdot\tan(\theta)^2}\cdot dt$$

Hence
$$\int_{y_0}^{0} y^{\frac{3}{2}}\,dy = -\frac{\sqrt{2\cdot g}\cdot d}{4\cdot\tan(\theta)^2}\cdot t$$
or
$$\frac{2}{5}\cdot y_0^{\frac{5}{2}} = \frac{\sqrt{2\cdot g}\cdot d}{4\cdot\tan(\theta)^2}\cdot t$$

Solving for t
$$t = \frac{8}{5}\cdot\frac{\tan(\theta)^2\cdot y_0^{\frac{5}{2}}}{\sqrt{2\cdot g}\cdot d}$$
and using the given data
$$t = 2.55\cdot\text{min}$$

To find the time to drain from 12 in to 6 in., we use the time equation with the two depths; this finds the time to drain from 12 in and 6 in, so the difference is the time we want

$y_1 = 6\cdot in$

$$\Delta t_1 = \frac{8}{5}\cdot\frac{\tan(\theta)^2\cdot y_0^{\frac{5}{2}}}{\sqrt{2\cdot g}\cdot d^2} - \frac{8}{5}\cdot\frac{\tan(\theta)^2\cdot y_1^{\frac{5}{2}}}{\sqrt{2\cdot g}\cdot d^2}$$

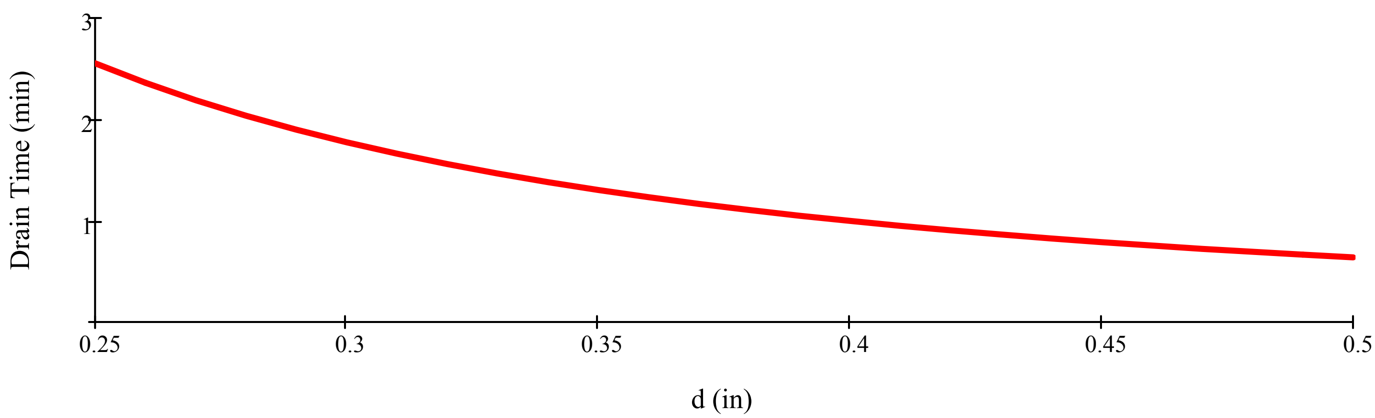
$\Delta t_1 = 2.1\cdot min$

$$\Delta t_2 = \frac{8}{5}\cdot\frac{\tan(\theta)^2\cdot y_1^{\frac{5}{2}}}{\sqrt{2\cdot g}\cdot d^2}$$

$\Delta t_2 = 0.451\cdot min$

Note that $\Delta t_1 + \Delta t_2 = 2.55\cdot min$

The second time is a bit longer because although the flow rate decreases, the area of the funnel does too.



Problem 4.56

[Difficulty: 4]

4.56 For the funnel of Problem 4.55, find the diameter d required if the funnel is to drain in $t = 1$ min. from an initial depth $y_0 = 12$ in. Plot the diameter d required to drain the funnel in 1 min as a function of initial depth y_0, for y_0 ranging from 1 in. to 24 in.

Given: Data on draining of a funnel

Find: Diameter that will drain in 1 min.; plot diamter versus depth y_0

Solution:

Basic equation

$$\frac{\partial}{\partial t}\int_{CV}\rho\, d\forall + \int_{CS}\rho\vec{V}\cdot d\vec{A} = 0$$

Assumptions: 1) Uniform flow 2) Incompressible flow 3) Neglect air density

Treating the funnel as the CV the basic equation becomes

$$\frac{\partial}{\partial t}\int_0^y \rho\cdot A_{funnel}\,dy + \rho\cdot V\cdot A_{opening} = 0$$

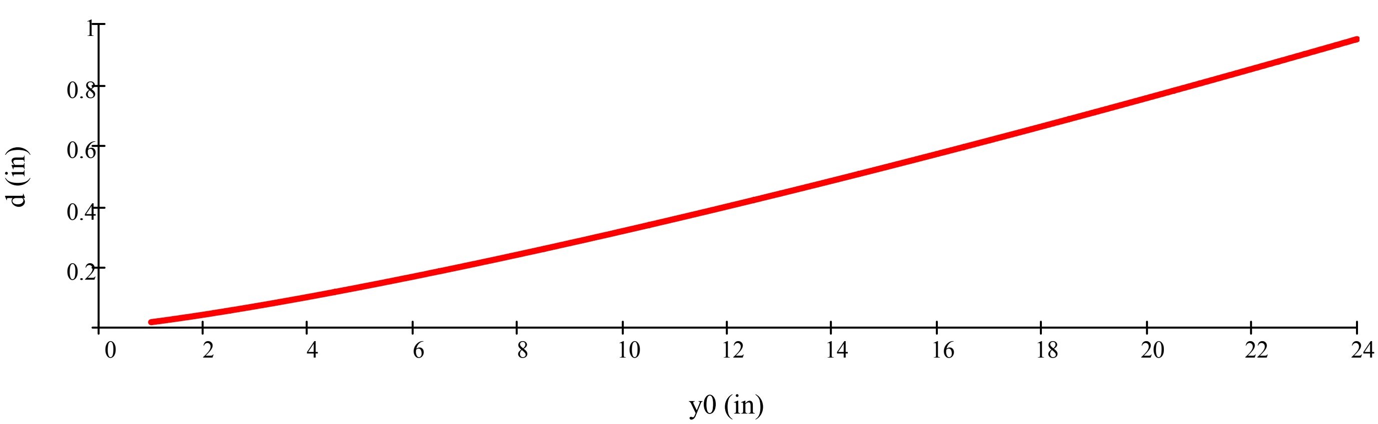
For the funnel $\quad A_{funnel} = \pi\cdot r^2 = \pi\cdot(y\cdot\tan(\theta))^2$

Hence $\quad \rho\cdot\pi\cdot(\tan(\theta))^2\cdot\frac{\partial}{\partial t}\int_0^y y^2\,dy + \rho\cdot V\cdot\frac{\pi}{4}\cdot d^2 = 0 \quad$ or $\quad (\tan(\theta))^2\cdot\frac{d}{dt}\left(\frac{y^3}{3}\right) = -\sqrt{2\cdot g\cdot y}\cdot\frac{d^2}{4}$

Then $\quad (\tan(\theta))^2\cdot y^2\cdot\frac{dy}{dt} = -\sqrt{2\cdot g\cdot y}\cdot\frac{d^2}{4} \quad$ Separating variables $\quad y^{\frac{3}{2}}\cdot dy = -\frac{\sqrt{2\cdot g\cdot d^2}}{4\cdot\tan(\theta)^2}\cdot dt$

Hence $\quad \int_{y_0}^0 y^{\frac{3}{2}}\,dy = -\frac{\sqrt{2\cdot g}\cdot d}{4\cdot\tan(\theta)^2}\cdot t \quad$ or $\quad \frac{2}{5}\cdot y_0^{\frac{5}{2}} = \frac{\sqrt{2\cdot g}\cdot d}{4\cdot\tan(\theta)^2}\cdot t$

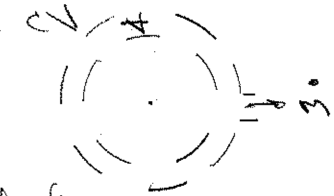
Solving for d $\quad d = \sqrt{\frac{8}{5}\cdot\frac{\tan(\theta)^2\cdot y_0^{\frac{5}{2}}}{\sqrt{2\cdot g}\cdot t}} \quad$ and using the given data, for $\quad t = 1\ \text{min} \qquad d = 0.399\ \text{in}$



4.57 Over time, air seeps through pores in the rubber of high-pressure bicycle tires. The saying is that a tire loses pressure at the rate of "a pound [1 psi] a day." The true rate of pressure loss is not constant; instead, the instantaneous leakage mass flow rate is proportional to the air density in the tire and to the gage pressure in the tire, $\dot{m} \propto \rho p$. Because the leakage rate is slow, air in the tire is nearly isothermal. Consider a tire that initially is inflated to 0.6 MPa (gage). Assume the initial rate of pressure loss is 1 psi per day. Estimate how long it will take for the pressure to drop to 500 kPa. How accurate is "a pound a day" over the entire 30 day period? Plot the pressure as a function of time over the 30 day period. Show the rule-of-thumb results for comparison.

Solution:

Apply conservation of mass to tire as the CV



Basic equation: $0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$

Assumptions: (1) uniform properties in tire
(2) air inside CV behaves as ideal gas
(3) $T = \text{constant}$ • $\forall = \text{constant}$
(4) $\dot{m} = c\,(p - p_{atm})\,\rho$

Then we can write

$$0 = \forall \frac{\partial \rho}{\partial t} + \dot{m} = \forall \frac{\partial \rho}{\partial t} + c\,(p - p_{atm})\rho \quad \ldots\ldots\ldots (1)$$

But $\rho = p/RT$ and $\frac{\partial \rho}{\partial t} = \frac{1}{RT}\frac{dp}{dt}$, so

$$0 = \frac{\forall}{RT}\frac{dp}{dt} + \frac{c\,p}{RT}(p - p_{atm})$$

At $t = 0$, $p = p_0$ and $dp/dt = dp/dt)_0$. Thus

$$0 = \forall \frac{dp}{dt}\Big)_0 + c\,p_0(p_0 - p_{atm}) \quad \text{and} \quad c = -\frac{\forall}{p_0(p_0 - p_{atm})}\frac{dp}{dt}\Big)_0$$

Substituting into Eq. 1 we obtain

$$0 = \frac{dp}{dt} - \frac{p(p - p_{atm})}{p_0(p_0 - p_{atm})}\frac{dp}{dt}\Big)_0$$

Separating variables and integrating

$$\int_{p_0}^{p} \frac{dp}{p(p - p_{atm})} = \frac{dp/dt)_0}{p_0(p_0 - p_{atm})}\int_0^t dt$$

$$\frac{1}{p_{atm}}\left[\ln \frac{p_0(p - p_{atm})}{p(p_0 - p_{atm})}\right] = \frac{dp/dt)_0}{p_0(p_0 - p_{atm})}\,t$$

$$\ln\left[\frac{1 - p_{atm}/p}{1 - p_{atm}/p_0}\right] = \frac{dp/dt)_0}{p_0(p_0/p_{atm} - 1)}\,t$$

Taking antilogs,

$$1-\frac{p_{atm}}{p}=\left(1-\frac{p_{atm}}{p_0}\right)e^{\left\{\frac{dp/dt)_0}{p_0(p_0/p_{atm}-1)}t\right\}}=\left(1-\frac{p_{atm}}{p_0}\right)e^{kt}$$

where

$$k=\frac{dp/dt)_0}{p_0(p_0/p_{atm}-1)}=-1\,psi\times\frac{6.895\,kPa}{psi}\times\frac{1}{701\,kPa}\,\frac{1}{(701/101-1)}$$

$$k=-0.00166\ day^{-1}$$

Then

$$\frac{p_{atm}}{p}=1-\left(\frac{p_0-p_{atm}}{p_0}\right)e^{kt}$$

and

$$p=\frac{p_{atm}}{1-\left(\frac{p_0-p_{atm}}{p_0}\right)e^{kt}} \qquad (2)$$

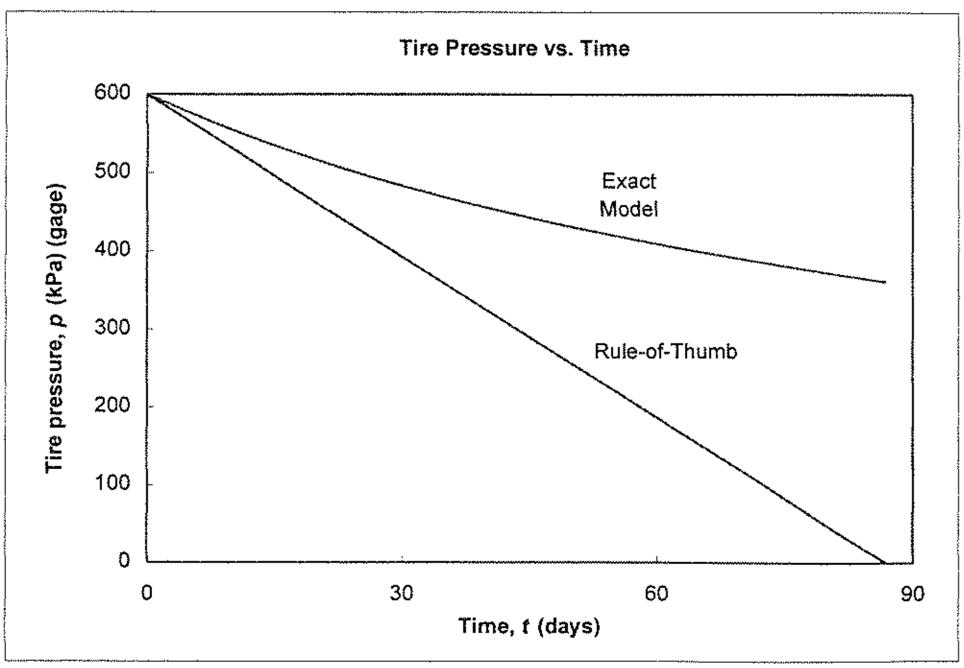
Evaluating at $t = 30$ days.

$$p=\frac{101\,kPa}{1-\frac{600}{701}e^{-30(0.00166)}}=544\,kPa \qquad p_{t=30\,days}$$

Rule of thumb gives $p = p_0 - 6.895\frac{kPa}{day}t$ ...... (3)

At $t = 30$ days $p = 600\,kPa - 207\,kPa = 493\,kPa$ $\quad p_{ROT}$

The rule of thumb predicts a larger pressure loss. Results for both models are presented below

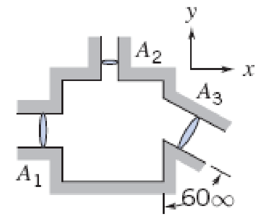


For p = 500 kPa, solving Eq. 2 for t we find t = 42.2 days

Problem 4.58

[Difficulty: 3]

4.58 Evaluate the net rate of flux of momentum out through the control surface of Problem 4.24.



Given: Data on flow through a control surface

Find: Net rate of momentum flux

Solution:

Basic equation: We need to evaluate $\int_{CS} \vec{V}\rho\vec{V}\cdot dA$

Assumptions: 1) Uniform flow at each section

From Problem 4.24 $V_1 = 10\cdot\frac{ft}{s}$ $A_1 = 0.5\cdot ft^2$ $V_2 = 20\cdot\frac{ft}{s}$ $A_2 = 0.1\cdot ft^2$ $A_3 = 0.6\cdot ft^2$ $V_3 = 5\cdot\frac{ft}{s}$ It is an outlet

Then for the control surface

$$\int_{CS} \vec{V}\rho\vec{V}\cdot dA = \vec{V}_1\rho\vec{V}_1\cdot\vec{A}_1 + \vec{V}_2\rho\vec{V}_2\cdot\vec{A}_2 + \vec{V}_3\rho\vec{V}_3\cdot\vec{A}_3$$
$$= V_1\hat{i}\rho(\vec{V}_1\cdot\vec{A}_1) + V_2\hat{j}\rho(\vec{V}_2\cdot\vec{A}_2) + \left[V_3\sin(60)\hat{i} - V_3\cos(60)\hat{j}\right]\rho(\vec{V}_3\cdot\vec{A}_3)$$
$$= -V_1\hat{i}\rho V_1 A_1 + V_2\hat{j}\rho V_2 A_2 + \left[V_3\sin(60)\hat{i} - V_3\cos(60)\hat{j}\right]\rho V_3 A_3$$
$$= \rho\left[-V_1^2 A_1 + V_3^2 A_3\sin(60)\right]\hat{i} + \rho\left[V_2^2 A_2 - V_3^2 A_3\cos(60)\right]\hat{j}$$

Hence the x component is $\rho\left[-V_1^2 A_1 + V_3^2 A_3\sin(60)\right] =$

$$65\cdot\frac{lbm}{ft^3}\times\left(-10^2\times 0.5 + 5^2\times 0.6\times\sin(60\cdot deg)\right)\cdot\frac{ft^4}{s^2}\times\frac{lbf\cdot s^2}{lbm\cdot ft} = -2406\cdot lbf$$

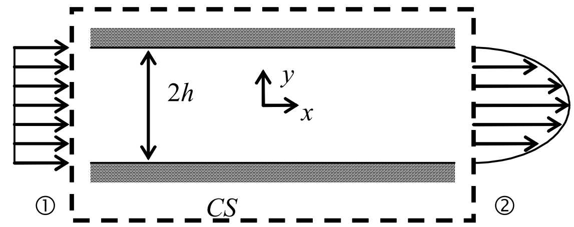
and the y component is $\rho\left[V_2^2 A_2 - V_3^2 A_3\cos(60)\right] =$

$$65\cdot\frac{lbm}{ft^3}\times\left(20^2\times 0.1 - 5^2\times 0.6\times\cos(60\cdot deg)\right)\cdot\frac{ft^4}{s^2}\times\frac{lbf\cdot s^2}{lbm\cdot ft} = 2113\cdot lbf$$

Problem 4.59

[Difficulty: 3]

4.59 For the conditions of Problem 4.34, evaluate the ratio of the x-direction momentum flux at the channel outlet to that at the inlet.



Given: Data on flow at inlet and outlet of channel

Find: Ratio of outlet to inlet momentum flux

Solution:

Basic equation: Momentum flux in x direction at a section $\quad \mathrm{mf}_x = \int_A u\rho\vec{V}\cdot dA$

Assumptions: 1) Steady flow 2) Incompressible flow

Evaluating at 1 and 2 $\quad \mathrm{mf}_{x1} = U\cdot\rho\cdot(-U\cdot 2\cdot h)\cdot w \qquad \left|\mathrm{mf}_{x1}\right| = 2\cdot\rho\cdot w\cdot U^2\cdot h$

Hence

$$\mathrm{mf}_{x2} = \int_{-h}^{h} \rho\cdot u^2\cdot w\,dy = \rho\cdot w\cdot u_{max}^{\ 2}\cdot\int_{-h}^{h}\left[1-\left(\frac{y}{h}\right)^2\right]^2 dy = \rho\cdot w\cdot u_{max}^{\ 2}\cdot\int_{-h}^{h}\left[1-2\cdot\left(\frac{y}{h}\right)^2+\left(\frac{y}{h}\right)^4\right]dy$$

$$\left|\mathrm{mf}_{x2}\right| = \rho\cdot w\cdot u_{max}^{\ 2}\cdot\left(2\cdot h-\frac{4}{3}\cdot h+\frac{2}{5}\cdot h\right) = \rho\cdot w\cdot u_{max}^{\ 2}\cdot\frac{16}{15}\cdot h$$

Then the ratio of momentum fluxes is

$$\frac{\left|\mathrm{mf}_{x2}\right|}{\left|\mathrm{mf}_{x1}\right|} = \frac{\frac{16}{15}\cdot\rho\cdot w\cdot u_{max}^{\ 2}\cdot h}{2\cdot\rho\cdot w\cdot U^2\cdot h} = \frac{8}{15}\cdot\left(\frac{u_{max}}{U}\right)^2$$

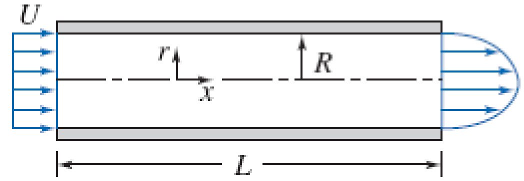
But, from Problem 4.34 $\quad u_{max} = \frac{3}{2}\cdot U \qquad \frac{\left|\mathrm{mf}_{x2}\right|}{\left|\mathrm{mf}_{x1}\right|} = \frac{8}{15}\cdot\left(\frac{\frac{3}{2}\cdot U}{U}\right)^2 = \frac{6}{5} = 1.2$

Hence the momentum increases as it flows in the entrance region of the channel. This appears to contradict common sense, as friction should reduce flow momentum. What happens is the pressure drops significantly along the channel so the net force on the CV is to the right.

Problem 4.60

[Difficulty: 3]

4.60 For the conditions of Problem 4.35, evaluate the ratio of the x-direction momentum flux at the pipe outlet to that at the inlet.



Given:

Data on flow at inlet and outlet of pipe

Find:

Ratio of outlet to inlet momentum flux

Solution:

Basic equation: Momentum flux in x direction at a section $$mf_x = \int_A u\rho\vec{V}\cdot dA$$

Assumptions: 1) Steady flow 2) Incompressible flow

Evaluating at 1 and 2 $\quad mf_{x1} = U\cdot\rho\cdot\left(-U\cdot\pi\cdot R^2\right) \qquad \left|mf_{x1}\right| = \rho\cdot\pi\cdot U^2\cdot R^2$

Hence $$mf_{x2} = \int_0^R \rho\cdot u^2\cdot 2\cdot\pi\cdot r\, dr = 2\cdot\rho\cdot\pi\cdot u_{max}^{\ 2}\cdot\int_0^R r\cdot\left[1-\left(\frac{r}{R}\right)^2\right]^2 dr = 2\cdot\rho\cdot\pi\cdot u_{max}^{\ 2}\cdot\int_0^R \left(r - 2\cdot\frac{r^3}{R^2} + \frac{r^5}{R^4}\right) dy$$

$$\left|mf_{x2}\right| = 2\cdot\rho\cdot\pi\cdot u_{max}^{\ 2}\cdot\left(\frac{R^2}{2} - \frac{R^2}{2} + \frac{R^2}{6}\right) = \rho\cdot\pi\cdot u_{max}^{\ 2}\cdot\frac{R^2}{3}$$

Then the ratio of momentum fluxes is

$$\frac{\left|mf_{x2}\right|}{\left|mf_{x1}\right|} = \frac{\frac{1}{3}\cdot\rho\cdot\pi\cdot u_{max}^{\ 2}\cdot R^2}{\rho\cdot\pi\cdot U^2\cdot R^2} = \frac{1}{3}\cdot\left(\frac{u_{max}}{U}\right)^2$$

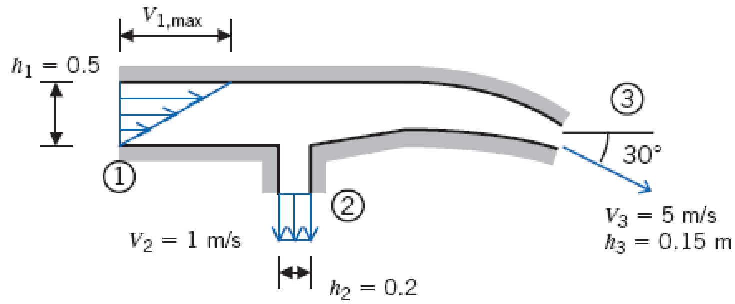
But, from Problem 4.35 $\quad u_{max} = 2\cdot U \qquad \frac{\left|mf_{x2}\right|}{\left|mf_{x1}\right|} = \frac{1}{3}\cdot\left(\frac{2\cdot U}{U}\right)^2 = \frac{4}{3} = 1.33$

Hence the momentum increases as it flows in the entrance region of the pipe This appears to contradict common sense, as friction should reduce flow momentum. What happens is the pressure drops significantly along the pipe so the net force on the CV is to the right.

Problem 4.61

[Difficulty: 3]

4.61 Evaluate the net momentum flux through the bend of Problem 4.38, if the depth normal to the diagram is $w = 1$ m.



Given: Data on flow through a bend

Find: Find net momentum flux

Solution:

Basic equations $\int_{CS} \rho \vec{V} \cdot d\vec{A} = 0$ Momentum fluxes: $mf_x = \int_{CS} u\,\rho \vec{V} \cdot d\vec{A}$ $mf_y = \int_{CS} v\,\rho \vec{V} \cdot d\vec{A}$

Assumptions: 1) Steady flow 2) Incompressible flow

Evaluating mass flux at 1, 2 and 3 $$-\int_0^{h_1} V_1(y)\cdot w\,dy + V_2\cdot w\cdot h_2 + V_3\cdot w\cdot h_3 = 0$$

or $$V_3\cdot h_3 = \int_0^{h_1} V_1(y)\,dy - V_2\cdot h_2 = \int_0^{h_1} V_{1max}\cdot\frac{y}{h_1}\,dy - V_2\cdot h_2 = \frac{V_{1max}}{h_1}\cdot\frac{h_1^{\,2}}{2} - V_2\cdot h_2$$

Hence $$V_{1max} = \frac{2}{h_1}\cdot\left(V_3\cdot h_3 + V_2\cdot h_2\right)$$ Using given data $$V_{1max} = 3.8\,\frac{m}{s}$$

For the x momentum, evaluating at 1, 2 and 3

$$mf_x = -\int_0^{h_1} V_1(y)\cdot\rho\cdot V_1(y)\cdot w\,dy + V_3\cdot\cos(\theta)\cdot\rho\cdot V_3\cdot h_3\cdot w$$

$$mf_x = -\int_0^{h_1}\left(V_{1max}\cdot\frac{y}{h_1}\right)^2\cdot\rho\cdot w\,dy + V_3^{\,2}\cdot\rho\cdot h_3\cdot\cos(\theta)\cdot w = -\frac{V_{1max}^{\,2}}{h_1^{\,2}}\cdot\frac{h_1^{\,3}}{3}\cdot\rho\cdot w + V_3^{\,2}\cdot\rho\cdot h_3\cdot w\cdot\cos(\theta)$$

$$mf_x = \rho\cdot w\cdot\left(-V_{1max}^{\,2}\cdot\frac{h_1}{3} + V_3^{\,2}\cdot\cos(\theta)\cdot h_3\right)$$ Using given data $$mf_x = 841\,N$$

For the y momentum, evaluating at 1, 2 and 3

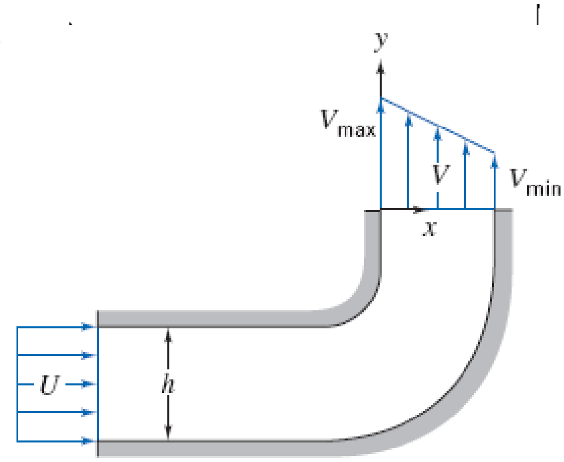
$$mf_y = -V_2\cdot\rho\cdot V_2\cdot h_2\cdot w + V_3\cdot\sin(\theta)\cdot\rho\cdot V_3\cdot h_3\cdot w$$

$$mf_y = \rho\cdot w\cdot\left(-V_2^{\,2}\cdot h_2 - V_3^{\,2}\cdot\sin(\theta)\cdot h_3\right)$$ Using given data $$mf_y = -2075\,N$$

Problem 4.62

[Difficulty: 2]

4.62 Evaluate the net momentum flux through the channel of Problem 4.39. Would you expect the outlet pressure to be higher, lower, or the same as the inlet pressure? Why?



Solution:

The momentum flux is defined as $m.f. = \int \vec{V}(\rho\vec{V}\cdot d\vec{A})$

The net momentum flux through the CV is

$$m.f. = \int_{A_1} \vec{V}(\rho\vec{V}\cdot d\vec{A}) + \int_{A_2} \vec{V}(\rho\vec{V}\cdot d\vec{A})$$

where $\vec{V}_1 = U\hat{i}$, $\vec{V}_2 = \left\{V_{max} - (V_{max} - V_{min})\frac{x}{h}\right\}\hat{j}$

$$\vec{V}_2 = \left\{2V_{min} - V_{min}\frac{x}{h}\right\}\hat{j} = V_{min}\left(2 - \frac{x}{h}\right)\hat{j}$$

Assumptions: (1) incompressible flow
(2) uniform flow at ① (given).

$$\int_{A_1} \vec{V}(\rho\vec{V}\cdot d\vec{A}) = \vec{V}_1\{-|\rho V_1 A_1|\} = -\rho U^2 h^2 \hat{i} \quad \text{.......(1)}$$

$$\int_{A_2} \vec{V}(\rho\vec{V}\cdot d\vec{A}) = \int_0^h V_{min}\left(2 - \frac{x}{h}\right)\hat{j}\, \rho V_{min}\left(2 - \frac{x}{h}\right) h\, dx$$

$$= \hat{j}\rho V_{min}^2 h \int_0^h \left(4 - 4\frac{x}{h} + \frac{x^2}{h^2}\right) dx$$

$$= \hat{j}\rho V_{min}^2 h\left[4x - 2\frac{x^2}{h} + \frac{x^3}{3h^2}\right]_0^h = \hat{j}\rho V_{min}^2 h\left[4h - 2h + \frac{h}{3}\right]$$

$$= \hat{j}\,\frac{7}{3}\rho V_{min}^2 h^2$$

$$\therefore\ m.f. = -\rho U^2 h^2 \hat{i} + \frac{7}{3}\rho V_{min}^2 h^2 \hat{j} = \rho h^2\left[-U^2\hat{i} + \frac{7}{3}V_{min}^2\hat{j}\right]$$

Evaluating

$$m.f. = 999\frac{kg}{m^3} \times (0.0755)^2 m^2\left[-(7.5)^2\frac{m^2}{s^2}\hat{i} + \frac{7}{3}(5)^2\frac{m^2}{s^2}\hat{j}\right] \times \frac{N\cdot s^2}{kg\cdot m}$$

$$m.f. = -320\hat{i} + 332\hat{j}\ N$$ m.f.

For viscous (real) flow friction causes a pressure drop in the direction of flow (Chapter 8)
For flow in a bend streamline curvature results in a pressure gradient normal to the flow (Chapter 6)

Problem 4.63

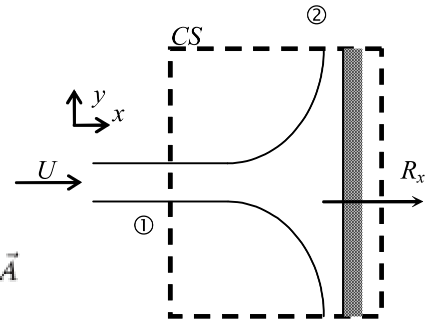
[Difficulty: 2]

4.63 Water jets are being used more and more for metal cutting operations. If a pump generates a flow of 1 gpm through an orifice of 0.01 in. diameter, what is the average jet speed? What force (lbf) will the jet produce at impact, assuming as an approximation that the water sprays sideways after impact?

Given: Water jet hitting object

Find: Jet speed; Force generated

Solution:



Basic equations: Continuity and Momentum flux in x direction

$$\int_{CS} \vec{V} \cdot d\vec{A} = 0 \quad F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\, dV + \int_{CS} u\,\rho\vec{V}\cdot d\vec{A}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) Atmospheric pressure throughout 4) Uniform flow

Given data $\quad Q = 1\cdot\text{gpm} \quad d = 0.01\cdot\text{in} \quad \rho = 1.94\cdot\frac{\text{slug}}{\text{ft}^3}$

Using continuity $\quad Q = V\cdot A = U\cdot\frac{\pi}{d}\cdot d^2 \quad$ Using data $\quad U = \frac{Q}{\frac{\pi}{4}\cdot d^2} \quad U = 4085\,\frac{\text{ft}}{\text{s}} \quad U = 2785\cdot\text{mph} \quad$ FAST!

Using momentum $\quad R_x = u_1\cdot\rho\cdot\left(-u_1\cdot A_1\right) = -\rho\cdot U^2\cdot A = -\rho\cdot U^2\cdot\frac{\pi\cdot D^2}{4}$

Hence $\quad R_x = -\rho\cdot U^2\cdot\frac{\pi\cdot d^2}{4}$

$$R_x = -1.94\cdot\frac{\text{slug}}{\text{ft}^3}\times\left(4085\cdot\frac{\text{ft}}{\text{s}}\right)^2\times\frac{\pi\cdot\left(\frac{.01}{12}\cdot\text{ft}\right)^2}{4}\times\frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}} \qquad R_x = -17.7\cdot\text{lbf}$$

Problem 4.64

[Difficulty: 1]

4.64 Considering that in the fully developed region of a pipe, the integral of the axial momentum is the same at all cross sections, explain the reason for the pressure drop along the pipe.

Given: Fully developed flow in pipe

Find: Why pressure drops if momentum is constant

Solution:

Basic equation: Momentum flux in x direction

$$F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\,d\forall + \int_{CS} u\,\rho\vec{V}\cdot d\vec{A}$$

Assumptions: 1) Steady flow 2) Fully developed flow

Hence $\qquad F_x = \frac{\Delta p}{L} - \tau_w \cdot A_s = 0 \qquad\qquad \Delta p = L \cdot \tau_w \cdot A_s$

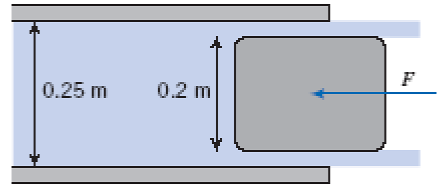
where Δp is the pressure drop over length L, τ_w is the wall friction and As is the pipe surface area

The sum of forces in the x direction is zero. The friction force on the fluid is in the negative x direction, so the net pressure force must be in the positive direction. Hence pressure drops in the x direction so that pressure and friction forces balance

Problem 4.65

[Difficulty: 2]

4.65 Find the force required to hold the plug in place at the exit of the water pipe. The flow rate is 1.5 m^3/s, and the upstream pressure is 3.5 MPa.



Given:

Data on flow and system geometry

Find:

Force required to hold plug

Solution:

Basic equation:

$$F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\,dV + \int_{CS} u\,\rho\vec{V}\cdot d\vec{A}$$

The given data is $D_1 = 0.25 \cdot m$ $\quad D_2 = 0.2 \cdot m$ $\quad Q = 1.5 \cdot \frac{m^3}{s}$ $\quad p_1 = 3500 \cdot kPa$ $\quad \rho = 999 \cdot \frac{kg}{m^3}$

Then $A_1 = \frac{\pi \cdot D_1^2}{4}$ $\quad A_1 = 0.0491\,m^2$ $\quad V_1 = \frac{Q}{A_1}$ $\quad V_1 = 30.6\,\frac{m}{s}$

$A_2 = \frac{\pi}{4}\cdot\left(D_1^2 - D_2^2\right)$ $\quad A_2 = 0.0177\,m^2$ $\quad V_2 = \frac{Q}{A_2}$ $\quad V_2 = 84.9\,\frac{m}{s}$

Applying the basic equation

$$-F + p_1 \cdot A_2 - p_2 \cdot A_2 = 0 + V_1 \cdot \left(-\rho \cdot V_1 \cdot A_1\right) + V_2 \cdot \left(\rho \cdot V_2 \cdot A_2\right) \quad \text{and} \quad p_2 = 0 \quad \text{(gage)}$$

Hence

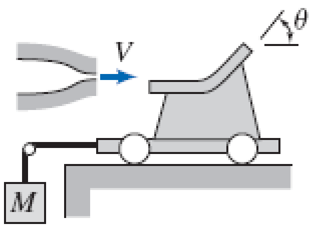
$$F = p_1 \cdot A_1 + \rho \cdot \left(V_1^2 \cdot A_1 - V_2^2 \cdot A_2\right)$$

$$F = 3500 \times \frac{kN}{m^2} \cdot 0.0491 \cdot m^2 + 999 \cdot \frac{kg}{m^3} \times \left[\left(30.6 \cdot \frac{m}{s}\right)^2 \cdot 0.0491 \cdot m^2 - \left(84.9 \cdot \frac{m}{s}\right)^2 \cdot 0.0177 \cdot m^2\right] \qquad F = 90.4 \cdot kN$$

Problem 4.66

[Difficulty: 2]

4.66 A jet of water issuing from a stationary nozzle at 10 m/s ($A_j = 0.1$ m^2) strikes a turning vane mounted on a cart as shown. The vane turns the jet through angle $\theta = 40°$. Determine the value of M required to hold the cart stationary. If the vane angle θ is adjustable, plot the mass, M, needed to hold the cart stationary versus θ for $0 \le \theta \le 180°$.



Given: Nozzle hitting stationary cart

Find: Value of M to hold stationary; plot M versu θ

Solution:

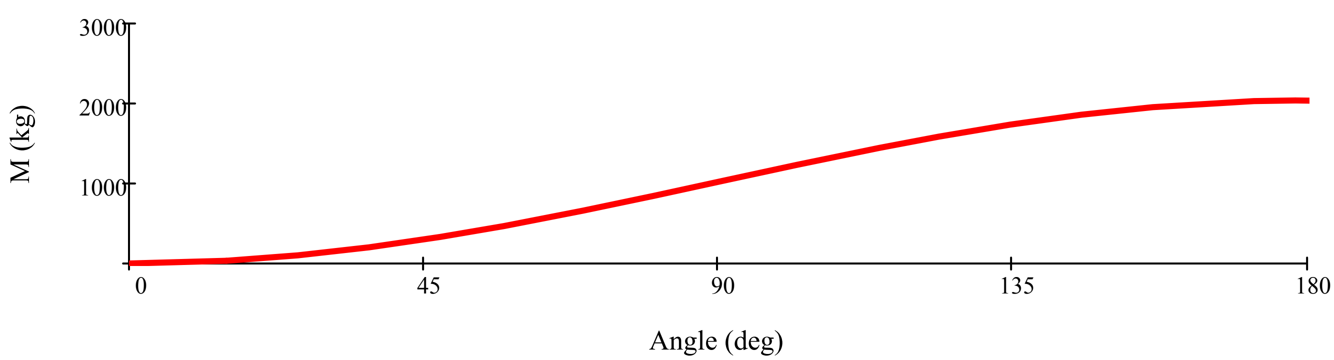
Basic equation: Momentum flux in x direction for the tank

$$F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\, dV\!\!\!\!- + \sum_{CS} u\,\rho \vec{V}\cdot\vec{A}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) Atmospheric pressure throughout 4) Uniform flow 5) Exit velocity is V

Hence $R_x = -M\cdot g = V\cdot\rho\cdot(-V\cdot A) + V\cdot\cos(\theta)\cdot(V\cdot A) = \rho\cdot V^2\cdot A\cdot(\cos(\theta) - 1)$ $\qquad M = \frac{\rho\cdot V^2\cdot A}{g}\cdot(1 - \cos(\theta))$

When $\theta = 40^o$ $\quad M = \frac{s^2}{9.81\cdot m} \times 1000\cdot\frac{kg}{m^3} \times \left(10\cdot\frac{m}{s}\right)^2 \times 0.1\cdot m^2 \times (1 - \cos(40\cdot deg))$ $\qquad M = 238\,kg$

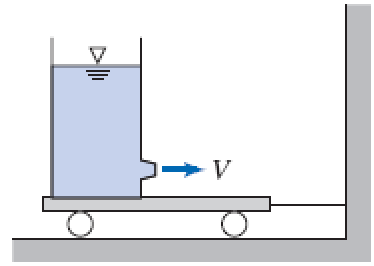


This graph can be plotted in *Excel*

Problem 4.67

[Difficulty: 2]

4.67 A large tank of height $h = 1$ m and diameter $D = 0.75$ m is affixed to a cart as shown. Water issues from the tank through a nozzle of diameter $d = 15$ mm. The speed of the liquid leaving the tank is approximately $V = \sqrt{2gy}$, where y is the height from the nozzle to the free surface. Determine the tension in the wire when $y = 0.9$ m. Plot the tension in the wire as a function of water depth for $0 \le y \le 0.9$ m.



Given: Large tank with nozzle and wire

Find: Tension in wire; plot for range of water depths

Solution:

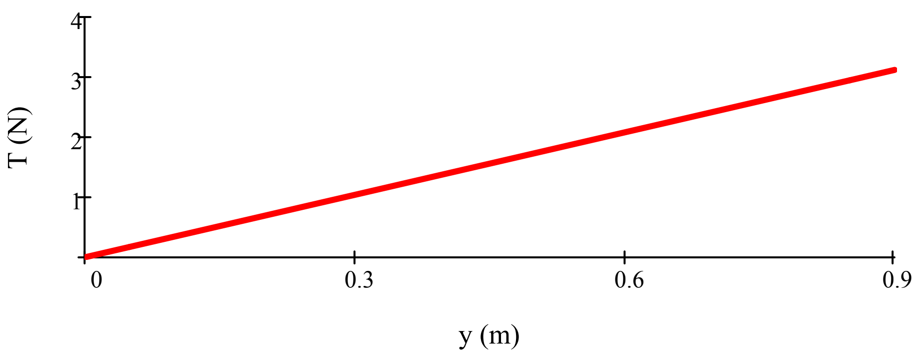
Basic equation: Momentum flux in x direction for the tank

$$F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\, dV\!\!\!/ + \sum_{CS} u\,\rho\vec{V}\cdot\vec{A}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) Atmospheric pressure throughout 4) Uniform flow

Hence $\quad R_x = T = V\cdot\rho\cdot(V\cdot A) = \rho\cdot V^2\cdot A = \rho\cdot(2\cdot g\cdot y)\cdot\frac{\pi\cdot d^2}{4} \qquad T = \frac{1}{2}\cdot\rho\cdot g\cdot y\cdot\pi\cdot d^2 \qquad$ T is linear with y!

When y = 0.9 m $\quad T = \frac{\pi}{2}\times 1000\cdot\frac{kg}{m^3}\times 9.81\cdot\frac{m}{s^2}\times 0.9\cdot m\times(0.015\cdot m)^2\times\frac{N\cdot s^2}{kg\cdot m} \qquad T = 3.12\,N$

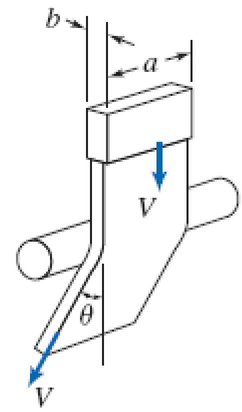


This graph can be plotted in *Excel*

Problem 4.68

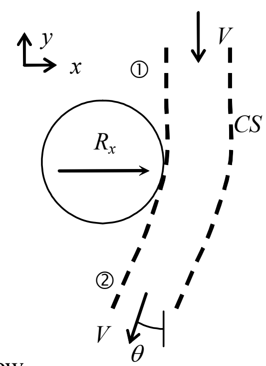
[Difficulty: 2]

4.68 A circular cylinder inserted across a stream of flowing water deflects the stream through angle θ, as shown. (This is termed the "Coanda effect.") For $a = 12.5$ mm, $b = 2.5$ mm, $V = 3$ m/s, and $\theta = 20°$, determine the horizontal component of the force on the cylinder caused by the flowing water.



Given: Water flowing past cylinder

Find: Horizontal force on cylinder



Solution:

Basic equation: Momentum flux in x direction

$$F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\,dV\!\!\!/ + \sum_{CS} u\,\rho\vec{V}\cdot\vec{A}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) Atmospheric pressure throughout 4) Uniform flow

Hence $\quad R_x = u_1\cdot\rho\cdot(-u_1\cdot A_1) + u_2\cdot\rho\cdot(u_2\cdot A_2) = 0 + \rho\cdot(-V\cdot\sin(\theta))\cdot(V\cdot a\cdot b) \qquad R_x = -\rho\cdot V^2\cdot a\cdot b\cdot\sin(\theta)$

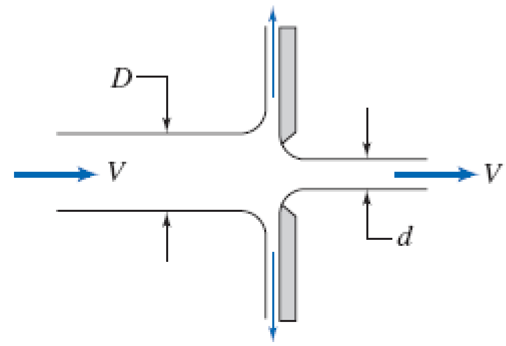
For given data $\quad R_x = -1000\cdot\frac{kg}{m^3}\times\left(3\cdot\frac{m}{s}\right)^2\times 0.0125\cdot m\times 0.0025\cdot m\times\sin(20\cdot deg)\times\frac{N\cdot s^2}{kg\cdot m} \qquad R_x = -0.0962\,N$

This is the force on the fluid (it is to the left). Hence the force on the cylinder is $\quad R_x = -R_x \qquad R_x = 0.0962\,N$

Problem 4.69

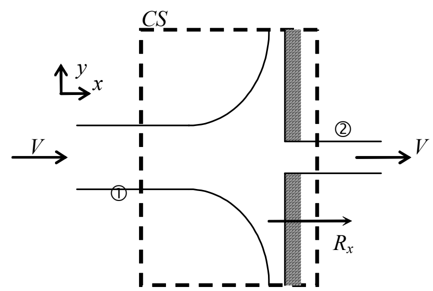
[Difficulty: 2]

4.69 A vertical plate has a sharp-edged orifice at its center. A water jet of speed V strikes the plate concentrically. Obtain an expression for the external force needed to hold the plate in place, if the jet leaving the orifice also has speed V. Evaluate the force for $V = 15$ ft/s, $D = 4$ in., and $d = 1$ in. Plot the required force as a function of diameter ratio for a suitable range of diameter d.



Given: Water jet hitting plate with opening

Find: Force generated on plate; plot force versus diameter d



Solution:

Basic equation: Momentum flux in x direction

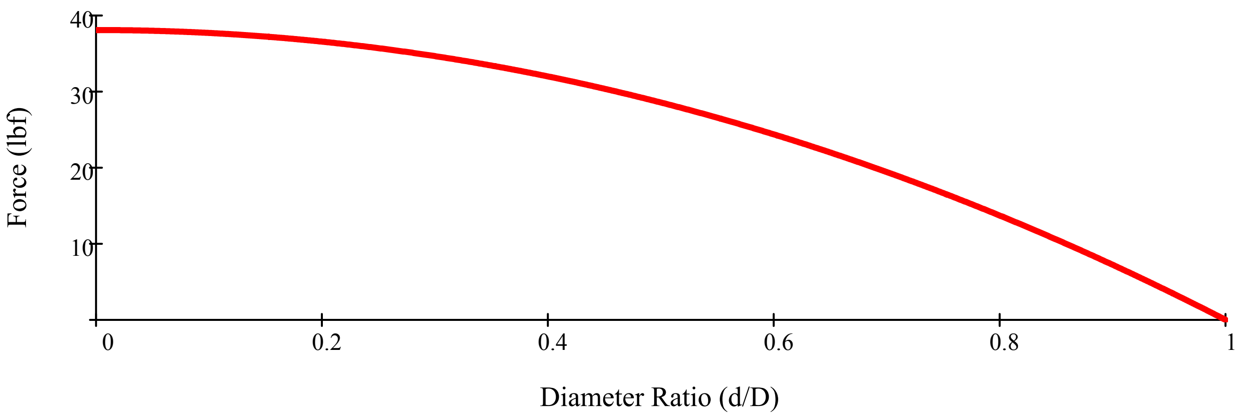
$$F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\, dV\!\!\!/ + \sum_{CS} u\,\rho \vec{V}\cdot\vec{A}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) Atmospheric pressure throughout 4) Uniform flow

Hence $$R_x = u_1\cdot\rho\cdot(-u_1\cdot A_1) + u_2\cdot\rho\cdot(u_2\cdot A_2) = -\rho\cdot V^2\cdot\frac{\pi\cdot D^2}{4} + \rho\cdot V^2\cdot\frac{\pi\cdot d^2}{4} \qquad R_x = -\frac{\pi\cdot\rho\cdot V^2\cdot D^2}{4}\cdot\left[1-\left(\frac{d}{D}\right)^2\right] \qquad (1)$$

For given data $$R_x = -\frac{\pi}{4}\cdot 1.94\cdot\frac{\text{slug}}{\text{ft}^3}\times\left(15\cdot\frac{\text{ft}}{\text{s}}\right)^2\times\left(\frac{1}{3}\cdot\text{ft}\right)^2\times\left[1-\left(\frac{1}{4}\right)^2\right]\times\frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}} \qquad R_x = -35.7\cdot\text{lbf}$$

From Eq 1 (using the absolute value of R_x)

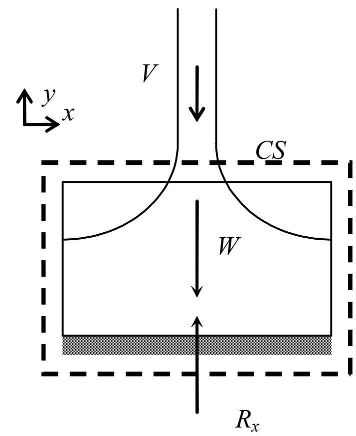


This graph can be plotted in *Excel*

Problem 4.70

[Difficulty: 4]

4.70 In a laboratory experiment, the water flow rate is to be measured catching the water as it vertically exits a pipe into an empty open tank that is on a zeroed balance. The tank is 10 m directly below the pipe exit, and the pipe diameter is 50 mm. One student obtains a flow rate by noting that after 60 s the volume of water (at 4°C) in the tank was 2 m^3. Another student obtains a flow rate by reading the instantaneous weight accumulated of 3150 kg indicated at the 60-s point. Find the mass flow rate each student computes. Why do they disagree? Which one is more accurate? Show that the magnitude of the discrepancy can be explained by any concept you may have.



Given: Water flowing into tank

Find: Mass flow rates estimated by students. Explain discrepancy

Solution:

Basic equation: Momentum flux in y direction

$$F_y = F_{S_y} + F_{B_y} = \frac{\partial}{\partial t}\int_{CV} v\,\rho\,dV\!\!\!\!- + \int_{CS} v\,\rho\vec{V}\cdot d\vec{A}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) Atmospheric pressure throughout 4) Uniform flow

For the first student $\quad m_1 = \dfrac{\rho\cdot V}{t} \quad$ where m_1 represents mass flow rate (software cannot render a dot above it!)

$$m_1 = 1000\cdot\frac{kg}{m^3}\times 3\cdot m^3\times\frac{1}{60\cdot s} \qquad m_1 = 50.0\,\frac{kg}{s}$$

For the second student $\quad m_2 = \dfrac{M}{t} \quad$ where m_2 represents mass flow rate

$$m_2 = 3150\cdot kg\times\frac{1}{60\cdot s} \qquad m_2 = 52.5\,\frac{kg}{s}$$

There is a discrepancy because the second student is measuring instantaneous weight PLUS the force generated as the pipe flow momentum is "killed".

There is a discrepancy because the second student is measuring instantaneous weight PLUS the force generated as the pipe flow momentum is "killed". To analyse this we first need to find the speed at which the water stream enters the tank, 10 m below the pipe exit. This would be a good place to use the Bernoulli equation, but this problem is in the set before Bernoulli is covered. Instead we use the simple concept that the fluid is falling under gravity (a conclusion supported by the Bernoulli equation). From the equations for falling under gravity:

$$V_{tank}^{\,2} = V_{pipe}^{\,2} + 2\cdot g\cdot h$$

where V_{tank} is the speed entering the tank, V_{pipe} is the speed at the pipe, and h = 10 m is the distance traveled. V_{pipe} is obtained from

$$V_{pipe} = \frac{m_1}{\rho\cdot\dfrac{\pi\cdot d_{pipe}^{\,2}}{4}} = \frac{4\cdot m_1}{\pi\cdot\rho\cdot d_{pipe}^{\,2}}$$

$$V_{pipe} = \frac{4}{\pi} \times 50 \cdot \frac{kg}{s} \times \frac{m^3}{1000 \cdot kg} \times \left(\frac{1}{0.05 \cdot m}\right)^2 \qquad V_{pipe} = 25.5 \frac{m}{s}$$

Then $\quad V_{tank} = \sqrt{V_{pipe}^2 + 2 \cdot g \cdot h} \qquad V_{tank} = \sqrt{\left(25.5 \cdot \frac{m}{s}\right)^2 + 2 \times 9.81 \cdot \frac{m}{s^2} \times 10m} \qquad V_{tank} = 29.1 \frac{m}{s}$

We can now use the y momentum equation for the CS shown above

$$R_y - W = -V_{tank} \cdot \rho \cdot \left(-V_{tank} \cdot A_{tank}\right)$$

where A_{tank} is the area of the water flow as it enters the tank. But for the water flow $\quad V_{tank} \cdot A_{tank} = V_{pipe} \cdot A_{pipe}$

Hence $\quad \Delta W = R_y - W = \rho \cdot V_{tank} \cdot V_{pipe} \cdot \frac{\pi \cdot d_{pipe}^2}{4}$

This equation indicate the instantaneous difference ΔW between the scale reading (R_y) and the actual weight of water (W) in the tank

$$\Delta W = 1000 \cdot \frac{kg}{m^3} \times 29.1 \cdot \frac{m}{s} \times 25.5 \cdot \frac{m}{s} \times \frac{\pi}{4} \times (0.05 \cdot m)^2 \qquad \Delta W = 1457\,N$$

Inducated as a mass, this is $\quad \Delta m = \frac{\Delta W}{g} \qquad \Delta m = 149\,kg$

Hence the scale overestimates the weight of water by 1457 N, or a mass of 149 kg

For the second student $\quad M = 3150 \cdot kg - 149 \cdot kg \qquad M = 3001\,kg$

Hence $\quad m_2 = \frac{M}{t} \quad$ where m_2 represents mass flow rate

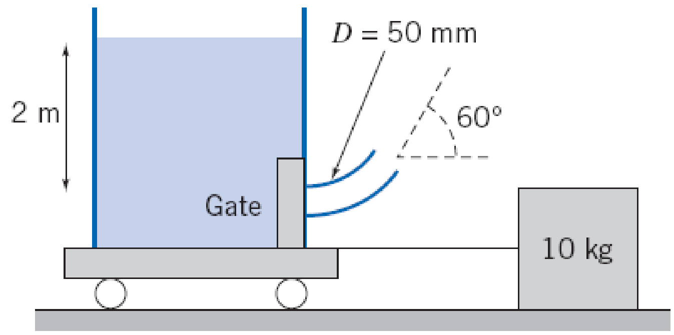
$$m_2 = 3001 \cdot kg \times \frac{1}{60 \cdot s} \qquad m_2 = 50.0 \frac{kg}{s}$$

Comparing with the answer obtained from student 1, we see the students now agree! The discrepancy was entirely caused by the fact that the second student was measuring the weight of tank water PLUS the momentum lost by the water as it entered the tank!

Problem 4.71

[Difficulty: 3]

4.71 A tank of water sits on a cart with frictionless wheels as shown. The cart is attached using a cable to a mass $M = 10$ kg, and the coefficient of static friction of the mass with the ground is $\mu = 0.55$. If the gate blocking the tank exit is removed, will the resulting exit flow be sufficient to start the tank moving? (Assume the water flow is frictionless, and that the jet velocity is $V = \sqrt{2gh}$, where $h = 2$ m is the water depth.) Find the mass M that is just sufficient to hold the tank in place.



Given: Water tank attached to mass

Find: Whether tank starts moving; Mass to just hold in place

Solution:

Basic equation: Momentum flux in x direction for the tank

$$F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\,d\forall + \int_{CS} u\,\rho\vec{V}\cdot d\vec{A}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) Atmospheric pressure at exit 4) Uniform flow

Hence $$R_x = V\cdot\cos(\theta)\cdot\rho\cdot(V\cdot A) = \rho\cdot V^2\cdot\frac{\pi\cdot D^2}{4}\cdot\cos(\theta)$$

We need to find V. We could use the Bernoulli equation, but here it is known that $V = \sqrt{2\cdot g\cdot h}$ where h = 2 m is the height of fluid in the tank

$$V = \sqrt{2\times 9.81\cdot\frac{m}{s^2}\times 2\cdot m} \qquad V = 6.26\,\frac{m}{s}$$

Hence $$R_x = 1000\cdot\frac{kg}{m^3}\times\left(6.26\cdot\frac{m}{s}\right)^2\times\frac{\pi}{4}\times(0.05\cdot m)^2\times\cos(60\cdot deg) \qquad R_x = 38.5\,N$$

This force is equal to the tension T in the wire $T = R_x$ $\qquad T = 38.5\,N$

For the block, the maximum friction force a mass of M = 10 kg can generate is $F_{max} = M\cdot g\cdot\mu$ where μ is static friction

$$F_{max} = 10\cdot kg\times 9.81\cdot\frac{m}{s^2}\times 0.55\times\frac{N\cdot s^2}{kg\cdot m} \qquad F_{max} = 54.0\,N$$

Hence the tension T created by the water jet is less than the maximum friction F_{max}; the tank is at rest

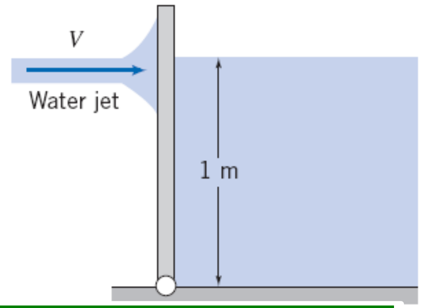
The mass that is just sufficient is given by $M\cdot g\cdot\mu = R_x$

$$M = \frac{R_x}{g\cdot\mu} \qquad M = 38.5\cdot N\times\frac{1}{9.81}\cdot\frac{s^2}{m}\times\frac{1}{0.55}\times\frac{kg\cdot m}{N\cdot s^2} \qquad M = 7.14\,kg$$

Problem 4.72

[Difficulty: 4]

4.72 A gate is 1 m wide and 1.2 m tall and hinged at the bottom. On one side the gate holds back a 1-m-deep body of water. On the other side, a 5-cm diameter water jet hits the gate at a height of 1 m. What jet speed V is required to hold the gate vertical? What will the required speed be if the body of water is lowered to 0.5 m? What will the required speed be if the water level is lowered to 0.25 m?



Given: Gate held in place by water jet

Find: Required jet speed for various water depths

Solution:

Basic equation: Momentum flux in x direction for the wall

$$F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\, d\mathcal{V} + \int_{CS} u\,\rho\vec{V}\cdot d\vec{A}$$

Note: We use this equation ONLY for the jet impacting the wall. For the hydrostatic force and location we use computing equations

$$F_R = p_c \cdot A \qquad y' = y_c + \frac{I_{xx}}{A \cdot y_c}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) Uniform flow

Hence $$R_x = V\cdot\rho\cdot\left(-V\cdot A_{jet}\right) = -\rho\cdot V^2\cdot\frac{\pi\cdot D^2}{4}$$

This force is the force generated by the wall on the jet; the force of the jet hitting the wall is then

$$F_{jet} = -R_x = \rho\cdot V^2\cdot\frac{\pi\cdot D^2}{4}$$ where D is the jet diameter

For the hydrostatic force $$F_R = p_c\cdot A = \rho\cdot g\cdot\frac{h}{2}\cdot h\cdot w = \frac{1}{2}\cdot\rho\cdot g\cdot w\cdot h^2 \qquad y' = y_c + \frac{I_{xx}}{A\cdot y_c} = \frac{h}{2} + \frac{\frac{w\cdot h^3}{12}}{w\cdot h\cdot\frac{h}{2}} = \frac{2}{3}\cdot h$$

where h is the water depth and w is the gate width

For the gate, we can take moments about the hinge to obtain $$-F_{jet}\cdot h_{jet} + F_R\cdot(h - y') = -F_{jet}\cdot h_{jet} + F_R\cdot\frac{h}{3} = 0$$

where h_{jet} is the height of the jet from the ground

Hence $$F_{jet} = \rho\cdot V^2\cdot\frac{\pi\cdot D^2}{4}\cdot h_{jet} = F_R\cdot\frac{h}{3} = \frac{1}{2}\cdot\rho\cdot g\cdot w\cdot h^2\cdot\frac{h}{3} \qquad V = \sqrt{\frac{2\cdot g\cdot w\cdot h^3}{3\cdot\pi\cdot D^2\cdot h_j}}$$

For the first case (h = 1 m) $$V = \sqrt{\frac{2}{3\cdot\pi}\times 9.81\cdot\frac{m}{s^2}\times 1\cdot m\times(1\cdot m)^3\times\left(\frac{1}{0.05\cdot m}\right)^2\times\frac{1}{1\cdot m}} \qquad V = 28.9\,\frac{m}{s}$$

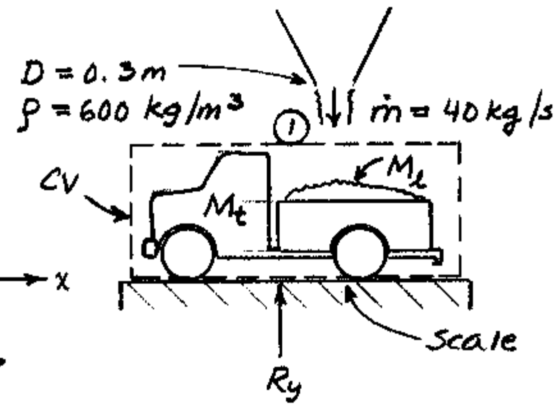
For the second case (h = 0.5 m) $$V = \sqrt{\frac{2}{3\cdot\pi}\times 9.81\cdot\frac{m}{s^2}\times 1\cdot m\times(0.5\cdot m)^3\times\left(\frac{1}{0.05\cdot m}\right)^2\times\frac{1}{1\cdot m}} \qquad V = 10.2\,\frac{m}{s}$$

For the first case (h = 0.25 m) $$V = \sqrt{\frac{2}{3\cdot\pi}\times 9.81\cdot\frac{m}{s^2}\times 1\cdot m\times(0.25\cdot m)^3\times\left(\frac{1}{0.05\cdot m}\right)^2\times\frac{1}{1\cdot m}} \qquad V = 3.61\,\frac{m}{s}$$

Problem 4.73

[Difficulty: 2]

4.73 A farmer purchases 675 kg of bulk grain from the local co-op. The grain is loaded into his pickup truck from a hopper with an outlet diameter of 0.3 m. The loading operator determines the payload by observing the indicated gross mass of the truck as a function of time. The grain flow from the hopper ($\dot{m} = 40$ kg/s) is terminated when the indicated scale reading reaches the desired gross mass. If the grain density is 600 kg/m³, determine the true payload.



Solution: Apply the y component of momentum equation using CV shown.

Basic equation:

$$F_{S_y} + F_{B_y} = \overbrace{\frac{\partial}{\partial t}\int_{CV} v\,\rho\, d\forall}^{\simeq 0(2)} + \int_{CS} v\,\rho\,\vec{V}\cdot d\vec{A}$$

Assumptions: (1) No net pressure force; $F_{S_y} = R_y$

(2) Neglect v inside CV

(3) Uniform flow of grain at inlet section ①

Then

$$R_y - (M_t + M_\ell)g = v_1\{-|\dot{m}|\}$$

$$v_1 = -V_1 = -\frac{\dot{m}}{\rho A}$$

or

$$R_y = (M_t + M_\ell)g + \frac{\dot{m}^2}{\rho A} \quad \text{(indicated during grain flow)}$$

Loading is terminated when

$$\frac{R_y}{g} - M_t = M_\ell + \frac{\dot{m}^2}{\rho g A} = 675 \text{ kg}$$

Thus

$$M_\ell = 675\text{ kg} - \frac{\dot{m}^2}{\rho g A}$$

$$= 675\text{ kg} - (40)^2\frac{\text{kg}^2}{\text{s}^2}\times\frac{\text{m}^3}{600\text{ kg}}\times\frac{\text{s}^2}{9.81\text{ m}}\times\frac{4}{\pi}\frac{1}{(0.3)^2\text{ m}^2}$$

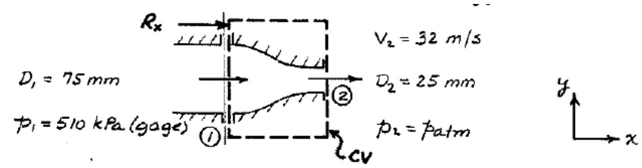
$$M_\ell = 671\text{ kg}$$

M_ℓ

Problem 4.74

[Difficulty: 2]

4.74 Water flows steadily through a fire hose and nozzle. The hose is 75 mm inside diameter, and the nozzle tip is 25 mm ID; water gage pressure in the hose is 510 kPa, and the stream leaving the nozzle is uniform. The exit speed and pressure are 32 m/s and atmospheric, respectively. Find the force transmitted by the coupling between the nozzle and hose. Indicate whether the coupling is in tension or compression.



Solution: Apply continuity and x component of momentum equation to inertial CV shown; use gage pressures to cancel p_{atm}.

Basic equations:
$$0 = \underbrace{\frac{\partial}{\partial t}\int_{CV} \rho\, d\forall}_{=0(1)} + \int_{CS} \rho \vec{V}\cdot d\vec{A}$$

$$F_{Sx} + \underbrace{F_{Bx}}_{=0(4)} = \underbrace{\frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall}_{=0(1)} + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$$

Assumptions: (1) Steady flow
(2) Uniform flow at each section
(3) Incompressible flow
(4) $F_{Bx} = 0$

Then
$$0 = \{-|\rho V_1 A_1|\} + \{|\rho V_2 A_2|\} = -\rho V_1 A_1 + \rho V_2 A_2$$

$$V_1 = V_2 \frac{A_2}{A_1} = V_2\left(\frac{D_2}{D_1}\right)^2 = 32\,\frac{m}{s} \times \left(\frac{25\ mm}{75\ mm}\right)^2 = 3.56\ m/s$$

and
$$R_x + p_{1g} A_1 = u_1\{-|\rho V_1 A_1|\} + u_2\{|\rho V_2 A_2|\}$$

$$u_1 = V_1 \qquad u_2 = V_2$$

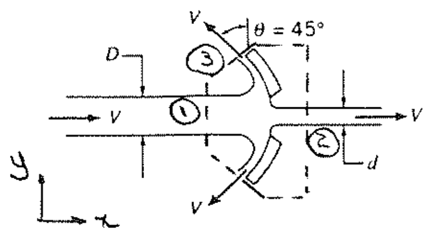
$$R_x = -p_{1g} A_1 - V_1 \rho V_1 A_1 + V_2 \rho V_2 A_2 = -p_{1g} A_1 + \rho V_2 A_2 (V_2 - V_1)$$

$$= -510\times 10^3\,\frac{N}{m^2} \times \frac{\pi}{4}(0.075)^2 m^2 + 999\,\frac{kg}{m^3} \times 32\,\frac{m}{s} \times \frac{\pi}{4}(0.025)^2 m^2 (32.0 - 3.56)\frac{m}{s} \times \frac{N\cdot s^2}{kg\cdot m}$$

$R_x = -1.81\ kN$ (i.e. force on CV is to the left) R_x

Thus the coupling must be in tension.

4.75 A shallow circular dish has a sharp-edged orifice at its center. A water jet, of speed V, strikes the dish concentrically. Obtain an expression for the external force needed to hold the dish in place if the jet issuing from the orifice also has speed V. Evaluate the force for $V = 5$ m/s, $D = 100$ mm, and $d = 25$ mm. Plot the required force as a function of the angle θ $(0 \le \theta \le 90°)$ with diameter ratio as a parameter for a suitable range of diameter d.



Solution:

Apply the x component of the momentum equation to the inertial CV shown.

Basic equation: $F_{S_x} + \cancel{F_{B_x}}^{=0(2)} = \cancel{\frac{\partial}{\partial t}\int_{CV} u \rho \, d\forall}^{=0(3)} + \int_{CS} u (\rho \vec{V} \cdot d\vec{A})$

Assumptions: (1) atmospheric pressure acts on all CV surfaces
(2) $F_{B_x} = 0$
(3) steady flow
(4) uniform flow at each section
(5) incompressible flow
(6) no change in jet speed on dish: $V_1 = V_2 = V_3 = V$

Then,

$$R_x = u_1\{-|\rho V_1 A_1|\} + u_2\{|\rho V_2 A_2|\} + u_3\{|\rho V_3 A_3|\}$$

$u_1 = V \quad A_1 = \frac{\pi D^2}{4} \quad u_2 = V \quad A_2 = \frac{\pi d^2}{4} \quad u_3 = -V\sin\theta \quad A_3 = A_1 - A_2$

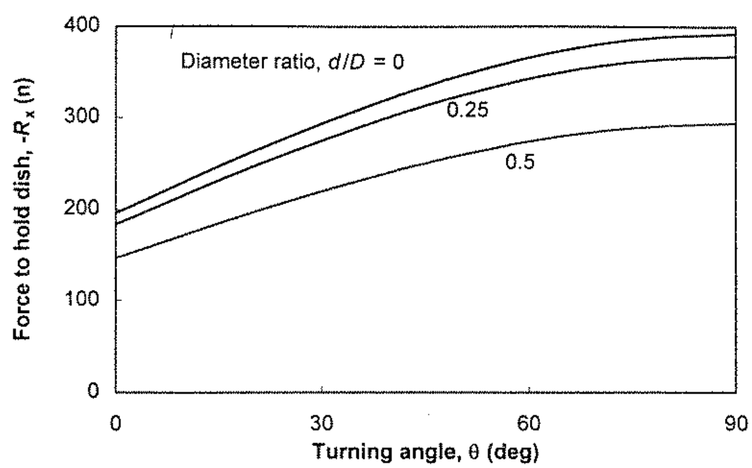
$$R_x = -\rho V^2 \frac{\pi D^2}{4} + \rho V^2 \frac{\pi d^2}{4} - \rho V^2 \sin\theta \frac{\pi}{4}(D^2 - d^2) = \rho V^2 \frac{\pi}{4}(1+\sin\theta)(d^2 - D^2)$$

$$R_x = -\rho V^2 \frac{\pi D^2}{4}(1+\sin\theta)\left[1 - \left(\frac{d}{D}\right)^2\right] \qquad R_x$$

Evaluating for $d = 25$ mm

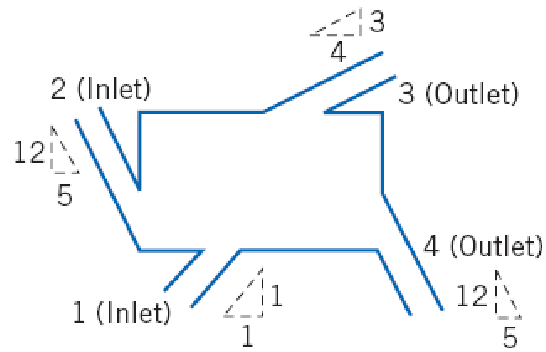
$$R_x = -\frac{\pi}{4} \times 999\frac{kg}{m^3} \times (5)^2\frac{m^2}{s^2} \times (0.10)^2 m^2 \, (1+\sin 45°)\left[1 - \left(\frac{25}{100}\right)^2\right]\frac{N \cdot s^2}{kg \cdot m} = -314 \text{ N} \qquad R_x$$

Since $R_x < 0$, it must be applied to the left. R_x is plotted as a function of θ for different values of d/D



Problem 4.76 [Difficulty: 3]

4.76 Obtain expressions for the rate of change in mass of the control volume shown, as well as the horizontal and vertical forces required to hold it in place, in terms of p_1, A_1, V_1, p_2, A_2, V_2, p_3, A_3, V_3, p_4, A_4, V_4, and the constant density ρ.



Given: Flow into and out of CV

Find: Expressions for rate of change of mass, and force

Solution:

Basic equations: Mass and momentum flux

$$\frac{\partial}{\partial t}\int_{CV} \rho \, dV + \int_{CS} \rho \vec{V} \cdot d\vec{A} = 0$$

$$F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u \, \rho \, dV + \int_{CS} u \, \rho \vec{V} \cdot d\vec{A}$$

$$F_y = F_{S_y} + F_{B_y} = \frac{\partial}{\partial t}\int_{CV} v \, \rho \, dV + \int_{CS} v \, \rho \vec{V} \cdot d\vec{A}$$

Assumptions: 1) Incompressible flow 2) Uniform flow

For the mass equation
$$\frac{dM_{CV}}{dt} + \sum_{CS} \left(\rho \cdot \vec{V} \cdot \vec{A}\right) = \frac{dM_{CV}}{dt} + \rho \cdot \left(-V_1 \cdot A_1 - V_2 \cdot A_2 + V_3 \cdot A_3 + V_4 \cdot A_4\right) = 0$$

$$\frac{dM_{CV}}{dt} = \rho \cdot \left(V_1 \cdot A_1 + V_2 \cdot A_2 - V_3 \cdot A_3 - V_4 \cdot A_4\right)$$

For the x momentum
$$F_x + \frac{p_1 \cdot A_1}{\sqrt{2}} + \frac{5}{13} \cdot p_2 \cdot A_2 - \frac{4}{5} \cdot p_3 \cdot A_3 - \frac{5}{13} \cdot p_4 \cdot A_4 = 0 + \frac{V_1}{\sqrt{2}} \cdot \left(-\rho \cdot V_1 \cdot A_1\right) + \frac{5}{13} \cdot V_2 \cdot \left(-\rho \cdot V_2 \cdot A_2\right) \dots$$
$$+ \frac{4}{5} \cdot V_3 \cdot \left(\rho \cdot V_3 \cdot A_3\right) + \frac{5}{13} \cdot V_3 \cdot \left(\rho \cdot V_3 \cdot A_3\right)$$

$$F_x = -\frac{p_1 \cdot A_1}{\sqrt{2}} - \frac{5}{13} \cdot p_2 \cdot A_2 + \frac{4}{5} \cdot p_3 \cdot A_3 + \frac{5}{13} \cdot p_4 \cdot A_4 + \rho \cdot \left(-\frac{1}{\sqrt{2}} \cdot V_1^2 \cdot A_1 - \frac{5}{13} \cdot V_2^2 \cdot A_2 + \frac{4}{5} \cdot V_3^2 \cdot A_3 + \frac{5}{13} \cdot V_3^2 \cdot A_3\right)$$

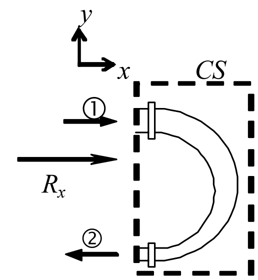
For the y momentum
$$F_y + \frac{p_1 \cdot A_1}{\sqrt{2}} - \frac{12}{13} \cdot p_2 \cdot A_2 - \frac{3}{5} \cdot p_3 \cdot A_3 + \frac{12}{13} \cdot p_4 \cdot A_4 = 0 + \frac{V_1}{\sqrt{2}} \cdot \left(-\rho \cdot V_1 \cdot A_1\right) - \frac{12}{13} \cdot V_2 \cdot \left(-\rho \cdot V_2 \cdot A_2\right) \dots$$
$$+ \frac{3}{5} \cdot V_3 \cdot \left(\rho \cdot V_3 \cdot A_3\right) - \frac{12}{13} \cdot V_3 \cdot \left(\rho \cdot V_3 \cdot A_3\right)$$

$$F_y = -\frac{p_1 \cdot A_1}{\sqrt{2}} + \frac{12}{13} \cdot p_2 \cdot A_2 + \frac{3}{5} \cdot p_3 \cdot A_3 - \frac{12}{13} \cdot p_4 \cdot A_4 + \rho \cdot \left(-\frac{1}{\sqrt{2}} \cdot V_1^2 \cdot A_1 - \frac{12}{13} \cdot V_2^2 \cdot A_2 + \frac{3}{5} \cdot V_3^2 \cdot A_3 - \frac{12}{13} \cdot V_3^2 \cdot A_3\right)$$

Problem 4.77

[Difficulty: 2]

4.77 A 180° elbow takes in water at an average velocity of 0.8 m/s and a pressure of 350 kPa (gage) at the inlet, where the diameter is 0.2 m. The exit pressure is 75 kPa, and the diameter is 0.04 m. What is the force required to hold the elbow in place?



Given: Water flow through elbow

Find: Force to hold elbow

Solution:

Basic equation: Momentum flux in x direction for the elbow $F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\,dV + \int_{CS} u\,\rho\vec{V}\cdot d\vec{A}$

Assumptions: 1) Steady flow 2) Incompressible flow 3) Uniform flow

Hence $R_x + p_{1g}\cdot A_1 + p_{2g}\cdot A_2 = V_1\cdot(-\rho\cdot V_1\cdot A_1) - V_2\cdot(\rho\cdot V_2\cdot A_2)$ $\quad R_x = -p_{1g}\cdot A_1 - p_{2g}\cdot A_2 - \rho\cdot\left(V_1^2\cdot A_1 + V_2^2\cdot A_2\right)$

From continuity $V_2\cdot A_2 = V_1\cdot A_1$ so $V_2 = V_1\cdot\frac{A_1}{A_2} = V_1\cdot\left(\frac{D_1}{D_2}\right)^2$ $\quad V_2 = 0.8\cdot\frac{m}{s}\cdot\left(\frac{0.2}{0.04}\right)^2$ $\quad V_2 = 20\cdot\frac{m}{s}$

Hence

$$R_x = -350\times 10^3\cdot\frac{N}{m^2}\times\frac{\pi\cdot(0.2\cdot m)^2}{4} - 75\times 10^3\cdot\frac{N}{m^2}\times\frac{\pi\cdot(0.04\cdot m)^2}{4}\ \ldots$$

$$+ -1000\cdot\frac{kg}{m^3}\times\left[\left(0.8\cdot\frac{m}{s}\right)^2\times\frac{\pi\cdot(0.2\cdot m)^2}{4} + \left(20\cdot\frac{m}{s}\right)^2\times\frac{\pi\cdot(.04\cdot m)^2}{4}\right]\times\frac{N\cdot s^2}{kg\cdot m}$$

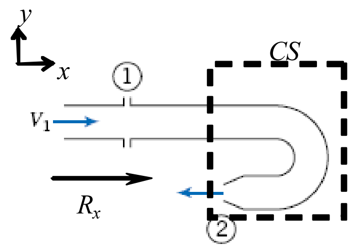
$$R_x = -11.6\cdot kN$$

The force is to the left: It is needed to hold the elbow on against the high pressures, plus it generates the large change in x momentum

Problem 4.78

[Difficulty: 2]

4.78 Water is flowing steadily through the 180° elbow shown. At the inlet to the elbow the gage pressure is 15 psi. The water discharges to atmospheric pressure. Assume properties are uniform over the inlet and outlet areas: $A_1 = 4$ in.^{2}, $A_2 = 1$ in.^{2}, and $V_1 = 10$ ft/s. Find the horizontal component of force required to hold the elbow in place.



Given: Water flow through elbow

Find: Force to hold elbow

Solution:

Basic equation: Momentum flux in x direction for the elbow

$$F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\,dV\!\!\!/ + \int_{CS} u\,\rho\vec{V}\cdot d\vec{A}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) Atmospheric pressure at exit 4) Uniform flow

Hence $\quad R_x + p_{1g}\cdot A_1 = V_1\cdot(-\rho\cdot V_1\cdot A_1) - V_2\cdot(\rho\cdot V_2\cdot A_2) \qquad R_x = -p_{1g}\cdot A_1 - \rho\cdot\left(V_1^2\cdot A_1 + V_2^2\cdot A_2\right)$

From continuity $\quad V_2\cdot A_2 = V_1\cdot A_1 \quad$ so $\quad V_2 = V_1\cdot\frac{A_1}{A_2} \qquad V_2 = 10\cdot\frac{ft}{s}\cdot\frac{4}{1} \qquad V_2 = 40\cdot\frac{ft}{s}$

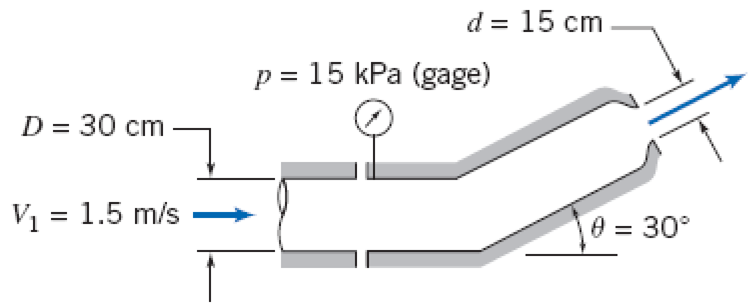
Hence $\quad R_x = -15\cdot\frac{lbf}{in^2}\times 4\cdot in^2 - 1.94\frac{slug}{ft^3}\times\left[\left(10\cdot\frac{ft}{s}\right)^2\cdot 4\cdot in^2 + \left(40\cdot\frac{ft}{s}\right)^2\cdot 1\cdot in^2\right]\times\left(\frac{1\cdot ft}{12\cdot in}\right)^2\times\frac{lbf\cdot s^2}{slug\cdot ft} \qquad R_x = -86.9\cdot lbf$

The force is to the left: It is needed to hold the elbow on against the high pressure, plus it generates the large change in x momentum

Problem 4.79

[Difficulty: 2]

4.79 Water flows steadily through the nozzle shown, discharging to atmosphere. Calculate the horizontal component of force in the flanged joint. Indicate whether the joint is in tension or compression.



Given: Water flow through nozzle

Find: Force to hold nozzle

Solution:

Basic equation: Momentum flux in x direction for the elbow
$$F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\, dV + \int_{CS} u\,\rho\vec{V}\cdot d\vec{A}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) Uniform flow

Hence $R_x + p_{1g}\cdot A_1 + p_{2g}\cdot A_2 = V_1\cdot(-\rho\cdot V_1\cdot A_1) + V_2\cdot\cos(\theta)\cdot(\rho\cdot V_2\cdot A_2)$ $\qquad R_x = -p_{1g}\cdot A_1 + \rho\cdot\left(V_2^{\,2}\cdot A_2\cdot\cos(\theta) - V_1^{\,2}\cdot A_1\right)$

From continuity $V_2\cdot A_2 = V_1\cdot A_1$ so $V_2 = V_1\cdot\frac{A_1}{A_2} = V_1\cdot\left(\frac{D_1}{D_2}\right)^2$ $\qquad V_2 = 1.5\cdot\frac{m}{s}\cdot\left(\frac{30}{15}\right)^2$ $\qquad V_2 = 6\cdot\frac{m}{s}$

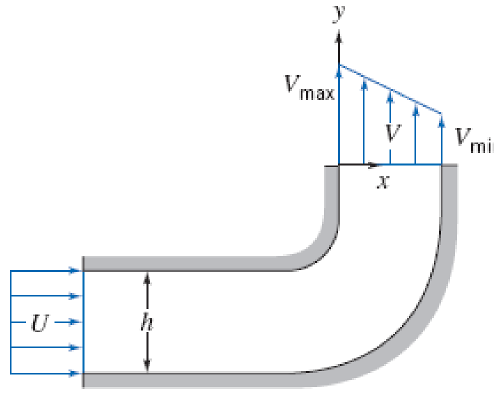
Hence $R_x = -15\times 10^3\cdot\frac{N}{m^2}\times\frac{\pi\cdot(0.3\cdot m)^2}{4} + 1000\cdot\frac{kg}{m^3}\times\left[\left(6\cdot\frac{m}{s}\right)^2\times\frac{\pi\cdot(0.15\cdot m)^2}{4}\cdot\cos(30\cdot deg) - \left(1.5\cdot\frac{m}{s}\right)^2\times\frac{\pi\cdot(.3\cdot m)^2}{4}\right]\times\frac{N\cdot s^2}{kg\cdot m}$

$R_x = -668\cdot N$ The joint is in tension: It is needed to hold the elbow on against the high pressure, plus it generates the large change in x momentum

Problem 4.80

[Difficulty: 2]

4.80 Assume the bend of Problem 4.39 is a segment of a larger channel and lies in a horizontal plane. The inlet pressure is 170 kPa (abs), and the outlet pressure is 130 kPa (abs). Find the force required to hold the bend in place.



Solution:

Basic equation: $\vec{F}_S + \vec{F}_B = \cancel{\frac{\partial}{\partial t}\int_{CV} \vec{V}\rho\, d\forall}^{=0(1)} + \int_{CS} \vec{V}(\rho \vec{V}\cdot d\vec{A})$

Assumptions: (1) steady flow
(2) $F_{B_x} = F_{B_y} = 0$
(3) incompressible flow
(4) atmospheric pressure acts on outside surfaces.

The x-momentum equation becomes

$$R_x + p_1 A_1 + \cancel{F_{B_x}}^{=0(2)} = \int_{CS} u(\rho\vec{V}\cdot d\vec{A}) = U\{-|\rho U A_1|\}$$

$$R_x = -p_1A_1 - \rho U^2 A_1 = -h^2(p_1 + \rho U^2)$$

$$R_x = -(0.0755)^2\,m^2\left[(170-101)10^3\,\frac{N}{m^2} + 999\,\frac{kg}{m^3}\times(7.5)^2\,\frac{m^2}{s^2}\times\frac{N\cdot s^2}{kg\cdot m}\right] = -714\ N$$

The y-momentum equation becomes

$$R_y - p_2A_2 + \cancel{F_{B_y}}^{=0(2)} = \int_{CS} v(\rho\vec{V}\cdot d\vec{A})$$

$$v_2 = V_2 = V_{max} - (V_{max}-V_{min})\frac{x}{h} = 2V_{min} - V_{min}\frac{x}{h} = V_{min}\left(2-\frac{x}{h}\right)$$

$$R_y - p_2A_2 = \int_0^h V_{min}\left(2-\frac{x}{h}\right)\rho V_{min}\left(2-\frac{x}{h}\right)h\,dx$$

$$R_y = p_2A_2 + \rho V_{min}^2 h\int_0^h\left(4 - 4\frac{x}{h} + \frac{x^2}{h^2}\right)dx$$

$$= p_2A_2 + \rho V_{min}^2 h\left[4x - 2\frac{x^2}{h} + \frac{x^3}{3h^2}\right]_0^h$$

$$R_y = p_2A_2 + \rho V_{min}^2 h\left[4h - 2h + \frac{h}{3}\right] = p_2A_2 + \frac{7}{3}\rho V_{min}^2 h^2$$

$$R_y = h^2\left(p_2 + \frac{7}{3}\rho V_{min}^2\right)$$

$$= (0.0755)^2\,m^2\left[(130-101)10^3\,\frac{N}{m^2} + \frac{7}{3}\times 999\,\frac{kg}{m^3}\times(5.0)^2\,\frac{m^2}{s^2}\times\frac{N\cdot s^2}{kg\cdot m}\right]$$

$$R_y = 498\ N$$

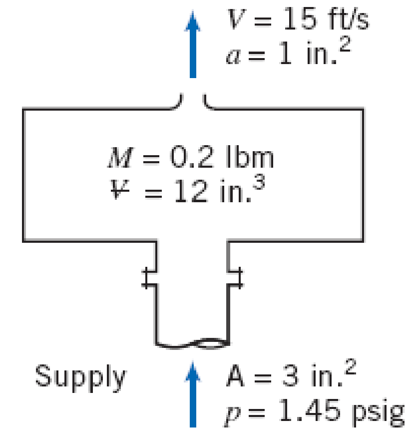
$$\therefore\ \vec{R} = -714\,\hat{i} + 498\,\hat{j}\ N$$

$\vec{R}$

Problem 4.81

[Difficulty: 2]

4.81 A spray system is shown in the diagram. Water is supplied at $p = 1.45$ psig, through the flanged opening of area $A = 3$ in.^{2} The water leaves in a steady free jet at atmospheric pressure. The jet area and speed are $a = 1.0$ in.^{2} and $V = 15$ ft/s. The mass of the spray system is 0.2 lbm and it contains $\forall = 12$ in.^{3} of water. Find the force exerted on the supply pipe by the spray system.



Solution:

Apply the y component of the momentum equation to the fixed control volume shown.

Basic Equation:

$$F_{S_y} + F_{B_y} = \cancel{\frac{\partial}{\partial t}\int_{CV} v \rho \, d\forall}^{\,0(1)} + \int_{CS} v \rho \vec{V} \cdot d\vec{A} \qquad (1)$$

Assumptions: (1) steady flow
(2) incompressible flow
(3) uniform flow at each section
(4) calculation of surface forces is simplified through use of gage pressures

From continuity, $0 = \cancel{\frac{\partial}{\partial t}\int \rho \, d\forall} + \int_{CS} \rho \vec{V} \cdot d\vec{A}$, for given conditions

$$0 = -|\rho V_1 A_1| + |\rho V_2 A_2| \quad \text{and} \quad V_1 = V_2 \frac{A_2}{A_1} = V \frac{a}{A}$$

The momentum flux is

$$\int_{CS} v \rho \vec{V} \cdot d\vec{A} = v_1 \{-|\rho V_1 A_1|\} + v_2 \{|\rho V_2 A_2|\} = V_1(-\rho V_1 A_1) + V(\rho V a)$$

$$= V \frac{a}{A}(-\rho V a) + V(\rho V a) = \rho V^2 a \left(1 - \frac{a}{A}\right)$$

Then from eq (1) we can write

$$R_y + P_{1g} A - \rho \forall g - M g = \rho V^2 a \left(1 - \frac{a}{A}\right). \quad \text{Solving for } R_y,$$

$$R_y = -P_{1g} A + \rho \forall g + M g + \rho V^2 a \left(1 - \frac{a}{A}\right)$$

$$= -1.45 \frac{lbf}{in^2} \times 3 in^2 + 1.94 \frac{slug}{ft^3} \times 12 in^3 \times 32.2 \frac{ft}{s^2} \times \frac{ft^3}{1728 in^3} \times \frac{lbf \cdot s^2}{slug \cdot ft}$$

$$+ 0.2 \, lbm \times 32.2 \frac{ft}{s^2} \times \frac{slug}{32.2 \, lbm} \times \frac{lbf \cdot s^2}{slug \cdot ft}$$

$$+ 1.94 \frac{slug}{ft^3} \times (15)^2 \frac{ft^2}{s^2} \times 1 in^2 \times \frac{ft^2}{144 in^2} \times \frac{lbf \cdot s^2}{slug \cdot ft} \left(1 - \frac{1 in^2}{3 in^2}\right)$$

$$R_y = -1.70 \, lbf$$

The force of the spray system on the supply pipe is

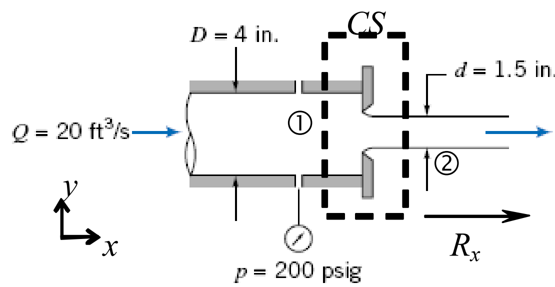
$$K_y = -R_y = 1.70 \, lbf \ (\text{upward})$$

K_y

Problem 4.82

[Difficulty: 2]

4.82 A flat plate orifice of 2 in. diameter is located at the end of a 4-in.-diameter pipe. Water flows through the pipe and orifice at 20 ft³/s. The diameter of the water jet downstream from the orifice is 1.5 in. Calculate the external force required to hold the orifice in place. Neglect friction on the pipe wall.



Given: Water flow through orifice plate

Find: Force to hold plate

Solution:

Basic equation: Momentum flux in x direction for the elbow
$$F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\,dV + \int_{CS} u\,\rho\vec{V}\cdot d\vec{A}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) Uniform flow

Hence
$$R_x + p_{1g}\cdot A_1 - p_{2g}\cdot A_2 = V_1\cdot\left(-\rho\cdot V_1\cdot A_1\right) + V_2\cdot\left(\rho\cdot V_2\cdot A_2\right) \qquad R_x = -p_{1g}\cdot A_1 + \rho\cdot\left(V_2^{\,2}\cdot A_2 - V_1^{\,2}\cdot A_1\right)$$

From continuity $Q = V_1\cdot A_1 = V_2\cdot A_2$

so
$$V_1 = \frac{Q}{A_1} = 20\cdot\frac{\text{ft}^3}{\text{s}} \times \frac{4}{\pi\cdot\left(\frac{1}{3}\cdot\text{ft}\right)^2} = 229\cdot\frac{\text{ft}}{\text{s}} \quad \text{and} \quad V_2 = V_1\cdot\frac{A_1}{A_2} = V_1\cdot\left(\frac{D}{d}\right)^2 = 229\cdot\frac{\text{ft}}{\text{s}} \times \left(\frac{4}{1.5}\right)^2 = 1628\cdot\frac{\text{ft}}{\text{s}}$$

NOTE: problem has an error: Flow rate should be 2 ft³/s not 20 ft³/s! We will provide answers to both

Hence
$$R_x = -200\cdot\frac{\text{lbf}}{\text{in}^2} \times \frac{\pi\cdot(4\cdot\text{in})^2}{4} + 1.94\cdot\frac{\text{slug}}{\text{ft}^3} \times \left[\left(1628\cdot\frac{\text{ft}}{\text{s}}\right)^2 \times \frac{\pi\cdot(1.5\cdot\text{in})^2}{4} - \left(229\cdot\frac{\text{ft}}{\text{s}}\right)^2 \times \frac{\pi\cdot(4\cdot\text{in})^2}{4}\right] \times \left(\frac{1\cdot\text{ft}}{12\cdot\text{in}}\right)^2 \times \frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}}$$

$R_x = 51707\cdot\text{lbf}$

With more realistic velocities

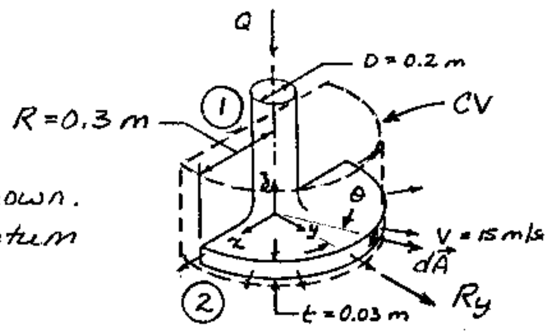
Hence
$$R_x = -200\cdot\frac{\text{lbf}}{\text{in}^2} \times \frac{\pi\cdot(4\cdot\text{in})^2}{4} + 1.94\cdot\frac{\text{slug}}{\text{ft}^3} \times \left[\left(163\cdot\frac{\text{ft}}{\text{s}}\right)^2 \times \frac{\pi\cdot(1.5\cdot\text{in})^2}{4} - \left(22.9\cdot\frac{\text{ft}}{\text{s}}\right)^2 \times \frac{\pi\cdot(4\cdot\text{in})^2}{4}\right] \times \left(\frac{1\cdot\text{ft}}{12\cdot\text{in}}\right)^2 \times \frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}}$$

$R_x = -1970\cdot\text{lbf}$

Problem 4.83

[Difficulty: 2]

4.83 The nozzle shown discharges a sheet of water through a 180° arc. The water speed is 15 m/s and the jet thickness is 30 mm at a radial distance of 0.3 m from the centerline of the supply pipe. Find (a) the volume flow rate of water in the jet sheet and (b) the y component of force required to hold the nozzle in place.



Solution: Choose CV and coordinates shown. Apply continuity and momentum equation in y-direction.

Basic equations: $Q = \int_A \vec{V} \cdot d\vec{A}$

$$F_{S_y} + \cancel{F_{B_y}}^{=0(2)} = \cancel{\frac{\partial}{\partial t}}^{=0(3)} \int_{CV} v \rho \, d\forall + \int_{CS} v \rho \vec{V} \cdot d\vec{A}$$

Assumptions: (1) Flow uniform across exit section
(2) $F_{B_y} = 0$
(3) Steady flow

At section (2), $\vec{V} \cdot d\vec{A} = VRt\,d\theta$, since flow out of CV. Then

$$Q = \int_{-\pi/2}^{\pi/2} VRt\,d\theta = VRt\left[\theta\right]_{-\pi/2}^{\pi/2} = VRt\pi$$

$$Q = 15\frac{m}{s} \times 0.3\,m \times 0.03\,m \times \pi = 0.424\ m^3/s$$ Q

From momentum

$$R_y = \int_{CS} v \rho \vec{V} \cdot d\vec{A} = \int_{A_1} v_1 \{-|\rho V_1 dA_1|\} + \int_{A_2} v_2 \{+|\rho V_2 dA_2|\}$$

with $\quad v_1 = 0 \quad\quad v_2 = V\cos\theta$

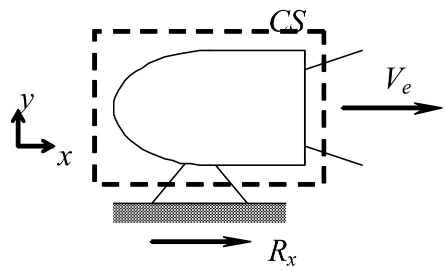
$$R_y = \int_{-\pi/2}^{\pi/2} V\cos\theta\, \rho V R t\, d\theta = \rho V^2 R t\left[\sin\theta\right]_{-\pi/2}^{\pi/2} = 2\rho V^2 R t$$

$$R_y = 2 \times 999\frac{kg}{m^3} \times (15)^2 \frac{m^2}{s^2} \times 0.3\,m \times 0.03\,m \times \frac{N \cdot s^2}{kg \cdot m} = 4.05\ kN$$ Ry

Problem 4.84

[Difficulty: 2]

4.84 At rated thrust, a liquid-fueled rocket motor consumes 80 kg/s of nitric acid as oxidizer and 32 kg/s of aniline as fuel. Flow leaves axially at 180 m/s relative to the nozzle and at 110 kPa. The nozzle exit diameter is $D = 0.6$ m. Calculate the thrust produced by the motor on a test stand at standard sea-level pressure.



Given: Data on rocket motor

Find: Thrust produced

Solution:

Basic equation: Momentum flux in x direction for the elbow $$F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\, dV\!\!\!\!- + \int_{CS} u\,\rho\vec{V}\cdot d\vec{A}$$

Assumptions: 1) Steady flow 2) Neglect change of momentum within CV 3) Uniform flow

Hence $$R_x - p_{eg}\cdot A_e = V_e\cdot\left(\rho_e\cdot V_e\cdot A_e\right) = m_e\cdot V_e \qquad R_x = p_{eg}\cdot A_e + m_e\cdot V_e$$

where p_{eg} is the exit pressure (gage), m_e is the mass flow rate at the exit (software cannot render dot over m!) and V_e is the exit velocity

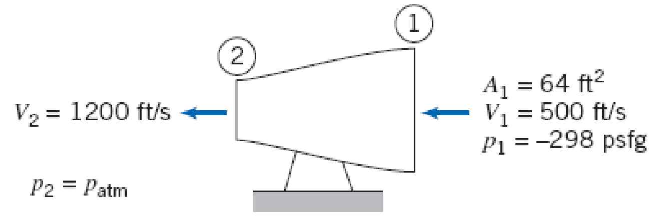
For the mass flow rate $$m_e = m_{nitricacid} + m_{aniline} = 80\cdot\frac{kg}{s} + 32\cdot\frac{kg}{s} \qquad m_e = 112\cdot\frac{kg}{s}$$

Hence $$R_x = (110 - 101)\times 10^3\cdot\frac{N}{m^2}\times\frac{\pi\cdot(0.6\cdot m)^2}{4} + 112\cdot\frac{kg}{s}\times 180\cdot\frac{m}{s}\times\frac{N\cdot s^2}{kg\cdot m} \qquad R_x = 22.7\cdot kN$$

Problem 4.85

[Difficulty: 2]

4.85 A typical jet engine test stand installation is shown, together with some test data. Fuel enters the top of the engine vertically at a rate equal to 2 percent of the mass flow rate of the inlet air. For the given conditions, compute the air flow rate through the engine and estimate the thrust.



Solution:

Apply x-component of the momentum equation to CV shown

Basic equations: $F_{S_x} + \cancel{F_{B_x}}^{=0(1)} = \cancel{\frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall}^{=0(2)} + \int_{CS} u\rho\vec{V}\cdot d\vec{A}$

$$\dot{m}_{air} = \rho_1 V_1 A_1 \quad , \quad \rho = p/RT$$

Assumptions: (1) $F_{B_x} = 0$
(2) steady flow
(3) uniform flow at inlet and outlet sections
(4) air behaves as ideal gas; $T = 70°F$
(5) fuel enters vertically (given)

$$\rho_1 = \frac{p_1}{RT_1} = \left(14.7\frac{lbf}{in^2} \times 144\frac{in^2}{ft^2} - 298\frac{lbf}{ft^2}\right) \times \frac{lbm\,°R}{53.3\,ft\cdot lbf} \times \frac{1}{530°R} = 0.0644\frac{lbm}{ft^3}$$

$$\dot{m}_{air} = \rho_1 V_1 A_1 = 0.0644\frac{lbm}{ft^3} \times 500\frac{ft}{s} \times 64\,ft^2 = 2060\,lbm/s \qquad \dot{m}$$

From the x-momentum equation

$$R_{1x} - p_{1g}A_1 + \cancel{p_{2g}A_2}^{=0} = u_1\{-\dot{m}_1\} + u_2\{\dot{m}_2\} + \cancel{u_f\{-\dot{m}_f\}}^{=0(5)}$$

$$u_1 = -V_1 \quad , u_2 = -V_2 \quad , \quad \dot{m}_2 = \dot{m}_1 + \dot{m}_f$$

Also thrust $T = K_x$ (force of engine on surroundings) $= -R_x$

So

$$-T - p_{1g}A_1 = \dot{m}_1 V_1 - \dot{m}_2 V_2 = \dot{m}_1 V_1 - (1.02\dot{m}_1)V_2$$

$$T = \dot{m}_1(1.02V_2 - V_1) - p_{1g}A_1$$

$$T = 2060\frac{lbm}{s}\left[1.02 \times 1200\frac{ft}{s} - 500\frac{ft}{s}\right] \times \frac{slug}{32.2\,lbm} \times \frac{lbf\cdot s^2}{ft\cdot slug} - \left(-298\frac{lbf}{ft^2}\right)64\,ft^2$$

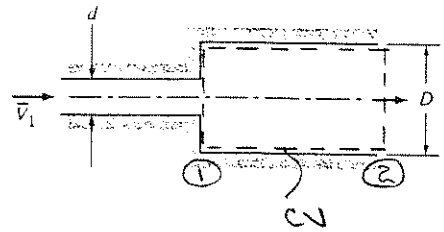
$$T = 65{,}400\,lbf \qquad T$$

Problem 4.86

[Difficulty: 3]

4.86 Consider flow through the sudden expansion shown. If the flow is incompressible and friction is neglected, show that the pressure rise, $\Delta p = p_2 - p_1$, is given by

$$\frac{\Delta p}{\frac{1}{2}\rho \bar{V}_1^2} = 2\left(\frac{d}{D}\right)^2\left[1-\left(\frac{d}{D}\right)^2\right]$$



Plot the nondimensional pressure rise versus diameter ratio to determine the optimum value of d/D and the corresponding value of the nondimensional pressure rise. *Hint:* Assume the pressure is uniform and equal to p_1 on the vertical surface of the expansion.

Solution:

Apply x component of momentum equation, using fixed CV shown

Basic equation: $F_{S_x} + \underbrace{\cancel{F_{B_x}}}_{=0(2)} = \underbrace{\cancel{\frac{\partial}{\partial t}}}_{=0(3)}\int_{CV} u\rho\, d\forall + \int_{CS} u(\rho \vec{V}\cdot d\vec{A})$

Assumptions: (1) no friction, so surface force due to pressure only
(2) $F_{B_x}=0$
(3) steady flow (4) incompressible flow (given).
(5) uniform flow at sections ① and ②
(6) uniform pressure p_1 on vertical surface of expansion

Then,

$$p_1A_2 - p_2A_2 = u_1\{-|\rho\bar{V}_1A_1|\} + u_2\{|\rho\bar{V}_2A_2|\} \qquad u_1=\bar{V}_1,\ u_2=\bar{V}_2$$

From continuity for uniform flow, $\dot{m}=\rho A_1\bar{V}_1 = \rho A_2\bar{V}_2$; $\bar{V}_2=\bar{V}_1\frac{A_1}{A_2}$

Thus,

$$p_2-p_1 = \rho\bar{V}_1\frac{A_1}{A_2}\bar{V}_1 - \rho\bar{V}_1\frac{A_1}{A_2}\bar{V}_2 = \rho\bar{V}_1\frac{A_1}{A_2}(\bar{V}_1-\bar{V}_2)$$

$$p_2-p_1 = \rho\bar{V}_1^2\frac{A_1}{A_2}\left(1-\frac{\bar{V}_2}{\bar{V}_1}\right) = \rho\bar{V}_1^2\frac{A_1}{A_2}\left(1-\frac{A_1}{A_2}\right)$$

and

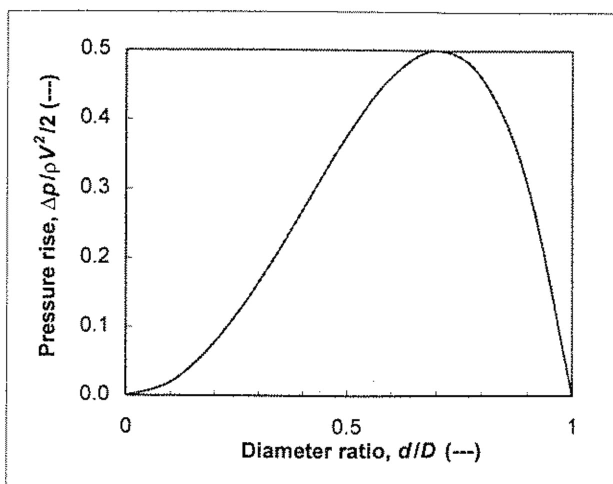
$$\frac{p_2-p_1}{\frac{1}{2}\rho\bar{V}_1^2} = 2\frac{A_1}{A_2}\left(1-\frac{A_1}{A_2}\right) = 2\left(\frac{d}{D}\right)^2\left[1-\left(\frac{d}{D}\right)^2\right]$$

Q.E.D.

From the plot below we see that $\frac{\Delta p}{\frac{1}{2}\rho\bar{V}_1^2}$ has an optimum value of ≈ 0.5 at $d/D = 0.70$

Note: As expected
· for $d=D$, $\Delta p=0$ for straight pipe
· for $\frac{d}{D}\to 0$, $\Delta p=0$ for free jet

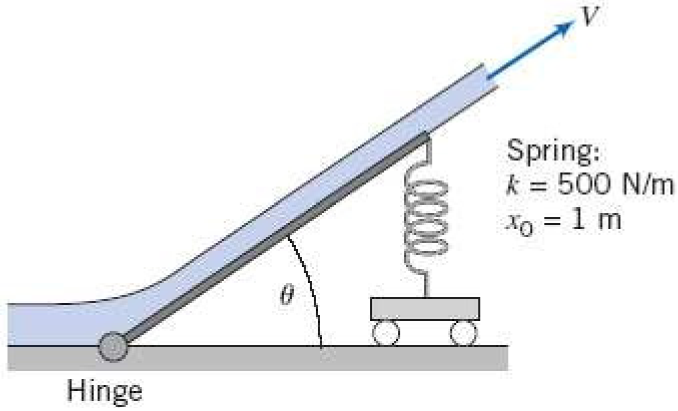
Also note that the location of section ② would have to be chosen with care to make assumption (5) reasonable.



Problem 4.87

[Difficulty: 2]

4.87 A free jet of water with constant cross-section area 0.01 m^2 is deflected by a hinged plate of length 2 m supported by a spring with spring constant $k = 500$ N/m and uncompressed length $x_0 = 1$ m. Find and plot the deflection angle θ as a function of jet speed V. What jet speed has a deflection of $\theta = 5°$?



Given: Data on flow and system geometry

Find: Deflection angle as a function of speed; jet speed for 10º deflection

Solution:

The given data is $\rho = 999 \cdot \frac{kg}{m^3}$ $\quad A = 0.01 \cdot m^2$ $\quad L = 2 \cdot m$ $\quad k = 500 \cdot \frac{N}{m}$ $\quad x_0 = 1 \cdot m$

Basic equation (y momentum): $$F_y = F_{S_y} + F_{B_y} = \frac{\partial}{\partial t}\int_{CV} v\,\rho\,dV\!\!\!\!- + \int_{CS} v\,\rho\vec{V}\cdot d\vec{A}$$

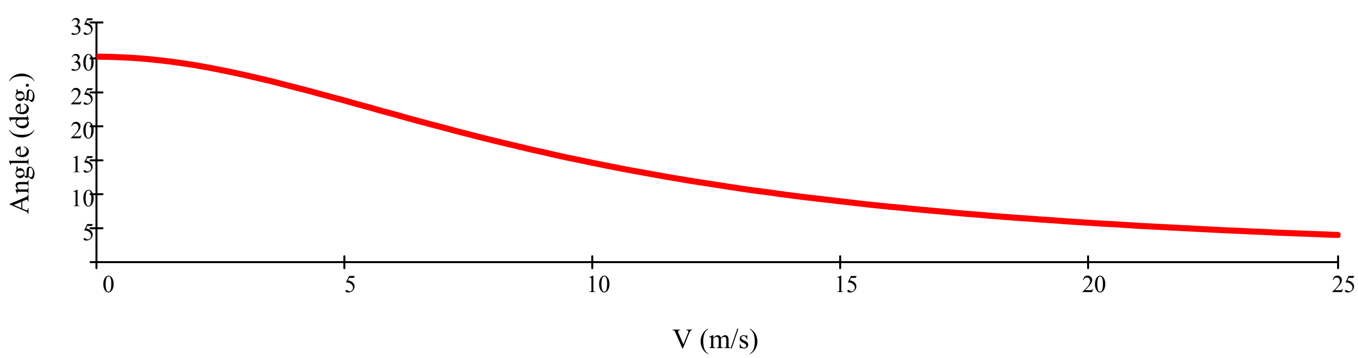
Applying this to the current system in the vertical direction

$$F_{spring} = V \cdot \sin(\theta) \cdot (\rho \cdot V \cdot A) \qquad \text{But} \qquad F_{spring} = k \cdot x = k \cdot \left(x_0 - L \cdot \sin(\theta)\right)$$

Hence $$k \cdot \left(x_0 - L \cdot \sin(\theta)\right) = \rho \cdot V^2 \cdot A \cdot \sin(\theta)$$

Solving for θ $$\theta = \operatorname{asin}\left(\frac{k \cdot x_0}{k \cdot L + \rho \cdot A \cdot V^2}\right)$$

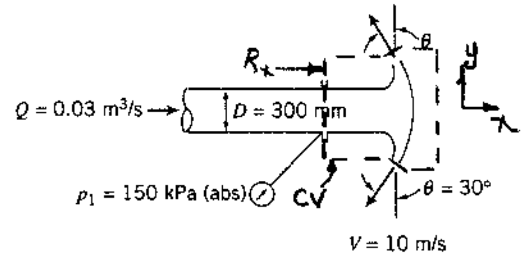
For the speed at which θ = 10º, solve $$V = \sqrt{\frac{k \cdot \left(x_0 - L \cdot \sin(\theta)\right)}{\rho \cdot A \cdot \sin(\theta)}} \qquad V = \sqrt{\frac{500 \cdot \frac{N}{m} \cdot (1 - 2 \cdot \sin(5 \cdot deg)) \cdot m}{999 \cdot \frac{kg}{m^3} \cdot 0.01 \cdot m^2 \cdot \sin(5 \cdot deg)} \cdot \frac{kg \cdot m}{N \cdot s^2}} \qquad V = 21.8 \frac{m}{s}$$



Problem 4.88

[Difficulty: 3]

4.88 A conical spray head is shown. The fluid is water and the exit stream is uniform. Evaluate (a) the thickness of the spray sheet at 400 mm radius and (b) the axial force exerted by the spray head on the supply pipe.



Solution: Apply continuity and the x component of the momentum equation, using the CV, CS shown.

Basic equation:

$$F_{S_x} + \underbrace{F_{B_x}}_{=0(1)} = \underbrace{\frac{\partial}{\partial t}\int_{CV} u\rho\, dV}_{=0(2)} + \int_{CS} u\rho\vec{V}\cdot d\vec{A}$$

Assumptions:
(1) $F_{B_x} = 0$
(2) Steady flow
(3) Incompressible flow
(4) Uniform flow at each section
(5) Use gage pressure to cancel p_{atm}

From continuity,

$$V_1 = \frac{Q}{A_1} = \frac{4Q}{\pi D_1^2} = \frac{4}{\pi} \times 0.03\frac{m^3}{sec} \times \frac{1}{(0.3)^2\, m^2} = 0.424\ m/s$$

Assume velocity in jet sheet is constant at $V = 10\ m/s$. Then

$$Q = 2\pi R t V;\quad t = \frac{Q}{2\pi R V} = \frac{1}{2\pi} \times 0.03\frac{m^3}{s} \times \frac{1}{0.4\ m} \times \frac{s}{10\ m} \times 1000\frac{mm}{m} = 1.19\ mm$$ t

From momentum,

$$R_x + p_{1g}A_1 = u_1\{-\rho Q\} + u_2\{+\rho Q\}$$

$$u_1 = V_1 \qquad u_2 = -V\sin\theta$$

$$R_x + p_{1g}A_1 = -(V_1 + V\sin\theta)\rho Q$$

or

$$R_x = -p_{1g}A_1 - (V_1 + V\sin\theta)\rho Q$$

$$= -(150-101)10^3\frac{N}{m^2} \times \frac{\pi}{4}(0.3)^2 m^2 - (0.424 + 10\sin 30°)\frac{m}{s} \times 999\frac{kg}{m^3} \times 0.03\frac{m^3}{s} \times \frac{N\cdot s^2}{kg\cdot m}$$

$$R_x = -3.63\ kN$$

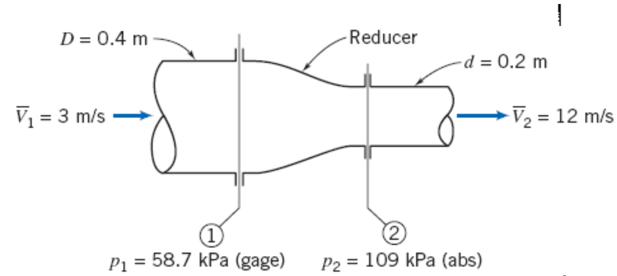
But R_x is force on CV; force on supply pipe is K_x,

$$K_x = -R_x = 3.63\ kN\ \text{(to the right)}$$ K_x

Problem 4.89

[Difficulty: 3]

4.89 A reducer in a piping system is shown. The internal volume of the reducer is 0.2 m^3 and its mass is 25 kg. Evaluate the total force that must be provided by the surrounding pipes to support the reducer. The fluid is gasoline.



Solution: Apply the x and y components of the momentum equation, using the CV and coordinates shown. cancel p_{atm}.

Basic equations:

$$F_{Sx} + \underbrace{F_{Bx}}_{=0}$$

$$F_{Sy} + F_{By} = \frac{\partial}{\partial t}\int_{CV} v \rho \, dV + \int_{CS} v \rho \vec{V} \cdot d\vec{A}$$

Assumptions: (1) $F_{Bx} = 0$
(2) Steady flow
(3) Uniform flow at each section
(4) Incompressible flow, $SG = 0.72$ {Table A.2, Appendix A}

From the x component of momentum,

$$R_x + p_{1g}A_1 - p_{2g}A_2 = u_1\{-|\rho V_1 A_1|\} + u_2\{+|\rho V_2 A_2|\} = (V_2 - V_1)\rho V_1 A_1$$

$$u_1 = V_1 \qquad u_2 = V_2$$

$$R_x = p_{2g}A_2 - p_{1g}A_1 + (\bar{V}_2 - \bar{V}_1)\rho \bar{V}_1 A_1$$

Note $\rho = SG\,\rho_{H_2O}$

$$= (109 - 101)10^3 \frac{N}{m^2} \times \frac{\pi}{4}(0.2)^2 m^2 - 58.7\times10^3 \frac{N}{m^2} \times \frac{\pi}{4}(0.4)^2 m^2$$
$$+ (12-3)\frac{m}{s} \times (0.72)1000\frac{kg}{m^3} \times 3\frac{m}{s} \times \frac{\pi}{4}(0.4)^2 m^2 \times \frac{N \cdot s^2}{kg \cdot m}$$

$R_x = -4.68$ kN (force must be applied to left) R_x

From the y component of momentum,

$$R_y - Mg - \rho g V = \underbrace{v_1}_{=0}\{-|\rho V_1 A_1|\} + \underbrace{v_2}_{=0}\{+|\rho V_2 A_2|\}$$

$$R_y = Mg + \rho g V$$

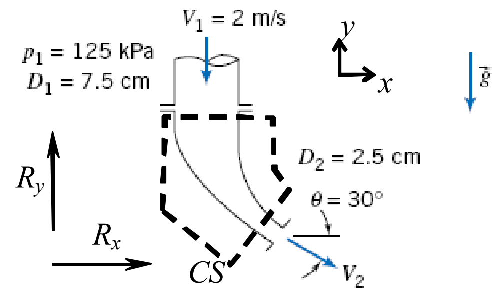
$$= 25\,kg \times 9.81\frac{m}{s^2} \times \frac{N \cdot s^2}{kg \cdot m} + (0.72)1000\frac{kg}{m^3} \times 9.81\frac{m}{s^2} \times 0.2\,m^3 \times \frac{N \cdot s^2}{kg \cdot m}$$

$R_y = 1.66$ kN (force must be applied up) R_y

Problem 4.90

[Difficulty: 2]

4.90 A curved nozzle assembly that discharges to the atmosphere is shown. The nozzle mass is 4.5 kg and its internal volume is 0.002 m^3. The fluid is water. Determine the reaction force exerted by the nozzle on the coupling to the inlet pipe.



Given: Data on nozzle assembly

Find: Reaction force

Solution:

Basic equation: Momentum flux in x and y directions

$$F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\, dV + \int_{CS} u\,\rho \vec{V}\cdot d\vec{A}$$

$$F_y = F_{S_y} + F_{B_y} = \frac{\partial}{\partial t}\int_{CV} v\,\rho\, dV + \int_{CS} v\,\rho \vec{V}\cdot d\vec{A}$$

Assumptions: 1) Steady flow 2) Incompressible flow CV 3) Uniform flow

For x momentum
$$R_x = V_2\cdot\cos(\theta)\cdot\left(\rho\cdot V_2\cdot A_2\right) = \rho\cdot V_2^{\,2}\cdot\frac{\pi\cdot D_2^{\,2}}{4}\cdot\cos(\theta)$$

From continuity
$$A_1\cdot V_1 = A_2\cdot V_2 \qquad V_2 = V_1\cdot\frac{A_1}{A_2} = V_1\cdot\left(\frac{D_1}{D_2}\right)^2 \qquad V_2 = 2\cdot\frac{m}{s}\times\left(\frac{7.5}{2.5}\right)^2 \qquad V_2 = 18\frac{m}{s}$$

Hence
$$R_x = 1000\cdot\frac{kg}{m^3}\times\left(18\cdot\frac{m}{s}\right)^2\times\frac{\pi}{4}\times(0.025\cdot m)^2\times\cos(30\cdot deg)\times\frac{N\cdot s^2}{kg\cdot m} \qquad R_x = 138\cdot N$$

For y momentum
$$R_y - p_1\cdot A_1 - W - \rho\cdot Vol\cdot g = -V_1\cdot\left(-\rho\cdot V_1\cdot A_1\right) - V_2\cdot\sin(\theta)\cdot\left(\rho\cdot V_2\cdot A_2\right)$$

$$R_y = p_1\cdot\frac{\pi\cdot D_1^{\,2}}{4} + W + \rho\cdot Vol\cdot g + \frac{\rho\cdot\pi}{4}\cdot\left(V_1^{\,2}\cdot D_1^{\,2} - V_2^{\,2}\cdot D_2^{\,2}\cdot\sin(\theta)\right)$$

where
$$W = 4.5\cdot kg\times 9.81\cdot\frac{m}{s^2}\times\frac{N\cdot s^2}{kg\cdot m} \qquad W = 44.1\,N \qquad Vol = 0.002\cdot m^3$$

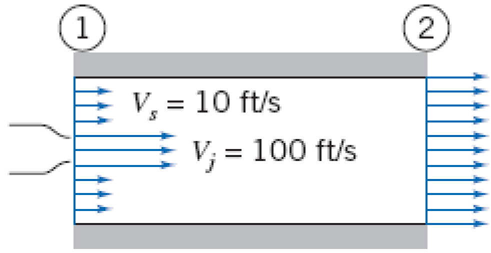
Hence
$$R_y = 125\times 10^3\cdot\frac{N}{m^2}\times\frac{\pi\cdot(0.075\cdot m)^2}{4} + 44.1\cdot N + 1000\cdot\frac{kg}{m^3}\times 0.002\cdot m^3\times 9.81\cdot\frac{m}{s^2}\times\frac{N\cdot s^2}{kg\cdot m}\ \ldots$$
$$+ 1000\cdot\frac{kg}{m^3}\times\frac{\pi}{4}\times\left[\left(2\cdot\frac{m}{s}\right)^2\times(0.075\cdot m)^2 - \left(18\cdot\frac{m}{s}\right)^2\times(0.025\cdot m)^2\times\sin(30\cdot deg)\right]\times\frac{N\cdot s^2}{kg\cdot m}$$

$$R_y = 554\cdot N$$

Problem 4.91

[Difficulty: 3]

4.91 A water jet pump has jet area 0.1 ft^2 and jet speed 100 ft/s. The jet is within a secondary stream of water having speed $V_s = 10$ ft/s. The total area of the duct (the sum of the jet and secondary stream areas) is 0.75 ft^2. The water is thoroughly mixed and leaves the jet pump in a uniform stream. The pressures of the jet and secondary stream are the same at the pump inlet. Determine the speed at the pump exit and the pressure rise, $p_2 - p_1$.



Given: Data on water jet pump

Find: Speed at pump exit; pressure rise

Solution:

Basic equation: Continuity, and momentum flux in x direction

$$\frac{\partial}{\partial t}\int_{CV} \rho \, dV + \int_{CS} \rho \vec{V} \cdot d\vec{A} = 0 \qquad F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u \, \rho \, dV + \int_{CS} u \, \rho \vec{V} \cdot d\vec{A}$$

Assumptions: 1) Steady flow 2) Incompressible flow CV 3) Uniform flow

From continuity
$$-\rho \cdot V_s \cdot A_s - \rho \cdot V_j \cdot A_j + \rho \cdot V_2 \cdot A_2 = 0 \qquad V_2 = V_s \cdot \frac{A_s}{A_2} + V_j \cdot \frac{A_j}{A_2} = V_s \cdot \left(\frac{A_2 - A_j}{A_2}\right) + V_j \cdot \frac{A_j}{A_2}$$

$$V_2 = 10 \cdot \frac{ft}{s} \times \left(\frac{0.75 - 0.1}{0.75}\right) + 100 \cdot \frac{ft}{s} \times \frac{0.1}{0.75} \qquad V_2 = 22 \cdot \frac{ft}{s}$$

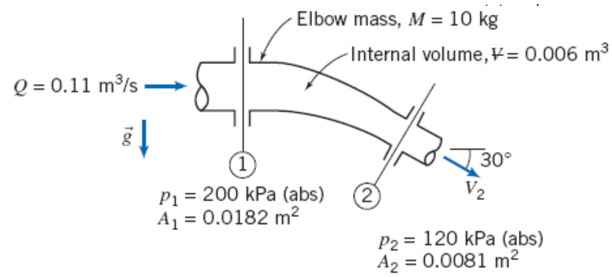
For x momentum
$$p_1 \cdot A_2 - p_2 \cdot A_2 = V_j \cdot \left(-\rho \cdot V_j \cdot A_j\right) + V_s \cdot \left(-\rho \cdot V_s \cdot A_s\right) + V_2 \cdot \left(\rho \cdot V_2 \cdot A_2\right)$$

$$\Delta p = p_2 - p_1 = \rho \cdot \left(V_j^2 \cdot \frac{A_j}{A_2} + V_s^2 \cdot \frac{A_s}{A_2} - V_2^2\right)$$

$$\Delta p = 1.94 \cdot \frac{slug}{ft^3} \times \left[\left(100 \cdot \frac{ft}{s}\right)^2 \times \frac{0.1}{0.75} + \left(10 \cdot \frac{ft}{s}\right)^2 \times \frac{(0.75 - 0.1)}{0.75} - \left(22 \cdot \frac{ft}{s}\right)^2\right] \times \frac{lbf \cdot s^2}{slug \cdot ft}$$

Hence
$$\Delta p = 1816 \cdot \frac{lbf}{ft^2} \qquad \Delta p = 12.6 \cdot psi$$

4.92 A 30° reducing elbow is shown. The fluid is water. Evaluate the components of force that must be provided by the adjacent pipes to keep the elbow from moving.



Solution: Apply the x and y components of the momentum equation using the cs and CV shown.

Basic equations:
$$F_{Sx} + \underbrace{F_{Bx}}_{=0(4)} = \underbrace{\frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall}_{=0(1)} + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$$
$$F_{Sy} + F_{By} = \underbrace{\frac{\partial}{\partial t}\int_{CV} v\rho\, d\forall}_{=0(1)} + \int_{CS} v\rho \vec{V}\cdot d\vec{A}$$

Assumptions: (1) Steady flow (3) Use gage pressures
(2) Uniform flow at each section (4) x horizontal

x comp: $R_x + p_{1g}A_1 - p_{2g}A_2\cos\theta = u_1\{-|\rho Q|\} + u_2\{+|\rho Q|\}$

$u_1 = V_1 \qquad u_2 = V_2\cos\theta$

$R_x = (-V_1 + V_2\cos\theta)\rho Q - p_{1g}A_1 + p_{2g}A_2\cos\theta$

$V_1 = \frac{Q}{A_1} = 0.11\frac{m^3}{s}\times\frac{1}{0.0182\ m^2} = 6.04\frac{m}{s}$

$V_2 = \frac{Q}{A_2} = 0.11\frac{m^3}{s}\times\frac{1}{0.0081\ m^2} = 13.6\frac{m}{s}$

$= \left(-6.04\frac{m}{s} + 13.6\frac{m}{s}\times\cos 30°\right)999\frac{kg}{m^3}\times 0.11\frac{m^3}{s}\times\frac{N\cdot s^2}{kg\cdot m} - (200-101)10^3\frac{N}{m^2}\times 0.0182\ m^2 + (120-101)10^3\frac{N}{m^2}\times 0.0081\ m^2\times\cos 30°$

$R_x = +631 - 1800 + 133\ N = -1040\ N$ $\quad R_x$

y comp: $R_y + p_{2g}A_2\sin\theta - Mg - \rho\forall g = v_1\{-|\rho Q|\} + v_2\{+|\rho Q|\}$

$v_1 = 0 \qquad v_2 = -V_2\sin\theta$

$R_y = -V_2\sin\theta\,\rho Q + Mg + \rho\forall g - p_{2g}A_2\sin\theta$

$= -13.6\frac{m}{s}\times\sin 30°\times 999\frac{kg}{m^3}\times 0.11\frac{m^3}{s}\times\frac{N\cdot s^2}{kg\cdot m} + 10\ kg\times 9.81\frac{m}{s^2}\times\frac{N\cdot s^2}{kg\cdot m}$

$+ 999\frac{kg}{m^3}\times 0.006\ m^3\times 9.81\frac{m}{s^2}\times\frac{N\cdot s^2}{kg\cdot m} - (120-101)10^3\frac{N}{m^2}\times 0.0081\ m^2\times\sin 30°$

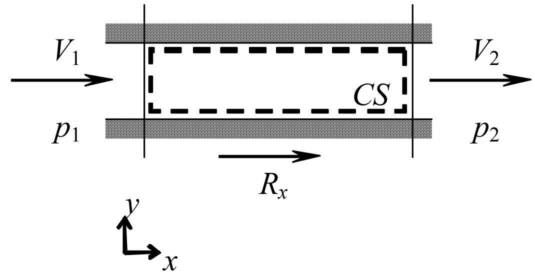
$R_y = -747 + 98.1 + 58.8 - 77 = -667\ N$ $\quad R_y$

{ R_x and R_y are the horizontal and vertical components of force that must be supplied by the adjacent pipes to keep the elbow (the control volume) from moving. }

Problem 4.93

[Difficulty: 3]

4.93 Consider the steady adiabatic flow of air through a long straight pipe with 0.05 m^2 cross-sectional area. At the inlet, the air is at 200 kPa (gage), 60°C, and has a velocity of 150 m/s. At the exit, the air is at 80 kPa and has a velocity of 300 m/s. Calculate the axial force of the air on the pipe. (Be sure to make the direction clear.)



Given: Data on adiabatic flow of air

Find: Force of air on pipe

Solution:

Basic equation: Continuity, and momentum flux in x direction, plus ideal gas equation

$$\frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A} = 0 \qquad F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\, d\forall + \int_{CS} u\,\rho \vec{V} \cdot d\vec{A} \qquad p = \rho \cdot R \cdot T$$

Assumptions: 1) Steady flow 2) Ideal gas CV 3) Uniform flow

From continuity $\quad -\rho_1 \cdot V_1 \cdot A_1 + \rho_2 \cdot V_2 \cdot A_2 = 0 \qquad \rho_1 \cdot V_1 \cdot A = \rho_2 \cdot V_2 \cdot A \qquad \rho_1 \cdot V_1 = \rho_2 \cdot V_2$

For x momentum $\quad R_x + p_1 \cdot A - p_2 \cdot A = V_1 \cdot (-\rho_1 \cdot V_1 \cdot A) + V_2 \cdot (\rho_2 \cdot V_2 \cdot A) = \rho_1 \cdot V_1 \cdot A \cdot (V_2 - V_1)$

$$R_x = (p_2 - p_1) \cdot A + \rho_1 \cdot V_1 \cdot A \cdot (V_2 - V_1)$$

For the air $\quad \rho_1 = \frac{P_1}{R_{air} \cdot T_1} \qquad \rho_1 = (200 + 101) \times 10^3 \cdot \frac{N}{m^2} \times \frac{kg \cdot K}{286.9 \cdot N \cdot m} \times \frac{1}{(60 + 273) \cdot K} \qquad \rho_1 = 3.15 \frac{kg}{m^3}$

$$R_x = (80 - 200) \times 10^3 \cdot \frac{N}{m^2} \times 0.05 \cdot m^2 + 3.15 \cdot \frac{kg}{m^3} \times 150 \cdot \frac{m}{s} \times 0.05 \cdot m^2 \times (300 - 150) \cdot \frac{m}{s} \times \frac{N \cdot s^2}{kg \cdot m}$$

Hence $\quad R_x = -2456\,N$

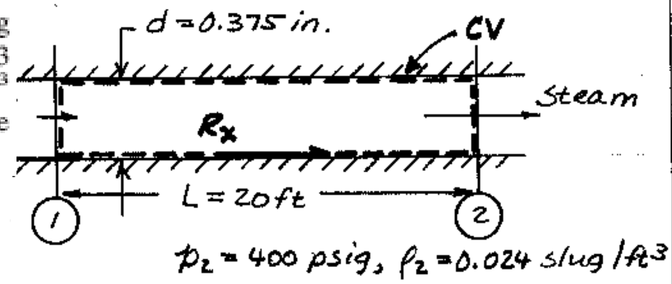
This is the force of the pipe on the air; the pipe is opposing flow. Hence the force of the air on the pipe is $\quad F_{pipe} = -R_x$

$F_{pipe} = 2456\,N \qquad$ The air is dragging the pipe to the right

Problem 4.94

[Difficulty: 3]

4.94 A monotube boiler consists of a 20 ft length of tubing with 0.375 in. inside diameter. Water enters at the rate of 0.3 lbm/s at 500 psia. Steam leaves at 400 psig with 0.024 slug/ft^3 density. Find the magnitude and direction of the force exerted by the flowing fluid on the tube.



Solution: Apply the x component of the momentum equation, using the CV and coordinates shown.

Basic equation:

$$F_{Sx} + \underbrace{F_{Bx}}_{=0(1)} = \underbrace{\frac{\partial}{\partial t}\int_{CV} u\rho\, dV}_{=0(2)} + \int_{CS} u\rho\vec{V}\cdot d\vec{A}$$

Assumptions: (1) $F_{Bx} = 0$
(2) Steady flow
(3) Uniform flow at each section
(4) Use gage pressures to cancel p_{atm}

From continuity,

$$\dot{m} = \rho_1 V_1 A_1 = \rho_2 V_2 A_2\ ;\ A = \text{constant, so } \rho_1 V_1 = \rho_2 V_2\text{. Thus}$$

$$V_1 = \frac{\dot{m}}{\rho_1 A} = 0.3\,\frac{\text{lbm}}{\text{s}} \times \frac{\text{ft}^3}{1.94\,\text{slug}} \times \frac{4}{\pi}\,\frac{1}{(0.375)^2\,\text{in.}^2} \times \frac{\text{slug}}{32.2\,\text{lbm}} \times 144\,\frac{\text{in.}^2}{\text{ft}^2} = 6.26\ \text{ft/s}$$

and

$$V_2 = V_1\frac{\rho_1}{\rho_2} = 6.26\,\frac{\text{ft}}{\text{s}} \times 1.94\,\frac{\text{slug}}{\text{ft}^3} \times \frac{\text{ft}^3}{0.024\,\text{slug}} = 506\ \text{ft/s}$$

From momentum,

$$R_x + p_{1g}A_1 - p_{2g}A_2 = u_1\{-\dot{m}\} + u_2\{+\dot{m}\} = (V_2 - V_1)\dot{m}$$

$$u_1 = V_1 \qquad u_2 = V_2$$

$$R_x = (p_{2g} - p_{1g})A + (V_2 - V_1)\dot{m}$$

$$= [400 - (500 - 14.7)]\,\frac{\text{lbf}}{\text{in.}^2} \times \frac{\pi}{4}(0.375)^2\,\text{in.}^2 + (506 - 6.26)\frac{\text{ft}}{\text{s}} \times 0.3\,\frac{\text{lbm}}{\text{s}} \times \frac{\text{slug}}{32.2\,\text{lbm}} \times \frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}}$$

$$R_x = -4.77\ \text{lbf}$$

But R_x is force on CV; force on pipe is K_x,

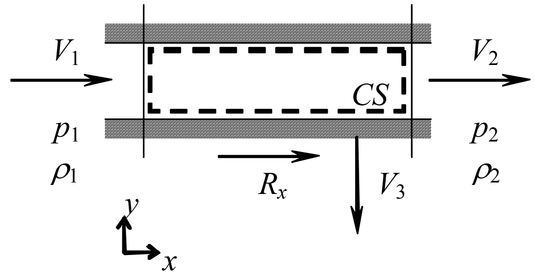
$$K_x = -R_x = 4.77\ \text{lbf (to right)}$$

K_x

Problem 4.95

[Difficulty: 3]

4.95 A gas flows steadily through a heated porous pipe of constant 0.15 m^2 cross-sectional area. At the pipe inlet, the absolute pressure is 400 kPa, the density is 6 kg/m^3, and the mean velocity is 170 m/s. The fluid passing through the porous wall leaves in a direction normal to the pipe axis, and the total flow rate through the porous wall is 20 kg/s. At the pipe outlet, the absolute pressure is 300 kPa and the density is 2.75 kg/m^3. Determine the axial force of the fluid on the pipe.



Given: Data on heated flow of gas

Find: Force of gas on pipe

Solution:

Basic equation: Continuity, and momentum flux in x direction

$$\frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A} = 0 \qquad F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\, d\forall + \int_{CS} u\,\rho \vec{V} \cdot d\vec{A} \qquad p = \rho \cdot R \cdot T$$

Assumptions: 1) Steady flow 2) Uniform flow

From continuity $\quad -\rho_1 \cdot V_1 \cdot A_1 + \rho_2 \cdot V_2 \cdot A_2 + m_3 = 0 \qquad V_2 = V_1 \cdot \frac{\rho_1}{\rho_2} - \frac{m_3}{\rho_2 \cdot A}$

where m_3 = 20 kg/s is the mass leaving through the walls (the software does not allow a dot)

$$V_2 = 170 \cdot \frac{m}{s} \times \frac{6}{2.75} - 20 \cdot \frac{kg}{s} \times \frac{m^3}{2.75 \cdot kg} \times \frac{1}{0.15 \cdot m^2} \qquad V_2 = 322\,\frac{m}{s}$$

For x momentum $\quad R_x + p_1 \cdot A - p_2 \cdot A = V_1 \cdot (-\rho_1 \cdot V_1 \cdot A) + V_2 \cdot (\rho_2 \cdot V_2 \cdot A)$

$$R_x = \left[(p_2 - p_1) + \rho_2 \cdot V_2^{\,2} - \rho_1 \cdot V_1^{\,2}\right] \cdot A$$

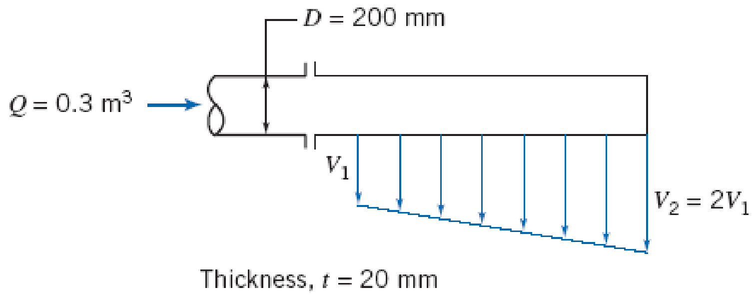
$$R_x = \left[(300 - 400) \times 10^3 \cdot \frac{N}{m^2} + \left[2.75 \cdot \frac{kg}{m^3} \times \left(322 \cdot \frac{m}{s}\right)^2 - 6 \cdot \frac{kg}{m^3} \times \left(170 \cdot \frac{m}{s}\right)^2\right] \times \frac{N \cdot s^2}{kg \cdot m}\right] \times 0.15 \cdot m^2$$

Hence $\quad R_x = 1760\,N$

Problem 4.96

[Difficulty: 3]

4.96 Water is discharged at a flow rate of 0.3 m^3/s from a narrow slot in a 200-mm-diameter pipe. The resulting horizontal two-dimensional jet is 1 m long and 20 mm thick, but of nonuniform velocity; the velocity at location (2) is twice that at location (1). The pressure at the inlet section is 50 kPa (gage). Calculate (a) the velocity in the pipe and at locations (1) and (2) and (b) the forces required at the coupling to hold the spray pipe in place. Neglect the mass of the pipe and the water it contains.



Given: Data on flow out of pipe device

Find: Velocities at 1 and 2; force on coupling

Solution:

Basic equations (continuity and x and y mom.):
$$\frac{\partial}{\partial t}\int_{CV}\rho\, dV + \int_{CS}\rho\vec{V}\cdot d\vec{A} = 0$$

$$F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\, dV + \int_{CS} u\,\rho\vec{V}\cdot d\vec{A} \qquad F_y = F_{S_y} + F_{B_y} = \frac{\partial}{\partial t}\int_{CV} v\,\rho\, dV + \int_{CS} v\,\rho\vec{V}\cdot d\vec{A}$$

The given data is $\rho = 999\cdot\frac{kg}{m^3}$ $\quad D = 20\cdot cm$ $\quad L = 1\cdot m$ $\quad t = 20\cdot mm$ $\quad p_{3g} = 50\cdot kPa$ $\quad Q = 0.3\cdot\frac{m^3}{s}$

From continuity $\quad Q = A\cdot V_{ave}$ $\quad$ due to linear velocity distribution $\quad V_{ave} = \frac{1}{2}\cdot(V_1 + V_2)$

Note that at the exit $\quad V(x) = V_1 + \frac{(V_2 - V_1)}{L}\cdot x$

Hence $\quad Q = \frac{1}{2}\cdot(V_1 + V_2)\cdot L\cdot t = \frac{1}{2}\cdot(V_1 + 2\cdot V_1)\cdot L\cdot t$

$V_1 = \frac{2\cdot Q}{3\cdot L\cdot t}$ $\quad V_1 = 10\frac{m}{s}$ $\quad V_2 = 2\cdot V_1$ $\quad V_2 = 20\frac{m}{s}$

At the inlet (location 3) $\quad V_3 = \frac{Q}{\frac{\pi}{4}\cdot D^2}$ $\quad V_3 = 9.549\frac{m}{s}$

Applying x momentum $\quad R_x + p_{3g}\cdot\frac{\pi}{4}\cdot D^2 = -V_3\cdot\rho\cdot Q$ $\quad R_x = -p_{3g}\cdot\frac{\pi}{4}\cdot D^2 - V_3\cdot\rho\cdot Q$ $\quad R_x = -4.43\cdot kN$

Applying y momentum $\quad R_y = -\int_0^L V(x)\cdot\rho\cdot V(x)\cdot t\, dx = -\rho\cdot t\cdot\int_0^L \left[V_1 + \frac{(V_2 - V_1)}{L}\cdot x\right]^2 dx$

Expanding and integrating $\quad R_y = -\rho\cdot t\cdot\left[V_1^{\,2}\cdot L + 2\cdot V_1\cdot\left(\frac{V_2 - V_1}{L}\right)\cdot\frac{L^2}{2} + \left(\frac{V_2 - V_1}{L}\right)^2\cdot\frac{L^3}{3}\right]$ $\quad R_y = -4.66\cdot kN$

Problem 4.97

[Difficulty: 3]

4.97 Water flows steadily through the square bend of Problem 4.39. Flow at the inlet is at $p_1 = 185$ kPa (abs). Flow at the exit is nonuniform, vertical, and at atmospheric pressure. The mass of the channel structure is $M_c = 2.05$ kg; the internal volume of the channel is $\forall = 0.00355\ \text{m}^3$. Evaluate the force exerted by the channel assembly on the supply duct.

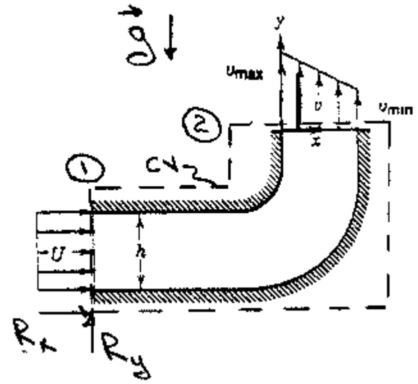
<ins>Solution:</ins> Apply conservation of mass & momentum equations to the CV shown.

Basic equations:

$$0 = \cancel{\frac{\partial}{\partial t}\int_{CV} \rho\, d\forall}^{\,0(1)} + \int_{CS} \rho \vec{V}\cdot d\vec{A} \qquad (1)$$

$$F_{S_x} + F_{B_x} = \cancel{\frac{\partial}{\partial t}\int_{CV} u \rho\, d\forall}^{\,0(1)} + \int_{CS} u \rho \vec{V}\cdot d\vec{A} \qquad (2)$$

$$F_{S_y} + F_{B_y} = \cancel{\frac{\partial}{\partial t}\int_{CV} v \rho\, d\forall}^{\,0(1)} + \int_{CS} v \rho \vec{V}\cdot d\vec{A} \qquad (3)$$



Assumptions:
(1) steady flow (2) incompressible flow
(3) uniform flow at inlet
(4) use gage pressures.

From continuity, $0 = \vec{V}_1 \cdot \vec{A}_1 + \int \vec{V}_2 \cdot d\vec{A}_2 = -Uwh + \int_0^h v w\, dx$

$\therefore\ Uh = \int_0^h v\, dx = \int_0^h v_{min}\left(2 - \frac{x}{h}\right) dx = v_{min}\left[2x - \frac{x^2}{2h}\right]_0^h = \frac{3}{2} v_{min} h$

and $\quad v_{min} = \frac{2}{3} U = \frac{2}{3} \times 7.5\ \frac{m}{s} = 5.0\ m/s$

From Eq. 2,

$R_x + p_{1g}A_1 = u_1\{-\rho U A_1\} + \int_0^h \cancel{u_2}^{\,0} \rho v_{min}\left(2 - \frac{x}{h}\right) w\, dx = -\rho U^2 A_1$

$R_x = -p_{1g}A_1 - \rho U^2 A_1 = -(185-101)10^3 \frac{N}{m^2} \times (0.0755)^2 m^2 - 999 \frac{kg}{m^3} \times (7.5)^2 \frac{m^2}{s^2} \times (0.0755)^2 m^2$

$R_x = -479\ N - 320 \frac{kg \cdot m}{s^2} \cdot \frac{N \cdot s^2}{kg \cdot m} = -479 N - 320 N = -799\ N$

$K_x = -R_x = 799\ N$ (on supply duct to the right) $\longleftarrow K_x$

From Eq. 3,

$R_y - M_c g - \rho \forall g = \cancel{v_1}^{\,v_1=0}\{-\rho U A_1\} + \int_0^h v_2\{\rho v_2 w\, dx\}$

$R_y - M_c g - \rho \forall g = \int_0^h v_{min}\left(2 - \frac{x}{h}\right) \rho v_{min}\left(2 - \frac{x}{h}\right) w\, dx$

$= \rho v_{min}^2 w \int_0^h \left(4 - 4\frac{x}{h} + \frac{x^2}{h^2}\right) dx$

$= \rho v_{min}^2 w \left[4x - 2\frac{x^2}{h} + \frac{x^3}{3h^2}\right]_0^h = \rho v_{min}^2 w h \frac{7}{3}$

$\therefore\ R_y = \left[2.05 kg \times 9.81 \frac{m}{s^2} + 999 \frac{kg}{m^3} \times (0.00355 m^3) 9.8 \frac{m}{s^2} + \frac{7}{3} \times 999 \frac{kg}{m^3} \times (5.0)^2 \frac{m^2}{s^2} (0.0755)^2 m^2\right] \frac{N \cdot s^2}{kg \cdot m}$

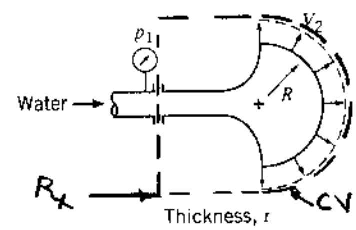
$R_y = (20.1 + 34.8 + 332)\ N = 387\ N$ (on CV)

$K_y = -R_y = -387\ N$ (on supply duct, down) $\longleftarrow K_y$

Problem 4.98

[Difficulty: 3]

4.98 A nozzle for a spray system is designed to produce a flat radial sheet of water. The sheet leaves the nozzle at $V_2 = 10$ m/s, covers 180° of arc, and has thickness $t = 1.5$ mm. The nozzle discharge radius is $R = 50$ mm. The water supply pipe is 35 mm in diameter and the inlet pressure is $p_1 = 150$ kPa (abs). Evaluate the axial force exerted by the spray nozzle on the coupling.



Solution: Apply the x component of momentum, using CV and coordinates shown.

Basic equation:

$$F_{S_x} + \underbrace{F_{B_x}}_{=0(1)} = \underbrace{\frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall}_{=0(2)} + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$$

Assumptions: (1) $F_{Bx} = 0$
(2) Steady flow
(3) Uniform flow at each section
(4) Use gage pressure to cancel p_{atm}

From continuity

$$Q = V_1 A_1 = V_2 A_2 = V_2 \pi R t = \pi \times 10\frac{m}{sec} \times 0.05\,m \times 0.0015\,m = 0.00236\ m^3/s$$

$$V_1 = \frac{Q}{A_1} = \frac{4Q}{\pi D_1^2} = \frac{4}{\pi} \times 0.00236\frac{m^3}{sec} \times \frac{1}{(0.035)^2 m^2} = 2.45\ m/s$$

From momentum $\left\{ \text{Note } A_1 = \frac{\pi D_1^2}{4} = 0.000962\ m^2 \right\}$

$$R_x + p_{1g} A_1 = u_1\{-\rho Q\} + \int_{A_2} u_2 \rho V_2\, dA_2$$

$$u_1 = V_1 \qquad u_2 = V_2 \cos\theta;\ dA_2 = Rt\,d\theta$$

$$\int_{A_2} = \int_{-\pi/2}^{\pi/2} V_2\cos\theta\, \rho V_2 R t\, d\theta = 2\rho V_2^2 R t \int_0^{\pi/2} \cos\theta\, d\theta = 2\rho V_2^2 R t$$

Thus

$$R_x = -p_{1g} A_1 - V_1 \rho Q + 2\rho V_2^2 R t$$

$$= -(150-101)10^3 \frac{N}{m^2} \times 0.000962\,m^2 - 2.45\frac{m}{sec} \times 999\frac{kg}{m^3} \times 0.00236\frac{m^3}{sec} \times \frac{N\cdot sec^2}{kg\cdot m}$$

$$+ 2 \times 999\frac{kg}{m^3} \times (10)^2 \frac{m^2}{sec^2} \times 0.05\,m \times 0.0015\,m \times \frac{N\cdot sec^2}{kg\cdot m}$$

$$R_x = -37.9\ N$$

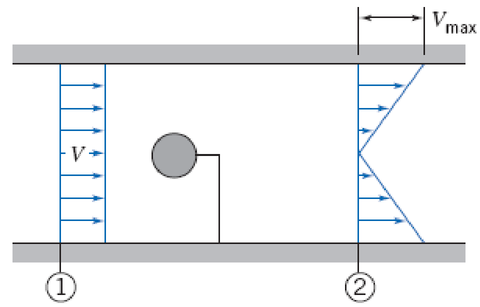
But R_x is force on CV; force on coupling is K_x,

$$K_x = -R_x = 37.9\ N \text{ (to right)}$$ K_x

Problem 4.99

[Difficulty: 4]

4.99 A small round object is tested in a 0.75-m diameter wind tunnel. The pressure is uniform across sections ① and ②. The upstream pressure is 30 mm H_2O (gage), the downstream pressure is 15 mm H_2O (gage), and the mean air speed is 12.5 m/s. The velocity profile at section ② is linear; it varies from zero at the tunnel centerline to a maximum at the tunnel wall. Calculate (a) the mass flow rate in the wind tunnel, (b) the maximum velocity at section ②, and (c) the drag of the object and its supporting vane. Neglect viscous resistance at the tunnel wall.



Given: Data on flow in wind tunnel

Find: Mass flow rate in tunnel; Maximum velocity at section 2; Drag on object

Solution: Basic equations: Continuity, and momentum flux in x direction; ideal gas equation

$$\frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A} = 0 \qquad F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\, d\forall + \int_{CS} u\,\rho\vec{V}\cdot d\vec{A} \qquad p = \rho\cdot R\cdot T$$

Assumptions: 1) Steady flow 2) Uniform density at each section

From continuity $\quad m_{flow} = \rho_1\cdot V_1\cdot A_1 = \rho_1\cdot V_1\cdot\frac{\pi\cdot D_1^{\,2}}{4}$ $\quad$ where m_{flow} is the mass flow rate

We take ambient conditions for the air density $\quad \rho_{air} = \frac{p_{atm}}{R_{air}\cdot T_{atm}} \qquad \rho_{air} = 101000\cdot\frac{N}{m^2}\times\frac{kg\cdot K}{286.9\cdot N\cdot m}\times\frac{1}{293\cdot K} \qquad \rho_{air} = 1.2\frac{kg}{m^3}$

$$m_{flow} = 1.2\cdot\frac{kg}{m^3}\times 12.5\cdot\frac{m}{s}\times\frac{\pi\cdot(0.75\cdot m)^2}{4} \qquad m_{flow} = 6.63\frac{kg}{s}$$

Also

$$m_{flow} = \int \rho_2\cdot u_2\, dA_2 = \rho_{air}\cdot\int_0^R V_{max}\cdot\frac{r}{R}\cdot 2\cdot\pi\cdot r\, dr = \frac{2\cdot\pi\cdot\rho_{air}\cdot V_{max}}{R}\cdot\int_0^R r^2\, dr = \frac{2\cdot\pi\cdot\rho_{air}\cdot V_{max}\cdot R^2}{3}$$

$$V_{max} = \frac{3\cdot m_{flow}}{2\cdot\pi\cdot\rho_{air}\cdot R^2} \qquad V_{max} = \frac{3}{2\cdot\pi}\times 6.63\cdot\frac{kg}{s}\times\frac{m^3}{1.2\cdot kg}\times\left(\frac{1}{0.375\cdot m}\right)^2 \qquad V_{max} = 18.8\frac{m}{s}$$

For x momentum $\quad R_x + p_1\cdot A - p_2\cdot A = V_1\cdot\left(-\rho_1\cdot V_1\cdot A\right) + \int \rho_2\cdot u_2\cdot u_2\, dA_2$

$$R_x = \left(p_2 - p_1\right)\cdot A - V_1\cdot m_{flow} + \int_0^R \rho_{air}\cdot\left(V_{max}\cdot\frac{r}{R}\right)^2\cdot 2\cdot\pi\cdot r\, dr = \left(p_2 - p_1\right)\cdot A - V_1\cdot m_{flow} + \frac{2\cdot\pi\cdot\rho_{air}\cdot V_{max}^{\,2}}{R^2}\cdot\int_0^R r^3\, dr$$

$$R_x = \left(p_2 - p_1\right)\cdot A - V_1\cdot m_{flow} + \frac{\pi}{2}\cdot\rho_{air}\cdot V_{max}^{\,2}\cdot R^2$$

We also have $\quad p_1 = \rho\cdot g\cdot h_1 \qquad p_1 = 1000\cdot\frac{kg}{m^3}\times 9.81\cdot\frac{m}{s^2}\times 0.03\cdot m \qquad p_1 = 294\,Pa \qquad p_2 = \rho\cdot g\cdot h_2 \qquad p_2 = 147\cdot Pa$

Hence $\quad R_x = (147 - 294)\cdot\frac{N}{m^2}\times\frac{\pi\cdot(0.75\cdot m)^2}{4} + \left[-6.63\cdot\frac{kg}{s}\times 12.5\cdot\frac{m}{s} + \frac{\pi}{2}\times 1.2\cdot\frac{kg}{m^3}\times\left(18.8\cdot\frac{m}{s}\right)^2\times(0.375\cdot m)^2\right]\times$

$R_x = -54\,N$ $\quad$ The drag on the object is equal and opposite $\quad F_{drag} = -R_x \qquad F_{drag} = 54.1\,N$

Problem 4.100

[Difficulty: 2]

4.100 The horizontal velocity in the wake behind an object in an air stream of velocity U is given by

$$u(r) = U\left[1 - \cos^2\left(\frac{\pi r}{2}\right)\right] \qquad |r| \leq 1$$

$$u(r) = U \qquad |r| > 1$$

where r is the nondimensional radial coordinate, measured perpendicular to the flow. Find an expression for the drag on the object.

Given: Data on wake behind object

Find: An expression for the drag

Solution:

Basic equation:

Momentum $$F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\,dV + \int_{CS} u\,\rho\vec{V}\cdot d\vec{A}$$

Applying this to the horizontal motion

$$-F = U\cdot\left(-\rho\cdot\pi\cdot 1^2\cdot U\right) + \int_0^1 u(r)\cdot\rho\cdot 2\cdot\pi\cdot r\cdot u(r)\,dr \qquad F = \pi\,\rho\cdot\left(U^2 - 2\cdot\int_0^1 r\cdot u(r)^2\,dr\right)$$

$$F = \pi\,\rho\cdot U^2\cdot\left[1 - 2\cdot\int_0^1 r\cdot\left(1 - \cos\left(\frac{\pi\cdot r}{2}\right)^2\right)^2 dr\right]$$

$$F = \pi\,\rho\cdot U^2\cdot\left(1 - 2\cdot\int_0^1 r - 2\cdot r\cdot\cos\left(\frac{\pi\cdot r}{2}\right)^2 + r\cdot\cos\left(\frac{\pi\cdot r}{2}\right)^4 dr\right)$$

Integrating and using the limits $$F = \pi\,\rho\cdot U^2\cdot\left[1 - \left(\frac{3}{8} + \frac{2}{\pi^2}\right)\right] \qquad F = \left(\frac{5\cdot\pi}{8} - \frac{2}{\pi}\right)\cdot\rho\cdot U^2$$

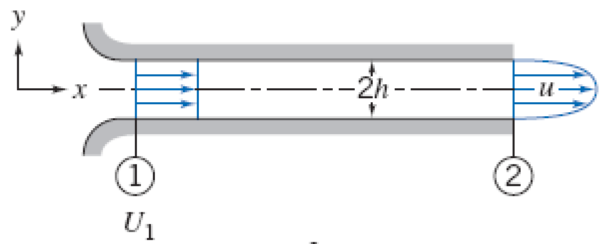
Problem 4.101

[Difficulty: 3]

4.101 An incompressible fluid flows steadily in the entrance region of a two-dimensional channel of height $2h = 100$ mm and width $w = 25$ mm. The flow rate is $Q = 0.025\ \mathrm{m^3/s}$. Find the uniform velocity U_1 at the entrance. The velocity distribution at a section downstream is

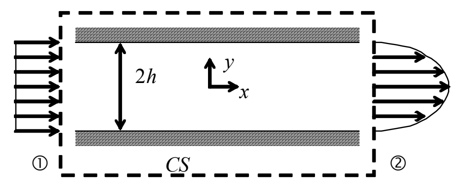
$$\frac{u}{u_{max}} = 1 - \left(\frac{y}{h}\right)^2$$

Evaluate the maximum velocity at the downstream section. Calculate the pressure drop that would exist in the channel if viscous friction at the walls could be neglected.



Given: Data on flow in 2D channel

Find: Maximum velocity; Pressure drop



Solution:

Basic equations: Continuity, and momentum flux in x direction

$$\frac{\partial}{\partial t}\int_{CV}\rho\, dV + \int_{CS}\rho \vec{V}\cdot d\vec{A} = 0 \qquad F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\, dV + \int_{CS} u\,\rho \vec{V}\cdot d\vec{A}$$

Assumptions: 1) Steady flow 2) Neglect friction

Given data $\quad w = 25\cdot mm \qquad h = 50\cdot mm \qquad Q = 0.025\cdot\frac{m^3}{s} \qquad \rho = 750\cdot\frac{kg}{m^3}$

From continuity $\quad Q = U_1\cdot 2\cdot h\cdot w \qquad U_1 = \frac{Q}{2\cdot w\cdot h} \qquad U_1 = 10.0\frac{m}{s}$

Also $\quad -\rho\cdot U_1\cdot A_1 + \int \rho\cdot u_2\, dA = 0$

$$U_1\cdot 2\cdot h\cdot w = w\cdot\int_{-h}^{h} u_{max}\cdot\left(1 - \frac{y^2}{h^2}\right) dy = w\cdot u_{max}\cdot\left[[h - (-h)] - \left[\frac{h}{3} - \left(-\frac{h}{3}\right)\right]\right] = w\cdot u_{max}\cdot\frac{4}{3}\cdot h$$

Hence $\quad u_{max} = \frac{3}{2}\cdot U_1 \qquad u_{max} = 15\frac{m}{s}$

For x momentum $\quad p_1\cdot A - p_2\cdot A = V_1\cdot\left(-\rho_1\cdot V_1\cdot A\right) + \int \rho_2\cdot u_2\cdot u_2\, dA_2 \qquad$ Note that there is no R_x (no friction)

$$p_1 - p_2 = -\rho\cdot U_1^2 + \frac{w}{A}\cdot\int_{-h}^{h}\rho\cdot u_{max}^2\cdot\left(1 - \frac{y^2}{h^2}\right)^2 dy = -\rho\cdot U_1^2 + \frac{\rho\cdot u_{max}^2}{h}\cdot\left[2\cdot h - 2\cdot\left(\frac{2}{3}\cdot h\right) + 2\cdot\left(\frac{1}{5}\cdot h\right)\right]$$

$$\Delta p = p_1 - p_2 = -\rho\cdot U_1^2 + \frac{8}{15}\cdot\rho\cdot u_{max}^2 = \rho\cdot U_1\cdot\left[\frac{8}{15}\cdot\left(\frac{3}{2}\right)^2 - 1\right]$$

Hence $\quad \Delta p = \frac{1}{5}\cdot\rho\cdot U_1^2 \qquad \Delta p = 15.0\cdot kPa$

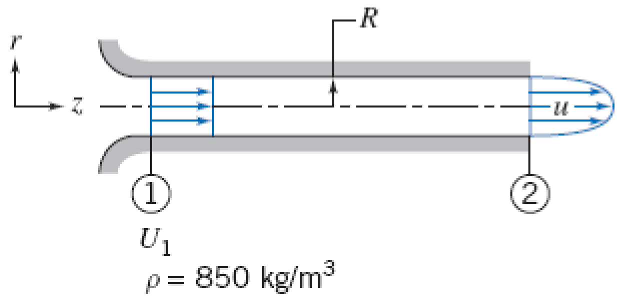
Problem 4.102

[Difficulty: 3]

4.102 An incompressible fluid flows steadily in the entrance region of a circular tube of radius $R = 75$ mm. The flow rate is $Q = 0.1\ \text{m}^3/\text{s}$. Find the uniform velocity U_1 at the entrance. The velocity distribution at a section downstream is

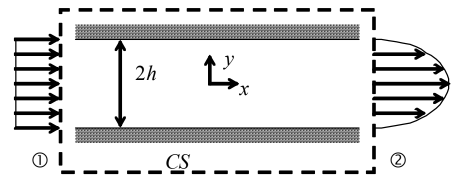
$$\frac{u}{u_{max}} = 1 - \left(\frac{r}{R}\right)^2$$

Evaluate the maximum velocity at the downstream section. Calculate the pressure drop that would exist in the channel if viscous friction at the walls could be neglected.



Given: Data on flow in 2D channel

Find: Maximum velocity; Pressure drop



Solution:

Basic equations: Continuity, and momentum flux in x direction

$$\frac{\partial}{\partial t}\int_{CV}\rho\, dV + \int_{CS}\rho\vec{V}\cdot d\vec{A} = 0 \qquad F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\, dV + \int_{CS} u\,\rho\vec{V}\cdot d\vec{A}$$

Assumptions: 1) Steady flow 2) Neglect friction

Given data $\quad R = 75\cdot\text{mm} \quad Q = 0.1\cdot\frac{\text{m}^3}{\text{s}} \quad \rho = 850\cdot\frac{\text{kg}}{\text{m}^3}$

From continuity $\quad Q = U_1\cdot\pi\cdot R^2 \quad U_1 = \frac{Q}{\pi\cdot R^2} \quad U_1 = 5.66\frac{\text{m}}{\text{s}}$

Also $\quad -\rho\cdot U_1\cdot A_1 + \int \rho\cdot u_2\, dA = 0$

$$U_1\cdot\pi\cdot R^2 = \int_0^R u_{max}\cdot\left(1 - \frac{r^2}{R^2}\right)\cdot 2\cdot\pi\cdot r\, dr = 2\cdot\pi\cdot u_{max}\cdot\left(\frac{R^2}{2} - \frac{R^4}{4\cdot R^2}\right) = 2\cdot\pi\cdot u_{max}\cdot\frac{R^2}{4} = \pi\cdot u_{max}\cdot\frac{R^2}{2}$$

Hence $\quad u_{max} = 2\cdot U_1 \quad u_{max} = 11.3\,\frac{\text{m}}{\text{s}}$

For x momentum $\quad p_1\cdot A - p_2\cdot A = V_1\cdot(-\rho_1\cdot V_1\cdot A) + \int \rho_2\cdot u_2\cdot u_2\, dA_2 \qquad$ Note that there is no R_x (no friction)

$$(p_1 - p_2)\cdot\pi\cdot R^2 = -\rho\cdot\pi\cdot R^2\cdot U_1^{\,2} + \int_0^R \rho\cdot u_{max}^{\,2}\cdot\left(1 - \frac{r^2}{R^2}\right)^2\cdot 2\cdot\pi\cdot r\, dr = -\rho\cdot\pi\cdot R^2\cdot U_1^{\,2} + 2\cdot\pi\cdot\rho\cdot u_{max}^{\,2}\cdot\left(\frac{R^2}{2} - 2\cdot\frac{R^4}{4\cdot R^2} + \frac{R^6}{6\cdot R^4}\right)$$

$$\Delta p = p_1 - p_2 = -\rho\cdot U_1^{\,2} + \frac{1}{3}\cdot\rho\cdot u_{max}^{\,2} = -\rho\cdot U_1^{\,2} + \frac{1}{3}\cdot\rho\cdot(2\cdot U_1)^2 = \rho\cdot U_1\cdot\left[\frac{1}{3}\cdot(2)^2 - 1\right] = \frac{1}{3}\cdot\rho\cdot U_1^{\,2}$$

Hence $\quad \Delta p = \frac{1}{3}\times 850\cdot\frac{\text{kg}}{\text{m}^3}\times\left(5.66\cdot\frac{\text{m}}{\text{s}}\right)^2\times\frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}} \qquad \Delta p = 9.08\cdot\text{kPa}$

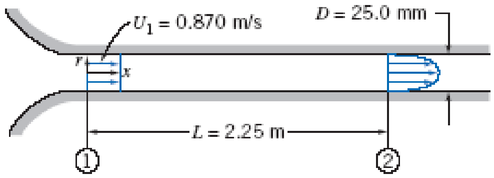
Problem 4.103

[Difficulty: 3]

4.103 Air enters a duct, of diameter $D = 25.0$ mm, through a well-rounded inlet with uniform speed, $U_1 = 0.870$ m/s. At a downstream section where $L = 2.25$ m, the fully developed velocity profile is

$$\frac{u(r)}{U_c} = 1 - \left(\frac{r}{R}\right)^2$$

The pressure drop between these sections is $p_1 - p_2 = 1.92$ N/m^2. Find the total force of friction exerted by the tube on the air.



Solution: Apply continuity and momentum to CV, CS

Basic equations: $$0 = \underbrace{\cancel{\frac{\partial}{\partial t}}}_{=0(1)}\int_{CV} \rho\, dV + \int_{CS} \rho \vec{V}\cdot d\vec{A}$$

$$F_{S_x} + \underbrace{\cancel{F_{B_x}}}_{=0(4)} = \underbrace{\cancel{\frac{\partial}{\partial t}}}_{=0(1)}\int_{CV} u\rho\, dV + \int_{CS} u\rho\vec{V}\cdot d\vec{A}$$

Assumptions: (1) Steady flow (2) Incompressible flow (3) Uniform flow at inlet (4) $F_{B_x} = 0$

Then $$0 = \{-|\rho U_1 A_1|\} + \int_{2} \rho u\, dA = -\rho U_1 \pi R^2 + \int_0^R \rho U_c\left[1-\left(\frac{r}{R}\right)^2\right] 2\pi r\, dr$$

$$0 = -\rho U_1 \pi R^2 + 2\rho\pi R^2 U_c \int_0^1 (1-\lambda^2)\lambda\, d\lambda \quad \text{or} \quad 0 = -U_1 + 2U_c\left[\frac{\lambda^2}{2} - \frac{\lambda^4}{4}\right]_0^1$$

Thus $0 = -U_1 + \frac{1}{2}U_c$ or $U_c = 2U_1$ $\qquad (\lambda = r/R)$

From momentum $$R_x + p_1 A_1 - p_2 A_2 = u_1\{-|\rho U_1 A_1|\} + \int_{2} u_2\{+\rho u_2\, dA_2\}$$

$$u_1 = U_1 \qquad u_2 = U_c\left[1-\left(\frac{r}{R}\right)^2\right]$$

So $$\int_{2} = \int_0^R U_c\left[1-\left(\frac{r}{R}\right)^2\right]\rho U_c\left[1-\left(\frac{r}{R}\right)^2\right] 2\pi r\, dr = 2\pi\rho U_c^2 R^2 \int_0^1 (1-\lambda^2)(1-\lambda^2)\lambda\, d\lambda$$

$$= 2\pi\rho U_c^2 R^2 \int_0^1 (1-2\lambda^2+\lambda^4)\lambda\, d\lambda = 2\pi\rho U_c^2 R^2\left[\frac{\lambda^2}{2} - \frac{\lambda^4}{2} + \frac{\lambda^6}{6}\right]_0^1 = \frac{1}{3}\pi\rho U_c^2 R^2$$

Substituting,

$$R_x + (p_1 - p_2)\pi R^2 = -\pi\rho U_1^2 R^2 + \frac{1}{3}\pi\rho U_c^2 R^2 = -\pi\rho U_1^2 R^2 + \frac{1}{3}\pi\rho(2U_1)^2 R^2$$

$$R_x = -(p_1 - p_2)\frac{\pi D^2}{4} + \frac{1}{3}\rho U_1^2 \frac{\pi D^2}{4}$$

$$= -1.92\frac{\text{N}}{\text{m}^2} \times \frac{\pi}{4}(0.025)^2\text{m}^2 + \frac{1}{3} \times 1.23\frac{\text{kg}}{\text{m}^3} \times (0.870)^2\frac{\text{m}^2}{\text{s}^2} \times \frac{\pi}{4}(0.025)^2\text{m}^2 \times \frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}}$$

$R_x = -7.90 \times 10^{-4}$ N (to left on CV, since < 0) $\qquad R_x$

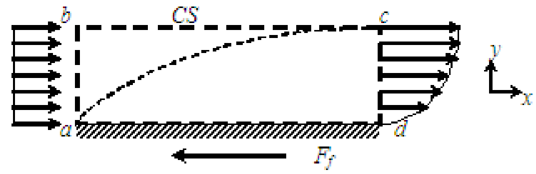
Problem 4.104

[Difficulty: 3]

4.104 Consider the incompressible flow of fluid in a boundary layer as depicted in Example 4.2. Show that the friction drag force of the fluid on the surface is given by

$$F_f = \int_0^\delta \rho u(U-u)w\,dy$$

Evaluate the drag force for the conditions of Example 4.2.



Solution: Apply continuity and x component of momentum using CV

Basic equations: $0 = \cancelto{=0(1)}{\frac{\partial}{\partial t}}\int_{CV}\rho\, d\forall + \int_{CS}\rho\vec{V}\cdot d\vec{A}$

$$F_{S_x} + \cancelto{=0(3)}{F_{B_x}} = \cancelto{=0(1)}{\frac{\partial}{\partial t}}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho\vec{V}\cdot d\vec{A}$$

Assumptions: (1) Steady flow
(2) No net pressure force; $F_{S_x} = -F_f$
(3) $F_{B_x} = 0$
(4) Uniform flow at section ab
(5) Incompressible flow

Then from continuity

$$0 = \{-|\rho U w\delta|\} + \{|\int_0^\delta \rho u w\,dy|\} + \dot{m}_{BC}\ ;\ \delta = \int_0^\delta dy\ ;\ \dot{m}_{BC} = \rho\int_0^\delta (U-u)w\,dy$$

From momentum

$$-F_f = U\{-|\rho U w\delta|\} + \{\int_0^\delta \rho u^2 w\,dy|\} + U\dot{m}_{BC} = \rho\int_0^\delta[-U^2+u^2+U(U-u)]w\,dy$$

$$\text{Drag} = F_f = \int_0^\delta \rho u(U-u)w\,dy$$

Drag

At CD, $\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2 = 2\eta - \eta^2$; $dy = \delta\, d\left(\frac{y}{\delta}\right) = \delta\, d\eta$

$$\text{Drag} = \int_0^\delta \rho U\left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right]\left(U - U\left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right]\right)w\,dy = \rho U^2 w\delta\int_0^1 (2\eta-\eta^2)(1-2\eta+\eta^2)\,d\eta$$

$$= \rho U^2 w\delta\int_0^1(2\eta - 5\eta^2 + 4\eta^3 - \eta^4)\,d\eta = \rho U^2 w\delta\left[\eta^2 - \frac{5}{3}\eta^3 + \eta^4 - \frac{1}{5}\eta^5\right]_0^1$$

$$= \frac{2}{15}\rho U^2 w\delta$$

$$\text{Drag} = \frac{2}{15}\times 1.24\,\frac{kg}{m^3}\times(30)^2\frac{m^2}{s^2}\times 0.6\,m\times 0.005\,m\times\frac{N\cdot s^2}{kg\cdot m} = 0.446\,N$$

Drag

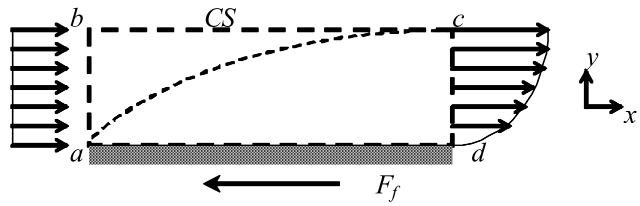
Problem 4.105

[Difficulty: 4]

4.105 A fluid with density $\rho = 750\ \mathrm{kg/m^3}$ flows along a flat plate of width 1 m. The undisturbed freestream speed is $U_0 = 10\ \mathrm{m/s}$. At $L = 1$ m downstream from the leading edge of the plate, the boundary-layer thickness is $\delta = 5$ mm. The velocity profile at this location is

$$\frac{u}{U_0} = \frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$$

Plot the velocity profile. Calculate the horizontal component of force required to hold the plate stationary.



Given: Data on flow of boundary layer

Find: Plot of velocity profile; force to hold plate

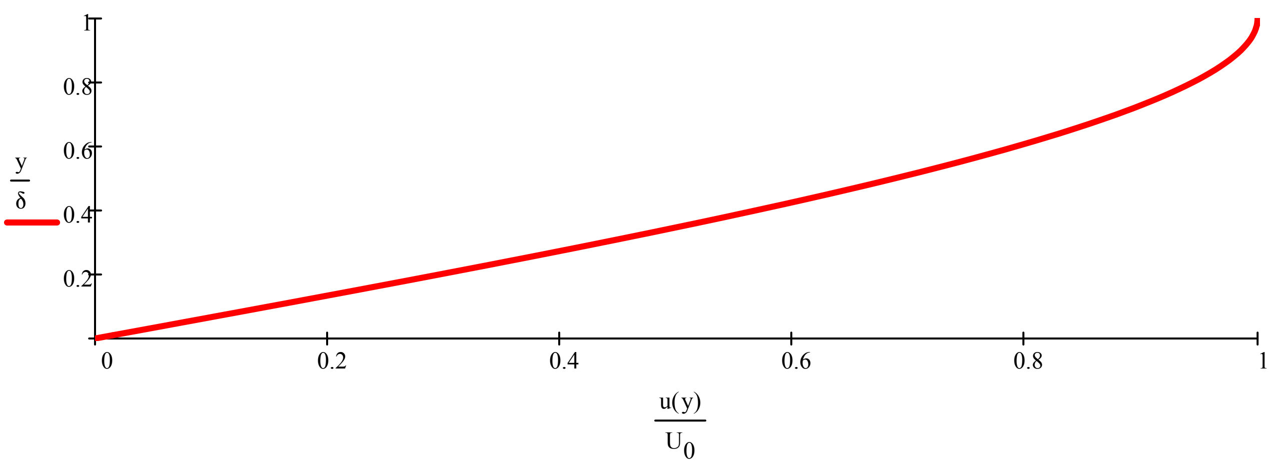
Solution:

Basic equations: Continuity, and momentum flux in x direction

$$\frac{\partial}{\partial t}\int_{CV}\rho\, d\mathcal{V} + \int_{CS}\rho\vec{V}\cdot d\vec{A} = 0 \qquad F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\, d\mathcal{V} + \int_{CS} u\,\rho\vec{V}\cdot d\vec{A}$$

Assumptions: 1) Steady flow 2) Incompressible 3) No net pressure force

Given data $\quad \rho = 750\cdot\frac{\mathrm{kg}}{\mathrm{m}^3} \quad w = 1\cdot\mathrm{m} \quad U_0 = 10\cdot\frac{\mathrm{m}}{\mathrm{s}} \quad L = 1\cdot\mathrm{m} \quad \delta = 5\cdot\mathrm{mm}$



From continuity $\quad -\rho\cdot U_0\cdot w\cdot\delta + m_{bc} + \int_0^{\delta}\rho\cdot u\cdot w\, dy = 0 \quad$ where m_{bc} is the mass flow rate across bc (Note: sotware cannot render a dot!)

Hence $\quad m_{bc} = \int_0^{\delta}\rho\cdot\left(U_0 - u\right)\cdot w\, dy$

For x momentum $\quad -F_f = U_0\cdot\left(-\rho\cdot U_0\cdot w\cdot\delta\right) + U_0\cdot m_{bc} + \int_0^{\delta} u\cdot\rho\cdot u\cdot w\, dy = \int_0^{\delta}\left[-U_0^{\,2} + u^2 + U_0\cdot\left(U_0 - u\right)\right]\cdot w\, dy$

Then the drag force is $\quad F_f = \int_0^{\delta}\rho\cdot u\cdot\left(U_0 - u\right)\cdot w\, dy = \int_0^{\delta}\rho\cdot U_0^{\,2}\cdot\frac{u}{U_0}\cdot\left(1 - \frac{u}{U_0}\right) dy$

But we have $\frac{u}{U_0} = \frac{3}{2}\cdot\eta - \frac{1}{2}\cdot\eta^3$ where we have used substitution $y = \delta\cdot\eta$

$$\frac{F_f}{w} = \int_0^{\eta=1} \rho\cdot U_0^{\,2}\cdot\delta\cdot\frac{u}{U_0}\cdot\left(1 - \frac{u}{U_0}\right) d\eta = \rho\cdot U_0^{\,2}\cdot\delta\cdot\int_0^1 \left(\frac{3}{2}\cdot\eta - \frac{9}{4}\cdot\eta^2 - \frac{1}{2}\cdot\eta^3 + \frac{3}{2}\cdot\eta^4 - \frac{1}{4}\cdot\eta^6\right) d\eta$$

$$\frac{F_f}{w} = \rho\cdot U_0^{\,2}\cdot\delta\cdot\left(\frac{3}{4} - \frac{3}{4} - \frac{1}{8} + \frac{3}{10} - \frac{1}{28}\right) = 0.139\cdot\rho\cdot U_0^{\,2}\cdot\delta$$

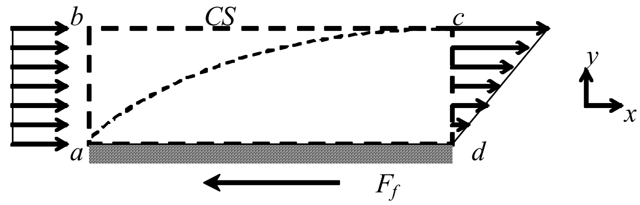
Hence $$\frac{F_f}{w} = 0.139 \times 750\cdot\frac{kg}{m^3} \times \left(10\cdot\frac{m}{s}\right)^2 \times 0.05\cdot m \times \frac{N\cdot s^2}{kg\cdot m}$$

$$\frac{F_f}{w} = 52.1\,\frac{N}{m}$$

Problem 4.106

[Difficulty: 4]

4.106 Air at standard conditions flows along a flat plate. The undisturbed freestream speed is $U_0 = 20$ m/s. At $L = 0.4$ m downstream from the leading edge of the plate, the boundary-layer thickness is $\delta = 2$ mm. The velocity profile at this location is approximated as $u/U_0 = y/\delta$. Calculate the horizontal component of force per unit width required to hold the plate stationary.



Given: Data on flow of boundary layer

Find: Force on plate per unit width

Solution:

Basic equations: Continuity, and momentum flux in x direction

$$\frac{\partial}{\partial t}\int_{CV}\rho\, d\forall + \int_{CS}\rho\vec{V}\cdot d\vec{A} = 0 \qquad F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\, d\forall + \int_{CS} u\,\rho\vec{V}\cdot d\vec{A}$$

Assumptions: 1) Steady flow 2) Incompressible 3) No net pressure force

From continuity $\quad -\rho\cdot U_0\cdot w\cdot\delta + m_{bc} + \int_0^\delta \rho\cdot u\cdot w\, dy = 0 \quad$ where m_{bc} is the mass flow rate across bc (Note: sotware cannot render a dot!)

Hence $\quad m_{bc} = \int_0^\delta \rho\cdot\left(U_0 - u\right)\cdot w\, dy$

For x momentum $\quad -F_f = U_0\cdot\left(-\rho\cdot U_0\cdot w\cdot\delta\right) + U_0\cdot m_{bc} + \int_0^\delta u\cdot\rho\cdot u\cdot w\, dy = \int_0^\delta \left[-U_0^{\,2} + u^2 + U_0\cdot\left(U_0 - u\right)\right]\cdot w\, dy$

Then the drag force is $\quad F_f = \int_0^\delta \rho\cdot u\cdot\left(U_0 - u\right)\cdot w\, dy = \int_0^\delta \rho\cdot U_0^{\,2}\cdot\frac{u}{U_0}\cdot\left(1 - \frac{u}{U_0}\right) dy$

But we have $\quad \frac{u}{U_0} = \frac{y}{\delta} \quad$ where we have used substitution $\quad y = \delta\cdot\eta$

$$\frac{F_f}{w} = \int_0^{\eta=1} \rho\cdot U_0^{\,2}\cdot\delta\cdot\frac{u}{U_0}\cdot\left(1 - \frac{u}{U_0}\right) d\eta = \rho\cdot U_0^{\,2}\cdot\delta\cdot\int_0^1 \eta\cdot(1-\eta)\, d\eta$$

$$\frac{F_f}{w} = \rho\cdot U_0^{\,2}\cdot\delta\cdot\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{6}\cdot\rho\cdot U_0^{\,2}\cdot\delta$$

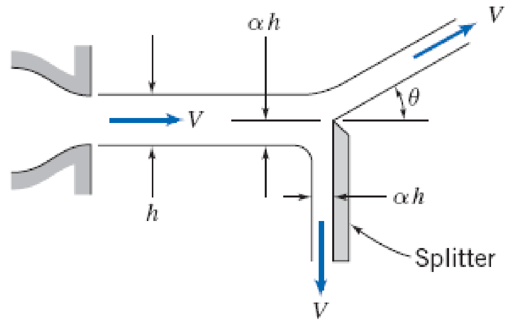
Hence $\quad \frac{F_f}{w} = \frac{1}{6}\times 1.225\cdot\frac{kg}{m^3}\times\left(20\cdot\frac{m}{s}\right)^2\times\frac{2}{1000}\cdot m\times\frac{N\cdot s^2}{kg\cdot m} \quad$ (using standard atmosphere density)

$$\frac{F_f}{w} = 0.163\cdot\frac{N}{m}$$

Problem 4.107

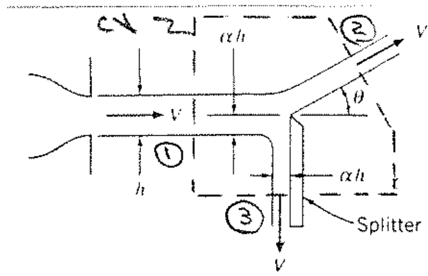
Difficulty: 4] Part 1/2

4.107 A sharp-edged splitter plate inserted part way into a flat stream of flowing water produces the flow pattern shown. Analyze the situation to evaluate θ as a function of α, where $0 \le \alpha < 0.5$. Evaluate the force needed to hold the splitter plate in place. (Neglect any friction force between the water stream and the splitter plate.) Plot both θ and R_x as functions of α.



Solution

Apply the x and y components of the momentum equation to the CV shown.



Basic equations:

$$F_{S_x} + \underbrace{\cancel{F_{B_x}}}_{=0(3)} = \underbrace{\cancel{\frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall}}_{=0(4)} + \int_{CS} u(\rho\vec{V}\cdot d\vec{A})$$

$$\underbrace{\cancel{F_{S_y}}}_{=0(2)} + \underbrace{\cancel{F_{B_y}}}_{=0(3)} = \underbrace{\cancel{\frac{\partial}{\partial t}\int_{CV} v\rho\, d\forall}}_{=0(4)} + \int_{CS} v(\rho\vec{V}\cdot d\vec{A})$$

Assumptions:
(1) no net pressure forces on CV
(2) no friction in y direction, so $F_{S_y} = 0$
(3) neglect body forces
(4) steady flow
(5) no change in jet speed: $V_1 = V_2 = V_3 = V$
(6) uniform flow at each section

Then from the y equation

$$0 = v_1\{-|\rho V_1 A_1|\} + v_2\{|\rho V_2 A_2|\} + v_3\{|\rho V_3 A_3|\}$$

$v_1 = 0$ $\quad$ $v_2 = V\sin\theta$ $\quad$ $v_3 = -V$

{w is depth} $\quad$ $A_1 = wh$ $\quad$ $A_2 = w(1-\alpha)h$ $\quad$ $A_3 = w\alpha h$

$$0 = 0 + \rho V^2 \sin\theta\, w(1-\alpha)h - \rho V^2 w\alpha h$$

Thus

$$\sin\theta = \frac{\rho V^2 w\alpha h}{\rho V^2 w(1-\alpha)h} = \frac{\alpha}{(1-\alpha)}\ ; \quad \theta = \sin^{-1}\left(\frac{\alpha}{1-\alpha}\right)$$

$\theta(\alpha)$

From the x equation

$$R_x = u_1\{-|\rho_1 V_1 A_1|\} + u_2\{|\rho_2 V_2 A_2|\} + u_3\{|\rho_3 V_3 A_3|\}$$

$u_1 = V$ $\quad$ $u_2 = V\cos\theta$ $\quad$ $u_3 = 0$

$$R_x = -\rho V^2 wh + \rho V^2\cos\theta\, w(1-\alpha)h = \rho V^2 wh\left[\cos\theta(1-\alpha) - 1\right]$$

$$\text{But } \cos\theta = (1-\sin^2\theta)^{1/2} = \left(1 - \frac{\alpha^2}{(1-\alpha)^2}\right)^{1/2} = \frac{(1-2\alpha)^{1/2}}{(1-\alpha)}$$

$$\therefore R_x = -\rho V^2 wh\left[1 - (1-2\alpha)^{1/2}\right]$$

($R_x < 0$; so to left) $\quad$ R_x

{ Check: $\alpha = 0$, $R_x = 0$ ✓ ; $\alpha = \frac{1}{2}$, $R_x = -\rho V^2 wh$ ✓ }

Plots of: $\theta = \sin^{-1}\left(\frac{\alpha}{1-\alpha}\right)$ and

$$\frac{R_x}{R_{x\,\alpha=0.5}} = 1 - \sqrt{1-2\alpha}$$

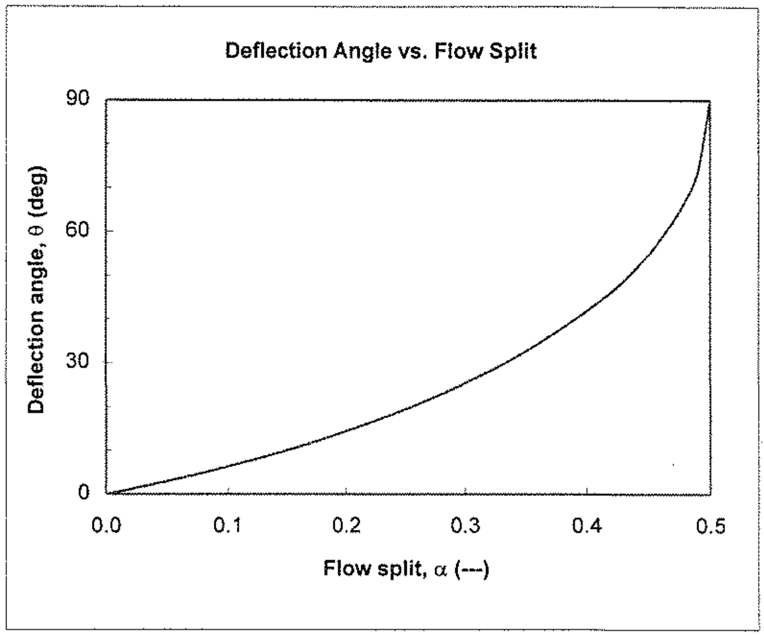
are presented below

Flow deflection by sharp-edged splitter:

α = fraction of jet intercepted by splitter

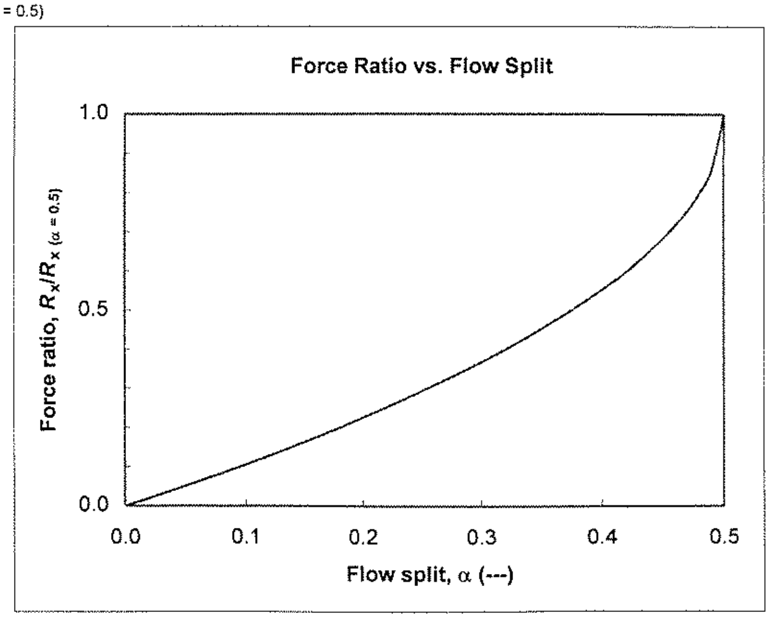
Calculated Results: Deflection angle

α (---)	θ (deg)
0	0
0.05	3.02
0.10	6.38
0.15	10.2
0.20	14.5
0.25	19.5
0.30	25.4
0.35	32.6
0.40	41.8
0.425	47.7
0.45	54.9
0.470	62.5
0.480	67.4
0.490	73.9
0.50	90.0



Calculated Results: Force over maximum force

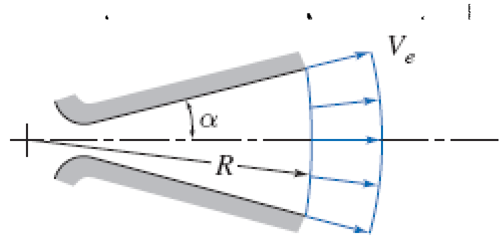
α (---)	$R_x/R_{x\,(\alpha=0.5)}$
0	0
0.05	0.0513
0.10	0.106
0.15	0.163
0.20	0.225
0.25	0.293
0.30	0.368
0.35	0.452
0.40	0.553
0.425	0.613
0.45	0.684
0.470	0.755
0.480	0.800
0.490	0.859
0.50	1.00



Problem 4.108

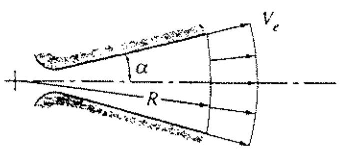
[Difficulty: 4]

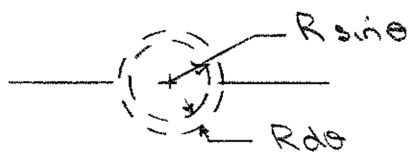
4.108 Gases leaving the propulsion nozzle of a rocket are modeled as flowing radially outward from a point upstream from the nozzle throat. Assume the speed of the exit flow, V_e, has constant magnitude. Develop an expression for the axial thrust, T_a, developed by flow leaving the nozzle exit plane. Compare your result to the one-dimensional approximation, $T = \dot{m}V_e$. Evaluate the percent error for $\alpha = 15°$. Plot the percent error versus α for $0 \le \alpha \le 22.5°$.



Solution:

Apply definitions $\dot{m} = \int_A \rho V dA$, $T_a = \int_A u \rho V dA$. Use spherically symmetric flow.





The mass flow rate is [assuming $\rho_e \neq \rho_e(\theta)$]

$$\dot{m} = \int_A \rho V dA = \int_0^\alpha \rho_e V_e (2\pi R \sin\theta) R d\theta = 2\pi \rho_e V_e R^2 \left[-\cos\theta\right]_0^\alpha = 2\pi \rho_e V_e R^2 (1 - \cos\alpha)$$

The one-dimensional approximation for thrust is then

$$T = \dot{m} V_e = 2\pi \rho_e V_e^2 R^2 (1 - \cos\alpha) \qquad T_{1\text{-}D}$$

The axial thrust is given by

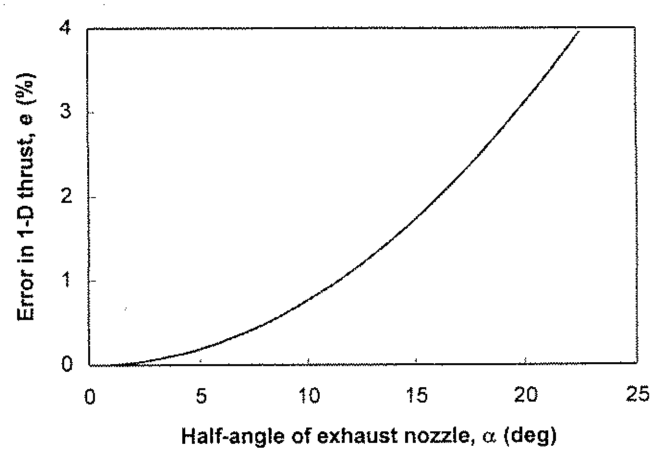
$$T_a = \int u \rho V dA = \int_0^\alpha V_e \cos\theta \, \rho_e V_e (2\pi R \sin\theta) R d\theta = 2\pi \rho_e V_e^2 R^2 \int_0^\alpha \sin\theta \cos\theta \, d\theta$$

$$T_a = 2\pi \rho_e V_e^2 R^2 \left[\frac{\sin^2\theta}{2}\right]_0^\alpha = \pi \rho_e V_e^2 R^2 \sin^2\alpha \qquad T_a$$

The error in the one-dimensional approximation is

$$e = \frac{T_{1\text{-}D} - T_a}{T_a} = \frac{T_{1\text{-}D}}{T_a} - 1 = \frac{2\pi \rho_e V_e^2 R^2 (1 - \cos\alpha)}{\pi \rho_e V_e^2 R^2 \sin^2\alpha} - 1 = \frac{2(1 - \cos\alpha)}{\sin^2\alpha} - 1 \quad \ldots (1)$$

The percent error is plotted as a function of α



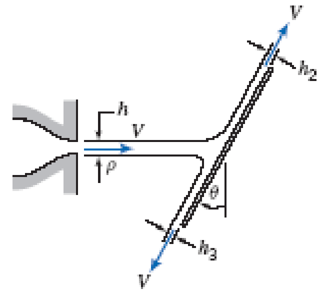
For $\alpha = 15°$

$$e_{15} = \frac{2(1 - \cos 15)}{\sin^2 15°} - 1$$

$$e_{15} = 0.0173 \text{ or } 1.73\% \qquad e_{15}$$

Problem 4.109

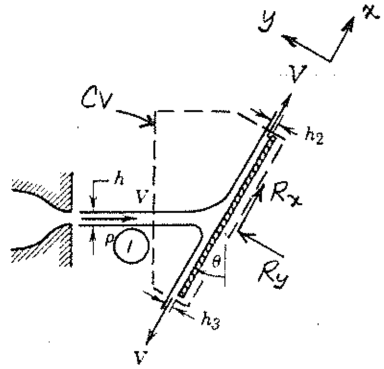
[Difficulty: 4]



4.109 When a plane liquid jet strikes an inclined flat plate, it splits into two streams of equal speed but unequal thickness. For frictionless flow there can be no tangential force on the plate surface. Use this assumption to develop an expression for h_2/h as a function of plate angle, θ. Plot your results and comment on the limiting cases, $\theta = 0$ and $\theta = 90°$.

Solution: Apply the x component of the momentum equation using the CV and coordinates shown.

Basic equation:



$$\underbrace{F_{S_x}}_{=0(1)} + \underbrace{F_{B_x}}_{=0(2)} = \underbrace{\frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall}_{=0(3)} + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$$

Assumptions: (1) No surface force on CV
(2) Neglect body forces
(3) Steady flow
(4) No change in jet speed: $V_1 = V_2 = V_3 = V$
(5) Uniform flow at each section

From continuity for uniform incompressible flow $0 = -\rho V w h + \rho V w h_2 + \rho V w h_3$

or

$$h = h_2 + h_3 = h_1 \quad \text{or} \quad h_3 = h_1 - h_2$$

From momentum

$$0 = u_1\{-|\rho V w h_1|\} + u_2\{+|\rho V w h_2|\} + u_3\{+|\rho V w h_3|\}$$

$$u_1 = V\sin\theta \qquad u_2 = V \qquad u_3 = -V$$

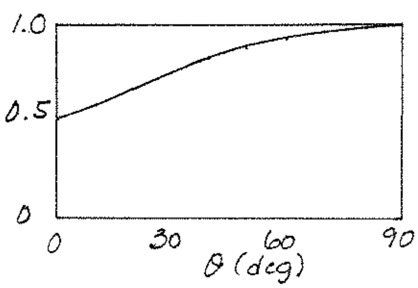
$$0 = -\rho V^2 \sin\theta\, w h_1 + \rho V^2 w h_2 - \rho V^2 w h_3$$

Substituting from continuity and simplifying

$$0 = -\sin\theta\, h_1 + h_2 - (h_1 - h_2) \quad \text{so} \quad \frac{h_2}{h} = \frac{h_2}{h_1} = \frac{1+\sin\theta}{2}$$

$\frac{h_2}{h}$

Plot:



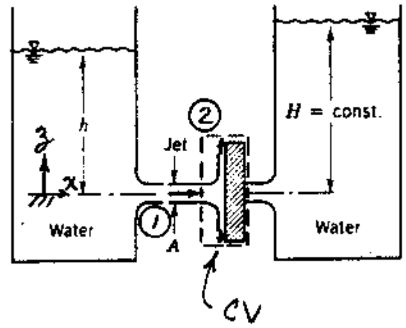
At $\theta = 0$, $\frac{h_2}{h} = 0.5$; flow is equally split when plate is $\perp$ to jet.

At $\theta = 90°$, $\frac{h_2}{h} = 1.0$; plate has no effect on flow.

Problem *4.110

[Difficulty: 4]

***4.110** Two large tanks containing water have small smoothly contoured orifices of equal area. A jet of liquid issues from the left tank. Assume the flow is uniform and unaffected by friction. The jet impinges on a vertical flat plate covering the opening of the right tank. Determine the minimum value for the height, h, required to keep the plate in place over the opening of the right tank.



Solution: Apply Bernoulli and momentum equations. Use CV enclosing plate, as shown.

Basic equations: $\dfrac{p}{\rho} + \dfrac{V^2}{2} + gz = \text{constant}$

$$F_{S_x} + \underbrace{F_{B_x}}_{=0(5)} = \underbrace{\frac{\partial}{\partial t}\int_{CV} u \rho \, d\forall}_{=0(1)} + \int_{CS} u \rho \vec{V} \cdot d\vec{A}$$

Assumptions:
(1) Steady flow
(2) Incompressible flow
(3) Flow along a streamline
(4) No friction
(5) $F_{Bx} = 0$

Apply Bernoulli from water surface to jet

$$\cancel{\frac{p}{\rho}} + \underbrace{\cancel{\frac{V_s^2}{2}}}_{\ll V^2} + gh = \cancel{\frac{p}{\rho}} + \frac{V^2}{2} + g(0) \quad \text{so that } V^2 = 2gh \text{ or } V = \sqrt{2gh}$$

From fluid statics, $p_{3g} = \rho g H$

From momentum

$$-p_{3g} A = -\rho g H A = u_1\{-\rho V A\} + u_2\{+\rho V A\} = -\rho V^2 A$$

$$u_1 = V \qquad u_2 = 0$$

Thus, using Bernoulli,

$$\rho g H A = \rho V^2 A = \rho(2gh)A = 2\rho g h A$$

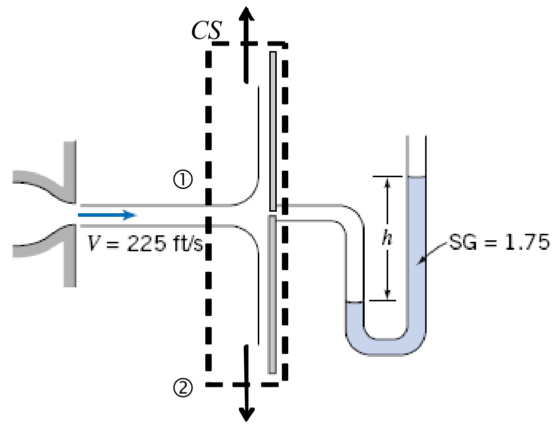
and

$$h = \frac{H}{2}$$ h

Problem *4.111

[Difficulty: 4]

*4.111 A horizontal axisymmetric jet of air with 0.5 in. diameter strikes a stationary vertical disk of 8 in. diameter. The jet speed is 225 ft/s at the nozzle exit. A manometer is connected to the center of the disk. Calculate (a) the deflection, h, if the manometer liquid has SG = 1.75 and (b) the force exerted by the jet on the disk.



Given: Air jet striking disk

Find: Manometer deflection; Force to hold disk

Solution:

Basic equations: Hydrostatic pressure, Bernoulli, and momentum flux in x direction

$$\frac{p}{\rho} + \frac{V^2}{2} + g\cdot z = \text{constant} \qquad F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\, d\forall + \int_{CS} u\,\rho \vec{V}\cdot d\vec{A}$$

Assumptions: 1) Steady flow 2) Incompressible 3) No friction 4) Flow along streamline 5) Uniform flow 6) Horizontal flow ($g_x = 0$)

Applying Bernoulli between jet exit and stagnation point

$$\frac{p}{\rho_{air}} + \frac{V^2}{2} = \frac{p_0}{\rho_{air}} + 0 \qquad p_0 - p = \frac{1}{2}\cdot\rho_{air}\cdot V^2$$

But from hydrostatics $\quad p_0 - p = SG\cdot\rho\cdot g\cdot\Delta h \quad$ so $\quad \Delta h = \dfrac{\frac{1}{2}\cdot\rho_{air}\cdot V^2}{SG\cdot\rho\cdot g} = \dfrac{\rho_{air}\cdot V^2}{2\cdot SG\cdot\rho\cdot g}$

$$\Delta h = 0.002377\cdot\frac{slug}{ft^3}\times\left(225\cdot\frac{ft}{s}\right)^2\times\frac{1}{2\cdot 1.75}\times\frac{ft^3}{1.94\cdot slug}\times\frac{s^2}{32.2\cdot ft} \qquad \Delta h = 0.55\cdot ft \qquad \Delta h = 6.6\cdot in$$

For x momentum $\quad R_x = V\cdot\left(-\rho_{air}\cdot A\cdot V\right) = -\rho_{air}\cdot V^2\cdot\dfrac{\pi\cdot D^2}{4}$

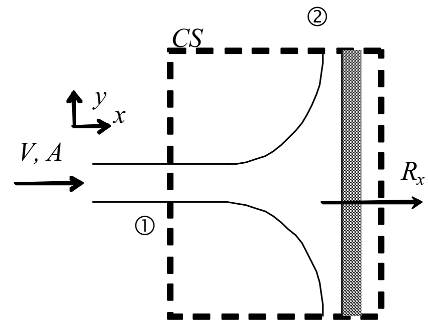
$$R_x = -0.002377\cdot\frac{slug}{ft^3}\times\left(225\cdot\frac{ft}{s}\right)^2\times\frac{\pi\cdot\left(\frac{0.5}{12}\cdot ft\right)^2}{4}\times\frac{lbf\cdot s^2}{slug\cdot ft} \qquad R_x = -0.164\cdot lbf$$

The force of the jet on the plate is then $\quad F = -R_x \qquad F = 0.164\cdot lbf$

Problem *4.112

[Difficulty: 3]

***4.112** Students are playing around with a water hose. When they point it straight up, the water jet just reaches one of the windows of Professor Pritchard's office, 10 m above. If the hose diameter is 1 cm, estimate the water flow rate (L/min). Professor Pritchard happens to come along and places his hand just above the hose to make the jet spray sideways axisymmetrically. Estimate the maximum pressure, and the total force, he feels. The next day the students again are playing around, and this time aim at Professor Fox's window, 15 m above. Find the flow rate (L/min) and the total force and maximum pressure when he, of course, shows up and blocks the flow.



Given: Water jet shooting upwards; striking surface

Find: Flow rate; maximum pressure; Force on hand

Solution:

Basic equations: Bernoulli and momentum flux in x direction

$$\frac{p}{\rho} + \frac{V^2}{2} + g \cdot z = \text{constant} \qquad F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\,dV + \int_{CS} u\,\rho\vec{V}\cdot d\vec{A}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) Atmospheric pressure throughout 4) Uniform flow

Given data $\quad h = 10 \cdot m \quad \rho = 1000 \cdot \frac{kg}{m^3} \quad D = 1 \cdot cm$

Using Bernoulli between the jet exit and its maximum height h $\quad \frac{p_{atm}}{\rho} + \frac{V^2}{2} = \frac{p_{atm}}{\rho} + g \cdot h$

or $\quad V = \sqrt{2 \cdot g \cdot h} \quad V = 14.0\,\frac{m}{s}$

Then $\quad Q = \frac{\pi}{4} \cdot D^2 \cdot V \quad Q = 66.0 \cdot \frac{L}{min}$

For Dr. Pritchard the maximum pressure is obtained from Bernoulli $\quad \frac{p_{atm}}{\rho} + \frac{V^2}{2} = \frac{p_{max}}{\rho} \quad p = \frac{1}{2} \cdot \rho \cdot V^2 \quad p = 98.1 \cdot kPa$ (gage)

For Dr. Pritchard blocking the jet, from x momentum applied to the CV $\quad R_x = u_1 \cdot \left(-\rho \cdot u_1 \cdot A_1\right) = -\rho \cdot V^2 \cdot A$

Hence $\quad F = \rho \cdot V^2 \cdot \frac{\pi}{4} \cdot D^2 \quad F = 15.4\,N$

Repeating for Dr. Fox $\quad h = 15 \cdot m \quad V = \sqrt{2 \cdot g \cdot h} \quad V = 17.2\,\frac{m}{s} \quad Q = \frac{\pi}{4} \cdot D^2 \cdot V \quad Q = 80.8 \cdot \frac{L}{min}$

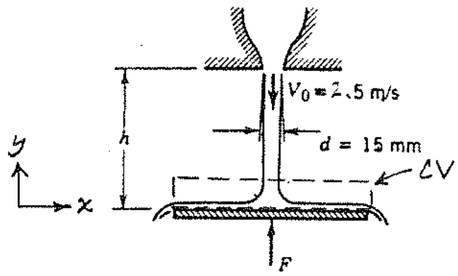
$p = \frac{1}{2} \cdot \rho \cdot V^2 \quad p = 147.1 \cdot kPa$ (gage)

$F = \rho \cdot V^2 \cdot \frac{\pi}{4} \cdot D^2 \quad F = 23.1\,N$

Problem *4.113

[Difficulty: 3]

***4.113** A uniform jet of water leaves a 15-mm-diameter nozzle and flows directly downward. The jet speed at the nozzle exit plane is 2.5 m/s. The jet impinges on a horizontal disk and flows radially outward in a flat sheet. Obtain a general expression for the velocity the liquid stream would reach at the level of the disk. Develop an expression for the force required to hold the disk stationary, neglecting the mass of the disk and water sheet. Evaluate for $h = 3$ m.



Solution: Apply Bernoulli and momentum equations. Use CV shown.

Basic equations: $\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$ (5)

$$\underbrace{F_{S_z}}_{=0(6)} + \cancel{F_{B_z}} = \underbrace{\cancel{\frac{\partial}{\partial t}}}_{=0(1)}\int_{CV} w\rho\, d\forall + \int_{CS} w\rho \vec{V}\cdot d\vec{A}$$

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Flow along a streamline
(4) Frictionless flow
(5) Atmospheric pressure along jet
(6) Neglect water on plate; $F_{B_z} = 0$
(7) Uniform flow at each section

The Bernoulli equation becomes

$$\frac{V_0^2}{2} + gh = \frac{V^2}{2} + g(0) \quad \text{or} \quad V^2 = V_0^2 + 2gh\,; \quad V = \sqrt{V_0^2 + 2gh}$$ V

From the momentum equation

$$R_z = w_1\{-\rho V A\} + w_2\{+\rho V_0 A_0\} = +\rho V^2 A$$

$$w_1 = -V \qquad w_2 = 0$$

But from continuity, $\dot{m} = \rho V_0 A_0 = \rho V A$. Thus $VA = V_0 A_0$, and

$$R_z = \rho V_0 A_0 V = \rho V_0 A_0 \sqrt{V_0^2 + 2gh}$$ R_z

At $h = 3.0\,\text{m}$,

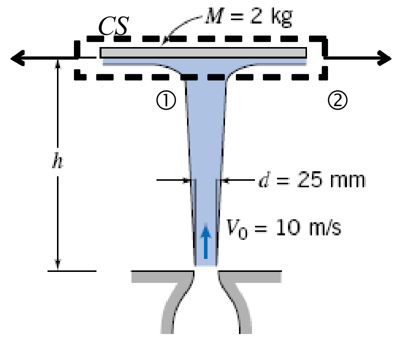
$$R_z = 999\,\frac{\text{kg}}{\text{m}^3} \times 2.5\,\frac{\text{m}}{\text{s}} \times \frac{\pi}{4}(0.015)^2\,\text{m}^2 \left[(2.5)^2\,\frac{\text{m}^2}{\text{s}^2} + 2 \times 9.81\,\frac{\text{m}}{\text{s}^2} \times 3.0\,\text{m}\right]^{1/2} \frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}}$$

$R_z = 3.56$ N (upward force) R_z

Problem *4.114

[Difficulty: 3]

*4.114 A 2-kg disk is constrained horizontally but is free to move vertically. The disk is struck from below by a vertical jet of water. The speed and diameter of the water jet are 10 m/s and 25 mm at the nozzle exit. Obtain a general expression for the speed of the water jet as a function of height, h. Find the height to which the disk will rise and remain stationary.



Given: Water jet striking disk

Find: Expression for speed of jet as function of height; Height for stationary disk

Solution:

Basic equations: Bernoulli; Momentum flux in z direction

$$\frac{p}{\rho} + \frac{V^2}{2} + g \cdot z = \text{constant} \qquad F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\,dV\!\!\!\!- + \int_{CS} u\,\rho\vec{V}\cdot d\vec{A}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) Atmospheric pressure in jet 4) Uniform flow

The Bernoulli equation becomes $\quad \frac{V_0^2}{2} + g \cdot 0 = \frac{V^2}{2} + g \cdot h \qquad V^2 = V_0^2 - 2 \cdot g \cdot h \qquad V = \sqrt{V_0^2 - 2 \cdot g \cdot h}$

Hence $\quad -M \cdot g = w_1 \cdot \left(-\rho \cdot w_1 \cdot A_1\right) = -\rho \cdot V^2 \cdot A$

But from continuity $\quad \rho \cdot V_0 \cdot A_0 = \rho \cdot V \cdot A \qquad$ so $\qquad V \cdot A = V_0 \cdot A_0$

Hence we get $\quad M \cdot g = \rho \cdot V \cdot V \cdot A = \rho \cdot V_0 \cdot A_0 \cdot \sqrt{V_0^2 - 2 \cdot g \cdot h}$

Solving for h $\quad h = \frac{1}{2 \cdot g} \cdot \left[V_0^2 - \left(\frac{M \cdot g}{\rho \cdot V_0 \cdot A_0}\right)^2\right]$

$$h = \frac{1}{2} \times \frac{s^2}{9.81 \cdot m} \times \left[\left(10 \cdot \frac{m}{s}\right)^2 - \left[2 \cdot kg \times \frac{9.81 \cdot m}{s^2} \times \frac{m^3}{1000 \cdot kg} \times \frac{s}{10 \cdot m} \times \frac{4}{\pi \cdot \left(\frac{25}{1000} \cdot m\right)^2}\right]^2\right]$$

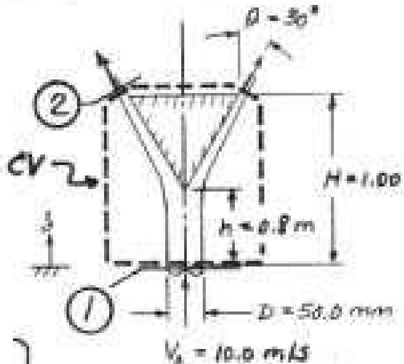
$h = 4.28\,m$

Problem *4.115

[Difficulty: 4] Part 1/2

***4.115** Water from a jet of diameter D is used to support the cone-shaped object shown. Derive an expression for the combined mass of the cone and water, M, that can be supported by the jet, in terms of parameters associated with a suitably chosen control volume. Use your expression to calculate M when $V_0 = 10$ m/s, $H = 1$ m, $h = 0.8$ m, $D = 50$ mm, and $\theta = 30°$. Estimate the mass of water in the control volume.

Solution: Apply continuity, Bernoulli, and momentum equations using CV shown.



Basic equations:
$$0 = \underbrace{\frac{\partial}{\partial t}\int_{CV} \rho\, d\forall}_{=0(1)} + \int_{CS} \rho \vec{V}\cdot d\vec{A}$$

$$\frac{\overset{(6)}{p_1}}{\rho} + \frac{V_1^2}{2} + g z_1 = \frac{\overset{(6)}{p_2}}{\rho} + \frac{V_2^2}{2} + g z_2$$

$$\underbrace{F_{S_z}}_{=0(6)} + F_{B_z} = \underbrace{\frac{\partial}{\partial t}\int_{CV} w \rho\, d\forall}_{=0(1)} + \int_{CS} w \rho \vec{V}\cdot d\vec{A}$$

Assumptions: (1) Steady flow
(2) No friction
(3) Flow along a streamline
(4) Incompressible flow
} required for Bernoulli
(5) Uniform flow at each cross-section
(6) $F_{S_z} = 0$ since p_{atm} acts everywhere

Then
$$0 = \{-|\rho V_1 A_1|\} + \{+|\rho V_2 A_2|\} \quad \text{so} \quad V_1 A_1 = V_2 A_2$$

From Bernoulli
$$\frac{V_1^2}{2} + g z_1 = \frac{V_2^2}{2} + g z_2 = \frac{V_0^2}{2} = \frac{V_2^2}{2} + gH \;; \quad V_2^2 = V_0^2 - 2gH$$

From momentum
$$F_{B_z} = \int_{CS} w \rho \vec{V}\cdot d\vec{A} = -Mg = w_1\{-|\rho V_1 A_1|\} + w_2\{+|\rho V_2 A_2|\}$$
$$w_1 = V_0 \qquad w_2 = V_2 \cos\theta$$

or
$$-Mg = -V_0 \rho V_1 A_1 + V_2\cos\theta\, \rho V_2 A_2 = \rho V_0 A_1 (V_2 \cos\theta - V_0)$$

so
$$M = \frac{(V_0 - V_2\cos\theta)\rho V_0 A_1}{g}$$

From Bernoulli
$$V_2 = (V_0^2 - 2gH)^{1/2} = \left[(10)^2 \frac{m^2}{s^2} - 2 \times 9.81 \frac{m}{s^2} \times 1\,m\right]^{1/2} = 8.97 \text{ m/s}$$

Substituting
$$M = \left(10.0 \frac{m}{s} - 8.97 \frac{m}{s} \times \cos 30°\right) 999 \frac{kg}{m^3} \times 10 \frac{m}{s} \times \frac{\pi}{4}(0.050)^2 m^2 \times \frac{s^2}{9.81\,m}$$

$M = 4.46$ kg (total mass in CV: water + object)

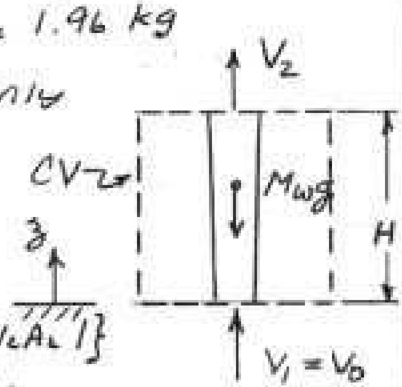
M

To find mass of water in CV, we have 3 options:

(1) assume area of jet is constant

$$M = \rho V\!\!\!/ \simeq \rho A_1 H = 999\,\frac{kg}{m^3} \times \frac{\pi}{4}(0.05)^2\,m^2 \times 1\,m = 1.96\,kg$$

(2) Use a CV that encloses the free jet only



Continuity $\quad V_1 A_1 = V_2 A_2$

Bernoulli $\quad V_2 = (V_1^2 - 2gH)^{1/2}$

Momentum $\quad -M_w g = u_1\{-|\rho V_1 A_1|\} + u_2\{+|\rho V_2 A_2|\}$

$$u_1 = V_1 = V_0 \qquad u_2 = V_2$$

Substituting in momentum

$$-M_w g = V_0(-\rho V_0 A_1) + V_2(+\rho V_0 A_1) = \rho V_0 A_1 (V_2 - V_0)$$

$$M_w = \frac{\rho V_0 A_1 (V_0 - V_2)}{g}$$

$$= 999\,\frac{kg}{m^3} \times 10\,\frac{m}{s} \times \frac{\pi}{4}(0.05)^2\,m^2\,(10 - 8.97)\,\frac{m}{s} \times \frac{s^2}{9.81\,m}$$

$$M_w = 2.06\,kg$$ M_w

(3) Evaluate the area at each cross-section using Bernoulli and continuity, then integrate to find $V\!\!\!/$.

$$VA = V_0 A_1 = (V_0^2 - 2gz)^{1/2} A = V_0 A_1 \quad \text{so} \quad A = \frac{V_0 A_1}{(V_0^2 - 2gz)^{1/2}}$$

$$V\!\!\!/ = \int_0^H A\,dz = \int_0^H \frac{V_0 A_1}{(V_0^2 - 2gz)^{1/2}}\,dz = A_1 \int_0^H \frac{V_0^2}{2g}\,\frac{1}{\left(1 - \frac{2gz}{V_0^2}\right)^{1/2}}\,d\left(\frac{2gz}{V_0^2}\right)$$

This can be integrated. Let $u = 1 - 2gz/V_0^2$, so $\int = \int \frac{-du}{u^{1/2}}$

$$\text{Then} \quad V\!\!\!/ = A_1 \frac{V_0^2}{2g}\left[-2\left(1 - \frac{2gz}{V_0^2}\right)^{1/2}\right]_{z=0}^{z=h} = \frac{A_1}{g}\left[V_0^2 - V_0(V_0^2 - 2gh)^{1/2}\right]$$

and

$$M_w = \rho V\!\!\!/ = \frac{\rho A_1 V_0 (V_0 - V_2)}{g} = 2.06\,kg \quad \text{(same as (2) above)}$$

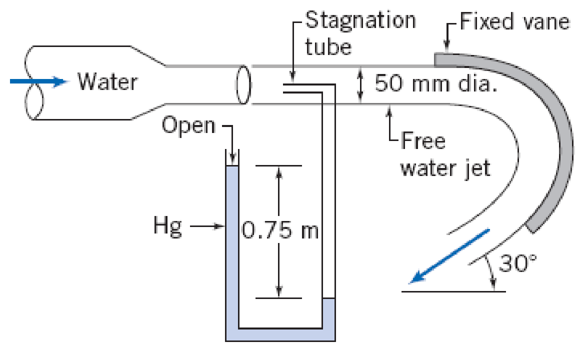
Thus the mass of the cone is $M_c = M - M_w = 2.40\,kg$. M_c

{Note: If V_0 were smaller or H larger, V_2 would differ more from V_0 and the jet area would increase significantly. Option (2) would still give the correct result with little effort.}

Problem *4.116

[Difficulty: 3]

***4.116** A stream of water from a 50-mm-diameter nozzle strikes a curved vane, as shown. A stagnation tube connected to a mercury-filled U-tube manometer is located in the nozzle exit plane. Calculate the speed of the water leaving the nozzle. Estimate the horizontal component of force exerted on the vane by the jet. Comment on each assumption used to solve this problem.



Given: Stream of water striking a vane

Find: Water speed; horizontal force on vane

Solution:

Basic equations: Bernoulli; Momentum flux in x direction

$$\frac{p}{\rho} + \frac{V^2}{2} + g \cdot z = \text{constant} \qquad F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\, dV + \int_{CS} u\,\rho \vec{V}\cdot d\vec{A}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) Atmospheric pressure in jet 4) Uniform flow

Given or available data $\quad D = 50 \cdot mm \quad \rho_{water} = 1000 \cdot \frac{kg}{m^3} \quad \rho_{Hg} = 13.6 \cdot \rho_{water} \quad \theta = 30 \cdot deg \quad \Delta h = 0.75 \cdot m$

From Bernoulli $\quad p_0 = p + \frac{1}{2} \cdot \rho_{water} \cdot V^2 \quad$ and for the manometer $\quad p_0 - p = \rho_{Hg} \cdot g \cdot \Delta h$

Combining $\quad \frac{1}{2} \cdot \rho_{water} \cdot V^2 = \rho_{Hg} \cdot g \cdot \Delta h \quad$ or $\quad V = \sqrt{\frac{2 \cdot \rho_{Hg} \cdot g \cdot \Delta h}{\rho_{water}}} \quad V = 14.1 \frac{m}{s}$

Applying x momentum to the vane $\quad R_x = \rho_{water} \cdot V \cdot \left(-V \cdot \frac{\pi}{4} \cdot D^2\right) + \rho_{water} \cdot (-V \cdot \cos(\theta)) \cdot \left(V \cdot \frac{\pi}{4} \cdot D^2\right)$

$$R_x = -\rho_{water} \cdot V^2 \cdot \frac{\pi}{4} \cdot D^2 \cdot (1 + \cos(\theta)) \qquad R_x = -733\,N$$

Assuming frictionless, incompressible flow with no net pressure force is realistic, except along the vane where friction will reduce flow momentum at the exit.

Problem *4.117

[Difficulty: 2]

***4.117** A Venturi meter installed along a water pipe consists of a convergent section, a constant-area throat, and a divergent section. The pipe diameter is $D = 100$ mm, and the throat diameter is $d = 50$ mm. Find the net fluid force acting on the convergent section if the water pressure in the pipe is 200 kPa (gage) and the flow rate is 1000 L/min. For this analysis, neglect viscous effects.

Given: Data on flow and venturi geometry

Find: Force on convergent section; water pressure

Solution:

Basic equations:

Bernoulli equation and x momentum

$$\frac{p}{\rho} + \frac{V^2}{2} + g\cdot z = \text{const} \qquad F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\,dV\!\!\!/ + \int_{CS} u\,\rho\vec{V}\cdot d\vec{A}$$

The given data is $\rho = 999\cdot\frac{kg}{m^3}$ $\quad D = 100\cdot mm$ $\quad d = 50\cdot mm$ $\quad p_1 = 200\cdot kPa$ $\quad Q = 1000\cdot\frac{L}{min}$

For pressure we first need the velocities

$$A_1 = \frac{\pi\cdot D^2}{4} \qquad A_1 = 0.00785\,m^2 \qquad A_2 = \frac{\pi}{4}\cdot d^2 \qquad A_2 = 0.00196\,m^2$$

Then

$$V_1 = \frac{Q}{\frac{\pi}{4}\cdot D^2} \qquad V_1 = 2.12\,\frac{m}{s} \qquad V_2 = \frac{Q}{A_2} \qquad V_2 = 8.49\,\frac{m}{s}$$

Applying Bernoulli between inlet and throat

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} = \frac{p_2}{\rho} + \frac{V_2^2}{2}$$

Solving for p_2

$$p_2 = p_1 + \frac{\rho}{2}\cdot\left(V_1^2 - V_2^2\right)$$

$$p_2 = 200\cdot kPa + \frac{1}{2}\cdot 999\cdot\frac{kg}{m^3} \times \left(2.12^2 - 8.49^2\right)\cdot\frac{m^2}{s^2} \times \frac{N\cdot s^2}{kg\cdot m} \times \frac{kN}{1000\cdot N} \qquad p_2 = 166\cdot kPa$$

Applying the horizontal component of momentum

$$-F + p_1\cdot A_2 - p_2\cdot A_2 = V_1\cdot\left(-\rho\cdot V_1\cdot A_1\right) + V_2\cdot\left(\rho\cdot V_2\cdot A_2\right)$$

or

$$F = p_1\cdot A_1 - p_2\cdot A_2 + \rho\cdot\left(V_1^2\cdot A_1 - V_2^2\cdot A_2\right)$$

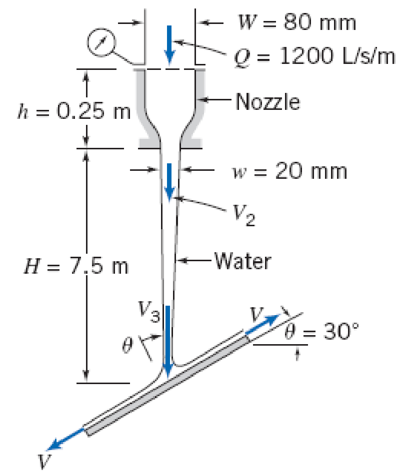
$$F = 200\cdot\frac{kN}{m^2} \times 0.00785\cdot m^2 - 166\cdot\frac{kN}{m^2} \times 0.00196\cdot m^2 + 999\cdot\frac{kg}{m^3} \times \left[\left(2.12\cdot\frac{m}{s}\right)^2\cdot 0.00785\cdot m^2 - \left(8.49\cdot\frac{m}{s}\right)^2\cdot 0.00196\cdot m^2\right]\cdot\frac{N\cdot s^2}{kg \times m}$$

$$F = 1.14\,kN$$

Problem *4.118

[Difficulty: 3]

***4.118** A plane nozzle discharges vertically 1200 L/s per unit width downward to atmosphere. The nozzle is supplied with a steady flow of water. A stationary, inclined, flat plate, located beneath the nozzle, is struck by the water stream. The water stream divides and flows along the inclined plate; the two streams leaving the plate are of unequal thickness. Frictional effects are negligible in the nozzle and in the flow along the plate surface. Evaluate the minimum gage pressure required at the nozzle inlet.



Given: Nozzle flow striking inclined plate

Find: Mimimum gage pressure

Solution:

Basic equations: Bernoulli and y momentum

$$\frac{p}{\rho} + \frac{V^2}{2} + g\cdot z = \text{const} \qquad F_y = F_{S_y} + F_{B_y} = \frac{\partial}{\partial t}\int_{CV} v\,\rho\,dV\!\!\!/ + \int_{CS} v\,\rho\vec{V}\cdot d\vec{A}$$

The given data is $\rho = 999\cdot\frac{kg}{m^3}$ $\quad q = 1200\cdot\frac{L}{s\cdot m}$ $\quad W = 80\cdot mm$ $\quad h = 0.25\cdot m$ $\quad w = 20\cdot mm$ $\quad H = 7.5\cdot m$ $\quad \theta = 30\cdot deg$

For the exit velocity and nozzle velocity $\quad V_2 = \frac{q}{w}$ $\quad V_2 = 15.0\frac{m}{s}$ $\quad V_1 = V_2\cdot\frac{w}{W}$ $\quad V_1 = 3.75\frac{m}{s}$

Then from Bernoulli $\quad p_1 + \frac{\rho}{2}\cdot V_1{}^2 = p_{atm} + \frac{\rho}{2}\cdot V_2{}^2$ $\quad$ or $\quad p_1 = \frac{\rho}{2}\cdot\left(V_2{}^2 - V_1{}^2\right) - \rho\cdot g\cdot h$ $\quad p_1 = 103\cdot kPa$ (gage)

Applying Bernoulli between 2 and the plate (state 3)

$$p_{atm} + \frac{\rho}{2}\cdot V_2{}^2 = p_{atm} + \frac{\rho}{2}\cdot V_3{}^2 - \rho\cdot g\cdot H \qquad V_3 = \sqrt{V_2{}^2 + 2\cdot g\cdot H} \qquad V_3 = 19.3\frac{m}{s}$$

For the plate there is no force along the plate (x momentum) as there is no friction. For the force normal to the plate (y momentum) we have

$$R_y = -V_3\cdot\cos(\theta)\cdot\left(-\rho\cdot V_3\cdot A_3\right) = -V_3\cdot\cos(\theta)\cdot(-\rho\cdot q) \qquad R_y = V_3\cdot\cos(\theta)\cdot\rho\cdot q \qquad R_y = 20.0\cdot\frac{kN}{m}$$

Problem *4.119

[Difficulty: 3]

***4.119** You turn on the kitchen faucet very slightly, so that a very narrow stream of water flows into the sink. You notice that it is "glassy" (laminar flow) and gets narrower and remains "glassy" for about the first 50 mm of descent. When you measure the flow, it takes three min to fill a 1-L bottle, and you estimate the stream of water is initially 5 mm in diameter. Assuming the speed at any cross section is uniform and neglecting viscous effects, derive expressions for and plot the variations of stream speed and diameter as functions of z (take the origin of coordinates at the faucet exit). What are the speed and diameter when it falls to the 50-mm point?

Given: Water faucet flow

Find: Expressions for stream speed and diameter; plot

Solution:

Basic equation: Bernoulli $\quad \dfrac{p}{\rho} + \dfrac{V^2}{2} + g \cdot z = \text{const}$

Assumptions: Laminar, frictionless, uniform flow

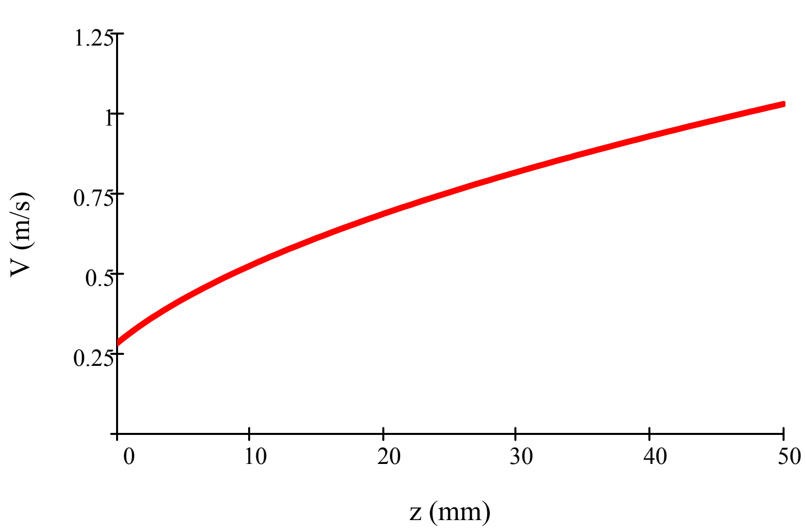
The given data is $\quad D_0 = 5 \cdot \text{mm} \quad h = 50 \cdot \text{mm} \quad Q = \dfrac{1 \cdot L}{3 \cdot \text{min}} \quad Q = 0.333 \dfrac{L}{\text{min}}$

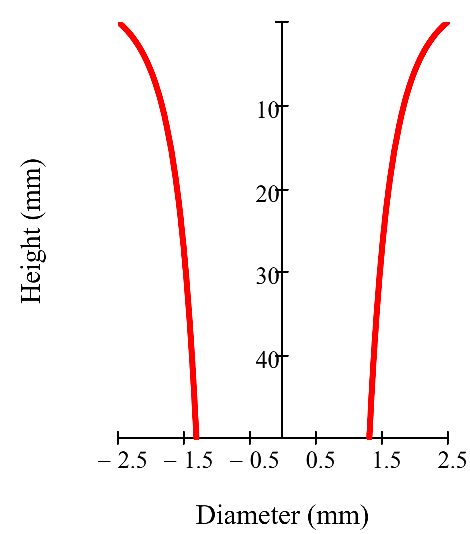
The initial velocity is $\quad V_0 = \dfrac{Q}{\dfrac{\pi}{4} \cdot D_0^{\,2}} \quad V_0 = 0.283 \dfrac{m}{s}$

Then applying Bernoulli between the exit and any other location $\quad \dfrac{p_{atm}}{\rho} + \dfrac{V_0^{\,2}}{2} = \dfrac{p_{atm}}{\rho} + \dfrac{V^2}{2} - g \cdot z \quad$ (z downwards)

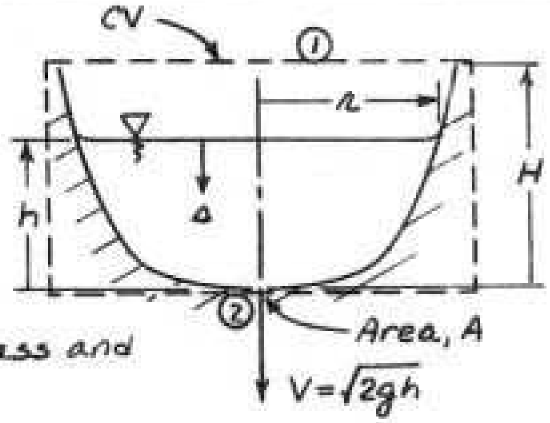
Then $\quad V(z) = \sqrt{V_0^{\,2} + 2 \cdot g \cdot z} \quad$ Also $\quad V_0 \cdot \dfrac{\pi}{4} \cdot D_0^{\,2} = V \cdot \dfrac{\pi}{4} \cdot D^2 \quad$ so $\quad D(z) = \dfrac{D_0}{\left(1 + \dfrac{2 \cdot g \cdot z}{V_0^{\,2}}\right)^{\frac{1}{4}}}$

Evaluating at h $\quad V(h) = 1.03 \dfrac{m}{s} \quad D(h) = 2.62 \cdot \text{mm}$





***4.120** In ancient Egypt, circular vessels filled with water sometimes were used as crude clocks. The vessels were shaped in such a way that, as water drained from the bottom, the surface level dropped at constant rate, s. Assume that water drains from a small hole of area A. Find an expression for the radius of the vessel, r, as a function of the water level, h. Obtain an expression for the volume of water needed so that the clock will operate for n hours.



Solution: Apply conservation of mass and the Bernoulli equation.

Basic equations: $0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$$

Assumptions: (1) Quasi-steady flow; $\frac{\partial}{\partial t}$ small
(2) Incompressible flow
(3) Uniform flow at each cross section
(4) Flow along a streamline
(5) No friction
(6) $\rho_{air} \ll \rho_{H_2O}$

Writing Bernoulli from the liquid surface to the jet exit,

$$\cancel{\frac{p_{atm}}{\rho}} + \frac{s^2}{2} + gh = \cancel{\frac{p_{atm}}{\rho}} + \frac{V^2}{2} + g(0);$$

For $s \ll V$, then $V = \sqrt{2gh}$.

For the CV,

$$0 = \frac{\partial}{\partial t}\int_{\forall_{air}} \cancel{\rho_{air}}^{\ll \rho_{H_2O}(6)} \, d\forall + \frac{\partial}{\partial t}\int_{\forall_{H_2O}} \rho_{H_2O} \, d\forall + \left\{-|\cancel{\rho_{air}}^{\ll \rho_{H_2O}(6)} V_1 A_1|\right\} + \left\{|\rho_{H_2O} V A|\right\}$$

or

$$0 = \rho \frac{d\forall}{dt} + \rho V A = \rho \pi r^2 \frac{dh}{dt} + \rho\sqrt{2gh}\, A = 0$$

But h decreases, so $\frac{dh}{dt} = -s$. Thus

$$\pi r^2 s = \sqrt{2gh}\, A \quad \text{or} \quad r = \sqrt[4]{2g}\sqrt{\frac{A}{\pi s}}\, h^{1/4}$$

r

For n hours' operation, $H = ns$, and

$$\forall = \int_0^H \pi r^2 dh = \int_0^{ns} \sqrt{2gh}\,\frac{A}{s}\, dh = \frac{2A}{3s}\sqrt{2g}\,(ns)^{3/2}$$

or

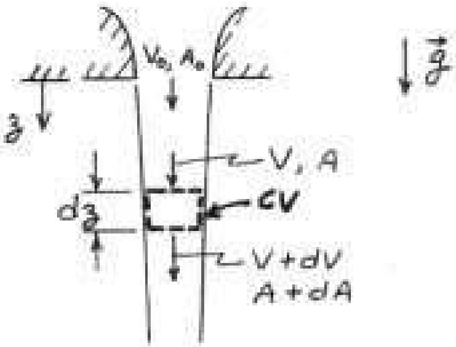
$$\forall = \frac{2A\sqrt{2g}\, n^{3/2} s^{1/2}}{3}$$

$\forall$

Check dimensions:

$$[\forall] = L^3 = \left[A\sqrt{g}\, n^{3/2} s^{1/2}\right] = L^2 \frac{L^{1/2}}{t} t^{3/2} \frac{L^{1/2}}{t^{1/2}} = L^3 \checkmark$$

***4.121** A stream of incompressible liquid moving at low speed leaves a nozzle pointed directly downward. Assume the speed at any cross section is uniform and neglect viscous effects. The speed and area of the jet at the nozzle exit are V_0 and A_0, respectively. Apply conservation of mass and the momentum equation to a differential control volume of length dz in the flow direction. Derive expressions for the variations of jet speed and area as functions of z. Evaluate the distance at which the jet area is half its original value. (Take the origin of coordinates at the nozzle exit.)



Solution: Apply continuity and momentum equations using CV shown.

Basic equations:

$$0 = \underbrace{\frac{\partial}{\partial t}\int_{CV} \rho \, d\forall}_{=0(1)} + \int_{CS} \rho \vec{V}\cdot d\vec{A}$$

$$\underbrace{F_{S_z}}_{=0(4,5)} + F_{B_z} = \underbrace{\frac{\partial}{\partial t}\int_{CV} w\rho \, d\forall}_{=0(1)} + \int_{CS} w\rho \vec{V}\cdot d\vec{A}$$

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Uniform flow at each section
(4) Patm acts everywhere
(5) No friction

(4) and (5): $F_{S_z} = 0$

Then

$$0 = \int_{CS} \vec{V}\cdot d\vec{A} = \{-VA\} + \{+(V+dV)(A+dA)\}; \quad VA = V_0A_0 = \text{constant}$$

From momentum,

$$\rho g \left(A + \frac{dA}{2}\right)dz = V\{-\rho VA\} + (V+dV)\{-\rho(V+dV)(A+dA)\} = \rho VA\,dV$$

since $dV\,dA \ll dA$. Also, since $dA\,dz \ll dz$, the left side is $\rho g A\,dz$.

Thus

$$\rho g A\,dz = \rho V A\,dV \quad \text{or} \quad V\,dV = g\,dz$$

Integrating from V_0 at $z_0 = 0$ to V at z,

$$\int_{V_0}^{V} V\,dV = \frac{V^2}{2}\Big]_{V_0}^{V} = \frac{V^2}{2} - \frac{V_0^2}{2} = \int_{z_0}^{z} g\,dz = g(z - z_0) = gz$$

Thus

$$V^2 = V_0^2 + 2gz \quad \text{or} \quad V(z) = \sqrt{V_0^2 + 2gz}$$ $V(z)$

Since $VA = V_0A_0$, $A = A_0\dfrac{V_0}{V}$

$$A(z) = A_0\frac{V_0}{\sqrt{V_0^2 + 2gz}} = \frac{A_0}{\sqrt{1 + 2gz/V_0^2}}$$ $A(z)$

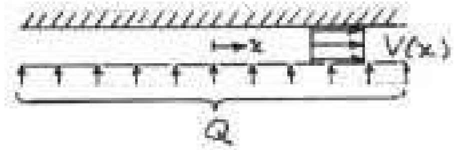
Solving for z,

$$z = \frac{V_0^2}{2g}\left[\left(\frac{A_0}{A}\right)^2 - 1\right]; \quad \text{for } \frac{A}{A_0} = \frac{1}{2}, \ \frac{A_0}{A} = 2, \text{ and } z_{1/2} = \frac{3V_0^2}{2g}$$ $z_{1/2}$

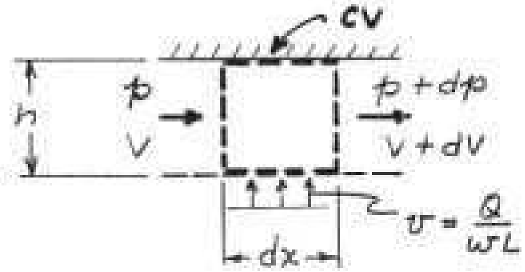
Problem *4.122

[Difficulty: 4]

*4.122 Incompressible fluid of negligible viscosity is pumped, at total volume flow rate Q, through a porous surface into the small gap between closely spaced parallel plates as shown. The fluid has only horizontal motion in the gap. Assume uniform flow across any vertical section. Obtain an expression for the pressure variation as a function of x. *Hint:* Apply conservation of mass and the momentum equation to a differential control volume of thickness dx, located at position x.



Solution: Apply continuity and x component of momentum equation.



Basic equations:

$$0 = \cancelto{=0(1)}{\frac{\partial}{\partial t}}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A}$$

$$F_{Sx} + \cancelto{=0(5)}{F_{Bx}} = \cancelto{=0(1)}{\frac{\partial}{\partial t}}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$$

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Uniform flow at each section
(4) Neglect friction
(5) $F_{Bx} = 0$

Then

$$0 = \int_{CS} \vec{V}\cdot d\vec{A} = \{-Vwh\} + \{-\frac{Q}{wL} w\,dx\} + \{(V+dV)wh\}; \quad wh\,dV = \frac{Q}{L}dx$$

$$V = \frac{Q}{wh}\frac{x}{L} + c \;;\; c=0 \text{ since } V(0)=0 \;;\; V(x) = \frac{Q}{wh}\frac{x}{L}$$

From momentum,

$$p\,wh - (p+dp)wh = u_x\{-\rho V wh\} + u_{dx}\{-\rho \frac{Q}{wh} w\,dx\} + u_{x+dx}\{+\rho(V+dV)wh\}$$

$$u_x = V \qquad u_{dx} = 0 \qquad u_{x+dx} = V+dV$$

From continuity, $(V+dV)wh = Vwh + Q\frac{dx}{L}$, so

$$-dp\,wh = -\rho V^2 wh + 0 + (V+dV)(Vwh + Q\frac{dx}{L})\rho$$

$$= -\cancel{\rho V^2 wh} + \cancel{\rho V^2 wh} + \rho V wh\,dV + V\rho Q\frac{dx}{L} + \rho Q\,dV\frac{dx}{L}$$

Neglecting products of differentials ($dV dx \ll dx$), and with $dV = \frac{Q}{wh}\frac{dx}{L}$

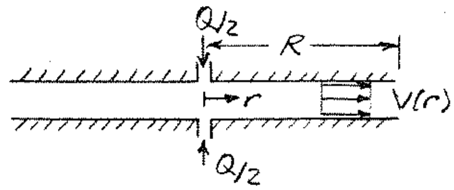
$$-dp = \rho V dV + \frac{V\rho Q}{wh}\frac{dx}{L} = \rho V\frac{Q}{wh}\frac{dx}{L} + \frac{V\rho Q}{wh}\frac{dx}{L} = 2\rho\frac{Q}{wh}\frac{x}{L}\frac{Q}{wh}\frac{dx}{L}$$

$$-dp = 2\rho\left(\frac{Q}{whL}\right)^2 x\,dx \qquad p(x) = -\rho\left(\frac{Q}{whL}\right)^2 x^2 + c$$

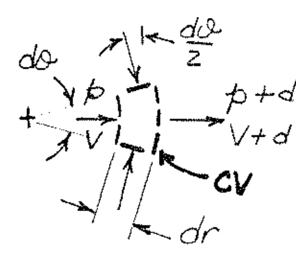
If $p(0) = p_0$, then $p(x) = p_0 - \rho\left(\frac{Q}{wh}\right)^2\left(\frac{x}{L}\right)^2$

$p(x)$

***4.123** Incompressible liquid of negligible viscosity is pumped, at total volume flow rate Q, through two small holes into the narrow gap between closely spaced parallel disks as shown. The liquid flowing away from the holes has only radial motion. Assume uniform flow across any vertical section and discharge to atmospheric pressure at $r = R$. Obtain an expression for the pressure variation and plot as a function of radius. *Hint:* Apply conservation of mass and the momentum equation to a differential control volume of thickness dr located at radius r.



Solution: Apply continuity and momentum equations to the differential CV shown.



Basic equations:

$$0 = \underbrace{\frac{\partial}{\partial t}\int_{CV} \rho\, d\forall}_{=0(1)} + \int_{CS} \rho \vec{V}\cdot d\vec{A}$$

$$F_{S_r} + \underbrace{F_{B_r}}_{=0(5)} = \underbrace{\frac{\partial}{\partial t}\int_{CV} V_r \rho\, d\forall}_{=0(1)} + \int_{CS} V_r \rho \vec{V}\cdot d\vec{A}$$

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Uniform flow at each section
(4) Neglect friction
(5) $F_{B_r} = 0$
(6) No flow in θ direction
(7) $\sin\frac{d\theta}{2} = \frac{d\theta}{2}$

Then

$$0 = \int_{CS} \vec{V}\cdot d\vec{A} = \{-\rho V h r d\theta\} + \{\rho(V+dV)h(r+dr)d\theta\}\ ;\ Vr = \text{constant}$$

For $r = R$, $Q = V_R 2\pi r h$, so $V_R = Q/2\pi R h$

From momentum,

$$p\, h r d\theta + 2\left(p + \frac{dp}{2}\right) h\, dr \sin\frac{d\theta}{2} - (p+dp)h(r+dr)d\theta$$

$$= V\{-\rho V h r d\theta\} + (V+dV)\{\rho(V+dV)h(r+dr)d\theta\}$$

$$\cancel{p h r d\theta} + \cancel{p h dr d\theta} + \tfrac{1}{2}dp\, h\, dr\, d\theta - (\cancel{pr} + \cancel{p dr} + r dp + dr\, dp)h\, d\theta$$

$$= dV(\rho V h r d\theta) \quad \{\text{Note terms in braces are equal.}\}$$

Assuming products of differentials are much smaller than single differentials,

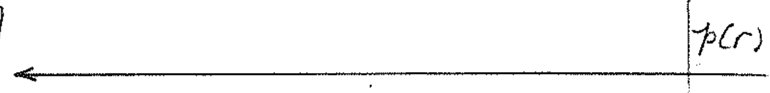
$$-r\, dp\, h\, d\theta = dV(\rho V h r d\theta) \quad \text{or} \quad dp = -\rho V dV$$

Integrating, $p(r) - p(R) = -\rho\frac{V^2}{2} + \rho\frac{V_R^2}{2}$ or $p(r) - p_{atm} = \frac{\rho}{2}(V_R^2 - V^2)$

$$= \frac{\rho V_R^2}{2}\left[1 - \left(\frac{V}{V_R}\right)^2\right]$$

Since $V_R = \dfrac{Q}{2\pi R h}$, and $Vr = \text{constant}$, $\dfrac{V}{V_R} = \dfrac{R}{r}$, so

$$p(r) - p_{atm} = \frac{\rho}{2}\left(\frac{Q}{2\pi R h}\right)^2\left[1 - \left(\frac{R}{r}\right)^2\right]$$



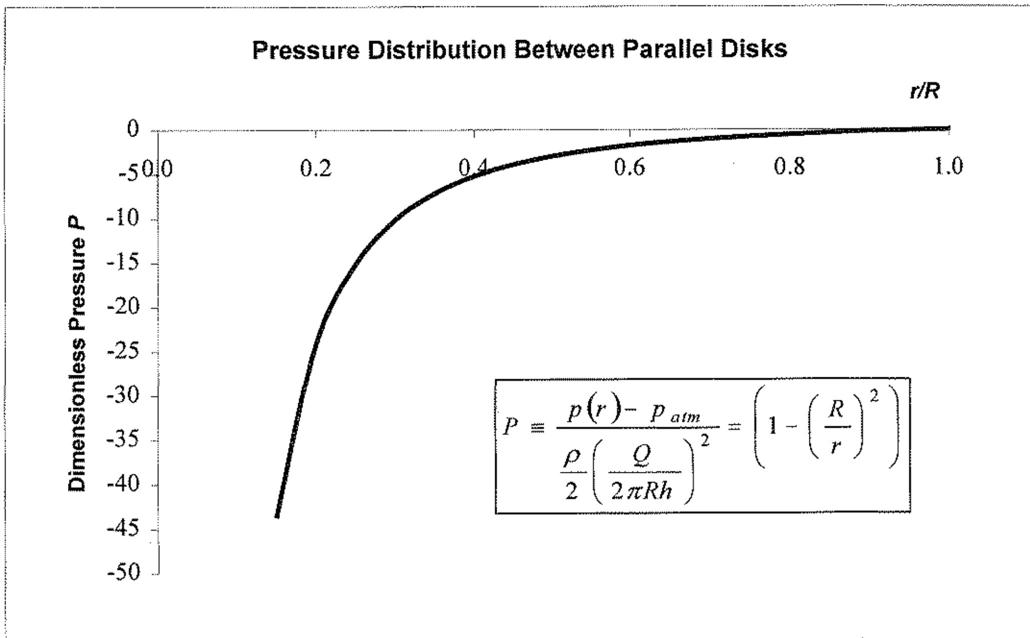
Note since $r < R$, that $p(r) < p_{atm}$ between the disks.

Problem *4.123

[Difficulty: 4] Part 2/2

The pressure distribution is computed and plotted in *Excel*:

r/R	*P*
0.15	-43.4
0.20	-24.0
0.25	-15.0
0.30	-10.1
0.35	-7.16
0.40	-5.25
0.45	-3.94
0.50	-3.00
0.55	-2.31
0.60	-1.78
0.65	-1.37
0.70	-1.04
0.75	-0.78
0.80	-0.563
0.85	-0.384
0.90	-0.235
0.95	-0.108
1.00	0.000



$$P \equiv \frac{p(r) - p_{atm}}{\frac{\rho}{2}\left(\frac{Q}{2\pi Rh}\right)^2} = \left(1 - \left(\frac{R}{r}\right)^2\right)$$

Problem *4.124

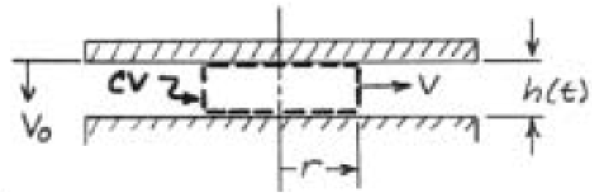
[Difficulty: 5]

***4.124** The narrow gap between two closely spaced circular plates initially is filled with incompressible liquid. At $t = 0$ the upper plate, initially h_0 above the lower plate, begins to move downward toward the lower plate with constant speed, V_0, causing the liquid to be squeezed from the narrow gap. Neglecting viscous effects and assuming uniform flow in the radial direction, develop an expression for the velocity field between the parallel plates. *Hint*: Apply conservation of mass to a control volume with the outer surface located at radius r. Note that even though the speed of the upper plate is constant, the flow is unsteady. For $V_0 = 0.01$ m/s and $h_0 = 2$ mm, find the velocity at the exit radius $R = 100$ mm at $t = 0$ and $t = 0.1$ s. Plot the exit velocity as a function of time, and explain the trend.

Given: Plates coming together

Find: Expression for velcoity field; exit velocity; plot

Solution: Apply continuity using deformable CV as shown



Basic equation: $$\frac{\partial}{\partial t}\int_{CV}\rho\, d\forall + \int_{CS}\rho\vec{V}\cdot d\vec{A} = 0$$

Assumptions: Incompressible, uniform flow

Given data: $V_0 = 0.01\cdot\frac{m}{s}$ $\qquad h_0 = 2\cdot mm$ $\qquad R = 100\cdot mm$

Continuity becomes $$\frac{\partial\forall}{\partial t} + \int_{CS}\vec{V}\cdot d\vec{A} = 0$$ or $$\frac{\partial}{\partial t}\left(\pi\cdot r^2\cdot h\right) + V\cdot 2\cdot\pi\cdot r\cdot h = 0$$

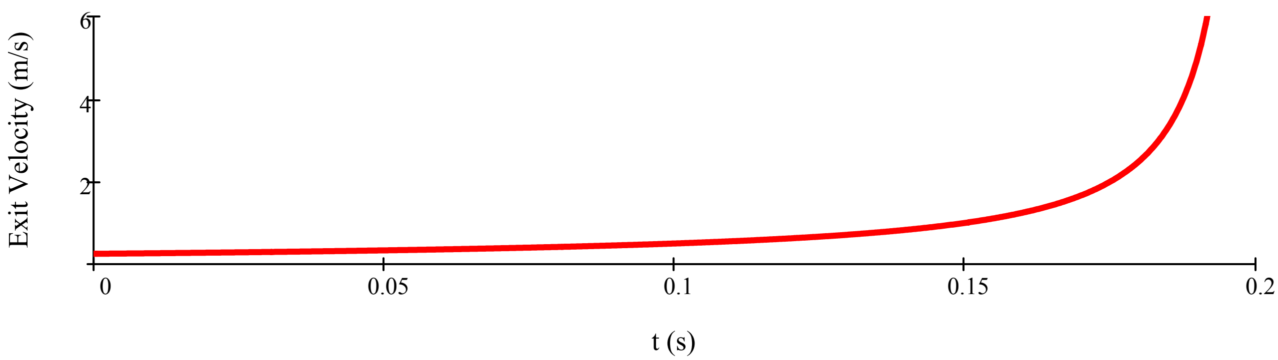
or $$\pi\cdot r^2\cdot\frac{dh}{dt} + V\cdot 2\cdot\pi\cdot r\cdot h = \pi\cdot r^2\cdot V_0 + V\cdot 2\cdot\pi\cdot r\cdot h = 0$$

Hence $$V(r) = V_0\cdot\frac{r}{2\cdot h}$$

If V_0 is constant $\quad h = h_0 - V_0\cdot t \quad$ so $$V(r,t) = \frac{V_0\cdot r}{2\cdot\left(h_0 - V_0\cdot t\right)}$$

Note that $t_{max} = \frac{h_0}{V_0}$ $\qquad t_{max} = 0.200\ s$

Evaluating $\quad V(R,0) = 0.250\ \frac{m}{s} \qquad V(R,0.1\cdot s) = 0.500\ \frac{m}{s}$



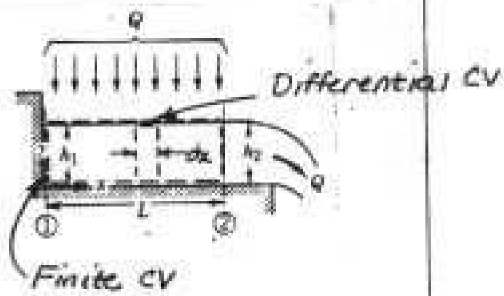
The velocity greatly increases as the constant flow rate exits through a gap that becomes narrower with time.

Problem *4.125

[Difficulty: 5] Part 1/2

***4.125** Liquid falls vertically into a short horizontal rectangular open channel of width b. The total volume flow rate, Q, is distributed uniformly over area bL. Neglect viscous effects. Obtain an expression for h_1 in terms of h_2, Q, and b. *Hint:* Choose a control volume with outer boundary located at $x = L$. Sketch the surface profile, $h(x)$. *Hint:* Use a differential control volume of width dx.

Solution: Apply continuity and momentum equations to (i) finite CV, and (ii) differential CV, as shown.



Basic equations:

$$0 = \underbrace{\frac{\partial}{\partial t}\int_{CV} \rho \, d\forall}_{=0(1)} + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$

$$F_{Sx} + \underbrace{F_{Bx}}_{=0(6)} = \underbrace{\frac{\partial}{\partial t}\int_{CV} u\rho \, d\forall}_{=0(1)} + \int_{CS} u\rho \vec{V} \cdot d\vec{A}$$

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Uniform flow at each section
(4) Hydrostatic pressure distribution; $F_p(h) = \rho g b \frac{h^2}{2}$
(5) No friction on bed
(6) Horizontal bed; $F_{Bx} = 0$

Then for finite CV shown,

$$0 = \int_{CS} \vec{V} \cdot d\vec{A} = -Q + V_2 b h_2 \; ; \quad V_2 = \frac{Q}{bh_2}$$

From momentum

$$\rho g b \frac{h_1^2}{2} - \rho g b \frac{h_2^2}{2} = u_1\{0\} + u_2\{+\rho Q\} + u_3\{-\rho Q\}$$

$$u_2 = V_2 \qquad u_3 = 0$$

$$\frac{\rho g b}{2}(h_1^2 - h_2^2) = V_2 \rho Q = \frac{Q}{bh_2}\rho Q = \frac{\rho Q^2}{bh_2} \; ; \quad h_1 = \sqrt{h_2^2 + \frac{2Q^2}{gb^2 h_2}}$$

h_1

For differential CV shown,

$$0 = \int_{CS} \vec{V} \cdot d\vec{A} = \{-Vbh\} + \left\{-\frac{Q}{bL} b\,dx\right\} + \{+(V+dV)b(h+dh)\}$$

$$0 = -\frac{Q}{L}dx + b(h\,dV + V\,dh) = -\frac{Q}{L}dx + b\,d(hV) \; ; \quad \frac{d(hV)}{dx} = \frac{Q}{L}$$

From momentum,

$$\rho g b \frac{h^2}{2} - \rho g b \frac{(h+dh)^2}{2} = V\{-\rho V b h\} + 0\left\{-\frac{\rho Q}{L}dx\right\} + (V+dV)\{+\rho(V+dV)b(h+dh)\}$$

Using continuity,

$$\frac{\rho g b}{2}(-2h\,dh + \underbrace{dh\,dh}_{\ll dh}) = -\rho V^2 bh + (V+dV)\left\{\rho V b h + \frac{\rho Q}{L}dx\right\}$$

Cont'd. →

$$-\rho g\, b\, h\, dh = -\rho \cancel{V^2} bh + \rho \cancel{V^2} bh + \rho V b h\, dV + \frac{\rho Q}{L} V dx + \frac{\rho Q}{L} dV \underbrace{\cancel{dx}}_{\ll dx}$$

or

$$-g h\, dh = V h\, dV + \frac{Q}{bL} V dx$$

From continuity, $V h\, dV = -V^2 dh + \frac{Q}{bL} V dx$, so

$$-g h\, dh = -V^2 dh + \frac{2Q}{bL} V dx$$

Solving,

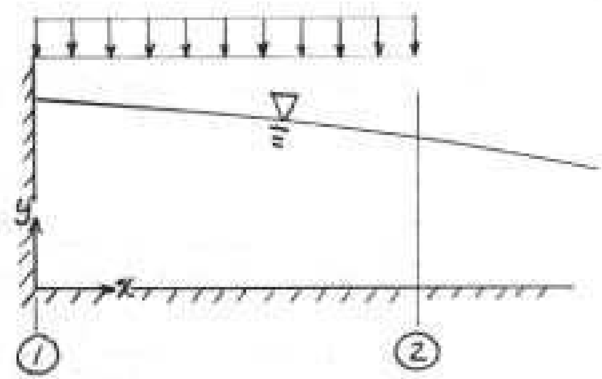
$$\frac{dh}{dx}\left(V^2 - gh\right) = \frac{2Q}{bL} V$$

$$\frac{dh}{dx} = \frac{2QV}{bL(V^2 - gh)} = \frac{2QV}{bLgh\left(V^2/gh - 1\right)}$$

From finite CV analysis, $h_1 > h_2$, so $\frac{dh}{dx} < 0$. Thus $V^2/gh < 1$. As x increases, $V \uparrow$ and $h \downarrow$. Therefore

$$\frac{V^2}{gh} \uparrow, \quad \frac{V}{h} \uparrow, \quad \text{and} \quad \left|\frac{dh}{dx}\right| \uparrow.$$

Sketch:

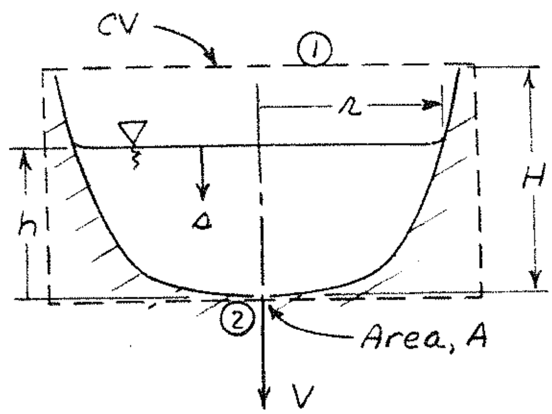


***4.126** Design a clepsydra (Egyptian water clock)—a vessel from which water drains by gravity through a hole in the bottom and which indicates time by the level of the remaining water. Specify the dimensions of the vessel and the size of the drain hole; indicate the amount of water needed to fill the vessel and the interval at which it must be filled. Plot the vessel radius as a function of elevation.

Discussion: The original Egyptian water clock was an open water-filled vessel with an orifice in the bottom. The vessel shape was designed so that the water level dropped at a constant rate during use.

Water leaves the orifice at higher speed when the water level within the vessel is high, and at lower speed when the water level within the vessel is low. The size of the orifice is constant. Thus the instantaneous volume flow rate depends on the water level in the vessel.

The rate at which the water level falls in the vessel depends on the volume flow rate and the area of the water surface. The surface area at each water level must be chosen so that the water level within the vessel decreases at a constant rate. The continuity and Bernoulli equations can be applied to determine the required vessel shape so that the water surface level drops at a constant rate.



Use the CV and notation shown (Problem 4.97):

Solution: Basic equations are

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A}$$

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$$

Assumptions: (1) Quasi-steady flow
(2) Incompressible flow
(3) Uniform flow at each cross-section
(4) Flow along a streamline
(5) No friction
(6) $\rho_{air} \ll \rho_{H_2O}$

Writing Bernoulli from the liquid surface to the jet exit,

$$\cancel{\frac{p_{atm}}{\rho}} + \frac{\dot{s}^2}{2} + gh = \cancel{\frac{p_{atm}}{\rho}} + \frac{V^2}{2} + g(0)$$

For $\dot{s} \ll V$, then $V = \sqrt{2gh}$

For the CV,

$$0 = \frac{\partial}{\partial t}\int_{\forall_{air}} \overset{\ll \rho_{H_2O}(6)}{\cancel{\rho_{air}}}\, d\forall + \frac{\partial}{\partial t}\int_{\forall_{H_2O}} \rho_{H_2O}\, d\forall + \left\{-|\overset{\ll \rho_{H_2O}(6)}{\cancel{\rho_{air}}} V_1 A_1|\right\} + \left\{|\rho_{H_2O} V A|\right\}$$

or $\quad 0 = \rho \dfrac{d\forall}{dt} + \rho V A = \rho \pi r^2 \dfrac{dh}{dt} + \rho \sqrt{2gh}\, A$

But h decreases, so $\dfrac{dh}{dt} = -s$. Thus

$$\pi r^2 s = \sqrt{2gh}\, A \qquad \text{or} \qquad r = \sqrt[4]{2g}\sqrt{\frac{A}{\pi s}}\, h^{1/4}$$ r

For n hours operation, $H = ns$, and

$$\forall = \int_0^H \pi r^2\, dh = \int_0^{ns} \sqrt{2gh}\,\frac{A}{s}\, dh = \frac{2A}{3s}\sqrt{2g}\,(ns)^{3/2}$$

or

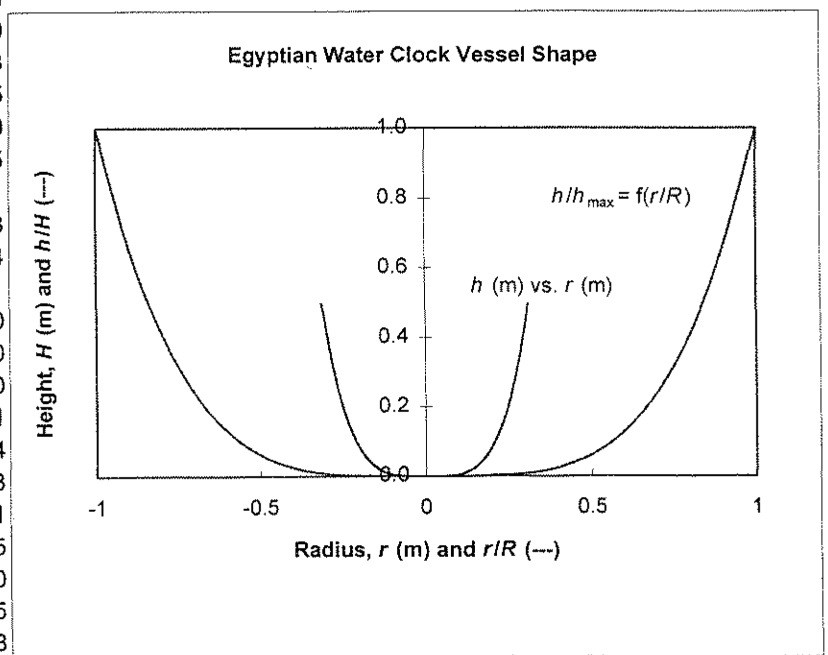
$$\forall = \frac{2A\sqrt{2g}\, n^{3/2}}{3} s^{1/2}$$ $\forall$

Evaluating and plotting:

Input Parameters:

Maximum water height:	H =	0.5	m
Number of hours' duration:	n =	24	hr

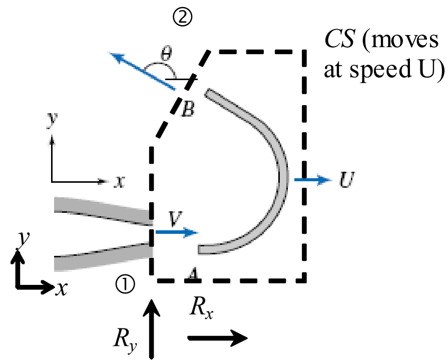
Dimensionless Shape		Actual Shape	
r/R	*h/H*	*r* (m)	*h* (m)
-1	1.00	-0.309	0.500
-0.9	0.656	-0.278	0.328
-0.8	0.410	-0.247	0.205
-0.7	0.240	-0.216	0.120
-0.6	0.130	-0.185	0.065
-0.5	0.063	-0.155	0.031
-0.4	0.026	-0.124	0.013
-0.3	0.008	-0.093	0.004
-0.2	0.002	-0.062	0.001
-0.1	0.000	-0.031	0.000
0	0	0	0
0.1	0.000	0.031	0.000
0.2	0.002	0.062	0.001
0.3	0.008	0.093	0.004
0.4	0.026	0.124	0.013
0.5	0.063	0.155	0.031
0.6	0.130	0.185	0.065
0.7	0.240	0.216	0.120
0.8	0.410	0.247	0.205
0.9	0.656	0.278	0.328
1	1.000	0.309	0.500



Problem *4.127

[Difficulty: 3]

4.127 A jet of water is directed against a vane, which could be a blade in a turbine or in any other piece of hydraulic machinery. The water leaves the stationary 40-mm-diameter nozzle with a speed of 25 m/s and enters the vane tangent to the surface at A. The inside surface of the vane at B makes angle $\theta = 150°$ with the x direction. Compute the force that must be applied to maintain the vane speed constant at $U = 5$ m/s.



Given: Water jet striking moving vane

Find: Force needed to hold vane to speed U = 5 m/s

Solution:

Basic equations: Momentum flux in x and y directions

$$F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\, dV\!\!\!/ + \int_{CS} u\,\rho \vec{V}\cdot d\vec{A} \qquad F_y = F_{S_y} + F_{B_y} = \frac{\partial}{\partial t}\int_{CV} v\,\rho\, dV\!\!\!/ + \int_{CS} v\,\rho \vec{V}\cdot d\vec{A}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) Atmospheric pressure in jet 4) Uniform flow 5) Jet relative velocity is constant

Then $\quad R_x = u_1\cdot\left(-\rho\cdot V_1\cdot A_1\right) + u_2\cdot\left(\rho\cdot V_2\cdot A_2\right) = -(V-U)\cdot[\rho\cdot(V-U)\cdot A] + (V-U)\cdot\cos(\theta)\cdot[\rho\cdot(V-U)\cdot A]$

$$R_x = \rho(V-U)^2\cdot A\cdot(\cos(\theta) - 1) \qquad A = \frac{\pi}{4}\cdot\left(\frac{40}{1000}\cdot m\right)^2 \qquad A = 1.26\times 10^{-3}\,m^2$$

Using given data

$$R_x = 1000\cdot\frac{kg}{m^3}\times\left[(25-5)\cdot\frac{m}{s}\right]^2 \times 1.26\times 10^{-3}\cdot m^2 \times (\cos(150\cdot deg) - 1)\times\frac{N\cdot s^2}{kg\cdot m} \qquad R_x = -940\,N$$

Then $\quad R_y = v_1\cdot\left(-\rho\cdot V_1\cdot A_1\right) + v_2\cdot\left(\rho\cdot V_2\cdot A_2\right) = -0 + (V-U)\cdot\sin(\theta)\cdot[\rho\cdot(V-U)\cdot A]$

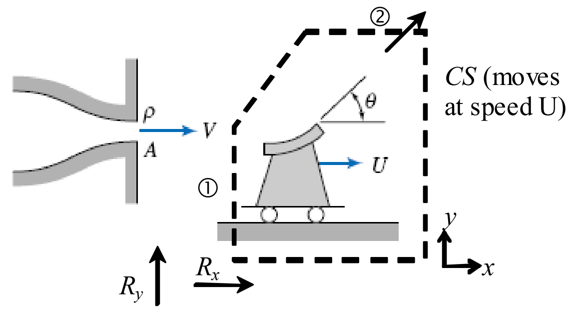
$$R_y = \rho(V-U)^2\cdot A\cdot\sin(\theta) \qquad R_y = 1000\cdot\frac{kg}{m^3}\times\left[(25-5)\cdot\frac{m}{s}\right]^2 \times 1.26\times 10^{-3}\cdot m^2 \times \sin(150\cdot deg)\times\frac{N\cdot s^2}{kg\cdot m} \qquad R_y = 252\,N$$

Hence the force required is 940 N to the left and 252 N upwards to maintain motion at 5 m/s

Problem 4.128

[Difficulty: 3]

4.128 Water from a stationary nozzle impinges on a moving vane with turning angle $\theta = 120°$. The vane moves away from the nozzle with constant speed, $U = 10$ m/s, and receives a jet that leaves the nozzle with speed $V = 30$ m/s. The nozzle has an exit area of 0.004 m^2. Find the force that must be applied to maintain the vane speed constant.



Given: Water jet striking moving vane

Find: Force needed to hold vane to speed U = 10 m/s

Solution:

Basic equations: Momentum flux in x and y directions

$$F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\,dV\!\!\!\!- + \int_{CS} u\,\rho\vec{V}\cdot d\vec{A} \qquad F_y = F_{S_y} + F_{B_y} = \frac{\partial}{\partial t}\int_{CV} v\,\rho\,dV\!\!\!\!- + \int_{CS} v\,\rho\vec{V}\cdot d\vec{A}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) Atmospheric pressure in jet 4) Uniform flow 5) Jet relative velocity is constant

Then $\quad R_x = u_1\cdot(-\rho\cdot V_1\cdot A_1) + u_2\cdot(\rho\cdot V_2\cdot A_2) = -(V-U)\cdot[\rho\cdot(V-U)\cdot A] + (V-U)\cdot\cos(\theta)\cdot[\rho\cdot(V-U)\cdot A]$

$$R_x = \rho(V-U)^2\cdot A\cdot(\cos(\theta)-1)$$

Using given data

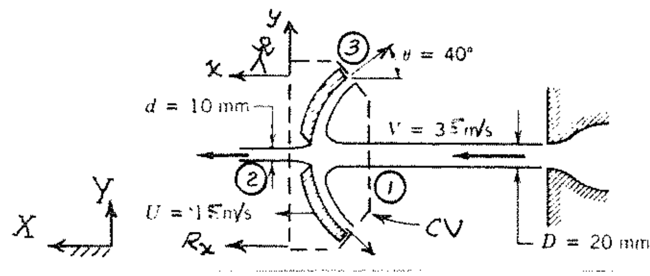
$$R_x = 1000\cdot\frac{kg}{m^3}\times\left[(30-10)\cdot\frac{m}{s}\right]^2\times 0.004\cdot m^2\times(\cos(120\cdot deg)-1)\times\frac{N\cdot s^2}{kg\cdot m} \qquad R_x = -2400\,N$$

Then $\quad R_y = v_1\cdot(-\rho\cdot V_1\cdot A_1) + v_2\cdot(\rho\cdot V_2\cdot A_2) = -0 + (V-U)\cdot\sin(\theta)\cdot[\rho\cdot(V-U)\cdot A]$

$$R_y = \rho(V-U)^2\cdot A\cdot\sin(\theta) \qquad R_y = 1000\cdot\frac{kg}{m^3}\times\left[(30-10)\cdot\frac{m}{s}\right]^2\times 0.004\cdot m^2\times\sin(120\cdot deg)\times\frac{N\cdot s^2}{kg\cdot m} \qquad R_y = 1386\,N$$

Hence the force required is 2400 N to the left and 1390 N upwards to maintain motion at 10 m/s

4.129 The circular dish, whose cross section is shown, has an outside diameter of 0.20 m. A water jet with speed of 35 m/s strikes the dish concentrically. The dish moves to the left at 15 m/s. The jet diameter is 20 mm. The dish has a hole at its center that allows a stream of water 10 mm in diameter to pass through without resistance. The remainder of the jet is deflected and flows along the dish. Calculate the force required to maintain the dish motion.



Solution: Apply continuity and x momentum equation to CV moving with dish as shown.

Basic equations:

$$0 = \underbrace{\frac{\partial}{\partial t}\int_{CV} \rho\, d\forall}_{=0(1)} + \int_{CS} \rho \vec{V}_{xyz} \cdot d\vec{A}$$

$$F_{Sx} + \underbrace{F_{Bx}}_{=0(3)} = \underbrace{\frac{\partial}{\partial t}\int_{CV} u_{xyz}\, \rho\, d\forall}_{=0(1)} + \int_{CS} u_{xyz}\, \rho \vec{V} \cdot d\vec{A}$$

Assumptions: (1) Steady flow w.r.t. CV
(2) No pressure forces on CV
(3) Horizontal; $F_{Bx} = 0$
(4) Uniform flow at each section
(5) No change in speed of jet relative to vane
(6) Incompressible flow

Then

$$0 = \int_{CS} \vec{V}_{xyz} \cdot d\vec{A} = (V-U)\left(-\frac{\pi D^2}{4} + \frac{\pi d^2}{4} + A_{3,4}\right)$$

$$A_{3,4} = \frac{\pi}{4}(D^2 - d^2) = \frac{\pi}{4}\left[(0.020)^2 - (0.010)^2\right] m^2 = 2.36 \times 10^{-4}\, m^2$$

From the momentum equation

$$R_x = u_1\left\{-\rho(V-U)\frac{\pi D^2}{4}\right\} + u_2\left\{+\rho(V-U)\frac{\pi d^2}{4}\right\} + u_3\left\{+\rho(V-U)A_{3,4}\right\}$$

$$u_1 = V - U \qquad u_2 = V - U \qquad u_3 = -(V-U)\cos 40°$$

$$R_x = -\rho(V-U)^2\frac{\pi D^2}{4} + \rho(V-U)^2\frac{\pi d^2}{4} - \rho(V-U)^2\frac{\pi}{4}(D^2 - d^2)\cos 40°$$

$$= -\rho(V-U)^2\frac{\pi}{4}(D^2 - d^2)(1 + \cos 40°)$$

$$= -999\,\frac{kg}{m^3} \times (35 - 15)^2\frac{m^2}{s^2} \times 2.36 \times 10^{-4}\, m^2 (1 + \cos 40°) \times \frac{N \cdot s^2}{kg \cdot m}$$

$R_x = -167$ N (force must be applied to right) R_x

{ Note: $R_y = Mg$, since there is no net momentum flux in the y-direction. }

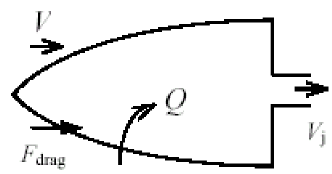
Problem 4.130

[Difficulty: 3]

4.130 A freshwater jet boat takes in water through side vents and ejects it through a nozzle of diameter $D = 75$ mm; the jet speed is V_j. The drag on the boat is given by $F_{drag} \propto kV^2$, where V is the boat speed. Find an expression for the steady speed, V, in terms of water density ρ, flow rate through the system of Q, constant k, and jet speed V_j. A jet speed $V_j = 15$ m/s produces a boat speed of $V = 10$ m/s.
(a) Under these conditions, what is the new flow rate Q?
(b) Find the value of the constant k.
(c) What speed V will be produced if the jet speed is increased to $V_j = 25$ m/s?
(d) What will be the new flow rate?

Given: Data on jet boat

Find: Formula for boat speed; flow rate; value of k; new speed and flow rate



CV in boat coordinates

Solution:

Basic equation:

Momentum $$\vec{F} = \vec{F}_S + \vec{F}_B = \frac{\partial}{\partial t}\int_{CV} \vec{V}_{xyz}\,\rho\, dV\!\!\!\!- + \int_{CS} \vec{V}_{xyz}\,\rho \vec{V}_{xyz}\cdot d\vec{A}$$

Given data $\quad D = 75\cdot mm \quad V_j = 15\cdot\frac{m}{s} \quad V = 10\cdot\frac{m}{s} \quad \rho = 1000\cdot\frac{kg}{m^3}$

Applying the horizontal component of momentum

$$F_{drag} = V\cdot(-\rho\cdot Q) + V_j\cdot(\rho\cdot Q) \qquad \text{or, with} \qquad F_{drag} = k\cdot V^2 \qquad k\cdot V^2 = \rho\cdot Q\cdot V_j - \rho\cdot Q\cdot V$$

$$k\cdot V^2 + \rho\cdot Q\cdot V - \rho\cdot Q\cdot V_j = 0$$

Solving for V $$V = -\frac{\rho\cdot Q}{2\cdot k} + \sqrt{\left(\frac{\rho\cdot Q}{2\cdot k}\right)^2 + \frac{\rho\cdot Q\cdot V_j}{k}} \qquad (1)$$

For the flow rate $$Q = V_j\cdot\frac{\pi}{4}\cdot D^2 \qquad Q = 0.0663\,\frac{m^3}{s}$$

To find k from Eq 1, let $$\alpha = \frac{\rho\cdot Q}{2\cdot k} \qquad \text{then} \qquad V = -\alpha + \sqrt{\alpha^2 + 2\cdot\alpha\cdot V_j}$$

$$(V+\alpha)^2 = V^2 + 2\cdot\alpha\cdot V + \alpha^2 = \alpha^2 + 2\cdot\alpha\cdot V_j \qquad \text{or} \qquad \alpha = \frac{V^2}{2\cdot\left(V_j - V\right)} \qquad \alpha = 10\,\frac{m}{s}$$

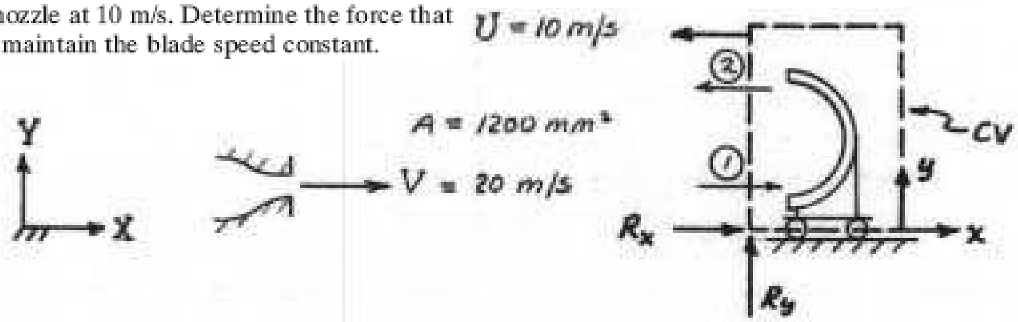
Hence $$k = \frac{\rho\cdot Q}{2\cdot\alpha} \qquad k = 3.31\,\frac{N}{\left(\frac{m}{s}\right)^2}$$

For $$V_j = 25\cdot\frac{m}{s} \qquad Q = V_j\cdot\frac{\pi}{4}\cdot D^2 \qquad Q = 0.11\,\frac{m^3}{s} \qquad V = \left[-\frac{\rho\cdot Q}{2\cdot k} + \sqrt{\left(\frac{\rho\cdot Q}{2\cdot k}\right)^2 + \frac{\rho\cdot Q\cdot V_j}{k}}\right] \qquad V = 16.7\,\frac{m}{s}$$

Problem 4.131

[Difficulty: 2]

4.131 A jet of oil (SG = 0.8) strikes a curved blade that turns the fluid through angle $\theta = 180°$. The jet area is 1200 mm^2 and its speed relative to the stationary nozzle is 20 m/s. The blade moves toward the nozzle at 10 m/s. Determine the force that must be applied to maintain the blade speed constant.



Find: Force needed to maintain vane speed constant.

Solution: Apply x component of momentum equation to moving CV shown.

Basic equation:
$$F_{S_x} + \cancel{F_{B_x}}^{=0(2)} = \cancel{\frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\,dV}^{=0(3)} + \int_{CS} u_{xyz}\,\rho\,\vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions:
(1) No net pressure force on CV; $F_{S_x} = R_x$
(2) $F_{B_x} = 0$
(3) Steady flow
(4) Flow uniform at each section
(5) Jet area and speed relative to vane are constant

The subscript xyz is a reminder that all velocities must be evaluated relative to the CV. Then

$$R_x = u_1\{-|\rho(V+U)A|\} + u_2\{|\rho(V+U)A|\}$$

$$u_1 = V + U \qquad\qquad u_2 = -(V+U)$$

and $R_x = -\rho(V+U)^2A - \rho(V+U)^2A = -2\rho(V+U)^2A = -2\,SG\,\rho_{H_2O}(V+U)^2A$

$$R_x = -2(0.8)\,999\,\frac{kg}{m^3}\,(20+10)^2\frac{m^2}{s^2}\times 1200\,mm^2\times\frac{m^2}{10^6\,mm^2}\times\frac{N\cdot s^2}{kg\cdot m} = -1.73\text{ kN}$$

R_x

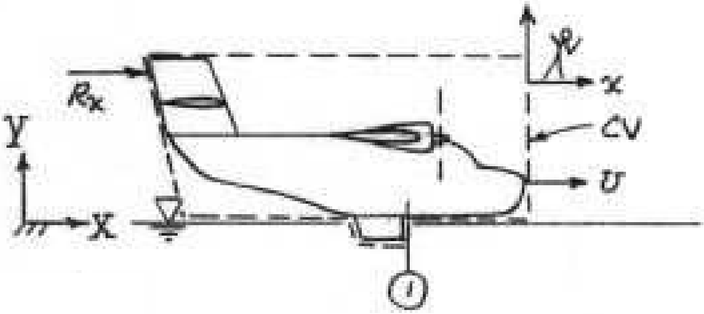
This force must be applied to the left on the vane.

{Note $R_y = mg$, since there are no vertical components of velocity.}

Problem 4.132

[Difficulty: 2]

4.132 The Canadair CL-215T amphibious aircraft is specially designed to fight fires. It is the only production aircraft that can scoop water—1620 gallons in 12 seconds—from any lake, river, or ocean. Determine the added thrust required during water scooping, as a function of aircraft speed, for a reasonable range of speeds.



Solution: Use CV moving with aircraft, as shown. Apply momentum.

Basic equation: $$F_{S_x} + \cancel{F_{B_x}}^{\approx 0(1)} = \cancel{\frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\, d\forall}^{\approx 0(2)} + \int_{CS} u_{xyz}\,\rho \vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: (1) Horizontal motion, so $F_{B_x} = 0$
(2) Neglect u_{xyz} within the CV
(3) Uniform flow at inlet cross-section
(4) Neglect hydrostatic pressure

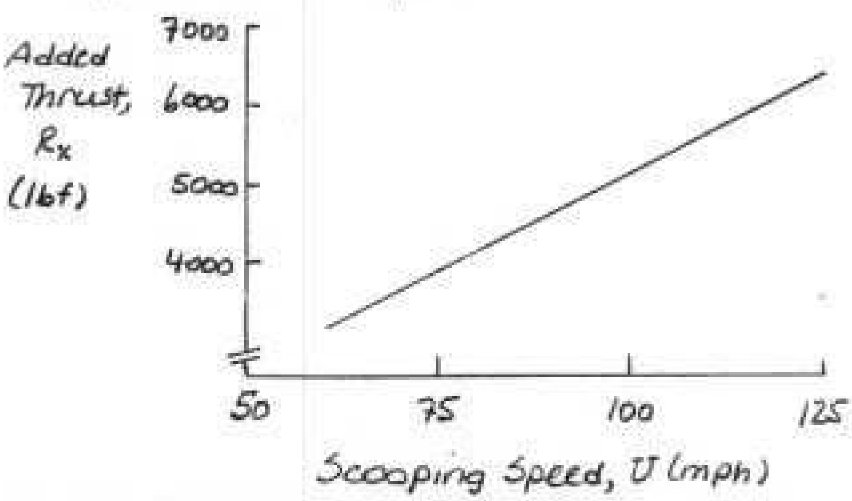
Then
$$R_x = u_1\{-|\rho Q_1|\} = -U(-\rho Q) = +U\rho Q$$
$$u_1 = -U$$

From data given
$$Q = \frac{\Delta\forall}{\Delta t} = \frac{1620\ \text{gal}}{12\ \text{sec}} \times \frac{ft^3}{7.48\ \text{gal}} = 18.0\ ft^3/s$$

For an aircraft speed of $U = 75$ mph (110 ft/s)
$$R_x = 110\,\frac{ft}{s} \times 1.94\,\frac{slug}{ft^3} \times 18.0\,\frac{ft^3}{s} \times \frac{lbf\cdot s^2}{slug\cdot ft} = 3{,}840\ lbf$$

R_x

For a range of aircraft speeds:

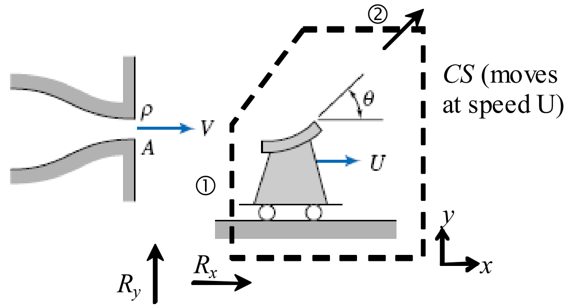


{Thus at 60 mph the added thrust is 3,070 lbf, while at 125 mph the added thrust is 6,400 lbf.}

Problem 4.133

[Difficulty: 3]

4.133 Consider a single vane, with turning angle θ, moving horizontally at constant speed, U, under the influence of an impinging jet as in Problem 4.128. The absolute speed of the jet is V. Obtain general expressions for the resultant force and power that the vane could produce. Show that the power is maximized when $U = V/3$.



Given: Water jet striking moving vane

Find: Expressions for force and power; Show that maximum power is when U = V/3

Solution:

Basic equation: Momentum flux for inertial CV

$$\vec{F} = \vec{F}_S + \vec{F}_B = \frac{\partial}{\partial t}\int_{CV} \vec{V}_{xyz}\,\rho\, d\forall + \int_{CS} \vec{V}_{xyz}\,\rho \vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) Atmospheric pressure in jet 4) Uniform flow
5) Jet relative velocity is constant

Then

$$R_x = u_1\cdot(-\rho\cdot V_1\cdot A_1) + u_2\cdot(\rho\cdot V_2\cdot A_2) = -(V-U)\cdot[\rho\cdot(V-U)\cdot A] + (V-U)\cdot\cos(\theta)\cdot[\rho\cdot(V-U)\cdot A]$$

$$R_x = \rho(V-U)^2\cdot A\cdot(\cos(\theta) - 1)$$

This is force on vane; Force exerted by vane is equal and opposite

$$F_x = \rho\cdot(V-U)^2\cdot A\cdot(1-\cos(\theta))$$

The power produced is then

$$P = U\cdot F_x = \rho\cdot U\cdot(V-U)^2\cdot A\cdot(1-\cos(\theta))$$

To maximize power wrt to U

$$\frac{dP}{dU} = \rho\cdot(V-U)^2\cdot A\cdot(1-\cos(\theta)) + \rho\cdot(2)\cdot(-1)\cdot(V-U)\cdot U\cdot A\cdot(1-\cos(\theta)) = 0$$

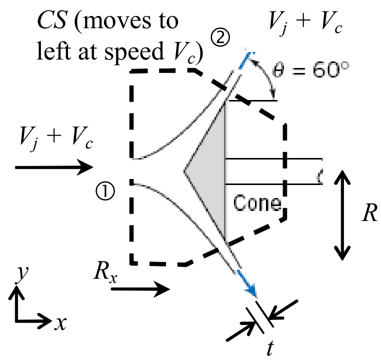
Hence $\quad V - U - 2\cdot U = V - 3\cdot U = 0 \qquad U = \frac{V}{3}\quad$ for maximum power

Note that there is a vertical force, but it generates no power

Problem 4.134

[Difficulty: 3]

4.134 Water, in a 4-in. diameter jet with speed of 100 ft/s to the right, is deflected by a cone that moves to the left at 45 ft/s. Determine (a) the thickness of the jet sheet at a radius of 9 in. and (b) the external horizontal force needed to move the cone.



Given: Water jet striking moving cone

Find: Thickness of jet sheet; Force needed to move cone

Solution:

Basic equations: Mass conservation; Momentum flux in x direction

$$\frac{\partial}{\partial t}\int_{CV}\rho\, dV + \int_{CS}\rho\vec{V}\cdot d\vec{A} = 0 \qquad F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\, dV + \int_{CS} u\,\rho\vec{V}\cdot d\vec{A}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) Atmospheric pressure in jet 4) Uniform flow 5) Jet relative velocity is constant

Then $\quad -\rho\cdot V_1\cdot A_1 + \rho\cdot V_2\cdot A_2 = 0 \qquad -\rho\cdot(V_j + V_c)\cdot\frac{\pi\cdot D_j^2}{4} + \rho\cdot(V_j + V_c)\cdot 2\cdot\pi\cdot R\cdot t = 0$ (Refer to sketch)

Hence $\quad t = \frac{D_j^2}{8\cdot R} \qquad t = \frac{1}{8}\times(4\cdot\text{in})^2\times\frac{1}{9\cdot\text{in}} \qquad t = 0.222\cdot\text{in}$

Using relative velocities, x momentum is

$$R_x = u_1\cdot(-\rho\cdot V_1\cdot A_1) + u_2\cdot(\rho\cdot V_2\cdot A_2) = -(V_j + V_c)\cdot\left[\rho\cdot(V_j + V_c)\cdot A_j\right] + (V_j + V_c)\cdot\cos(\theta)\cdot\left[\rho\cdot(V_j + V_c)\cdot A_j\right]$$

$$R_x = \rho(V_j + V_c)^2\cdot A_j\cdot(\cos(\theta) - 1)$$

Using given data

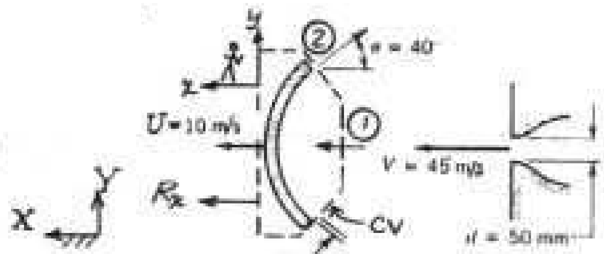
$$R_x = 1.94\cdot\frac{\text{slug}}{\text{ft}^3}\times\left[(100 + 45)\cdot\frac{\text{ft}}{\text{s}}\right]^2\times\frac{\pi\cdot\left(\frac{4}{12}\cdot\text{ft}\right)^2}{4}\times(\cos(60\cdot\text{deg}) - 1)\times\frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}} \qquad R_x = -1780\cdot\text{lbf}$$

Hence the force is 1780 lbf to the left; the upwards equals the weight

Problem 4.135

[Difficulty: 3]

4.135 The circular dish, whose cross section is shown, has an outside diameter of 0.15 m. A water jet strikes the dish concentrically and then flows outward along the surface of the dish. The jet speed is 45 m/s and the dish moves to the left at 10 m/s. Find the thickness of the jet sheet at a radius of 75 mm from the jet axis. What horizontal force on the dish is required to maintain this motion?



t = sheet thickness

Solution: Apply the momentum equation to a CV moving with the dish, as shown.

Basic equation:

$$F_{S_x} + \overset{=0(2)}{\cancel{F_{B_x}}} = \overset{=0(3)}{\cancel{\frac{\partial}{\partial t}\int_{CV} u_{xyz}\rho\, dV}} + \int_{CS} u_{xyz}\,\rho\,\vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: (1) No pressure forces
(2) Horizontal; $F_{B_x} = 0$
(3) Steady flow w.r.t. CV
(4) Uniform flow at each section
(5) Use relative velocities
(6) No change in relative velocity on the dish

Then

$$R_x = u_1\{-\rho(V-U)A\} + u_2\{+\rho(V-U)A\}$$

$$u_1 = V-U \qquad u_2 = -(V-U)\cos\theta$$

$$R_x = -\rho(V-U)^2A - \rho(V-U)^2A\cos\theta = -\rho(V-U)^2A(1+\cos\theta)$$

$$= -999\,\frac{kg}{m^3}\times(45-10)^2\frac{m^2}{s^2}\times\frac{\pi}{4}(0.050)^2 m^2\,(1+\cos 40°)\,\frac{N\cdot s^2}{kg\cdot m}$$

$R_x = -4.24$ kN (force must act to right) $\qquad R_x$

Apply conservation of mass to determine the jet sheet thickness:

Basic equation: $0 = \frac{\partial}{\partial t}\int_{CV}\rho\, dV + \int_{CS}\rho\vec{V}\cdot d\vec{A}$

Using the above assumptions, then

$$0 = -\rho V_1A_1 + \rho V_2A_2$$

$$V_1 = V-U \;;\; V_2 = V-U \;;\; A_1 = \frac{\pi d^2}{4} \;;\; A_2 = 2\pi Rt$$

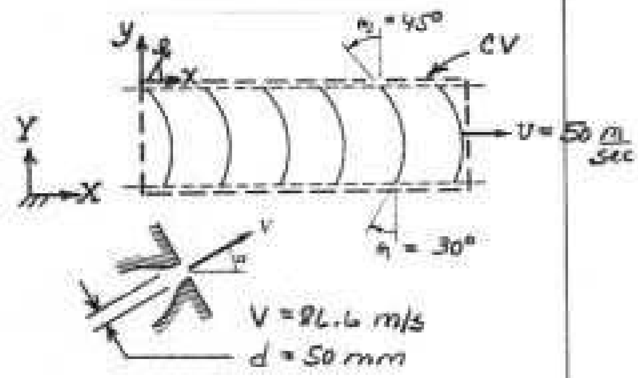
Therefore $A_1 = A_2 = \frac{\pi d^2}{4} = 2\pi Rt$, and $t = \frac{d^2}{8R}$

$t = \frac{1}{8}\times(0.050)^2 m^2 \times \frac{1}{0.075\ m} = 4.17\times10^{-3}$ m or 4.17 mm $\qquad t$

Problem 4.136

[Difficulty: 3]

4.136 Consider a series of turning vanes struck by a continuous jet of water that leaves a 50-mm diameter nozzle at constant speed, $V = 86.6$ m/s. The vanes move with constant speed, $U = 50$ m/s. Note that all the mass flow leaving the jet crosses the vanes. The curvature of the vanes is described by angles $\theta_1 = 30°$ and $\theta_2 = 45°$, as shown. Evaluate the nozzle angle, α, required to ensure that the jet enters tangent to the leading edge of each vane. Calculate the force that must be applied to maintain the vane speed constant.



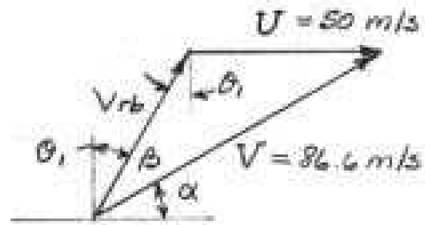
Solution: Apply momentum equation using CV moving with vanes, as shown.

Basic equation:

$$F_{S_x} + \cancel{F_{B_x}}^{=0(2)} = \cancel{\frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\, d\forall}^{=0(3)} + \int_{CS} u_{xyz}\,\rho\,\vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: (1) No pressure forces
(2) Horizontal; $F_{B_x} = 0$
(3) Steady flow w.r.t. CV
(4) Uniform flow at each section
(5) No change in relative velocity on vane
(6) Flow enters and leaves tangent to vanes

The nozzle angle may be obtained from trigonometry. The inlet velocity relationship is shown in the sketch:



From the law of sines,

$$\frac{\sin\alpha}{V_{rb}} = \frac{\sin(90+\theta_1)}{V} = \frac{\sin\beta}{U}$$

$$\beta = \sin^{-1}\left[\frac{U}{V}\sin(90+\theta_1)\right] = \sin^{-1}\left[\frac{50}{86.6}\sin(120°)\right] = 30°$$

From the sketch, $90° = \alpha + \beta + \theta_1$, so $\alpha = 90° - \beta - \theta_1 = 90° - 30° - 30° = 30°$ α

Also $V_{rb}\cos\theta_1 = V\sin\alpha$; $V_{rb} = V\dfrac{\sin\alpha}{\cos\theta_1} = 86.6\,\dfrac{m}{s} \times \dfrac{\sin 30°}{\cos 30°} = 50.0$ m/s

From momentum equation (note all of $\dot{m}$ flows across vanes)

$$R_x = u_1\{-\dot{m}\} + u_2\{\dot{m}\} = V_{rb}\sin\theta_1(-\dot{m}) - V_{rb}\sin\theta_2(\dot{m}) = V_{rb}\dot{m}(-\sin\theta_1 - \sin\theta_2)$$

$u_1 = V_{rb}\sin\theta_1$, $u_2 = -V_{rb}\sin\theta_2$; $R_y = \dot{m}V_{rb}(-\cos\theta_1 + \cos\theta_2)$

Thus, since $\dot{m} = \rho Q$,

$$R_x = V_{rb}\,\rho Q(-\sin\theta_1 - \sin\theta_2)$$

$$= 50\,\frac{m}{s} \times 999\,\frac{kg}{m^3} \times 0.170\,\frac{m^3}{s}(-\sin 30° - \sin 45°) \times \frac{N\cdot s^2}{kg\cdot m}$$

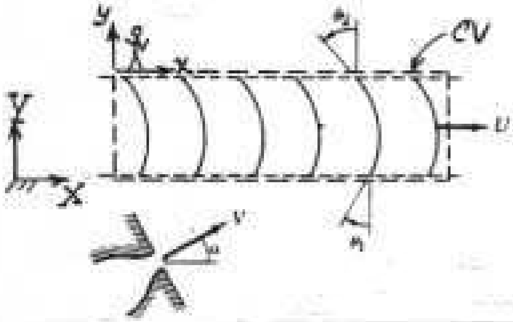
$R_x = -10.3$ kN (to left) R_x

{Note: The net force on the CV in the y-direction is $R_y = -1.35$ kN.}

Problem 4.137

[Difficulty: 3]

4.137 Consider again the moving multiple-vane system described in Problem 4.136. Assuming that a way could be found to make α nearly zero (and thus, θ_1 nearly 90°), evaluate the vane speed, U, that would result in maximum power output from the moving vane system.



Solution: Apply momentum equation using CV moving with vanes, as shown.

Basic equation:

$$F_{S_x} + \underbrace{F_{B_x}}_{=0(2)} = \underbrace{\frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\, d\forall}_{=0(3)} + \int_{CS} u_{xyz}\,\rho \vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: (1) No pressure forces
(2) Horizontal; $F_{B_x} = 0$
(3) Steady flow w.r.t. CV
(4) Uniform flow at each section
(5) No change in relative velocity on vane
(6) Flow enters and leaves tangent to vanes

For $\alpha \approx 0$, $V_{rb} \approx V-U$; the momentum equation becomes

$$R_x = u_1\{-\dot{m}\} + u_2\{+\dot{m}\} = -\dot{m}(V-U) - \dot{m}(V-U)\sin\theta_2 = -\dot{m}(V-U)(1+\sin\theta_2)$$

$$u_1 \approx V_{rb} \approx V-U\,;\; u_2 \approx -V_{rb}\sin\theta_2 \approx -(V-U)\sin\theta_2$$

The vane system produces force, $K_x = -R_x$, and power $\mathcal{P} = K_x U$. Thus

$$\mathcal{P} = K_x U = -R_x U = \dot{m}(V-U)U(1+\sin\theta_2) \qquad (1)$$

To find maximum power, set $\frac{d\mathcal{P}}{dU} = 0$

$$\frac{d\mathcal{P}}{dU} = \dot{m}(-1)U(1+\sin\theta_2) + \dot{m}(V-U)(1)(1+\sin\theta_2) = \dot{m}(V-2U)(1+\sin\theta_2)$$

Thus power is maximized when $V-2U = 0$, or $U = \frac{V}{2}$ (for $\mathcal{P}_{max}$) $\qquad U$

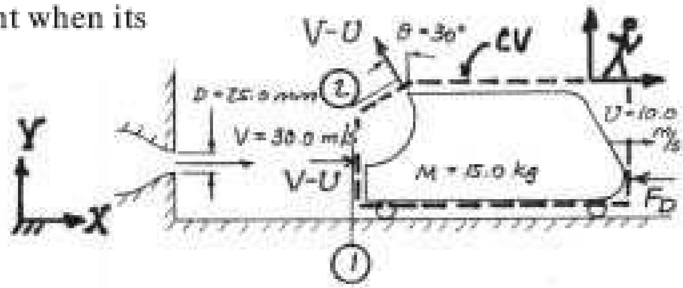
{Note from Eq. 1 that $\theta_2 \to 90°$ increases power also.}

{Note also that $K_y = -R_y = -\dot{m}V_{rb}\cos\theta_2$ but this force does not produce power.}

Problem 4.138

[Difficulty: 2]

4.138 A steady jet of water is used to propel a small cart along a horizontal track as shown. Total resistance to motion of the cart assembly is given by $F_D = kU^2$, where $k = 0.92\ \text{N}\cdot\text{s}^2/\text{m}^2$. Evaluate the acceleration of the cart at the instant when its speed is $U = 10$ m/s.



Solution: Apply the momentum equation using CV and CS shown.

Basic equation: $$F_{S_x} + \underbrace{F_{B_x}}_{=0(2)} - \int_{CV} a_{rf_x} \rho\, d\forall = \underbrace{\frac{\partial}{\partial t}\int_{CV} u_{xyz} \rho\, d\forall}_{\approx 0(3)} + \int_{CS} u_{xyz} \rho \vec{V}\cdot d\vec{A}$$

Assumptions: (1) Only resistance is F_D; $F_{S_x} = -F_D = -kU^2$
(2) Horizontal; $F_{B_x} = 0$
(3) Neglect $\partial/\partial t$ of mass of water in CV
(4) No change in speed w.r.to vane
(5) Uniform flow at each cross-section

Then
$$-kU^2 - a_{rf_x} M_{cv} = u_1\{-\rho(V-U)A\} + u_2\{+\rho(V-U)A\}$$

Measure u w.r.to CV: $\quad u_1 = V-U \qquad u_2 = -(V-U)\sin\theta$

$$-kU^2 - a_{rf_x} M_{cv} = -\rho(V-U)^2 A - \rho(V-U)^2 A\sin\theta = -\rho(V-U)^2 A(1+\sin\theta)$$

so
$$a_{rf_x} = \frac{1}{M}\left[\rho(V-U)^2 A(1+\sin\theta) - kU^2\right]$$

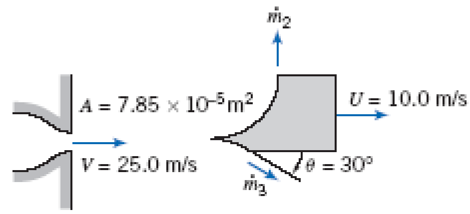
$$= \frac{1}{15\ \text{kg}}\left[999\frac{\text{kg}}{\text{m}^3}(30-10)^2\frac{\text{m}^2}{\text{s}^2}\frac{\pi}{4}(0.025)^2\text{m}^2(1+\sin 30°) - 0.92\frac{\text{N}\cdot\text{s}^2}{\text{m}^2}\times(10)^2\frac{\text{m}^2}{\text{sec}^2}\times\frac{\text{kg}\cdot\text{m}}{\text{N}\cdot\text{s}^2}\right]$$

$a_{rf_x} = 13.5\ \text{m/s}^2$ (to right) $\qquad a_{rf_x}$

Problem 4.139

[Difficulty: 4]

4.139 A plane jet of water strikes a splitter vane and divides into two flat streams, as shown. Find the mass flow rate ratio, $\dot{m}_2/\dot{m}_3$, required to produce zero net vertical force on the splitter vane. If there is a resistive force of 16 N applied to the splitter vane, find the steady speed U of the vane.



Given: Jet impacting a splitter vane

Find: Mass flow rate ratio; new speed U

Solution: Apply momentum equation to inertial CV

Assumptions: No pressure force; neglect water mass on vane; steady flow wrt vane; uniform flow; no change of speed wrt the vane

Basic equation $$\vec{F} = \vec{F}_S + \vec{F}_B = \frac{\partial}{\partial t}\int_{CV} \vec{V}_{xyz}\,\rho\, dV + \int_{CS} \vec{V}_{xyz}\,\rho \vec{V}_{xyz}\cdot d\vec{A}$$

Given data $\quad V = 25\cdot\frac{m}{s} \qquad A = 7.85\cdot 10^{-5}\cdot m^2 \qquad U = 10\cdot\frac{m}{s} \qquad \theta = 30\cdot deg \qquad \rho = 999\cdot\frac{kg}{m^3}$

For constant speed wrt the vane, the jet velocity at each location is $\quad V - U$

For no vertical force, y momentum becomes $\quad 0 = v_1\cdot(-m_1) + v_2\cdot m_2 + v_3\cdot m_3 \quad$ where v_i and m_i are the vertical components of velocity and mass flow rates, respectively, at the inlet and exits, wrt the vane coordinates

Hence $\quad 0 = 0 + (V - U)\cdot m_2 - (V - U)\cdot\sin(\theta)\cdot m_3 \quad$ or $\quad m_2 = m_3\cdot\sin(\theta) \qquad \frac{m_2}{m_3} = \sin(\theta) = \frac{1}{2}$

Note that $\quad m_1 = \rho\cdot A\cdot(V - U) \qquad m_1 = 1.18\,\frac{kg}{s}$

and $\quad m_1 = m_2 + m_3 \quad$ so $\quad m_1 = m_3\cdot\sin(\theta) + m_3 \qquad \frac{m_3}{m_1} = \frac{1}{1+\sin(\theta)} \qquad \frac{m_3}{m_1} = \frac{2}{3} \qquad m_3 = 0.784\,\frac{kg}{s} \qquad m_2 = 0.392\,\frac{kg}{s}$

and using x momentum $\quad R_x = u_1\cdot(-m_1) + u_2\cdot m_2 + u_3\cdot m_3 = (V - U)\cdot(-m_1) + 0 + (V - U)\cdot\cos(\theta)\cdot m_3$

Writing in terms of m_1 $\quad R_x = (V - U)\cdot m_1\cdot\left(\frac{\cos(\theta)}{1+\sin(\theta)} - 1\right) \qquad R_x = -7.46\,N$

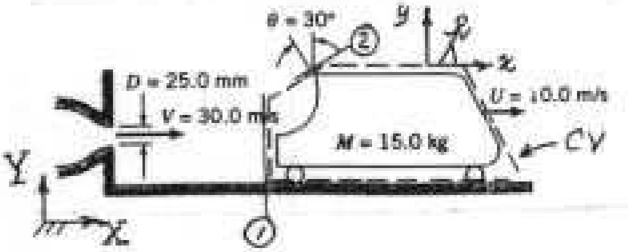
Instead, the force is now $\quad R_x = -16\cdot N \quad$ but $\quad R_x = (V - U)\cdot m_1\cdot\left(\frac{\cos(\theta)}{1+\sin(\theta)} - 1\right) \quad$ and $\quad m_1 = \rho\cdot A\cdot(V - U)$

Hence $\quad R_x = (V - U)^2\cdot\rho\cdot A\cdot\left(\frac{\cos(\theta)}{1+\sin(\theta)} - 1\right)$

Solving for U $\quad U = V - \sqrt{\frac{R_x}{\left[\rho\cdot A\cdot\left(\frac{\cos(\theta)}{1+\sin(\theta)} - 1\right)\right]}} \qquad U = 3.03\,\frac{m}{s}$

Problem 4.140 [Difficulty: 3]

4.140 The hydraulic catapult of Problem 4.138 is accelerated by a jet of water that strikes the curved vane. The cart moves along a level track with negligible resistance. At any time its speed is U. Calculate the time required to accelerate the cart from rest to $U = V/2$.



Solution: Apply x component of momentum equation to accelerating CV.

Basic equation: $$\overset{=0(1)}{\cancel{F_{S_x}}} + \overset{=0(2)}{\cancel{F_{B_x}}} - \int_{CV} a_{rf_x} \rho \, dV = \overset{\approx 0(3)}{\cancel{\frac{\partial}{\partial t}}} \int_{CV} u_{xyz} \rho \, dV + \int_{CS} u_{xyz} \rho \vec{V}_{xyz} \cdot d\vec{A}$$

Assumptions: (1) $F_{S_x} = 0$, since no pressure forces, no resistance
(2) $F_{B_x} = 0$, since horizontal
(3) Neglect mass of water on vane
(4) Uniform flow in jet
(5) No change in relative velocity on vane

Then

$$-a_{rf_x} M_{cv} = u_1\{-\rho(V-U)A\} + u_2\{+\rho(V-U)A\} = -(1+\sin\theta)\rho(V-U)^2 A$$

$$u_1 = V-U \qquad u_2 = -(V-U)\sin\theta$$

so $$\frac{dU}{dt} = \frac{\rho A(1+\sin\theta)}{M}(V-U)^2$$

To integrate, note since V = constant, $d(V-U) = -dU$, so

$$-\int_0^{V/2} \frac{d(V-U)}{(V-U)^2} = \int_0^t \frac{\rho A(1+\sin\theta)}{M} dt$$

or $$\frac{1}{V-U}\Big]_{U=0}^{U=V/2} = \frac{2}{V} - \frac{1}{V} = \frac{1}{V} = \frac{\rho A(1+\sin\theta)}{M} t$$

Thus

$$t = \frac{M}{\rho V A(1+\sin\theta)}$$

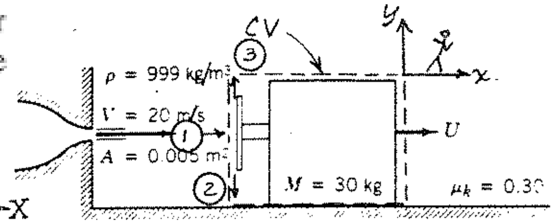
$$= 15.0\,\text{kg} \times \frac{m^3}{999\,\text{kg}} \times \frac{s}{30.0\,m} \times \frac{4}{\pi(0.025)^2\,m^2} \times \frac{1}{(1+\sin 30°)}$$

$$t = 0.680\ \text{s}$$

t

Problem 4.141 [Difficulty: 2]

4.141 A vane/slider assembly moves under the influence of a liquid jet as shown. The coefficient of kinetic friction for motion of the slider along the surface is $\mu_k = 0.30$. Calculate the terminal speed of the slider.



Solution: Apply x momentum equation to linearly accelerating CV.

Basic equation:

$$\cancelto{=0(1)}{F_{S_x}} + \cancelto{}{F_{B_x}} - \int_{CV} a_{rf_x} \rho \, dV = \cancelto{\approx 0(2)}{\frac{\partial}{\partial t}} \int_{CV} u_{xyz} \rho \, dV + \int_{CS} u_{xyz} \rho \vec{V}_{xyz} \cdot d\vec{A}$$

Assumptions: (1) Horizontal motion, so $F_{B_x} = 0$
(2) Neglect mass of liquid on vane, $u \approx 0$ on vane
(3) Uniform flow at each section
(4) Measure velocities relative to CV

Then

$$-Mg\mu_k - a_{rf_x} M = u_1\{-|\rho(V-U)A|\} + u_2\{+\dot{m}_2\} + u_3\{+\dot{m}_3\}$$

$$u_1 = V - U \qquad u_2 = 0 \qquad u_3 = 0$$

$$-Mg\mu_k - M\frac{dU}{dt} = -\rho(V-U)^2 A$$

or

$$\frac{dU}{dt} = \frac{\rho(V-U)^2 A}{M} - g\mu_k$$

At terminal speed, $dU/dt = 0$ and $U = U_t$, so

$$0 = \frac{\rho(V-U_t)^2 A}{M} - g\mu_k \quad \text{or} \quad V - U_t = \sqrt{\frac{Mg\mu_k}{\rho A}}$$

and

$$U_t = V - \sqrt{\frac{Mg\mu_k}{\rho A}}$$

$$= 20\,\frac{m}{s} - \left[30\,kg \times 9.81\,\frac{m}{s^2} \times 0.3 \times \frac{m^3}{999\,kg} \times \frac{1}{0.005\,m^2}\right]^{1/2}$$

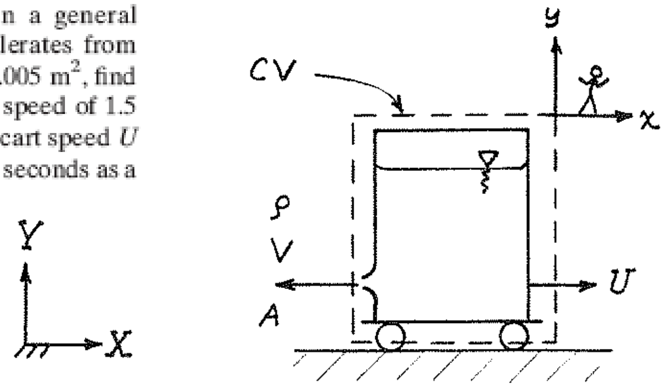
$$U_t = 15.8\ m/s$$

U_t

Problem 4.142

[Difficulty: 3]

4.142 A cart is propelled by a liquid jet issuing horizontally from a tank as shown. The track is horizontal; resistance to motion may be neglected. The tank is pressurized so that the jet speed may be considered constant. Obtain a general expression for the speed of the cart as it accelerates from rest. If $M_0 = 100$ kg, $\rho = 999$ kg/m^3, and $A = 0.005$ m^2, find the jet speed V required for the cart to reach a speed of 1.5 m/s after 30 seconds. For this condition, plot the cart speed U as a function of time. Plot the cart speed after 30 seconds as a function of jet speed.



Solution:

a) Apply x component of momentum equation using linearly accelerating CV shown.

Basic equation:
$$\overset{=0(1)}{F_{S_x}} + \overset{=0(2)}{F_{B_x}} - \int_{CV} a_{rf_x}\rho\, d\forall = \overset{\approx 0(3)}{\frac{\partial}{\partial t}\int_{CV} u_{xyz}\rho\, d\forall} + \int_{CS} u_{xyz}\rho \vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: (1) No resistance
(2) $F_{B_x} = 0$ since track is horizontal
(3) Neglect u_{xyz} within CV
(4) Uniform flow at jet exit

Then
$$-a_{rf_x} M = u\{|\rho V A|\} = -\rho V^2 A$$
$$u = -V$$

From continuity, $M = M_0 - \dot{m}t = M_0 - \rho V A\, t$. Using $a_{rf_x} = \frac{dU}{dt}$,
$$\frac{dU}{dt} = \frac{\rho V^2 A}{M_0 - \rho V A t}$$

Separating variables and integrating,
$$\int_0^U dU = U = \int_0^t \frac{\rho V^2 A\, dt}{M_0 - \rho V A t} = -V \ln(M_0 - \rho V A t)\Big|_0^t = V \ln\left(\frac{M_0}{M_0 - \rho V A t}\right)$$

or
$$\frac{U}{V} = \ln\left(\frac{M_0}{M_0 - \rho V A t}\right)$$

$\frac{U}{V}$

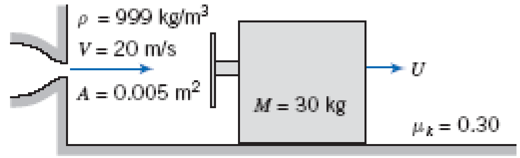
Check dimensions: $[\rho V A t] = \frac{M}{L^3}\frac{L}{t}L^2 t = M$ ✓

b) Using the given data in Excel (with Solver) the jet speed for $U = 1.5$ m/s @ $t = 30$ s is $V = 0.61$ m/s

Problem 4.143

[Difficulty: 4]

4.143 For the vane/slider problem of Problem 4.141, find and plot expressions for the acceleration and speed of the slider as a function of time.



Given: Data on vane/slider

Find: Formula for acceleration and speed; plot

Solution:

The given data is $\rho = 999\cdot\frac{kg}{m^3}$ $\quad M = 30\cdot kg$ $\quad A = 0.005\cdot m^2$ $\quad V = 20\cdot\frac{m}{s}$ $\quad \mu_k = 0.3$

The equation of motion, from Problem 4.141, is $$\frac{dU}{dt} = \frac{\rho\cdot(V-U)^2\cdot A}{M} - g\cdot\mu_k$$

The acceleration is thus $$a = \frac{\rho\cdot(V-U)^2\cdot A}{M} - g\cdot\mu_k$$

Separating variables $$\frac{dU}{\frac{\rho\cdot(V-U)^2\cdot A}{M} - g\cdot\mu_k} = dt$$

Substitute $u = V - U$ $\quad dU = -du$ $$\frac{du}{\frac{\rho\cdot A\cdot u^2}{M} - g\cdot\mu_k} = -dt$$

But $$\int \frac{1}{\left(\frac{\rho\cdot A\cdot u^2}{M} - g\cdot\mu_k\right)}\,du = -\sqrt{\frac{M}{g\cdot\mu_k\cdot\rho\cdot A}}\cdot\operatorname{atanh}\left(\sqrt{\frac{\rho\cdot A}{g\cdot\mu_k\cdot M}}\cdot u\right)$$

and u = V - U so $$-\sqrt{\frac{M}{g\cdot\mu_k\cdot\rho\cdot A}}\cdot\operatorname{atanh}\left(\sqrt{\frac{\rho\cdot A}{g\cdot\mu_k\cdot M}}\cdot u\right) = -\sqrt{\frac{M}{g\cdot\mu_k\cdot\rho\cdot A}}\cdot\operatorname{atanh}\left[\sqrt{\frac{\rho\cdot A}{g\cdot\mu_k\cdot M}}\cdot(V-U)\right]$$

Using initial conditions $$-\sqrt{\frac{M}{g\cdot\mu_k\cdot\rho\cdot A}}\cdot\operatorname{atanh}\left[\sqrt{\frac{\rho\cdot A}{g\cdot\mu_k\cdot M}}\cdot(V-U)\right] + \sqrt{\frac{M}{g\cdot\mu_k\cdot\rho\cdot A}}\cdot\operatorname{atanh}\left(\sqrt{\frac{\rho\cdot A}{g\cdot\mu_k\cdot M}}\cdot V\right) = -t$$

$$V - U = \sqrt{\frac{g\cdot\mu_k\cdot M}{\rho\cdot A}}\cdot\tanh\left(\sqrt{\frac{g\cdot\mu_k\cdot\rho\cdot A}{M}}\cdot t + \operatorname{atanh}\left(\sqrt{\frac{\rho\cdot A}{g\cdot\mu_k\cdot M}}\cdot V\right)\right)$$

$$U = V - \sqrt{\frac{g\cdot\mu_k\cdot M}{\rho\cdot A}}\cdot\tanh\left(\sqrt{\frac{g\cdot\mu_k\cdot\rho\cdot A}{M}}\cdot t + \operatorname{atanh}\left(\sqrt{\frac{\rho\cdot A}{g\cdot\mu_k\cdot M}}\cdot V\right)\right)$$

Note that $$\operatorname{atanh}\left(\sqrt{\frac{\rho\cdot A}{g\cdot\mu_k\cdot M}}\cdot V\right) = 0.213 - \frac{\pi}{2}\cdot i$$

which is complex and difficult to handle in *Excel*, so we use the identity $\quad \operatorname{atanh}(x) = \operatorname{atanh}\left(\frac{1}{x}\right) - \frac{\pi}{2}\cdot i \quad$ for x > 1

so
$$U = V - \sqrt{\frac{g\cdot\mu_k\cdot M}{\rho\cdot A}}\cdot\tanh\left(\sqrt{\frac{g\cdot\mu_k\cdot\rho\cdot A}{M}}\cdot t + \operatorname{atanh}\left(\frac{1}{\sqrt{\frac{\rho\cdot A}{g\cdot\mu_k\cdot M}}\cdot V}\right) - \frac{\pi}{2}\cdot i\right)$$

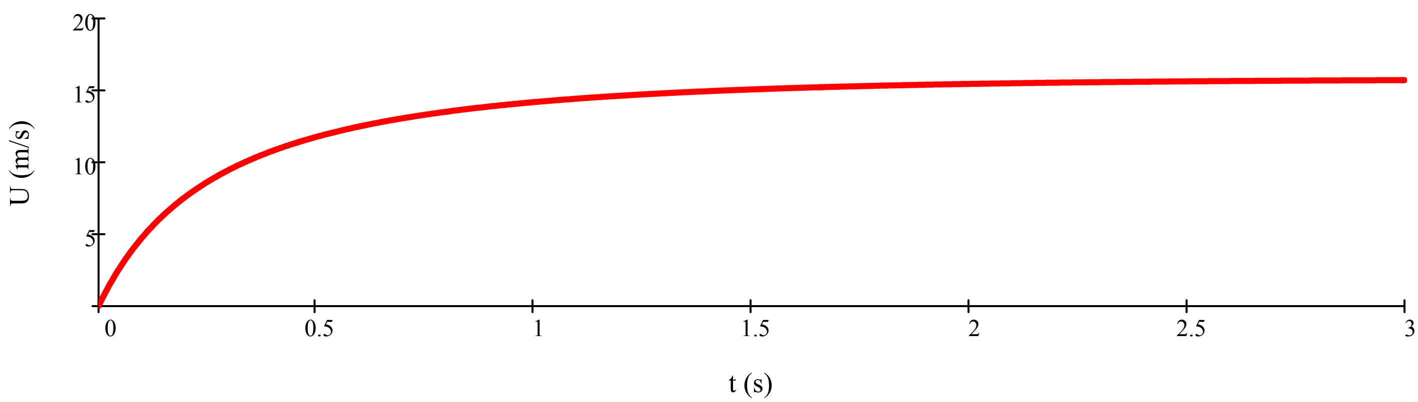
and finally the identity
$$\tanh\left(x - \frac{\pi}{2}\cdot i\right) = \frac{1}{\tanh(x)}$$

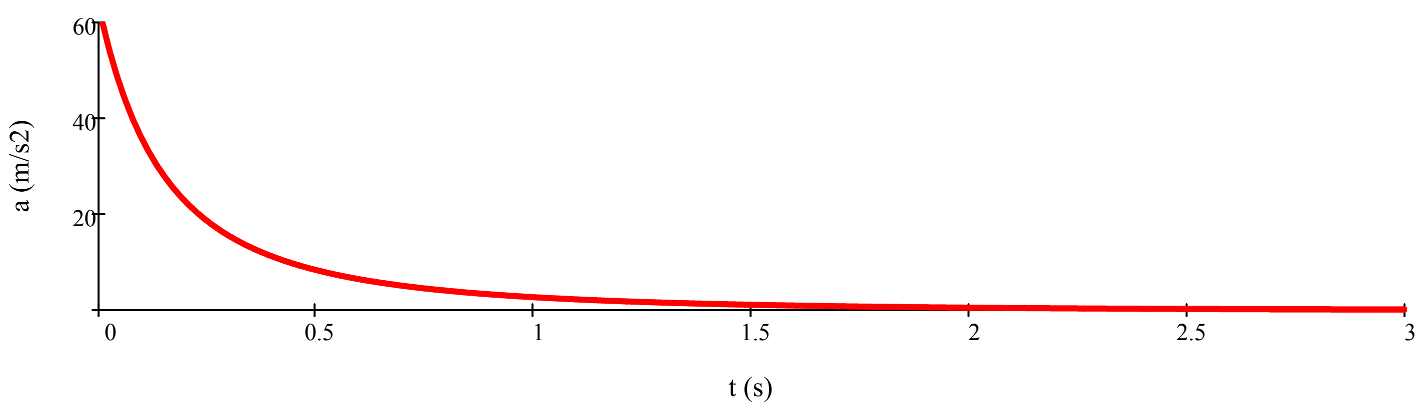
to obtain
$$U(t) = V - \frac{\sqrt{\frac{g\cdot\mu_k\cdot M}{\rho\cdot A}}}{\tanh\left(\sqrt{\frac{g\cdot\mu_k\cdot\rho\cdot A}{M}}\cdot t + \operatorname{atanh}\left(\sqrt{\frac{g\cdot\mu_k\cdot M}{\rho\cdot A}}\cdot\frac{1}{V}\right)\right)}$$

Note that
$$a = \frac{\rho\cdot(V-U)^2\cdot A}{M} - g\cdot\mu_k$$
and
$$V - U = \frac{\sqrt{\frac{g\cdot\mu_k\cdot M}{\rho\cdot A}}}{\tanh\left(\sqrt{\frac{g\cdot\mu_k\cdot\rho\cdot A}{M}}\cdot t + \operatorname{atanh}\left(\sqrt{\frac{g\cdot\mu_k\cdot M}{\rho\cdot A}}\cdot\frac{1}{V}\right)\right)}$$

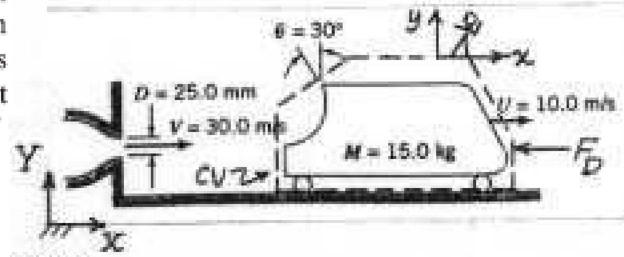
Hence
$$a(t) = \frac{g\cdot\mu_k}{\tanh\left(\sqrt{\frac{g\cdot\mu_k\cdot\rho\cdot A}{M}}\cdot t + \operatorname{atanh}\left(\sqrt{\frac{g\cdot\mu_k\cdot M}{\rho\cdot A}}\cdot\frac{1}{V}\right)\right)^2} - g\cdot\mu_k$$

The plots are presented below





4.144 If the cart of Problem 4.138 is released at $t = 0$, when would you expect the acceleration to be maximum? Sketch what you would expect for the curve of acceleration versus time. What value of θ would maximize the acceleration at any time? Why? Will the cart speed ever equal the jet speed? Explain briefly.



Solution: Apply x component of momentum equation to accelerating CV

Basic equation:
$$\overset{\approx 0(1)}{F_{S_x}} + \overset{=0(2)}{F_{B_x}} - \int_{CV} a_{rf_x}\rho\, d\forall = \overset{\approx 0(3)}{\frac{\partial}{\partial t}\int_{CV} u_{xyz}\rho\, d\forall} + \int_{CS} u_{xyz}\rho \vec{V}_{xyz}\cdot d\vec{A}$$

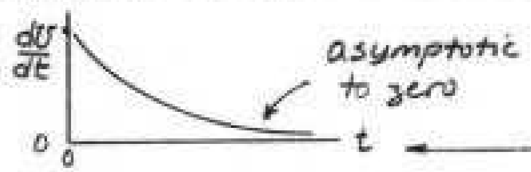
Assumptions:
(1) $F_{S_x} = -F_D = -kU^2$, where $k = 0.92\ \mathrm{N\cdot s^2/m^2}$
(2) $F_{B_x} = 0$, since horizontal
(3) Neglect mass of water on vane
(4) Uniform flow in jet
(5) No change in relative velocity on vane

Then
$$-kU^2 - a_{rf_x}M_{CV} = u_1\{-\rho(V-U)A\} + u_2\{+\rho(V-U)A\} = -(1+\sin\theta)\rho(V-U)^2 A$$
$$u_1 = V-U \qquad u_2 = -(V-U)\sin\theta$$

so
$$\frac{dU}{dt} = \frac{\rho A(1+\sin\theta)}{M}(V-U)^2 - kU^2/M \qquad (1)$$

(a) Acceleration is maximum at $t = 0$, when $U = 0$

(b) Acceleration vs. time will be



(c) From Eq. 1, dU/dt is maximum when $\theta = \pi/2$ and $\sin\theta = 1$

(d) From Eq. 1, $\frac{dU}{dt}$ will go to zero when $U < V$; this will be the terminal speed for the cart, U_t. From Eq. 1, $\frac{dU}{dt} = 0$ when

$$\rho A(1+\sin\theta)(V-U)^2 = kU^2$$

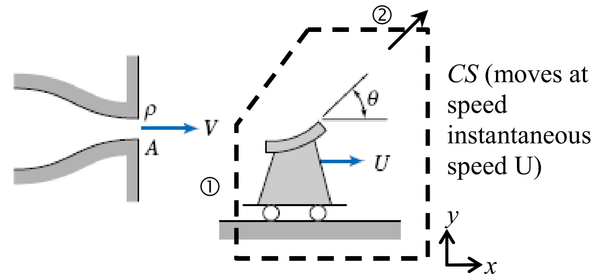
or
$$U = \frac{\left[\frac{\rho A(1+\sin\theta)}{k}\right]^{1/2}}{1+\left[\frac{\rho A(1+\sin\theta)}{k}\right]^{1/2}}V = 0.472V$$

U will be asymptotic to V.

Problem 4.145

[Difficulty: 4]

4.145 The acceleration of the vane/cart assembly of Problem 4.128 is to be controlled as it accelerates from rest by changing the vane angle, θ. A constant acceleration, $a = 1.5\ \text{m/s}^2$, is desired. The water jet leaves the nozzle of area $A = 0.025\ \text{m}^2$, with speed $V = 15$ m/s. The vane/cart assembly has a mass of 55 kg; neglect friction. Determine θ at $t = 5$ s. Plot $\theta(t)$ for the given constant acceleration over a suitable range of t.



Given: Water jet striking moving vane/cart assembly

Find: Angle θ at t = 5 s; Plot θ(t)

Solution:

Basic equation: Momentum flux in x direction for accelerating CV

$$F_{S_x} + F_{B_x} - \int_{CV} a_{rf_x}\,\rho\,d\forall = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\,d\forall + \int_{CS} u_{xyz}\,\rho\vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: 1) No changes in CV 2) Incompressible flow 3) Atmospheric pressure in jet 4) Uniform flow 5) Constant jet relative velocity

Then $\quad -M\cdot a_{rfx} = u_1\cdot\left(-\rho\cdot V_1\cdot A_1\right) + u_2\cdot\left(\rho\cdot V_2\cdot A_2\right) = -(V-U)\cdot[\rho\cdot(V-U)\cdot A] + (V-U)\cdot\cos(\theta)\cdot[\rho\cdot(V-U)\cdot A]$

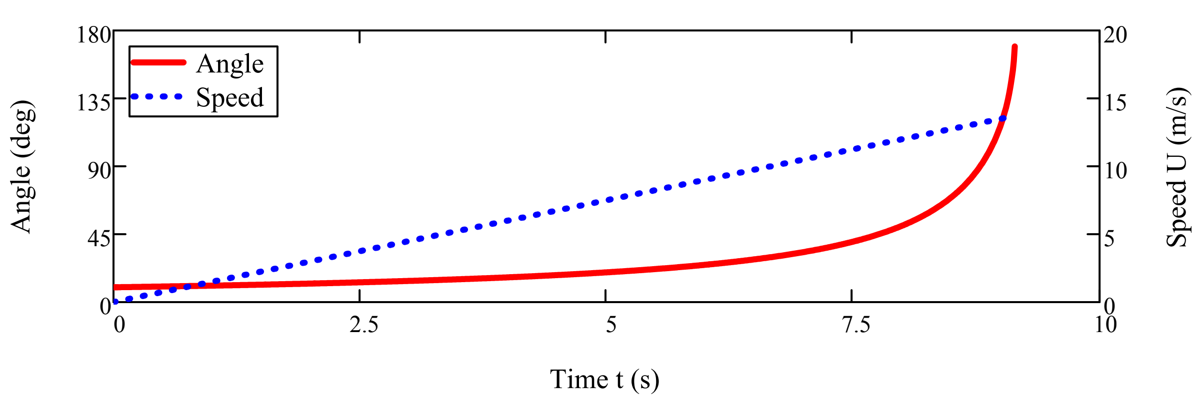
$-M\cdot a_{rfx} = \rho(V-U)^2\cdot A\cdot(\cos(\theta)-1)$ or $\cos(\theta) = 1 - \dfrac{M\cdot a_{rfx}}{\rho\cdot(V-U)^2\cdot A}$

Since $\quad a_{rfx} = \text{constant}$ then $U = a_{rfx}\cdot t \qquad \cos(\theta) = 1 - \dfrac{M\cdot a_{rfx}}{\rho\cdot\left(V - a_{rfx}\cdot t\right)^2\cdot A}$

$$\theta = \operatorname{acos}\left[1 - \frac{M\cdot a_{rfx}}{\rho\cdot\left(V - a_{rfx}\cdot t\right)^2\cdot A}\right]$$

Using given data

$$\theta = \operatorname{acos}\left[1 - 55\cdot\text{kg}\times 1.5\cdot\frac{\text{m}}{\text{s}^2}\times\frac{\text{m}^3}{1000\cdot\text{kg}}\times\frac{1}{\left(15\cdot\frac{\text{m}}{\text{s}} - 1.5\cdot\frac{\text{m}}{\text{s}^2}\times 5\cdot\text{s}\right)^2}\times\frac{1}{0.025\cdot\text{m}^2}\right] \qquad \theta = 19.7\cdot\text{deg} \quad \text{at } t = 5\text{ s}$$

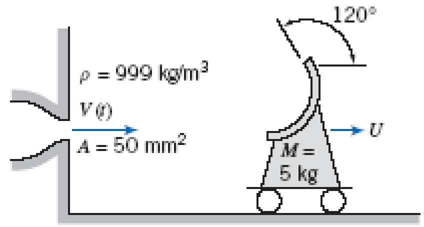


The solution is only valid for θ up to 180° (when t = 9.14 s). This graph can be plotted in *Excel*

Problem 4.146

[Difficulty: 3]

4.146 The wheeled cart shown rolls with negligible resistance. The cart is to accelerate to the right at a constant rate of 2.5 m/s^2. This is to be accomplished by "programming" the water jet speed, $V(t)$, that hits the cart. The jet area remains constant at 50 mm^2. Find the initial jet speed, and the jet speed and cart speeds after 2.5 s and 5 s. Theoretically, what happens to the value of $(V - U)$ over time?



Given: Vaned cart with negligible resistance

Find: Initial jet speed; jet and cart speeds at 2.5 s and 5 s; what happens to V - U?

Solution: Apply x momentum

$$F_{S_x} + F_{B_x} - \int_{CV} a_{rf_x}\,\rho\,dV\!\!\!/ = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\,dV\!\!\!/ + \int_{CS} u_{xyz}\,\rho\vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: 1) All changes wrt CV 2) Incompressible flow 3) Atmospheric pressure in jet 4) Uniform flow 5) Constant jet area

Given data $\rho = 999\cdot\frac{kg}{m^3}$ $M = 5\cdot kg$ $A = 50\cdot mm^2$ $a = 2.5\cdot\frac{m}{s^2}$ $\theta = 120\cdot deg$

Then $-a\cdot M = u_1\cdot[-\rho\cdot(V-U)\cdot A] + u_1\cdot[\rho\cdot(V-U)\cdot A]$ where $u_1 = V - U$ and $u_2 = (V-U)\cdot\cos(\theta)$

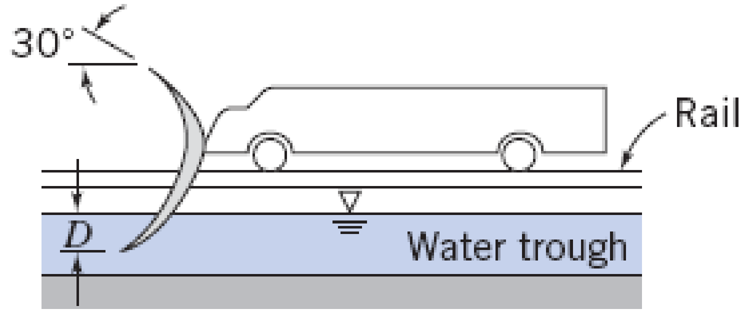
Hence $a\cdot M = \rho\cdot(V-U)^2\cdot(1-\cos(\theta))\cdot A$ From this equation we can see that for constant acceleration V and U must increase at the same rate!

Solving for V $V(t) = a\cdot t + \sqrt{\frac{M\cdot a}{\rho\cdot(1-\cos(\theta))\cdot A}}$

Hence, evaluating $V(0) = 12.9\frac{m}{s}$ $V(2.5\cdot s) = 19.2\frac{m}{s}$ $V(5\cdot s) = 25.4\frac{m}{s}$

Also, for constant acceleration $U(t) = a\cdot t$ so $V - U = \sqrt{\frac{M\cdot a}{\rho\cdot(1-\cos(\theta))\cdot A}} = const!$

4.147 A rocket sled, weighing 10,000 lbf and traveling 600 mph, is to be braked by lowering a scoop into a water trough. The scoop is 6 in. wide. Determine the time required (after lowering the scoop to a depth of 3 in. into the water) to bring the sled to a speed of 20 mph. Plot the sled speed as a function of time.



Solution: Apply x component of momentum equation to linearly accelerating CV. Basic equation is

$$\underbrace{F_{S_x}}_{=0(1)} + \underbrace{F_{B_x}}_{=0(2)} - \int_{CV} a_{rf_x}\,\rho\, d\forall = \underbrace{\frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\, d\forall}_{\approx 0(3)} + \int_{CS} u_{xyz}\,\rho \vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: (1) $F_{S_x} = 0$
(2) $F_{B_x} = 0$
(3) Neglect u_{xyz} and its rate of change in CV
(4) Uniform flow at each section
(5) Speed of water relative to sled is constant

Then

$$-a_{rf_x} M = u_1\{-|\rho U w h|\} + u_2\{|\rho U w h|\}\ ;\ u_1 = U,\ u_2 = -U\cos\theta$$

$$-a_{rf_x}\frac{W}{g} = -\rho U^2 w h(1+\cos\theta),\ \text{or}\quad a_{rf_x} = \frac{\rho g U^2 w h(1+\cos\theta)}{W}$$

Now $a_{rf_x} = -\dfrac{dU}{dt}$, because of coordinate choice. Thus

$$\frac{dU}{U^2} = -\frac{\gamma w h}{W}(1+\cos\theta)\,dt$$

and

$$\int_{U_i}^{U}\frac{dU}{U^2} = -\frac{1}{U} + \frac{1}{U_i} = -\frac{\gamma w h}{W}(1+\cos\theta)\,t \qquad (1)$$

Solving for t,

$$t = \left[\frac{1}{U} - \frac{1}{U_i}\right]\frac{W}{\gamma w h(1+\cos\theta)}$$

$$= \left[\frac{1}{20} - \frac{1}{600}\right]\frac{hr}{mi}\times\frac{mi}{5280\,ft}\times\frac{3600\,s}{hr}\times\frac{ft^3}{62.4\,lbf}\times\frac{1}{6\,in.}\times\frac{1}{3\,in.}\times\frac{144\,in.^2}{ft^2}\times\frac{10{,}000\,lbf}{1+\cos 30^\circ}$$

$$t = 22.6\ s \qquad t$$

The plot is presented on the next page.

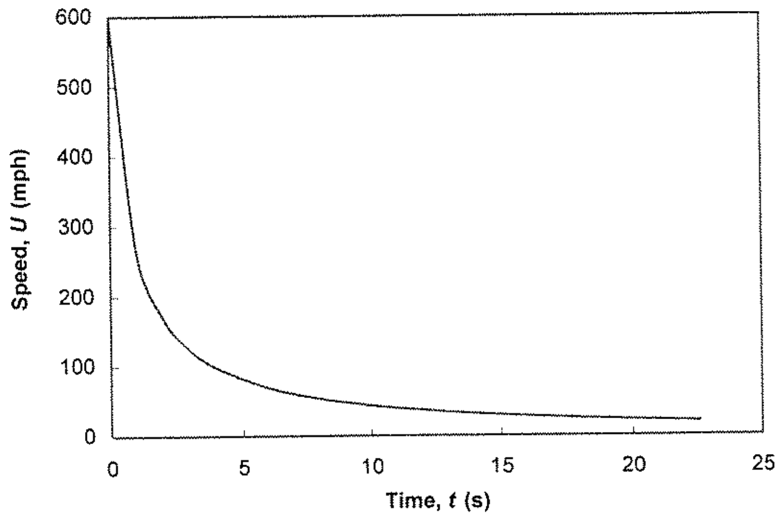
Problem 4.147

Solving Eq. 1 for U,

$$\frac{1}{U} = \frac{1}{U_i} + \frac{\gamma w h}{W}(1+\cos\theta)t = \frac{W + \gamma w h U_i (1+\cos\theta)t}{W U_i}$$

or $$U = \frac{W U_i}{W + \gamma w h U_i (1+\cos\theta)t} \qquad (2)$$

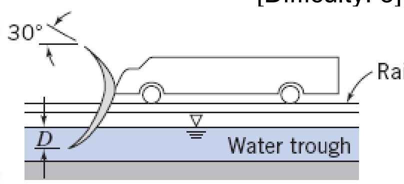
Plotting,

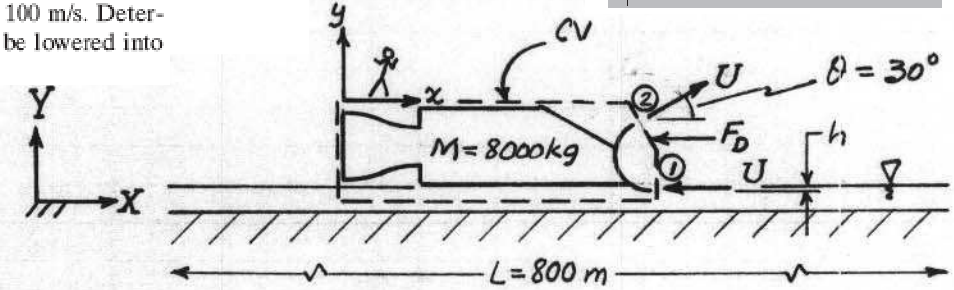


Problem 4.148

[Difficulty: 3]

4.148 A rocket sled is to be slowed from an initial speed of 300 m/s by lowering a scoop into a water trough. The scoop is 0.3 m wide; it deflects the water through 150°. The trough is 800 m long. The mass of the sled is 8000 kg. At the initial speed it experiences an aerodynamic drag force of 90 kN. The aerodynamic force is proportional to the square of the sled speed. It is desired to slow the sled to 100 m/s. Determine the depth D to which the scoop must be lowered into the water.





Solution: Apply x component of momentum equation using linearly accelerating CV shown.

Basic equation: $$F_{S_x} + \overset{=0(1)}{F_{B_x}} - \int_{CV} a_{rf_x}\,\rho\, dV = \overset{\approx 0(2)}{\frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\, dV} + \int_{CS} u_{xyz}\,\rho\,\vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: (1) $F_{B_x} = 0$
(2) Neglect rate of change of u in CV
(3) Uniform flow at each section
(4) No change in relative speed of liquid crossing scoop

Then

$$-F_D - Ma_{rf_x} = u_1\{-|\rho U w h|\} + u_2\{|\rho U w h|\}\ ; \ h = \text{scoop immersion}$$

$$u_1 = -U \qquad u_2 = U\cos\theta$$

But $F_D = kU^2$; $k = \dfrac{F_{D_0}}{U_0^2} = 90\ \text{kN} \times \dfrac{s^2}{(300)^2 m^2} \times \dfrac{10^3 N}{kN} \times \dfrac{kg\cdot m}{N\cdot s^2} = 1.00\ kg/m$

$$-kU^2 - M\frac{dU}{dt} = \rho U^2 w h(1+\cos\theta)\ ,\ \text{since } a_{rf_x} = dU/dt,$$

Thus $$-M\frac{dU}{dt} = [k + \rho w h(1+\cos\theta)]\,U^2 = -MU\frac{dU}{dX}$$

or $$\frac{dU}{U} = -c\,dX\ ,\ \text{where } c = \frac{k + \rho w h(1+\cos\theta)}{M}$$

Integrating, $\ln\dfrac{U}{U_0} = -cX$, so $c = -\dfrac{1}{X}\ln\dfrac{U}{U_0}$

$$c = -\frac{1}{800\,m}\ln\left(\frac{100}{300}\right) = 1.37\times 10^{-3}\,m^{-1}$$

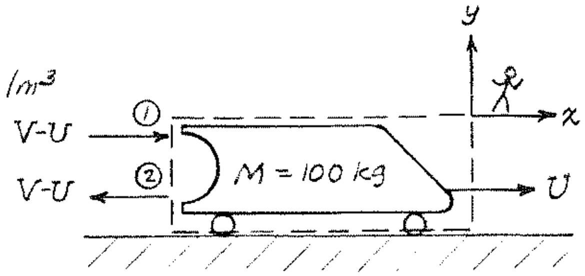
Solving for h, $$h = \frac{Mc - k}{\rho w(1+\cos\theta)}$$

$$h = \left[8000\,kg \times \frac{1.37\times10^{-3}}{m} - 1.00\,\frac{kg}{m}\right]\frac{m^3}{999\,kg} \times \frac{1}{0.3\,m} \times \frac{1}{(1+\cos 30°)} = 0.0179\,m$$

$h = 17.9$ mm ← h

4.149 Starting from rest, the cart shown is propelled by a hydraulic catapult (liquid jet). The jet strikes the curved surface and makes a 180° turn, leaving horizontally. Air and rolling resistance may be neglected. If the mass of the cart is 100 kg and the jet of water leaves the nozzle (of area 0.001 m^2) with a speed of 35 m/s, determine the speed of the cart 5 s after the jet is directed against the cart. Plot the cart speed as a function of time.

:g /m³



Solution: Apply x component of momentum equation using the linearly accelerating CV shown above.

Basic equation: $\overset{=0(1)}{F_{S_x}} + \overset{=0(2)}{F_{B_x}} - \int_{CV} a_{rf_x}\rho\, d\forall = \overset{\approx 0(3)}{\frac{\partial}{\partial t}\int_{CV} u_{xyz}\rho\, d\forall} + \int_{CS} u_{xyz}\rho \vec{V}_{xyz}\cdot d\vec{A}$

Assumptions: (1) $F_{S_x} = 0$
(2) $F_{B_x} = 0$
(3) Neglect mass of liquid and rate of change of u in CV
(4) Uniform flow at each section
(5) Jet area and speed with respect to vehicle are constant

Then

$$-Ma_{rf_x} = -M\frac{dU}{dt} = u_1\{-|\rho(V-U)A|\} + u_2\{|\rho(V-U)A|\}$$

$$u_1 = V-U \qquad u_2 = -(V-U)$$

or

$$\frac{dU}{dt} = \frac{2\rho(V-U)^2A}{M}$$

Note that $dU = -d(V-U)$, and separate variables to obtain

$$-\frac{d(V-U)}{(V-U)^2} = \frac{2\rho A}{M}\,dt$$

Integrate from $U=0$ at $t=0$ to U at t,

$$\int_{V-U=V}^{V-U} -\frac{d(V-U)}{(V-U)^2} = \left.\frac{1}{V-U}\right]_V^{V-U} = \frac{1}{V-U} - \frac{1}{V} = \frac{V-(V-U)}{V(V-U)} = \frac{U}{V(V-U)} = \frac{2\rho A}{M}t$$

Solving,

$$U = (V-U)\frac{2\rho VA}{M}t \quad \text{or} \quad U = V\left[\frac{\frac{2\rho VA}{M}t}{1+\frac{2\rho VA}{M}t}\right] \qquad (1)$$

For the given conditions at $t = 5$ s,

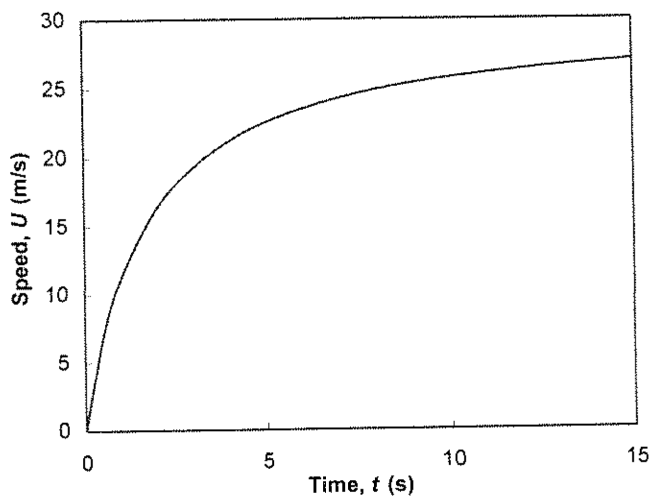
$$\frac{2\rho VA}{M}t = 2\times 999\frac{kg}{m^3}\times 30\frac{m}{s}\times 0.001\,m^2\times 5\,s \times\frac{1}{100\,kg} = 3.00$$

$$U = 30\frac{m}{s}\left[\frac{3.00}{1+3.00}\right] = 22.5\ m/s \qquad U$$

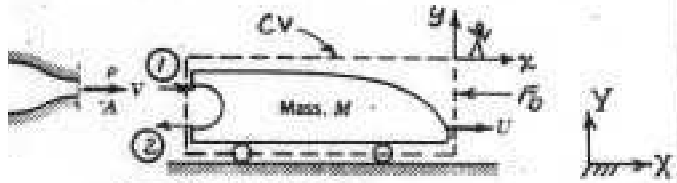
The plot is on the next page.

Problem 4.149

The speed vs. time plot is



4.150 Consider the jet and cart of Problem 4.149 again, but include an aerodynamic drag force proportional to the square of cart speed, $F_D = kU^2$, with $k = 2.0\ \text{N}\cdot\text{s}^2/\text{m}^2$. Derive an expression for the cart acceleration as a function of cart speed and other given parameters. Evaluate the acceleration of the cart at $U = 10$ m/s. What fraction is this speed of the terminal speed of the cart?



$\rho = 999\ \text{kg/m}^3$

$V = 30\ \text{m/s}$ $\quad M = 100\ \text{kg}$

$A = 0.001\ \text{m}^2$

Solution: Apply x momentum for CV with linear acceleration.

Basic equation:

$$F_{S_x} + F_{B_x} - \int_{CV} a_{rf_x}\rho\, d\forall = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\rho\, d\forall + \int_{CS} u_{xyz}\rho \vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: (1) Horizontal, $F_{B_x} = 0$
(2) Neglect mass of liquid in CV (components of u cancel)
(3) Uniform flow at each section
(4) Measure all velocities relative to the CV
(5) No change in stream area or speed on vane

Then

$$-kU^2 - a_{rf_x}M = u_1\{-|\rho(V-U)A|\} + u_2\{+|\rho(V-U)A|\} = -2\rho(V-U)^2A$$

$$u_1 = V - U \qquad u_2 = -(V-U)$$

or

$$a_{rf_x} = \frac{dU}{dt} = \frac{2\rho(V-U)^2A - kU^2}{M}$$ $a(U)$

At $U = 10$ m/sec

$$a_{rf_x} = \frac{2\times 999\frac{\text{kg}}{\text{m}^3}(30-10)^2\frac{\text{m}^2}{\text{s}^2}\times 0.001\,\text{m}^2 - 2.0\frac{\text{N}\cdot\text{s}^2}{\text{m}^2}\times(10)^2\frac{\text{m}^2}{\text{s}^2}\times\frac{\text{kg}\cdot\text{m}}{\text{N}\cdot\text{s}^2}}{100\,\text{kg}} = 5.99\ \frac{\text{m}}{\text{s}^2}$$ a_{rf_x}

At terminal speed, $a_{rf_x} = 0$. Then $2\rho(V-U_t)^2A = kU_t^2$, or

$$V - U_t = U_t\sqrt{\frac{k}{2\rho A}}$$

Solving, $$U_t = \frac{V}{1+\sqrt{k/2\rho A}}$$

$$U_t = 30\frac{\text{m}}{\text{s}}\times\frac{1}{1+\left[\frac{1}{2}\times 2.0\frac{\text{N}\cdot\text{s}^2}{\text{m}^2}\cdot\frac{\text{m}^3}{999\,\text{kg}}\cdot\frac{1}{0.001\,\text{m}^2}\times\frac{\text{kg}\cdot\text{m}}{\text{N}\cdot\text{s}^2}\right]^{1/2}} = 15.0\ \text{m/s}$$

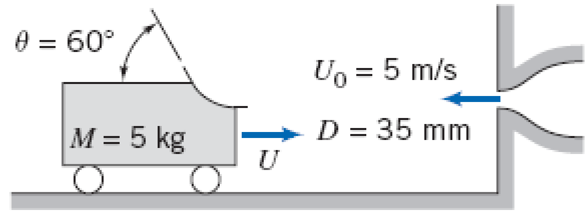
Finally,

$$\text{Fraction} = \frac{U}{U_t} = 10.0\frac{\text{m}}{\text{s}}\times\frac{\text{s}}{15.0\,\text{m}} = 0.667$$ Fraction

Problem 4.151

[Difficulty: 4]

4.151 A small cart that carries a single turning vane rolls on a level track. The cart mass is $M = 5$ kg and its initial speed is $U_0 = 5$ m/s. At $t = 0$, the vane is struck by an opposing jet of water, as shown. Neglect any external forces due to air or rolling resistance. Determine the jet speed V required to bring the cart to rest in (a) 1 s and (b) 2 s. In each case find the total distance traveled.



Given: Vaned cart being hit by jet

Find: Jet speed to stop cart in 1s and 2 s; distance traveled

Solution: Apply x momentum

$$F_{S_x} + F_{B_x} - \int_{CV} a_{rf_x}\,\rho\, dV = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\, dV + \int_{CS} u_{xyz}\,\rho \vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: 1) All changes wrt CV 2) Incompressible flow 3) Atmospheric pressure in jet 4) Uniform flow 5) Constant jet area

Given data $\quad \rho = 999\cdot\frac{kg}{m^3} \quad M = 5\cdot kg \quad D = 35\cdot mm \quad \theta = 60\cdot deg \quad U_0 = 5\cdot\frac{m}{s}$

$$A = \frac{\pi}{4}\cdot D^2 \qquad A = 962\cdot mm^2$$

Then $\quad -a_{rf}\cdot M = u_1\cdot[-\rho\cdot(V+U)\cdot A] + u_2\cdot[\rho\cdot(V+U)\cdot A]$

where $\quad a_{rf} = \frac{dU}{dt} \qquad u_1 = -(V+U) \qquad$ and $\qquad u_2 = -(V+U)\cdot\cos(\theta)$

Hence $\quad -\frac{dU}{dt}\cdot M = \rho\cdot(V+U)^2\cdot A - \rho\cdot(V+U)^2\cdot A\cdot\cos(\theta) = \rho\cdot(V+U)^2\cdot A\cdot(1-\cos(\theta))$

or $\quad -\frac{dU}{dt}\cdot M = \rho\cdot(V+U)^2\cdot A\cdot(1-\cos(\theta)) \qquad (1)$

Note that V is constant, so dU = d(V+U), separating variables $\quad -\frac{d(V+U)}{(V+U)^2} = \frac{\rho\cdot(1-\cos(\theta))\cdot A}{M}\cdot dt$

Integrating from U_0 at t = 0 to U = 0 at t $\quad \frac{1}{V} - \frac{1}{V+U_0} = \frac{\rho\cdot(1-\cos(\theta))\cdot A}{M}\cdot t$

Solving for V $\quad \frac{U_0}{V\cdot\left(V+U_0\right)} = \frac{\rho\cdot(1-\cos(\theta))\cdot A\cdot t}{M} \qquad$ or $\qquad V^2 + V\cdot U_0 - \frac{M\cdot U_0}{\rho\cdot(1-\cos(\theta))\cdot A\cdot t}$

Hence $\quad V = -\frac{U_0}{2} + \sqrt{\frac{U_0^{\,2}}{4} + \frac{U_0\cdot M}{\rho\cdot A\cdot(1-\cos(\theta))\cdot t}}$

To find distances note that $\frac{dU}{dt} = \frac{dU}{dx} \cdot \frac{dx}{dt} = U \cdot \frac{dU}{dx}$

so Eq. 1 can be rewritten as $-U \cdot \frac{dU}{dx} \cdot M = \rho \cdot (V + U)^2 \cdot A \cdot (1 - \cos(\theta))$

Separating variables $\frac{U \cdot dU}{(V + U)^2} = -\frac{\rho \cdot A \cdot (1 - \cos(\theta))}{M} \cdot dx$

It can be shown that $\int_{U_0}^{0} \frac{U}{(V + U)^2} \, dU = \ln\left(\frac{V}{V + U_0}\right) + \frac{V}{V} - \frac{V}{V + U_0}$ (Remember that V is constant)

$$\ln\left(\frac{V}{V + U_0}\right) + 1 - \frac{V}{V + U_0} = -\frac{\rho \cdot A \cdot (1 - \cos(\theta))}{M} \cdot x$$

Solving for x $x = -\frac{M}{\rho \cdot A \cdot (1 - \cos(\theta))} \cdot \left(\ln\left(\frac{V}{V + U_0}\right) + 1 - \frac{V}{V + U_0}\right)$

Substituting values:

To stop in $t = 1 \cdot s$ $V = -\frac{U_0}{2} + \sqrt{\frac{U_0^2}{4} + \frac{U_0 \cdot M}{\rho \cdot A \cdot (1 - \cos(\theta)) \cdot t}}$ $V = 5.13 \frac{m}{s}$

and $x = -\frac{M}{\rho \cdot A \cdot (1 - \cos(\theta))} \cdot \left(\ln\left(\frac{V}{V + U_0}\right) + 1 - \frac{V}{V + U_0}\right)$ $x = 1.94\,m$

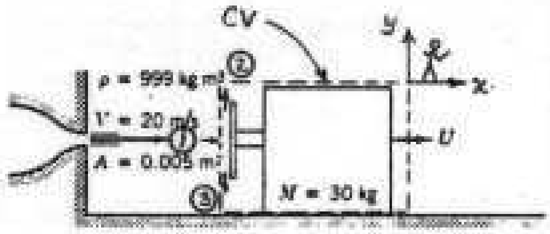
To stop in $t = 2 \cdot s$ $V = -\frac{U_0}{2} + \sqrt{\frac{U_0^2}{4} + \frac{U_0 \cdot M}{\rho \cdot A \cdot (1 - \cos(\theta)) \cdot t}}$ $V = 3.18 \frac{m}{s}$

and $x = -\frac{M}{\rho \cdot A \cdot (1 - \cos(\theta))} \cdot \left(\ln\left(\frac{V}{V + U_0}\right) + 1 - \frac{V}{V + U_0}\right)$ $x = 3.47\,m$

Problem 4.152

[Difficulty: 3]

4.152 Solve Problem 4.141 if the vane and slider ride on a film of oil instead of sliding in contact with the surface. Assume motion resistance is proportional to speed, $F_R = kU$, with $k = 7.5\ \text{N}\cdot\text{s/m}$.



Solution: Apply x momentum equation to linearly accelerating CV.

Basic equation:

$$\cancel{F_{S_x}}^{=0(1)} + F_{B_x} - \int_{CV} a_{rf_x}\rho\, d\forall = \cancel{\frac{\partial}{\partial t}\int_{CV} u_{xyz}\rho\, d\forall}^{\approx 0(2)} + \int_{CS} u_{xyz}\rho \vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: (1) Horizontal, so $F_{B_x} = 0$
(2) Neglect mass of liquid on vane, $u \approx 0$ on vane
(3) Uniform flow at each section
(4) Measure velocities relative to CV

Then

$$-kU - a_{rf_x}M = u_1\{-|\rho(V-U)A|\} + u_2\{+\dot{m}_2\} + u_3\{+\dot{m}_3\}$$

$$u_1 = V-U \qquad u_2 = 0 \qquad u_3 = 0$$

$$-kU - M\frac{dU}{dt} = -\rho(V-U)^2A$$

or

$$\frac{dU}{dt} = \frac{\rho(V-U)^2A}{M} - \frac{kU}{M}$$

$$= 999\frac{kg}{m^3}(20-10)^2\frac{m^2}{s^2}\times 0.005\, m^2 \times \frac{1}{30\, kg} - 7.5\frac{N\cdot s}{m}\times 10\frac{m}{s}\times\frac{1}{30\, kg}\times\frac{kg\cdot m}{N\cdot s^2}$$

$$\frac{dU}{dt} = 14.2\ m/s^2 \qquad (\text{at } U = 10\ m/s)$$ $\frac{dU}{dt}$

At terminal speed, $U = U_t$ and $dU/dt = 0$, so

$$0 = \frac{\rho(V-U)^2A}{M} - \frac{kU}{M} \quad \text{or} \quad V^2 - 2UV + U^2 - \frac{k}{\rho A}U = 0$$

$$U^2 - \left(2V + \frac{k}{\rho A}\right)U + V^2 = 0$$

$$U = \frac{2V + k/\rho A \pm\sqrt{(2V + k/\rho A)^2 - 4V^2}}{2} = V\left\{\left(1 + \frac{k}{2\rho VA}\right) \pm\sqrt{\left(1 + \frac{k}{2\rho VA}\right)^2 - 1}\right\}$$

$$1 + \frac{k}{2\rho VA} = 1 + \frac{1}{2}\times 7.5\frac{N\cdot s}{m}\times\frac{m^3}{999\, kg}\times\frac{s}{20\, m}\times\frac{1}{0.005\, m^2}\times\frac{kg\cdot m}{N\cdot s^2} = 1.0375$$

$$U = V\left\{1.0375 \pm\sqrt{(1.0375)^2 - 1}\right\} = 0.761\, V = 0.761\times 20\frac{m}{s} = 15.2\ m/s$$ U_t

{The negative root was chosen so $U_t < V$, as required.}

Problem 4.153

[Difficulty: 4]

4.153 For the vane/slider problem of Problem 4.152, plot the acceleration, speed, and position of the slider as functions of time. (Consider numerical integration.)

Given: Data on vane/slider

Find: Formula for acceleration, speed, and position; plot

Solution: Apply x momentum

$$F_{S_x} + F_{B_x} - \int_{CV} a_{rf_x}\,\rho\,dV = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\,dV + \int_{CS} u_{xyz}\,\rho\vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: 1) All changes wrt CV 2) Incompressible flow 3) Atmospheric pressure in jet 4) Uniform flow 5) Constant jet area

The given data is $\rho = 999\cdot\frac{kg}{m^3}$ $\quad M = 30\cdot kg$ $\quad A = 0.005\cdot m^2$ $\quad V = 20\cdot\frac{m}{s}$ $\quad k = 7.5\cdot\frac{N\cdot s}{m}$

Then $\quad -k\,U - M\cdot a_{rf} = u_1\cdot[-\rho\cdot(V-U)\cdot A] + u_2\cdot m_2 + u_3\cdot m_3$

where $\quad a_{rf} = \frac{dU}{dt} \qquad u_1 = V - U \qquad u_2 = 0 \qquad u_3 = 0$

Hence $\quad -k\cdot U - M\cdot\frac{dU}{dt} = -\rho\cdot(V-U)^2\cdot A$

or $\quad \frac{dU}{dt} = \frac{\rho\cdot(V-U)^2\cdot A}{M} - \frac{k\cdot U}{M}$

The acceleration is thus $\quad a = \frac{\rho\cdot(V-U)^2\cdot A}{M} - \frac{k\cdot U}{M}$

The differential equation for *U* can be solved analytically, but is quite messy. Instead we use a simple numerical method - Euler's method

$$U(n+1) = U(n) + \left[\frac{\rho\cdot(V-U(n))^2\cdot A}{M} - \frac{k\cdot U(n)}{M}\right]\cdot\Delta t$$

where Δt is the time step

For the position x $\quad \frac{dx}{dt} = U$

so $\quad x(n+1) = x(n) + U(n)\cdot\Delta t$

The final set of equations is

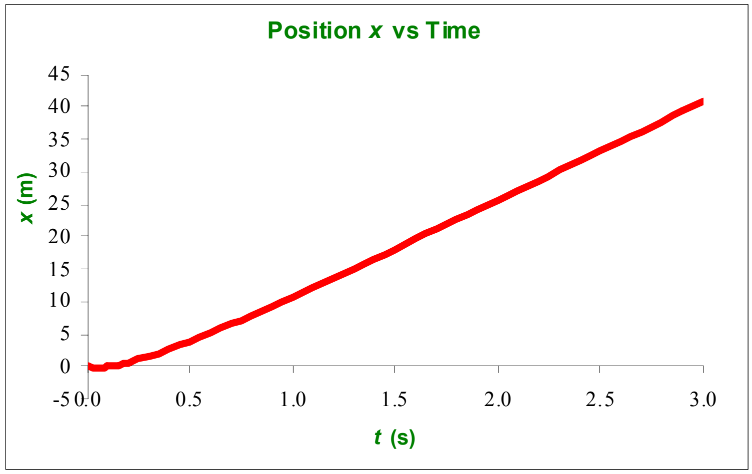
$$U(n+1) = U(n) + \left[\frac{\rho\cdot(V-U(n))^2\cdot A}{M} - \frac{k\cdot U(n)}{M}\right]\cdot\Delta t$$

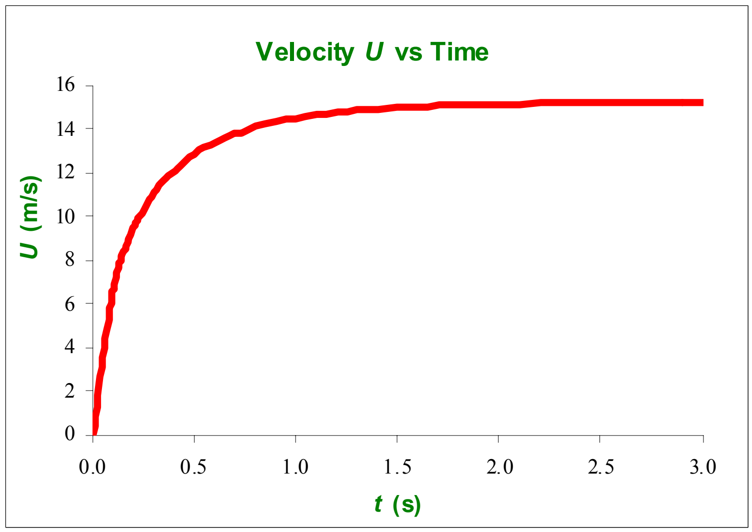
$$a(n) = \frac{\rho\cdot(V-U(n))^2\cdot A}{M} - \frac{k\cdot U(n)}{M}$$

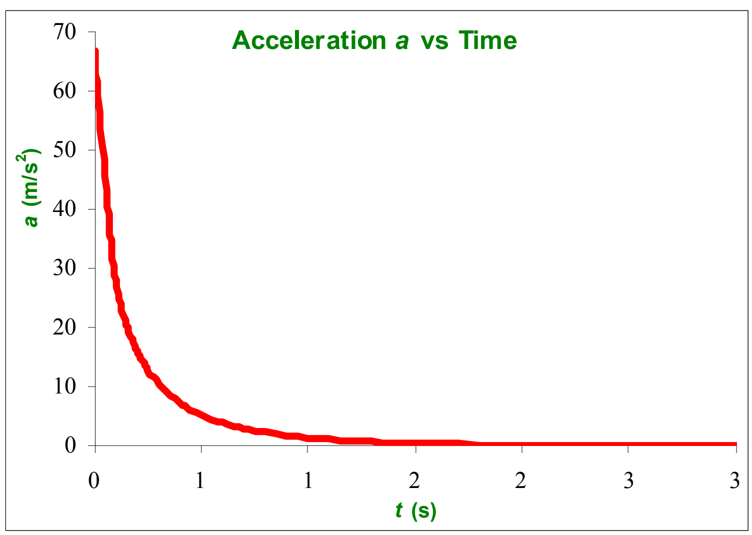
$$x(n+1) = x(n) + U(n)\cdot\Delta t$$

The results can be plotted in *Excel*

t (s)	x (m)	U (m/s)	a (m/s^2)
0.0	0.0	0.0	66.6
0.1	0.0	6.7	28.0
0.2	0.7	9.5	16.1
0.3	1.6	11.1	10.5
0.4	2.7	12.1	7.30
0.5	3.9	12.9	5.29
0.6	5.2	13.4	3.95
0.7	6.6	13.8	3.01
0.8	7.9	14.1	2.32
0.9	9.3	14.3	1.82
1.0	10.8	14.5	1.43
1.1	12.2	14.6	1.14
1.2	13.7	14.7	0.907
1.3	15.2	14.8	0.727
1.4	16.6	14.9	0.585
1.5	18.1	15.0	0.472
1.6	19.6	15.0	0.381
1.7	21.1	15.1	0.309
1.8	22.6	15.1	0.250
1.9	24.1	15.1	0.203
2.0	25.7	15.1	0.165
2.1	27.2	15.1	0.134
2.2	28.7	15.2	0.109
2.3	30.2	15.2	0.0889
2.4	31.7	15.2	0.0724
2.5	33.2	15.2	0.0590
2.6	34.8	15.2	0.0481
2.7	36.3	15.2	0.0392
2.8	37.8	15.2	0.0319
2.9	39.3	15.2	0.0260
3.0	40.8	15.2	0.0212



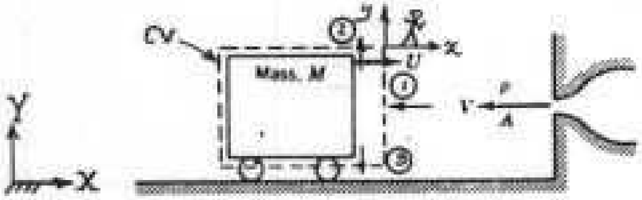




Problem 4.154

[Difficulty: 3]

4.154 A rectangular block of mass M, with vertical faces, rolls without resistance along a smooth horizontal plane as shown. The block travels initially at speed U_0. At $t = 0$ the block is struck by a liquid jet and its speed begins to slow. Obtain an algebraic expression for the acceleration of the block for $t > 0$. Solve the equation to determine the time at which $U = 0$.



Solution: Apply x momentum equation to linearly accelerating CV.

Basic equation:

$$\underbrace{F_{S_x}}_{=0(1)} + \underbrace{F_{B_x}}_{=0(2)} - \int_{CV} a_{rf_x} \rho \, d\forall = \underbrace{\frac{\partial}{\partial t}\int_{CV} u_{xyz} \rho \, d\forall}_{\approx 0(3)} + \int_{CS} u_{xyz} \rho \vec{V}_{xyz} \cdot d\vec{A}$$

Assumptions: (1) No pressure or friction forces, so $F_{S_x} = 0$
(2) Horizontal, so $F_{B_x} = 0$
(3) Neglect mass of liquid in CV, $u \approx 0$ in CV
(4) Uniform flow at each section
(5) Measure velocities relative to CV

Then

$$-Ma_{rf_x} = -M\frac{dU}{dt} = u_1\{-|\rho(V+U)A|\} + u_2\{+\dot{m}_2\} + u_3\{+\dot{m}_3\}$$

$$u_1 = -(V+U) \qquad u_2 = 0 \qquad u_3 = 0$$

or

$$\frac{dU}{dt} = -\frac{\rho(V+U)^2 A}{M}$$ $\frac{dU}{dt}$

But, since V = constant, $dU = d(V+U)$, so

$$\frac{d(V+U)}{(V+U)^2} = -\frac{\rho A}{M}dt$$

Integrating from U_0 at $t = 0$ to $U = 0$ at t

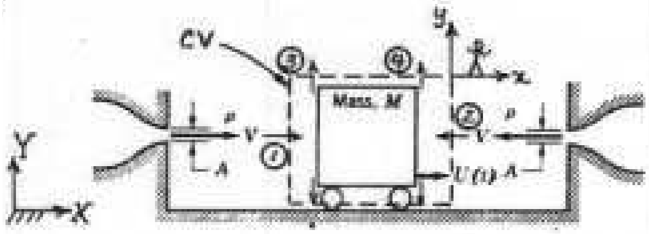
$$\int_{V+U_0}^{V} \frac{d(V+U)}{(V+U)^2} = -\frac{1}{(V+U)}\Bigg]_{V+U_0}^{V} = -\frac{1}{V} + \frac{1}{V+U_0} = \frac{-U_0}{V(V+U_0)} = -\frac{\rho A t}{M}$$

Solving, $$t = \frac{MU_0}{\rho VA(V+U_0)} = \frac{M}{\rho VA(1+V/U_0)}$$ t

Problem 4.155

[Difficulty: 3]

4.155 A rectangular block of mass M, with vertical faces, rolls on a horizontal surface between two opposing jets as shown. At $t = 0$ the block is set into motion at speed U_0. Subsequently, it moves without friction parallel to the jet axes with speed $U(t)$. Neglect the mass of any liquid adhering to the block compared with M. Obtain general expressions for the acceleration of the block, $a(t)$, and the block speed, $U(t)$.



Solution: Apply x momentum to linearly accelerating CV.

Basic equation:

$$\overset{=0(1)}{F_{S_x}} + \overset{=0(2)}{F_{B_x}} - \int_{CV} a_{rf_x} \rho \, d\forall = \overset{\approx 0(3)}{\frac{\partial}{\partial t}\int_{CV} u_{xyz} \rho \, d\forall} + \int_{CS} u_{xyz} \rho \vec{V}_{xyz} \cdot d\vec{A}$$

Assumptions: (1) No pressure or friction forces, so $F_{S_x} = 0$
(2) Horizontal, so $F_{B_x} = 0$
(3) Neglect mass of liquid in CV; $u \approx 0$ in CV
(4) Uniform flow at each section
(5) Measure velocities relative to CV

Then

$$-a_{rf_x} M = -M\frac{dU}{dt} = u_1\{-|\rho(V-U)A|\} + u_2\{-|\rho(V+U)A|\} + u_3\{\dot{m}_3\} + u_4\{\dot{m}_4\}$$

$$u_1 = V - U \qquad u_2 = -(V+U) \qquad u_3 = 0 \qquad u_4 = 0$$

or

$$-M\frac{dU}{dt} = \rho A\left[-(V-U)^2 + (V+U)^2\right] = \rho A[4UV] = 4\rho VAU$$

Thus

$$\frac{dU}{U} = -\frac{4\rho VA}{M}\,dt$$

Integrating

$$\int_{U_0}^{U}\frac{dU}{U} = \ln U\Big]_{U_0}^{U} = \ln\frac{U}{U_0} = -\frac{4\rho VA}{M}t$$

or

$$U(t) = U_0 e^{-\frac{4\rho VA}{M}t}$$

U(t)

Also

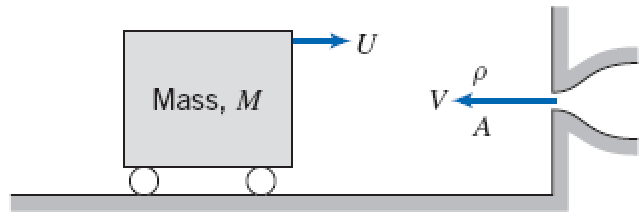
$$a(t) = \frac{dU}{dt} = -\frac{4\rho VA}{M}U_0 e^{-\frac{4\rho VA}{M}t}$$

a(t)

Problem 4.156

[Difficulty: 3]

4.156 Consider the diagram of Problem 4.154. If $M = 100$ kg, $\rho = 999$ kg/m^3, and $A = 0.01$ m^2, find the jet speed V required for the cart to be brought to rest after one second if the initial speed of the cart is $U_0 = 5$ m/s. For this condition, plot the speed U and position x of the cart as functions of time. What is the maximum value of x, and how long does the cart take to return to its initial position?



Given: Data on system

Find: Jet speed to stop cart after 1 s; plot speed & position; maximum x; time to return to origin

Solution: Apply x momentum

$$F_{S_x} + F_{B_x} - \int_{CV} a_{rf_x}\,\rho\,dV = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\,dV + \int_{CS} u_{xyz}\,\rho\vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: 1) All changes wrt CV 2) Incompressible flow 3) Atmospheric pressure in jet 4) Uniform flow 5) Constant jet area

The given data is $\rho = 999\cdot\frac{kg}{m^3}$ $\quad M = 100\cdot kg$ $\quad A = 0.01\cdot m^2$ $\quad U_0 = 5\cdot\frac{m}{s}$

Then $\quad -a_{rf}\cdot M = u_1\cdot[-\rho\cdot(V+U)\cdot A] + u_2\cdot m_2 + u_3\cdot m_3$

where $\quad a_{rf} = \frac{dU}{dt}$ $\quad u_1 = -(V+U)$ $\quad$ and $\quad u_2 = u_3 = 0$

Hence $\quad -\frac{dU}{dt}\cdot M = \rho\cdot(V+U)^2\cdot A$ $\quad$ or $\quad \frac{dU}{dt} = -\frac{\rho\cdot(V+U)^2\cdot A}{M}$ $\quad$ which leads to $\quad \frac{d(V+U)}{(V+U)^2} = -\left(\frac{\rho\cdot A}{M}\cdot dt\right)$

Integrating and using the IC $U = U_0$ at $t = 0$

$$U = -V + \frac{V+U_0}{1 + \frac{\rho\cdot A\cdot\left(V+U_0\right)}{M}\cdot t}$$

To find the jet speed V to stop the cart after 1 s, solve the above equation for V, with $U = 0$ and $t = 1$ s. (The equation becomes a quadratic in V). Instead we use *Excel*'s *Goal Seek* in the associated workbook

From *Excel* $\quad V = 5\cdot\frac{m}{s}$

For the position x we need to integrate

$$\frac{dx}{dt} = U = -V + \frac{V+U_0}{1 + \frac{\rho\cdot A\cdot\left(V+U_0\right)}{M}\cdot t}$$

The result is

$$x = -V\cdot t + \frac{M}{\rho\cdot A}\cdot\ln\left[1 + \frac{\rho\cdot A\cdot\left(V+U_0\right)}{M}\cdot t\right]$$

This equation (or the one for U with $U = 0$) can be easily used to find the maximum value of x by differentiating, as well as the time for x to be zero again. Instead we use *Excel*'s *Goal Seek* and *Solver* in the associated workbook

From *Excel* $\quad x_{max} = 1.93\cdot m$ $\quad t(x=0) = 2.51\cdot s$

The complete set of equations is

$$U = -V + \frac{V+U_0}{1 + \frac{\rho\cdot A\cdot\left(V+U_0\right)}{M}\cdot t} \qquad x = -V\cdot t + \frac{M}{\rho\cdot A}\cdot\ln\left[1 + \frac{\rho\cdot A\cdot\left(V+U_0\right)}{M}\cdot t\right]$$

The plots are presented in the *Excel* workbook:

t (s)	x (m)	U (m/s)
0.0	0.00	5.00
0.2	0.82	3.33
0.4	1.36	2.14
0.6	1.70	1.25
0.8	1.88	0.56
1.0	1.93	0.00
1.2	1.88	-0.45
1.4	1.75	-0.83
1.6	1.56	-1.15
1.8	1.30	-1.43
2.0	0.99	-1.67
2.2	0.63	-1.88
2.4	0.24	-2.06
2.6	-0.19	-2.22
2.8	-0.65	-2.37
3.0	-1.14	-2.50

To find V for $U = 0$ in 1 s, use *Goal Seek*

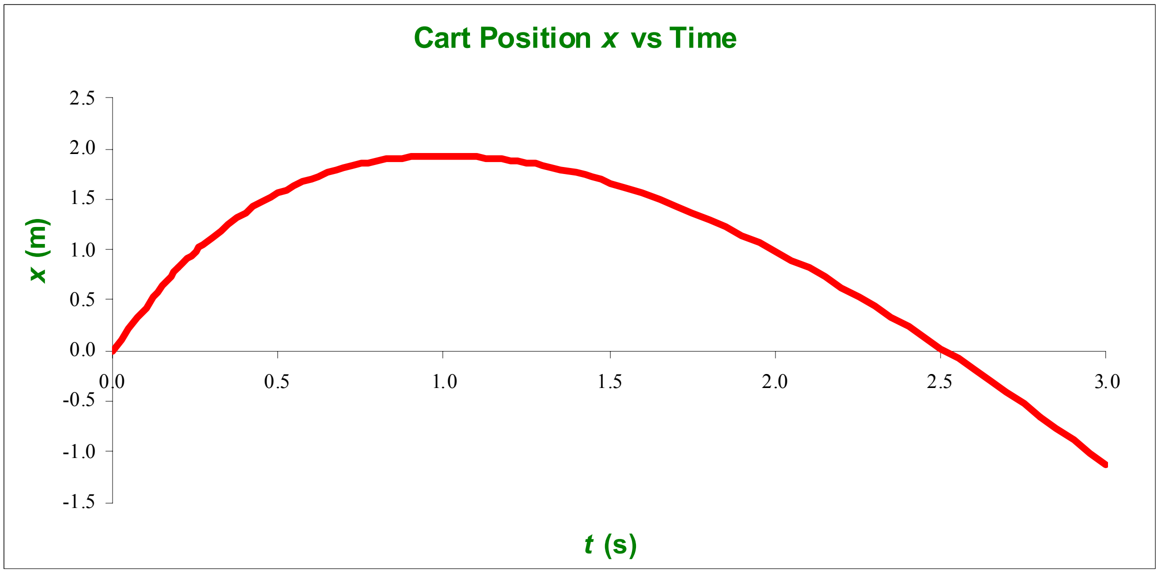
t (s)	U (m/s)	V (m/s)
1.0	0.00	5.00

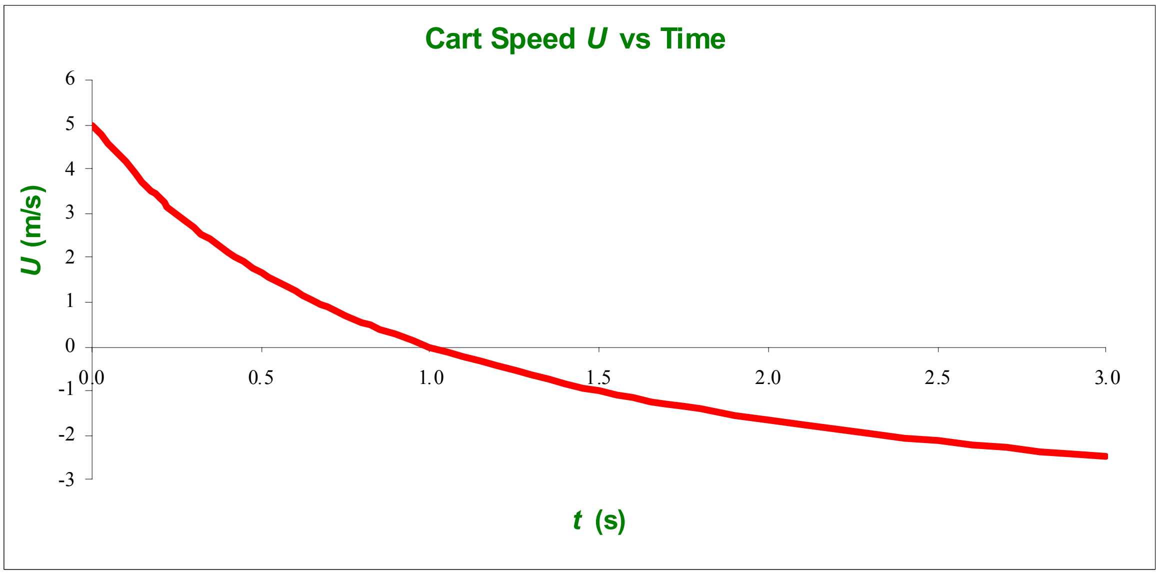
To find the maximum x, use *Solver*

t (s)	x (m)
1.0	1.93

To find the time at which $x = 0$ use *Goal Seek*

t (s)	x (m)
2.51	0.00

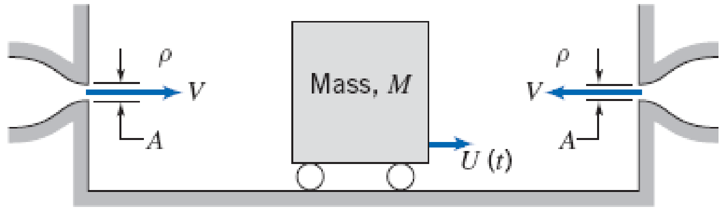




Problem 4.157

[Difficulty: 2]

4.157 Consider the statement and diagram of Problem 4.155. Assume that at $t = 0$, when the block of mass $M = 5$ kg is at $x = 0$, it is set into motion at speed $U_0 = 10$ m/s, to the right. The water jets have speed $V = 20$ m/s and area $A = 100$ mm^2. Calculate the time required to reduce the block speed to $U = 2.5$ m/s. Plot the block position x versus time. Compute the final rest position of the block. Explain why it comes to rest.



Given: Mass moving betweem two jets

Find: Time st slow to 2.5 m/s; plot position; rest position; explain

Solution: Apply x momentum

$$F_{S_x} + F_{B_x} - \int_{CV} a_{rf_x}\,\rho\, dV\!\!\!\!- = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\, dV\!\!\!\!- + \int_{CS} u_{xyz}\,\rho \vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: 1) All changes wrt CV 2) Incompressible flow 3) Atmospheric pressure in jet 4) Uniform flow 5) Constant jet area

The given data is $\rho = 999\cdot\frac{kg}{m^3}$ $M = 5\cdot kg$ $V = 20\cdot\frac{m}{s}$ $U_0 = 10\cdot\frac{m}{s}$ $U = 2.5\cdot\frac{m}{s}$ $A = 100\,mm^2$

Then $$-a_{rf}\cdot M = u_1\cdot[-\rho\cdot(V-U)\cdot A] + u_2\cdot[-\rho\cdot(V+U)\cdot A] + u_3\cdot m_3$$

where $a_{rf} = \frac{dU}{dt}$ $\quad u_1 = V - U \quad u_2 = -(V+U)$ and $u_3 = 0$

Hence $$-\frac{dU}{dt}\cdot M = \rho\cdot A\cdot\left[-(V-U)^2 + (V+U)^2\right] = 4\cdot\rho\cdot A\cdot V\cdot U$$

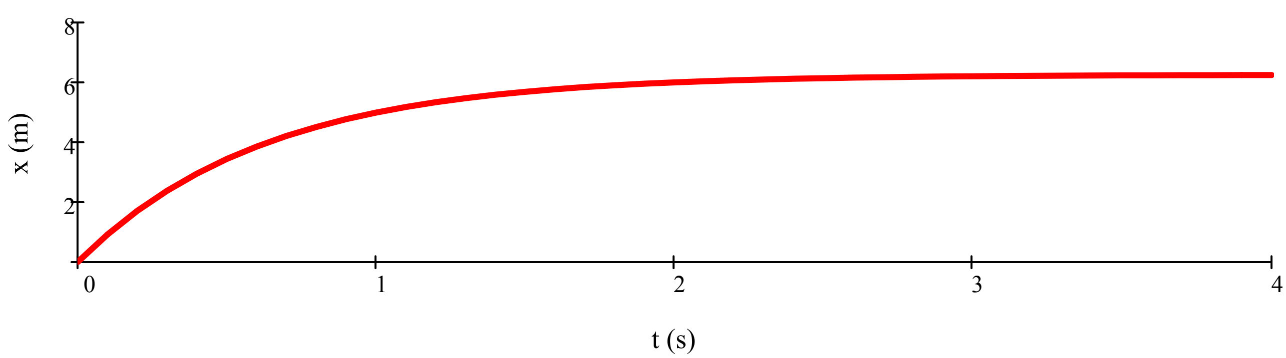
Separating and integrating $\frac{dU}{U} = -\frac{4\cdot\rho\cdot A\cdot V}{M}\cdot dt$ or $\ln(U) - \ln(U_0) = -\frac{4\cdot\rho\cdot A\cdot V}{M}\cdot t$ $\quad U = U_0\cdot e^{-\frac{4\cdot\rho\cdot A\cdot V}{M}\cdot t}$ (1)

Solving for t $t = -\frac{M}{4\cdot\rho\cdot V\cdot A}\cdot\ln\left(\frac{U}{U_0}\right)$ and using given data $t = 0.867\,s$ for $U = 2.5\frac{m}{s}$

For position x $$\frac{dx}{dt} = U = U_0\cdot e^{-\frac{4\cdot\rho\cdot A\cdot V}{M}\cdot t}$$

and a straightforward integration leads to $x(t) = \frac{M\cdot U_0}{4\cdot\rho\cdot V\cdot A}\cdot\left(1 - e^{-\frac{4\cdot\rho\cdot V\cdot A}{M}\cdot t}\right)$ For $t = 0.867\,s$ $\quad x(t) = 4.69\,m$

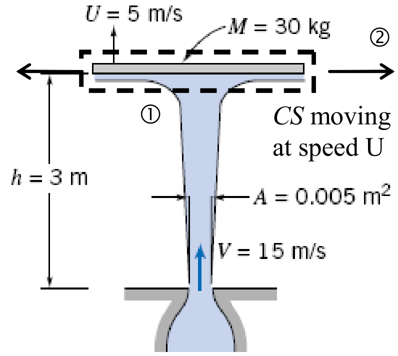
For large time $x_{final} = \frac{M\cdot U_0}{4\cdot\rho\cdot V\cdot A}$ $\quad x_{final} = 6.26\,m$



Problem *4.158

[Difficulty: 3]

***4.158** A vertical jet of water impinges on a horizontal disk as shown. The disk assembly mass is 30 kg. When the disk is 3 m above the nozzle exit, it is moving upward at $U = 5$ m/s. Compute the vertical acceleration of the disk at this instant.



Given: Water jet striking moving disk

Find: Acceleration of disk when at a height of 3 m

Solution:

Basic equations: Bernoulli; Momentum flux in z direction (treated as upwards) for linear accelerating CV

$$\frac{p}{\rho} + \frac{V^2}{2} + g\cdot z = \text{constant} \qquad F_{S_z} + F_{B_z} - \int_{CV} a_{rf_z}\,\rho\,d\mathcal{V} = \frac{\partial}{\partial t}\int_{CV} w_{xyz}\,\rho\,d\mathcal{V} + \int_{CS} w_{xyz}\,\rho\vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) Atmospheric pressure in jet 4) Uniform flow (All in jet)

The Bernoulli equation becomes $\quad \frac{V_0^{\,2}}{2} + g\cdot 0 = \frac{V_1^{\,2}}{2} + g\cdot(z - z_0) \qquad V_1 = \sqrt{V_0^{\,2} + 2\cdot g\cdot(z_0 - z)}$

$$V_1 = \sqrt{\left(15\cdot\frac{m}{s}\right)^2 + 2\times 9.81\cdot\frac{m}{s^2}\cdot(0 - 3)\cdot m} \qquad V_1 = 12.9\,\frac{m}{s}$$

The momentum equation becomes

$$-W - M\cdot a_{rfz} = w_1\cdot(-\rho\cdot V_1\cdot A_1) + w_2\cdot(\rho\cdot V_2\cdot A_2) = (V_1 - U)\cdot\left[-\rho\cdot(V_1 - U)\cdot A_1\right] + 0$$

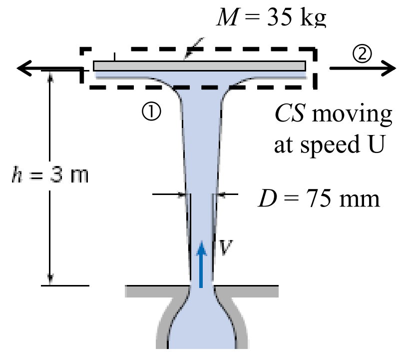
Hence $\quad a_{rfz} = \frac{\rho\cdot(V_1 - U)^2\cdot A_1 - W}{M} = \frac{\rho\cdot(V_1 - U)^2\cdot A_1}{M} - g = \frac{\rho\cdot(V_1 - U)^2\cdot A_0\cdot\frac{V_0}{V_1}}{M} - g \qquad$ using $\qquad V_1\cdot A_1 = V_0\cdot A_0$

$$a_{rfz} = 1000\cdot\frac{kg}{m^3}\times\left[(12.9 - 5)\cdot\frac{m}{s}\right]^2\times 0.005\cdot m^2\times\frac{15}{12.9}\times\frac{1}{30\cdot kg} - 9.81\cdot\frac{m}{s^2} \qquad a_{rfz} = 2.28\,\frac{m}{s^2}$$

Problem 4.159

[Difficulty: 4]

4.159 A vertical jet of water leaves a 75-mm diameter nozzle. The jet impinges on a horizontal disk (see Problem 4.158). The disk is constrained horizontally but is free to move vertically. The mass of the disk is 35 kg. Plot disk mass versus flow rate to determine the water flow rate required to suspend the disk 3 m above the jet exit plane.



Given: Water jet striking disk

Find: Plot mass versus flow rate to find flow rate for a steady height of 3 m

Solution:

Basic equations: Bernoulli; Momentum flux in z direction (treated as upwards)

$$\frac{p}{\rho} + \frac{V^2}{2} + g\cdot z = \text{constant} \qquad F_z = F_{S_z} + F_{B_z} = \frac{\partial}{\partial t}\int_{CV} w\,\rho\, dV\!\!\!/ + \int_{CS} w\,\rho \vec{V}\cdot d\vec{A}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) Atmospheric pressure in jet 4) Uniform flow (All in jet)

The Bernoulli equation becomes $\dfrac{V_0^{\,2}}{2} + g\cdot 0 = \dfrac{V_1^{\,2}}{2} + g\cdot h \qquad V_1 = \sqrt{V_0^{\,2} - 2\cdot g\cdot h}$

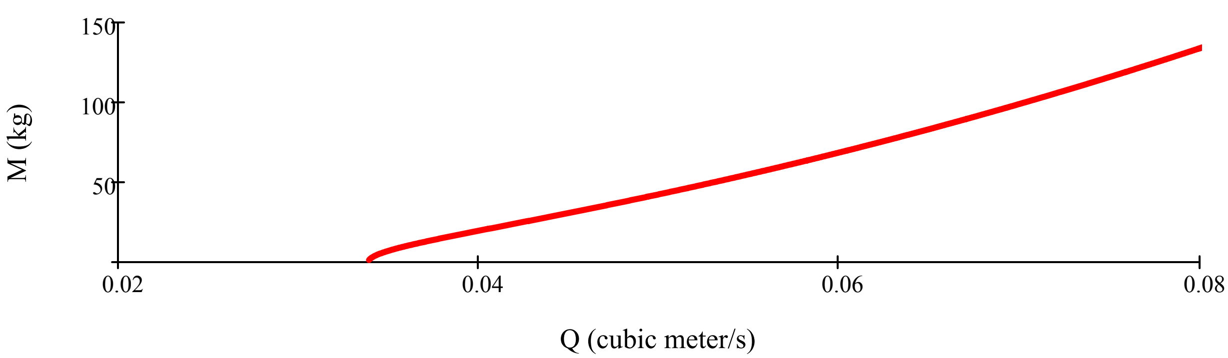
The momentum equation becomes

$$-M\cdot g = w_1\cdot\left(-\rho\cdot V_1\cdot A_1\right) + w_2\cdot\left(\rho\cdot V_2\cdot A_2\right) = V_1\cdot\left(-\rho\cdot V_1\cdot A_1\right) + 0$$

Hence $M = \dfrac{\rho\cdot V_1^{\,2}\cdot A_1}{g}$ but from continuity $V_1\cdot A_1 = V_0\cdot A_0$

$$M = \frac{\rho\cdot V_1\cdot V_0\cdot A_0}{g} = \frac{\pi}{4}\cdot\frac{\rho\cdot V_0\cdot D_0^{\,2}}{g}\cdot\sqrt{V_0^{\,2} - 2\cdot g\cdot h} \qquad \text{and also} \qquad Q = V_0\cdot A_0$$

This equation is difficult to solve for V_0 for a given M. Instead we plot first:



Goal Seek or Solver in *Excel* feature can be used to find Q when M = 35 kg $\qquad Q = 0.0469\cdot\dfrac{m^3}{s}$

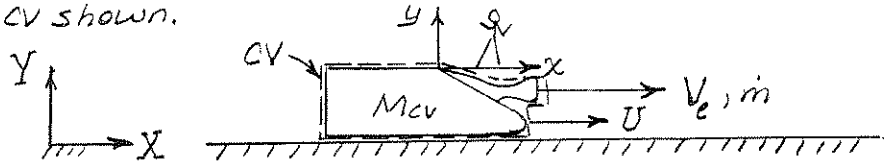
Problem 4.160

[Difficulty: 3]

4.160 A rocket sled traveling on a horizontal track is slowed by a retro-rocket fired in the direction of travel. The initial speed of the sled is $U_0 = 500$ m/s. The initial mass of the sled is $M_0 = 1500$ kg. The retro-rocket consumes fuel at the rate of 7.75 kg/s, and the exhaust gases leave the nozzle at atmospheric pressure and a speed of 2500 m/s relative to the rocket. The retro-rocket fires for 20 s. Neglect aerodynamic drag and rolling resistance. Obtain and plot an algebraic expression for sled speed U as a function of firing time. Calculate the sled speed at the end of retro-rocket firing.

Solution: Apply x-component of momentum equation to the linearly accelerating CV shown.

From continuity,



$$M_{cv} = M_0 - \dot{m}t, \quad t < t_{bo}$$

Basic equation:

$$\cancel{F_{S_x}}^{=0(1)} + \cancel{F_{B_x}}^{=0(2)} - \int_{CV} a_{rf_x} \rho \, d\forall = \cancel{\frac{\partial}{\partial t}\int_{CV} u_{xyz} \rho \, d\forall}^{\approx 0(3)} + \int_{CS} u_{xyz} \rho \vec{V}_{xyz} \cdot d\vec{A}$$

Assumptions: (1) No pressure, drag, or rolling resistance, so $F_{S_x} = 0$

(2) Horizontal motion, so $F_{B_x} = 0$

(3) Neglect unsteady effects within CV

(4) Uniform flow at nozzle exit plane

(5) $p_e = p_{atm}$

Then $\quad -a_{rf_x} M_{cv} = u_e\{+\dot{m}\} = +V_e\dot{m} \quad$ or $\quad \dfrac{dU}{dt} = -\dfrac{V_e\dot{m}}{M_{cv}} = -\dfrac{V_e\dot{m}}{M_0 - \dot{m}t}$

$u_e = V_e$

Thus $\quad dU = V_e\left(\dfrac{-\dot{m}\,dt}{M_0 - \dot{m}t}\right) \quad$ and $\quad U - U_0 = V_e \ln\left(M_0 - \dot{m}t\right)\Big|_0^t = V_e \ln\left(1 - \dfrac{\dot{m}t}{M_0}\right)$

$$U(t) = U_0 + V_e \ln\left(1 - \frac{\dot{m}t}{M_0}\right) \quad ; \quad t < t_{bo}$$

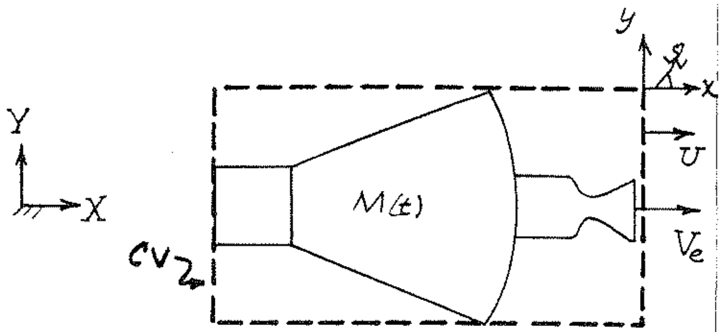
$U(t)$

At t_{bo}, $\quad U(t_{bo}) = 500\,\dfrac{m}{s} + 2500\,\dfrac{m}{s} \times \ln\left(1 - 7.75\,\dfrac{kg}{s} \times 20.0\,s \times \dfrac{1}{1500\,kg}\right)$

$$U(t_{bo}) = 227 \text{ m/s}$$

$U(t_{bo})$

4.161 A manned space capsule travels in level flight above the Earth's atmosphere at initial speed $U_0 = 8.00$ km/s. The capsule is to be slowed by a retro-rocket to $U = 5.00$ km/s in preparation for a reentry maneuver. The initial mass of the capsule is $M_0 = 1600$ kg. The rocket consumes fuel at $\dot{m} = 8.0$ kg/s, and exhaust gases leave at $V_e = 3000$ m/s relative to the capsule and at negligible pressure. Evaluate the duration of the retro-rocket firing needed to accomplish this. Plot the final speed as a function of firing duration for a time range $\pm 10\%$ of this firing time.



Solution: Apply x component of momentum to CV with linear acceleration.

Basic equation:

$$\underbrace{F_{S_x}}_{=0(1)} + \underbrace{F_{B_x}}_{=0(2)} - \int_{CV} a_{rf_x}\rho\, d\forall = \frac{\partial}{\partial t}\int_{CV} \underbrace{u_{xyz}}_{\approx 0(4)}\rho\, d\forall + \int_{CS} u_{xyz}\,\rho \vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: (1) No resistance; $F_{S_x} = 0$
(2) Horizontal; $F_{B_x} = 0$
(3) Use velocities measured relative to CV
(4) Neglect velocity within CV
(5) Uniform flow at exit plane with negligible p_e (given)

From continuity,
$$\frac{dM}{dt} = \frac{\partial}{\partial t}\int_{CV}\rho\, d\forall = -\int_{CS}\rho\vec{V}_{xyz}\cdot d\vec{A} = -\dot{m}; \quad M(t) = M_0 - \dot{m}t$$

From momentum,
$$-a_{rf_x} M = -\frac{dU}{dt}(M_0 - \dot{m}t) = u_e\{+\dot{m}\} = V_e\dot{m}$$

$$u_e = V_e$$

Thus
$$\frac{dU}{dt} = -\frac{V_e\dot{m}}{M_0 - \dot{m}t}$$

Integrating,
$$U - U_0 = V_e\int_0^t \frac{-\dot{m}\,dt}{M_0 - \dot{m}t} = V_e \ln(M_0 - \dot{m}t)\Big]_0^t = V_e \ln\left(\frac{M_0 - \dot{m}t}{M_0}\right)$$

Solving for t,
$$\frac{M_0 - \dot{m}t}{M_0} = e^{\frac{U-U_0}{V_e}}; \quad M_0 - \dot{m}t = M_0 e^{U-U_0/V_e}$$

$$t = \frac{M_0}{\dot{m}}\left(1 - e^{U-U_0/V_e}\right)$$

$$= 1600\ \text{kg} \times \frac{\text{s}}{8\ \text{kg}}\left\{1 - e^{\left[(5.00-8.0)\frac{\text{km}}{\text{s}} \times \frac{\text{s}}{3000\ \text{m}} \times 1000\frac{\text{m}}{\text{km}}\right]}\right\}$$

$$t = 126\ \text{s}$$

Problem 4.162

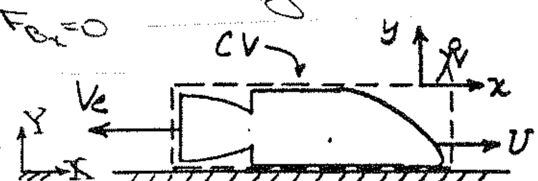
4.162 A rocket sled accelerates from rest on a level track with negligible air and rolling resistances. The initial mass of the sled is $M_0 = 600$ kg. The rocket initially contains 150 kg of fuel. The rocket motor burns fuel at constant rate $\dot{m} = 15$ kg/s. Exhaust gases leave the rocket nozzle uniformly and axially at $V_e = 2900$ m/s relative to the nozzle, and the pressure is atmospheric. Find the maximum speed reached by the rocket sled. Calculate the maximum acceleration of the sled during the run.

Solution:

Apply the momentum equation to linearly accelerating CV shown.

Basic equation: $\cancel{F_{S_x}}^{=0(1)} + \cancel{F_{B_x}}^{=0(2)} - \int_{CV} a_{rf_x} \rho \, d\forall = \cancel{\frac{\partial}{\partial t}\int_{CV} u_{xyz} \rho \, d\forall}^{=0(3)} + \int_{CS} u_{xyz} \rho \vec{V}_{xyz} \cdot d\vec{A}$

Assumptions: (1) no net pressure forces ($p_e = p_{atm}$, given)
(2) horizontal motion, $F_{B_x} = 0$
(3) neglect $\partial/\partial t$ in CV
(4) uniform axial jet



From continuity, $M = M_0 - \dot{m}t$. Then

$$-a_{rf_x} M = -\frac{dU}{dt}(M_0 - \dot{m}t) = u_e\{\dot{m}\} = -V_e \dot{m} \quad \text{....(1)}$$

Separating variables,

$$dU = V_e \frac{\dot{m}\,dt}{M_0 - \dot{m}t}$$

Integrating from $U = 0$ at $t = 0$ to U at t gives

$$U = -V_e \ln(M_0 - \dot{m}t)\Big]_0^t = -V_e \ln \frac{(M_0 - \dot{m}t)}{M_0} = V_e \ln \frac{M_0}{(M_0 - \dot{m}t)} \quad \text{---(2)}$$

The speed is a maximum at burnout. At burnout $M_f = 0$ and $M = M_0 - \dot{m}t = 450$ kg

At burnout, $t = \frac{M_f)_{initial}}{\dot{m}_{fuel}} = 150\,\text{kg} \times \frac{s}{15\,\text{kg}} = 10\,\text{s}$

Then from Eq. 2

$$U_{max} = 2900 \frac{m}{s} \ln \frac{600\,\text{kg}}{450\,\text{kg}} = 834\,\text{m/s}$$ $\leftarrow$ U_{max}

From Eq. 1 the acceleration is $\frac{dU}{dt} = \frac{\dot{m} V_e}{M_0 - \dot{m}t}$

The maximum acceleration occurs at the instant prior to burn out

$$\left.\frac{dU}{dt}\right)_{max} = 15 \frac{kg}{s} \times 2900 \frac{m}{s} \times \frac{1}{450\,\text{kg}} = 96.7\,\text{m/s}^2$$ $\leftarrow$ $\left.\frac{dU}{dt}\right)_{max}$

Problem 4.162

The sled speed as a function of time is

$$U = V_e \ln \frac{M_0}{(M_0 - \dot{m}t)} \quad \text{for } 0 \le t \le 10\text{s}$$

$$U = \text{constant} = 834 \text{ m/s} \quad \text{for } t > 10 \quad \text{(neglecting resistance)}$$

The sled acceleration is given by

$$\frac{dU}{dt} = \frac{\dot{m} V_e}{(M_0 - \dot{m}t)} \quad \text{for } 0 \le t \le 10\text{s}$$

$$\frac{dU}{dt} = 0 \quad \text{for } t \ge 10\text{s}.$$

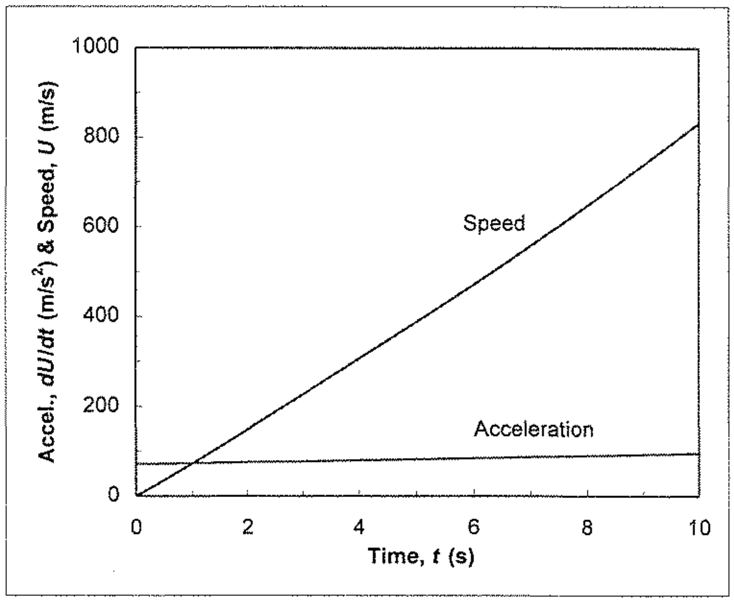
Acceleration and Velocity vs. Time for Rocket Sled:

Input Data:

$M_0 =$	600	kg
m(dot) =	15	kg/s
$V_e =$	2900	m/s

Calculated Results:

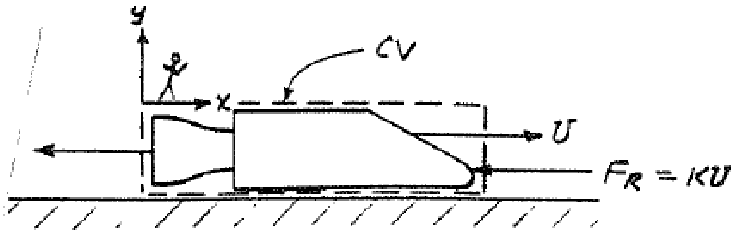
Time, t (s)	Acceleration, dU/dt (m/s^2)	Velocity, U (m/s)
0	72.5	0
1	74.4	73.4
2	76.3	149
3	78.4	226
4	80.6	306
5	82.9	387
6	85.3	471
7	87.9	558
8	90.6	647
9	93.5	739
10	96.7	834



Problem 4.163

4.163 A rocket sled has mass of 5000 kg, including 1000 kg of fuel. The motion resistance in the track on which the sled rides and that of the air total kU, where k is 50 N · s/m and U is the speed of the sled in m/s. The exit speed of the exhaust gas relative to the rocket is 1750 m/s, and the exit pressure is atmospheric. The rocket burns fuel at the rate of 50 kg/s.

(a) Plot the sled speed as a function of time.
(b) Find the maximum speed.
(c) What percentage increase in maximum speed would be obtained by reducing k by 10 percent?



Given:
Rocket sled on track

Find:
Plot speed versus time; maximum speed; effect of reducing k

Solution:

Basic equation: Momentum flux in x direction

$$F_{S_z} + F_{B_z} - \int_{CV} a_{rf_z}\,\rho\,dV = \frac{\partial}{\partial t}\int_{CV} w_{xyz}\,\rho\,dV + \int_{CS} w_{xyz}\,\rho\vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) Atmospheric pressure in jet 4) Uniform flow (All in jet)

Given data $M_0 = 5000\cdot kg$ $\quad k = 50\cdot\frac{N\cdot s}{m}$ $\quad V_e = 1750\cdot\frac{m}{s}$ $\quad M_{fuel} = 1000\cdot kg$ $\quad m_{rate} = 50\cdot\frac{kg}{s}$

The momentum equation becomes $-F_R - a_{rf}\cdot M = u_e\cdot m_{rate} = -V_e\cdot m_{rate}$ and $F_R = k\cdot U$

From continuity $M = M_0 - m_{rate}\cdot t$

Hence, combining $-k\cdot U - \left(M_0 - m_{rate}\cdot t\right)\cdot\frac{dU}{dt} = -V_e\cdot m_{rate}$ or $\frac{dU}{dt} = \frac{V_e\cdot m_{rate} - k\cdot U}{M_0 - m_{rate}\cdot t}$

Separating variables $\frac{dU}{V_e\cdot m_{rate} - k\cdot U} = \frac{dt}{\left(M_0 - m_{rate}\cdot t\right)}$

Integrating $\frac{1}{k}\cdot\left(\left(\ln\left(V_e\cdot m_{rate} - k\cdot U\right) - \ln\left(V_e\cdot m_{rate}\right)\right)\right) = \frac{1}{m_{rate}}\cdot\left(\ln\left(M_0 - m_{rate}\cdot t\right) - \ln\left(M_0\right)\right)$

Simplifying $\frac{1}{k}\cdot\ln\left(\frac{V_e\cdot m_{rate} - k\cdot U}{V_e\cdot m_{rate}}\right) = \frac{1}{k}\cdot\ln\left(1 - \frac{k\cdot U}{V_e\cdot m_{rate}}\right) = \frac{1}{m_{rate}}\cdot\ln\left(\frac{M_0 - m_{rate}\cdot t}{M_0}\right) = \frac{1}{m_{rate}}\cdot\ln\left(1 - \frac{m_{rate}\cdot t}{M_0}\right)$

Solving for U

$$U(t) = \frac{V_e\cdot m_{rate}}{k}\cdot\left[1 - \left(1 - \frac{m_{rate}\cdot t}{M_0}\right)^{\frac{k}{m_{rate}}}\right]$$

Using given data $U(10\cdot s) = 175\,\frac{m}{s}$ and fuel is used up when $t_{fuel} = \frac{M_{fuel}}{m_{rate}}$ $\quad t_{fuel} = 20\,s$

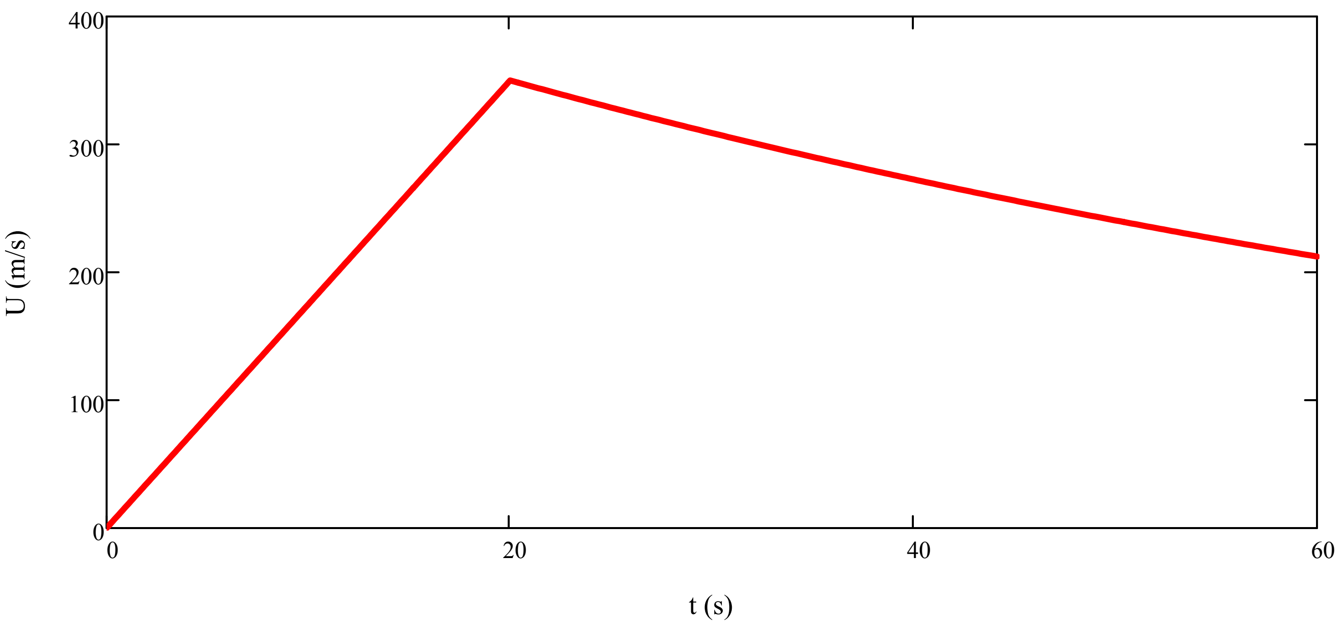
This is when the speed is maximum $\quad U_{max} = U(t_{fuel}) \quad U_{max} = 350\,\frac{m}{s}$

With 10% reduction in k $\quad k_2 = 0.9 \cdot k \quad U_{max2} = \frac{V_e \cdot m_{rate}}{k_2} \cdot \left[1 - \left(1 - \frac{m_{rate} \cdot t_{fuel}}{M_0}\right)^{\frac{k_2}{m_{rate}}}\right] \quad U_{max2} = 354\,\frac{m}{s}$

The percent improvement is $\quad \frac{U_{max2} - U_{max}}{U_{max}} = 1.08 \cdot \%$

When the fuel runs out the momentum equation simplifies from $\quad -k \cdot U - (M_0 - m_{rate} \cdot t) \cdot \frac{dU}{dt} = -V_e \cdot m_{rate} \quad$ to $\quad -k \cdot U - \frac{dU}{dt} = 0$

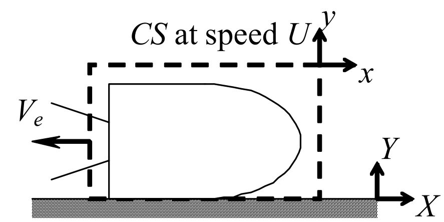
The solution to this (with $U = U_{max}$ when $t = t_{fuel}$) $\quad U_{empty}(t) = U_{max} \cdot e^{-\frac{k \cdot (t - t_{fuel})}{M_0 - M_{fuel}}}$



Problem 4.164

[Difficulty: 3]

4.164 A rocket sled with initial mass of 900 kg is to be accelerated on a level track. The rocket motor burns fuel at constant rate $\dot{m} = 13.5\,\text{kg/s}$. The rocket exhaust flow is uniform and axial. Gases leave the nozzle at 2750 m/s relative to the nozzle, and the pressure is atmospheric. Determine the minimum mass of rocket fuel needed to propel the sled to a speed of 265 m/s before burnout occurs. As a first approximation, neglect resistance forces.



Given: Data on rocket sled

Find: Minimum fuel to get to 265 m/s

Solution:

Basic equation: Momentum flux in x direction

$$F_{S_x} + F_{B_x} - \int_{CV} a_{rf_x}\,\rho\, d\mathcal{V} = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\, d\mathcal{V} + \int_{CS} u_{xyz}\,\rho \vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: 1) No resistance 2) $p_e = p_{atm}$ 3) Uniform flow 4) Use relative velocities

From continuity $\quad \frac{dM}{dt} = m_{rate} = \text{constant} \quad$ so $\quad M = M_0 - m_{rate}\cdot t \quad$ (Note: Software cannot render a dot!)

Hence from momentum $\quad -a_{rfx}\cdot M = -\frac{dU}{dt}\cdot\left(M_0 - m_{rate}\cdot t\right) = u_e\cdot\left(\rho_e\cdot V_e\cdot A_e\right) = -V_e\cdot m_{rate}$

Separating variables $\quad dU = \frac{V_e\cdot m_{rate}}{M_0 - m_{rate}\cdot t}\cdot dt$

Integrating $\quad U = V_e\cdot\ln\left(\frac{M_0}{M_0 - m_{rate}\cdot t}\right) = -V_e\cdot\ln\left(1 - \frac{m_{rate}\cdot t}{M_0}\right) \quad$ or $\quad t = \frac{M_0}{m_{rate}}\cdot\left(1 - e^{-\frac{U}{V_e}}\right)$

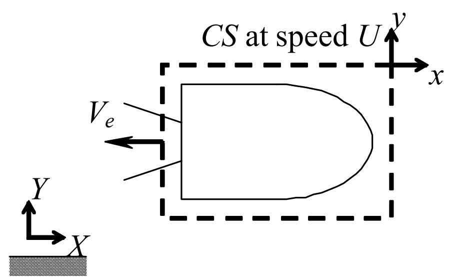
The mass of fuel consumed is $\quad m_f = m_{rate}\cdot t = M_0\cdot\left(1 - e^{-\frac{U}{V_e}}\right)$

Hence $\quad m_f = 900\cdot\text{kg}\times\left(1 - e^{-\frac{265}{2750}}\right) \qquad m_f = 82.7\,\text{kg}$

Problem 4.165

[Difficulty: 3]

4.165 A rocket motor is used to accelerate a kinetic energy weapon to a speed of 3500 mph in horizontal flight. The exit stream leaves the nozzle axially and at atmospheric pressure with a speed of 6000 mph relative to the rocket. The rocket motor ignites upon release of the weapon from an aircraft flying horizontally at $U_0 = 600$ mph. Neglecting air resistance, obtain an algebraic expression for the speed reached by the weapon in level flight. Determine the minimum fraction of the initial mass of the weapon that must be fuel to accomplish the desired acceleration.



Given: Data on rocket weapon

Find: Expression for speed of weapon; minimum fraction of mass that must be fuel

Solution:

Basic equation: Momentum flux in x direction

$$F_{S_x} + F_{B_x} - \int_{CV} a_{rf_x}\,\rho\, d\forall = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\, d\forall + \int_{CS} u_{xyz}\,\rho \vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: 1) No resistance 2) $p_e = p_{atm}$ 3) Uniform flow 4) Use relative velocities 5) Constant mass flow rate

From continuity $\quad \frac{dM}{dt} = m_{rate} = \text{constant} \quad$ so $\quad M = M_0 - m_{rate}\cdot t \quad$ (Note: Software cannot render a dot!)

Hence from momentum $\quad -a_{rfx}\cdot M = -\frac{dU}{dt}\cdot\left(M_0 - m_{rate}\cdot t\right) = u_e\cdot\left(\rho_e\cdot V_e\cdot A_e\right) = -V_e\cdot m_{rate}$

Separating variables $\quad dU = \frac{V_e\cdot m_{rate}}{M_0 - m_{rate}\cdot t}\cdot dt$

Integrating from $U = U_0$ at $t = 0$ to $U = U$ at $t = t$

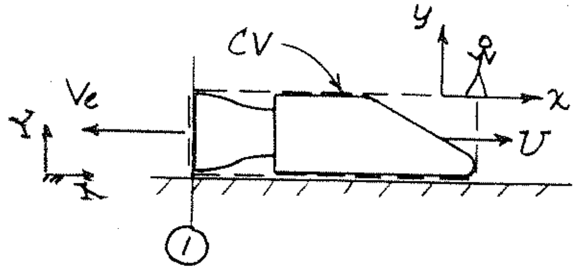
$$U - U_0 = -V_e\cdot\left(\ln\left(M_0 - m_{rate}\cdot t\right) - \ln\left(M_0\right)\right) = -V_e\cdot\ln\left(1 - \frac{m_{rate}\cdot t}{M_0}\right)$$

$$U = U_0 - V_e\cdot\ln\left(1 - \frac{m_{rate}\cdot t}{M_0}\right)$$

Rearranging $\quad MassFractionConsumed = \frac{m_{rate}\cdot t}{M_0} = 1 - e^{-\frac{\left(U-U_0\right)}{V_e}} = 1 - e^{-\frac{(3500-600)}{6000}} = 0.383$

Hence 38.3% of the mass must be fuel to accomplish the task. In reality, a much higher percentage would be needed due to drag effects

4.166 A rocket sled with initial mass of 3 metric tons, including 1 ton of fuel, rests on a level section of track. At $t = 0$, the solid fuel of the rocket is ignited and the rocket burns fuel at the rate of 75 kg/s. The exit speed of the exhaust gas relative to the rocket is 2500 m/s, and the pressure is atmospheric. Neglecting friction and air resistance, calculate the acceleration and speed of the sled at $t = 10$ s.



Solution:

Apply x component of momentum to linearly accelerating CV; use continuity to find M(t)

Basic equations: $0 = \frac{\partial}{\partial t}\int_{CV} \rho\, dV + \int_{CS} \rho \vec{V}_{xyz} \cdot d\vec{A}$

$$\cancel{F_{S_x}}^{=0(1)} + \cancel{F_{B_x}}^{=0(2)} - \int_{CV} a_{rf}\, \rho\, dV = \cancel{\frac{\partial}{\partial t}}^{\approx 0(3)} \int_{CV} u_{xyz}\, \rho\, dV + \int_{CS} u_{xyz} (\rho \vec{V}_{xyz} \cdot d\vec{A})$$

Assumptions: (1) $F_{S_x} = 0$, no resistance (given)
(2) $F_{B_x} = 0$, horizontal
(3) neglect $\partial/\partial t$ inside CV
(4) uniform flow at nozzle exit
(5) $p_e = p_{atm}$ (given)

From continuity, $0 = \frac{\partial M}{\partial t} + \{+|\dot{m}|\} = \frac{dM}{dt} + \dot{m}$ or $dM = -\dot{m}\,dt$

Integrating, $\int_{M_0}^{M} dM = M - M_0 = \int_0^t -\dot{m}\,dt = -\dot{m}t$ or $M = M_0 - \dot{m}t$

From the momentum equation

$$-a_{rf_x} M = -a_{rf_x}(M_0 - \dot{m}t) = u_1 \{+|\dot{m}|\} = -V_e \dot{m} \qquad \{u_1 = -V_e\}$$

Thus

$$a_{rf_x} = \frac{dU}{dt} = \frac{V_e \dot{m}}{(M_0 - \dot{m}t)} \quad \text{--------- (1)}$$

At $t = 10$ s

$$\frac{dU}{dt} = 2500\frac{m}{s} \times 75\frac{kg}{s} \times \frac{1}{3000\,kg - 75\frac{kg}{s} \times 10\,s} = 83.3\ m/s^2 \longleftarrow a_{rf_x}$$

From Eq. 1, $dU = V_e \frac{\dot{m}\,dt}{M_0 - \dot{m}t}$

Integrating from $U = 0$ at $t = 0$ to U at t gives

$$U = -V_e \ln(M_0 - \dot{m}t)\Big]_0^t = -V_e \ln\frac{(M_0 - \dot{m}t)}{M_0}$$

$$U = V_e \ln\frac{M_0}{(M_0 - \dot{m}t)} \quad \text{--------- (2)}$$

Problem 4.166

[Difficulty: 3] Part 2/2

At $t = 10\text{ s}$

$$U = 2500\,\frac{\text{m}}{\text{s}}\ln\frac{3000\text{ kg}}{3000\text{ kg} - 75\,\frac{\text{kg}}{\text{s}}\times 10\text{ s}} = 719\text{ m/s} \quad U.$$

Note that all fuel would be expended at $t_{bo} = \frac{M_f}{\dot{m}} = 1000\text{ kg}\cdot\frac{\text{s}}{75\text{ kg}}$

i.e. at $t_{b.o} = 13.3\text{ s}$.

The sled speed as a function of time is then

$$U = V_e \ln\frac{M_0}{(M_0 - \dot{m}t)} \quad \text{for } t \le 13.3\text{ s}$$

$$U = U_{max} = 1010\text{ m/s} \quad \text{for } t \ge 13.3\text{ s}$$

The sled acceleration is given by

$$\frac{dU}{dt} = \frac{\dot{m}V_e}{(M_0 - \dot{m}t)} \quad \text{for } 0 \le t \le 13.3\text{ s}$$

$$\frac{dU}{dt} = 0 \quad \text{for } t \ge 13.3\text{ s}$$

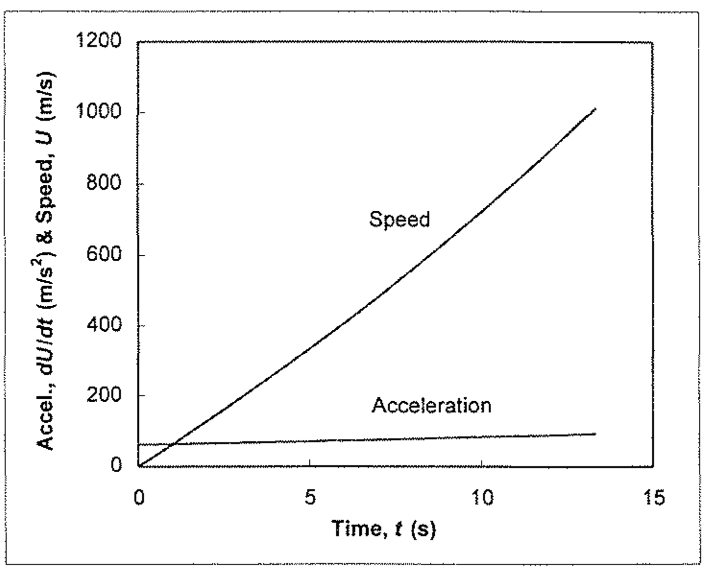
Acceleration and Speed vs. Time for Rocket Sled:

Input Data:

M_0 =	3000	kg
m(dot) =	75	kg/s
V_e =	2500	m/s

Calculated Results:

Time, t (s)	Acceleration, dU/dt (m/s^2)	Speed, U (m/s)
0	62.5	0
1	64.1	63.3
2	65.8	128
3	67.6	195
4	69.4	263
5	71.4	334
6	73.5	406
7	75.8	481
8	78.1	558
9	80.6	637
10	83.3	719
11	86.2	804
12	89.3	892
13	92.6	983
13.33	93.8	1014

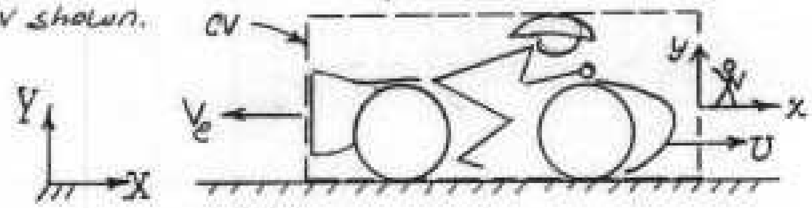


Problem 4.167

[Difficulty: 3]

4.167 A daredevil considering a record attempt—for the world's longest motorcycle jump—asks for your consulting help: He must reach 875 km/hr (from a standing start on horizontal ground) to make the jump, so he needs rocket propulsion. The total mass of the motorcycle, the rocket motor without fuel, and the rider is 375 kg. Gases leave the rocket nozzle horizontally, at atmospheric pressure, with a speed of 2510 m/s. Evaluate the minimum amount of rocket fuel needed to accelerate the motorcycle and rider to the required speed.

<u>Solution</u>: Apply x-component of momentum equation to linearly accelerating CV shown.



From continuity,

$$M_{cv} = M_0 - \dot{m}t$$

Basic equation:

$$\overset{\approx 0(1)}{F_{S_x}} + \overset{\approx 0(2)}{F_{B_x}} - \int_{CV} a_{rf_x}\,\rho\,d\forall = \overset{\approx 0(3)}{\frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\,d\forall} + \int_{CS} u_{xyz}\,\rho\vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: (1) Neglect air and rolling resistance
(2) Level track, so $F_{B_x} = 0$
(3) Neglect unsteady effects within CV
(4) Uniform flow at nozzle exit plane
(5) $p_e = p_{atm}$

Then

$$-a_{rf_x} M_{cv} = u_e\{+\dot{m}\} = -V_e\dot{m} \quad \text{or} \quad \frac{dU}{dt} = \frac{V_e\dot{m}}{M_{cv}} = \frac{V_e\dot{m}}{M_0 - \dot{m}t}$$

$$u_e = -V_e$$

Separating variables and integrating,

$$dU = -V_e\left(\frac{-\dot{m}dt}{M_0 - \dot{m}t}\right) \quad \text{or} \quad U_j = -V_e \ln(M_0 - \dot{m}t)\Big|_0^t = V_e \ln\left(\frac{M_0}{M_0 - \dot{m}t}\right)$$

But $M_0 = M_B + M_F$ and $M_F = \dot{m}t$, so

$$\frac{U_j}{V_e} = \ln\left(\frac{M_B + M_F}{M_B}\right) = \ln\left(1 + \frac{M_F}{M_B}\right); \quad 1 + \frac{M_F}{M_B} = e^{U_j/V_e}; \quad \frac{M_F}{M_B} = e^{U_j/V_e} - 1$$

Finally, $M_F = M_B\left(e^{U_j/V_e} - 1\right)$

$$M_F = 375\ \text{kg} \times \exp\left[875\ \frac{\text{km}}{\text{hr}} \times \frac{\text{s}}{2510\ \text{m}} \times 1000\ \frac{\text{m}}{\text{km}} \times \frac{\text{hr}}{3600\ \text{s}} - 1\right]$$

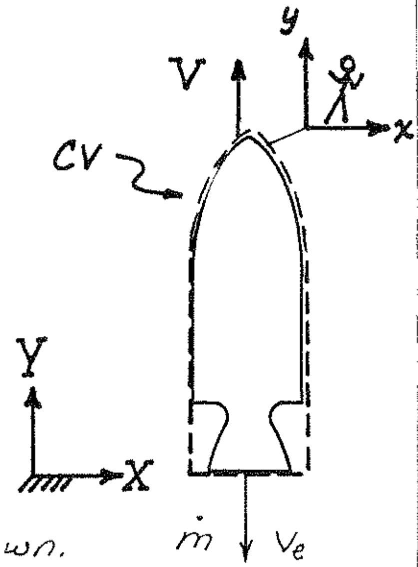
$$M_F = 38.1\ \text{kg}$$

M_F

The fuel mass required is about 10 percent of the mass of the motorcycle and rider.

Problem 4.168

4.168 A "home-made" solid propellant rocket has an initial mass of 20 lbm; 15 lbm of this is fuel. The rocket is directed vertically upward from rest, burns fuel at a constant rate of 0.5 lbm/s, and ejects exhaust gas at a speed of 6500 ft/s relative to the rocket. Assume that the pressure at the exit is atmospheric and that air resistance may be neglected. Calculate the rocket speed after 20 s and the distance traveled by the rocket in 20 s. Plot the rocket speed and the distance traveled as functions of time.



Solution: Apply y-component of momentum equation to accelerating CV using CS shown.

Basic equation:

$$\cancel{F_{S_y}}^{=0(1)} + F_{B_y} - \int_{CV} a_{rf_y}\,\rho\, d\forall = \cancel{\frac{\partial}{\partial t}\int_{CV} v_{xyz}\,\rho\, d\forall}^{\approx 0(2)} + \int_{CS} v_{xyz}\,\rho \vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: (1) Neglect air resistance; $p_e = p_{atm}$ (given)
(2) Neglect v_{xyz} and $\partial/\partial t$ within CV
(3) Uniform flow at nozzle exit section

Then

$$F_{B_y} - a_{rf_y} M = -Mg - M a_{rf_y} = v_e\{+\dot{m}\} = -V_e \dot{m}$$

and

$$v_e = -V_e$$

$$a_{rf_y} = \frac{dV}{dt} = \frac{V_e \dot{m}}{M} - g$$

Introducing $M = M_0 - \dot{m}t$ and separating variables,

$$dV = \left(\frac{V_e \dot{m}}{M_0 - \dot{m}t} - g\right) dt$$

Integrating from rest at $t = 0$

$$V = \int_0^t \left(\frac{V_e \dot{m}}{M_0 - \dot{m}t} - g\right) dt = -V_e \ln(M_0 - \dot{m}t)\Big]_0^t - gt$$

or

$$V = V_e \ln\left(\frac{M_0}{M_0 - \dot{m}t}\right) - gt \qquad (1)$$

At $t = 20$ sec,

$$V = 6500\,\frac{ft}{s}\,\ln\left(\frac{20\ lbm}{20\ lbm - 0.5\,\frac{lbm}{s}\times 20\,s}\right) - 32.2\,\frac{ft}{s^2}\times 20\,s$$

$$V(20\,s) = 3{,}860\ ft/s \qquad V_{20}$$

To find height, note $V = \frac{dY}{dt}$. Substitute into Eq. 1 to obtain

Problem 4.168

$$\frac{dY}{dt} = V_e \ln\left(\frac{M_0}{M_0 - \dot{m}t}\right) - gt = -V_e \ln\left(1 - \frac{\dot{m}t}{M_0}\right) - gt$$

Let $r = 1 - \frac{\dot{m}t}{M_0}$, and $dr = -\frac{\dot{m}}{M_0}dt$, then

$$dY = -V_e \ln r \, dt - gt\,dt = +\frac{V_e M_0}{\dot{m}} \ln r \, dr - gt\,dt$$

Integrating from $Y=0$ at $t=0$,

$$Y = \int_0^t \frac{V_e M_0}{\dot{m}} \ln r \, dr - \frac{1}{2}gt^2 = \frac{V_e M_0}{\dot{m}}\Big[r \ln r - r\Big]_0^t - \frac{1}{2}gt^2$$

$$= \frac{V_e M_0}{\dot{m}}\left\{\left(1 - \frac{\dot{m}t}{M_0}\right)\left[\ln\left(1 - \frac{\dot{m}t}{M_0}\right) - 1\right]\right\}\Big|_0^t - \frac{1}{2}gt^2$$

$$Y = \frac{V_e M_0}{\dot{m}}\left\{\left(1 - \frac{\dot{m}t}{M_0}\right)\left[\ln\left(1 - \frac{\dot{m}t}{M_0}\right) - 1\right] + 1\right\} - \frac{1}{2}gt^2$$

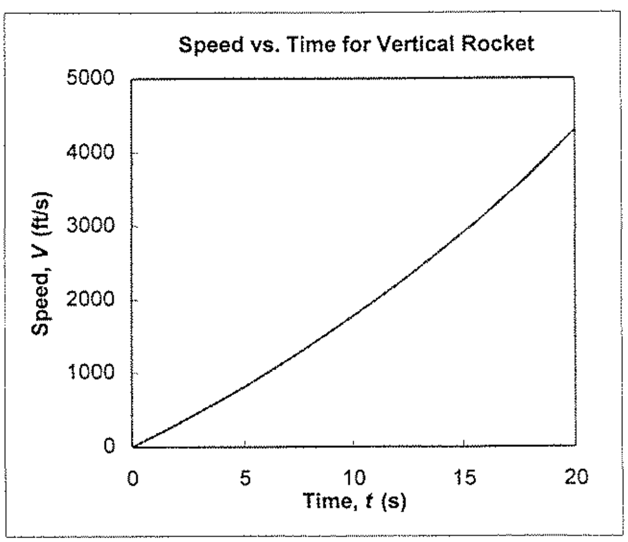
At $t = 20\text{ s}$,

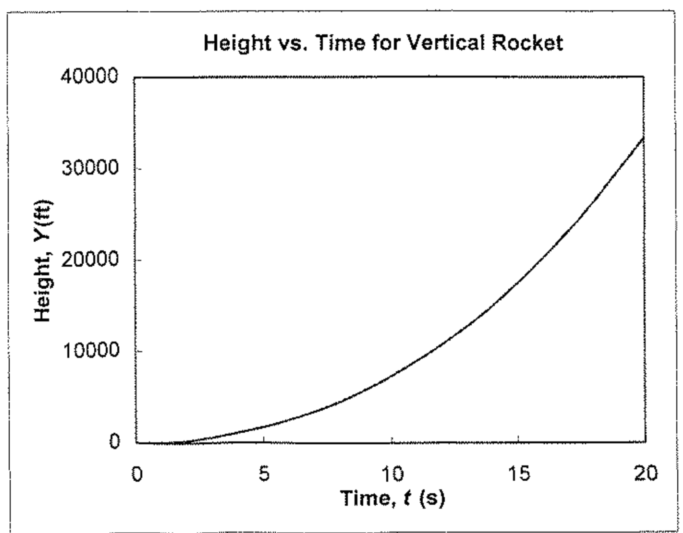
$$1 - \frac{\dot{m}t}{M_0} = 1 - 0.5\,\frac{\text{lbm}}{\text{s}} \times 20\text{ s} \times \frac{1}{20\text{ lbm}} = \frac{1}{2}$$

so

$$Y = 6500\,\frac{\text{ft}}{\text{s}} \times 20\text{ lbm} \times \frac{\text{s}}{0.5\text{ lbm}}\left\{\left(\frac{1}{2}\right)\left[\ln\left(\frac{1}{2}\right) - 1\right] + 1\right\} - \frac{1}{2} \times 32.2\,\frac{\text{ft}}{\text{s}^2}\,(20)^2\text{s}^2$$

$$Y = 33{,}500\text{ ft}$$ Y

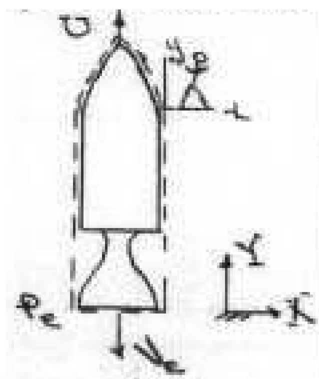




Problem 4.169

[Difficulty: 3]

4.169 A large two-stage liquid rocket with mass of 30,000 kg is to be launched from a sea-level launch pad. The main engine burns liquid hydrogen and liquid oxygen in a stoichiometric mixture at 2450 kg/s. The thrust nozzle has an exit diameter of 2.6 m. The exhaust gases exit the nozzle at 2270 m/s and an exit plane pressure of 66 kPa absolute. Calculate the acceleration of the rocket at liftoff. Obtain an expression for speed as a function of time, neglecting air resistance.



Solution: Apply y component of momentum equation to CV with linear acceleration

Basic equation: $$F_{S_y} + F_{B_y} - \int_{CV} a_{rf_y}\,\rho\, dV = \cancel{\frac{\partial}{\partial t}\int_{CV} v_{xyz}\,\rho\, dV}^{\,0(2)} + \int_{CS} v_{xyz}\,\rho \vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: (1) F_{S_y} due to pressure, p_{atm} assumed constant, neglect air resistance
(2) neglect rate of change of momentum inside CV
(3) uniform flow at exit

Then, $$(p_e - p_{atm})A_e - Mg - a_{rf_y}M = v_e\{+\dot{m}\} = -\dot{m}V_e$$

Solving for a_{rf_y},
$$a_{rf_y} = \frac{dU}{dt} = \frac{1}{M}\left[\dot{m}V_e + (p_e - p_{atm})A_e\right] - g \quad \ldots(1)$$

$M = M(t)$. From conservation of mass $\frac{\partial}{\partial t}\int_{CV}\rho\, dV + \int_{CS}\rho\vec{V}\cdot d\vec{A} = 0$

Then $\frac{\partial}{\partial t}\int_{CV}\rho\, dV = \frac{dM}{dt} = -\int_{CS}\rho\vec{V}\cdot d\vec{A} = -\dot{m}_e$ (constant)

Hence $M(t) = M_0 - \dot{m}t$, and

$$a_{rf_y} = \frac{dU}{dt} = \frac{\dot{m}V_e}{M_0 - \dot{m}t} + \frac{(p_e - p_{atm})A_e}{M_0 - \dot{m}t} - g$$

$$U = \int_0^U dU = \int_0^t \frac{\dot{m}V_e}{M_0 - \dot{m}t}\,dt + \int_0^t \frac{(p_e - p_{atm})A_e}{M_0 - \dot{m}t}\,dt - \int_0^t g\,dt$$

$$U = -V_e \ln\left[\frac{M_0 - \dot{m}t}{M_0}\right] - \frac{(p_e - p_{atm})A_e}{\dot{m}}\ln\left[\frac{M_0 - \dot{m}t}{M_0}\right] - gt$$

$$U = -\left[V_e + \frac{(p_e - p_{atm})A_e}{\dot{m}}\right]\ln\left[\frac{M_0 - \dot{m}t}{M_0}\right] - gt \quad \longleftarrow U(t)$$

At lift-off, $t = 0$, $M = M_0$

$$a_{rf_y} = \frac{1}{M}\left[\dot{m}V_e + (p_e - p_{atm})A_e\right] - g$$

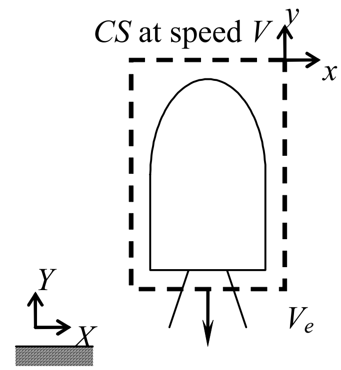
$$= \frac{1}{3\times10^4\ \text{kg}}\left[2450\,\frac{\text{kg}}{\text{s}} \times 2270\,\frac{\text{m}}{\text{s}} + (66 - 101)10^3\,\frac{\text{N}}{\text{m}^2} \times \frac{\pi}{4}(2.6)^2\,\text{m}^2 \cdot \frac{\text{kg}\cdot\text{m}}{\text{N}\cdot\text{s}^2}\right] - 9.81\,\frac{\text{m}}{\text{s}^2}$$

$$a_{rf_y} = 169\ \text{m/s}^2 \quad \longleftarrow a_{rf_y}$$

Problem 4.170

[Difficulty: 3]

4.170 Neglecting air resistance, what speed would a vertically directed rocket attain in 5 s if it starts from rest, has initial mass of 350 kg, burns 10 kg/s, and ejects gas at atmospheric pressure with a speed of 2500 m/s relative to the rocket? What would be the maximum velocity? Plot the rocket speed as a function of time for the first minute of flight.



Given: Data on rocket

Find: Speed after 5 s; Maximum velocity; Plot of speed versus time

Solution:

Basic equation: Momentum flux in y direction
$$F_{S_y} + F_{B_y} - \int_{CV} a_{rf_y}\,\rho\, dV\!\!\!\!- = \frac{\partial}{\partial t}\int_{CV} v_{xyz}\,\rho\, dV\!\!\!\!- + \int_{CS} v_{xyz}\,\rho \vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: 1) No resistance 2) $p_e = p_{atm}$ 3) Uniform flow 4) Use relative velocities 5) Constant mass flow rate

From continuity $\dfrac{dM}{dt} = m_{rate} = \text{constant}$ so $M = M_0 - m_{rate}\cdot t$ (Note: Software cannot render a dot!)

Hence from momentum $-M\cdot g - a_{rfy}\cdot M = u_e\cdot(\rho_e\cdot V_e\cdot A_e) = -V_e\cdot m_{rate}$ or $a_{rfy} = \dfrac{dV}{dt} = \dfrac{V_e\cdot m_{rate}}{M} - g = \dfrac{V_e\cdot m_{rate}}{M_0 - m_{rate}\cdot t} - g$

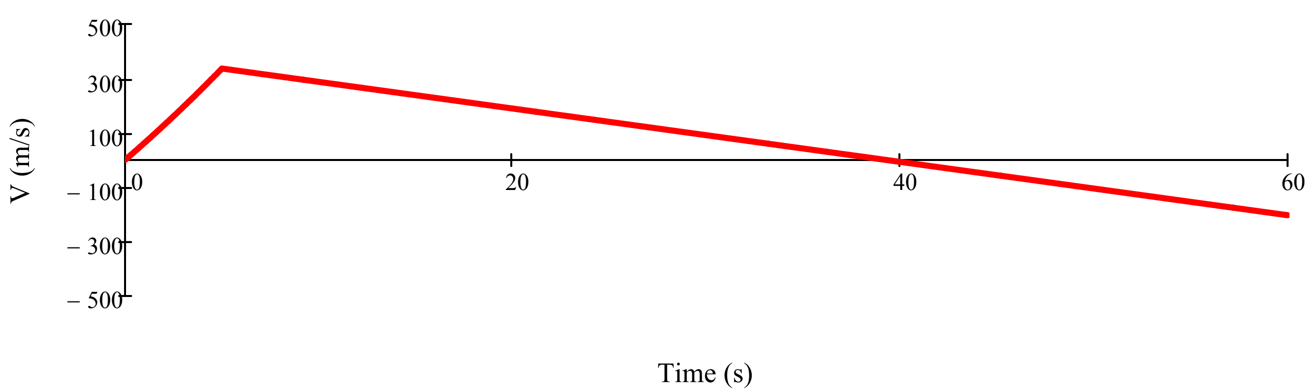
Separating variables $dV = \left(\dfrac{V_e\cdot m_{rate}}{M_0 - m_{rate}\cdot t} - g\right)\cdot dt$

Integrating from V = at t = 0 to V = V at t = t

$$V = -V_e\cdot\left(\ln(M_0 - m_{rate}\cdot t) - \ln(M_0)\right) - g\cdot t = -V_e\cdot\ln\left(1 - \frac{m_{rate}\cdot t}{M_0}\right) - g\cdot t \qquad V = -V_e\cdot\ln\left(1 - \frac{m_{rate}\cdot t}{M_0}\right) - g\cdot t$$

At t = 5 s $V_{max} = -2500\cdot\dfrac{m}{s}\cdot\ln\left(1 - 10\cdot\dfrac{kg}{s}\times\dfrac{1}{350\cdot kg}\times 5\cdot s\right) - 9.81\cdot\dfrac{m}{s^2}\times 5\cdot s$ $\quad V_{max} = 336\,\dfrac{m}{s}$

For the motion after 5 s, assuming the fuel is used up, the equation of motion becomes $a = -M\cdot g$



Problem 4.171

[Difficulty: 3]

4.171 Inflate a toy balloon with air and release it. Watch as the balloon darts about the room. Explain what causes the phenomenon you see.

Discussion: Air blown into a balloon to inflate it must be compressed to overcome the skin's resistance to stretching. (Remember how hard it is to create enough pressure to "start" the inflation process!) After decreasing briefly, the required pressure seems to increase as inflation of the balloon continues.

As the balloon is inflated, the skin stretches and stores energy. When the inflated balloon is released, the stored energy in the skin forces the compressed air out the open mouth of the balloon. The expansion of the compressed air to the lower surrounding atmospheric pressure creates a high-speed jet of air, which propels the relatively light balloon initially at a high speed.

The moving balloon is unstable because it has a poor aerodynamic shape. Therefore it darts about in a random pattern. The balloon keeps moving as long as it contains pressurized air to act as a propulsion jet. However, it is not long before the energy stored in the skin is exhausted and the air in the balloon is reduced to atmospheric pressure.

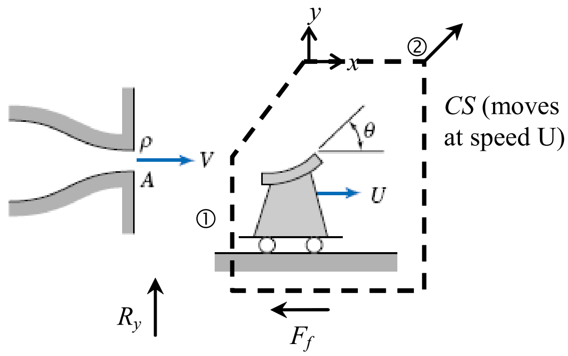
When the balloon reaches atmospheric pressure it is slowed by aerodynamic drag. Finally the empty, wrinkled balloon simply falls to the floor.

Some toys that use a balloon for propulsion are available. Most have stabilizing surfaces. It is instructive to study these toys carefully to understand how each works, and why each toy is shaped the way it is.

Problem 4.172

[Difficulty: 4]

4.172 The vane/cart assembly of mass $M = 30$ kg, shown in Problem 4.128, is driven by a water jet. The water leaves the stationary nozzle of area $A = 0.02\text{ m}^2$, with a speed of 20 m/s. The coefficient of kinetic friction between the assembly and the surface is 0.10. Plot the terminal speed of the assembly as a function of vane turning angle, θ, for $0 \le \theta \le \pi/2$. At what angle does the assembly begin to move if the coefficient of static friction is 0.15?



Given: Water jet striking moving vane

Find: Plot of terminal speed versus turning angle; angle to overcome static friction

Solution:

Basic equations: Momentum flux in x and y directions

$$F_{S_x} + F_{B_x} - \int_{CV} a_{rf_x}\,\rho\, d\mathcal{V} = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\, d\mathcal{V} + \int_{CS} u_{xyz}\,\rho \vec{V}_{xyz}\cdot d\vec{A}$$

$$F_{S_y} + F_{B_y} - \int_{CV} a_{rf_y}\,\rho\, d\mathcal{V} = \frac{\partial}{\partial t}\int_{CV} v_{xyz}\,\rho\, d\mathcal{V} + \int_{CS} v_{xyz}\,\rho \vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: 1) Incompressible flow 2) Atmospheric pressure in jet 3) Uniform flow 4) Jet relative velocity is constant

Then $$-F_f - M\cdot a_{rfx} = u_1\cdot\left(-\rho\cdot V_1\cdot A_1\right) + u_2\cdot\left(\rho\cdot V_2\cdot A_2\right) = -(V-U)\cdot[\rho\cdot(V-U)\cdot A] + (V-U)\cdot\cos(\theta)\cdot[\rho\cdot(V-U)\cdot A]$$

$$a_{rfx} = \frac{\rho(V-U)^2\cdot A\cdot(1-\cos(\theta)) - F_f}{M} \qquad (1)$$

Also $$R_y - M\cdot g = v_1\cdot\left(-\rho\cdot V_1\cdot A_1\right) + v_2\cdot\rho\cdot V_2\cdot A_2 = 0 + (V-U)\cdot\sin(\theta)\cdot[\rho\cdot(V-U)\cdot A]$$

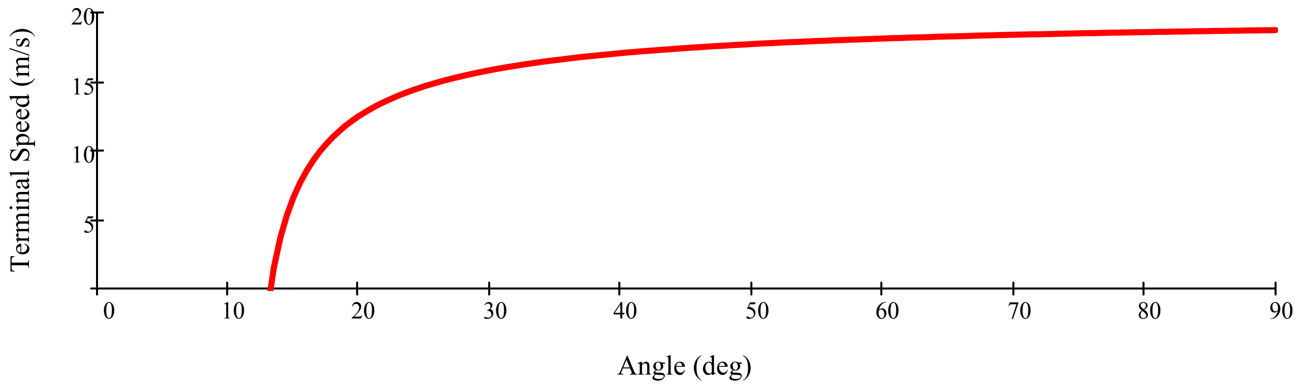
$$R_y = M\cdot g + \rho(V-U)^2\cdot A\cdot\sin(\theta)$$

At terminal speed $a_{rfx} = 0$ and $F_f = \mu_k R_y$. Hence in Eq 1

$$0 = \frac{\rho\cdot\left(V-U_t\right)^2\cdot A\cdot(1-\cos(\theta)) - \mu_k\cdot\left[M\cdot g + \rho\cdot\left(V-U_t\right)^2\cdot A\cdot\sin(\theta)\right]}{M} = \frac{\rho\cdot\left(V-U_t\right)^2\cdot A\cdot\left(1-\cos(\theta)-\mu_k\cdot\sin(\theta)\right)}{M} - \mu_k\cdot g$$

or $$V - U_t = \sqrt{\frac{\mu_k\cdot M\cdot g}{\rho\cdot A\cdot\left(1-\cos(\theta)-\mu_k\cdot\sin(\theta)\right)}} \qquad U_t = V - \sqrt{\frac{\mu_k\cdot M\cdot g}{\rho\cdot A\cdot\left(1-\cos(\theta)-\mu_k\cdot\sin(\theta)\right)}}$$

The terminal speed as a function of angle is plotted below; it can be generated in *Excel*



For the static case $F_f = \mu_s \cdot R_y$ and $a_{rfx} = 0$ (the cart is about to move, but hasn't)

Substituting in Eq 1, with U = 0

$$0 = \frac{\rho \cdot V^2 \cdot A \cdot \left[1 - \cos(\theta) - \mu_s \cdot \left(\rho \cdot V^2 \cdot A \cdot \sin(\theta) + M \cdot g\right)\right.}{M}$$

or $$\cos(\theta) + \mu_s \cdot \sin(\theta) = 1 - \frac{\mu_s \cdot M \cdot g}{\rho \cdot V^2 \cdot A}$$

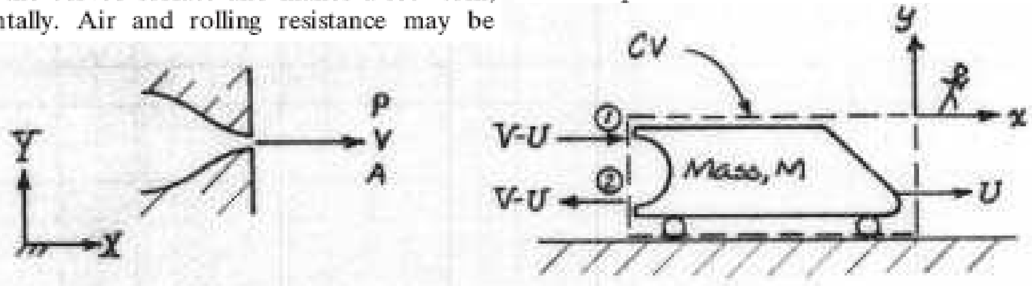
We need to solve this for θ! This can be done by hand or by using Excel's Goal Seek or Solver $\theta = 19.0 \cdot deg$

Note that we need $\theta = 19^o$, but once started we can throttle back to about $\theta = 12.5^o$ and still keep moving!

Problem 4.173

[Difficulty: 3]

4.173 Consider the vehicle shown in Problem 4.149. Starting from rest, it is propelled by a hydraulic catapult (liquid jet). The jet strikes the curved surface and makes a 180° turn, leaving horizontally. Air and rolling resistance may be neglected. Using the notation shown, obtain an equation for the acceleration of the vehicle at any time and determine the time required for the vehicle to reach $U = V/2$.



Solution: Apply x component of momentum equation using linearly accelerating CV shown above.

Basic equation:
$$\underbrace{F_{S_x}}_{=0(1)} + \underbrace{F_{B_x}}_{=0(2)} - \int_{CV} a_{rf_x}\,\rho\, d\forall = \underbrace{\frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\, d\forall}_{\approx 0(3)} + \int_{CS} u_{xyz}\,\rho \vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: (1) $F_{S_x} = 0$

(2) $F_{B_x} = 0$

(3) Neglect mass of liquid and rate of change of u in CV

(4) Uniform flow at each section

(5) Jet area and speed with respect to vehicle are constant

Then
$$-Ma_{rf_x} = -M\frac{dU}{dt} = u_1\{-|\rho(V-U)A|\} + u_2\{|\rho(V-U)A|\}$$

$$u_1 = V-U \qquad u_2 = -(V-U)$$

or
$$a_{rf_x} = \frac{dU}{dt} = \frac{2\rho(V-U)^2 A}{M} \quad ; \quad \frac{dU}{(V-U)^2} = \frac{2\rho A}{M}dt \quad ; -\frac{d(V-U)}{(V-U)^2} = \frac{2\rho A}{M}dt$$

To obtain $a_{rf_x}(t)$, we must first find $U(t)$. Integrating from $U=0$ at $t=0$ to U at t,

$$\int_{V-U=V}^{V-U} -\frac{d(V-U)}{(V-U)^2} = \left.\frac{1}{V-U}\right]_V^{V-U} = \frac{1}{V-U} - \frac{1}{V} = \frac{V-(V-U)}{V(V-U)} = \frac{2\rho A}{M}t \; ; \; \frac{U}{V-U} = \frac{2\rho VA}{M}t$$

Solving,
$$U = (V-U)\frac{2\rho VA}{M}t \;, \quad U = V\frac{\frac{2\rho VA}{M}t}{1+\frac{2\rho VA}{M}t} \quad \text{and} \quad V-U = V\left[1-\frac{\frac{2\rho VA}{M}t}{1+\frac{2\rho VA}{M}t}\right]$$

Substituting,
$$a_{rf_x} = \frac{2\rho V^2 A}{M}\left[1-\frac{\frac{2\rho VA}{M}t}{1+\frac{2\rho VA}{M}t}\right]^2 = \frac{2\rho V^2 A}{M}\left[\frac{1}{1+\frac{2\rho VA}{M}t}\right]^2$$ $a_{rf_x}(t)$

The time to reach $U = V/2$ is

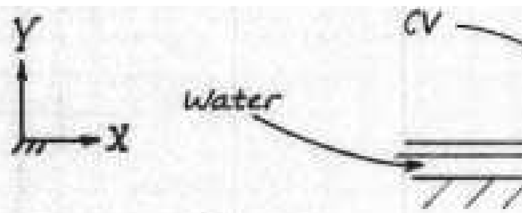
$$\frac{U}{V} = \frac{1}{2} = \frac{\frac{2\rho VA}{M}t}{1+\frac{2\rho VA}{M}t} \quad \text{or} \quad t = \frac{M}{2\rho VA}$$ $t\left(\frac{V}{2}\right)$

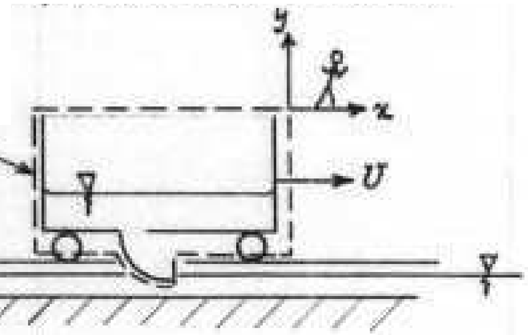
Check: $\left[\frac{M}{\rho VA}\right] = M\frac{L^3}{M}\frac{t}{L}\frac{1}{L^2} = t$ ✓ ; $\left[\frac{\rho V^2 A}{M}\right] = \frac{M}{L^3}\frac{L^2}{t^2}L^2\frac{1}{M} = \frac{L}{t^2}$ ✓

Problem 4.174

[Difficulty: 3]

4.174 The moving tank shown is to be slowed by lowering a scoop to pick up water from a trough. The initial mass and speed of the tank and its contents are M_0 and U_0, respectively. Neglect external forces due to pressure or friction and assume that the track is horizontal. Apply the continuity and momentum equations to show that at any instant $U = U_0 M_0 / M$. Obtain a general expression for U/U_0 as a function of time.





Solution: Apply continuity and momentum equations to linearly accelerating CV shown.

Basic equations: $0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V}_{xyz} \cdot d\vec{A}$

$$\overbrace{\cancel{F_{S_x}}}^{=0(1)} + \overbrace{\cancel{F_{B_x}}}^{=0(2)} - \int_{CV} a_{rf_x} \rho \, d\forall = \frac{\partial}{\partial t}\int_{CV} \overbrace{\cancel{u_{xyz}}}^{\approx 0(3)} \rho \, d\forall + \int_{CS} u_{xyz} \rho \vec{V}_{xyz} \cdot d\vec{A}$$

Assumptions: (1) $F_{S_x} = 0$

(2) $F_{B_x} = 0$

(3) Neglect u within CV

(4) Uniform flow across inlet section

From continuity

$$0 = \frac{\partial}{\partial t} M_{cv} + \{-|\rho U A|\} \quad \text{or} \quad \frac{dM}{dt} = \rho U A$$

From momentum

$$-a_{rf_x} M = -\frac{dU}{dt} M = u\{-|\rho U A|\} = U \rho U A, \text{ since } u = -U$$

But from continuity, $\rho U A = \frac{dM}{dt}$, so

$$M \frac{dU}{dt} + U \frac{dM}{dt} = 0 \quad \text{or} \quad UM = \text{constant} = U_0 M_0 \,; \quad U = U_0 M_0 / M$$ U

Substituting $M = M_0 U_0 / U$ into momentum, $-\frac{dU}{dt}\frac{M_0 U_0}{U} = \rho U^2 A$, or

$$\frac{dU}{U^3} = -\frac{\rho A}{U_0 M_0} dt$$

Integrating, $\int_{U_0}^{U} \frac{dU}{U^3} = -\frac{1}{2}\frac{1}{U^2}\Big]_{U_0}^{U} = -\frac{1}{2}\left(\frac{1}{U^2} - \frac{1}{U_0^2}\right) = -\int_0^t \frac{\rho A}{U_0 M_0} dt = -\frac{\rho A}{U_0 M_0} t$

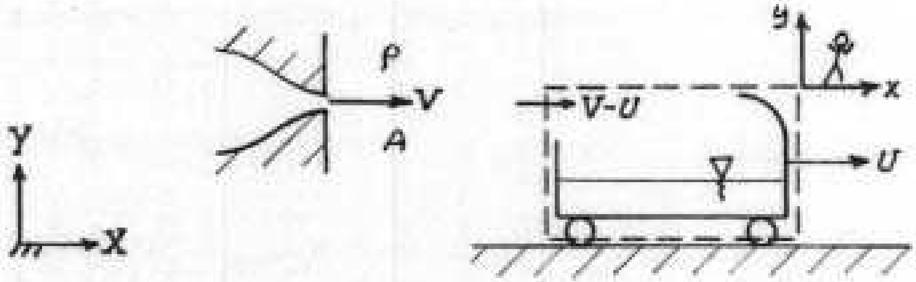
Solving for U,

$$U = \frac{U_0}{\left[1 + \frac{2\rho U_0 A}{M_0} t\right]^{\frac{1}{2}}}$$ U(t)

Problem 4.175

[Difficulty: 3]

4.175 The tank shown rolls with negligible resistance along a horizontal track. It is to be accelerated from rest by a liquid jet that strikes the vane and is deflected into the tank. The initial mass of the tank is M_0. Use the continuity and momentum equations to show that at any instant the mass of the vehicle and liquid contents is $M = M_0V/(V - U)$. Obtain a general expression for U/V as a function of time.



Find: (a) Apply continuity and momentum to show $M = M_0 V/(V-U)$
(b) General expression for U/V as a function of time.

Solution: Apply continuity and x component of momentum equation to linearly accelerating CV shown.

Basic equations: $0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V}_{xyz} \cdot d\vec{A}$

$$\underbrace{F_{S_x}}_{=0(1)} + \underbrace{F_{B_x}}_{=0(2)} - \int_{CV} a_{rf_x} \rho \, d\forall = \underbrace{\frac{\partial}{\partial t}\int_{CV} u_{xyz} \rho \, d\forall}_{\approx 0(3)} + \int_{CS} u_{xyz} \rho \vec{V}_{xyz} \cdot d\vec{A}$$

Assumptions: (1) $F_{S_x} = 0$
(2) $F_{B_x} = 0$
(3) Neglect u within CV
(4) Uniform flow in jet

From continuity

$$0 = \frac{\partial}{\partial t} M_{cv} + \{-|\rho(V-U)A|\} \quad \text{or} \quad \frac{dM}{dt} = \rho(V-U)A$$

From momentum

$$-a_{rf_x} M = -\frac{dU}{dt} M = u\{-|\rho(V-U)A|\} = (V-U)[-\rho(V-U)A]; \; u = V-U$$

But from continuity, $\rho(V-U)A = \frac{dM}{dt}$, and $dU = -d(V-U)$, so

$$-\frac{dU}{dt} M = \frac{d(V-U)}{dt} M = -(V-U)\frac{dM}{dt} \quad \text{or} \quad M(V-U) = \text{constant} = M_0 V$$

Thus $M = M_0 V/(V-U)$ — **M**

Substituting into momentum, $-\frac{dU}{dt} M = \frac{d(V-U)}{dt} \frac{M_0 V}{(V-U)} = -\rho(V-U)^2 A$, or

$$\frac{d(V-U)}{(V-U)^3} = -\frac{\rho A}{V M_0} dt$$

Integrating, $$\int_V^{V-U} \frac{d(V-U)}{(V-U)^3} = -\frac{1}{2}\left[\frac{1}{(V-U)^2} - \frac{1}{V^2}\right] = -\int_0^t \frac{\rho A}{V M_0} dt = -\frac{\rho A}{V M_0} t$$

Solving,

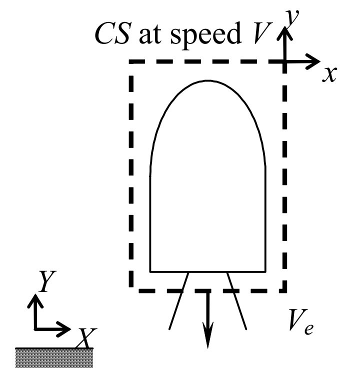
$$\frac{U}{V} = \left\{1 - \frac{1}{\left[1 + \frac{2\rho V A}{M_0} t\right]^{1/2}}\right\}$$

— $\frac{U}{V}$

Problem 4.176

[Difficulty: 4]

4.176 A model solid propellant rocket has a mass of 69.6 g, of which 12.5 g is fuel. The rocket produces 5.75 N of thrust for a duration of 1.7 s. For these conditions, calculate the maximum speed and height attainable in the absence of air resistance. Plot the rocket speed and the distance traveled as functions of time.



Given: Data on rocket

Find: Maximum speed and height; Plot of speed and distance versus time

Solution:

Basic equation: Momentum flux in y direction

$$F_{S_y} + F_{B_y} - \int_{CV} a_{rf_y}\,\rho\,dV = \frac{\partial}{\partial t}\int_{CV} v_{xyz}\,\rho\,dV + \int_{CS} v_{xyz}\,\rho \vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: 1) No resistance 2) $p_e = p_{atm}$ 3) Uniform flow 4) Use relative velocities 5) Constant mass flow rate

From continuity $\quad \dfrac{dM}{dt} = m_{rate} = \text{constant} \quad$ so $\quad M = M_0 - m_{rate}\cdot t \qquad$ (Note: Software cannot render a dot!)

Hence from momentum $\quad -M\cdot g - a_{rfy}\cdot M = u_e\cdot\left(\rho_e\cdot V_e\cdot A_e\right) = -V_e\cdot m_{rate}$

Hence $\quad a_{rfy} = \dfrac{dV}{dt} = \dfrac{V_e\cdot m_{rate}}{M} - g = \dfrac{V_e\cdot m_{rate}}{M_0 - m_{rate}\cdot t} - g$

Separating variables $\quad dV = \left(\dfrac{V_e\cdot m_{rate}}{M_0 - m_{rate}\cdot t} - g\right)\cdot dt$

Integrating from V = at t = 0 to V = V at t = t

$$V = -V_e\cdot\left(\ln\left(M_0 - m_{rate}\cdot t\right) - \ln\left(M_0\right)\right) - g\cdot t = -V_e\cdot\ln\left(1 - \frac{m_{rate}\cdot t}{M_0}\right) - g\cdot t$$

$$V = -V_e\cdot\ln\left(1 - \frac{m_{rate}\cdot t}{M_0}\right) - g\cdot t \qquad \text{for} \qquad t \le t_b \qquad \text{(burn time)} \qquad (1)$$

To evaluate at t_b = 1.7 s, we need V_e and m_{rate} $\quad m_{rate} = \dfrac{m_f}{t_b} \quad m_{rate} = \dfrac{12.5\cdot gm}{1.7\cdot s} \quad m_{rate} = 7.35\times 10^{-3}\,\dfrac{kg}{s}$

Also note that the thrust F_t is due to momentum flux from the rocket $\quad F_t = m_{rate}\cdot V_e \quad V_e = \dfrac{F_t}{m_{rate}} \quad V_e = \dfrac{5.75\cdot N}{7.35\times 10^{-3}\cdot\dfrac{kg}{s}}\times\dfrac{kg\cdot m}{s^2\cdot N} \quad V_e = 782\,\dfrac{m}{s}$

Hence $\quad V_{max} = -V_e\cdot\ln\left(1 - \dfrac{m_{rate}\cdot t_b}{M_0}\right) - g\cdot t_b$

$$V_{max} = -782\cdot\frac{m}{s}\cdot\ln\left(1 - 7.35\times 10^{-3}\cdot\frac{kg}{s}\times\frac{1}{0.0696\cdot kg}\times 1.7\cdot s\right) - 9.81\cdot\frac{m}{s^2}\times 1.7\cdot s \qquad V_{max} = 138\,\frac{m}{s}$$

To obtain Y(t) we set V = dY/dt in Eq 1, and integrate to find

$$Y = \frac{V_e \cdot M_0}{m_{rate}} \cdot \left[\left(1 - \frac{m_{rate} \cdot t}{M_0}\right) \cdot \left(\ln\left(1 - \frac{m_{rate} \cdot t}{M_0}\right) - 1 \right) + 1 \right] - \frac{1}{2} \cdot g \cdot t^2 \qquad t \le t_b \qquad t_b = 1.7 \cdot s \qquad (2)$$

At $t = t_b$

$$Y_b = 782 \cdot \frac{m}{s} \times 0.0696 \cdot kg \times \frac{s}{7.35 \times 10^{-3} \cdot kg} \cdot \left[\left(1 - \frac{0.00735 \cdot 1.7}{0.0696}\right) \left(\ln\left(1 - \frac{.00735 \cdot 1.7}{.0696}\right) - 1 \right) + 1 \right] \ldots$$
$$+ -\frac{1}{2} \times 9.81 \cdot \frac{m}{s^2} \times (1.7 \cdot s)^2$$

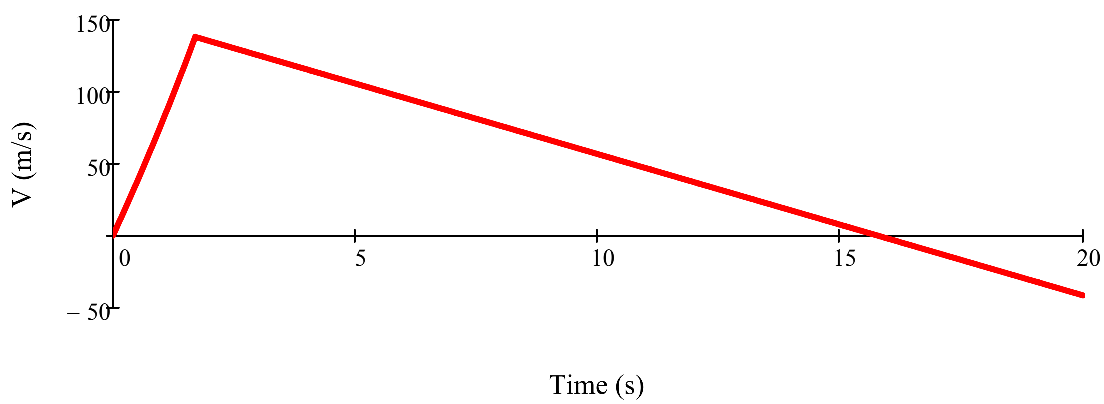
$$Y_b = 113\ m$$

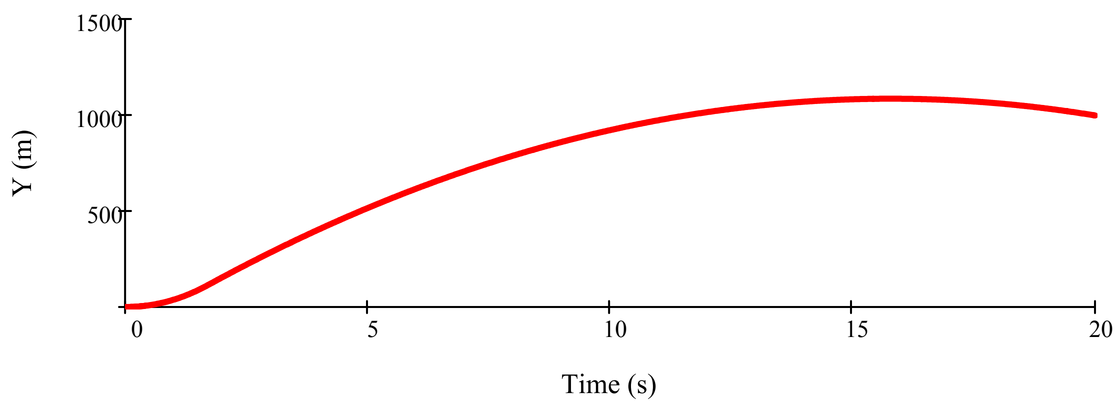
After burnout the rocket is in free assent. Ignoring drag

$$V(t) = V_{max} - g \cdot (t - t_b) \qquad (3)$$

$$Y(t) = Y_b + V_{max} \cdot (t - t_b) - \frac{1}{2} \cdot g \cdot (t - t_b)^2 \qquad t > t_b \qquad (4)$$

The speed and position as functions of time are plotted below. These are obtained from Eqs 1 through 4, and can be plotted in *Excel*





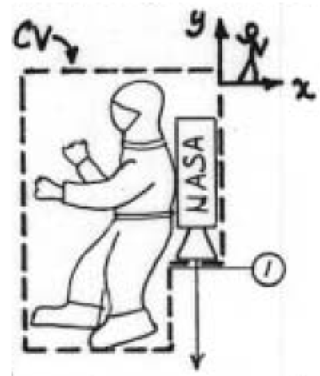
Using Solver, or by differentiating y(t) and setting to zero, or by setting V(t) = 0, we find for the maximum y

$$t = 15.8\ s \qquad y_{max} = 1085\ m$$

Problem 4.177

[Difficulty: 3]

4.177 A small rocket motor is used to power a "jet pack" device to lift a single astronaut above the Moon's surface. The rocket motor produces a uniform exhaust jet with constant speed, $V_e = 3000$ m/s, and the thrust is varied by changing the jet size. The total initial mass of the astronaut and the jet pack is $M_0 = 200$ kg, 100 kg of which is fuel and oxygen for the rocket motor. Find (a) the exhaust mass flow rate required to just lift off initially, (b) the mass flow rate just as the fuel and oxygen are used up, and (c) the maximum anticipated time of flight. Note that the Moon's gravity is about 17 percent of Earth's.



Given: Data on "jet pack" rocket

Find: Initial exhaust mass flow rate; mass flow rate at end; maximum time of flight

Solution:

Basic equation: Momentum flux in y direction $$F_{S_y} + F_{B_y} - \int_{CV} a_{rf_y}\,\rho\, dV = \frac{\partial}{\partial t}\int_{CV} v_{xyz}\,\rho\, dV + \int_{CS} v_{xyz}\,\rho \vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: 1) Jet pack just hovers 2) Steady flow 3) Uniform flow 4) Use relative velocities

Given data $\quad V_e = 3000\frac{m}{s} \quad M_0 = 200\,kg \quad M_{fuel} = 100\,kg \quad g_{moon} = 0.17\cdot g \quad g_{moon} = 1.67\frac{m}{s^2}$

At all instants, the momentum becomes $\quad -M\cdot g_{moon} = -v_1\cdot m_{rate} = -V_e\cdot m_{rate} \quad$ or $\quad m_{rate} = \frac{M\cdot g_{moon}}{V_e}$

Hence, initially $\quad m_{rateinit} = \frac{M_0\cdot g_{moon}}{V_e} \quad m_{rateinit} = 0.111\frac{kg}{s}$

Finally, when all the fuel is just used up, the mass is $\quad M_f = M_0 - M_{fuel} \quad M_f = 100\,kg$

Then $\quad m_{ratefinal} = \frac{M_f\cdot g_{moon}}{V_e} \quad m_{ratefinal} = 0.0556\frac{kg}{s}$

Flight ends as fuel is used up. To find this, from continuity $\quad \frac{dM}{dt} = m_{rate} \quad$ but $\quad m_{rate} = \frac{M\cdot g_{moon}}{V_e}$

Hence $\quad \frac{dM}{dt} = \frac{M\cdot g_{moon}}{V_e} \quad$ so $\quad \frac{dM}{M} = \frac{g_{moon}}{V_e}\cdot dt$

Integrating $\quad \ln\left(\frac{M_0}{M}\right) = \frac{g_{moon}}{V_e}\cdot t \quad$ or $\quad M = M_0\cdot e^{-\frac{g_{moon}}{V_e}\cdot t}$

Solving for t $\quad t = -\frac{V_e}{g_{moon}}\cdot \ln\left(\frac{M}{M_0}\right) \quad$ so when $\quad M = M_f \quad t_{final} = -\frac{V_e}{g_{moon}}\cdot \ln\left(\frac{M_f}{M_0}\right) \quad t_{final} = 20.8\,min$

4.178 Several toy manufacturers sell water "rockets" that consist of plastic tanks to be partially filled with water and then pressurized with air. Upon release, the compressed air forces water out of the nozzle rapidly, propelling the rocket. You are asked to help specify optimum conditions for this water-jet propulsion system. To simplify the analysis, consider horizontal motion only. Perform the analysis and design needed to define the acceleration performance of the compressed air/water-propelled rocket. Identify the fraction of tank volume that initially should be filled with compressed air to achieve optimum performance (i.e., maximum speed from the water charge). Describe the effect of varying the initial air pressure in the tank.

Discussion: The process may be modeled as a polytropic expansion of the trapped air which forces water out the jet nozzle, causing the "rocket" to accelerate. The polytropic exponent may be varied to model anything from an isothermal expansion process ($n = 1$) to an adiabatic expansion process ($n = k$), which is more likely to be an accurate model for the sudden expansion of the air.

Speed of the water jet leaving the "rocket" is proportional to the square root of the pressure difference between the tank and atmosphere.

Qualitatively it is apparent that the smaller the initial volume fraction of trapped air, the larger will be the expansion ratio of the air, and the more rapid will be the pressure reduction as the air expands. This will cause the water jet speed to drop rapidly. The combination of low water jet speed and relatively large mass of water will produce sluggish acceleration.

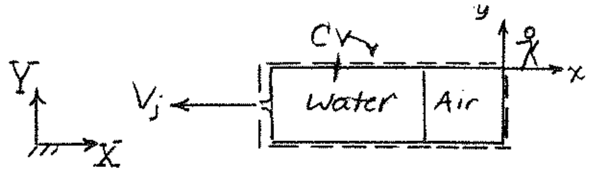
Increasing the initial volume fraction of air will reduce the expansion ratio, so higher pressure will be maintained longer in the tank and the water jet will maintain higher speed longer. This combined with the relatively small mass of water in the tank will produce rapid acceleration.

If the initial volume fraction of air is too large, all water will be expended before the air pressure is reduced significantly. In this situation, some of the stored energy of the air will be dissipated in a relatively ineffective air jet. Consequently, for any initial pressure in the tank, there is an optimum initial air fraction.

This problem cannot be solved in closed form because of the varying air pressure, mass flow rate, and mass of water in the tank;. it can only be solved numerically. One possible integration scheme is to increment time and solve for all properties of the system at each instant. The drawback to this scheme is that the water is unlikely to be exhausted at an even increment of time. A second scheme is to increment the volume of water remaining and solve for properties using the average flow rate during the interval. This scheme is outlined below.

Model the air/water jet-propelled "rocket" using the CV and coordinates shown.

First choose dimensions and mass of "rocket" to be simulated:



Input Data:

Jet diameter:	$D_j =$	0.003	m
Tank diameter:	$D_t =$	0.035	m
Tank length:	$L =$	0.1	m
Tank mass:	$M_t =$	0.01	kg
Polytropic exponent:	$n =$	1.4	---

Next choose initial conditions for the simulation (see sample calculations below):

Initial Conditions:

Air fraction in tank:	$\alpha =$	0.5	---
Tank pressure:	$p_0 =$	200	kPa (gage)
Volume increment:	$\Delta\alpha =$	0.02	---

Compute reference parameters:

Calculated Parameters:

Jet area:	$A_j =$	7.07E-06	m^2
Tank volume:	$V_t =$	9.62E-05	m^3
Initial air volume:	$V_0 =$	4.81E-05	m^3
Initial water mass:	$M_0 =$	0.0481	kg

(These are used in the spreadsheet below.)

Then decrease the water fraction in the tank by $\Delta\alpha$:

Calculated Results:

Water Fraction, V_w/V_t (---)	Gage Pressure, p (kPa)	Water Mass, M_w (kg)	Jet Speed, V_j (m/s)	Flow Rate, dm/dt (kg/s)	Time Interval, Δt (s)	Current Time, t (s)	"Rocket" Accel., a (m/s^2)	"Rocket" Speed, U (m/s)
0.50	200	0.0481	20.0	0.141	0	0	48.7	0
0.48	184	0.0461	19.2	0.135	0.0139	0.0139	47.5	0.668

The computation is made as follows:

(1) Decrease α by $\Delta\alpha$

(2) Compute p from $p = p_0\left(\frac{V_0}{V}\right)^n$

$$p = (200+101.325)\,kPa\left(\frac{0.50}{0.52}\right)^{1.4} - 101.325 = 183.9 \ kPa\,(gage)$$

(3) Use Bernoulli to calculate jet speed

$$V_j = \sqrt{\frac{2\Delta p}{\rho}} = \left[2 \times 183.9\times10^3 \frac{N}{m^2} \times \frac{m^3}{999\,kg} \times \frac{kg\cdot m}{N\cdot s^2}\right]^{\frac{1}{2}} = 19.10 \ m/s^*$$

(4) Calculate water mass using α.

(5) Use conservation of mass to compute mass flow rate

$$\dot{m} = \rho V_j A_j = 999\frac{kg}{m^3} \times 19.10\frac{m}{s} \times 7.07\times10^{-6} m^2 = 0.1349 \ kg/s$$

(6) Use the average mass flow rate during the interval to approximate Δt:

$$\Delta t = \frac{\Delta m}{dm/dt} = \frac{\Delta m}{\bar{\dot{m}}} = (0.0481 - 0.0461)\,kg \times \frac{s}{0.138\,kg} = 0.01449 \ s^*$$

(7) Use momentum to compute acceleration (note $M = M_w + M_t$):

$$a_{rf_x} = \frac{\dot{m}V_j}{M} = 0.135\frac{kg}{s} \times 19.2\frac{m}{s} \times \frac{1}{0.0461 + 0.0100\,kg} = 46.2 \ m/s^{2\,*}$$

(8) Finally: use average acceleration to get speed

$$U = U_0 + \bar{a}\Delta t = 0 + 48.1\frac{m}{s^2} \times 0.0139\,s = 0.669 \ m/s^*$$

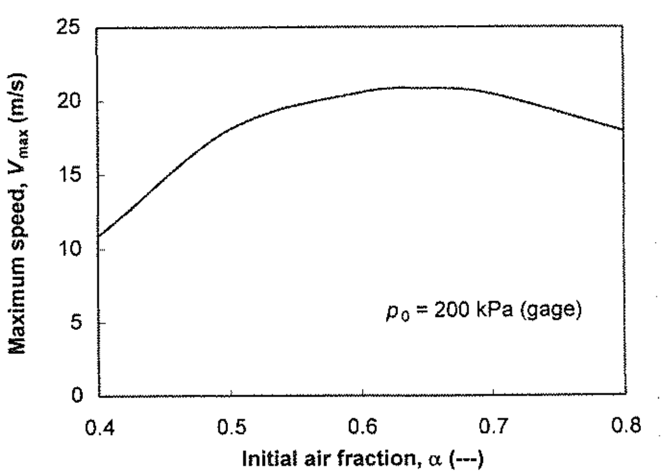
* Note effect of roundoff error.

Repeat these calculations until water is depleted or air pressure falls to zero, as shown below:

Water Fraction, V_w/V_t (---)	Gage Pressure, p (kPa)	Water Mass, M_w (kg)	Jet Speed, V_j (m/s)	Flow Rate, dm/dt (kg/s)	Time Interval, Δt (s)	Current Time, t (s)	"Rocket" Accel., a (m/s^2)	"Rocket" Speed, U (m/s)
0.50	200	0.0481	20.0	0.141	0	0	48.7	0
0.48	184	0.0461	19.2	0.135	0.0139	0.0139	47.5	0.668
0.46	169	0.0442	18.4	0.130	0.0145	0.0284	45.2	1.34
0.44	156	0.0423	17.7	0.125	0.0151	0.0435	43.1	2.01
0.42	143	0.0404	16.9	0.120	0.0157	0.0592	41.2	2.67
0.40	132	0.0384	16.3	0.115	0.0164	0.0756	39.4	3.33
0.38	122	0.0365	15.6	0.110	0.0171	0.0927	37.8	3.99
0.36	112	0.0346	15.0	0.106	0.0178	0.110	36.2	4.65
0.34	103	0.0327	14.4	0.101	0.0186	0.129	34.8	5.31
0.32	94.6	0.0308	13.8	0.0972	0.0194	0.148	33.5	5.97
0.30	86.8	0.0288	13.2	0.0931	0.0202	0.169	32.2	6.63
0.28	79.5	0.0269	12.6	0.0891	0.0211	0.190	31.0	7.30
0.26	72.7	0.0250	12.1	0.0852	0.0221	0.212	29.9	7.97
0.24	66.3	0.0231	11.5	0.0814	0.0231	0.235	28.9	8.65
0.22	60.4	0.0211	11.0	0.0776	0.0242	0.259	27.9	9.34
0.20	54.7	0.0192	10.5	0.0739	0.0254	0.284	26.9	10.0
0.18	49.4	0.0173	9.95	0.0702	0.0267	0.311	26.0	10.7
0.16	44.4	0.0154	9.43	0.0666	0.0281	0.339	25.2	11.5
0.14	39.7	0.0135	8.92	0.0630	0.0297	0.369	24.3	12.2
0.12	35.2	0.0115	8.40	0.0593	0.0314	0.400	23.5	12.9
0.10	31.0	0.00961	7.88	0.0556	0.0334	0.434	22.7	13.7
0.08	27.0	0.00769	7.35	0.0519	0.0357	0.469	22.0	14.5
0.06	23.2	0.00577	6.81	0.0481	0.0384	0.508	21.2	15.3
0.04	19.6	0.00384	6.26	0.0442	0.0416	0.550	20.4	16.2
0.02	16.1	0.00192	5.68	0.0401	0.0456	0.595	19.5	17.1
0.00	12.9	0.0000	5.07	0.0358	0.0506	0.646	18.6	18.1

In this simulation, the water is depleted when $t \approx 0.65$ s; $V_{max} = 18.1$ m/s.

Varying the initial air fraction produces the following:

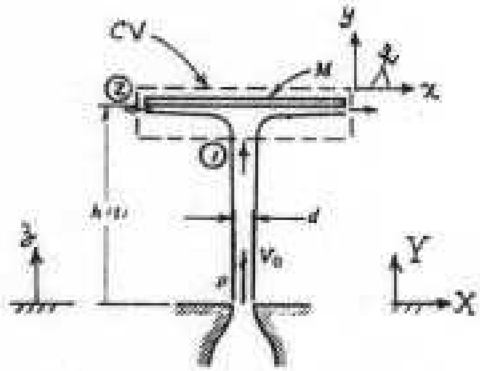


For this combination of parameters, a peak speed of about 20.8 m/s is attained with an initial air fraction of about 0.66.

4.179 A disk, of mass M, is constrained horizontally but is free to move vertically. A jet of water strikes the disk from below. The jet leaves the nozzle at initial speed V_0. Obtain a differential equation for the disk height, $h(t)$, above the jet exit plane if the disk is released from large height, H. (You will not be able to solve this ODE, as it is highly nonlinear!) Assume that when the disk reaches equilibrium, its height above the jet exit plane is h_0.

(a) Sketch $h(t)$ for the disk released at $t = 0$ from $H > h_0$.

(b) Explain why the sketch is as you show it.



Solution: Apply Bernoulli equation to jet, then y momentum equation to CV with linear acceleration.

Basic equations:

$$\frac{\cancel{p_0}}{\rho} + \frac{V_0^2}{2} + g\underbrace{\cancel{z_0}}_{=0} = \frac{\cancel{p_1}}{\rho} + \frac{V_1^2}{2} + g z_1$$

$$\underbrace{\cancel{F_{S_y}}}_{=0(6)} + F_{B_y} - \int_{CV} a_{rf_y}\,\rho\, d\forall = \underbrace{\cancel{\frac{\partial}{\partial t}\int_{CV} v_{xyz}\,\rho\, d\forall}}_{\approx 0(7)} + \int_{CS} v_{xyz}\,\rho\,\vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) No friction
(4) Flow along a streamline
} in jet
(5) $p_1 = p_0 = p_{atm}$

(6) No pressure force on CV, so $F_{S_y} = 0$
(7) Neglect mass of liquid in CV and $v \approx 0$ in CV
(8) Uniform flow at each section
(9) Measure velocities relative to CV

From momentum

$$-(M + \underbrace{\cancel{m_{LCV}}}_{\approx 0(7)})g - a_{rf_y}(M + \underbrace{\cancel{m_{LCV}}}_{\approx 0(7)}) = v_1\{-|\rho(V_1 - U)A_1|\} + \cancel{v_2}\{\dot{m}_2\}$$

$$v_1 = V_1 - U \qquad\qquad v_2 \approx 0$$

With $a_{rf_y} = \frac{d^2h}{dt^2}$, $U = \frac{dh}{dt}$, then

$$-Mg - M\frac{d^2h}{dt^2} = -\rho\left(V_1 - \frac{dh}{dt}\right)^2 A_1$$

But from Bernoulli, $\frac{V_1^2}{2} = \frac{V_0^2}{2} - g z_1$, so $V_1 = \sqrt{V_0^2 - 2gh}$, since $z_1 = h(t)$

Also from continuity, $V_1 A_1 = V_0 A_0$, so $A_1 = A_0 V_0 / V_1$. Substituting

$$\frac{d^2h}{dt^2} = \rho\left(\sqrt{V_0^2 - 2gh} - \frac{dh}{dt}\right)^2 \frac{A_0 V_0}{M\sqrt{V_0^2 - 2gh}} - g \qquad \ddot{h}(t)$$

At equilibrium height, $h = h_0$, $\frac{dh}{dt} = 0$, and $\frac{d^2h}{dt^2} = 0$. Then

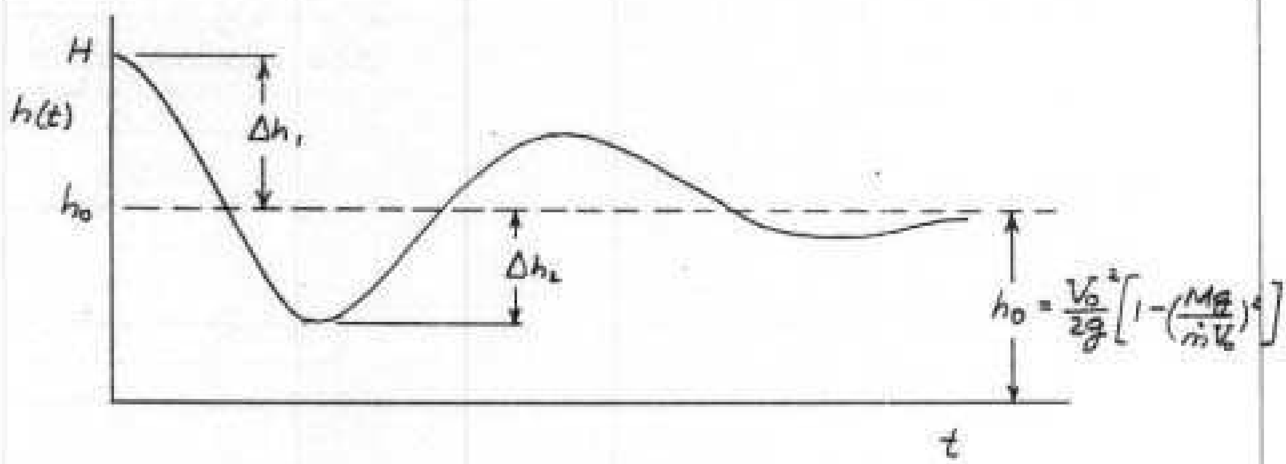
$$\rho\sqrt{V_0^2 - 2gh_0}\,A_0 V_0 - Mg = 0$$

Thus $$V_0^2 - 2gh_0 = \left(\frac{Mg}{\rho V_0 A_0}\right)^2$$

This may be solved to obtain

$$h_0 = \frac{V_0^2}{2g}\left[1-\left(\frac{Mg}{\rho V_0^2 A_0}\right)^2\right] = \frac{V_0^2}{2g}\left[1-\left(\frac{Mg}{\dot{m}V_0}\right)^2\right]$$

When released, $H > h_0$, and $dh/dt = 0$. Because the equation for d^2h/dt^2 is nonlinear, oscillations will occur. The expected behavior is sketched below:

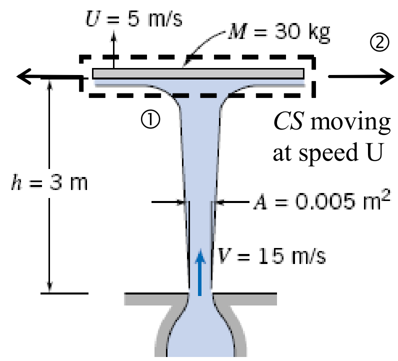


Notes: (1) Expect oscillations
(2) $\Delta h_3 < \Delta h_2 < \Delta h_1$ due to nonlinear equation

Problem 4.180

[Difficulty: 5]

4.180 Consider the configuration of the vertical jet impinging on a horizontal disk shown in Problem 4.158. Assume the disk is released from rest at an initial height of 2 m above the jet exit plane. Using a numerical method such as the Euler method (see Section 5.5), solve for the subsequent motion of this disk. Identify the steady-state height of the disk.



Given: Water jet striking moving disk

Find: Motion of disk; steady state height

Solution:

Basic equations: Bernoulli; Momentum flux in z direction (treated as upwards) for linear accelerating CV

$$\frac{p}{\rho} + \frac{V^2}{2} + g\cdot z = \text{constant} \qquad F_{S_z} + F_{B_z} - \int_{CV} a_{rf_z}\,\rho\, dV = \frac{\partial}{\partial t}\int_{CV} w_{xyz}\,\rho\, dV + \int_{CS} w_{xyz}\,\rho\vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) Atmospheric pressure 4) Uniform flow 5) velocities wrt CV (All in jet)

The Bernoulli equation becomes $\dfrac{V_0^2}{2} + g\cdot 0 = \dfrac{V_1^2}{2} + g\cdot h \qquad V_1 = \sqrt{V_0^2 - 2\cdot g\cdot h}$ (1)

$$V_1 = \sqrt{\left(15\cdot\frac{m}{s}\right)^2 + 2\times 9.81\cdot\frac{m}{s^2}\cdot(0-3)\cdot m} \qquad V_1 = 12.9\,\frac{m}{s}$$

The momentum equation becomes $-M\cdot g - M\cdot a_{rfz} = w_1\cdot(-\rho\cdot V_1\cdot A_1) + w_2\cdot(\rho\cdot V_2\cdot A_2) = (V_1 - U)\cdot\left[-\rho\cdot(V_1 - U)\cdot A_1\right] + 0$

With $a_{rfz} = \dfrac{d^2h}{dt^2}$ and $U = \dfrac{dh}{dt}$ we get $-M\cdot g - M\cdot\dfrac{d^2h}{dt^2} = -\rho\cdot\left(V_1 - \dfrac{dh}{dt}\right)^2\cdot A_1$

Using Eq 1, and from continuity $V_1\cdot A_1 = V_0\cdot A_0 \qquad \dfrac{d^2h}{dt^2} = \left(\sqrt{V_0^2 - 2\cdot g\cdot h} - \dfrac{dh}{dt}\right)^2\cdot\dfrac{\rho\cdot A_0\cdot V_0}{M\cdot\sqrt{V_0^2 - 2\cdot g\cdot h}} - g$ (2)

This must be solved numerically! One approach is to use Euler's method (see the *Excel* solution)

At equilibrium $h = h_0 \qquad \dfrac{dh}{dt} = 0 \qquad \dfrac{d^2h}{dt^2} = 0$ so

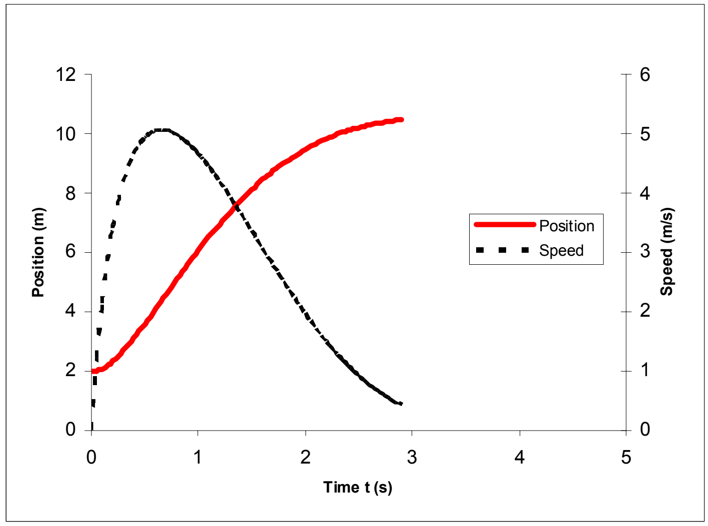
$$\sqrt{\left(V_0^2 - 2\cdot g\cdot h_0\right)}\cdot\rho\cdot A_0\cdot V_0 - M\cdot g = 0 \qquad \text{and} \qquad h_0 = \frac{V_0^2}{2\cdot g}\cdot\left[1 - \left(\frac{M\cdot g}{\rho\cdot V_0^2\cdot A_0}\right)^2\right]$$

Hence $h_0 = \dfrac{1}{2}\times\left(15\cdot\dfrac{m}{s}\right)^2\times\dfrac{s^2}{9.81\cdot m}\times\left[1 - \left[30\cdot kg\times 9.81\cdot\dfrac{m}{s^2}\times\dfrac{m^3}{1000\cdot kg}\times\left(\dfrac{s}{15\cdot m}\right)^2\times\dfrac{1}{.005\cdot m^2}\right]^2\right] \qquad h_0 = 10.7\,m$

In *Excel*:

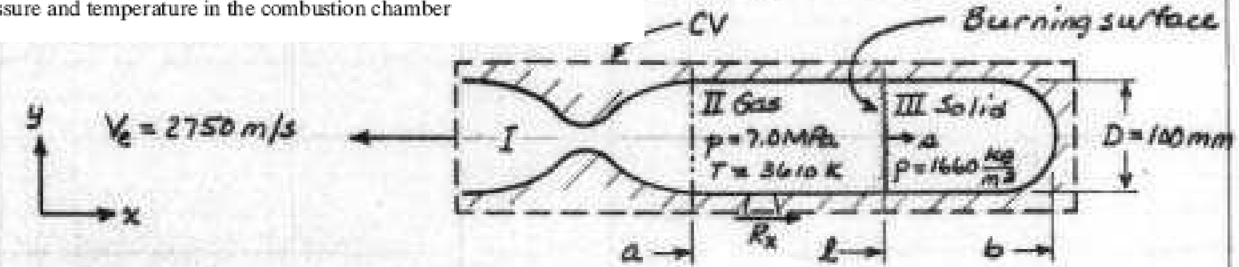
Ξt = 0.05 s
A_0 = 0.005 m^2
g = 9.81 m/s^2
V = 15 m/s
M = 30 kg
Ξ = 1000 kg/m^3

t (s)	*h* (m)	*dh/dt* (m/s)	d^2h/dt^2 (m/s^2)
0.000	2.000	0.000	24.263
0.050	2.000	1.213	18.468
0.100	2.061	2.137	14.311
0.150	2.167	2.852	11.206
0.200	2.310	3.412	8.811
0.250	2.481	3.853	6.917
0.300	2.673	4.199	5.391
0.350	2.883	4.468	4.140
0.400	3.107	4.675	3.100
0.450	3.340	4.830	2.227
0.500	3.582	4.942	1.486
0.550	3.829	5.016	0.854
0.600	4.080	5.059	0.309
0.650	4.333	5.074	-0.161
0.700	4.587	5.066	-0.570
0.750	4.840	5.038	-0.926
0.800	5.092	4.991	-1.236
0.850	5.341	4.930	-1.507
0.900	5.588	4.854	-1.744
0.950	5.830	4.767	-1.951
1.000	6.069	4.669	-2.130
1.050	6.302	4.563	-2.286
1.100	6.530	4.449	-2.420
1.150	6.753	4.328	-2.535
1.200	6.969	4.201	-2.631
1.250	7.179	4.069	-2.711
1.300	7.383	3.934	-2.776
1.350	7.579	3.795	-2.826
1.400	7.769	3.654	-2.864
1.450	7.952	3.510	-2.889
1.500	8.127	3.366	-2.902
1.550	8.296	3.221	-2.904
1.600	8.457	3.076	-2.896
1.650	8.611	2.931	-2.878
1.700	8.757	2.787	-2.850
1.750	8.896	2.645	-2.814
1.800	9.029	2.504	-2.769
1.850	9.154	2.365	-2.716
1.900	9.272	2.230	-2.655
1.950	9.384	2.097	-2.588
2.000	9.488	1.967	-2.514
2.050	9.587	1.842	-2.435
2.100	9.679	1.720	-2.350
2.150	9.765	1.602	-2.261
2.200	9.845	1.489	-2.167
2.250	9.919	1.381	-2.071
2.300	9.989	1.278	-1.972
2.350	10.052	1.179	-1.871
2.400	10.111	1.085	-1.769
2.450	10.166	0.997	-1.666



t (s)	*h* (m)	*dh/dt* (m/s)	d^2h/dt^2 (m/s^2)
2.950	10.506	0.380	-0.766
3.000	10.525	0.341	-0.698
3.050	10.542	0.307	-0.634
3.100	10.558	0.275	-0.574
3.150	10.571	0.246	-0.519
3.200	10.584	0.220	-0.469
3.250	10.595	0.197	-0.422
3.300	10.604	0.176	-0.380
3.350	10.613	0.157	-0.341
3.400	10.621	0.140	-0.306
3.450	10.628	0.124	-0.274
3.500	10.634	0.111	-0.245
3.550	10.640	0.098	-0.219
3.600	10.645	0.087	-0.195
3.650	10.649	0.078	-0.174
3.700	10.653	0.069	-0.155
3.750	10.656	0.061	-0.138
3.800	10.659	0.054	-0.123
3.850	10.662	0.048	-0.109
3.900	10.665	0.043	-0.097
3.950	10.667	0.038	-0.086
4.000	10.669	0.033	-0.077
4.050	10.670	0.030	-0.068
4.100	10.672	0.026	-0.060
4.150	10.673	0.023	-0.053
4.200	10.674	0.021	-0.047
4.250	10.675	0.018	-0.042
4.300	10.676	0.016	-0.037
4.350	10.677	0.014	-0.033
4.400	10.678	0.013	-0.029
4.450	10.678	0.011	-0.026
4.500	10.679	0.010	-0.023
4.550	10.679	0.009	-0.020

4.181 A small solid-fuel rocket motor is fired on a test stand. The combustion chamber is circular, with 100 mm diameter. Fuel, of density 1660 kg/m^3, burns uniformly at the rate of 12.7 mm/s. Measurements show that the exhaust gases leave the rocket at ambient pressure, at a speed of 2750 m/s. The absolute pressure and temperature in the combustion chamber are 7.0 MPa and 3610 K, respectively. Treat the combustion products as an ideal gas with molecular mass of 25.8. Evaluate the rate of change of mass and of linear momentum within the rocket motor. Express the rate of change of linear momentum within the motor as a percentage of the motor thrust.



Treat combustion products as ideal gas with molecular mass, $M_m = 25.8$.

Find: (a) Evaluate rate of change of mass and of linear momentum within rocket motor.

(b) Express rate of change of momentum as a percentage of thrust.

<u>Solution</u>: Apply continuity and x component of momentum equations using fixed CV shown.

Basic equations:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$

$$F_{S_x} + \underbrace{F_{B_x}}_{=0(2)} = \frac{\partial}{\partial t}\int_{CV} u\rho \, d\forall + \int_{CS} u\rho \vec{V} \cdot d\vec{A}$$

Assumptions: (1) No net pressure force; $F_{S_x} = R_x$

(2) $F_{B_x} = 0$

(3) All properties constant at each point, except at surface where combustion takes place

(4) Uniform flow at exit section

The continuity equation becomes

$$0 = \underbrace{\frac{\partial}{\partial t}\int_{I} \rho \, d\forall}_{=0(3)} + \frac{\partial}{\partial t}\int_a^{\ell} \rho_g A \, dx + \frac{\partial}{\partial t}\int_{\ell}^{b} \rho_f A \, dx + \{|\rho_e V_e A_e|\}$$

$$0 = \frac{\partial}{\partial t}[\rho_g A(\ell - a)] + \frac{\partial}{\partial t}[\rho_f A(b-\ell)] + \dot{m}_e = (\rho_g - \rho_f) A \frac{d\ell}{dt} + \dot{m}_e$$

or

$$\dot{m}_e = (\rho_f - \rho_g) A \frac{d\ell}{dt} = (\rho_f - \rho_g) A r$$

For an ideal gas,

$$\rho_g = \frac{p_g}{R T_g} = \frac{p_g M_m}{R_u T_g} = 7.0 \times 10^6 \frac{N}{m^2} \times \frac{25.8\ kg}{mol} \times \frac{mol \cdot K}{8314\ N \cdot m} \times \frac{1}{3610\ K} = 6.02\ kg/m^3$$

so

$$\dot{m}_e = (1660 - 6)\frac{kg}{m^3} \times \frac{\pi}{4}(0.1)^2 m^2 \times 0.0127 \frac{m}{s} = 0.165\ kg/s$$

Mass flow is out, so $\frac{\partial M_{cv}}{\partial t} = -0.165$ kg/s $\quad \frac{\partial M_{cv}}{\partial t}$

From the momentum equation,

$$R_x = \frac{\partial}{\partial t}\int_{I}^{\approx 0(3)} u\rho\, d\forall + \frac{\partial}{\partial t}\int_a^{\ell} u_g \rho_g A\, dx + \frac{\partial}{\partial t}\int_{\ell}^{b} \overset{\approx 0}{u_f} \rho_f A\, dx + u_e\{|\rho_e V_e A_e|\}$$

$$= \frac{\partial}{\partial t}\left[u_g \rho_g A(\ell - a)\right] + u_e \dot{m}_e \quad ; \quad u_g = -V_g \text{ and } u_e = -V_e$$

$$R_x = -\rho_g V_g A \frac{d\ell}{dt} - V_e \dot{m}_e = -\rho_g V_g A \dot{s} - V_e \dot{m}_e$$

But from continuity, $\rho_g V_g A = \dot{m}_e$, since no mass accumulates in region I of the CV. Thus

$$R_x = -\dot{m}_e (V_e + \dot{s})$$

R_x is the force on the CV. The thrust is

$$K_x = \text{Thrust} = -R_x = \dot{m}_e (V_e + \dot{s})$$

$$K_x = 0.165 \frac{kg}{s} (2750 + 0.0127)\frac{m}{s} \times \frac{N \cdot s^2}{kg \cdot m} = 454 \text{ N}$$

The rate of change of linear momentum within the CV is

$$\frac{\partial P_{x_{cv}}}{\partial t} = -\dot{m}_e \dot{s} = -0.165\frac{kg}{s} \times 0.0127\frac{m}{s} \times \frac{N \cdot s^2}{kg \cdot m} = -2.10 \text{ mN}$$ $\frac{\partial P_{x_{cv}}}{\partial t}$

The ratio of rate of change of linear momentum to thrust is

$$\frac{\frac{\partial P_{x_{cv}}}{\partial t}}{K_x} = \frac{-\dot{m}_e \dot{s}}{\dot{m}_e (V_e + \dot{s})} = -\frac{\dot{s}}{(V_e + \dot{s})} = -\frac{0.0127 \frac{m}{s}}{(2750 + 0.0127)\frac{m}{s}} = -4.62 \times 10^{-6}$$

or

$$\frac{\frac{\partial P_{x_{cv}}}{\partial t}}{K_x} = -4.62 \times 10^{-4} \text{ percent}$$ Ratio

{ Neglecting the unsteady momentum term in the analysis of this rocket motor would cause an error of approximately 1 part in 217,000. The assumption that $\partial P_{x_{cv}}/\partial t \approx 0$ is certainly justified for engineering work. }

4.182 The capability of the Aircraft Landing Loads and Traction Facility at NASA's Langley Research Center is to be upgraded. The facility consists of a rail-mounted carriage propelled by a jet of water issuing from a pressurized tank. (The setup is identical in concept to the hydraulic catapult of Problem 4.138.) Specifications require accelerating the carriage with 49,000 kg mass to a speed of 220 knots in a distance of 122 m. (The vane turning angle is 170°.) Identify a range of water jet sizes and speeds needed to accomplish this performance. Specify the recommended operating pressure for the water-jet system and determine the shape and estimated size of tankage to contain the pressurized water.

Discussion: The analysis of Example 4.11 forms the basis for the solution outlined below. Use a control volume attached to and moving with the carriage to analyze the motion. Neglect aerodynamic and rolling resistance to obtain a best-case solution. Solve the resulting differential equation of motion for carriage speed and position as functions of time, and for speed as a function of position along the rails.

Computing equations are summarized and results tabulated below. As shown in Example 4.11, analysis of the carriage motion results in the differential equation

$$\frac{dU}{dt} = \frac{\rho (V_j - U)^2 (1-\cos\theta)}{M} \tag{1}$$

Integrating with respect to time gives carriage speed versus time

$$U = V_j \frac{bt}{1+bt} \tag{2}$$

where parameter b is

$$b = \frac{\rho V_j A_j (1-\cos\theta)}{M} \tag{3}$$

Equation 2 is integrated to obtain carriage position versus time

$$x = V_j \left[t - \frac{\ln(1+bt)}{b} \right] \tag{4}$$

Substitute $dU/dt = UdU/dx$ and integrate Eq. 1 for distance traveled versus carriage speed

$$x = \frac{V_j}{b} \left[\ln(1-U/V_j) + \frac{1}{1-U/V_j} - 1 \right] \tag{5}$$

Relate jet speed to water tank pressure using the Bernoulli equation

$$V_j = \sqrt{2\Delta p/\rho} \tag{6}$$

The required volume of water is computed as follows:

1. Assume a range of tank pressures.
2. Compute the jet speed corresponding to each tank pressure from Eq. 6.
3. Solve for parameter b from Eq. 5 using the known maximum speed and specified distance.
4. Obtain jet area from Eq. 3.
5. Compute the time required to accelerate the carriage from Eq. 2.
6. Calculate jet diameter from jet area.
7. Compute the required volume of water from the product of mass flow rate and acceleration time.

The optimum operating pressure requires the least costly tankage. (Assume the most efficient spherical shape for pressurized tankage and constant tank pressure during acceleration.) Tankage calculations are organized as follows:

1. Obtain tank diameter from tank volume.
2. Calculate wall thickness from a force balance on the thin wall of the tank.
3. Calculate steel volume from tank surface area and wall thickness.
4. Assume steel cost is proportional to steel volume.

<u>Sample Calculation</u>: assume $p = 6000$ psig

$$V_j = \left[2 \times 6000 \frac{lbf}{in.^2} \times \frac{ft^3}{1.94\, slug} \times 144 \frac{in.^2}{ft^2} \times \frac{slug \cdot ft}{lbf \cdot s^2}\right]^{\frac{1}{2}} = 944\ ft/s \ ; \ \frac{U}{V_j} = \frac{371}{944} = 0.393$$

$$b = 944 \frac{ft}{s} \times \frac{1}{400\, ft}\left[\ln(1-0.393) + \frac{1}{1-0.393} - 1\right] = 0.350\ s^{-1}$$

$$A_j = \frac{bM}{\rho V_j (1-\cos\theta)} = \frac{0.350}{s} \times 3350\, slug \times \frac{ft^3}{1.94\, slug} \times \frac{s}{944\, ft} \times \frac{1}{(1-\cos 170°)} = 0.323\ ft^2$$

$$D = \sqrt{\frac{4A}{\pi}} = \left[\frac{4}{\pi} \times 0.323\, ft^2 \times 144 \frac{in.^2}{ft^2}\right]^{\frac{1}{2}} = 7.69\ in.$$

$$t = \frac{1}{b}\left(\frac{U/V_j}{1-U/V_j}\right) = \frac{s}{0.350} \times \frac{0.393}{1-0.393} = 1.85\ s$$

$$Q = V_j A = 944 \frac{ft}{s} \times 0.323\, ft^2 \times 7.48 \frac{gal}{ft^3} = 2280\ gal/s$$

$$\forall = Qt = 2280 \frac{gal}{s} \times 1.85\, s = 4220\ gal$$

$$D = (6\forall/\pi)^{1/3} = \left(\frac{6}{\pi} \times 4220\, gal \times \frac{ft^3}{7.48\, gal}\right)^{1/3} = 10.3\ ft$$

$$\Delta p \frac{\pi D^2}{4} = \pi D t \ ; \ t = \frac{pD}{4\sigma} = \frac{1}{4} \times 6000 \frac{lbf}{in.^2} \times 10.3\, ft \times \frac{in.^2}{40{,}000\, lbf} \times 12 \frac{in.}{ft} = 4.64\ in.$$

$$\forall_{steel} = \pi D^2 t = \pi \times (10.3)^2 ft^2 \times 4.64\, in. \times \frac{ft}{12\, in.} = 129\ ft^3$$

<u>Discussion</u>: The results show the steel volume plummets as tank pressure is raised, with a broad minimum between 3,000 and 4000 psig.

Problem 4.182

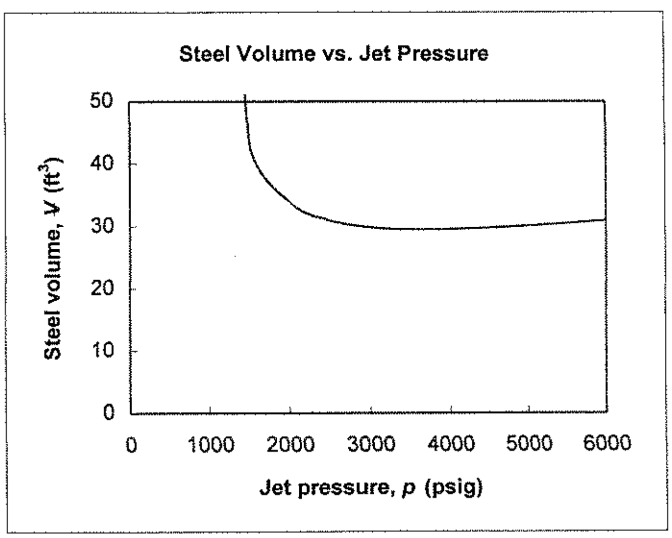
Input Data:

M =	49000	kg	3355	slug
U =	220	kt	371.3	ft/s
X =	122	m	400.3	ft
θ =	170	degrees		

Calculated Results:

Jet Pressure (psig)	Jet Speed (ft/s)	Parameter b (s^{-1})	Jet Area (ft^2)	Jet Diameter (in.)	Flow Rate (gal/s)	Flow Time (s)	Water Volume (gal)
6000	944	0.351	0.324	7.70	2285	1.85	4227
5500	904	0.380	0.367	8.20	2477	1.84	4546
5000	862	0.417	0.421	8.79	2715	1.82	4936
4500	817	0.463	0.494	9.51	3019	1.80	5426
4000	771	0.525	0.593	10.4	3419	1.77	6061
3500	721	0.610	0.737	11.6	3973	1.74	6924
3000	667	0.736	0.961	13.3	4797	1.70	8174
2500	609	0.944	1.35	15.7	6155	1.65	10172
2000	545	1.35	2.17	19.9	8830	1.58	13942
1500	472	2.53	4.67	29.3	16490	1.46	24061
1000	385	22.4	50.6	96.3	145835	1.19	173113

Jet Pressure (psig)	Water Volume (gal)	Tank Diameter (ft)	Wall Thickness (in.)	Steel Volume (ft^3)	Steel Mass (ton)
6000	4227	10.3	4.6	127.2	30.9
5500	4546	10.5	4.3	125.4	30.5
5000	4936	10.8	4.1	123.7	30.1
4500	5426	11.1	3.8	122.4	29.8
4000	6061	11.6	3.5	121.5	29.6
3500	6924	12.1	3.2	121.5	29.6
3000	8174	12.8	2.9	122.9	29.9
2500	10172	13.7	2.6	127.5	31.0
2000	13942	15.3	2.3	139.8	34.0
1500	24061	18.3	2.1	180.9	44.0
1000	173113	35.4	2.7	867.9	211.2



4.183 A classroom demonstration of linear momentum is planned, using a water-jet propulsion system for a cart traveling on a horizontal linear air track. The track is 5 m long, and the cart mass is 155 g. The objective of the design is to obtain the best performance for the cart, using 1 L of water contained in an open cylindrical tank made from plastic sheet with density of 0.0819 g/cm^2. For stability, the maximum height of the water tank cannot exceed 0.5 m. The diameter of the smoothly rounded water jet may not exceed 10 percent of the tank diameter. Determine the best dimensions for the tank and the water jet by modeling the system performance. Using a numerical method such as the Euler method (see Section 5.5), plot acceleration, velocity, and distance as functions of time. Find the optimum dimensions of the water tank and jet opening from the tank. Discuss the limitations on your analysis. Discuss how the assumptions affect the predicted performance of the cart. Would the actual performance of the cart be better or worse than predicted? Why? What factors account for the difference(s)?

Discussion: This solution is an extension of Problem 4.184. The analyses for tank level, acceleration, and velocity are identical; please refer to the solution for Problem 4.184 for equations describing each of these variables as functions of time.

One new feature of this problem is computation of distance traveled. Equation 7 of Problem 4.184 could be integrated in closed form to provide an equation for distance traveled as a function of time. However, the integral would be messy, and it would provide little insight into the dependence on key parameters. Consequently, a numerical analysis has been chosen in this problem. The results are presented in the plots and spreadsheet on the next page.

We have chosen to define velocity as the output to be maximized.

A second new feature of this problem is the geometric constraints: the maximum track length is 5 m. Intuitively jet diameter should be chosen as the largest possible fraction of tank diameter for optimum performance. Using the spreadsheet to vary $\beta = d/D$ verifies that this is the case. Therefore we have used the maximum allowable ratio, $\beta = 0.1$, for all computations.

Tank height should be a factor in performance. Intuition suggests that increasing tank height should improve performance. Using the spreadsheet shows a very weak dependence on tank height. Performance is best at smaller tank heights, corresponding to the minimum tank mass.

As tank height is decreased, diameter increases because tank volume is held constant. Since diameter ratio is constant, then jet diameter increases with decreasing tank height. This effect almost overshadows the effect of tank height.

The principal limitations on the analysis are the assumptions of negligible motion resistance and no slope to the free surface of water in the tank. Actual performance of the cart would likely be less than predicted because of motion resistance.

Distance is modeled as

$$x_{i+1} = x_i + U_i \Delta t + \frac{1}{2} a_{x,i} \Delta t^2$$

The accuracy of this model for position is consistent with the accuracy of modeling the water-jet propulsion system.

Problem 4.183

Analysis of Cart Propelled by Gravity-Driven Water Jet:

Input Data:

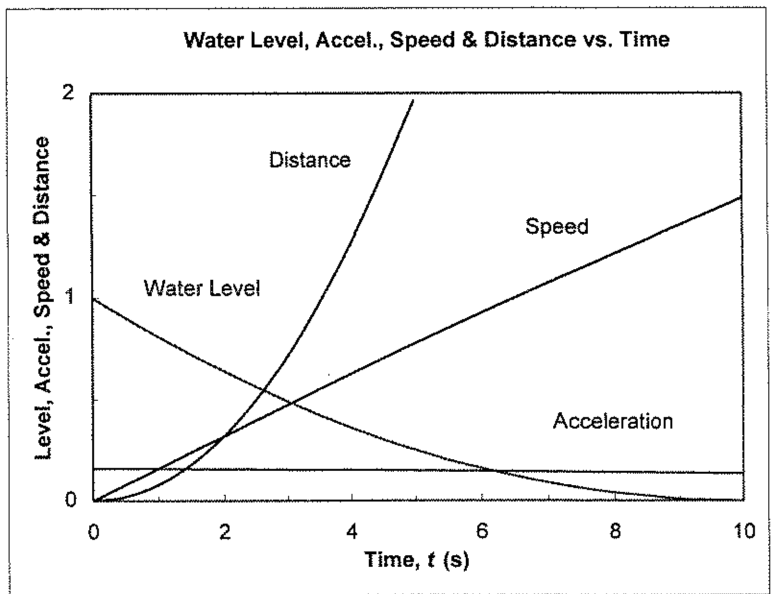
g =	9.81	m/s^2	Acceleration of gravity
H =	500	mm	Height of tank
M_c =	0.155	kg	Mass of cart
V =	1.00	L	Tank volume
β =	0.100	(---)	Ratio of jet diameter to tank diameter
ρ =	999	kg/m^3	Density of water
ρ'' =	0.819	kg/m^2	(Area) density of tank material

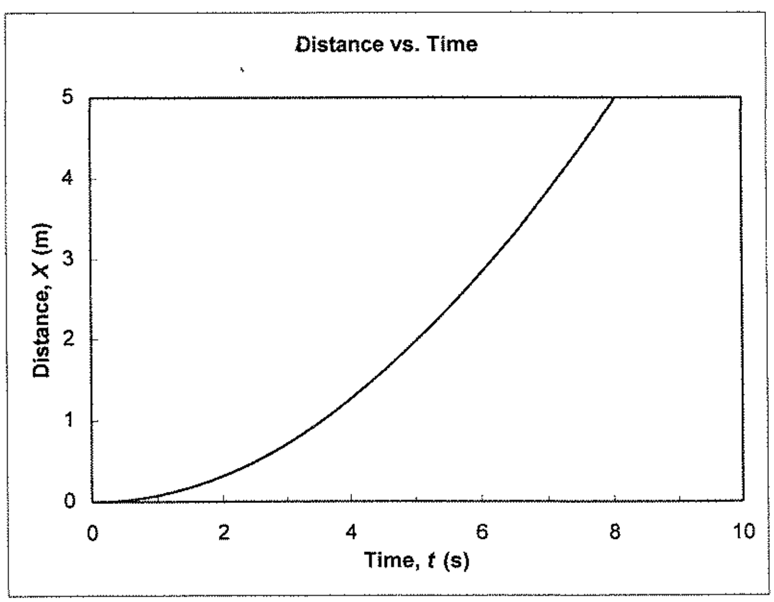
Calculated Parameters:

a =	0.471	(---)	(a^2 =) Ratio of mass of tank to initial mass of water
b =	0.0313	s^{-1}	Geometric parameter of solution
d =	5.05	mm	Diameter of water jet
D =	50.5	mm	Diameter of tank
M_0 =	1.00	kg	Initial mass of water in tank
M_p =	0.0666	kg	Mass of plastic in tank
M_t =	0.222	kg	Mass of plastic tank plus cart

Calculated Results:

Time, t	Level, y/H	Accel., a_x	Velocity, U	Position, X
(s)	(---)	(m/s^2)	(m/s)	(m)
0	1	0.161	0	0
0.5	0.903	0.160	0.080	0.0201
1.0	0.810	0.159	0.160	0.080
1.5	0.723	0.158	0.239	0.180
2.0	0.640	0.157	0.317	0.319
2.5	0.563	0.156	0.395	0.497
3.0	0.490	0.154	0.473	0.714
3.5	0.423	0.153	0.550	0.97
4.0	0.360	0.152	0.626	1.26
4.5	0.303	0.151	0.702	1.60
5.0	0.250	0.150	0.777	1.97
5.5	0.203	0.148	0.852	2.37
6.0	0.160	0.147	0.925	2.82
6.5	0.123	0.145	1.00	3.30
7.0	0.0900	0.144	1.07	3.82
7.5	0.0625	0.142	1.14	4.37
8.0	0.0400	0.141	1.21	4.96
8.03	0.0388	0.141	1.22	5.00
9.0	0.0100	0.137	1.35	
9.5	0.0025	0.135	1.42	
10.0	0.0000	0.133	1.49	





4.184 Analyze the design and optimize the performance of a cart propelled along a horizontal track by a water jet that issues under gravity from an open cylindrical tank carried on board the cart. (A water-jet-propelled cart is shown in the diagram for Problem 4.142.) Neglect any change in slope of the liquid free surface in the tank during acceleration. Analyze the motion of the cart along a horizontal track, assuming it starts from rest and begins to accelerate when water starts to flow from the jet. Derive algebraic equations or solve numerically for the acceleration and speed of the cart as functions of time. Present results as plots of acceleration and speed versus time, neglecting the mass of the tank. Determine the dimensions of a tank of minimum mass required to accelerate the cart from rest along a horizontal track to a specified speed in a specified time interval.

Discussion: This problem solution consists of two parts. The first is to analyze the acceleration and velocity of a cart propelled by a gravity-driven water jet. The second is to optimize the dimensions of the cart and jet to accelerate to a specified speed in a specified time interval.

To analyze the problem, apply conservation of mass and the Bernoulli equation to the draining of the tank, then apply the x component of the momentum equation for a control volume to analyze the resulting linear acceleration. A representative plot of the results is presented below.

To optimize the performance of the water-jet-propelled cart, manipulate the solution dimensions until the best performance is attained.

Input Data:

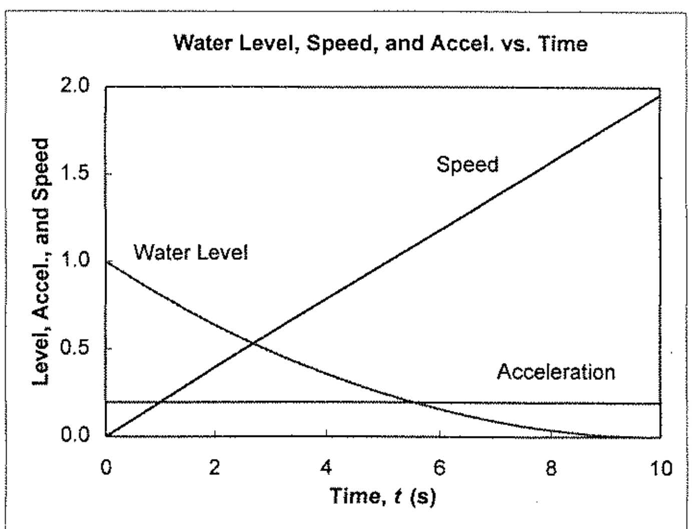
$d =$	10	mm	Diameter of water jet
$D =$	100	mm	Diameter of tank
$g =$	9.81	ft/s^2	Acceleration of gravity
$H =$	150	mm	Height of tank
$M_t =$	0.001	kg	Mass of tank
$\rho =$	999	kg/m^3	Density of water

Calculated Parameters:

$a =$	0.029	(---)	($a^2 =$) Ratio of mass of tank to initial mass of water
$b =$	0.0572	s^{-1}	Geometric parameter of solution
$M_0 =$	1.18	kg	Initial mass of water in tank
$\beta =$	0.1	(---)	Ratio of jet diameter to tank diameter

Calculated Results:

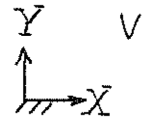
Time, t (s)	Level Ratio, y/H (---)	Accel., a_x (m/s^2)	Velocity, U (m/s)
0	1	0.196	0
1	0.810	0.196	0.196
2	0.640	0.196	0.392
3	0.490	0.196	0.588
4	0.360	0.196	0.784
5	0.250	0.196	0.980
6	0.160	0.196	1.176
7	0.0900	0.196	1.37
8	0.0400	0.196	1.57
9	0.0100	0.196	1.76
10	0	0.195	1.96

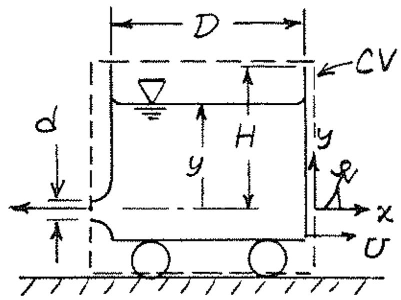


Given: Cart, propelled by water jet, accelerates along horizontal track.

Find: (a) Analyze motion, derive algebraic equations for acceleration and speed of cart as functions of time
(b) Plot acceleration and speed vs. time.

Solution: Apply conservation of mass, Bernoulli, and momentum equations.





Basic equations:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$

$$\cancel{\frac{p_j}{\rho}} + \frac{V_j^2}{2} + \overset{=0(8)}{\cancel{g y_j}} = \cancel{\frac{p}{\rho}} + \frac{V^2}{2} + gy$$

M_t = mass of tank, cart

$\beta = \frac{d}{D}$

$$\overset{=0(9)}{\cancel{F_{Sx}}} + \overset{=0(10)}{\cancel{F_{Bx}}} - \int_{CV} a_{rf_x} \rho \, d\forall = \overset{\approx 0(11)}{\cancel{\frac{\partial}{\partial t}}} \int_{CV} u \rho \, d\forall + \int_{CS} u \rho \vec{V} \cdot d\vec{A}$$

Assumptions: (1) Uniform flow from exit jet (2) Neglect air in CV

$$0 = \frac{\partial}{\partial t}(\rho A_t y) + \{+|\rho V_j A_j|\} = \rho A_t \frac{dy}{dt} + \rho V_j A_j = -\rho A_t V + \rho V_j A_j \qquad (1)$$

Thus $V = V_j \frac{A_j}{A_t} = V_j \left(\frac{d}{D}\right)^2 = \beta^2 V_j$ (2)

(3) No slope to free surface (given)
(4) Quasi-steady flow
(5) Frictionless flow
(6) Incompressible flow
(7) Flow along a streamline
(8) $p = p_j = p_{atm}$
(9) $y_j = 0$

From Bernoulli, $\frac{V_j^2}{2} = \frac{V^2}{2} + gy$ or $V_j^2 - V^2 = 2gy$

Substituting from (2), $V_j^2 - \beta^4 V_j^2 = V_j^2(1-\beta^4) = 2gy$; $V_j^2 = \frac{2gy}{(1-\beta^4)}$ (3)

Substituting into (1), $\frac{dy}{dt} = -\beta^2 V_j = -\beta^2 \frac{\sqrt{2gy}}{(1-\beta^4)}$ or $\frac{dy}{y^{1/2}} = -\frac{\beta^2\sqrt{2g}}{1-\beta^4} dt$

Integrating, $2y^{1/2}\Big]_{y_0}^{y} = -\frac{\beta^2\sqrt{2g}}{(1-\beta^4)} t$ or $y^{1/2} - y_0^{1/2} = -\frac{\beta^2\sqrt{2g}}{2(1-\beta^4)} t$

Thus $\left(\frac{y}{y_0}\right)^{1/2} = 1 - \left[\frac{g\beta^4}{2y_0(1-\beta^4)}\right]^{1/2} t = 1 - bt$; $b = \left[\frac{g\beta^4}{2y_0(1-\beta^4)}\right]^{1/2}$ (4)

From momentum (10) $F_{S_x} = 0$; no resistance

(11) $F_{B_x} = 0$; horizontal motion

(12) $u \approx 0$ in CV, so $\partial/\partial t \approx 0$

Then

$$-a_{rf_x} M(t) = u_j \{+|\rho V_j A_j|\} = -\rho V_j^2 A_j \qquad (5)$$

$$a_{rf_x} = \frac{dU}{dt} \qquad u_j = -V_j$$

But from (4), $M(t) = M_t + \rho A_t y = M_t + \rho A_t y_0 (1-bt)^2$

From (3), $V_j^2 = \dfrac{2gy}{1-\beta^4} = \dfrac{2g}{1-\beta^4} y_0 (1-bt)^2$

Substituting into (5)

$$\frac{dU}{dt}\left[M_t + \rho A_t y_0 (1-bt)^2\right] = \rho A_j \frac{2g}{1-\beta^4} y_0 (1-bt)^2 = \rho A_t y_0 \frac{2g\beta^2}{1-\beta^4}(1-bt)^2$$

Define M_0 = initial mass of water $= \rho A_t y_0$. Then

$$\frac{dU}{dt}\left[M_t + M_0 (1-bt)^2\right] = M_0 \frac{2g\beta^2}{1-\beta^4}(1-bt)^2$$

or

$$\frac{dU}{dt} = \frac{2g\beta^2}{1-\beta^4} \frac{M_0(1-bt)^2}{M_t + M_0(1-bt)^2} \qquad (6) \qquad \frac{dU}{dt}(t)$$

To integrate, let $r = 1-bt$, $dr = -b\,dt$, and $a^2 = M_t/M_0$. Then

$$U = \int_0^U dU = \frac{2g\beta^2}{1-\beta^4}\left(-\frac{1}{b}\right)\int_0^t \frac{r^2}{a^2+r^2}dr = -\frac{2g\beta^2}{1-\beta^4}\frac{1}{b}\left[r - a\tan^{-1}\left(\frac{r}{a}\right)\right]_0^t$$

$$= -\frac{2g\beta^2}{1-\beta^4}\frac{1}{b}\left[(1-bt) - a\tan^{-1}\left(\frac{1-bt}{a}\right)\right]_0^t$$

$$U = -\frac{2g\beta^2}{1-\beta^4}\frac{1}{b}\left[(1-bt) - a\tan^{-1}\left(\frac{1-bt}{a}\right) - 1 + a\tan^{-1}\left(\frac{1}{a}\right)\right]$$

Simplifying, then

$$U = \frac{2g\beta^2}{1-\beta^4}\left\{t + \frac{a}{b}\left[\tan^{-1}\left(\frac{1-bt}{a}\right) - \tan^{-1}\left(\frac{1}{a}\right)\right]\right\} \qquad (7) \qquad U(t)$$

$$a^2 = \frac{M_t}{M_0}\ ; \quad b = \left[\frac{g\beta^4}{2y_0(1-\beta^4)}\right]^{1/2}$$

Given: Cart, propelled by water jet, accelerating on horizontal track.

$$\frac{dU}{dt} = \frac{2g\beta^2}{1-\beta^4}\frac{(1-bt)^2}{a^2+(1-bt)^2} \qquad (1)$$

$$U(t) = \frac{2g\beta^2}{1-\beta^4}\left\{t + \frac{a}{b}\left[\tan^{-1}\left(\frac{1-bt}{a}\right) - \tan^{-1}\left(\frac{1}{a}\right)\right]\right\} \qquad (2)$$

$$\beta = \frac{d}{D}, \quad a^2 = \frac{M_t}{M_0}, \quad b = \left[\frac{g\beta^4}{2y_0(1-\beta^4)}\right]^{1/2}$$

Find: (a) Shape for tank of minimum mass for given volume.
(b) Minimum water volume to reach $U = 2.5$ m/sec in $t = 25$ sec.

Solution: Mass of tank is $M = \rho_t A_s t$, where t = thickness of wall

$$A_s = A_{bottom} + A_{cylinder} = \pi\frac{D^2}{4} + \pi DH$$

Since volume is $\forall = \frac{\pi D^2}{4}H$, then $H = \frac{4\forall}{\pi D^2}$, and

$$A_s = \frac{\pi D^2}{4} + \pi D\left(\frac{4\forall}{\pi D^2}\right) = \frac{\pi D^2}{4} + \frac{4\forall}{D}$$

To minimize, set $dA_s/dD = 0$

$$\frac{dA_s}{dD} = \frac{\pi D}{2} + (-1)\frac{4\forall}{D^2} = 0 \quad \text{so} \quad D^3 = \frac{8\forall}{\pi} \quad \text{or} \quad D = \left(\frac{8\forall}{\pi}\right)^{1/3} \qquad (3) \quad D_{opt}$$

$$\text{Then} \quad \forall = \frac{\pi D^2 H}{4} = \frac{\pi D^3}{8} \quad \text{so} \quad \frac{H}{D} = \frac{1}{2} \qquad (4) \quad \left.\frac{H}{D}\right)_{opt}$$

The tank mass per volume for optimum H/D is

$$m = \frac{M}{\forall} = \rho_t\frac{\left(\frac{\pi D^2}{4} + \pi DH\right)t}{\frac{\pi D^2}{4}H} = \rho_t\left(\frac{t}{H} + \frac{4t}{D}\right) = \rho_t\frac{t}{H}\left(1 + 4\frac{H}{D}\right) = 3\rho_t\frac{t}{H}$$

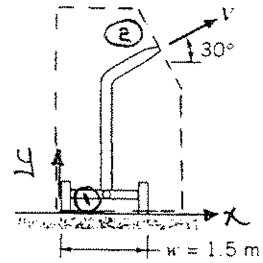
Therefore mass depends on $\rho_t t$ for a given volume. The minimum mass is achieved for the smallest combination of ρ_t and t.

$$a^2 = \frac{M_t}{M_0} = \frac{M_t}{\rho\forall} = \frac{3\rho_t}{\rho}\frac{t}{H} = 3\,SG\left(\frac{t}{H}\right) \qquad (5)$$

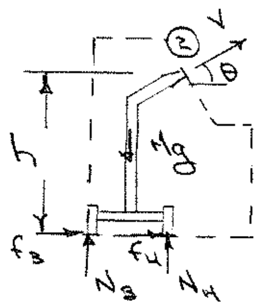
which still depends on Volume, since it contains H.

The best solution strategy seems to be: pick $\forall$, calculate H, D, β, a, and b, then plot $U(t)$.

4.185 A large irrigation sprinkler unit, mounted on a cart, discharges water with a speed of 40 m/s at an angle of 30° to the horizontal. The 50-mm-diameter nozzle is 3 m above the ground. The mass of the sprinkler and cart is $M = 350$ kg. Calculate the magnitude of the moment that tends to overturn the cart. What value of V will cause impending motion? What will be the nature of the impending motion? What is the effect of the angle of jet inclination on the results? For the case of impending motion, plot the jet velocity as a function of the angle of jet inclination over an appropriate range of the angles.



Solution:



Apply moment of momentum equation, using fixed CV shown at left. Origin of coordinates is on ground at left wheel of cart. With this coordinate system counterclockwise moments are positive (about the z axis).

Basic equation:

$$\vec r \times \vec F_s + \int_{CV} \vec r \times \vec g \rho \, dV + \cancel{\vec T_s}^{=0(1)} = \cancel{\frac{\partial}{\partial t}\int_{CV} \vec r \times \vec V \rho \, dV}^{=0(2)} + \int_{CS} \vec r \times \vec V (\rho \vec V \cdot d\vec A)$$

Assumptions:
(1) $T_s = 0$
(2) steady flow
(3) uniform flow at nozzle outlet
(4) neglect $\vec r \times \vec V$ of inlet flow
(5) center of mass located at $x = w/2$
(6) nozzle length is short; coordinates of nozzle exit at $(x_2, y_2) = (w/2, h)$

Then

$$\vec r \times \vec F_s + \vec r \times M\vec g = \cancel{\vec r_1 \times \vec V_1}^{=0(4)}\{-|\rho V_1 A_1|\} + \vec r_2 \times \vec V_2 \{|\rho_2 V_2 A_2|\}$$

$$r_2 = \frac{w}{2}\hat i + h\hat j \qquad V_2 = V(\cos\theta\,\hat i + \sin\theta\,\hat j)$$

and

$$wN_4\hat k - \frac{w}{2}Mg\hat k = \frac{w}{2}V\sin\theta\,\dot m_2\hat k - hV\cos\theta\,\dot m\,\hat k$$

$$wN_4 - \frac{w}{2}Mg = \dot m_2 V\left[\frac{w}{2}\sin\theta - h\cos\theta\right] \quad \text{------(1)}$$

Rewriting Eq. 1 in the form $\Sigma M_z = 0$ {for static equilibrium}

$$wN_4 - \frac{w}{2}Mg + \dot m_2 V\left[h\cos\theta - \frac{w}{2}\sin\theta\right] = 0 \quad \text{---(2)}$$

The last term in Eq 2 is the moment (due to the jet) which tends to overturn the cart.

Evaluating, $\dot{m}_2 = \rho A_2 V_2 = \rho \frac{\pi D^2}{4} V_2$

$$\dot{m}_2 = 999 \frac{kg}{m^3} \times \frac{\pi}{4} (0.05)^2 m^2 \times 40 \frac{m}{s} = 78.5 \, kg/s$$

Then with $V_2 = 40 \, m/s$

$$\text{Moment from jet} = 78.5 \frac{kg}{s} \times 40 \frac{m}{s} \times \frac{N \cdot s^2}{kg \cdot m} \left[3m \cos 30° - \frac{1.5m}{2} \sin 30° \right]$$

$$\text{Moment}_{jet} = 6.98 \, kN \cdot m \qquad \text{Moment}_{jet}$$

For the case of impending tipping (about point 3)
$N_4 \to 0$ and from Eq. 2

$$-\frac{W}{2} Mg + \dot{m}_2 V \left[h \cos\theta - \frac{W}{2} \sin\theta \right] = 0$$

To solve for V_2, write $\dot{m} = \rho A_2 V_2$

$$V_2^2 = \frac{W Mg}{2 \rho A_2 \left[h \cos\theta - \frac{W}{2} \sin\theta \right]} \qquad (3)$$

$$V_2^2 = \frac{1.5m}{2} \times 350 kg \times 9.81 \frac{m}{s^2} \times \frac{m^3}{999 \, kg} \times \frac{1}{1.96 \times 10^{-3} m^2} \times \frac{1}{(3 \cos 30° - 0.75 \sin 30°) m}$$

$$V_2^2 = 592 \, m^2 s^{-2} \qquad \therefore V_2 = 24.3 \, m/s \qquad V_2$$

Thus, the maximum speed allowable without tipping is less than the value suggested.

The impending motion will be tipping since $f_3 < \mu N_3$

From the x momentum equation $\quad f_3 = \dot{m} V_2 \cos\theta$

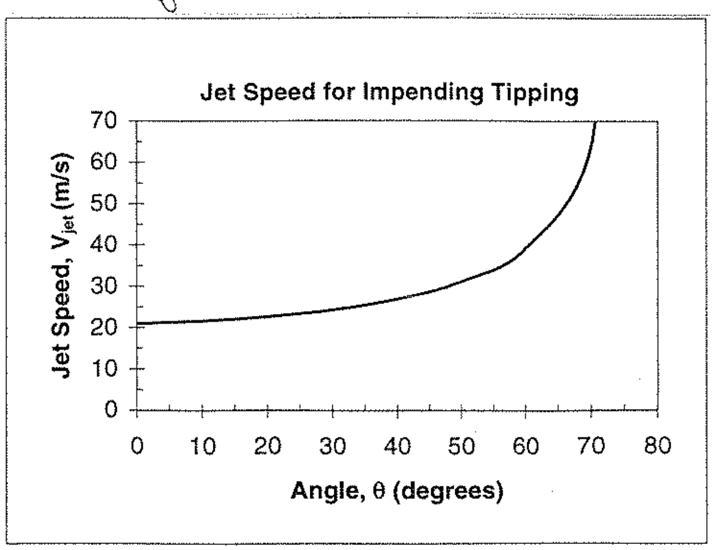
From the y momentum equation $\quad N_3 = Mg + \dot{m} V_2 \sin\theta$

For tipping $\mu > 0.377$

From Eq. 2 we see that as θ increases the tendency to tip decreases

For impending motion from Eq. 3.

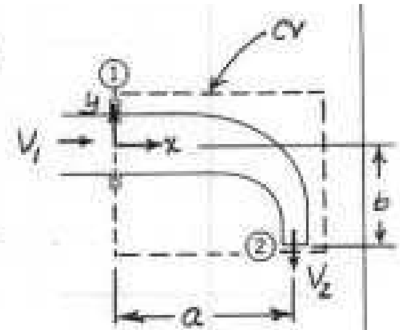
$$V = \left\{ \frac{W Mg}{2 \rho A_2 \left[h \cos\theta - \frac{W}{2} \sin\theta \right]} \right\}^{1/2}$$



Problem 4.186

[Difficulty: 2]

4.186 The 90° reducing elbow of Example 4.6 discharges to atmosphere. Section ② is located 0.3 m to the right of Section ①. Estimate the moment exerted by the flange on the elbow.



Solution: Apply moment of momentum, using the CV and CS shown.

From Example Problem 4.6 $\vec{V}_2 = -16\hat{j}$ m/s, $A_1 = 0.01$ m²

Steady flow, $A_2 = 0.0025$ m²

Basic equation (fixed CV):

$$\vec{r}\times\vec{F}_s + \int_{CV} \underbrace{\vec{r}\times\vec{g}\,\rho\, d\forall}_{=0(1)} + \underbrace{\vec{T}_{shaft}}_{=0(2)} = \underbrace{\frac{\partial}{\partial t}\int_{CV} \vec{r}\times\vec{V}\rho\, d\forall}_{=0(3)} + \int_{CS} \vec{r}\times\vec{V}\rho\vec{V}\cdot d\vec{A}$$

Assumptions: (1) Neglect body forces
(2) No shafts, so $\vec{T}_{shaft} = 0$
(3) Steady flow (given)
(4) Uniform flow at each cross section
(5) Incompressible flow

Then

$$\vec{M}_{flange} = \vec{r}\times\vec{F}_s\big)_{flange} = \vec{r}_1\times\vec{V}_1\{-\rho V_1 A_1\} + \vec{r}_2\times\vec{V}_2\{+\rho V_2 A_2\} \qquad (1)$$

$$\vec{r}_1 = 0 \qquad \left.\begin{array}{l}\vec{r}_2 = a\hat{i} - b\hat{j}\\ \vec{V}_2 = -V_2\hat{j}\end{array}\right\} \vec{r}_2\times\vec{V}_2 = -aV_2\hat{k} + 0$$

Substituting into Eq. 1,

$$\vec{M}_{flange} = -aV_2\hat{k}\{+\rho V_2 A_2\} = -a\rho V_2^2 A_2\hat{k}$$

$$= 0.3\text{ m}\times 999\frac{\text{kg}}{\text{m}^3}\times(16)^2\frac{\text{m}^2}{\text{s}^2}\times 0.0025\text{ m}^2\times\frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}}(-\hat{k})$$

$$\vec{M}_{flange} = -192\,\hat{k}\text{ N}\cdot\text{m} \longleftarrow \vec{M}_{flange}$$

This is the torque that must be exerted on the CV by the flange.

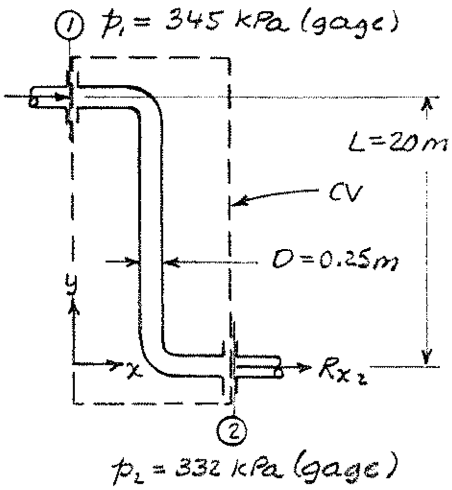
{ Since $\vec{M}_{flange}$ is in the $-\hat{k}$ direction, it must act cw in the xy-plane. }

Problem 4.187

[Difficulty: 3]

4.187 Crude oil (SG = 0.95) from a tanker dock flows through a pipe of 0.25 m diameter in the configuration shown. The flow rate is 0.58 m³/s, and the gage pressures are shown in the diagram. Determine the force and torque that are exerted by the pipe assembly on its supports.

$Q = 0.58\ m^3/s$



Solution: No momentum components exist in the y direction. Apply x component of linear momentum and the moment of momentum equations using the CV shown. Location of coordinates is arbitrary; for simplicity, choose as shown.

Basic equations: $$F_{S_x} + \overset{=0(1)}{F_{B_x}} = \overset{=0(2)}{\frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall} + \int_{CS} u\rho\vec V\cdot d\vec A$$

$$\vec r\times\vec F_s + \overset{=0(4)}{\int_{CV}\vec r\times\vec g\,\rho\, d\forall} + \overset{=0(5)}{\vec T_{shaft}} = \overset{=0(2)}{\frac{\partial}{\partial t}\int_{CV}\vec r\times\vec V\rho\, d\forall} + \int_{CS}\vec r\times\vec V\,\rho\vec V\cdot d\vec A$$

Assumptions: (1) $F_{B_x} = 0$; $\vec g$ acts in z direction
(2) Steady flow
(3) Uniform flow at each section
(4) No z component of $\vec r\times\vec g$
(5) $\vec T_{shaft} = 0$

$$A = \frac{\pi D^2}{4} = \frac{\pi}{4}(0.25)^2 m^2 = 0.049\ m^2$$

From momentum equation,

$$R_{x_1} + R_{x_2} + p_1A - p_2A = u_1\{-\dot m\} + u_2\{\dot m\} = 0\ ;\quad R_{x_1} + R_{x_2} = (p_2 - p_1)A$$

From moment of momentum,

$$\vec r_1\times(R_{x_1}+p_1A)\hat i + \overset{=0}{\vec r_2}\times(R_{x_2}-p_2A)\hat i = \vec r_1\times V_1\hat i\{-\dot m\} + \overset{=0}{\vec r_2}\times V_2\hat i\{\dot m\}\ ;\quad \vec r_1 = L\hat j,\ \vec r_1\times\hat i = -L\hat k$$

$$-L(R_{x_1}+p_1A)\hat k = -LV_1(-\dot m)\hat k = LV_1\dot m\,\hat k = L\frac{Q}{A}(\rho Q)\hat k = \frac{L\rho Q^2}{A}\hat k$$

$$R_{x_1} = -\frac{\rho Q^2}{A} - p_1A = -0.95\times 999\frac{kg}{m^3}\times(0.58)^2\frac{m^6}{s^2}\times\frac{1}{0.049\,m^2}\times\frac{N\cdot s^2}{kg\cdot m} - 3.45\times10^5\frac{N}{m^2}\times 0.049\,m^2 = -23.4\ kN$$

$$R_{x_2} = (p_2-p_1)A - R_{x_1} = p_2A - p_1A + \frac{\rho Q^2}{A} + p_1A = p_2A + \frac{\rho Q^2}{A}$$

$$= 3.32\times10^5\frac{N}{m^2}\times0.049\,m^2 + 0.95\times999\frac{kg}{m^3}\times(0.58)^2\frac{m^6}{s^2}\times\frac{1}{0.049\,m^2}\times\frac{N\cdot s^2}{kg\cdot m} = 22.8\ kN$$

$$\vec r\times\vec F_s = \vec r_1\times R_{x_1}\hat i = L\hat j\times R_{x_1}\hat i = -LR_{x_1}\hat k = -20\,m\times(-46.0)\,kN\,\hat k = 468\,\hat k\ kN\cdot m$$

These are forces and torque on CV. The corresponding reactions are:

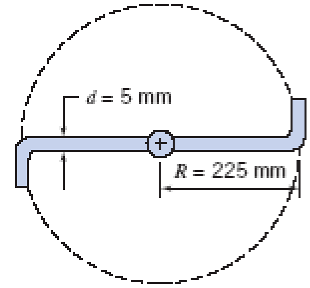
$K_{x_1} = -R_{x_1} = 23.4\ kN\ ,\quad K_{x_2} = -R_{x_2} = -22.8\ kN$ ← Force

$\vec M = -\vec r\times\vec F_s = -468\,\hat k\ kN\cdot m$ (i.e. clockwise) ← Torque

Problem 4.188

[Difficulty: 3]

4.188 The simplified lawn sprinkler shown rotates in the horizontal plane. At the center pivot, $Q = 15$ L/min of water enters vertically. Water discharges in the horizontal plane from each jet. If the pivot is frictionless, calculate the torque needed to keep the sprinkler from rotating. Neglecting the inertia of the sprinkler itself, calculate the angular acceleration that results when the torque is removed.



Given: Data on rotating spray system

Find: Torque required to hold stationary; steady-state speed

Solution:

Basic equation: Rotating CV

$$\vec{r} \times \vec{F}_s + \int_{CV} \vec{r} \times \vec{g}\,\rho\, d\forall + \vec{T}_{shaft} - \int_{CV} \vec{r} \times [2\vec{\omega} \times \vec{V}_{xyz} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \dot{\vec{\omega}} \times \vec{r}]\,\rho\, d\forall = \frac{\partial}{\partial t}\int_{CV} \vec{r} \times \vec{V}_{xyz}\,\rho\, d\forall + \int_{CS} \vec{r} \times \vec{V}_{xyz}\,\rho \vec{V}_{xyz} \cdot d\vec{A} \qquad (4.52)$$

Assumptions: 1) No surface force; 2) Body torques cancel; 3) Sprinkler stationary; 4) Steady flow; 5) Uniform flow; 6) L<<r

The given data is $Q = 15 \cdot \frac{L}{min}$ $\quad R = 225 \cdot mm$ $\quad d = 5 \cdot mm$ $\quad \rho = 999 \cdot \frac{kg}{m^3}$

For each branch $V = \frac{1}{2} \cdot \frac{Q}{\frac{\pi}{4} \cdot d^2}$ $\quad V = 6.37 \frac{m}{s}$

The basic equation reduces to a single scalar equation (FOR EACH BRANCH)

$$T_{shaft} - \int \vec{r} \times (\vec{\alpha} \times \vec{r}) \cdot \rho\, dV = \int \vec{r} \times \overrightarrow{V_{xyz}} \cdot \rho \cdot \overrightarrow{V_{xyz}}\, d\vec{A}$$

where α is the angular acceleration

But $\vec{r} \times (\vec{\alpha} \times \vec{r}) = r^2 \cdot \alpha$ (r and α perpendicular); the volume integral is $\int \vec{r} \times (\vec{\alpha} \times \vec{r}) \cdot \rho\, dV = \int r^2 \cdot \alpha \cdot \rho\, dV = \frac{R^3}{3} \cdot \alpha \cdot \frac{\pi}{4} \cdot d^2$

For the surface integral (FOR EACH BRANCH) $\int \vec{r} \times \overrightarrow{V_{xyz}} \cdot \rho \cdot \overrightarrow{V_{xyz}}\, d\vec{A} = R \cdot V \cdot \rho \cdot \frac{Q}{2}$

Combining $\quad T_{shaft} - \frac{R^3}{3} \cdot \alpha \cdot \rho \cdot \frac{\pi}{4} \cdot d^2 = R \cdot V \cdot \rho \cdot \frac{Q}{2} \qquad (1)$

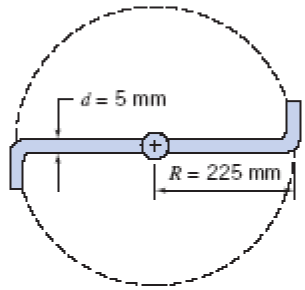
When the sprayer is at rest, $\alpha = 0$, so $\quad T_{shaft} = R \cdot V \cdot \rho \cdot \frac{Q}{2}$ $\quad T_{shaft} = 0.179\ N \cdot m$

The total torque is then $\quad T_{total} = 2 \cdot T_{shaft}$ $\quad T_{total} = 0.358\ N \cdot m$

When the device is released is released ($T_{shaft} = 0$ in Eq 1), we can solve for α $\quad \alpha = \frac{6 \cdot \rho \cdot Q \cdot V}{\rho \cdot \pi \cdot d^2 \cdot R^2}$ $\quad \alpha = 2.402 \times 10^3 \frac{1}{s^2}$

Problem 4.189 [Difficulty: 3]

4.189 Consider the sprinkler of Problem 4.188 again. Derive a differential equation for the angular speed of the sprinkler as a function of time. Evaluate its steady-state speed of rotation if there is no friction in the pivot.



Given: Data on rotating spray system

Find: Differential equation for motion; steady speed

Solution:

Basic equation: Rotating CV

$$\vec{r}\times\vec{F}_s + \int_{CV}\vec{r}\times\vec{g}\,\rho\,dV\!\!\!- + \vec{T}_{shaft} - \int_{CV}\vec{r}\times[2\vec{\omega}\times\vec{V}_{xyz} + \vec{\omega}\times(\vec{\omega}\times\vec{r}) + \dot{\vec{\omega}}\times\vec{r}]\,\rho\,dV\!\!\!- = \frac{\partial}{\partial t}\int_{CV}\vec{r}\times\vec{V}_{xyz}\,\rho\,dV\!\!\!- + \int_{CS}\vec{r}\times\vec{V}_{xyz}\,\rho\vec{V}_{xyz}\cdot d\vec{A} \qquad (4.52)$$

Assumptions: 1) No surface force; 2) Body torques cancel; 3) Steady flow; 5) Uniform flow; 6) L<<r

The given data is $Q = 15\cdot\frac{L}{min}$ $\quad R = 225\cdot mm$ $\quad d = 5\cdot mm$ $\quad \rho = 999\cdot\frac{kg}{m^3}$

For each branch $V = \frac{1}{2}\cdot\frac{Q}{\frac{\pi}{4}\cdot d^2}$ $\quad V = 6.37\frac{m}{s}$ $\quad A = \frac{\pi}{4}\cdot d^2$ $\quad A = 19.6\,mm^2$

The basic equation reduces to a single scalar equation (FOR EACH BRANCH)

$$-\int \vec{r}\times\left(2\cdot\vec{\omega}\times\vec{V}\times\vec{r} + \vec{\alpha}\times\vec{r}\right)\cdot\rho\,dV = \int \vec{r}\times\overrightarrow{V_{xyz}}\cdot\rho\cdot\overrightarrow{V_{xyz}}\,d\vec{A} \qquad \text{where } \alpha \text{ is the angular acceleration}$$

But $\vec{r}\times\left(2\cdot\vec{\omega}\times\vec{V}\times\vec{r} + \vec{\alpha}\times\vec{r}\right) = 2\cdot\omega\cdot r\cdot V + \alpha\cdot r^2$ (r and α perpendicular)

The volume integral is then

$$-\int \vec{r}\times\left(2\cdot\vec{\omega}\times\vec{V}\times\vec{r} + \vec{\alpha}\times\vec{r}\right)\cdot\rho\,dV = -\left(\omega\cdot R^2\cdot V + \alpha\cdot\frac{R^3}{3}\right)\cdot\rho\cdot A$$

For the surface integral (FOR EACH BRANCH)

$$\int \vec{r}\times\overrightarrow{V_{xyz}}\cdot\rho\cdot\overrightarrow{V_{xyz}}\,d\vec{A} = R\cdot V\cdot\rho\cdot\frac{Q}{2}$$

Combining

$$-\left(\omega\cdot R^2\cdot V + \alpha\cdot\frac{R^3}{3}\right)\cdot\rho\cdot A = R\cdot V\cdot\rho\cdot\frac{Q}{2} \qquad \text{or} \qquad \alpha = \frac{3}{A\cdot R^2}\cdot\left(-\omega\cdot V\cdot A\cdot R - \frac{Q\cdot V}{2}\right) \qquad (1)$$

The steady state speed (α = 0 in Eq 1) is then when

$$-\omega_{max}\cdot V\cdot A\cdot R - \frac{Q\cdot V}{2} = 0 \qquad \text{or} \qquad \omega_{max} = -\frac{Q}{2\cdot A\cdot R}$$

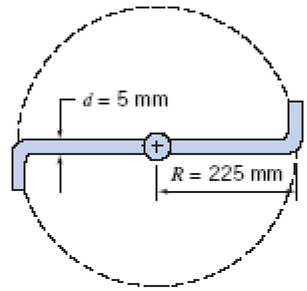
$$\omega_{max} = -28.3\frac{1}{s} \qquad \omega_{max} = -270\,rpm$$

Problem 4.190

[Difficulty: 3]

4.190 Repeat Problem 4.189, but assume a constant retarding torque in the pivot of 0.5 N · m. At what retarding torque would the sprinkler not be able to rotate?

NOTE ERROR: Retarding torque is 0.05 N.m!



Given:
Data on rotating spray system

Find:
Differential equation for motion; steady speed; troque to stop

Solution:

Basic equation: Rotating CV

$$\vec{r}\times\vec{F}_s + \int_{CV}\vec{r}\times\vec{g}\,\rho\, dV + \vec{T}_{shaft} - \int_{CV}\vec{r}\times[2\vec{\omega}\times\vec{V}_{xyz} + \vec{\omega}\times(\vec{\omega}\times\vec{r}) + \dot{\vec{\omega}}\times\vec{r}]\,\rho\, dV = \frac{\partial}{\partial t}\int_{CV}\vec{r}\times\vec{V}_{xyz}\,\rho\, dV + \int_{CS}\vec{r}\times\vec{V}_{xyz}\,\rho\vec{V}_{xyz}\cdot d\vec{A} \qquad (4.52)$$

Assumptions: 1) No surface force; 2) Body torques cancel; 3) Steady flow; 5) Uniform flow; 6) L<<r

The given data is $\quad Q = 15\cdot\frac{L}{min} \quad R = 225\cdot mm \quad d = 5\cdot mm \quad \rho = 999\cdot\frac{kg}{m^3} \quad T = 0.05\cdot N\cdot m$

For each branch $\quad V = \frac{1}{2}\cdot\frac{Q}{\frac{\pi}{4}\cdot d^2} \quad V = 6.37\,\frac{m}{s} \quad A = \frac{\pi}{4}\cdot d^2 \quad A = 19.6\cdot mm^2$

The basic equation reduces to a single scalar equation (FOR EACH BRANCH)

$$\frac{T}{2} - \int \vec{r}\times\left(2\cdot\vec{\omega}\times\vec{V}\times\vec{r} + \vec{\alpha}\times\vec{r}\right)\cdot\rho\, dV = \int \vec{r}\times\overrightarrow{V_{xyz}}\rho\cdot\overrightarrow{V_{xyz}}d\vec{A}$$

where T is the retarding torque α is the angular acceleration

But $\quad \vec{r}\times\left(2\cdot\vec{\omega}\times\vec{V}\times\vec{r} + \vec{\alpha}\times\vec{r}\right) = 2\cdot\omega\cdot r\cdot V + \alpha\cdot r^2 \quad$ (r and α perpendicular)

The volume integral is then $\quad -\int \vec{r}\times\left(2\cdot\vec{\omega}\times\vec{V}\times\vec{r} + \vec{\alpha}\times\vec{r}\right)\cdot\rho\, dV = -\left(\omega\cdot R^2\cdot V + \alpha\cdot\frac{R^3}{3}\right)\cdot\rho\cdot A$

For the surface integral (FOR EACH BRANCH) $\quad \int \vec{r}\times\overrightarrow{V_{xyz}}\rho\cdot\overrightarrow{V_{xyz}}d\vec{A} = R\cdot V\cdot\rho\cdot\frac{Q}{2}$

Combining $\quad \frac{T}{2} - \left(\omega\cdot R^2\cdot V + \alpha\cdot\frac{R^3}{3}\right)\cdot\rho\cdot A = R\cdot V\cdot\rho\cdot\frac{Q}{2} \quad$ or $\quad \alpha = \frac{3}{2\cdot\rho\cdot A\cdot R^3}\cdot\left(T - 2\cdot\rho\cdot A\cdot\omega\cdot R^2\cdot V - \rho\cdot R\cdot Q\cdot V\right) \quad (1)$

The steady state speed (α = 0 in Eq 1) is then when $\quad T - 2\cdot\rho\cdot A\cdot\omega_{max}\cdot R^2\cdot V - \rho\cdot R\cdot Q\cdot V = 0 \quad$ or $\quad \omega_{max} = \frac{T - \rho\cdot R\cdot Q\cdot V}{2\cdot\rho\cdot A\cdot R^2\cdot V}$

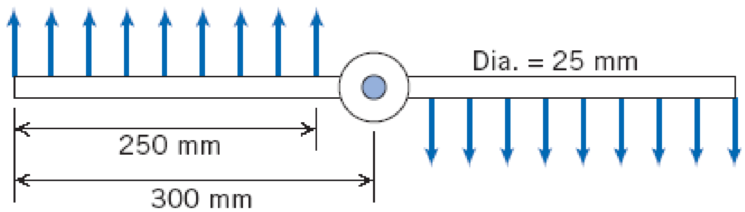
$$\omega_{max} = -24.3\,\frac{1}{s} \qquad \omega_{max} = -232\cdot rpm$$

For no rotation use α = ω = 0 in Eq 1, and solve for T_{max} $\quad T_{max} = \rho\cdot Q\cdot R\cdot V \quad T_{max} = 0.358\cdot N\cdot m$

Problem 4.191

[Difficulty: 4]

4.191 Water flows in a uniform flow out of the 2.5-mm slots of the rotating spray system, as shown. The flow rate is 3 L/s. Find (a) the torque required to hold the system stationary and (b) the steady-state speed of rotation after it is released.



Given: Data on rotating spray system

Find: Torque required to hold stationary; steady-state speed

Solution:

Basic equation: Rotating CV

$$\vec{r}\times\vec{F}_s+\int_{CV}\vec{r}\times\vec{g}\,\rho\,dV\!\!\!\!-+\vec{T}_{shaft}-\int_{CV}\vec{r}\times[2\vec{\omega}\times\vec{V}_{xyz}+\vec{\omega}\times(\vec{\omega}\times\vec{r})+\dot{\vec{\omega}}\times\vec{r}]\,\rho\,dV\!\!\!\!-=\frac{\partial}{\partial t}\int_{CV}\vec{r}\times\vec{V}_{xyz}\,\rho\,dV\!\!\!\!-+\int_{CS}\vec{r}\times\vec{V}_{xyz}\,\rho\vec{V}_{xyz}\cdot d\vec{A} \quad (4.52)$$

The given data is $\rho = 999\cdot\frac{kg}{m^3}$ $\quad\delta = 2.5\cdot mm$ $\quad r_o = 300\cdot mm$ $\quad Q_{in} = 3\cdot\frac{L}{s}$ $\quad r_i = (300-250)\cdot mm$

For no rotation ($\omega = 0$) the basic equation reduces to a single scalar equation

$$T_{shaft} = \int \vec{r}\times\overrightarrow{V_{xyz}}\cdot\rho\cdot\overrightarrow{V_{xyz}}\,d\vec{A} \qquad \text{or} \qquad T_{shaft} = 2\cdot\delta\cdot\int_{r_i}^{r_o} r\cdot V\cdot\rho\cdot V\,dr = 2\cdot\rho\cdot V^2\cdot\delta\cdot\int_{r_i}^{r_o} r\,dr = \rho\cdot V^2\cdot\delta\cdot\left(r_o^{\,2}-r_i^{\,2}\right)$$

where V is the exit velocity with respect to the CV $\qquad V = \frac{Q_{in}}{2\cdot\delta\cdot\left(r_o-r_i\right)} \qquad V = 2.40\,\frac{m}{s}$

Hence $\qquad T_{shaft} = \rho\cdot\left[\frac{Q_{in}}{2\cdot\delta\cdot\left(r_o-r_i\right)}\right]^2\cdot\delta\cdot\left(r_o^{\,2}-r_i^{\,2}\right) \qquad T_{shaft} = \frac{\rho\cdot Q_{in}^{\,2}}{4\cdot\delta}\cdot\frac{\left(r_o+r_i\right)}{\left(r_o-r_i\right)}$

$$T_{shaft} = \frac{1}{4}\times\left(3\cdot\frac{L}{s}\times\frac{10^{-3}\cdot m^3}{L}\right)^2\times\frac{999\cdot kg}{m^3}\times\frac{1}{0.0025\cdot m}\times\frac{(0.3+0.05)}{(0.3-0.05)} \qquad T_{shaft} = 1.26\cdot N\cdot m$$

For the steady rotation speed the equation becomes $\qquad -\int \vec{r}\times\left(2\cdot\vec{\omega}\times\overrightarrow{V_{xyz}}\right)\cdot\rho\,dV = \int \vec{r}\times\overrightarrow{V_{xyz}}\cdot\rho\cdot\overrightarrow{V_{xyz}}\,d\vec{A}$

The volume integral term $-\int \vec{r}\times\left(2\cdot\vec{\omega}\times\overrightarrow{V_{xyz}}\right)\cdot\rho\,dV$ must be evaluated for the CV. The velocity in the CV varies with r. This variation can be found from mass conservation

For an infinitesimal CV of length dr and cross-section A at radial position r, if the flow in is Q, the flow out is $Q + dQ$, and the loss through the slot is $V\delta dr$. Hence mass conservation leads to

$$(Q+dQ)+V\cdot\delta\cdot dr-Q=0 \qquad dQ = -V\cdot\delta\cdot dr \qquad Q(r) = -V\cdot\delta\cdot r + const$$

At the inlet ($r = r_i$) $\quad Q = Q_i = \dfrac{Q_{in}}{2}$

Hence $\quad Q = Q_i + V\cdot\delta\cdot(r_i - r) = \dfrac{Q_{in}}{2} + \dfrac{Q_{in}}{2\cdot\delta\cdot(r_o - r_i)}\cdot\delta\cdot(r_i - r) \qquad Q = \dfrac{Q_{in}}{2}\cdot\left(1 + \dfrac{r_i - r}{r_o - r_i}\right) = \dfrac{Q_{in}}{2}\cdot\left(\dfrac{r_o - r}{r_o - r_i}\right)$

and along each rotor the water speed is $\quad v(r) = \dfrac{Q}{A} = \dfrac{Q_{in}}{2\cdot A}\cdot\left(\dfrac{r_o - r}{r_o - r_i}\right)$

Hence the term $-\int \vec{r}\times\left(2\cdot\vec{\omega}\times\overrightarrow{V_{xyz}}\right)\cdot\rho\, dV$ becomes

$$-\int \vec{r}\times\left(2\cdot\vec{\omega}\times\overrightarrow{V_{xyz}}\right)\cdot\rho\, dV = 4\cdot\rho\cdot A\cdot\omega\cdot\int_{r_i}^{r_o} r\cdot v(r)\, dr = 4\cdot\rho\cdot\omega\cdot\int_{r_i}^{r_o} r\cdot\frac{Q_{in}}{2}\cdot\left(\frac{r_o - r}{r_o - r_i}\right) dr$$

or

$$-\int \vec{r}\times\left(2\cdot\vec{\omega}\times\overrightarrow{V_{xyz}}\right)\cdot\rho\, dV = 2\cdot\rho\cdot Q_{in}\cdot\omega\cdot\int_{r_i}^{r_o} r\cdot\left(\frac{r_o - r}{r_o - r_i}\right) dr = \rho\cdot Q_{in}\cdot\omega\cdot\frac{r_o^{\,3} + r_i^{\,2}\cdot(2\cdot r_i - 3\cdot r_o)}{3\cdot(r_o - r_i)}$$

Recall that $\quad \int \vec{r}\times\overrightarrow{V_{xyz}}\cdot\rho\cdot\overrightarrow{V_{xyz}}\, d\vec{A} = \rho\cdot V^2\cdot\delta\cdot\left(r_o^{\,2} - r_i^{\,2}\right)$

Hence equation $\quad -\int \vec{r}\times\left(2\cdot\vec{\omega}\times\overrightarrow{V_{xyz}}\right)\cdot\rho\, dV = \int \vec{r}\times\overrightarrow{V_{xyz}}\cdot\rho\cdot\overrightarrow{V_{xyz}}\, d\vec{A} \quad$ becomes

$$\rho\cdot Q_{in}\cdot\omega\cdot\frac{r_o^{\,3} + r_i^{\,2}\cdot(2\cdot r_i - 3\cdot r_o)}{3\cdot(r_o - r_i)} = \rho\cdot V^2\cdot\delta\cdot\left(r_o^{\,2} - r_i^{\,2}\right)$$

Solving for ω $\quad \omega = \dfrac{3\cdot(r_o - r_i)\cdot V^2\cdot\delta\cdot\left(r_o^{\,2} - r_i^{\,2}\right)}{Q_{in}\cdot\left[r_o^{\,3} + r_i^{\,2}\cdot(2\cdot r_i - 3\cdot r_o)\right]} \qquad \omega = 120\cdot\text{rpm}$

Problem 4.192

[Difficulty: 4]

4.192 If the same flow rate in the rotating spray system of Problem 4.191 is not uniform but instead varies linearly from a maximum at the outer radius to zero at the inner radius, find (a) the torque required to hold it stationary and (b) the steady-state speed of rotation.

Given: Data on rotating spray system

Find: Torque required to hold stationary; steady-state speed

Solution:

Governing equation: Rotating CV

$$\vec{r} \times \vec{F}_s + \int_{CV} \vec{r} \times \vec{g}\, \rho\, dV + \vec{T}_{shaft} - \int_{CV} \vec{r} \times [2\vec{\omega} \times \vec{V}_{xyz} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \dot{\vec{\omega}} \times \vec{r}]\, \rho\, dV = \frac{\partial}{\partial t}\int_{CV} \vec{r} \times \vec{V}_{xyz}\, \rho\, dV + \int_{CS} \vec{r} \times \vec{V}_{xyz}\, \rho \vec{V}_{xyz} \cdot d\vec{A} \qquad (4.52)$$

The given data is $\rho = 999 \cdot \frac{kg}{m^3}$ $\quad \delta = 2.5 \cdot mm \quad r_o = 300 \cdot mm \quad r_i = (300 - 250) \cdot mm \quad Q_{in} = 3 \cdot \frac{L}{s}$

For no rotation ($\omega = 0$) this equation reduces to a single scalar equation

$$T_{shaft} = \int \vec{r} \times \overrightarrow{V_{xyz}} \cdot \rho \cdot \overrightarrow{V_{xyz}}\, d\vec{A} \qquad \text{or} \qquad T_{shaft} = 2 \cdot \delta \cdot \int_{r_i}^{r_o} r \cdot V \cdot \rho \cdot V\, dr$$

where V is the exit velocity with respect to the CV. We need to find $V(r)$. To do this we use mass conservation, and the fact that the distribution is linear

$$V(r) = V_{max} \cdot \frac{(r - r_i)}{(r_o - r_i)} \qquad \text{and} \qquad 2 \cdot \frac{1}{2} \cdot V_{max} \cdot (r_o - r_i) \cdot \delta = Q_{in}$$

so

$$V(r) = \frac{Q_{in}}{\delta} \cdot \frac{(r - r_i)}{(r_o - r_i)^2}$$

Hence

$$T_{shaft} = 2 \cdot \rho \cdot \delta \cdot \int_{r_i}^{r_o} r \cdot V^2\, dr = 2 \cdot \frac{\rho \cdot Q_{in}^2}{\delta} \cdot \int_{r_i}^{r_o} r \cdot \left[\frac{(r - r_i)}{(r_o - r_i)^2}\right]^2 dr \qquad T_{shaft} = \frac{\rho \cdot Q_{in}^2 \cdot (r_i + 3 \cdot r_o)}{6 \cdot \delta \cdot (r_o - r_i)}$$

$$T_{shaft} = \frac{1}{6} \times \left(3 \cdot \frac{L}{s} \times \frac{10^{-3} \cdot m^3}{L}\right)^2 \times \frac{999 \cdot kg}{m^3} \times \frac{1}{0.0025 \cdot m} \times \frac{(0.05 + 3 \cdot 0.3)}{(0.3 - 0.05)} \qquad T_{shaft} = 2.28 \cdot N \cdot m$$

For the steady rotation speed the equation becomes

$$-\int \vec{r} \times \left(2 \cdot \vec{\omega} \times \overrightarrow{V_{xyz}}\right) \cdot \rho\, dV = \int \vec{r} \times \overrightarrow{V_{xyz}} \cdot \rho \cdot \overrightarrow{V_{xyz}}\, d\vec{A}$$

The volume integral term $-\int \vec{r} \times \left(2\cdot\vec{\omega} \times \overrightarrow{V_{xyz}}\right)\cdot\rho\, dV$ must be evaluated for the CV. The velocity in the CV varies with *r*. This variation can be found from mass conservation

For an infinitesmal CV of length *dr* and cross-section *A* at radial position *r*, if the flow in is *Q*, the flow out is *Q* + *dQ*, and the loss through the slot is $V\delta dr$ Hence mass conservation leads to

$$(Q + dQ) + V\cdot\delta\cdot dr - Q = 0 \qquad dQ = -V\cdot\delta\cdot dr \qquad Q(r) = Q_i - \delta\cdot\int_{r_i}^{r} \frac{Q_{in}}{\delta}\cdot\frac{(r - r_i)}{(r_o - r_i)^2}\, dr = Q_i - \int_{r_i}^{r} Q_{in}\cdot\frac{(r - r_i)}{(r_o - r_i)^2}\, dr$$

At the inlet ($r = r_i$) $\qquad Q = Q_i = \dfrac{Q_{in}}{2}$

Hence $\qquad Q(r) = \dfrac{Q_{in}}{2}\cdot\left[1 - \dfrac{(r - r_i)^2}{(r_o - r_i)^2}\right]$

and along each rotor the water speed is $\qquad v(r) = \dfrac{Q}{A} = \dfrac{Q_{in}}{2\cdot A}\cdot\left[1 - \dfrac{(r - r_i)^2}{(r_o - r_i)^2}\right]$

Hence the term $-\int \vec{r} \times \left(2\cdot\vec{\omega} \times \overrightarrow{V_{xyz}}\right)\cdot\rho\, dV$ becomes

$$4\cdot\rho\cdot A\cdot\omega\cdot\left(\int_{r_i}^{r_o} r\cdot v(r)\, dr\right) = 4\cdot\rho\cdot\omega\cdot\int_{r_i}^{r_o} \frac{Q_{in}}{2}\cdot r\cdot\left[1 - \frac{(r - r_i)^2}{(r_o - r_i)^2}\right] dr$$

or

$$2\cdot\rho\cdot Q_{in}\cdot\omega\cdot\int_{r_i}^{r_o} r\cdot\left[1\cdot - \frac{(r_o - r)^2}{(r_o - r_i)^2}\right] dr = \rho\cdot Q_{in}\cdot\omega\cdot\left(\frac{1}{6}\cdot r_o^{\,2} + \frac{1}{3}\cdot r_i\cdot r_o - \frac{1}{2}\cdot r_i^{\,2}\right)$$

Recall that $\qquad \int \vec{r} \times \overrightarrow{V_{xyz}}\cdot\rho\cdot\overrightarrow{V_{xyz}}\, d\vec{A} = \dfrac{\rho\cdot Q_{in}^{\,2}\cdot(r_i + 3\cdot r_o)}{6\cdot(r_o - r_i)\cdot\delta}$

Hence equation $\qquad -\int \vec{r} \times \left(2\cdot\vec{\omega} \times \overrightarrow{V_{xyz}}\right)\cdot\rho\, dV = \int \vec{r} \times \overrightarrow{V_{xyz}}\cdot\rho\cdot\overrightarrow{V_{xyz}}\, d\vec{A}$

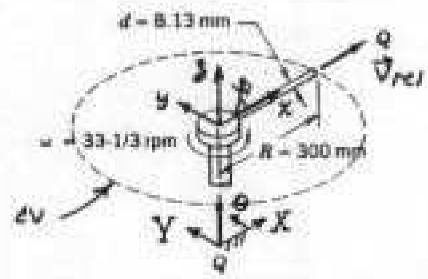
becomes $\qquad \rho\cdot Q_{in}\cdot\omega\cdot\left(\dfrac{1}{6}\cdot r_o^{\,2} + \dfrac{1}{3}\cdot r_i\cdot r_o - \dfrac{1}{2}\cdot r_i^{\,2}\right) = \dfrac{\rho\cdot Q_{in}^{\,2}\cdot(r_i + 3\cdot r_o)}{6\cdot(r_o - r_i)\cdot\delta}$

Solving for ω $\qquad \omega = \dfrac{\rho\cdot Q_{in}\cdot(r_i + 3\cdot r_o)}{\left(r_o^{\,2} + 2\cdot r_i\cdot r_o - 3\cdot r_i^{\,2}\right)\cdot(r_o - r_i)\cdot\rho\cdot\delta} \qquad \omega = 387\cdot rpm$

Problem 4.193

[Difficulty: 3]

4.193 A single tube carrying water rotates at constant angular speed, as shown. Water is pumped through the tube at volume flow rate $Q = 13.8$ L/min. Find the torque that must be applied to maintain the steady rotation of the tube using two methods of analysis: (a) a rotating control volume and (b) a fixed control volume.



Solution: Apply angular momentum principle. $\left\{ \omega = 33\tfrac{1}{3}\ \frac{\text{rev}}{\text{min}} = 3.49\ \text{rad/s} \right\}$

(a) Rotating CV: use relative velocities, Eq. 4.53:

Basic equation: $$\underbrace{\vec{r}\times\vec{F}_s}_{=0(1)} + \underbrace{\int_{CV}\vec{r}\times\vec{g}\,\rho\, d\forall}_{=0(2)} + \vec{T}_{shaft} - \int_{CV}\vec{r}\times\left[2\vec{\omega}\times\vec{V}_{xyz} + \underbrace{\vec{\omega}\times(\vec{\omega}\times\vec{r})}_{=0(3)} + \underbrace{\dot{\vec{\omega}}\times\vec{r}}_{=0(4)}\right]\rho\, d\forall$$
$$= \underbrace{\frac{\partial}{\partial t}\int_{CV}\vec{r}\times\vec{V}_{xyz}\,\rho\, d\forall}_{=0(5)} + \int_{CS}\underbrace{\vec{r}\times\vec{V}_{xyz}}_{=0(6)}\,\rho\vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: (1) $\vec{F}_s = 0$, (2) Body torques cancel, (3) No $\hat{k}$ in centripetal accel, (4) $\dot{\vec{\omega}} = 0$, (5) Steady flow, (6) $\vec{r}\times\vec{V} = 0$

Then

$$T_{shaft}\,\hat{k} = \int_{CV}\vec{r}\times(2\vec{\omega}\times\vec{V})\rho\, d\forall = \int_0^R r\hat{i}\times(2\omega\hat{k}\times V\hat{i})\rho A\, dr = \omega\rho V A R^2\hat{k} = \omega\rho Q R^2\hat{k}$$

$$T_{shaft} = 3.49\ \frac{\text{rad}}{\text{s}} \times 999\ \frac{\text{kg}}{\text{m}^3} \times 13.8\times10^{-3}\ \frac{\text{m}^3}{\text{min}} \times (0.3)^2\ \text{m}^2 \times \frac{\text{min}}{60\ \text{s}} \times \frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}} = 0.0722\ \text{N}\cdot\text{m}$$ T

(b) Fixed control volume: use absolute velocities, Eq. 4.47:

Basic equation: $$\underbrace{\vec{r}\times\vec{F}_s}_{=0(1)} + \underbrace{\int_{CV}\vec{r}\times\vec{g}\,\rho\, d\forall}_{=0(2)} + \vec{T}_{shaft} = \frac{\partial}{\partial t}\int_{CV}\vec{r}\times\vec{V}\rho\, d\forall + \int_{CS}\vec{r}\times\vec{V}\rho\vec{V}_{xyz}\cdot d\vec{A}$$

Relative to fixed coordinates XY, $\vec{r} = r(\cos\theta\,\hat{I} + \sin\theta\,\hat{J})$

$$\vec{V} = V(\cos\theta\,\hat{I} + \sin\theta\,\hat{J}) + r\omega(-\sin\theta\,\hat{I} + \cos\theta\,\hat{J})$$

$$\vec{r}\times\vec{V} = \begin{vmatrix} \hat{I} & \hat{J} & \hat{k} \\ r\cos\theta & r\sin\theta & 0 \\ V\cos\theta - r\omega\sin\theta & V\sin\theta + r\omega\cos\theta & 0 \end{vmatrix} = \hat{k}\left(rV\sin\theta\cos\theta + \omega r^2\cos^2\theta - rV\sin\theta\cos\theta + \omega r^2\sin^2\theta\right) = \omega r^2\hat{k}$$

Thus $\partial/\partial t = 0$ and $\int_{CS}\vec{r}\times\vec{V}\rho\vec{V}_{xyz}\cdot d\vec{A} = \omega R^2\hat{k}\{+\rho Q\} = \omega\rho Q R^2\hat{k}$ and

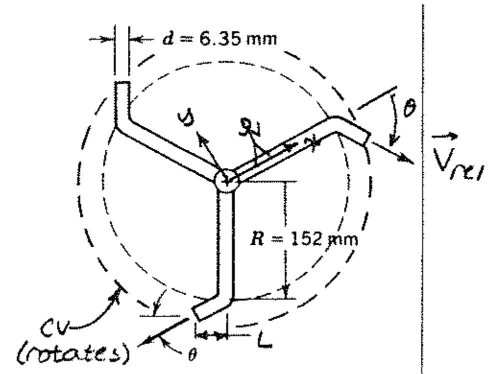
$T_{shaft}\,\hat{k} = \omega\rho Q R^2\hat{k}$ (as before); $T = 0.0722\ \text{N}\cdot\text{m}$ T

{ Note that when applied correctly, either choice of CV produces the same result. }

Problem 4.194

[Difficulty: 3]

4.194 The lawn sprinkler shown is supplied with water at a rate of 68 L/min. Neglecting friction in the pivot, determine the steady-state angular speed for $\theta = 30°$. Plot the steady-state angular speed of the sprinkler for $0 \le \theta \le 90°$.



Solution: Choose rotating CV. Apply angular momentum principle, Eq. 4.53.

Basic equation:
$$\vec{r}\times\cancelto{=0(1)}{\vec{F}_s} + \int_{CV}\vec{r}\times\cancelto{=0(2)}{\vec{g}}\,\rho\,d\forall + \cancelto{=0(3)}{\vec{T}_{shaft}}$$
$$-\int_{CV}\vec{r}\times\left[2\vec{\omega}\times\vec{V}_{xyz} + \vec{\omega}\times(\vec{\omega}\times\cancelto{=0(5)}{\vec{r}}) + \cancelto{=0(6)}{\dot{\vec{\omega}}}\times\vec{r}\right]\rho\,d\forall$$
$$= \cancelto{=0(6)}{\frac{\partial}{\partial t}}\int_{CV}\vec{r}\times\vec{V}_{xyz}\,\rho\,d\forall + \int_{CS}\vec{r}\times\vec{V}_{xyz}\,\rho\vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: (1) $\vec{F}_s = 0$, (2) Body torques cancel, (3) $\vec{T}_{shaft} = 0$, (4) Neglect aerodynamic drag, (5) No $\hat{k}$ component of centripetal acceleration, (6) Steady flow, (7) $L \ll R$

Analyze one arm of sprinkler. From geometry, $\vec{r} = r\hat{i}$ in CV, $\vec{r} = R\hat{i}$ at jet. Then

$$-\int_{CV}\vec{r}\times\left[2\vec{\omega}\times\vec{V}_{xyz}\right]\rho\,d\forall = R\hat{i}\times(-V\sin\theta\,\hat{j})\,\rho\frac{Q}{3} = -\rho\frac{QRV}{3}\sin\theta\,\hat{k}$$

$$r\hat{i}\times(2\omega\hat{k}\times V\hat{i}) = 2\omega V r\,\hat{k}\;;\; -\int_{CV} = -\omega V R^2\rho A\,\hat{k}$$

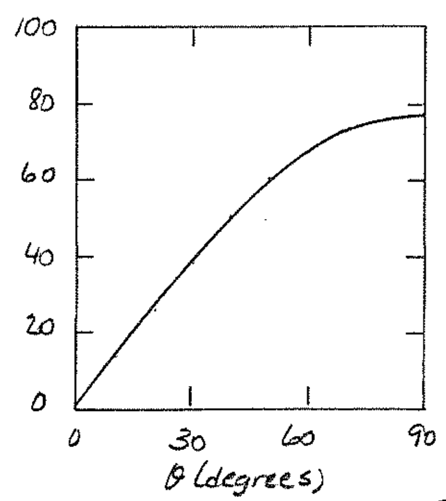
Dropping $\hat{k}$, $\quad -\omega V R^2 \rho A = -\rho\frac{QRV}{3}\sin\theta$, so with $VA = Q/3$,

$$\omega = \frac{V}{R}\sin\theta \;;\; V = \frac{Q}{3A} = \frac{4Q}{3\pi d^2} = \frac{4}{3\pi}\times 68\times10^{-3}\frac{m^3}{min}\times\frac{1}{(0.00635)^2\,m^2}\times\frac{min}{60\,s} = 11.9\ m/s$$

$$\omega = 11.9\,\frac{m}{s}\times\frac{1}{0.152\,m}\times\sin\theta = 78.3\sin\theta\ \text{rad/s}$$

ω

Plotting:



For $\theta = 30°$,

$\omega = 78.3\sin 30°$

$\omega = 39.1$ rad/s

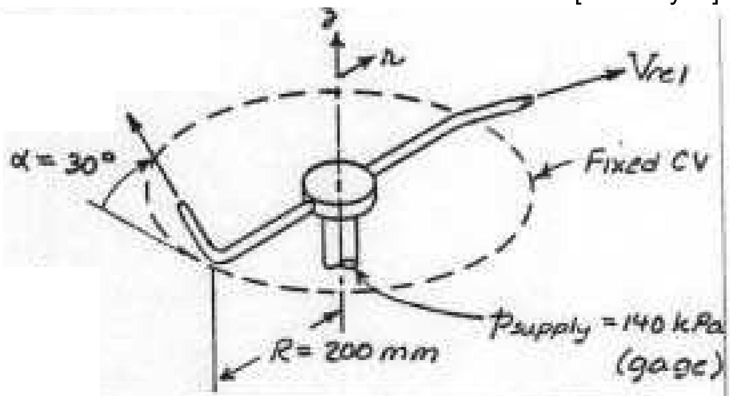
ω (θ = 30°)

Plot

Problem 4.195

[Difficulty: 3]

4.195 A small lawn sprinkler is shown. The sprinkler operates at a gage pressure of 140 kPa. The total flow rate of water through the sprinkler is 4 L/min. Each jet discharges at 17 m/s (relative to the sprinkler arm) in a direction inclined 30° above the horizontal. The sprinkler rotates about a vertical axis. Friction in the bearing causes a torque of 0.18 N · m opposing rotation. Evaluate the torque required to hold the sprinkler stationary.



Solution: Apply moment of momentum using fixed CV enclosing sprinkler arms.

Basic equation:

$$\underbrace{\vec{r}\times\vec{F}_s}_{=0(1)} + \underbrace{\int_{CV} \vec{r}\times\vec{g}\,\rho\, d\forall}_{=0(2)} + \vec{T}_{shaft} = \underbrace{\frac{\partial}{\partial t}\int_{CV} \vec{r}\times\vec{V}\,\rho\, d\forall}_{=0(3)} + \int_{CS} \vec{r}\times\vec{V}\,\rho\,\vec{V}\cdot d\vec{A}$$

Assumptions: (1) Neglect torque due to surface forces
(2) Torques due to body forces cancel by symmetry
(3) Steady flow
(4) Uniform flow leaving each jet

Then

$$-T_f\,\hat{k} = (\vec{r}\times\vec{V})_{in}\{-\rho Q\} + 2(\vec{r}\times\vec{V})_{jet}\{\tfrac{1}{2}\rho Q\}$$

$$(\vec{r}\times\vec{V})_{in} \approx 0 \qquad \vec{r} = R\hat{e}_r$$
$$\vec{V} = (R\omega - V_{rel}\cos\alpha)\hat{e}_\theta + V_{rel}\sin\alpha\,\hat{e}_z$$

The absolute velocity of the jet leaving sprinkler is $\vec{V} = V_{rel}\left[\cos\alpha(-\hat{e}_\theta)+\sin\alpha(\hat{e}_z)\right]$

Then $(\vec{r}\times\vec{V})_z = \left\{R\hat{e}_r\times V_{rel}\left[\cos\alpha(-\hat{e}_\theta)+\sin\alpha(\hat{e}_z)\right]\right\}_z = \left\{RV_{rel}\cos\alpha(-\hat{e}_z)+RV_{rel}\sin\alpha(-\hat{e}_\theta)\right\}_z$

$$(\vec{r}\times\vec{V})_z = -RV_{rel}\cos\alpha$$

Substituting, $T_{shaft} = T_{ext} - T_f = 2(-RV_{rel}\cos\alpha)(\tfrac{1}{2}\rho Q)$

Thus $T_{ext} = T_f - \rho Q R V_{rel}\cos\alpha$

$$= 0.18\ N\cdot m - 999\frac{kg}{m^3}\times 4\frac{L}{min}\times 0.2\,m\times 17\frac{m}{s}\times 0.866\times\frac{m^3}{1000\,L}\times\frac{min}{60\,s}\times\frac{N\,s^2}{kg\cdot m}$$

$$T_{ext} = -0.0161\ N\cdot m \quad \text{(to hold sprinkler stationary)}$$

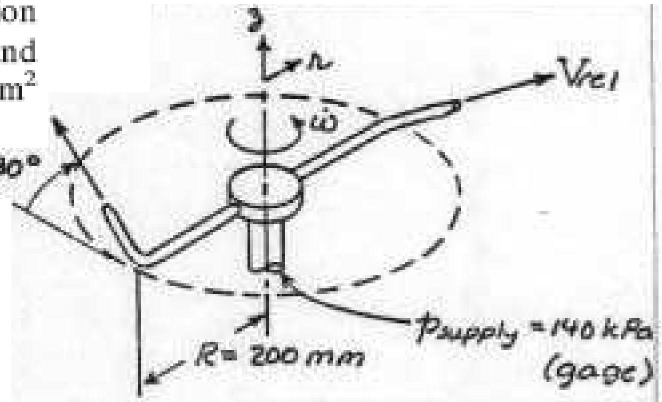
T_{ext}

{Since $T_{ext} < 0$, it must be applied in the minus z direction to oppose motion}

Problem 4.196

[Difficulty: 3]

4.196 In Problem 4.195, calculate the initial acceleration of the sprinkler from rest if no external torque is applied and the moment of inertia of the sprinkler head is 0.1 kg · m² when filled with water.



Solution: Apply moment of momentum using fixed CV enclosing sprinkler arms.

Basic equation:

$$\vec{r} \times \overset{\approx 0(1)}{\vec{F}_s} + \int_{CV} \overset{=0(2)}{\vec{r} \times \vec{g}}\, \rho\, d\forall + \vec{T}_{shaft} = \overset{=0(3)}{\frac{\partial}{\partial t}} \int_{CV} \vec{r} \times \vec{V} \rho\, d\forall + \int_{CS} \vec{r} \times \vec{V} \rho \vec{V} \cdot d\vec{A}$$

Assumptions: (1) Neglect torque due to surface forces
(2) Torques due to body forces cancel by symmetry
(3) Steady flow
(4) Uniform flow leaving each jet

Then

$$-T_f \hat{k} = (\vec{r} \times \vec{V})_{in} \{-\rho Q\} + 2(\vec{r} \times \vec{V})_{jet} \{\tfrac{1}{2} \rho Q\}$$

$$(\vec{r} \times \vec{V})_{in} \approx 0 \qquad \vec{r} = R\hat{e}_r$$

$$\vec{V} = (R\omega - V_{rel} \cos\alpha)\hat{e}_\theta + V_{rel} \sin\alpha\, \hat{e}_z$$

The jet leaves the sprinkler at $\vec{V}(abs) = V_{rel}[\cos\alpha(-\hat{e}_\theta) + \sin\alpha(\hat{e}_z)]$

Then $\vec{r} \times \vec{V} = R\hat{e}_r \times V_{rel}[\cos\alpha(-\hat{e}_\theta) + \sin\alpha(\hat{e}_z)] = RV_{rel}[\cos\alpha(-\hat{e}_z) + \sin\alpha(-\hat{e}_\theta)]$

Summing moments on the rotor, $\Sigma \vec{M} = I\dot{\vec{\omega}}$. Thus

$$\dot{\omega} = \frac{\Sigma T}{I} = \frac{\rho Q R V_{rel} \cos\alpha - T_f}{I}$$

$$= \left[999\frac{kg}{m^3} \times 4\frac{L}{min} \times 0.2\,m \times 17\frac{m}{s} \times 0.866 \times \frac{m^3}{1000\,L} \times \frac{min}{60\,s} - 0.18\,N\cdot m \times \frac{kg\cdot m}{N\cdot s^2} \right] \frac{1}{0.1\,kg\cdot m^2}$$

$$\dot{\omega} = 0.161\ rad/s^2$$

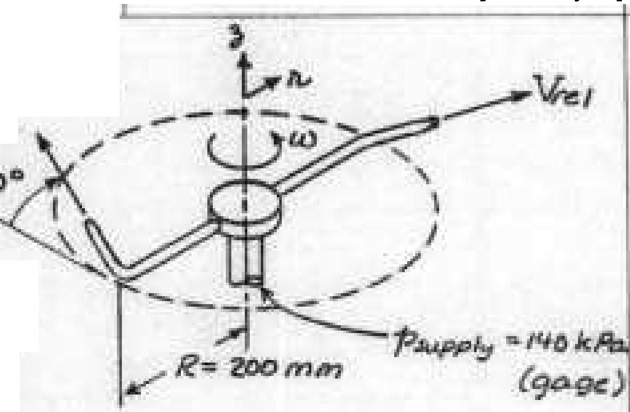
$\dot{\omega}$

{It is not necessary to use a rotating CV, because at the instant considered, $\vec{\omega} = 0$ and I is known.}

Problem 4.197

[Difficulty: 4]

4.197 A small lawn sprinkler is shown (Problem 4.196). The sprinkler operates at an inlet gage pressure of 140 kPa. The total flow rate of water through the sprinkler is 4.0 L/min. Each jet discharges at 17 m/s (relative to the sprinkler arm) in a direction inclined 30° above the horizontal. The sprinkler rotates about a vertical axis. Friction in the bearing causes a torque of 0.18 N · m opposing rotation. Determine the steady speed of rotation of the sprinkler and the approximate area covered by the spray.



Solution: Apply moment of momentum using fixed CV enclosing sprinkler arms.

Basic equation:

$$\overset{=0(1)}{\vec{r}\times\vec{F}_s} + \overset{=0(2)}{\int_{CV}\vec{r}\times\vec{g}\,\rho\, dV} + \vec{T}_{shaft} = \overset{=0(3)}{\frac{\partial}{\partial t}\int_{CV}\vec{r}\times\vec{V}\,\rho\, dV} + \int_{CS}\vec{r}\times\vec{V}\rho\vec{V}\cdot d\vec{A}$$

Assumptions: (1) Neglect torque due to surface forces
(2) Torques due to body forces cancel by symmetry
(3) Steady flow
(4) Uniform flow leaving each jet

Then

$$-T_f\hat{k} = (\vec{r}\times\vec{V})_{in}\{-\rho Q\} + 2(\vec{r}\times\vec{V})_{jet}\{\tfrac{1}{2}\rho Q\}$$

$$(\vec{r}\times\vec{V})_{in} \approx 0 \qquad \vec{r} = R\hat{e}_r$$

$$\vec{V} = (R\omega - V_{rel}\cos\alpha)\hat{e}_\theta + V_{rel}\sin\alpha\,\hat{e}_z$$

or

$$(\vec{r}\times\vec{V})_z = R(R\omega - V_{rel}\cos\alpha)$$

$$-T_f = R(R\omega - V_{rel}\cos\alpha)\rho Q$$

Thus

$$\omega = \frac{V_{rel}\cos\alpha}{R} - \frac{T_f}{\rho Q R^2}$$

$$= 17\frac{m}{s}\times\frac{\cos 30°}{0.2\,m} - 0.18\,N\cdot m\times\frac{m^3}{999\,kg}\times\frac{min}{4.0\,L}\times\frac{1}{(0.2)^2 m^2}\times\frac{60\,s}{min}\times\frac{10^3\,L}{m^3}\times\frac{kg\cdot m}{N\cdot s^2}$$

$$\omega = 6.04\,\frac{rad}{s} \quad \text{or} \quad 57.7\,rpm$$

ω

Treat the spray outside each nozzle as moving without air resistance:

For each particle, $\frac{dV_{az}}{dt} = -g$, so $V_z = V_{z_0} - gt$

At z_{max}, $V_z = 0$, so $t = \frac{V_{z0}}{g}$; flight time is 2t.

$$L = 2t\,V_{a\theta} = \frac{2V_{az}V_{a\theta}}{g} = \frac{2V_{rel}\sin\alpha(V_{rel}\cos\alpha - R\omega)}{g}$$

$$L = 2\times 17\frac{m}{s}\times\sin 30°\left(17\frac{m}{s}\times\cos 30° - 0.2\,m\times 6.04\frac{rad}{s}\right)\frac{s^2}{9.81\,m} = 23.4\,m$$

$$R_{spray} = \sqrt{R^2+L^2} = 23.4\,m\,;\quad A_{spray} = \pi R_{spray}^2 = \pi\times(23.4)^2 m^2 = 1720\,m^2$$

Aspray

Problem 4.198

[Difficulty: 4]

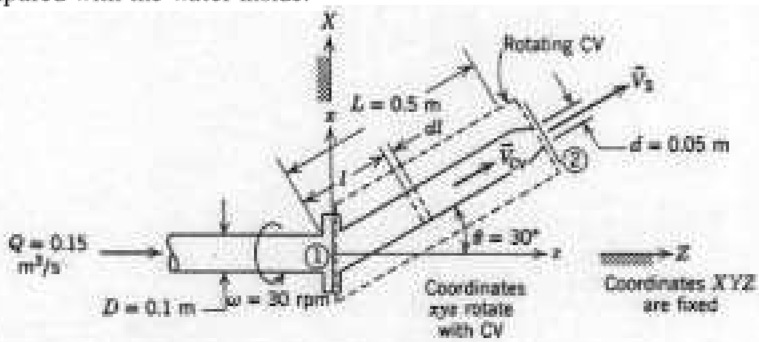
4.198 When a garden hose is used to fill a bucket, water in the bucket may develop a swirling motion. Why does this happen? How could the amount of swirl be calculated approximately?

Discussion: Frequently when filling a bucket the hose is held so that the water stream entering the bucket is not vertical. If, in addition, the water stream is off-center in the bucket, then flow entering the bucket has a tangential component of velocity, a swirl component.

The tangential component of the water velocity entering the bucket has a moment-of-momentum (swirl) with respect to a control volume drawn around the stationary bucket. This entering swirl can only be reduced by a torque acting to oppose it. Viscous forces among fluid layers will tend to transfer swirl to other layers so that eventually all of the water in the bucket has a swirling motion.

Swirl in the bucket may be influenced by viscosity. The swirl may tend to nearly a rigid-body motion to minimize viscous forces between annular layers of water in the bucket. The rigid-body motion assumption may be a reasonable model to calculate the total angular momentum (moment-of-momentum) of the water in the bucket.

4.199 Water flows at the rate of 0.15 m³/s through a nozzle assembly that rotates steadily at 30 rpm. The arm and nozzle masses are negligible compared with the water inside. Determine the torque required to drive the device and the reaction torques at the flange.



Find: (a) Torque required to drive the nozzle assembly
(b) Reaction torques at the flange.

<u>Solution</u>: Apply the moment of momentum equation to the rotating CV shown.

Basic equation:

$$\vec{r}\times\vec{F}_s + \int_{CV} \vec{r}\times\vec{g}\,\rho\,d\forall + \vec{T}_{shaft} \quad (\approx 0(2))$$

$$-\int_{CV} \vec{r}\times\left[2\vec{\omega}\times\vec{V}_{xyz} + \vec{\omega}\times(\vec{\omega}\times\vec{r}) + \underbrace{\dot{\vec{\omega}}\times\vec{r}}_{=0(3)}\right]\rho\,d\forall = \underbrace{\frac{\partial}{\partial t}\int_{CV}\vec{r}\times\vec{V}_{xyz}\,\rho\,d\forall}_{=0(5)} + \underbrace{\int_{CS}\vec{r}\times\vec{V}_{xyz}}_{=0(7)}\,\rho\vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: (1) Let $\vec{T}_{cv}$ represent all torques acting on the cv
(2) Neglect torque due to body force
(3) Constant angular speed
(4) Neglect mass of arm compared to water inside
(5) Steady flow in CV
(6) Neglect nozzle length compared to L
(7) $\vec{r}$ colinear with $\vec{V}$, so $\vec{r}\times\vec{V}_{xyz} = 0$

Then

$$\vec{T}_{cv} = \int_{CV} \vec{r}\times\left[2\vec{\omega}\times\vec{V}_{xyz} + \vec{\omega}\times(\vec{\omega}\times\vec{r})\right]\rho\,d\forall$$

Since $\vec{\omega} = \omega\hat{k}$ and $\vec{r} = l(\sin\theta\hat{i} + \cos\theta\hat{k})$, then

$$\vec{\omega}\times\vec{r} = \omega l \sin\theta\,\hat{j}$$

$$\vec{\omega}\times(\vec{\omega}\times\vec{r}) = \omega\hat{k}\times\omega l\sin\theta\hat{j} = \omega^2 l\sin\theta(-\hat{i})$$

and $\vec{r}\times[\vec{\omega}\times(\vec{\omega}\times\vec{r})] = l(\sin\theta\hat{i} + \cos\theta\hat{k})\times\omega^2 l\sin\theta(-\hat{i}) = \omega^2 l^2\sin\theta\cos\theta(-\hat{j})$

Since $\vec{V}_{xyz} = V_{cv}(\sin\theta\hat{i} + \cos\theta\hat{k})$, then

$$2\vec{\omega}\times\vec{V}_{xyz} = 2\omega\hat{k}\times V_{cv}(\sin\theta\hat{i} + \cos\theta\hat{k}) = 2\omega V_{cv}\sin\theta\hat{j}$$

and $\vec{r}\times[2\vec{\omega}\times\vec{V}_{xyz}] = l(\sin\theta\hat{i} + \cos\theta\hat{k})\times 2\omega V_{cv}\sin\theta\hat{j} = 2\omega l V_{cv}\sin^2\theta\,\hat{k}$
$+ 2\omega l V_{cv}\sin\theta\cos\theta(-\hat{i})$

Substituting and introducing $d\forall = A\,dl$,

$$\vec{T}_{cv} = \int_0^L \left(-2\omega l V_{cv} \sin\theta\cos\theta\,\hat{i} - \omega^2 l^2 \sin\theta\cos\theta\,\hat{j} + 2\omega l V_{cv}\sin^2\theta\,\hat{k}\right)\rho A\,dl$$

$$\vec{T}_{cv} = \left[-\omega L^2 V_{cv}\sin\theta\cos\theta\,\hat{i} - \frac{\omega^2 L^3}{3}\sin\theta\cos\theta\,\hat{j} + \omega L^2 V_{cv}\sin^2\theta\,\hat{k}\right]\rho A$$

The shaft torque needed to maintain steady rotation of the assembly is

$$T_{shaft} = T_{cv_z} = \omega L^2 V_{cv}\sin^2\theta\,\rho A = \omega L^2 \frac{Q}{A}\sin^2\theta\,\rho A = \rho Q \omega L^2 \sin^2\theta$$

$$= 999\,\frac{kg}{m^3} \times 0.15\,\frac{m^3}{s} \times 30\,\frac{rev}{min} \times (0.5)^2 m^2 \times (0.5)^2 \times 2\pi\,\frac{rad}{rev} \times \frac{min}{60\,s} \times \frac{N\cdot s^2}{kg\cdot m}$$

$$T_{shaft} = 29.4\ N\cdot m$$ T_{shaft}

The reaction moments acting on the flange are

$$M_x = -T_{cv_x} = \omega L^2 V_{cv}\sin\theta\cos\theta\,\rho A = \rho Q \omega L^2 \sin\theta\cos\theta$$

$$= 999\,\frac{kg}{m^3} \times 0.15\,\frac{m^3}{s} \times 30\,\frac{rev}{min} \times (0.5)^2 m^2 \times (0.5)(0.866)\,2\pi\,\frac{rad}{rev} \times \frac{min}{60\,s} \times \frac{N\cdot s^2}{kg\cdot m}$$

$M_x = 51.0\ N\cdot m$ (applied to flange by CV) M_x

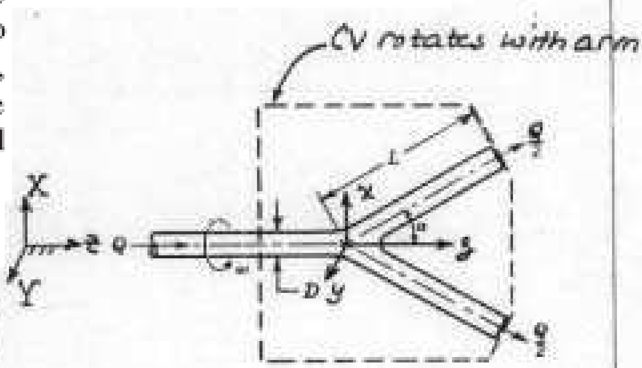
$$M_y = -T_{cv_y} = \frac{1}{3}\rho \omega^2 L^3 A \sin\theta\cos\theta$$

$$= \frac{1}{3} \times 999\,\frac{kg}{m^3}\left[30\,\frac{rev}{min} \times 2\pi\,\frac{rad}{rev} \times \frac{min}{60\,s}\right]^2 (0.5)^3 m^3 \times \frac{\pi}{4}(0.1)^2 m^2 \times (0.5)(0.866) \times \frac{N\cdot s^2}{kg\cdot m}$$

$M_y = 1.40\ N\cdot m$ (applied to flange by CV) M_y

{ Torques due to the masses of water, tube, and nozzle must be considered in the overall design. }

4.200 A pipe branches symmetrically into two legs of length L, and the whole system rotates with angular speed ω around its axis of symmetry. Each branch is inclined at angle α to the axis of rotation. Liquid enters the pipe steadily, with zero angular momentum, at volume flow rate Q. The pipe diameter, D, is much smaller than L. Obtain an expression for the external torque required to turn the pipe. What additional torque would be required to impart angular acceleration $\dot{\omega}$?



Solution: Apply moment of momentum equation using rotating CV.

Basic equation:

$$\underbrace{\vec{r}\times\vec{F}_s}_{=0(1)} + \underbrace{\int_{CV}\vec{r}\times\vec{g}\,\rho\,d\forall}_{=0(2)} + \vec{T}_{shaft} - \int_{CV}\vec{r}\times[2\vec{\omega}\times\vec{V}_{xyz} + \vec{\omega}\times(\vec{\omega}\times\vec{r}) + \dot{\vec{\omega}}\times\vec{r}]\rho\,d\forall = \underbrace{\frac{\partial}{\partial t}\int_{CV}\vec{r}\times\vec{V}_{xyz}\,\rho\,d\forall}_{=0(3)} + \int_{CS}\underbrace{\vec{r}\times\vec{V}_{xyz}}_{=0(4)}\,\rho\vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: (1) No surface forces
(2) Body-forces produce no torque about axis (symmetry)
(3) Flow steady in rotating frame
(4) $\vec{r}$ and $\vec{V}_{xyz}$ are colinear: $\vec{r}\times\vec{V}_{xyz} = 0$

Then

$$\vec{T}_{shaft} = \int_{CV}\vec{r}\times[2\vec{\omega}\times\vec{V}_{xyz} + \vec{\omega}\times(\vec{\omega}\times\vec{r}) + \dot{\vec{\omega}}\times\vec{r}]\rho\,d\forall$$

Using the coordinates above, $\vec{\omega} = \omega\hat{k}$ $\qquad \dot{\vec{\omega}} = \dot{\omega}\hat{k}$

$$\vec{r} = r(\cos\alpha\,\hat{k} + \sin\alpha\,\hat{i}) \quad \text{(upper tube)}$$

$$\vec{V}_{xyz} = \frac{Q}{2A}(\cos\alpha\,\hat{k} + \sin\alpha\,\hat{i}) \quad \text{(upper tube)};\; A = \frac{\pi D^2}{4}$$

and

$$\dot{\vec{\omega}}\times\vec{r} = \dot{\omega}r\sin\alpha\,\hat{j}$$

$$\vec{\omega}\times(\vec{\omega}\times\vec{r}) = \omega\hat{k}\times\omega r\sin\alpha\,\hat{j} = -\omega^2 r\sin\alpha\,\hat{i}$$

$$2\vec{\omega}\times\vec{V}_{xyz} = 2\omega\frac{Q}{2A}\sin\alpha\,\hat{j} = \frac{\omega Q}{A}\sin\alpha\,\hat{j}$$

Thus for the upper tube,

$$\vec{T}_{shaft} = \int_0^L\left\{r(\cos\alpha\,\hat{k} + \sin\alpha\,\hat{i})\times\left[\left(\frac{\omega Q}{A} + \dot{\omega}r\right)\sin\alpha\,\hat{j} - \omega^2 r\sin\alpha\,\hat{i}\right]\right\}\rho A\,dr$$

$$= \int_0^L\left[\left(\frac{r\omega Q}{A} + \dot{\omega}r^2\right)(\sin\alpha\cos\alpha)\hat{i} + \left(\frac{r\omega Q}{A} + \dot{\omega}r^2\right)\sin^2\alpha\,\hat{k} + \omega^2 r^2\sin\alpha\cos\alpha(-\hat{j})\right]\rho A\,dr$$

$$\vec{T}_{shaft}(\text{upper}) = \left[\left(\frac{L^2\omega Q}{2A} + \frac{\dot{\omega}L^3}{3}\right)\sin\alpha\cos\alpha\,\hat{i} + \left(\frac{L^2\omega Q}{2A} + \frac{\dot{\omega}L^3}{3}\right)\sin^2\alpha\,\hat{k} + \frac{\omega^2 L^3}{3}\sin\alpha\cos\alpha(-\hat{j})\right]\rho A$$

For the lower tube, $\vec{\omega} = \omega\hat{k}$ $\qquad \dot{\vec{\omega}} = \dot{\omega}\hat{k}$

$$\vec{r} = r(\cos\alpha\,\hat{k} - \sin\alpha\,\hat{i}) \quad \text{(lower tube)}$$

$$\vec{V}_{xyz} = \frac{Q}{2A}(\cos\alpha\,\hat{k} - \sin\alpha\,\hat{i}) \quad \text{(lower tube)}$$

and

$$\dot{\vec{\omega}} \times \vec{r} = -r\dot{\omega}\sin\alpha\,\hat{j}$$

$$\vec{\omega} \times (\vec{\omega} \times \vec{r}) = \omega\hat{k} \times (-r\omega\sin\alpha\,\hat{j}) = r\omega^2\sin\alpha\,\hat{i}$$

$$2\vec{\omega} \times \vec{V}_{xyz} = 2\omega\frac{Q}{2A}(-\sin\alpha)(\hat{j}) = -\frac{\omega Q}{A}\sin\alpha\,\hat{j}$$

Thus for the lower tube,

$$\vec{T}_{shaft} = \int_0^L \left\{ r(\cos\alpha\,\hat{k} - \sin\alpha\,\hat{i}) \times \left[\left(\frac{\omega Q}{A} + r\dot{\omega}\right)\sin\alpha(-\hat{j}) + r\omega^2\sin\alpha \right] \right\} \rho A\,dr$$

$$= \int_0^L \left[\left(\frac{r\omega Q}{A} + r^2\dot{\omega}\right)\sin\alpha\cos\alpha(\hat{i}) + \left(\frac{r\omega Q}{A} + r^2\dot{\omega}\right)\sin^2\alpha\,\hat{k} + r^2\omega^2\sin\alpha\cos\alpha\,\hat{j} \right] \rho A\,dr$$

$$\vec{T}_{shaft}(\text{lower}) = \left[\left(\frac{L^2\omega Q}{2A} + \frac{L^3\dot{\omega}}{3}\right)\sin\alpha\cos\alpha\,\hat{i} + \left(\frac{L^2\omega Q}{2A} + \frac{L^3\dot{\omega}}{3}\right)\sin^2\alpha\,\hat{k} + \frac{L^3\omega^2}{3}\sin\alpha\cos\alpha\,\hat{j} \right] \rho A$$

Summing these expressions gives

$$\vec{T}_{shaft}(\text{total}) = \left(\frac{L^2\omega Q}{A} + \frac{2L^3\dot{\omega}}{3}\right)\sin^2\alpha\,\rho A\,\hat{k}$$

Thus the steady-state portion of the torque is

$$\vec{T}_{shaft}(\text{steady state}) = \left(\frac{L^2\omega Q}{A}\right)\sin^2\alpha\,\rho A\,\hat{k} = L^2\rho\omega Q\sin^2\alpha\,\hat{k}$$

Stead

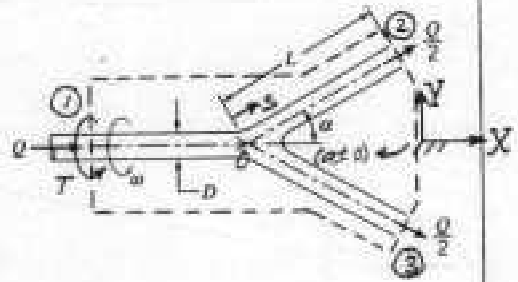
The additional torque needed to provide angular acceleration, $\dot{\omega}$, is

$$\vec{T}_{shaft}(\text{acceleration}) = \frac{2L^3\rho\dot{\omega}A}{3}\sin^2\alpha\,\hat{k}$$

Accel

$\left\{ \begin{array}{l} \text{Torques of individual tubes about the } x \text{ and } y \text{ axes are reacted} \\ \text{internally; they must be considered in design of the tube.} \end{array} \right\}$

(b) Using fixed CV:



Basic equation: $\cancel{\vec{r}\times\vec{F}_s}^{=0(1)} + \int_{CV} \cancel{\vec{r}\times\vec{g}}^{=0(2)}\rho\, d\forall + \vec{T}_{shaft}$

$$= \cancel{\frac{\partial}{\partial t}\int_{CV} \vec{r}\times\vec{V}\rho\, d\forall}^{=0(3)} + \int_{CS} \vec{r}\times\vec{V}\rho\vec{V}\cdot d\vec{A}$$

Assumptions: (1) No surface forces
(2) Body forces symmetric (no moment about X axis)
(3) No change in angular momentum within CV w.r.to time
(4) Symmetry in two branches
(5) Uniform flow at each cross-section

Then $\vec{T}_s = T\hat{I} = \cancel{\vec{r}_1}\times\vec{V}_1\{-\rho Q\} + \vec{r}_2\times\vec{V}_2\{+\rho\frac{Q}{2}\} + \vec{r}_3\times\vec{V}_3\{+\rho\frac{Q}{2}\} = 2\vec{r}_2\times\vec{V}_2\{\rho\frac{Q}{2}\}$

$\vec{r}_1 = 0$ $\quad\quad \vec{r}_2 = L\sin\alpha\,\hat{J}$; $\vec{V}_2 = \omega r_2\,\hat{K}$; $\vec{r}_2\times\vec{V}_2 = \omega L^2\sin^2\alpha\,\hat{I}$

or

$T_{ss} = \rho\omega Q L^2\sin^2\alpha$ (steady-state torque) $\quad T_{ss}$

The torque required for acceleration is $T_{acc} = I\dot{\omega}$, where $I = \int r^2 dm$

For one leg of the branch, $I = \int r^2 dm = \int_0^L (s\sin\alpha)^2\rho A\, ds = \frac{\rho A L^3}{3}\sin^2\alpha$

(6) Neglect mass of pipe

For both sides, $I = \frac{2\rho A L^3}{3}\sin^2\alpha$.

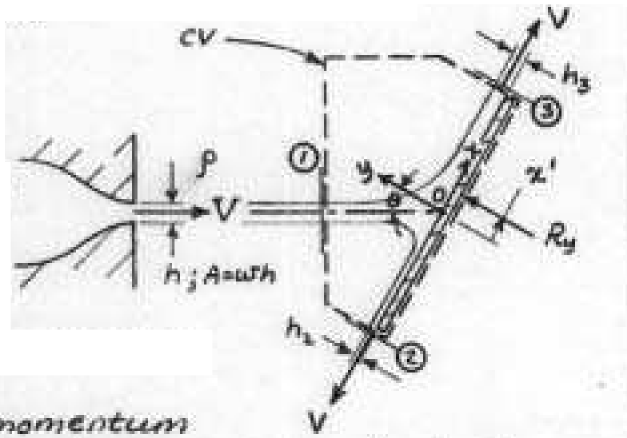
Thus

$T_{acc} = \frac{2\rho\dot{\omega}AL^3}{3}\sin^2\alpha$ (torque required for angular acceleration) $\quad T_{acc}$

The total torque that must be applied is

$T = T_{ss} + T_{acc} = \rho\omega Q L^2\sin^2\alpha + \frac{2\rho\dot{\omega}AL^3}{3}\sin^2\alpha$ $\quad T_{total}$

4.201 Liquid in a thin sheet, of width w and thickness h, flows from a slot and strikes a stationary inclined flat plate, as shown. Experiments show that the resultant force of the liquid jet on the plate does not act through point O, where the jet centerline intersects the plate. Determine the magnitude and line of application of the resultant force as functions of θ. Evaluate the equilibrium angle of the plate if the resultant force is applied at point O. Neglect any viscous effects.



Solution: Apply continuity, linear momentum and moment of momentum using CV and coordinates shown.

Basic equations:

$$0 = \overset{=0(1)}{\frac{\partial}{\partial t}\int_{CV} \rho\, d\forall} + \int_{CS} \rho \vec{V}\cdot d\vec{A}$$

$$\overset{=0(4)}{F_{Sx}} + \overset{=0(5)}{F_{Bx}} = \overset{=0(1)}{\frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall} + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$$

$$F_{Sy} + \overset{=0(5)}{F_{By}} = \overset{=0(1)}{\frac{\partial}{\partial t}\int_{CV} v\rho\, d\forall} + \int_{CS} v\rho \vec{V}\cdot d\vec{A}$$

$$\vec{r}\times\vec{F}_S + \overset{=0(5)}{\int_{CV} \vec{r}\times\vec{g}\,\rho\, d\forall} + \overset{=0(6)}{\vec{T}_{shaft}} = \overset{=0(1)}{\frac{\partial}{\partial t}\int_{CV} \vec{r}\times\vec{V}\rho\, d\forall} + \int_{CS} \vec{r}\times\vec{V}\rho\vec{V}\cdot d\vec{A}$$

Assumptions:
(1) Steady flow
(2) Uniform flow at each section
(3) No net pressure forces; $F_{Sx} = R_x$, $F_{Sy} = R_y$
(4) No viscous effects; $R_x = 0$ and $V_1 = V_2 = V_3 = V$
(5) Neglect body forces and torques
(6) $\vec{T}_{shaft} = 0$
(7) Incompressible flow, ρ = constant

Then from continuity,

$$0 = \{-|\rho V w h_1|\} + \{|\rho V w h_2|\} + \{|\rho V w h_3|\} \quad \text{or} \quad h_1 = h_2 + h_3 = h \qquad (1)$$

From x momentum

$$0 = u_1\{-|\rho V w h_1|\} + u_2\{|\rho V w h_2|\} + u_3\{|\rho V w h_3|\}$$

$$u_1 = V\sin\theta \qquad u_2 = -V \qquad u_3 = V$$

$$0 = \rho V^2 w\,(-h_1\sin\theta - h_2 + h_3) \quad \text{or} \quad h_3 - h_2 = h_1\sin\theta = h\sin\theta \qquad (2)$$

Combining Eqs. 1 and 2,

$$h_2 = h\left(\frac{1-\sin\theta}{2}\right) \qquad (3)$$

$$h_3 = h\left(\frac{1+\sin\theta}{2}\right) \qquad (4)$$

Problem 4.201

From y momentum, $R_y = v_1\{-|\rho V w h_1|\} + v_2\{|\rho V w h_2|\} + v_3\{|\rho V w h_3|\}$

$$v_1 = -V\cos\theta \qquad v_2 = 0 \qquad v_3 = 0$$

$$R_y = \rho V^2 w h \cos\theta \qquad (5) \qquad R_y$$

From moment of momentum,

$$\vec{r}\,' \times \vec{F}_s = \vec{r}_1 \times \vec{V}_1\{-|\rho V w h_1|\} + \vec{r}_2 \times \vec{V}_2\{|\rho V w h_2|\} + \vec{r}_3 \times \vec{V}_3\{|\rho V w h_3|\}$$

$\vec{r}\,' = x'\hat{i}$ $\quad \vec{r}_1 \times \vec{V}_1 = 0 \quad$ $\vec{r}_2 = \frac{h_2}{2}\hat{j}$ $\quad$ $\vec{r}_3 = -\frac{h_3}{2}\hat{j}$

$\vec{F}_s = R_y\hat{j}$ $\quad$ $\vec{V}_2 = -V\hat{i}$ $\quad$ $\vec{V}_3 = V\hat{i}$

$\vec{r}\,' \times \vec{F}_s = x' R_y \hat{k}$ $\quad$ $\vec{r}_2 \times \vec{V}_2 = \frac{h_2 V}{2}\hat{k}$ $\quad$ $\vec{r}_3 \times \vec{V}_3 = -\frac{h_3 V}{2}\hat{k}$

Combining and dropping $\hat{k}$,

$$x' R_y = \frac{1}{2}\rho V^2 w h_2^2 - \frac{1}{2}\rho V^2 w h_3^2 = \frac{1}{2}\rho V^2 w (h_2^2 - h_3^2)$$

or

$$x' = \frac{\rho V^2 w (h_2^2 - h_3^2)}{2R_y} = \frac{\rho V^2 w (h_2 + h_3)(h_2 - h_3)}{2R_y}$$

Substituting from Eqs. 3, 4 and 5,

$$x' = \frac{\rho V^2 w h^2 \left(\frac{1-\sin\theta}{2} + \frac{1+\sin\theta}{2}\right)\left(\frac{1-\sin\theta}{2} - \frac{1+\sin\theta}{2}\right)}{2\rho V^2 w h \cos\theta} = \frac{h(-\sin\theta)}{2\cos\theta}$$

or

$$x' = -\frac{h}{2}\tan\theta \qquad (6) \qquad x'$$

Note that $x' < 0$. This means that R_y must be applied <u>below</u> point 0.

If R_y is applied <u>at</u> point 0, then $x' = 0$. For equilibrium, from Eq. 6, $\theta = 0$. Thus if force is applied at point 0, plate will be in equilibrium when <u>perpendicular</u> to jet.

Problem 4.202

[Difficulty: 5] Part 1/2

4.202 For the rotating sprinkler of Example 4.14, what value of α will produce the maximum rotational speed? What angle will provide the maximum area of coverage by the spray? Draw a velocity diagram (using an r, θ, z coordinate system) to indicate the absolute velocity of the water jet leaving the nozzle. What governs the steady rotational speed of the sprinkler? Does the rotational speed of the sprinkler affect the area covered by the spray? How would you estimate the area? For fixed α, what might be done to increase or decrease the area covered by the spray?

Solution: The results of Example Problem 4.14 were computed assuming steady flow of water and constant frictional retarding torque at the sprinkler pivot.

$$T_f = R(V_{rel}\cos\alpha - \omega R)\rho Q$$

From these results,

$$\omega = \frac{V_{rel}\cos\alpha}{R} - \frac{T_f}{\rho Q R^2}$$

Thus rotational speed of the sprinkler increases as cos α increases, i.e., as α decreases. The maximum rotational speed occurs when α = 0. Then cos α = 1 and the rotational speed is

$$\omega = \frac{V_{rel}}{R} - \frac{T_f}{\rho Q R^2}$$

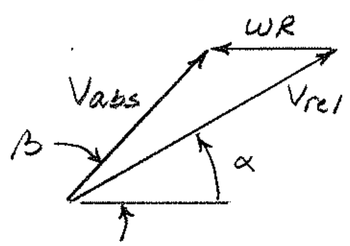
For the conditions of Example Problem 4.14 the maximum rotational speed is

$$\omega = 4.97\,\frac{m}{s} \times \frac{1}{0.150\,m} - 0.0718\,N\cdot m \times \frac{m^3}{999\,kg} \times \frac{min}{7.5\,L} \times \frac{1}{(0.150)^2\,m^2} \times \frac{1000\,L}{m^3} \times \frac{60\,s}{min} = 7.58\ rad/s$$

ω_{max}

The steady rotation speed ω of the sprinkler is governed by torque T_f and angle α.

Maximum coverage by the spray occurs when the "carry" of each jet stream is the longest. When aerodynamic drag on the stream is neglected, maximum carry occurs when the absolute velocity of the stream leaves the sprinkler at β = 45°, as shown in the velocity diagram below.



Note $\vec{V}_{abs} = \vec{V}_{rel} - \omega R\,\hat{i}_\theta$

Both the magnitude and direction of $\vec{V}_{abs}$ vary with ω!

For ω = 0, the relative velocity angle α and absolute velocity angle β are equal. Therefore maximum carry occurs when α = 45° (see graph on next page).

Any rotation rate ω reduces the magnitude V_{abs} and increases the angle β of the absolute velocity leaving the sprinkler jet. When ω > 0, then β > α, so for maximum carry α must be less than 45°. Consequently rotation reduces the carry of the stream and the area of coverage; at specified α the area of coverage decreases with increasing ω.

For the conditions of Example Problem 4.14 (ω = 30 rpm), optimum carry occurs at α ≈ 42°, and the coverage area is reduced from approximately 20 m² with a fixed sprinkler to 15 m² with 30 rpm rotation. If the rotation speed is increased (by decreasing pivot friction or decreasing nozzle angle α), coverage area may be reduced still further, to 9 m² or less.

$$A \approx \pi (x_{max})^2$$

Problem 4.202

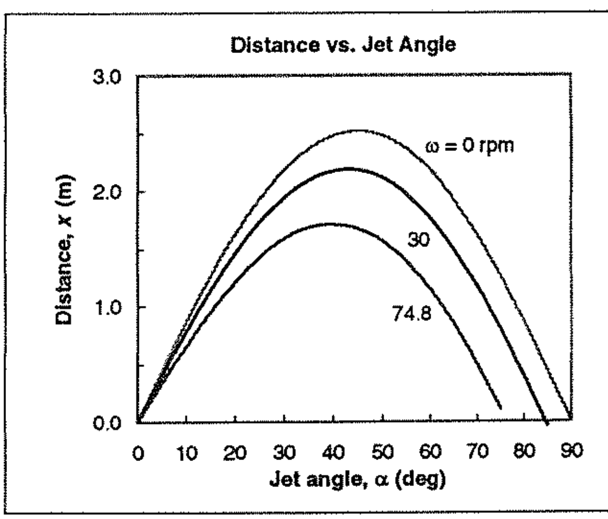
Analysis of Ground Area Covered by Rotating Lawn Sprinkler:

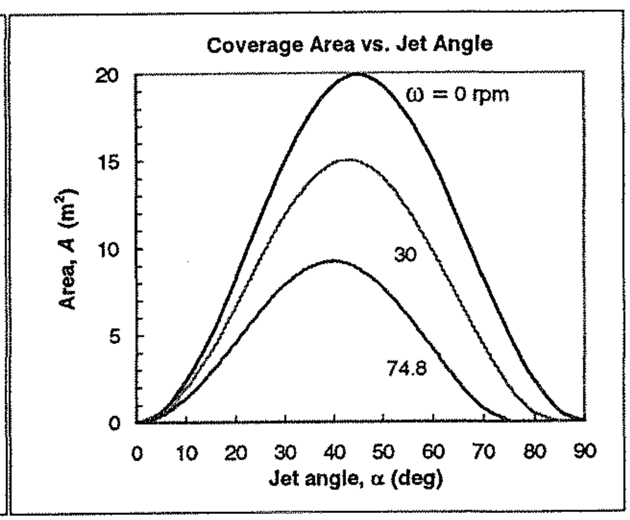
Variables:
A = ground area covered by spray stream
x = ground distance reached by spray stream
α = angle of jet above ground plane
β = angle of absolute velocity above ground plane

Input Data:
R = 0.150 m
V_{rel} = 4.97 m/s (Q = 7.5 L/min)

Results:

	ω (rpm) =	0		30		74.8
	ωR (m/s) =	0		0.471		1.17
α (deg)	x_{max} (m)	A (m^2)	x_{max} (m)	A (m^2)	x_{max} (m)	A (m^2)
0	0.00	0.00	0.00	0.00	0.00	0.00
5	0.437	0.601	0.396	0.492	0.333	0.349
10	0.861	2.33	0.778	1.90	0.654	1.35
15	1.26	4.98	1.14	4.05	0.951	2.84
20	1.62	8.23	1.46	6.65	1.21	4.61
25	1.93	11.7	1.73	9.37	1.43	6.39
30	2.18	14.9	1.94	11.8	1.59	7.90
35	2.37	17.6	2.09	13.8	1.68	8.90
40	2.48	19.3	2.17	14.8	1.71	9.23
45	2.52	19.9	2.18	14.9	1.68	8.83
50	2.48	19.3	2.11	14.0	1.57	7.72
55	2.37	17.6	1.97	12.3	1.39	6.08
60	2.18	14.9	1.77	9.81	1.15	4.15
65	1.93	11.7	1.50	7.03	0.850	2.269
70	1.62	8.23	1.17	4.30	0.500	0.785
75	1.26	4.98	0.798	2.00	0.109	0.037
78	1.02	3.30	0.557	0.975		
80	0.861	2.33	0.391	0.480		
85	0.437	0.601	-0.04	0.00		
90	0.00	0.00				

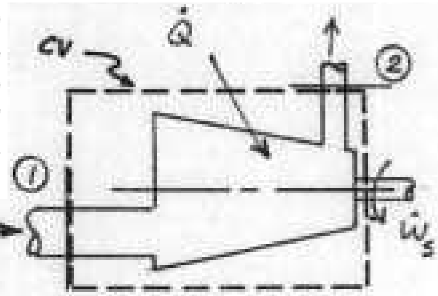




Problem 4.203

[Difficulty: 2]

4.203 Air at standard conditions enters a compressor at 75 m/s and leaves at an absolute pressure and temperature of 200 kPa and 345 K, respectively, and speed $V = 125$ m/s. The flow rate is 1 kg/s. The cooling water circulating around the compressor casing removes 18 kJ/kg of air. Determine the power required by the compressor.



Solution: Apply first law of thermodynamics, using CV shown.

B.E.
$$\dot{Q} - \dot{W}_s - \cancel{\dot{W}_{shear}}^{=0(1)} = \cancel{\frac{\partial}{\partial t}}^{=0(2)}\int_{CV} e\rho\, d\forall + \int_{CS} (e + pv)\rho \vec{V}\cdot d\vec{A}$$

Assume: (1) $\dot{W}_{shear} = 0$
(2) Steady flow
(3) Uniform flow at each section
(4) Neglect Δz
(5) Ideal gas, $p = \rho RT$, $\Delta h = c_p \Delta T$; $c_p = 1.00$ kJ/kg·K
(6) From continuity, $\dot{m}_1 = \dot{m}_2 = \dot{m}$

Then
$$\dot{Q} - \dot{W}_s = \left(u_2 + \frac{V_2^2}{2} + \cancel{gz_2}^{\approx 0(4)} + p_2 v_2\right)\{|\dot{m}|\} + \left(u_1 + \frac{V_1^2}{2} + \cancel{gz_1}^{=0(4)} + p_1 v_1\right)\{-|\dot{m}|\}$$

Note that $h = u + pv$, and $\dot{Q} = \dot{m}\frac{dQ}{dm}$, so
$$\dot{W}_{in} = -\dot{W}_s = \dot{m}\left(\frac{V_2^2 - V_1^2}{2} + h_2 - h_1 - \frac{dQ}{dm}\right) = \dot{m}\left[\frac{V_2^2 - V_1^2}{2} + c_p(T_2 - T_1) - \frac{dQ}{dm}\right]$$

or
$$\dot{W}_{in} = 1.0\,\frac{kg}{s}\left\{\frac{1}{2}\left[(125)^2 - (75)^2\right]\frac{m^2}{s^2} \times \frac{N\cdot s^2}{kg\cdot m} \times \frac{kJ}{1000\,N\cdot m} + 1.00\,\frac{kJ}{kg\cdot K}(345 - 288)\,K - \left(-18\,\frac{kJ}{kg}\right)\right\}\frac{kW\cdot s}{kJ}$$

$$\dot{W}_{in} = 80.0\ kW$$

$\dot{W}_{in}$

Problem 4.204

[Difficulty: 3]

4.204 Compressed air is stored in a pressure bottle with a volume of 100 L, at 500 kPa and 20°C. At a certain instant, a valve is opened and mass flows from the bottle at $\dot{m} = 0.01$ kg/s. Find the rate of change of temperature in the bottle at this instant

Given: Compressed air bottle

Find: Rate of temperature change

Solution:

Basic equations: Continuity; First Law of Thermodynamics for a CV

$$\frac{\partial}{\partial t}\int_{CV} \rho\, d\mathcal{V} + \int_{CS} \rho \vec{V}\cdot d\vec{A} = 0 \qquad \dot{Q} - \dot{W}_s - \dot{W}_{shear} - \dot{W}_{other} = \frac{\partial}{\partial t}\int_{CV} e\,\rho\, d\mathcal{V} + \int_{CS}\left(u + pv + \frac{V^2}{2} + gz\right)\rho\vec{V}\cdot d\vec{A}$$

Assumptions: 1) Adiabatic 2) No work 3) Neglect KE 4) Uniform properties at exit 5) Ideal gas

Given data $\quad p = 500\cdot kPa \qquad T = 20\,°C \qquad T = 293\,K \qquad V = 100\cdot L \qquad m_{exit} = 0.01\cdot\frac{kg}{s}$

Also $\quad R_{air} = \frac{286.9\,N\cdot m}{kg\cdot K} \qquad c_v = 717.4\cdot\frac{N\cdot m}{kg\cdot K}$

From continuity $\quad \frac{\partial}{\partial t}M_{CV} + m_{exit} = 0 \quad$ where m_{exit} is the mass flow rate at the exit (Note: Software does not allow a dot!)

$$\frac{\partial}{\partial t}M_{CV} = -m_{exit}$$

From the 1st law

$$0 = \frac{\partial}{\partial t}\int u\,dM + \left(u + \frac{p}{\rho}\right)\cdot m_{exit} = u\cdot\left(\frac{\partial}{\partial t}M\right) + M\cdot\left(\frac{\partial}{\partial t}u\right) + \left(u + \frac{p}{\rho}\right)\cdot m_{exit}$$

Hence

$$u\cdot\left(-m_{exit}\right) + M\cdot c_v\cdot\frac{dT}{dt} + u\cdot m_{exit} + \frac{p}{\rho}\cdot m_{exit} = 0 \qquad \frac{dT}{dt} = -\frac{m_{exit}\cdot p}{M\cdot c_v\cdot\rho}$$

But $\quad M = \rho\cdot V \quad$ (where V is volume) so $\quad \frac{dT}{dt} = -\frac{m_{exit}\cdot p}{V\cdot c_v\cdot\rho^2}$

For air

$$\rho = \frac{p}{R_{air}\cdot T} \qquad \rho = 500\times 10^3\cdot\frac{N}{m^2}\times\frac{kg\cdot K}{286.9\cdot N\cdot m}\times\frac{1}{(20+273)\cdot K} \qquad \rho = 5.95\,\frac{kg}{m^3}$$

Hence

$$\frac{dT}{dt} = -0.01\cdot\frac{kg}{s}\times 500\times 10^3\cdot\frac{N}{m^2}\times\frac{1}{100\cdot L}\times\frac{L}{10^{-3}\cdot m^3}\times\frac{kg\cdot K}{717.4\cdot N\cdot m}\times\left(\frac{m^3}{5.95\cdot kg}\right)^2 = -1.97\cdot\frac{K}{s} = -1.97\cdot\frac{C}{s}$$

Problem 4.205

[Difficulty: 3]

4.205 A centrifugal water pump with a 0.1-m-diameter inlet and a 0.1-m-diameter discharge pipe has a flow rate of 0.02 m^3/s. The inlet pressure is 0.2 m Hg vacuum and the exit pressure is 240 kPa. The inlet and outlet sections are located at the same elevation. The measured power input is 6.75 kW. Determine the pump efficiency.

Given: Data on centrifugal water pump

Find: Pump efficiency

Solution:

Basic equations:

$$\dot{Q} - \dot{W}_s - \dot{W}_{shear} - \dot{W}_{other} = \frac{\partial}{\partial t}\int_{CV} e\,\rho\, d\forall + \int_{CS}\left(u + pv + \frac{V^2}{2} + gz\right)\rho \vec{V}\cdot d\vec{A} \qquad (4.56)$$

$$\Delta p = SG_{Hg}\cdot\rho\cdot g\cdot\Delta h \qquad \eta = \frac{W_s}{P_{in}}$$

Available data:

$D_1 = 0.1\cdot m$ $\quad D_2 = 0.1\cdot m$ $\quad Q = 0.02\cdot\frac{m^3}{s}$ $\quad P_{in} = 6.75\cdot kW$

$\rho = 1000\frac{kg}{m^3}$ $\quad SG_{Hg} = 13.6$ $\quad h_1 = -0.2\cdot m$ $\quad p_2 = 240\cdot kPa$

Assumptions: 1) Adiabatic 2) Only shaft work 3) Steady 4) Neglect Δu 5) Δz = 0 6) Incompressible 7) Uniform flow

Then

$$-W_s = \left(p_1\cdot v_1 + \frac{V_1^{\,2}}{2}\right)\cdot\left(-m_{rate}\right) + \left(p_2\cdot v_2 + \frac{V_2^{\,2}}{2}\right)\cdot\left(m_{rate}\right)$$

Since $m_{rate} = \rho\cdot Q$ and $V_1 = V_2$ (from continuity)

$$-W_s = \rho\cdot Q\cdot\left(p_2\cdot v_2 - p_1\cdot v_1\right) = Q\cdot\left(p_2 - p_1\right)$$

$p_1 = \rho_{Hg}\cdot g\cdot h$ or $p_1 = SG_{Hg}\cdot\rho\cdot g\cdot h_1$ $\quad p_1 = -26.7\cdot kPa$

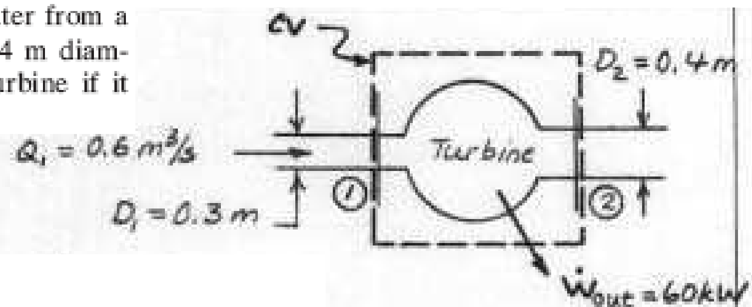
$W_s = Q\cdot\left(p_1 - p_2\right)$ $\quad W_s = -5.33\cdot kW$ $\quad$ The negative sign indicates work in

$$\eta = \frac{\left|W_s\right|}{P_{in}} \qquad \eta = 79.0\cdot\%$$

Problem 4.206

[Difficulty: 2]

4.206 A turbine is supplied with 0.6 m^3/s of water from a 0.3-m-diameter pipe; the discharge pipe has a 0.4 m diameter. Determine the pressure drop across the turbine if it delivers 60 kW.



Solution: Apply continuity, energy equations, using CV shown.

Basic equations:

$$0 = \underbrace{\frac{\partial}{\partial t}\int_{CV} \rho\, d\forall}_{=0(1)} + \int_{CS} \rho \vec{V}\cdot d\vec{A}$$

$$\underbrace{\dot{Q}}_{=0(4)} - \dot{W}_s - \underbrace{\dot{W}_{shear}}_{=0(5)} - \underbrace{\dot{W}_{other}}_{=0(5)} = \underbrace{\frac{\partial}{\partial t}\int_{CV} e\rho\, d\forall}_{=0(1)} + \int_{CS}\Big(\underbrace{u}_{\approx 0(6)} + \frac{V^2}{2} + \underbrace{gz}_{\approx 0(7)} + pv\Big)\rho\vec{V}\cdot d\vec{A}$$

Assumptions:
(1) Steady flow
(2) Uniform flow at each section
(3) Incompressible flow
(4) $\dot{Q} = 0$
(5) $\dot{W}_{shear} = 0$ by choice of CV; $\dot{W}_{other} = 0$
(6) Neglect Δu
(7) Neglect Δz

Then

$$0 = \{-|\rho V_1 A_1|\} + \{|\rho V_2 A_2|\} \quad \text{or} \quad V_2 = V_1 \frac{A_1}{A_2} = V_1\left(\frac{D_1}{D_2}\right)^2$$

and

$$-\dot{W}_s = \left(\frac{V_1^2}{2} + p_1 v\right)\{-|\rho V_1 A_1|\} + \left(\frac{V_2^2}{2} + p_2 v\right)\{|\rho V_2 A_2|\}$$

$$-\dot{W}_s = -\left[\frac{V_1^2 - V_2^2}{2} + (p_1 - p_2)v\right]\rho Q = -\left\{\frac{V_1^2}{2}\left[1 - \left(\frac{D_1}{D_2}\right)^4\right] + (p_1 - p_2)v\right\}\rho Q$$

or

$$p_1 - p_2 = \frac{1}{v}\left\{\frac{\dot{W}_s}{\rho Q} - \frac{V_1^2}{2}\left[1 - \left(\frac{D_1}{D_2}\right)^4\right]\right\} = \frac{\dot{W}_s}{Q} - \frac{\rho V_1^2}{2}\left[1 - \left(\frac{D_1}{D_2}\right)^4\right]$$

But $V_1 = \dfrac{Q}{A_1} = 0.6\dfrac{m^3}{s} \times \dfrac{4}{\pi} \dfrac{1}{(0.3)^2 m^2} = 8.49\ m/s$, and $\dot{W}_s = \dot{W}_{out} = 60\ kW$, so

$$p_1 - p_2 = (60\ kW)\,10^3\frac{N\cdot m}{kW\cdot s} \times \frac{s}{0.6\ m^3} - \frac{1}{2} \times 999\frac{kg}{m^3} \times (8.49)^2\frac{m^2}{s^2}\left[1 - \left(\frac{0.3}{0.4}\right)^4\right]\frac{N\cdot s^2}{kg\cdot m}$$

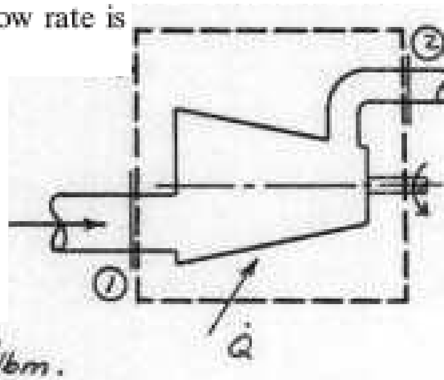
$$p_1 - p_2 = 75.4\ kPa$$

$p_1 - p_2$

Problem 4.207

[Difficulty: 2]

4.207 Air enters a compressor at 14 psia, 80°F with negligible speed and is discharged at 70 psia, 500°F with a speed of 500 ft/s. If the power input is 3200 hp and the flow rate is 20 lbm/s, determine the rate of heat transfer.



Find: Heat transfer, in Btu/lbm.

Solution: Apply energy equation to CV shown.

Basic equations: $p = \rho RT$, $\Delta h = c_p \Delta T$

$$\dot{Q} - \dot{W}_s - \underbrace{\dot{W}_{shear}}_{=0(2)} - \underbrace{\dot{W}_{other}}_{=0(2)} = \underbrace{\frac{\partial}{\partial t}\int_{CV} e\rho\, d\forall}_{=0(3)} + \int_{CS} \left(u + pv + \frac{V^2}{2} + \underbrace{gz}_{=0(5)}\right) \rho \vec{V} \cdot d\vec{A}$$

Assumptions: (1) Ideal gas, constant specific heat
(2) $\dot{W}_{shear} = 0$ by choice of CV; $\dot{W}_{other} = 0$
(3) Steady flow
(4) Uniform flow at each section
(5) Neglect Δz
(6) $V_1 \approx 0$

By definition $h \equiv u + pv$, so

$$\dot{Q} - \dot{W}_s = \left(h_1 + \underbrace{\frac{V_1^2}{2}}_{\approx 0(6)}\right)\{-|\dot{m}|\} + \left(h_2 + \frac{V_2^2}{2}\right)\{|\dot{m}|\} = \dot{m}\left[\frac{V_2^2}{2} + c_p (T_2 - T_1)\right]$$

or

$$\frac{\delta Q}{dm} = \frac{\dot{Q}}{\dot{m}} = \frac{\dot{W}_s}{\dot{m}} + \frac{V_2^2}{2} + c_p (T_2 - T_1)$$

Noting $\dot{W}_s = -3200$ hp, so

$$\frac{\delta Q}{dm} = -3200\,\text{hp} \times \frac{2545\,\text{Btu}}{\text{hp}\cdot\text{hr}} \times \frac{\text{s}}{20\,\text{lbm}} \times \frac{\text{hr}}{3600\,\text{s}} + 0.240\,\frac{\text{Btu}}{\text{lbm}\cdot{}^\circ\text{F}} \times (500-80)\,{}^\circ\text{F}$$

$$+ \frac{(500)^2}{2}\frac{\text{ft}^2}{\text{s}^2} \times \frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}} \times \frac{\text{slug}}{32.2\,\text{lbm}} \times \frac{\text{Btu}}{778\,\text{ft}\cdot\text{lbf}}$$

$$\frac{\delta Q}{dm} = -7.32\,\text{Btu/lbm}$$ $\frac{\delta Q}{dm}$

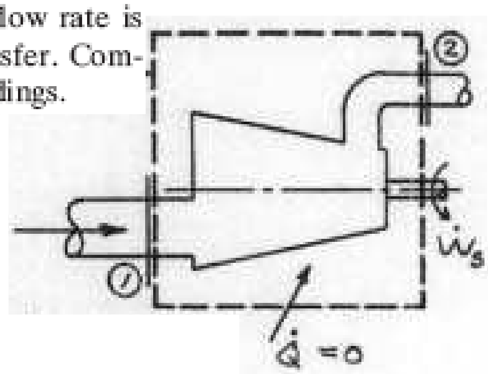
Therefore heat transfer is out of CV, since $\delta Q/dm < 0$. The rate of heat transfer is

$$\dot{Q} = -7.32\,\frac{\text{Btu}}{\text{lbm}} \times 20\,\frac{\text{lbm}}{\text{s}} = -146\,\text{Btu/s}$$ $\dot{Q}$

Problem 4.208

[Difficulty: 2]

4.208 Air is drawn from the atmosphere into a turbomachine. At the exit, conditions are 500 kPa (gage) and 130°C. The exit speed is 100 m/s and the mass flow rate is 0.8 kg/s. Flow is steady and there is no heat transfer. Compute the shaft work interaction with the surroundings.



Solution: Apply energy equation, using CV shown.

Basic equations: $p=\rho RT$, $\Delta h = c_p \Delta T$

$$\underbrace{\dot{Q}}_{=0(7)} - \dot{W}_s - \underbrace{\dot{W}_{shear}}_{=0(2)} - \underbrace{\dot{W}_{other}}_{=0(2)} = \underbrace{\frac{\partial}{\partial t}\int_{CV} e\rho\, d\forall}_{=0(3)} + \int_{CS}\left(u+pv+\frac{V^2}{2}+\underbrace{gz}_{\approx 0(5)}\right)\rho \vec{V}\cdot d\vec{A}$$

Assumptions: (1) Ideal gas, constant specific heat
(2) $\dot{W}_{shear}=0$ by choice of CV; $\dot{W}_{other}=0$
(3) Steady flow
(4) Uniform flow at each section
(5) Neglect Δz
(6) $V_1 \approx 0$
(7) $\dot{Q}=0$

By definition, $h \equiv u + pv$, so

$$-\dot{W}_s = \left(h_1+\underbrace{\frac{V_1^2}{2}}_{\approx 0(6)}\right)\{-|\dot{m}|\} + \left(h_2+\frac{V_2^2}{2}\right)\{|\dot{m}|\} = \dot{m}\left(h_2-h_1+\frac{V_2^2}{2}\right)$$

or

$$-\dot{W}_s = \dot{m}\left(h_2-h_1+\frac{V_2^2}{2}\right) = \dot{m}\left[c_p(T_2-T_1)+\frac{V_2^2}{2}\right]$$

$$= 0.8\,\frac{kg}{s}\left[1.00\,\frac{kJ}{kg\cdot K}(403-288)\,K + \frac{(100)^2}{2}\frac{m^2}{s^2}\times\frac{N\cdot s^2}{kg\cdot m}\times\frac{kJ}{10^3\,N\cdot m}\right]\frac{kW\cdot s}{kJ}$$

$-\dot{W}_s = 96.0$ kW or $\dot{W}_s = -96.0$ kW

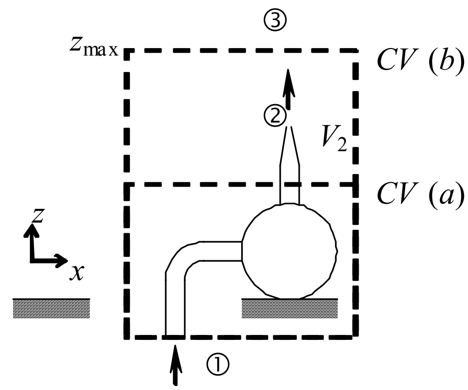
$\dot{W}_s$

{Power is <u>into</u> CV because $\dot{W}_s < 0$.}

Problem 4.209

[Difficulty: 3]

4.209 All major harbors are equipped with fire boats for extinguishing ship fires. A 3-in.-diameter hose is attached to the discharge of a 15-hp pump on such a boat. The nozzle attached to the end of the hose has a diameter of 1 in. If the nozzle discharge is held 10 ft above the surface of the water, determine the volume flow rate through the nozzle, the maximum height to which the water will rise, and the force on the boat if the water jet is directed horizontally over the stern.



Given: Data on fire boat hose system

Find: Volume flow rate of nozzle; Maximum water height; Force on boat

Solution:

Basic equation: First Law of Thermodynamics for a CV

$$\dot{Q} - \dot{W}_s - \dot{W}_{shear} - \dot{W}_{other} = \frac{\partial}{\partial t}\int_{CV} e\,\rho\,dV + \int_{CS}\left(u + pv + \frac{V^2}{2} + gz\right)\rho\vec{V}\cdot d\vec{A}$$

Assumptions: 1) Neglect losses 2) No work 3) Neglect KE at 1 4) Uniform properties at exit 5) Incompressible 6) p_{atm} at 1 and 2

Hence for CV (a) $\quad -W_s = \left(\frac{V_2^{\,2}}{2} + g\cdot z_2\right)\cdot m_{exit} \qquad m_{exit} = \rho\cdot V_2\cdot A_2 \qquad$ where m_{exit} is mass flow rate (Note: Software cannot render a dot!)

Hence, for V_2 (to get the flow rate) we need to solve $\quad \left(\frac{1}{2}\cdot V_2^{\,2} + g\cdot z_2\right)\cdot\rho\cdot V_2\cdot A_2 = -W_s \quad$ which is a cubic for V_2!

To solve this we could ignore the gravity term, solve for velocity, and then check that the gravity term is in fact minor. Alternatively we could manually iterate, or use a calculator or Excel, to solve. The answer is $\quad V_2 = 114\cdot\frac{ft}{s}$

Hence the flow rate is $\quad Q = V_2\cdot A_2 = V_2\cdot\frac{\pi\cdot D_2^{\,2}}{4} \qquad Q = 114\cdot\frac{ft}{s}\times\frac{\pi}{4}\times\left(\frac{1}{12}\cdot ft\right)^2 \qquad Q = 0.622\cdot\frac{ft^3}{s} \qquad Q = 279\cdot gpm$

To find z_{max}, use the first law again to (to CV (b)) to get $\quad -W_s = g\cdot z_{max}\cdot m_{exit}$

$$z_{max} = -\frac{W_s}{g\cdot m_{exit}} = -\frac{W_s}{g\cdot\rho\cdot Q} \qquad z_{max} = 15\cdot hp\times\frac{550\cdot\frac{ft\cdot lbf}{s}}{1\cdot hp}\times\frac{s^2}{32.2\cdot ft}\times\frac{ft^3}{1.94\cdot slug}\times\frac{s}{0.622\cdot ft^3}\times\frac{slug\cdot ft}{s^2\cdot lbf} \qquad z_{max} = 212\cdot ft$$

For the force in the x direction when jet is horizontal we need x momentum

$$F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\,dV + \int_{CS} u\,\rho\vec{V}\cdot d\vec{A}$$

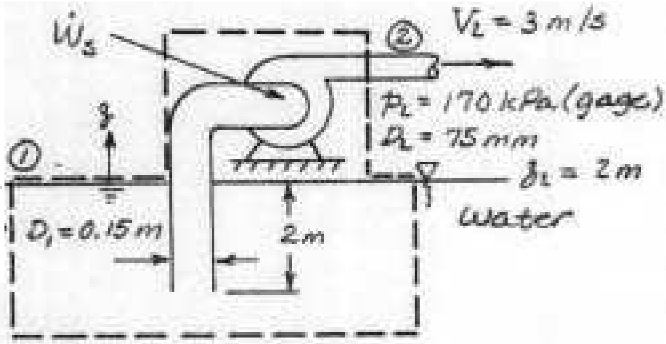
Then $\quad R_x = u_1\cdot\left(-\rho\cdot V_1\cdot A_1\right) + u_2\cdot\left(\rho\cdot V_2\cdot A_2\right) = 0 + V_2\cdot\rho\cdot Q \qquad R_x = \rho\cdot Q\cdot V_2$

$$R_x = 1.94\cdot\frac{slug}{ft^3}\times 0.622\cdot\frac{ft^3}{s}\times 114\cdot\frac{ft}{s}\times\frac{lbf\cdot s^2}{slug\cdot ft} \qquad R_x = 138\cdot lbf$$

Problem 4.210

[Difficulty: 3]

4.210 A pump draws water from a reservoir through a 150-mm-diameter suction pipe and delivers it to a 75-mm-diameter discharge pipe. The end of the suction pipe is 2 m below the free surface of the reservoir. The pressure gage on the discharge pipe (2 m above the reservoir surface) reads 170 kPa. The average speed in the discharge pipe is 3 m/s. If the pump efficiency is 75 percent, determine the power required to drive it.



Solution: Apply first law to CV shown, noting that flow enters with negligible velocity at section ①.

Basic equation:

$$\dot{Q} - \dot{W}_{shaft} - \underbrace{\dot{W}_{shear}}_{=0(1)} - \underbrace{\dot{W}_{other}}_{=0(1)} = \underbrace{\frac{\partial}{\partial t}\int_{CV} e\rho\, dV}_{=0(2)} + \int_{CS}\left(e + \frac{p}{\rho}\right)\rho\vec{V}\cdot d\vec{A}$$

$$e = u + \frac{V^2}{2} + gz$$

Assumptions: (1) $\dot{W}_{shear} = \dot{W}_{other} = 0$
(2) Steady flow
(3) $V_1 \simeq 0$
(4) $z_1 = 0$
(5) $p_1 = 0$ (gage)
(6) Uniform flow at each section
(7) Incompressible flow; $V_1 A_1 = V_2 A_2$

Then

$$\dot{Q} - \dot{W}_s = \Big(u_1 + \underbrace{\frac{V_1^2}{2}}_{\simeq 0(3)} + \underbrace{gz_1}_{=0(4)} + \underbrace{\frac{p_1}{\rho}}_{=0(5)}\Big)\{-\dot{m}\} + \left(u_2 + \frac{V_2^2}{2} + gz_2 + \frac{p_2}{\rho}\right)\{\dot{m}\}$$

or

$$-\dot{W}_s = \dot{m}\left[\frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2 + \left(u_2 - u_1 - \frac{\delta Q}{dm}\right)\right]$$

Obtain the ideal or minimum power input by neglecting thermal effects. Thus

$$-\dot{W}_{s,ideal} = \dot{m}\left[\frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2\right]$$

For the system,

$$\dot{m} = \rho V_2 A_2 = 999\frac{kg}{m^3}\times 3\frac{m}{s}\times\frac{\pi}{4}(0.075)^2 m^2 = 13.2\ kg/s$$

and

$$-\dot{W}_{s,ideal} = 13.2\frac{kg}{s}\left[1.70\times10^5\frac{N}{m^2}\times\frac{m^3}{999\ kg} + \frac{1}{2}(3)^2\frac{m^2}{s^2}\times\frac{N\cdot s^2}{kg\cdot m} + 9.81\frac{m}{s^2}\times 2m\times\frac{N\cdot s^2}{kg\cdot m}\right]$$

$$\dot{W}_{s,ideal} = -2560\frac{N\cdot m}{s}\times\frac{kW\cdot s}{10^3\ N\cdot m} = -2.56\ kW$$

Finally

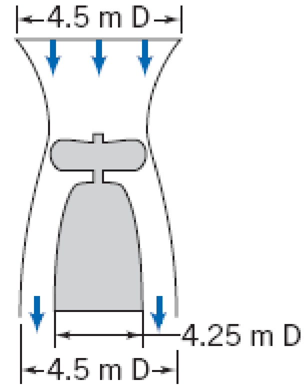
$$\dot{W}_{s,actual} = \frac{\dot{W}_{s,ideal}}{\eta} = \frac{-2.56\ kW}{0.75} = -3.41\ kW$$

$\dot{W}_{s,actua}$

Problem 4.211

[Difficulty: 4]

4.211 The total mass of the helicopter-type craft shown is 1000 kg. The pressure of the air is atmospheric at the outlet. Assume the flow is steady and one-dimensional. Treat the air as incompressible at standard conditions and calculate, for a hovering position, the speed of the air leaving the craft and the minimum power that must be delivered to the air by the propeller.



Given: Data on helicopter-type craft

Find: Air speed; Minimum power needed

Solution:

Basic equation: Contunity, z momentum; First Law of Thermodynamics for a CV; Bernoulli; Ideal gas

$$\frac{\partial}{\partial t}\int_{CV}\rho\, d\forall + \int_{CS}\rho\vec{V}\cdot d\vec{A} = 0 \qquad F_z = F_{S_z} + F_{B_z} = \frac{\partial}{\partial t}\int_{CV} w\,\rho\, d\forall + \int_{CS} w\,\rho\vec{V}\cdot d\vec{A}$$

$$\dot{Q} - \dot{W}_s - \dot{W}_{shear} - \dot{W}_{other} = \frac{\partial}{\partial t}\int_{CV} e\,\rho\, d\forall + \int_{CS}\left(u + pv + \frac{V^2}{2} + gz\right)\rho\vec{V}\cdot d\vec{A} \qquad \frac{p}{\rho} + \frac{V^2}{2} + g\cdot z = \text{const} \qquad p = \rho\cdot R_{air}\cdot T \qquad \Delta h = c_p\cdot\Delta T$$

Assumptions: 1) Atmospheric at exit 2) Standard air 3) Uniform properties at exit 4) Incompressible

Given data $\quad M = 1000\cdot kg \quad p = 101\cdot kPa \quad T = 15\,°C \quad D_o = 4.5\cdot m \quad D_i = 4.25\cdot m \quad R_{air} = \frac{286.9\cdot N\cdot m}{kg\cdot K}$

Then $\quad A_1 = \frac{\pi}{4}\cdot D_o^{\,2} \quad A_1 = 15.9\,m^2 \quad A_2 = \frac{\pi}{4}\cdot\left(D_o^{\,2} - D_i^{\,2}\right) \quad A_2 = 1.72\,m^2 \quad \rho = \frac{p}{R_{air}\cdot T} \quad \rho = 1.222\,\frac{kg}{m^3}$

From continuity $\quad 0 = \left(-\rho\cdot V_1\cdot A_1\right) + \left(\rho\cdot V_2\cdot A_2\right) \quad$ or $\quad V_1 = \frac{A_2}{A_1}\cdot V_1$

From momentum $\quad -p_{1g}\cdot A_1 - M\cdot g = w_1\cdot\left(-\rho\cdot V_1\cdot A_1\right) + w_2\cdot\left(\rho\cdot V_2\cdot A_2\right) \quad w_1 = -V_1 \quad w_2 = -V_1 \quad$ and $\quad \rho\cdot V_1\cdot A_1 = \rho\cdot V_2\cdot A_2$

Then $\quad -p_{1g}\cdot A_1 - M\cdot g = V_1\cdot\rho\cdot V_1\cdot A_1 - V_2\cdot\rho\cdot V_2\cdot A_2 = -\rho\cdot V_2\cdot A_2\cdot\left(V_2 - V_1\right)$

For this flow Bernoulli also applies between the atmosphere and location 1 $\quad p_{atm} = p_1 + \frac{1}{2}\cdot\rho\cdot V_1^{\,2} \quad p_{1g} = -\frac{1}{2}\cdot\rho\cdot V_1^{\,2}$

Using continuity $\quad p_{1g}\cdot A_1 = -\frac{1}{2}\cdot\rho\cdot V_1^{\,2}\cdot A_1 = -\frac{1}{2}\cdot\rho\cdot V_2\cdot A_2\cdot V_1 = -\frac{1}{2}\cdot\rho\cdot V_2^{\,2}\cdot A_2\cdot\frac{A_2}{A_1}$

Substituting into the momentum equation and using continuity

$$\frac{1}{2}\cdot\rho\cdot V_2^{\,2}\cdot A_2\cdot\frac{A_2}{A_1} - M\cdot g = -\rho\cdot V_2^{\,2}\cdot A_2\cdot\left(1 - \frac{V_1}{V_2}\right) = -\rho\cdot V_2^{\,2}\cdot A_2\cdot\left(1 - \frac{A_2}{A_1}\right) \quad \text{or} \qquad M\cdot g = \rho\cdot V_2^{\,2}\cdot A_2\cdot\left(1 - \frac{1}{2}\cdot\frac{A_2}{A_1}\right)$$

Hence $\quad V_2 = \sqrt{\dfrac{M\cdot g}{\rho\cdot A_2\cdot\left(1 - \dfrac{1}{2}\cdot\dfrac{A_2}{A_1}\right)}}$ $\quad$ Substituting values $\quad V_2 = 70.3\,\dfrac{m}{s}$

For power we use the First Law

$$\dot{Q} - \dot{W}_s - \dot{W}_{\text{shear}} - \dot{W}_{\text{other}} = \frac{\partial}{\partial t}\int_{CV} e\,\rho\,d\mathcal{V} + \int_{CS}\left(u + pv + \frac{V^2}{2} + gz\right)\rho\vec{V}\cdot d\vec{A}$$

We have additional assumptions 5) pv = const 6) Neglect Δz

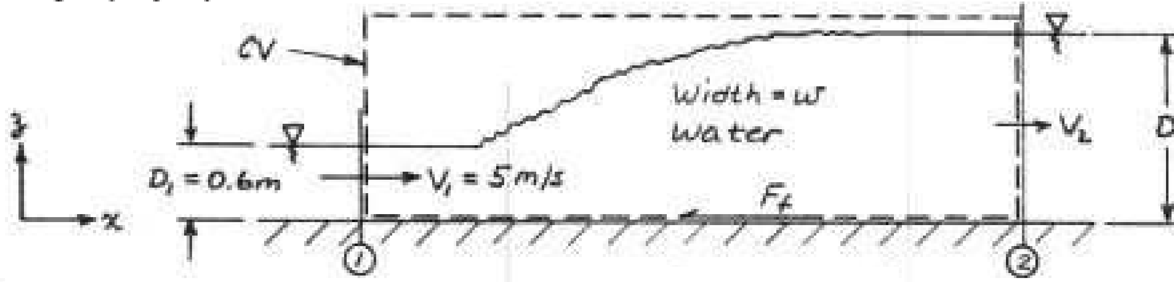
Then $\quad -W_s = m_{rate}\cdot\left(\dfrac{V_2^{\,2} - V_1^{\,2}}{2}\right) + m_{rate}\cdot\left(u_2 - u_1 - \dfrac{dQ}{dm}\right)$

The last term is non-mechanical energy; the minimum possible work is when this is zero. Hence

$$-W_s = -W_{min} = m_{rate}\cdot\left(\frac{V_2^{\,2} - V_1^{\,2}}{2}\right) = m_{rate}\cdot\frac{V_2^{\,2}}{2}\cdot\left[1 - \left(\frac{V_1}{V_2}\right)^2\right] = \frac{\rho\cdot A_2\cdot V_2^{\,2}}{2}\cdot\left[1 - \left(\frac{A_2}{A_1}\right)^2\right]$$

$W_{min} = \dfrac{\rho\cdot A_2\cdot V_2^{\,3}}{2}\cdot\left[1 - \left(\dfrac{A_2}{A_1}\right)^2\right]$ $\quad$ Using given data $\quad W_{min} = 360\cdot kW$

4.212 Liquid flowing at high speed in a wide, horizontal open channel under some conditions can undergo a hydraulic jump, as shown. For a suitably chosen control volume, the flows entering and leaving the jump may be considered uniform with hydrostatic pressure distributions (see Example 4.7). Consider a channel of width w, with water flow at $D_1 = 0.6$ m and $V_1 = 5$ m/s. Show that in general, $D_2 = D_1\left[\sqrt{1 + 8V_1^2/gD_1} - 1\right]/2$.



Evaluate the change in mechanical energy through the hydraulic jump. If heat transfer to the surroundings is negligible, determine the change in water temperature through the jump.

Solution: Apply continuity, x component of momentum, and energy equations using CV shown.

Basic equations: $0 = \overset{=0(1)}{\frac{\partial}{\partial t}\int_{CV}\rho\, d\forall} + \int_{CS}\rho\vec V\cdot d\vec A$

$$F_{Sx} + \overset{=0(8)}{F_{Bx}} = \overset{=0(1)}{\frac{\partial}{\partial t}\int_{CV} V_x\rho\, d\forall} + \int_{CS} V_x\rho\vec V\cdot d\vec A$$

$$\overset{=0(6)}{\dot Q} - \overset{=0(7)}{\dot W_s} - \overset{=0(7)}{\dot W_{shear}} - \overset{=0(7)}{\dot W_{other}} = \overset{=0(1)}{\frac{\partial}{\partial t}\int_{CV} e\rho\, d\forall} + \int_{CS}(e + pv)\rho\vec V\cdot d\vec A;$$

Assumptions: (1) Steady flow $\qquad e = u + \frac{V^2}{2} + gz$

(2) Incompressible flow

(3) Uniform flow at each section

(4) Hydrostatic pressure distribution at sections ①, ②, so $p = \rho g(D - z)$

(5) Neglect friction force, F_f, on CV

(6) $\dot Q = 0$

(7) $\dot W_s = \dot W_{shear} = \dot W_{other} = 0$

(8) $F_{Bx} = 0$, since channel is horizontal

From continuity,

$$0 = \{-|\rho V_1 A_1|\} + \{|\rho V_2 A_2|\} = -\rho V_1 w D_1 + \rho V_2 w D_2\ ; \quad V_1 D_1 = V_2 D_2$$

From momentum,

$$F_{Sx} = \underbrace{\rho g\frac{D_1}{2}wD_1 - \rho g\frac{D_2}{2}wD_2}_{\text{hydrostatic forces}} = V_{x_1}\{-|\rho V_1 w D_1|\} + V_{x_2}\{|\rho V_2 w D_2|\}$$

$V_{x_1} = V_1 \qquad V_{x_2} = V_2$

or

$$\frac{g}{2}(D_1^2 - D_2^2) = V_1 D_1(V_2 - V_1) = V_1^2 D_1\left(\frac{V_2}{V_1} - 1\right) = V_1^2 D_1\left(\frac{D_1}{D_2} - 1\right)$$

or

$$\frac{g}{2}(D_1 + D_2)\cancel{(D_1 - D_2)} = V_1^2\frac{D_1}{D_2}\cancel{(D_1 - D_2)}$$

Thus $\frac{gD_1}{2}\left(1+\frac{D_2}{D_1}\right) = V_1^2\frac{D_1}{D_2}$ or $\frac{D_2}{D_1}\left(1+\frac{D_2}{D_1}\right) = \frac{2V_1^2}{gD_1}$ or $\left(\frac{D_2}{D_1}\right)^2 + \frac{D_2}{D_1} - \frac{2V_1^2}{gD_1} = 0$

Using the quadratic equation,

$$\frac{D_2}{D_1} = \frac{1}{2}\left[-1 \pm \sqrt{1+\frac{8V_1^2}{gD_1}}\right] \quad \text{or} \quad D_2 = \frac{D_1}{2}\left[\sqrt{1+\frac{8V_1^2}{gD_1}} - 1\right]$$

D_2

Solving for D_2

$$D_2 = \frac{1}{2}\times 0.6\,m\left[\sqrt{1 + 8\times(5)^2\frac{m^2}{s^2}\times\frac{s^2}{9.81\,m}\times\frac{1}{0.6\,m}} - 1\right] = 1.47\,m$$

$$V_2 = \frac{D_1}{D_2}V_1 = \frac{0.6}{1.47}\times 5\frac{m}{s} = 2.04\,m/s$$

From the energy equation, with $e_{mech} = \frac{V^2}{2} + gz + \frac{p}{\rho}$, and $dA = w\,dz$, the mechanical energy fluxes are

$$mef_1 = \int_0^{D_1}\left[\frac{V_1^2}{2} + gz + \frac{1}{\rho}\rho g(D-z)\right]\rho V_1 w\,dz = \left(\frac{V_1^2}{2} + gD_1\right)\rho V_1 w D_1$$

$$mef_2 = \int_0^{D_2}\left[\frac{V_2^2}{2} + gz + \frac{1}{\rho}\rho g(D-z)\right]\rho V_2 w\,dz = \left(\frac{V_2^2}{2} + gD_2\right)\rho V_2 w D_2$$

and

$$\Delta mef = mef_2 - mef_1 = \left[\frac{V_2^2 - V_1^2}{2} + g(D_2 - D_1)\right]\rho V_1 w D_1, \text{ since } V_1 D_1 = V_2 D_2$$

Thus $\frac{\Delta mef}{\dot{m}} = \frac{1}{2}\left[V_2^2 - V_1^2 + 2g(D_2 - D_1)\right]$

$$\frac{\Delta mef}{\dot{m}} = \frac{1}{2}\left[(2.04)^2\frac{m^2}{s^2} - (5)^2\frac{m^2}{s^2} + 2\times 9.81\frac{m}{s^2}(1.47-0.6)m\right]\frac{N\cdot s^2}{kg\cdot m} = -1.88\,N\cdot m/kg$$

$\frac{\Delta mef}{\dot{m}}$

From the energy equation,

$$0 = \left[u_1 + \frac{V_1^2}{2} + gz + \frac{1}{\rho}\rho g(D-z)\right]\{-|\rho V_1 w D_1|\}$$
$$+\left[u_2 + \frac{V_2^2}{2} + gz + \frac{1}{\rho}\rho g(D-z)\right]\{|\rho V_2 w D_2|\}$$

or

$$0 = (u_2 - u_1)\dot{m} + \Delta mef$$

Thus

$$u_2 - u_1 = c_v(T_2 - T_1) = -\frac{\Delta mef}{\dot{m}}$$

$$\Delta T = T_2 - T_1 = -\frac{\Delta mef}{\dot{m}c_v} = -\left(-1.88\frac{N\cdot m}{kg}\right)\frac{kg\cdot K}{1\,kcal}\times\frac{kcal}{4187\,J} = 4.49\times 10^{-4}\,K$$

ΔT

{This small temperature change would be almost impossible to measure.}

Problem 5.1

[Difficulty: 1]

5.1 Which of the following sets of equations represent possible two-dimensional incompressible flow cases?

(a) $u = 2x^2 + y^2 - x^2y;\ v = x^3 + x(y^2 - 4y)$
(b) $u = 2xy - x^2y;\ v = 2xy - y^2 + x^2$
(c) $u = x^2t + 2y;\ v = xt^2 - yt$
(d) $u = (2x + 4y)xt;\ v = -3(x + y)yt$

Given: The list of velocity fields provided above

Find: Which of these fields possibly represent two-dimensional, incompressible flow

Solution: We will check these flow fields against the continuity equation

Governing Equations:

$$\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)+\frac{\partial \rho}{\partial t}=0 \quad \text{(Continuity equation)}$$

Assumptions:
(1) Incompressible flow (ρ is constant)
(2) Two dimensional flow (velocity is not a function of z)

Based on the two assumptions listed above, the continuity equation reduces to: $\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$

This is the criterion against which we will check all of the flow fields.

a) $u(x,y,t) = 2\cdot x^2 + y^2 - x^2\cdot y$ $\quad v(x,y,t) = x^3 + x\cdot\left(y^2 - 4\cdot y\right)$ $\quad \frac{\partial}{\partial x}u(x,y,t) = 4\cdot x - 2\cdot x\cdot y$ $\quad \frac{\partial}{\partial y}v(x,y,t) = x\cdot(2\cdot y - 4)$

Hence $\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v \neq 0$ INCOMPRESSIBLE

b) $u(x,y,t) = 2\cdot x\cdot y - x^2\cdot y$ $\quad v(x,y,t) = 2\cdot x\cdot y - y^2 + x^2$ $\quad \frac{\partial}{\partial x}u(x,y,t) = 2\cdot y - 2\cdot x\cdot y$ $\quad \frac{\partial}{\partial y}v(x,y,t) = 2\cdot x - 2\cdot y$

Hence $\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v \neq 0$ NOT INCOMPRESSIBLE

c) $u(x,y,t) = x^2\cdot t + 2\cdot y$ $\quad v(x,y,t) = x\cdot t^2 - y\cdot t$ $\quad \frac{\partial}{\partial x}u(x,y,t) = 2\cdot t\cdot x$ $\quad \frac{\partial}{\partial y}v(x,y,t) = -t$

Hence $\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v \neq 0$ NOT INCOMPRESSIBLE

d) $u(x,y,t) = (2\cdot x + 4\cdot y)\cdot x\cdot t$ $\quad v(x,y,t) = -3\cdot(x + y)\cdot y\cdot t$

$\frac{\partial}{\partial x}u(x,y,t) = t\cdot(2\cdot x + 4\cdot y) + 2\cdot t\cdot x$ $\quad \frac{\partial}{\partial y}v(x,y,t) = -t\cdot(3\cdot x + 3\cdot y) - 3\cdot t\cdot y$

Hence $\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v \neq 0$ NOT INCOMPRESSIBLE

Problem 5.2

[Difficulty: 2]

5.2 Which of the following sets of equations represent possible three-dimensional incompressible flow cases?

(a) $u = 2y^2 + 2xz$; $v = -2yz + 6x^2yz$; $w = 3x^2z^2 + x^3y^4$

(b) $u = xyzt$; $v = -xyzt^2$; $w = z^2(xt^2 - yt)$

(c) $u = x^2 + 2y + z^2$; $v = x - 2y + z$; $w = -2xz + y^2 + 2z$

Given: Velocity fields

Find: Which are 3D incompressible

Solution: We will check these flow fields against the continuity equation

Governing Equation:

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) + \frac{\partial \rho}{\partial t} = 0 \quad \text{(Continuity equation)}$$

Assumption: Incompressible flow (ρ is constant)

Based on the assumption, the continuity equation reduces to: $\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v + \frac{\partial}{\partial z}w = 0$

This is the criterion against which we will check all of the flow fields.

a) $u(x,y,z,t) = 2\cdot y^2 + 2\cdot x\cdot z$ $\quad v(x,y,z,t) = -2\cdot y\cdot z + 6\cdot x^2\cdot y\cdot z$ $\quad w(x,y,z,t) = 3\cdot x^2\cdot z^2 + x^3\cdot y^4$

$\frac{\partial}{\partial x}u(x,y,z,t) = 2\cdot z$ $\quad \frac{\partial}{\partial y}v(x,y,z,t) = 6\cdot x^2\cdot z - 2\cdot z$ $\quad \frac{\partial}{\partial z}w(x,y,z,t) = 6\cdot x^2\cdot z$

Hence $\quad \frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v + \frac{\partial}{\partial z}w \neq 0$ $\quad$ NOT INCOMPRESSIBLE

b) $u(x,y,z,t) = x\cdot y\cdot z\cdot t$ $\quad v(x,y,z,t) = -x\cdot y\cdot z\cdot t^2$ $\quad w(x,y,z,t) = z^2\cdot\left(x\cdot t^2 - y\cdot t\right)$

$\frac{\partial}{\partial x}u(x,y,z,t) = t\cdot y\cdot z$ $\quad \frac{\partial}{\partial y}v(x,y,z,t) = -t^2\cdot x\cdot z$ $\quad \frac{\partial}{\partial z}w(x,y,z,t) = 2\cdot z\cdot\left(t^2\cdot x - t\cdot y\right)$

Hence $\quad \frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v + \frac{\partial}{\partial z}w \neq 0$ $\quad$ NOT INCOMPRESSIBLE

c) $u(x,y,z,t) = x^2 + 2\cdot y + z^2$ $\quad v(x,y,z,t) = x - 2\cdot y + z$ $\quad w(x,y,z,t) = -2\cdot x\cdot z + y^2 + 2\cdot z$

$\frac{\partial}{\partial x}u(x,y,z,t) = 2\cdot x$ $\quad \frac{\partial}{\partial y}v(x,y,z,t) = -2$ $\quad \frac{\partial}{\partial z}w(x,y,z,t) = 2 - 2\cdot x$

Hence $\quad \frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v + \frac{\partial}{\partial z}w = 0$ $\quad$ INCOMPRESSIBLE

Problem 5.3

[Difficulty: 2]

5.3 For a flow in the xy plane, the x component of velocity is given by $u = Ax(y - B)$, where $A = 1\ \text{ft}^{-1} \cdot \text{s}^{-1}$, $B = 6$ ft, and x and y are measured in feet. Find a possible y component for steady, incompressible flow. Is it also valid for unsteady, incompressible flow? Why? How many y components are possible?

Given: x component of velocity

Find: y component for incompressible flow; Valid for unsteady?; How many y components?

Solution:

Basic equation: $$\frac{\partial}{\partial x}(\rho \cdot u) + \frac{\partial}{\partial y}(\rho \cdot v) + \frac{\partial}{\partial z}(\rho \cdot w) + \frac{\partial}{\partial t}\rho = 0$$

Assumption: Incompressible flow; flow in x-y plane

Hence $$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0 \qquad \text{or} \qquad \frac{\partial}{\partial y}v = -\frac{\partial}{\partial x}u = -\frac{\partial}{\partial x}[A \cdot x \cdot (y - B)] = -A \cdot (y - B)$$

Integrating $$v(x,y) = -\int A \cdot (y - B)\, dy = -A \cdot \left(\frac{y^2}{2} - B \cdot y\right) + f(x)$$

This basic equation is valid for steady and unsteady flow (t is not explicit)

There are an infinite number of solutions, since f(x) can be any function of x. The simplest is f(x) = 0

$$v(x,y) = -A \cdot \left(\frac{y^2}{2} - B \cdot y\right) \qquad v(x,y) = 6 \cdot y - \frac{y^2}{2}$$

Problem 5.4

[Difficulty: 1]

5.4 The three components of velocity in a velocity field are given by $u = Ax + By + Cz$, $v = Dx + Ey + Fz$, and $w = Gx + Hy + Jz$. Determine the relationship among the coefficients A through J that is necessary if this is to be a possible incompressible flow field.

Given: The velocity field provided above

Find: The conditions under which this fields could represent incompressible flow

Solution: We will check this flow field against the continuity equation

Governing Equations:

$$\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)+\frac{\partial \rho}{\partial t}=0 \quad \text{(Continuity equation)}$$

Assumptions: (1) Incompressible flow (ρ is constant)

Based on the assumption listed, the continuity equation reduces to:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

Calculating the partial derivatives of the velocity components:

$$\frac{\partial u}{\partial x}=A \qquad \frac{\partial v}{\partial y}=E \qquad \frac{\partial w}{\partial z}=J$$

Applying this information to the continuity equation we get the necessary condition for incompressible flow:

A + E + J = 0

(B, C, D, F, G, and H are arbitrary)

Problem 5.5

[Difficulty: 2]

5.5 For a flow in the xy plane, the x component of velocity is given by $u = 3x^2y - y^3$. Determine a possible y component for steady, incompressible flow. Is it also valid for unsteady, incompressible flow? Why? How many possible y components are there?

Given: x component of velocity

Find: y component for incompressible flow; Valid for unsteady? How many y components?

Solution:

Basic Equation:

$$\frac{\partial}{\partial x}(\rho \cdot u) + \frac{\partial}{\partial y}(\rho \cdot v) + \frac{\partial}{\partial z}(\rho \cdot w) + \frac{\partial}{\partial t}\rho = 0$$

Assumptions: Incompressible flow (ρ is constant)
Flow is only in the x-y plane

Hence $\quad \frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0 \quad$ or $\quad \frac{\partial}{\partial y}v = -\frac{\partial}{\partial x}u = -\frac{\partial}{\partial x}\left(3 \cdot x^2 \cdot y - y^3\right) = -6 \cdot x \cdot y$

Integrating $\quad v(x,y) = -\int 6 \cdot x \cdot y \, dy = -3 \cdot x \cdot y^2 + f(x)$

This basic equation is valid for steady and unsteady flow (t is not explicit)

There are an infinite number of solutions, since f(x) can be any function of x. The simplest is f(x) = 0 $\quad v(x,y) = -3 \cdot x \cdot y^2$

Problem 5.6

[Difficulty: 2]

5.6 The x component of velocity in a steady, incompressible flow field in the xy plane is $u = A/x$, where $A = 2\ \text{m}^2/\text{s}$, and x is measured in meters. Find the simplest y component of velocity for this flow field.

Given: The x-component of velocity in a steady, incompressible flow field

Find: The simplest y-component of velocity for this flow field

Solution: We will check this flow field against the continuity equation

Governing Equations:

$$\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)+\frac{\partial \rho}{\partial t}=0 \quad \text{(Continuity equation)}$$

Assumptions:
(1) Incompressible flow (ρ is constant)
(2) Two dimensional flow (velocity is not a function of z)

Based on the two assumptions listed above, the continuity equation reduces to: $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$

The partial of u with respect to x is: $\frac{\partial u}{\partial x}=-\frac{A}{x^2}$ Therefore from continuity, we have $\frac{\partial v}{\partial y}=-\frac{\partial u}{\partial x}=\frac{A}{x^2}$

Integrating this expression will yield the y-component of velocity: $v = \int \frac{A}{x^2}\,dy + f(x) = \frac{A\cdot y}{x^2} + f(x)$

The simplest version of this velocity component would result when f(x) = 0: $v = \frac{A\cdot y}{x^2}$

Problem 5.7

[Difficulty: 2]

5.7 The y component of velocity in a steady, incompressible flow field in the xy plane is $v = Axy(x^2 - y^2)$, where $A = 3\ m^{-3} \cdot s^{-1}$ and x and y are measured in meters. Find the simplest x component of velocity for this flow field.

Given: y component of velocity

Find: x component for incompressible flow; Simplest x components?

Solution:

Basic equation:
$$\frac{\partial}{\partial x}(\rho \cdot u) + \frac{\partial}{\partial y}(\rho \cdot v) + \frac{\partial}{\partial z}(\rho \cdot w) + \frac{\partial}{\partial t}\rho = 0$$

Assumptions: Incompressible flow (ρ is constant)
Flow is only in the x-y plane

Hence
$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0 \quad \text{or} \quad \frac{\partial}{\partial x}u = -\frac{\partial}{\partial y}v = -\frac{\partial}{\partial y}\left[A \cdot x \cdot y \cdot \left(x^2 - y^2\right)\right] = -\left[A \cdot x \cdot \left(x^2 - y^2\right) - A \cdot x \cdot y \cdot 2 \cdot y\right]$$

Integrating
$$u(x,y) = -\int A \cdot \left(x^3 - 3 \cdot x \cdot y^2\right) dx = -\frac{1}{4} \cdot A \cdot x^4 + \frac{3}{2} \cdot A \cdot x^2 \cdot y^2 + f(y)$$

This basic equation is valid for steady and unsteady flow (t is not explicit)

There are an infinite number of solutions, since f(y) can be any function of y. The simplest is f(y) = 0

$$u(x,y) = \frac{3}{2} \cdot A \cdot x^2 \cdot y^2 - \frac{1}{4} \cdot A \cdot x^4 \qquad u(x,y) = \frac{9}{2} \cdot x^2 \cdot y^2 - \frac{3}{4} \cdot x^4$$

Problem 5.8 [Difficulty: 3]

5.8 The y component of velocity in a steady incompressible flow field in the xy plane is

$$v = \frac{2xy}{(x^2+y^2)^2}$$

Show that the simplest expression for the x component of velocity is

$$u = \frac{1}{(x^2+y^2)} - \frac{2y^2}{(x^2+y^2)^2}$$

Given: y component of velocity

Find: x component for incompressible flow; Simplest x component

Solution:

Basic equation: $$\frac{\partial}{\partial x}(\rho \cdot u) + \frac{\partial}{\partial y}(\rho \cdot v) + \frac{\partial}{\partial z}(\rho \cdot w) + \frac{\partial}{\partial t}\rho = 0$$

Assumption: Incompressible flow; flow in x-y plane

Hence $$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0 \qquad \text{or} \qquad \frac{\partial}{\partial x}u = -\frac{\partial}{\partial y}v = -\frac{\partial}{\partial y}\left[\frac{2\cdot x\cdot y}{\left(x^2+y^2\right)^2}\right] = -\left[\frac{2\cdot x\cdot\left(x^2-3\cdot y^2\right)}{\left(x^2+y^2\right)^3}\right]$$

Integrating $$u(x,y) = -\int \left[\frac{2\cdot x\cdot\left(x^2-3\cdot y^2\right)}{\left(x^2+y^2\right)^3}\right] dx = \frac{x^2-y^2}{\left(x^2+y^2\right)^2} + f(y) = \frac{x^2+y^2-2\cdot y^2}{\left(x^2+y^2\right)^2} + f(y)$$

$$u(x,y) = \frac{1}{x^2+y^2} - \frac{2\cdot y^2}{\left(x^2+y^2\right)^2} + f(y)$$

The simplest form is $$u(x,y) = \frac{1}{x^2+y^2} - \frac{2\cdot y^2}{\left(x^2+y^2\right)^2}$$

Note: Instead of this approach we could have verified that u and v satisfy continuity

$$\frac{\partial}{\partial x}\left[\frac{1}{x^2+y^2} - \frac{2\cdot y^2}{\left(x^2+y^2\right)^2}\right] + \frac{\partial}{\partial y}\left[\frac{2\cdot x\cdot y}{\left(x^2+y^2\right)^2}\right] = 0$$

However, this does not verify the solution is the simplest.

Problem 5.9

[Difficulty: 2]

5.9 The x component of velocity in a steady incompressible flow field in the xy plane is $u = Ae^{x/b}\cos(y/b)$, where $A = 10$ m/s, $b = 5$ m, and x and y are measured in meters. Find the simplest y component of velocity for this flow field.

Given: x component of velocity

Find: y component for incompressible flow; Valid for unsteady? How many y components?

Solution:

Basic equation: $$\frac{\partial}{\partial x}(\rho\cdot u) + \frac{\partial}{\partial y}(\rho\cdot v) + \frac{\partial}{\partial z}(\rho\cdot w) + \frac{\partial}{\partial t}\rho = 0$$

Assumption: Incompressible flow; flow in x-y plane

Hence $$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0 \qquad \text{or} \qquad \frac{\partial}{\partial y}v = -\frac{\partial}{\partial x}u = -\frac{\partial}{\partial x}\left(A\cdot e^{\frac{x}{b}}\cdot\cos\left(\frac{y}{b}\right)\right) = -\left(\frac{A}{b}\cdot e^{\frac{x}{b}}\cdot\cos\left(\frac{y}{b}\right)\right)$$

Integrating $$v(x,y) = -\int \frac{A}{b}\cdot e^{\frac{x}{b}}\cdot\cos\left(\frac{y}{b}\right)dy = -A\cdot e^{\frac{x}{b}}\cdot\sin\left(\frac{y}{b}\right) + f(x)$$

This basic equation is valid for steady and unsteady flow (t is not explicit)

There are an infinite number of solutions, since f(x) can be any function of x. The simplest is f(x) = 0

$$v(x,y) = -A\cdot e^{\frac{x}{b}}\cdot\sin\left(\frac{y}{b}\right) \qquad v(x,y) = -10\cdot e^{\frac{x}{5}}\cdot\sin\left(\frac{y}{5}\right)$$

Problem 5.10

[Difficulty: 2]

5.10 A crude approximation for the x component of velocity in an incompressible laminar boundary layer is a linear variation from $u = 0$ at the surface ($y = 0$) to the freestream velocity, U, at the boundary-layer edge ($y = \delta$). The equation for the profile is $u = Uy/\delta$, where $\delta = cx^{1/2}$ and c is a constant. Show that the simplest expression for the y component of velocity is $v = uy/4x$. Evaluate the maximum value of the ratio v/U, at a location where $x = 0.5$ m and $\delta = 5$ mm.

Given: Approximate profile for a laminar boundary layer:

$$u = \frac{U \cdot y}{\delta} \qquad \delta = c \cdot \sqrt{x} \qquad \text{(c is constant)}$$

Find: (a) Show that the simplest form of v is

$$v = \frac{u}{4} \cdot \frac{y}{x}$$

(b) Evaluate maximum value of v/u where $\delta = 5$ mm and $x = 0.5$ m

Solution: We will check this flow field using the continuity equation

Governing Equations:

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) + \frac{\partial \rho}{\partial t} = 0 \qquad \text{(Continuity equation)}$$

Assumptions:
(1) Incompressible flow (ρ is constant)
(2) Two dimensional flow (velocity is not a function of z)

Based on the two assumptions listed above, the continuity equation reduces to: $\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0$

The partial of u with respect to x is: $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial \delta}\dfrac{d\delta}{dx} = -\dfrac{Uy}{\delta^2} \times \dfrac{1}{2}cx^{-\frac{1}{2}} = -\dfrac{Uy}{2cx^{\frac{3}{2}}}$ Therefore from continuity: $\dfrac{\partial v}{\partial y} = -\dfrac{\partial u}{\partial x} = \dfrac{Uy}{2cx^{\frac{3}{2}}}$

Integrating this expression will yield the y-component of velocity:

$$v = \int \frac{U \cdot y}{2 \cdot c \cdot x^{\frac{3}{2}}}\, dy + f(x) = \frac{U \cdot y^2}{4 \cdot c \cdot x^{\frac{3}{2}}} + f(x)$$

Now due to the no-slip condition at the wall (y = 0) we get f(x) = 0. Thus:

$$v = \frac{U \cdot y^2}{4 \cdot c \cdot x^{\frac{3}{2}}} = \frac{U \cdot y}{c \cdot x^{\frac{1}{2}}} \cdot \frac{y}{4 \cdot x} = \frac{u \cdot y}{4 \cdot x} \text{(Q.E.D.)}$$

$$v = \frac{u}{4} \cdot \frac{y}{x}$$

The maximum value of v/U is where $y = \delta$: $v_{ratmax} = \dfrac{v}{u} = \dfrac{\delta}{4 \cdot x}$ $\qquad v_{ratmax} = \dfrac{5 \times 10^{-3} \cdot m}{4 \times 0.5 \cdot m}$

$$v_{ratmax} = 0.0025$$

Problem 5.11

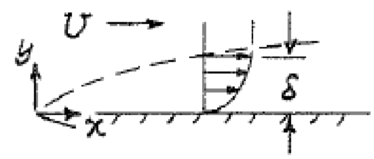
[Difficulty: 3]

5.11 A useful approximation for the x component of velocity in an incompressible laminar boundary layer is a parabolic variation from $u = 0$ at the surface ($y = 0$) to the freestream velocity, U, at the edge of the boundary layer ($y = \delta$). The equation for the profile is $u/U = 2(y/\delta) - (y/\delta)^2$, where $\delta = cx^{1/2}$ and c is a constant. Show that the simplest expression for the y component of velocity is

$$\frac{v}{U} = \frac{\delta}{x}\left[\frac{1}{2}\left(\frac{y}{\delta}\right)^2 - \frac{1}{3}\left(\frac{y}{\delta}\right)^3\right]$$

Plot v/U versus y/δ to find the location of the maximum value of the ratio v/U. Evaluate the ratio where $\delta = 5$ mm and $x = 0.5$ m.

Given: Approximate (parabolic) profile for a laminar boundary layer:



$$\frac{u}{U} = 2\cdot\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2 \qquad \delta = c\cdot\sqrt{x} \quad \text{(c is constant)}$$

Find: (a) Show that the simplest form of v for incompressible flow is

$$\frac{v}{U} = \frac{\delta}{x}\cdot\left[\frac{1}{2}\cdot\left(\frac{y}{\delta}\right)^2 - \frac{1}{3}\cdot\left(\frac{y}{\delta}\right)^3\right]$$

(b) Plot v/U versus y/δ

(c) Evaluate maximum value of v/U where δ = 5 mm and x = 0.5 m

Solution: We will check this flow field using the continuity equation

Governing Equations:

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) + \frac{\partial \rho}{\partial t} = 0 \quad \text{(Continuity equation)}$$

Assumptions:

(1) Incompressible flow (ρ is constant)

(2) Two dimensional flow (velocity is not a function of z)

Based on the two assumptions listed above, the continuity equation reduces to: $\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0$

The partial of u with respect to x is: $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial \delta}\dfrac{d\delta}{dx} = U\left[-\dfrac{2y}{\delta^2} + \dfrac{2y^2}{\delta^3}\right] \times \dfrac{1}{2}cx^{-\frac{1}{2}}$ Now since $\delta = c\cdot x^{\frac{1}{2}}$ $\quad x^{-\frac{1}{2}} = \dfrac{c}{\delta}$ and thus

$$\frac{\partial u}{\partial x} = \frac{Uc^2}{\delta}\left[-\frac{y}{\delta^2} + \frac{y^2}{\delta^3}\right] = \frac{Uc^2}{\delta^2}\left[-\left(\frac{y}{\delta}\right) + \left(\frac{y}{\delta}\right)^2\right]$$

Therefore from continuity: $\dfrac{\partial v}{\partial y} = -\dfrac{\partial u}{\partial x} = \dfrac{Uc^2}{\delta^2}\left[\left(\dfrac{y}{\delta}\right) - \left(\dfrac{y}{\delta}\right)^2\right]$

Integrating this expression will yield the y-component of velocity: $v = \displaystyle\int \frac{U\cdot c^2}{\delta}\cdot\left[\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right] dy + f(x)$ Evaluating:

$$v = \frac{U\cdot c^2}{\delta^2}\cdot\left(\frac{y^2}{2\cdot\delta} - \frac{y^3}{3\cdot\delta^2}\right) + f(x) = \frac{U\cdot c^2}{\delta}\cdot\left[\frac{1}{2}\cdot\left(\frac{y}{\delta}\right)^2 - \frac{1}{3}\cdot\left(\frac{y}{\delta}\right)^3\right] + f(x)$$

Since $\delta = c\cdot x^{\frac{1}{2}}$ $\quad c^2 = \dfrac{\delta^2}{x}$ Thus:

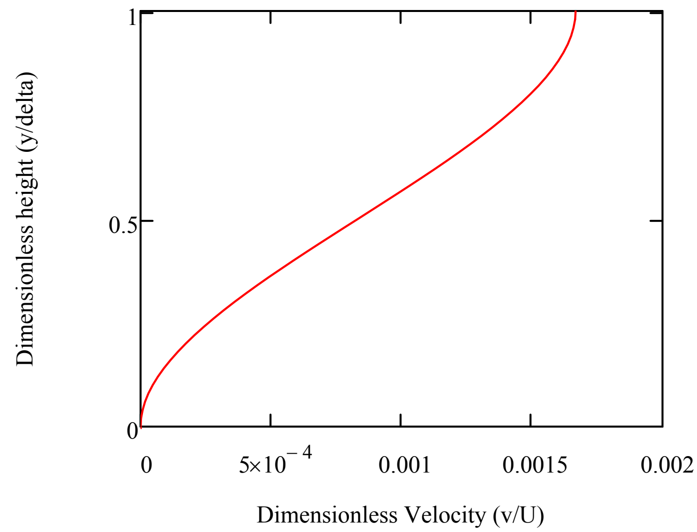
$$v = U\cdot\frac{\delta}{x}\cdot\left[\frac{1}{2}\cdot\left(\frac{y}{\delta}\right)^2 - \frac{1}{3}\cdot\left(\frac{y}{\delta}\right)^3\right] + f(x)$$

Now due to the no-slip condition at the wall (y = 0) we get f(x) = 0. Therefore:

$$\frac{v}{U} = \frac{\delta}{x}\cdot\left[\frac{1}{2}\cdot\left(\frac{y}{\delta}\right)^2 - \frac{1}{3}\cdot\left(\frac{y}{\delta}\right)^3\right] \quad \text{(Q.E.D.)} \quad \frac{v}{U} = \frac{\delta}{x}\cdot\left[\frac{1}{2}\cdot\left(\frac{y}{\delta}\right)^2 - \frac{1}{3}\cdot\left(\frac{y}{\delta}\right)^3\right]$$

Plotting this relationship shows:

Assuming x = 0.5 m and δ = 5 mm



The maximum value of v/U is where y = δ: $v_{ratmax} = \frac{v}{U} = \frac{\delta}{x}\cdot\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{\delta}{6\cdot x}$ $\quad v_{ratmax} = \frac{5\times 10^{-3}\cdot m}{6\times 0.5\cdot m}$ $\quad v_{ratmax} = 0.00167$

Problem 5.12

[Difficulty: 3]

5.12 A useful approximation for the x component of velocity in an incompressible laminar boundary layer is a sinusoidal variation from $u = 0$ at the surface ($y = 0$) to the freestream velocity, U, at the edge of the boundary layer ($y = \delta$). The equation for the profile is $u = U\sin(\pi y/2\delta)$, where $\delta = cx^{1/2}$ and c is a constant. Show that the simplest expression for the y component of velocity is

$$\frac{v}{U} = \frac{1}{\pi}\frac{\delta}{x}\left[\cos\left(\frac{\pi}{2}\frac{y}{\delta}\right) + \left(\frac{\pi}{2}\frac{y}{\delta}\right)\sin\left(\frac{\pi}{2}\frac{y}{\delta}\right) - 1\right]$$

Plot u/U and v/U versus y/δ, and find the location of the maximum value of the ratio v/U. Evaluate the ratio where $x = 0.5$ m and $\delta = 5$ mm.

Given: Approximate (sinusoidal) profile for a laminar boundary layer:

$$\frac{u}{U} = \sin\left(\frac{\pi\cdot y}{2\cdot\delta}\right) \qquad \delta = c\cdot\sqrt{x} \quad \text{(c is constant)}$$

Find: (a) Show that the simplest form of v for incompressible flow is

$$\frac{v}{U} = \frac{1}{\pi}\cdot\frac{\delta}{x}\cdot\left[\cos\left(\frac{\pi}{2}\cdot\frac{y}{\delta}\right) + \left(\frac{\pi}{2}\cdot\frac{y}{\delta}\right)\cdot\sin\left(\frac{\pi}{2}\cdot\frac{y}{\delta}\right) - 1\right]$$

(b) Plot v/U versus y/δ

(c) Evaluate maximum value of v/U where δ = 5 mm and x = 0.5 m

Solution: We will check this flow field using the continuity equation

Governing Equations:

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) + \frac{\partial \rho}{\partial t} = 0 \quad \text{(Continuity equation)}$$

Assumptions: (1) Incompressible flow (ρ is constant)

(2) Two dimensional flow (velocity is not a function of z)

Based on the two assumptions listed above, the continuity equation reduces to: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

The partial of u with respect to x is: $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \delta}\frac{d\delta}{dx} = U\left[-\frac{\pi y}{2\delta^2}\cos\left(\frac{\pi y}{2\delta}\right)\right]\times\frac{1}{2}cx^{-\frac{1}{2}} = -\frac{\pi U c y}{4\delta^2}x^{-\frac{1}{2}}\cos\left(\frac{\pi y}{2\delta}\right)$

Now since $\delta = c\cdot x^{\frac{1}{2}}$ $\quad x^{-\frac{1}{2}} = \frac{c}{\delta}$ and thus $\frac{\partial u}{\partial x} = -\frac{\pi U c^2 y}{4\delta^3}\cos\left(\frac{\pi y}{2\delta}\right)$ Therefore from continuity: $\frac{\partial v}{\partial y} = \frac{\pi U c^2 y}{4\delta^3}\cos\left(\frac{\pi y}{2\delta}\right)$

Integrating this expression will yield the y-component of velocity: $v = \int \frac{\pi\cdot U\cdot c^2\cdot y}{4\cdot\delta^3}\cdot\cos\left(\frac{\pi\cdot y}{2\cdot\delta}\right)dy + f(x)$ Evaluating:

$$v = \frac{\pi\cdot U\cdot c^2}{4\cdot\delta^3}\cdot\int y\cdot\cos\left(\frac{\pi\cdot y}{2\cdot\delta}\right)dy + f(x) = \frac{\pi\cdot U\cdot c^2}{4\cdot\delta^3}\cdot\left(\frac{2\cdot\delta\cdot y}{\pi}\cdot\sin\left(\frac{\pi}{2}\cdot\frac{y}{\delta}\right) + \frac{4\cdot\delta^2}{\pi^2}\cdot\cos\left(\frac{\pi}{2}\cdot\frac{y}{\delta}\right)\right) + f(x)$$

Simplifying this expression:

$$v = \frac{U\cdot c^2}{2\cdot\delta^2}\cdot\left(y\cdot\sin\left(\frac{\pi}{2}\cdot\frac{y}{\delta}\right) + \frac{2\cdot\delta}{\pi}\cdot\cos\left(\frac{\pi}{2}\cdot\frac{y}{\delta}\right)\right) + f(x)$$

Since $\delta = c\cdot x^{\frac{1}{2}}$ $\quad c^2 = \frac{\delta^2}{x}$ Thus:

$$v = \frac{U}{2}\cdot\frac{\delta}{x}\cdot\left(\frac{y}{\delta}\cdot\sin\left(\frac{\pi}{2}\cdot\frac{y}{\delta}\right) + \frac{2}{\pi}\cdot\cos\left(\frac{\pi}{2}\cdot\frac{y}{\delta}\right)\right) + f(x)$$

Now due to the no-slip condition at the wall (y = 0) we get:

$$0 = \frac{U}{2}\cdot\frac{\delta}{x}\cdot\left(\frac{2}{\pi}\cdot\cos(0)\right) + f(x) \qquad f(x) = -\frac{U\cdot\delta}{\pi\cdot x}$$

Therefore:

$$v = \frac{U}{2}\cdot\frac{\delta}{x}\cdot\left(\frac{y}{\delta}\cdot\sin\left(\frac{\pi}{2}\cdot\frac{y}{\delta}\right) + \frac{2}{\pi}\cdot\cos\left(\frac{\pi}{2}\cdot\frac{y}{\delta}\right)\right) - \frac{U\cdot\delta}{\pi\cdot x}$$

Simplifying:

$$v = \frac{U}{\pi}\cdot\frac{\delta}{x}\cdot\left(\frac{\pi}{2}\cdot\frac{y}{\delta}\cdot\sin\left(\frac{\pi}{2}\cdot\frac{y}{\delta}\right) + \cos\left(\frac{\pi}{2}\cdot\frac{y}{\delta}\right) - 1\right)$$

Thus:

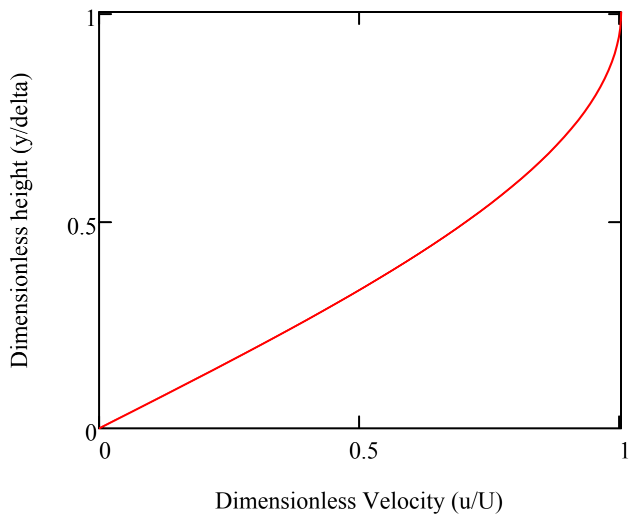
$$\frac{v}{U} = \frac{\delta}{\pi\cdot x}\cdot\left(\frac{\pi}{2}\cdot\frac{y}{\delta}\cdot\sin\left(\frac{\pi}{2}\cdot\frac{y}{\delta}\right) + \cos\left(\frac{\pi}{2}\cdot\frac{y}{\delta}\right) - 1\right)$$

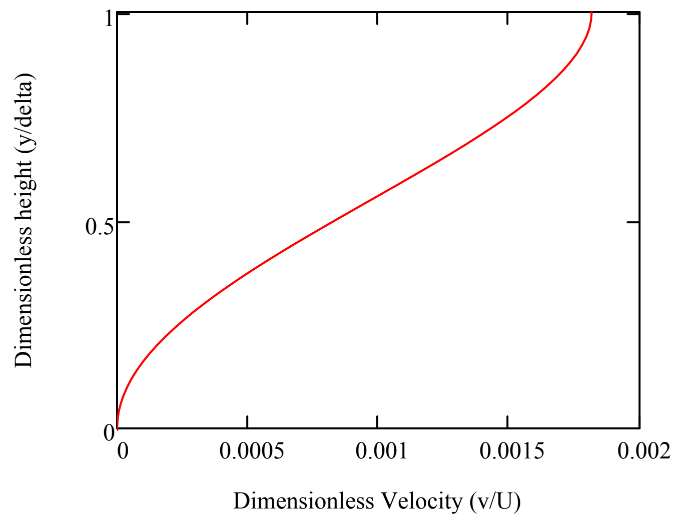
(Q.E.D.)

$$\frac{v}{U} = \frac{\delta}{\pi\cdot x}\cdot\left(\frac{\pi}{2}\cdot\frac{y}{\delta}\cdot\sin\left(\frac{\pi}{2}\cdot\frac{y}{\delta}\right) + \cos\left(\frac{\pi}{2}\cdot\frac{y}{\delta}\right) - 1\right)$$

Plotting this relationship shows:

Assuming x = 0.5 m and δ = 5 mm





The maximum value of v/U is where y = δ: $v_{ratmax} = \frac{v}{U} = \frac{\delta}{\pi\cdot x}\cdot\left(\frac{\pi}{2}\cdot\sin\left(\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{2}\right) - 1\right) = \frac{\delta}{\pi\cdot x}\cdot\left(\frac{\pi}{2} - 1\right)$

$$v_{ratmax} = \frac{5\times 10^{-3}\cdot m}{\pi\times 0.5\cdot m}\times\left(\frac{\pi}{2} - 1\right) \qquad v_{ratmax} = 0.00182$$

Problem 5.13

[Difficulty: 3]

5.13 A useful approximation for the x component of velocity in an incompressible laminar boundary layer is a cubic variation from $u = 0$ at the surface ($y = 0$) to the freestream velocity, U, at the edge of the boundary layer ($y = \delta$). The equation for the profile is $u/U = \frac{3}{2}(y/\delta) - \frac{1}{2}(y/\delta)^3$, where $\delta = cx^{1/2}$ and c is a constant. Derive the simplest expression for v/U, the y component of velocity ratio. Plot u/U and v/U versus y/δ, and find the location of the maximum value of the ratio v/U. Evaluate the ratio where $\delta = 5$ mm and $x = 0.5$ m.

Given: Data on boundary layer

Find: *y* component of velocity ratio; location of maximum value; plot velocity profiles; evaluate at particular point

Solution:

$$u(x,y) = U\cdot\left[\frac{3}{2}\cdot\left(\frac{y}{\delta(x)}\right) - \frac{1}{2}\cdot\left(\frac{y}{\delta(x)}\right)^3\right] \qquad \text{and} \qquad \delta(x) = c\cdot\sqrt{x}$$

so

$$u(x,y) = U\cdot\left[\frac{3}{2}\cdot\left(\frac{y}{c\cdot\sqrt{x}}\right) - \frac{1}{2}\cdot\left(\frac{y}{c\cdot\sqrt{x}}\right)^3\right]$$

For incompressible flow

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$$

Hence

$$v(x,y) = -\int \frac{d}{dx}u(x,y)\,dy \qquad \text{and} \qquad \frac{du}{dx} = \frac{3}{4}\cdot U\cdot\left(\frac{y^3}{c^3\cdot x^{\frac{5}{2}}} - \frac{y}{c\cdot x^{\frac{3}{2}}}\right)$$

so

$$v(x,y) = -\int \frac{3}{4}\cdot U\cdot\left(\frac{y^3}{c^3}\cdot\frac{x^5}{2} - \frac{y}{c}\cdot\frac{x^3}{2}\right) dy$$

$$v(x,y) = \frac{3}{8}\cdot U\cdot\left(\frac{y^2}{c\cdot x^{\frac{3}{2}}} - \frac{y^4}{2\cdot c^3\cdot x^{\frac{5}{2}}}\right) \qquad v(x,y) = \frac{3}{8}\cdot U\cdot\frac{\delta}{x}\cdot\left[\left(\frac{y}{\delta}\right)^2 - \frac{1}{2}\cdot\left(\frac{y}{\delta}\right)^4\right]$$

The maximum occurs at $y = \delta$ as seen in the *Excel* work shown below.

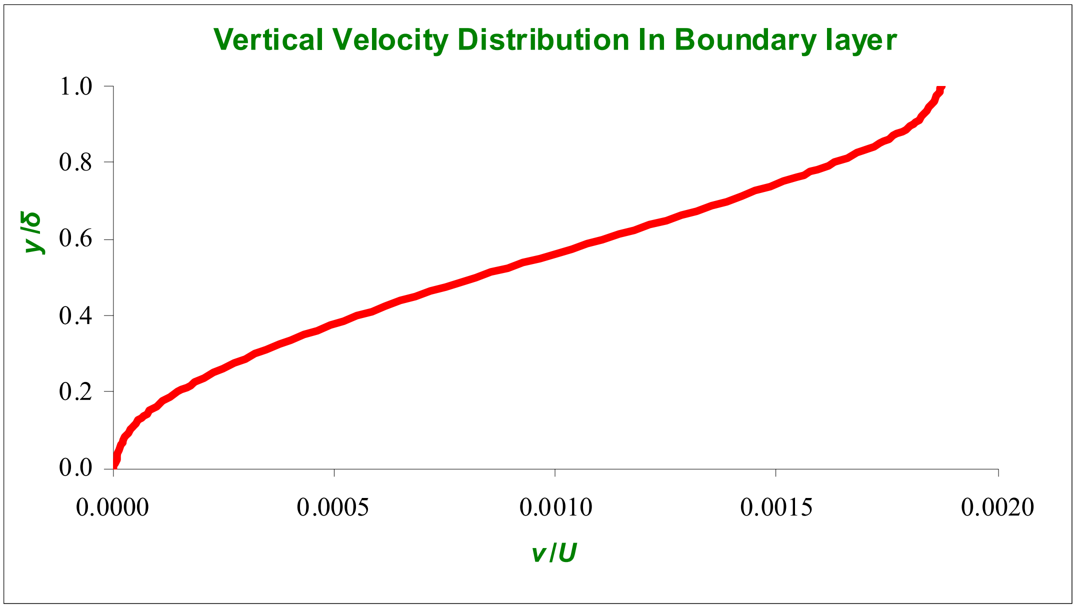
$$v_{max} = \frac{3}{8}\cdot U\cdot\frac{\delta}{x}\cdot\left(1 - \frac{1}{2}\cdot 1\right)$$

At $\delta = 5\cdot mm$ and $x = 0.5\cdot m$, the maximum vertical velocity is $\frac{v_{max}}{U} = 0.00188$

To find when v/U is maximum, use *Solver* in *Excel*

v/U	y/δ
0.00188	1.0

v/U	y/δ
0.000000	0.0
0.000037	0.1
0.000147	0.2
0.000322	0.3
0.000552	0.4
0.00082	0.5
0.00111	0.6
0.00139	0.7
0.00163	0.8
0.00181	0.9
0.00188	1.0



Problem 5.14

[Difficulty: 3]

5.14 For a flow in the xy plane, the x component of velocity is given by $u = Ax^2y^2$, where $A = 0.3\ \mathrm{m^{-3}\cdot s^{-1}}$, and x and y are measured in meters. Find a possible y component for steady, incompressible flow. Is it also valid for unsteady, incompressible flow? Why? How many possible y components are there? Determine the equation of the streamline for the simplest y component of velocity. Plot the streamlines through points (1, 4) and (2, 4).

Given: Steady, incompressible flow in x-y plane:

$$u = A\cdot x^2\cdot y^2 \qquad A = 0.3\cdot m^{-3}\cdot s^{-1}$$

Find:
(a) a possible y component of velocity for this flow field
(b) if the result is valid for unsteady, incompressible flow
(c) number of possible y components for velocity
(d) equation of the streamlines for the flow
(e) plot streamlines through points (1,4) and (2,4)

Solution: We will check this flow field using the continuity equation

Governing Equations:

$$\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)+\frac{\partial \rho}{\partial t}=0 \quad \text{(Continuity equation)}$$

Assumptions:
(1) Incompressible flow (ρ is constant)
(2) Two dimensional flow (velocity is not a function of z)

Based on the two assumptions listed above, the continuity equation reduces to: $\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}=0$

The partial of u with respect to x is: $\dfrac{\partial u}{\partial x}=2Axy^2$ Therefore from continuity: $\dfrac{\partial v}{\partial y}=-\dfrac{\partial u}{\partial x}=-2Axy^2$

Integrating this expression will yield the y-component of velocity: $v = \int -2\cdot A\cdot x\cdot y^2\,dx + f(x)$ $\qquad v = -\frac{2}{3}\cdot A\cdot x^2\cdot y^3 + f(x)$

The basic equation reduces for the same form for unsteady flow. Hence The result is valid for unsteady, incompressible flow.

Since f(x) is arbitrary: There are an infinite number of possible y-components of velocity.

The simplest version of v is when f(x) = 0. Therefore, the equation of the corresponding streamline is:

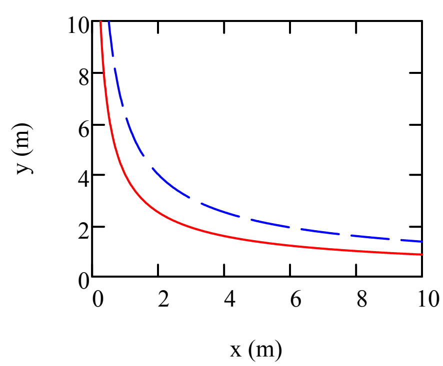
$$\frac{dy}{dx}=\frac{v}{u}=\frac{-\frac{2}{3}A\cdot x^2\cdot y^3}{A\cdot x^2\cdot y^2}=-\frac{2}{3}\cdot\frac{y}{x}$$ Separating variables and integrating: $\dfrac{dy}{y}=-\dfrac{2}{3}\cdot\dfrac{dx}{x}$ $\quad \ln(y)=-\frac{2}{3}\cdot\ln(x)$ $\quad$ Thus: $x\cdot y^{\frac{3}{2}} = \text{constant}$

are the equations of the streamlines of this flow field.

Plotting streamline for point (1, 4): $1\times 4^{\frac{3}{2}} = 8 \qquad x\cdot y^{\frac{3}{2}} = 8$

Plotting streamline for point (2, 4): $2\times 4^{\frac{3}{2}} = 16 \qquad x\cdot y^{\frac{3}{2}} = 16$

The two streamlines are plotted here in red (1,4) and blue (2,4):



Problem 5.15

[Difficulty: 3]

5.15 The y component of velocity in a steady, incompressible flow field in the xy plane is $v = -Bxy^3$, where $B = 0.2\ \mathrm{m}^{-3}\cdot\mathrm{s}^{-1}$, and x and y are measured in meters. Find the simplest x component of velocity for this flow field. Find the equation of the streamlines for this flow. Plot the streamlines through points (1, 4) and (2, 4).

Given: Steady, incompressible flow in x-y plane:

$$v = -B\cdot x\cdot y^3 \qquad B = 0.2\cdot \mathrm{m}^{-3}\cdot \mathrm{s}^{-1}$$

Find: (a) the simplest x component of velocity for this flow field
(b) equation of the streamlines for the flow
(c) plot streamlines through points (1,4) and (2,4)

Solution: We will check this flow field using the continuity equation

Governing Equations:

$$\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)+\frac{\partial \rho}{\partial t}=0 \quad \text{(Continuity equation)}$$

Assumptions: (1) Incompressible flow (ρ is constant)
(2) Two dimensional flow (velocity is not a function of z)

Based on the two assumptions listed above, the continuity equation reduces to: $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$

The partial of v with respect to y is: $\frac{\partial v}{\partial y} = -3Bxy^2$ Therefore from continuity: $\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} = 3Bxy^2$

Integrating this expression will yield the x-component of velocity: $u = \int 3\cdot B\cdot x\cdot y^2\, dx + f(y)$ Evaluating the integral:

$u = \frac{3}{2}\cdot B\cdot x^2\cdot y^2 + f(y)$ The simplest version of this equation is obtained when f(y) = 0: $u = \frac{3}{2}\cdot B\cdot x^2\cdot y^2$

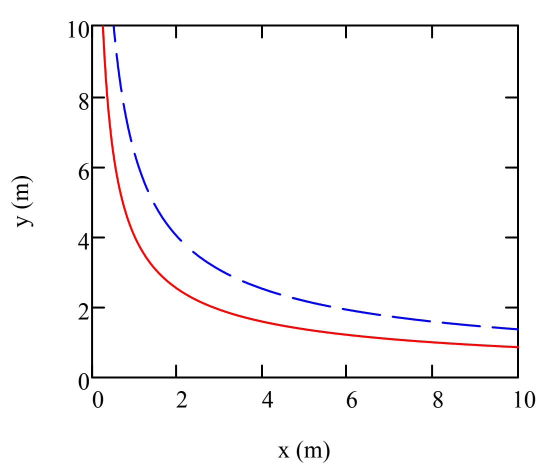
The equation of a streamline is: $\frac{dy}{dx} = \frac{v}{u} = \frac{-B\cdot x\cdot y^3}{\frac{3}{2}\cdot B\cdot x^2\cdot y^2} = -\frac{2}{3}\cdot\frac{y}{x}$ Separating variables and integrating: $\frac{dy}{y} = -\frac{2}{3}\cdot\frac{dx}{x}$

$\ln(y) = -\frac{2}{3}\cdot\ln(x)$ Thus: $x\cdot y^{\frac{3}{2}} = \text{constant}$ are the equations of the streamlines of this flow field. $x\cdot y^{\frac{3}{2}} = \text{constant}$

Plotting streamline for point (1, 4): $1 \times 4^{\frac{3}{2}} = 8 \qquad x\cdot y^{\frac{3}{2}} = 8$

Plotting streamline for point (2, 4): $2 \times 4^{\frac{3}{2}} = 16 \qquad x\cdot y^{\frac{3}{2}} = 16$

The two streamlines are plotted here in red (1,4) and blue (2,4):



Problem 5.16

[Difficulty: 5]

5.16 Consider a water stream from a jet of an oscillating lawn sprinkler. Describe the corresponding pathline and streakline.

Discussion: Refer back to the discussion of streamlines, pathlines, and streaklines in Section 2-2.

Because the sprinkler jet oscillates, this is an unsteady flow. Therefore pathlines and streaklines need not coincide.

A *pathline* is a line tracing the path of an individual fluid particle. The path of each particle is determined by the jet angle and the speed at which the particle leaves the jet.

Once a particle leaves the jet it is subject to gravity and drag forces. If aerodynamic drag were negligible, the path of each particle would be parabolic. The horizontal speed of the particle would remain constant throughout its trajectory. The vertical speed would be slowed by gravity until reaching peak height, and then it would become increasingly negative until the particle strikes the ground. The effect of aerodynamic drag is to reduce the particle speed. With drag the particle will not rise as high vertically nor travel as far horizontally. At each instant the particle trajectory will be lower and closer to the jet compared to the no-friction case. The trajectory after the particle reaches its peak height will be steeper than in the no-friction case.

A *streamline* is a line drawn in the flow that is tangent everywhere to the velocity vectors of the fluid motion. It is difficult to visualize the streamlines for an unsteady flow field because they move laterally. However, the streamline pattern may be drawn at an instant.

A *streakline* is the locus of the present locations of fluid particles that passed a reference point at previous times. As an example, choose the exit of a jet as the reference point. Imagine marking particles that pass the jet exit at a given instant and at uniform time intervals later. The first particle will travel farthest from the jet exit and on the lowest trajectory; the last particle will be located right at the jet exit. The curve joining the present positions of the particles will resemble a spiral whose radius increases with distance from the jet opening.

Problem 5.17

[Difficulty: 4]

5.17 Derive the differential form of conservation of mass in rectangular coordinates by expanding the *products* of density and the velocity components, ρu, ρv, and ρw, in a Taylor series about a point O. Show that the result is identical to Eq. 5.1a.

Given: Conservation of mass in rectangular coordinates

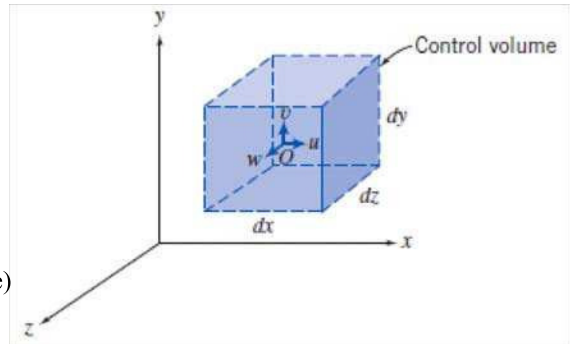
Find: Identical result to Equation 5.1a by expanding products of density and velocity in a Taylor Series.

Solution: We will use the diagram in Figure 5.1 (shown here). We will apply the conservation of mass evaluating the derivatives at point O:

Governing Equations:

$$\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)+\frac{\partial \rho}{\partial t}=0 \quad \text{(Continuity equation - Eqn 5.1a)}$$

Assumptions: Expansion of density and velocity via Taylor series is valid around point O.



In the x-direction, the mass flux is: $\dot{m}_x = \rho u dA = \rho u dydz$

At the right face: $\dot{m}_{x+dx/2} = \left[\rho u + \frac{\partial(\rho u)}{\partial x}\frac{dx}{2}\right]dydz$ (out of the volume)

At the left face: $\dot{m}_{x-dx/2} = \left[\rho u + \frac{\partial(\rho u)}{\partial x}\left(-\frac{dx}{2}\right)\right]dydz$ (into the volume)

The net mass flux out of the volume in the x-direction would then be:

$$\dot{m}_{x(net)} = \dot{m}_{x+dx/2} - \dot{m}_{x-dx/2} = \left[\rho u + \frac{\partial(\rho u)}{\partial x}\frac{dx}{2}\right]dydz - \left[\rho u + \frac{\partial(\rho u)}{\partial x}\left(-\frac{dx}{2}\right)\right]dydz = \frac{\partial(\rho u)}{\partial x}dxdydz$$

Similarly, the net mass fluxes in the y-direction and z-direction are: $\dot{m}_{y(net)} = \frac{\partial(\rho v)}{\partial x}dxdydz \quad \dot{m}_{z(net)} = \frac{\partial(\rho w)}{\partial x}dxdydz$

The rate of mass accumulation in the volume is: $\left.\frac{dm}{dt}\right)_{vol} = \frac{\partial \rho}{\partial t}dxdydz$ Now the net outflux must balance the accumulation:

$\dot{m}_{(net)} + \left.\frac{dm}{dt}\right)_{vol} = 0$ Therefore we may write: $\frac{\partial(\rho u)}{\partial x}dxdydz + \frac{\partial(\rho v)}{\partial x}dxdydz + \frac{\partial(\rho w)}{\partial x}dxdydz + \frac{\partial \rho}{\partial t}dxdydz = 0$

We may divide the volume out of all terms: $\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial x} + \frac{\partial(\rho w)}{\partial x} + \frac{\partial \rho}{\partial t} = 0$ (Q.E.D.)

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial x} + \frac{\partial(\rho w)}{\partial x} + \frac{\partial \rho}{\partial t} = 0$$

Problem 5.18

[Difficulty: 2]

5.18 Which of the following sets of equations represent possible incompressible flow cases?
(a) $V_r = U \cos\theta$; $V_\theta = -U \sin\theta$
(b) $V_r = -q/2\pi r$; $V_\theta = K/2\pi r$
(c) $V_r = U\cos\theta\,[1-(a/r)^2]$; $V_\theta = -U\sin\theta[1+(a/r)^2]$

Given: The list of velocity fields provided above

Find: Which of these fields possibly represent incompressible flow

Solution: We will check these flow fields against the continuity equation

Governing Equations:

$$\frac{1}{r}\frac{\partial}{\partial r}(r\rho V_r)+\frac{1}{r}\frac{\partial}{\partial\theta}(\rho V_\theta)+\frac{\partial}{\partial z}(\rho V_z)+\frac{\partial\rho}{\partial t}=0 \qquad \text{(Continuity equation)}$$

Assumptions:
(1) Incompressible flow (ρ is constant)
(2) Two dimensional flow (velocity is not a function of z)

Based on the two assumptions listed above, the continuity equation reduces to: $\frac{\partial(rV_r)}{\partial r}+\frac{\partial V_\theta}{\partial\theta}=0$

This is the criterion against which we will check all of the flow fields.

(a) $V_r = U\cdot\cos(\theta)$

$V_\theta = -U\cdot\sin(\theta)$

$$\frac{\partial(rV_r)}{\partial r}+\frac{\partial V_\theta}{\partial\theta}=(U\cos\theta)+(-U\cos\theta)=0$$

This could be an incompressible flow field.

(b) $V_r = -\frac{q}{2\cdot\pi\cdot r}$

$V_\theta = \frac{K}{2\cdot\pi\cdot r}$

$$\frac{\partial(rV_r)}{\partial r}+\frac{\partial V_\theta}{\partial\theta}=0+0=0$$

This could be an incompressible flow field.

(c) $V_r = U\cdot\cos(\theta)\cdot\left[1-\left(\frac{a}{r}\right)^2\right]$

$V_\theta = -U\cdot\sin(\theta)\cdot\left[1+\left(\frac{a}{r}\right)^2\right]$

$$\frac{\partial(rV_r)}{\partial r}+\frac{\partial V_\theta}{\partial\theta}=U\cos\theta\left[1+\left(\frac{a}{r}\right)^2\right]-U\cos\theta\left[1+\left(\frac{a}{r}\right)^2\right]=0$$

This could be an incompressible flow field.

Problem 5.19

[Difficulty: 2]

5.19 Which of the following sets of equations represent(s) possible incompressible flow cases?

(a) $V_r = -K/r$; $V_\theta = 0$

(b) $V_r = 0$; $V_\theta = K/r$

(c) $V_r = -K \cos\theta/r^2$; $V_\theta = -K \sin\theta/r^2$

Given: The list of velocity fields provided above

Find: Which of these fields possibly represent incompressible flow

Solution: We will check these flow fields against the continuity equation

Governing Equations:

$$\frac{1}{r}\frac{\partial}{\partial r}(r\rho V_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho V_\theta) + \frac{\partial}{\partial z}(\rho V_z) + \frac{\partial \rho}{\partial t} = 0 \quad \text{(Continuity equation)}$$

Assumptions:

(1) Incompressible flow (ρ is constant)

(2) Two dimensional flow (velocity is not a function of z)

Based on the two assumptions listed above, the continuity equation reduces to: $\frac{\partial}{\partial r}(r \cdot V_r) + \frac{\partial}{\partial \theta} V_\theta = 0$

This is the criterion against which we will check all of the flow fields.

a) $V_r(r,\theta,t) = -\frac{K}{r}$ $\quad V_\theta(r,\theta,t) = 0$ $\quad \frac{\partial}{\partial r}(r \cdot V_r(r,\theta,t)) = 0$ $\quad \frac{\partial}{\partial \theta} V_\theta(r,\theta,t) = 0$

Hence $\quad \frac{\partial}{\partial r}(r \cdot V_r) + \frac{\partial}{\partial \theta} V_\theta = 0$ $\quad$ INCOMPRESSIBLE

b) $V_r(r,\theta,t) = 0$ $\quad V_\theta(r,\theta,t) = \frac{K}{r}$ $\quad \frac{\partial}{\partial r}(r \cdot V_r(r,\theta,t)) = 0$ $\quad \frac{\partial}{\partial \theta} V_\theta(r,\theta,t) = 0$

Hence $\quad \frac{\partial}{\partial r}(r \cdot V_r) + \frac{\partial}{\partial \theta} V_\theta = 0$ $\quad$ INCOMPRESSIBLE

b) $V_r(r,\theta,t) = -\frac{K \cdot \cos(\theta)}{r^2}$ $\quad V_\theta(r,\theta,t) = -\frac{K \cdot \sin(\theta)}{r^2}$

$\frac{\partial}{\partial r}(r \cdot V_r(r,\theta,t)) = \frac{K \cdot \cos(\theta)}{r^2}$ $\quad \frac{\partial}{\partial \theta} V_\theta(r,\theta,t) = -\frac{K \cdot \cos(\theta)}{r^2}$

Hence $\quad \frac{\partial}{\partial r}(r \cdot V_r) + \frac{\partial}{\partial \theta} V_\theta = 0$ $\quad$ INCOMPRESSIBLE

Problem 5.20

[Difficulty: 3]

5.20 For an incompressible flow in the $r\theta$ plane, the r component of velocity is given as $V_r = U\cos\theta$.
(a) Determine a possible θ component of velocity.
(b) How many possible θ components are there?

Given: r component of velocity

Find: θ component for incompressible flow; How many θ components

Solution:

Basic equation: $$\frac{1}{r}\cdot\frac{\partial}{\partial r}(\rho\cdot r\cdot V_r) + \frac{1}{r}\cdot\frac{\partial}{\partial\theta}(\rho\cdot V_\theta) + \frac{\partial}{\partial z}(\rho\cdot V_z) + \frac{\partial}{\partial t}\rho = 0$$

Assumptions: Incompressible flow
Flow in r-θ plane

Hence $$\frac{1}{r}\cdot\frac{\partial}{\partial r}(r\cdot V_r) + \frac{1}{r}\cdot\frac{\partial}{\partial\theta}(V_\theta) = 0$$ or $$\frac{\partial}{\partial\theta}V_\theta = -\frac{\partial}{\partial r}(r\cdot V_r) = -\frac{\partial}{\partial r}(r\cdot U\cdot\cos(\theta)) = -U\cdot\cos(\theta)$$

Integrating $$V_\theta(r,\theta) = -\int U\cdot\cos(\theta)\,d\theta = -U\cdot\sin(\theta) + f(r)$$

$$V_\theta(r,\theta) = -U\cdot\sin(\theta) + f(r)$$

There are an infinite number of solutions as f(r) can be any function of r

The simplest form is $V_\theta(r,\theta) = -U\cdot\sin(\theta)$

Problem 5.21

[Difficulty: 3]

5.21 For an incompressible flow in the $r\theta$ plane, the r component of velocity is given as $V_r = -\Lambda \cos\theta / r^2$. Determine a possible θ component of velocity. How many possible θ components are there?

Given: r component of velocity

Find: θ component for incompressible flow; How many θ components

Solution:

Basic equation: $$\frac{1}{r}\cdot\frac{\partial}{\partial r}(\rho\cdot r\cdot V_r) + \frac{1}{r}\cdot\frac{\partial}{\partial \theta}(\rho\cdot V_\theta) + \frac{\partial}{\partial z}(\rho\cdot V_z) + \frac{\partial}{\partial t}\rho = 0$$

Assumption: Incompressible flow; flow in r-θ plane

Hence $$\frac{1}{r}\cdot\frac{\partial}{\partial r}(r\cdot V_r) + \frac{1}{r}\cdot\frac{\partial}{\partial \theta}(V_\theta) = 0$$ or $$\frac{\partial}{\partial \theta}V_\theta = -\frac{\partial}{\partial r}(r\cdot V_r) = -\frac{\partial}{\partial r}\left(-\frac{\Lambda\cdot\cos(\theta)}{r}\right) = -\frac{\Lambda\cdot\cos(\theta)}{r^2}$$

Integrating $$V_\theta(r,\theta) = -\int \frac{\Lambda\cdot\cos(\theta)}{r^2}\,d\theta = -\frac{\Lambda\cdot\sin(\theta)}{r^2} + f(r)$$

$$V_\theta(r,\theta) = -\frac{\Lambda\cdot\sin(\theta)}{r^2} + f(r)$$

There are an infinite number of solutions as f(r) can be any function of r

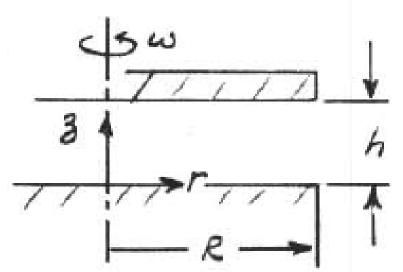
The simplest form is $$V_\theta(r,\theta) = -\frac{\Lambda\cdot\sin(\theta)}{r^2}$$

Problem 5.22

[Difficulty: 2]

5.22 A viscous liquid is sheared between two parallel disks of radius R, one of which rotates while the other is fixed. The velocity field is purely tangential, and the velocity varies linearly with z from $V_\theta = 0$ at $z = 0$ (the fixed disk) to the velocity of the rotating disk at its surface ($z = h$). Derive an expression for the velocity field between the disks.

Given: Flow between parallel disks as shown. Velocity is purely tangential. No-slip condition is satisfied, so velocity varies linearly with z.



Find: An expression for the velocity field

Solution: We will apply the continuity equation to this system.

Governing Equations:

$$\frac{1}{r}\frac{\partial}{\partial r}(r\rho V_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho V_\theta) + \frac{\partial}{\partial z}(\rho V_z) + \frac{\partial \rho}{\partial t} = 0 \quad \text{(Continuity equation)}$$

$$\vec{V} = V_r\hat{e}_r + V_\theta\hat{e}_\theta + V_z\hat{k} \quad \text{(Velocity flow field)}$$

Assumptions:

(1) Incompressible flow (ρ is constant)
(2) Purely tangential flow
(3) Linear velocity variation with z

Based on the first two assumptions, the continuity equation reduces to: $\frac{\partial V_\theta}{\partial \theta} = 0$ thus: $V_\theta = V_\theta(r,z)$

Since the velocity is linear with z, we may write: $V_\theta(r,z) = z\cdot f(r) + C$ Now we apply known boundary conditions:

1: $V_\theta(r,0) = 0 \quad 0\cdot f(r) + C = 0 \quad C = 0$ 2: $V_\theta(r,h) = r\cdot\omega \quad h\cdot f(r) = r\cdot\omega \quad f(r) = \frac{r\cdot\omega}{h}$

Therefore the tangential velocity is: $V_\theta = \omega\cdot r\cdot\frac{z}{h}$ Thus, the velocity field is:

$$\vec{V} = \omega r\frac{z}{h}\hat{e}_\theta$$

Problem 5.23

[Difficulty: 4]

5.23 Evaluate $\nabla\cdot\rho\vec{V}$ in cylindrical coordinates. Use the definition of ∇ in cylindrical coordinates. Substitute the velocity vector and perform the indicated operations, using the hint in footnote 1 on page 178. Collect terms and simplify; show that the result is identical to Eq. 5.2c.

Given: Definition of "del" operator in cylindrical coordinates, velocity vector

Find: (a) An expression for $\vec{\nabla}\cdot\left(\rho\vec{V}\right)$ in cylindrical coordinates.
(b) Show result is identical to Equation 5.2c.

Solution: We will apply the velocity field to the del operator and simplify.

Governing Equations:

$$\vec{\nabla} = \hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{k}\frac{\partial}{\partial z}$$ (Definition of "del" operator)

$$\vec{V} = V_r\hat{e}_r + V_\theta\hat{e}_\theta + V_z\hat{k}$$ (Velocity flow field)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\rho V_r\right) + \frac{1}{r}\frac{\partial}{\partial\theta}\left(\rho V_\theta\right) + \frac{\partial}{\partial z}\left(\rho V_z\right) = 0$$ (Equation 5.2c)

$$\frac{\partial\hat{e}_r}{\partial\theta} = \hat{e}_\theta \qquad \frac{\partial\hat{e}_\theta}{\partial\theta} = -\hat{e}_r$$ (Hints from footnote)

Substituting $\vec{\nabla}\cdot\left(\rho\vec{V}\right)$ using the governing equations yields:

$$\vec{\nabla}\cdot\left(\rho\vec{V}\right) = \left(\hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\rho\left(V_r\hat{e}_r + V_\theta\hat{e}_\theta + V_z\hat{k}\right)$$

$$= \hat{e}_r\frac{\partial}{\partial r}\cdot\rho\left(V_r\hat{e}_r + V_\theta\hat{e}_\theta + V_z\hat{k}\right) + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}\cdot\rho\left(V_r\hat{e}_r + V_\theta\hat{e}_\theta + V_z\hat{k}\right) + \hat{k}\frac{\partial}{\partial z}\cdot\rho\left(V_r\hat{e}_r + V_\theta\hat{e}_\theta + V_z\hat{k}\right)$$

$$= \hat{e}_r\cdot\hat{e}_r\frac{\partial}{\partial r}\left(\rho V_r\right) + \hat{e}_\theta\cdot\hat{e}_r\frac{1}{r}\frac{\partial}{\partial\theta}\left(\rho V_r\right) + \hat{e}_\theta\cdot\frac{1}{r}\frac{\partial\hat{e}_r}{\partial\theta}\left(\rho V_r\right)$$

$$+ \hat{e}_\theta\cdot\hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}\left(\rho V_\theta\right) + \hat{e}_\theta\cdot\frac{1}{r}\frac{\partial\hat{e}_\theta}{\partial\theta}\left(\rho V_\theta\right) + \hat{k}\cdot\hat{k}\frac{\partial}{\partial z}\left(\rho V_z\right)$$

Using the hints listed above, and knowing that: $\hat{e}_\theta\cdot\hat{e}_\theta = \hat{e}_r\cdot\hat{e}_r = \hat{k}\cdot\hat{k} = 1 \qquad \hat{e}_\theta\cdot\hat{e}_r = \hat{e}_r\cdot\hat{e}_\theta = 0$

$$\vec{\nabla}\cdot\left(\rho\vec{V}\right) = \frac{\partial}{\partial r}\left(\rho V_r\right) + \hat{e}_\theta\cdot\frac{1}{r}\frac{\partial\hat{e}_r}{\partial\theta}\left(\rho V_r\right) + \frac{1}{r}\frac{\partial}{\partial\theta}\left(\rho V_\theta\right) + \hat{e}_\theta\cdot\frac{1}{r}\frac{\partial\hat{e}_\theta}{\partial\theta}\left(\rho V_\theta\right) + \frac{\partial}{\partial z}\left(\rho V_z\right)$$

$$= \frac{\partial}{\partial r}\left(\rho V_r\right) + \hat{e}_\theta\cdot\hat{e}_\theta\frac{1}{r}\left(\rho V_r\right) + \frac{1}{r}\frac{\partial}{\partial\theta}\left(\rho V_\theta\right) - \hat{e}_\theta\cdot\hat{e}_r\frac{1}{r}\left(\rho V_\theta\right) + \frac{\partial}{\partial z}\left(\rho V_z\right)$$

$$= \frac{\partial}{\partial r}\left(\rho V_r\right) + \frac{1}{r}\left(\rho V_r\right) + \frac{1}{r}\frac{\partial}{\partial\theta}\left(\rho V_\theta\right) + \frac{\partial}{\partial z}\left(\rho V_z\right)$$

Combining the first two terms: $\frac{\partial}{\partial r}\left(\rho V_r\right) + \frac{1}{r}\left(\rho V_r\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\rho V_r\right)$ which can be verified through differentiation. Thus:

$$\vec{\nabla}\cdot\left(\rho\vec{V}\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\rho V_r\right) + \frac{1}{r}\frac{\partial}{\partial\theta}\left(\rho V_\theta\right) + \frac{\partial}{\partial z}\left(\rho V_z\right)$$ (Q.E.D.)

Problem 5.24

[Difficulty: 3]

*5.24 A velocity field in cylindrical coordinates is given as $\vec{V} = \hat{e}_r A/r + \hat{e}_\theta B/r$, where A and B are constants with dimensions of m^2/s. Does this represent a possible incompressible flow? Sketch the streamline that passes through the point $r_0 = 1$ m, $\theta = 90°$ if $A = B = 1$ m^2/s, if $A = 1$ m^2/s and $B = 0$, and if $B = 1$ m^2/s and $A = 0$.

Given: The velocity field

Find: Whether or not it is a incompressible flow; sketch various streamlines

Solution:

$$V_r = \frac{A}{r} \qquad V_\theta = \frac{B}{r}$$

For incompressible flow $\quad \frac{1}{r}\cdot\frac{d}{dr}(r\cdot V_r) + \frac{1}{r}\cdot\frac{d}{d\theta}V_\theta = 0 \qquad \frac{1}{r}\cdot\frac{d}{dr}(r\cdot V_r) = 0 \qquad \frac{1}{r}\cdot\frac{d}{d\theta}V_\theta = 0$

Hence $\quad \frac{1}{r}\cdot\frac{d}{dr}(r\cdot V_r) + \frac{1}{r}\cdot\frac{d}{d\theta}V_\theta = 0 \quad$ Flow is incompressible

For the streamlines $\quad \frac{dr}{V_r} = \frac{r\cdot d\theta}{V_\theta} \qquad \frac{r\cdot dr}{A} = \frac{r^2\cdot d\theta}{B}$

so $\quad \int \frac{1}{r}\,dr = \int \frac{A}{B}\,d\theta \quad$ Integrating $\quad \ln(r) = \frac{A}{B}\cdot\theta + \text{const}$

Equation of streamlines is $\quad r = C\cdot e^{\frac{A}{B}\cdot\theta}$

(a) For $A = B = 1$ m^2/s, passing through point (1m, $\pi/2$)

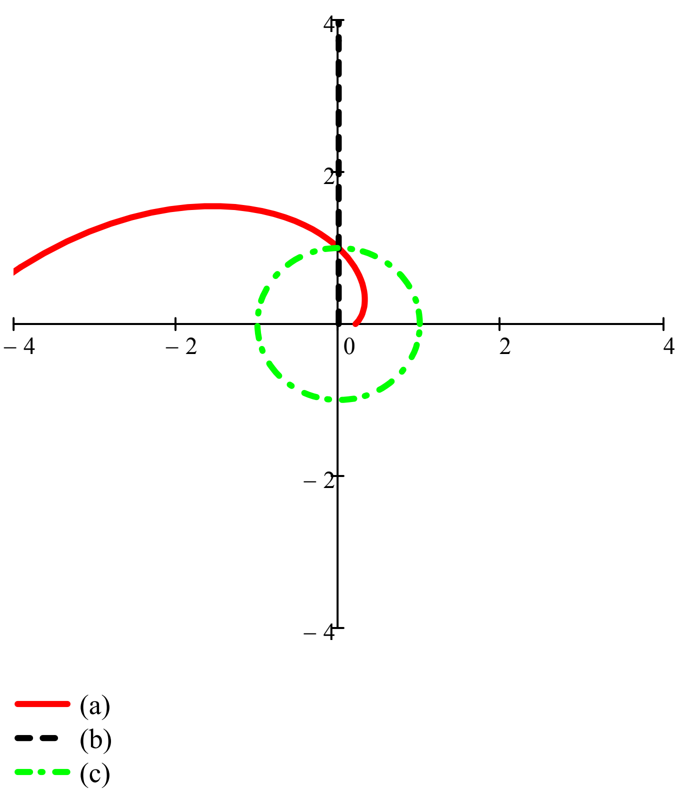
$$r = e^{\theta - \frac{\pi}{2}}$$

(b) For $A = 1$ m^2/s, $B = 0$ m^2/s, passing through point (1m, $\pi/2$)

$$\theta = \frac{\pi}{2}$$

(c) For $A = 0$ m^2/s, $B = 1$ m^2/s, passing through point (1m, $\pi/2$)

$$r = 1\cdot m$$



Problem *5.25

[Difficulty: 2]

*5.25 The velocity field for the viscometric flow of Example 5.7 is $\vec{V} = U(y/h)\hat{i}$. Find the stream function for this flow. Locate the streamline that divides the total flow rate into two equal parts.

Given: Velocity field for viscometric flow of Example 5.7: $\vec{V} = U\dfrac{y}{h}\hat{i}$

Find: (a) Stream function
(b) Locate streamline that divides flow rate equally

Solution: The flow is incompressible, so the stream function may be derived

Governing Equations:

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x} \qquad \text{(Definition of stream function)}$$

Integrating the velocity will result in the stream function: $\psi = \displaystyle\int u\,dy + f(x) = \int U\cdot\frac{y}{h}\,dy + f(x) = \frac{U\cdot y^2}{2\cdot h} + f(x)$

Let $\psi = 0$ at $y = 0$, so $f(x) = 0$:

$$\psi = \frac{U\cdot y^2}{2\cdot h}$$

The stream function is a maximum value at y = h: $\psi_{max} = \dfrac{U\cdot h^2}{2\cdot h} = \dfrac{U\cdot h}{2}$ The flow rate is: $\dfrac{Q}{w} = \psi_{max} - \psi_{min} = \dfrac{U\cdot h}{2} - 0 = \dfrac{U\cdot h}{2}$

So the streamline which splits the flow rate into two equal parts is: $\psi_{halfQ} = \dfrac{1}{2}\cdot\psi_{max} = \dfrac{1}{2}\cdot\dfrac{U\cdot h}{2} = \dfrac{U\cdot h}{4}$

Therefore, the equation of this streamline would be: $\dfrac{U\cdot y^2}{2\cdot h} = \dfrac{U\cdot h}{4}$ Simplifying this equation: $y^2 = \dfrac{h^2}{2}$ or: $y = \dfrac{h}{\sqrt{2}}$

$$y = \frac{h}{\sqrt{2}}$$

Problem *5.26

[Difficulty: 3]

***5.26** Determine the family of stream functions ψ that will yield the velocity field $\vec{V} = 2y(2x+1)\hat{i} + [x(x+1) - 2y^2]\hat{j}$.

Given: Velocity field

Find: Stream function ψ

Solution:

Basic equations: $\frac{\partial}{\partial x}(\rho \cdot u) + \frac{\partial}{\partial y}(\rho \cdot v) + \frac{\partial}{\partial z}(\rho \cdot w) + \frac{\partial}{\partial t}\rho = 0$ $\quad u = \frac{\partial}{\partial y}\psi$ $\quad v = -\frac{\partial}{\partial x}\psi$

Assumptions: Incompressible flow
Flow in x-y plane

Hence $\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$ or $\frac{\partial}{\partial x}[2 \cdot y \cdot (2x + 1)] + \frac{\partial}{\partial y}\left[x \cdot (x + 1) - 2 \cdot y^2\right] = 0$

Hence $u = 2 \cdot y \cdot (2 \cdot x + 1) = \frac{\partial}{\partial y}\psi$ $\quad \psi(x,y) = \int 2 \cdot y \cdot (2 \cdot x + 1)\, dy = 2 \cdot x \cdot y^2 + y^2 + f(x)$

and $v = x \cdot (x + 1) - 2 \cdot y^2 = -\frac{\partial}{\partial x}\psi$ $\quad \psi(x,y) = -\int \left[x \cdot (x + 1) - 2 \cdot y^2\right] dx = -\frac{x^3}{3} - \frac{x^2}{2} + 2 \cdot x \cdot y^2 + g(y$

Comparing these $f(x) = -\frac{x^3}{3} - \frac{x^2}{2}$ and $g(y) = y^2$

The stream function is $\psi(x,y) = y^2 + 2 \cdot x \cdot y^2 - \frac{x^2}{2} - \frac{x^3}{3}$

Checking $u(x,y) = \frac{\partial}{\partial y}\left(y^2 + 2 \cdot x \cdot y^2 - \frac{x^2}{2} - \frac{x^3}{3}\right) \rightarrow u(x,y) = 2 \cdot y + 4 \cdot x \cdot y$

$v(x,y) = -\frac{\partial}{\partial x}\left(y^2 + 2 \cdot x \cdot y^2 - \frac{x^2}{2} - \frac{x^3}{3}\right) \rightarrow v(x,y) = x^2 + x - 2 \cdot y^2$

Problem *5.27

[Difficulty: 3]

*5.27 Does the velocity field of Problem 5.24 represent a possible incompressible flow case? If so, evaluate and sketch the stream function for the flow. If not, evaluate the rate of change of density in the flow field.

Given: The velocity field

Find: Whether or not it is a incompressible flow; sketch stream function

Solution:

$$V_r = \frac{A}{r} \qquad V_\theta = \frac{B}{r}$$

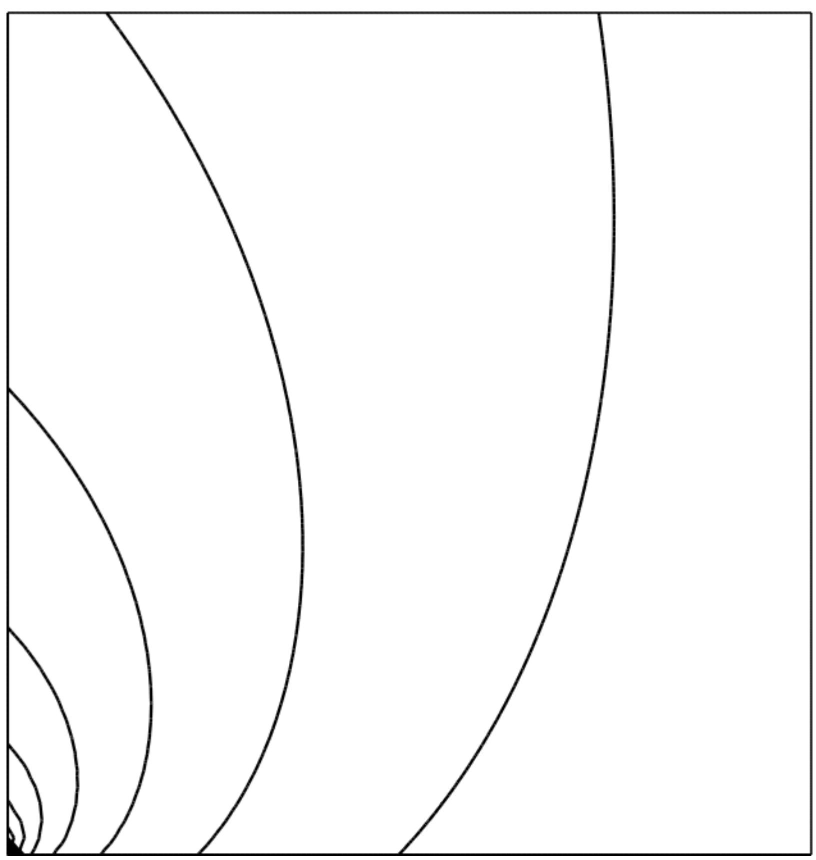
For incompressible flow $\quad \frac{1}{r}\cdot\frac{d}{dr}\left(r\cdot V_r\right) + \frac{1}{r}\cdot\frac{d}{d\theta}V_\theta = 0 \qquad \frac{1}{r}\cdot\frac{d}{dr}\left(r\cdot V_r\right) = 0 \qquad \frac{1}{r}\cdot\frac{d}{d\theta}V_\theta = 0$

Hence $\quad \frac{1}{r}\cdot\frac{d}{dr}\left(r\cdot V_r\right) + \frac{1}{r}\cdot\frac{d}{d\theta}V_\theta = 0 \qquad$ Flow is incompressible

For the stream function $\quad \frac{\partial}{\partial\theta}\psi = r\cdot V_r = A \qquad \psi = A\cdot\theta + f(r)$

Integrating $\quad \frac{\partial}{\partial r}\psi = -V_\theta = -\frac{B}{r} \qquad \psi = -B\cdot\ln(r) + g(\theta)$

Comparing, stream function is $\quad \psi = A\cdot\theta - B\cdot\ln(r)$



ψ

Problem *5.28

[Difficulty: 2]

***5.28** The stream function for a certain incompressible flow field is given by the expression $\psi = -Ur\sin\theta + q\theta/2\pi$. Obtain an expression for the velocity field. Find the stagnation point(s) where $|\vec{V}| = 0$, and show that $\psi = 0$ there.

Given: Stream function for an incompressible flow field:

$$\psi = -U\cdot r\cdot\sin(\theta) + \frac{q}{2\cdot\pi}\cdot\theta$$

Find:
(a) Expression for the velocity field
(b) Location of stagnation points
(c) Show that the stream function is equal to zero at the stagnation points.

Solution: We will generate the velocity field from the stream function.

Governing Equations:

$$V_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} \qquad V_\theta = -\frac{\partial\psi}{\partial r} \qquad \text{(Definition of stream function)}$$

Taking the derivatives of the stream function: $V_r = -U\cdot\cos(\theta) + \frac{q}{2\cdot\pi\cdot r}$ $\quad V_\theta = U\cdot\sin(\theta)$

So the velocity field is:

$$\vec{V} = \left(-U\cos\theta + \frac{q}{2\pi R}\right)\hat{e}_r + U\sin\theta\,\hat{e}_\theta$$

To find the stagnation points we must find the places where both velocity components are zero. When $V_r = 0$ $\quad r = \frac{q}{2\cdot\pi\cdot U\cdot\cos(\theta)}$

When $V_\theta = 0$ $\quad \sin(\theta) = 0$ therefore: $\quad \theta = 0, \pi$ $\quad$ Now we can apply these values of θ to the above relation to find r:

For θ = 0: $r = \frac{q}{2\cdot\pi\cdot U\cdot\cos(0)} = \frac{q}{2\cdot\pi\cdot U}$ $\quad$ For θ = π: $r = \frac{q}{2\cdot\pi\cdot\cos(\pi)} = -\frac{q}{2\cdot\pi\cdot U}$ $\quad$ These represent the same point:

Stagnation point at:

$$(r,\theta) = \left(\frac{q}{2\cdot\pi\cdot U}, 0\right)$$

At the stagnation point: $\quad \psi_{stagnation} = -U\cdot\frac{q}{2\cdot\pi\cdot U}\cdot\sin(0) + \frac{q}{2\cdot\pi}\cdot 0 = 0$

$$\psi_{stagnation} = 0$$

Problem *5.29

[Difficulty: 3]

***5.29** Consider a flow with velocity components $u = z(3x^2 - z^2)$, $v = 0$, and $w = x(x^2 - 3z^2)$.
(a) Is this a one-, two-, or three-dimensional flow?
(b) Demonstrate whether this is an incompressible flow.
(c) If possible, derive a stream function for this flow.

Given: Velocity field

Find: Whether it's 1D, 2D or 3D flow; Incompressible or not; Stream function ψ

Solution:

Basic equation: $\frac{\partial}{\partial x}(\rho\cdot u) + \frac{\partial}{\partial y}(\rho\cdot v) + \frac{\partial}{\partial z}(\rho\cdot w) + \frac{\partial}{\partial t}\rho = 0$ $\quad u = \frac{\partial}{\partial z}\psi$ $\quad w = -\frac{\partial}{\partial x}\psi$

Assumption: Incompressible flow; flow in x-z plane (v = 0)

Velocity field is a function of x and z only, so is 2D

Check for incompressible $\quad \frac{\partial}{\partial x}u + \frac{\partial}{\partial z}w = 0$

$$\frac{\partial}{\partial x}\left[z\cdot\left(3\cdot x^2 - z^2\right)\right] \rightarrow 6\cdot x\cdot z \qquad \frac{\partial}{\partial z}\left[x\cdot\left(x^2 - 3\cdot z^2\right)\right] \rightarrow -6\cdot x\cdot z$$

Hence $\quad \frac{\partial}{\partial x}u + \frac{\partial}{\partial z}w = 0 \qquad$ Flow is INCOMPRESSIBLE

Hence $\quad u = z\cdot\left(3\cdot x^2 - z^2\right) = \frac{\partial}{\partial z}\psi \qquad \psi(x,z) = \int z\cdot\left(3\cdot x^2 - z^2\right)dz = \frac{3}{2}\cdot x^2\cdot z^2 - \frac{1}{4}\cdot z^4 + f(x)$

and $\quad w = x\cdot\left(x^2 - 3\cdot z^2\right) = -\frac{\partial}{\partial x}\psi \qquad \psi(x,z) = -\int \left[x\cdot\left(x^2 - 3\cdot z^2\right)\right]dx = -\frac{x^4}{4} + \frac{3}{2}\cdot x^2\cdot z^2 + g(z)$

Comparing these $\quad f(x) = -\frac{x^4}{4} \quad$ and $\quad g(z) = -\frac{z^4}{4}$

The stream function is $\quad \psi(x,z) = -\frac{x^4}{4} + \frac{3}{2}\cdot x^2\cdot z^2 - \frac{z^4}{4}$

Checking $\quad u(x,z) = \frac{\partial}{\partial z}\left(-\frac{x^4}{4} + \frac{3}{2}\cdot x^2\cdot z^2 - \frac{z^4}{4}\right) \rightarrow u(x,z) = 3\cdot x^2\cdot z - z^3$

$$w(x,z) = -\frac{\partial}{\partial y}\left(z\cdot y^3 - z^3\cdot y\right) \rightarrow w(x,z) = z^3 - 3\cdot y^2\cdot z$$

Problem *5.30

[Difficulty: 3]

***5.30** An incompressible frictionless flow field is specified by the stream function $\psi = -5Ax - 2Ay$, where $A = 2$ m/s, and x and y are coordinates in meters.
(a) Sketch the streamlines $\psi = 0$ and $\psi = 5$, and indicate the direction of the velocity vector at the point (0, 0) on the sketch.
(b) Determine the magnitude of the flow rate between the streamlines passing through (2, 2) and (4, 1).

Given: Stream function for an incompressible flow field:

$$\psi = -5\cdot A\cdot x - 2\cdot A\cdot y \qquad A = 2\cdot\frac{m}{s}$$

Find:
(a) Sketch streamlines $\psi = 0$ and $\psi = 5$
(b) Velocity vector at (0, 0)
(c) Flow rate between streamlines passing through points (2, 2) and (4, 1)

Solution: We will generate the velocity field from the stream function.

Governing Equations:

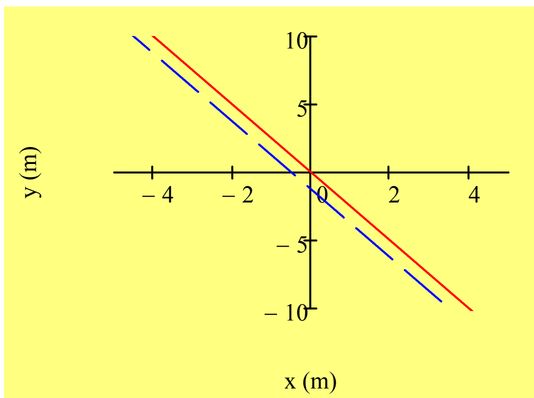
$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x} \qquad \text{(Definition of stream function)}$$

Assumptions:
Incompressible flow (ρ is constant)
Flow is only in the x-y plane

For $\psi = 0$: $0 = -5\cdot A\cdot x - 2\cdot A\cdot y$ Solving for y: $y = -\frac{5}{2}\cdot x$

For $\psi = 5$: $5 = -5\cdot A\cdot x - 2\cdot A\cdot y$ Solving for y: $y = -\frac{5}{2}\cdot x - \frac{5}{2}\cdot\frac{m^2}{s} \times \frac{s}{2\cdot m} = -\frac{5}{2}\cdot x - \frac{5}{2}\cdot m$

Here is the plot of the two streamlines: $\psi = 0$ is in red; $\psi = 5$ is in blue



Generating the velocity components from the stream function derivatives:

$u = -2\cdot A \qquad v = 5\cdot A$ Therefore, the velocity vector at (0, 0) is:

$$\vec{V} = -4\hat{i} + 10\hat{j}$$

At the point (2, 2) the stream function value is: $\psi_a = -5 \times 2\cdot\frac{m}{s} \times 2\cdot m - 2 \times 2\cdot\frac{m}{s} \times 2\cdot m \quad \psi_a = -28\frac{m^2}{s}$

At the point (4, 1) the stream function value is: $\psi_b = -5 \times 2\cdot\frac{m}{s} \times 4\cdot m - 2 \times 2\cdot\frac{m}{s} \times 1\cdot m \quad \psi_b = -44\frac{m^2}{s}$

The flow rate between these two streamlines is: $Q = \psi_b - \psi_a \qquad Q = \left(-44\cdot\frac{m^2}{s}\right) - \left(-28\cdot\frac{m^2}{s}\right) \qquad Q = -16\cdot\frac{m^3}{s\cdot m}$

Flow rate is 16 m^3/s per meter of depth

Problem *5.31

[Difficulty: 3]

*5.31 A linear velocity profile was used to model flow in a laminar incompressible boundary layer in Problem 5.10. Derive the stream function for this flow field. Locate streamlines at one-quarter and one-half the total volume flow rate in the boundary layer.

Given: Approximate profile for a laminar boundary layer:

$$u = \frac{U \cdot y}{\delta} \qquad \delta = c \cdot \sqrt{x} \qquad \text{(c is constant)}$$

Find:
(a) Stream function for the flow field
(b) Location of streamlines at one-quarter and one-half the total flow rate in the boundary layer.

Solution: We will generate the stream function from the velocity field.

Governing Equations:

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x} \qquad \text{(Definition of stream function)}$$

Integrating the x-component of velocity yields the stream function: $\psi = \int \frac{U \cdot y}{\delta} dy + f(x) = \frac{U \cdot y^2}{2 \cdot \delta} + f(x)$

If we set $\psi = 0$ at $y = 0$ then the stream function would be: $\psi = \frac{U \cdot y^2}{2 \cdot \delta}$

The total flow rate per unit depth within the boundary layer is: $Q = \psi(\delta) - \psi(0) = \frac{U \cdot \delta^2}{2 \cdot \delta} - 0 = \frac{1}{2} \cdot U \cdot \delta$

At one-quarter of the flow rate in the boundary layer: $Q = \frac{1}{4} \cdot \frac{1}{2} \cdot U \cdot \delta = \frac{1}{8} \cdot U \cdot \delta$ Therefore, the streamline would be located at:

$\frac{1}{8} \cdot U \cdot \delta = \frac{U \cdot y^2}{2 \cdot \delta}$ Solving for y: $y^2 = \frac{1}{4} \delta^2$ So at one-quarter of the flow rate: $\frac{y}{\delta} = \frac{1}{2}$

At one-half of the flow rate in the boundary layer: $Q = \frac{1}{2} \cdot \frac{1}{2} \cdot U \cdot \delta = \frac{1}{4} \cdot U \cdot \delta$ Therefore, the streamline would be located at:

$\frac{1}{4} \cdot U \cdot \delta = \frac{U \cdot y^2}{2 \cdot \delta}$ Solving for y: $y^2 = \frac{1}{2} \delta^2$ So at one-quarter of the flow rate: $\frac{y}{\delta} = \frac{1}{\sqrt{2}}$

Problem *5.32

*5.32 A parabolic velocity profile was used to model flow in a laminar incompressible boundary layer in Problem 5.11. Derive the stream function for this flow field. Locate streamlines at one-quarter and one-half the total volume flow rate in the boundary layer.

Given: Approximate profile for a laminar boundary layer:

$$\frac{u}{U} = 2\cdot\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2 \qquad \delta = c\cdot\sqrt{x} \quad \text{(c is constant)}$$

Find: (a) Stream function for the flow field
(b) Location of streamlines at one-quarter and one-half the total flow rate in the boundary layer.

Solution: We will generate the stream function from the velocity field.

Governing Equations:

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x} \qquad \text{(Definition of stream function)}$$

Integrating the x-component of velocity yields the stream function:

$$\psi = \int U\cdot\left[2\cdot\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right] dy + f(x) = U\cdot\delta\cdot\left[\left(\frac{y}{\delta}\right)^2 - \frac{1}{3}\cdot\left(\frac{y}{\delta}\right)^3\right] + f(x)$$

If we set $\psi = 0$ at $y = 0$ the stream function would be:

$$\psi = U\cdot\delta\cdot\left[\left(\frac{y}{\delta}\right)^2 - \frac{1}{3}\cdot\left(\frac{y}{\delta}\right)^3\right]$$

The total flow rate per unit depth within the boundary layer is: $Q = \psi(\delta) - \psi(0) = U\cdot\delta\cdot\left[\left(\frac{\delta}{\delta}\right)^2 - \frac{1}{3}\cdot\left(\frac{\delta}{\delta}\right)^3\right] - 0 = \frac{2}{3}\cdot U\cdot\delta$

At one-quarter of the flow rate in the boundary layer: $Q = \frac{1}{4}\cdot\frac{2}{3}\cdot U\cdot\delta = \frac{1}{6}\cdot U\cdot\delta$ Therefore, the streamline would be located at:

$$\frac{1}{6}\cdot U\cdot\delta = U\cdot\delta\cdot\left[\left(\frac{y}{\delta}\right)^2 - \frac{1}{3}\cdot\left(\frac{y}{\delta}\right)^3\right] \quad \text{or} \quad 2\cdot\left(\frac{y}{\delta}\right)^3 - 6\cdot\left(\frac{y}{\delta}\right)^2 + 1 = 0$$

We may solve this cubic for y/δ using several methods, including Goal Seek in Excel or polyroots in Mathcad. Once the roots are determined, only one root would make physical sense.

So at one-quarter of the flow rate: $\frac{y}{\delta} = 0.442$

At one-half of the flow rate in the boundary layer: $Q = \frac{1}{2}\cdot\frac{2}{3}\cdot U\cdot\delta = \frac{1}{3}\cdot U\cdot\delta$ Therefore, the streamline would be located at:

$$\frac{1}{3}\cdot U\cdot\delta = U\cdot\delta\cdot\left[\left(\frac{y}{\delta}\right)^2 - \frac{1}{3}\cdot\left(\frac{y}{\delta}\right)^3\right] \quad \text{or} \quad \left(\frac{y}{\delta}\right)^3 - 3\cdot\left(\frac{y}{\delta}\right)^2 + 1 = 0$$

We solve this cubic as we solved the previous one.

So at one-half of the flow rate: $\frac{y}{\delta} = 0.653$

Problem *5.33

[Difficulty: 3]

***5.33** Derive the stream function that represents the sinusoidal approximation used to model the x component of velocity for the boundary layer of Problem 5.12. Locate streamlines at one-quarter and one-half the total volume flow rate in the boundary layer.

Given: Approximate profile for a laminar boundary layer:

$$\frac{u}{U} = \sin\left(\frac{\pi \cdot y}{2 \cdot \delta}\right) \qquad \delta = c \cdot \sqrt{x} \quad \text{(c is constant)}$$

Find: (a) Stream function for the flow field
(b) Location of streamlines at one-quarter and one-half the total flow rate in the boundary layer.

Solution: We will generate the stream function from the velocity field.

Governing Equations:

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x} \qquad \text{(Definition of stream function)}$$

Integrating the x-component of velocity yields the stream function: $\psi = \int U \cdot \sin\left(\frac{\pi \cdot y}{2 \cdot \delta}\right) dy + f(x) = -\frac{2 \cdot U \cdot \delta}{\pi} \cdot \cos\left(\frac{\pi \cdot y}{2 \cdot \delta}\right) + f(x)$

If we set $\psi = 0$ at $y = 0$ the stream function would be: $\psi = -\frac{2 \cdot U \cdot \delta}{\pi} \cdot \cos\left(\frac{\pi \cdot y}{2 \cdot \delta}\right)$

The total flow rate per unit depth within the boundary layer is: $Q = \psi(\delta) - \psi(0) = -\frac{2 \cdot U \cdot \delta}{\pi} \cdot \left(\cos\left(\frac{\pi}{2}\right) - \cos(0)\right) = \frac{2 \cdot U \cdot \delta}{\pi}$

At one-quarter of the flow rate in the boundary layer: $Q = \frac{1}{4} \cdot \frac{2 \cdot U \cdot \delta}{\pi} = \frac{U \cdot \delta}{2 \cdot \pi}$ Therefore, the streamline would be located at:

$$\frac{U \cdot \delta}{2 \cdot \pi} = \frac{2 \cdot U \cdot \delta}{\pi} \cdot \left(1 - \cos\left(\frac{\pi \cdot y}{2 \cdot \delta}\right)\right) \quad \text{or} \quad \frac{1}{4} = 1 - \cos\left(\frac{\pi \cdot y}{2 \cdot \delta}\right) \quad \text{solving for } y/\delta\text{:} \quad \frac{y}{\delta} = \frac{2}{\pi} \cdot \text{acos}\left(\frac{3}{4}\right)$$

So at one-quarter of the flow rate: $\frac{y}{\delta} = 0.460$

At one-quarter of the flow rate in the boundary layer: $Q = \frac{1}{2} \cdot \frac{2 \cdot U \cdot \delta}{\pi} = \frac{U \cdot \delta}{\pi}$ Therefore, the streamline would be located at:

$$\frac{U \cdot \delta}{\pi} = \frac{2 \cdot U \cdot \delta}{\pi} \cdot \left(1 - \cos\left(\frac{\pi \cdot y}{2 \cdot \delta}\right)\right) \quad \text{or} \quad \frac{1}{2} = 1 - \cos\left(\frac{\pi \cdot y}{2 \cdot \delta}\right) \quad \text{solving for } y/\delta\text{:} \quad \frac{y}{\delta} = \frac{2}{\pi} \cdot \text{acos}\left(\frac{1}{2}\right)$$

So at one-half of the flow rate: $\frac{y}{\delta} = 0.667$

Problem *5.34

[Difficulty: 3]

***5.34** A cubic velocity profile was used to model flow in a laminar incompressible boundary layer in Problem 5.13. Derive the stream function for this flow field. Locate streamlines at one-quarter and one-half the total volume flow rate in the boundary layer.

Given:
Data on boundary layer

Find:
Stream function; locate streamlines at 1/4 and 1/2 of total flow rate

Solution:

$$u(x,y) = U\cdot\left[\frac{3}{2}\cdot\left(\frac{y}{\delta}\right) - \frac{1}{2}\cdot\left(\frac{y}{\delta}\right)^3\right] \qquad \text{and} \qquad \delta(x) = c\cdot\sqrt{x}$$

For the stream function $u = \frac{\partial}{\partial y}\psi = U\cdot\left[\frac{3}{2}\cdot\left(\frac{y}{\delta}\right) - \frac{1}{2}\cdot\left(\frac{y}{\delta}\right)^3\right]$

Hence $\psi = \int U\cdot\left[\frac{3}{2}\cdot\left(\frac{y}{\delta}\right) - \frac{1}{2}\cdot\left(\frac{y}{\delta}\right)^3\right] dy \qquad \psi = U\cdot\left(\frac{3}{4}\cdot\frac{y^2}{\delta} - \frac{1}{8}\cdot\frac{y^4}{\delta^3}\right) + f(x)$

Let $\psi = 0 = 0$ along $y = 0$, so $f(x) = 0$, so $\qquad \psi = U\cdot\delta\cdot\left[\frac{3}{4}\cdot\left(\frac{y}{\delta}\right)^2 - \frac{1}{8}\cdot\left(\frac{y}{\delta}\right)^4\right]$

The total flow rate in the boundary layer is

$$\frac{Q}{W} = \psi(\delta) - \psi(0) = U\cdot\delta\cdot\left(\frac{3}{4} - \frac{1}{8}\right) = \frac{5}{8}\cdot U\cdot\delta$$

At 1/4 of the total $\qquad \psi - \psi_0 = U\cdot\delta\cdot\left[\frac{3}{4}\cdot\left(\frac{y}{\delta}\right)^2 - \frac{1}{8}\cdot\left(\frac{y}{\delta}\right)^4\right] = \frac{1}{4}\cdot\left(\frac{5}{8}\cdot U\cdot\delta\right)$

$$24\cdot\left(\frac{y}{\delta}\right)^2 - 4\cdot\left(\frac{y}{\delta}\right)^4 = 5 \qquad \text{or} \qquad 4\cdot X^2 - 24\cdot X + 5 = 0 \qquad \text{where} \qquad X^2 = \frac{y}{\delta}$$

The solution to the quadratic is $\qquad X = \frac{24 - \sqrt{24^2 - 4\cdot 4\cdot 5}}{2\cdot 4} \qquad X = 0.216 \qquad$ Note that the other root is $\frac{24 + \sqrt{24^2 - 4\cdot 4\cdot 5}}{2\cdot 4} = 5.784$

Hence $\qquad \frac{y}{\delta} = \sqrt{X} = 0.465$

At 1/2 of the total flow $\qquad \psi - \psi_0 = U\cdot\delta\cdot\left[\frac{3}{4}\cdot\left(\frac{y}{\delta}\right)^2 - \frac{1}{8}\cdot\left(\frac{y}{\delta}\right)^4\right] = \frac{1}{2}\cdot\left(\frac{5}{8}\cdot U\cdot\delta\right)$

$$12\cdot\left(\frac{y}{\delta}\right)^2 - 2\cdot\left(\frac{y}{\delta}\right)^4 = 5 \qquad \text{or} \qquad 2\cdot X^2 - 12\cdot X + 5 = 0 \qquad \text{where} \qquad X^2 = \frac{y}{\delta}$$

The solution to the quadratic is $\qquad X = \frac{12 - \sqrt{12^2 - 4\cdot 2\cdot 5.}}{2\cdot 2} \qquad X = 0.450 \qquad$ Note that the other root is $\frac{12 + \sqrt{12^2 - 4\cdot 2\cdot 5}}{2\cdot 2} = 5.55$

Hence $\qquad \frac{y}{\delta} = \sqrt{X} = 0.671$

Problem *5.35

[Difficulty: 3]

*5.35 A rigid-body motion was modeled in Example 5.6 by the velocity field $\vec{V} = r\omega\hat{e}_\theta$. Find the stream function for this flow. Evaluate the volume flow rate per unit depth between $r_1 = 0.10$ m and $r_2 = 0.12$ m, if $\omega = 0.5$ rad/s. Sketch the velocity profile along a line of constant θ. Check the flow rate calculated from the stream function by integrating the velocity profile along this line.

Given: Rigid body motion in Example Problem 5.6

$$\vec{V} = r\omega\hat{e}_\theta \qquad \omega = 0.5\cdot\frac{\text{rad}}{\text{s}}$$

Find:
(a) Stream function for the flow field
(b) Volume flow rate per unit depth between r = 0.10 m and 0.12 m
(c) Sketch velocity profiles along a line of constant θ
(d) Check the volume flow rate calculated from the stream function by integrating the velocity profile along this line.

Solution: We will generate the stream function from the velocity field.

Governing Equations:

$$V_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} \qquad V_\theta = -\frac{\partial\psi}{\partial r}$$

(Definition of stream function)

Integrating the θ-component of velocity yields the stream function: $\psi = -\int r\cdot\omega\,dr + f(\theta) = -\frac{\omega\cdot r^2}{2} + f(\theta)$

Now take the derivative of the stream function: $V_r = \frac{1}{r}\cdot\frac{df}{d\theta} = 0$ Therefore, $f(\theta) = C$

$$\psi = -\frac{\omega\cdot r^2}{2} + C$$

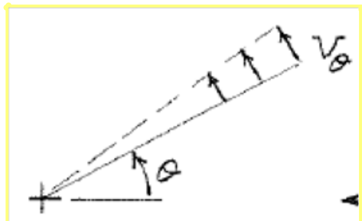
The volume flow rate per unit depth is: $Q = \psi(r_2) - \psi(r_1) = \left(-\frac{\omega\cdot r_2^{\,2}}{2} + C\right) - \left(-\frac{\omega\cdot r_1^{\,2}}{2} + C\right) = \frac{\omega}{2}\cdot\left(r_1^{\,2} - r_2^{\,2}\right)$

Substituting in known values: $Q = \frac{1}{2} \times 0.5\cdot\frac{\text{rad}}{\text{s}} \times \left(0.10^2 - 0.12^2\right)\cdot\text{m}^2$

$$Q = -0.001100\cdot\frac{\text{m}^3}{\text{s}\cdot\text{m}}$$

Because Q<0, the flow is in the direction of e_θ

Along a line of constant θ, the velocity varies linearly:



From the linear velocity variation, $V_\theta = \omega\cdot r$ Thus the flow rate is:

$$Q = \int_{r_1}^{r_2} V_\theta\,dr = \omega\cdot\int_{r_1}^{r_2} r\,dr = \frac{\omega}{2}\cdot\left(r_2^{\,2} - r_1^{\,2}\right)$$

Substituting known values: $Q = \frac{1}{2} \times 0.5\cdot\frac{\text{rad}}{\text{s}} \times \left(0.12^2 - 0.10^2\right)\cdot\text{m}^2$

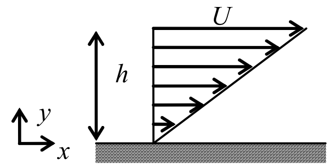
$$Q = 0.001100\cdot\frac{\text{m}^3}{\text{s}\cdot\text{m}}$$

These two expressions are the same with the exception of the sign.

Problem *5.36

[Difficulty: 3]

***5.36** In a parallel one-dimensional flow in the positive x direction, the velocity varies linearly from zero at $y = 0$ to 30 m/s at $y = 1.5$ m. Determine an expression for the stream function, ψ. Also determine the y coordinate above which the volume flow rate is half the total between $y = 0$ and $y = 1.5$ m.



Given:
Linear velocity profile

Find:
Stream function ψ; y coordinate for half of flow

Solution:

Basic equations: $u = \frac{\partial}{\partial y}\psi \qquad v = -\frac{\partial}{\partial x}\psi$ and we have $u = U\cdot\left(\frac{y}{h}\right) \qquad v = 0$

Assumption: Incompressible flow; flow in x-y plane

Check for incompressible $\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$

$$\frac{\partial}{\partial x}\left(U\cdot\frac{y}{h}\right) \rightarrow 0 \qquad \frac{\partial}{\partial y}0 \rightarrow 0$$

Hence $\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$ Flow is INCOMPRESSIBLE

Hence $u = U\cdot\frac{y}{h} = \frac{\partial}{\partial y}\psi \qquad \psi(x,y) = \int U\cdot\frac{y}{h}\,dy = \frac{U\cdot y^2}{2\cdot h} + f(x)$

and $v = 0 = -\frac{\partial}{\partial x}\psi \qquad \psi(x,y) = -\int 0\,dx = g(y)$

Comparing these $f(x) = 0$ and $g(y) = \frac{U\cdot y^2}{2\cdot h}$

The stream function is $\psi(x,y) = \frac{U\cdot y^2}{2\cdot h}$

For the flow (0 < y < h) $Q = \int_0^h u\,dy = \frac{U}{h}\cdot\int_0^h y\,dy = \frac{U\cdot h}{2}$

For half the flow rate $\frac{Q}{2} = \int_0^{h_{half}} u\,dy = \frac{U}{h}\cdot\int_0^{h_{half}} y\,dy = \frac{U\cdot h_{half}^2}{2\cdot h} = \frac{1}{2}\cdot\left(\frac{U\cdot h}{2}\right) = \frac{U\cdot h}{4}$

Hence $h_{half}^2 = \frac{1}{2}\cdot h^2 \qquad h_{half} = \frac{1}{\sqrt{2}}\cdot h = \frac{1.5\cdot m}{\sqrt{2}} = 1.06\cdot m$

Problem *5.37

[Difficulty: 3]

***5.37** Example 5.6 showed that the velocity field for a free vortex in the $r\theta$ plane is $\vec{V} = \hat{e}_\theta C/r$. Find the stream function for this flow. Evaluate the volume flow rate per unit depth between $r_1 = 0.20$ m and $r_2 = 0.24$ m, if $C = 0.3\ \text{m}^2/\text{s}$. Sketch the velocity profile along a line of constant θ. Check the flow rate calculated from the stream function by integrating the velocity profile along this line.

Given: Rigid body motion in Example Problem 5.6

$$\vec{V} = \frac{C}{r}\hat{e}_\theta \qquad C = 0.3\cdot\frac{\text{m}^2}{\text{s}}$$

Find:
(a) Stream function for the flow field
(b) Volume flow rate per unit depth between r = 0.20 m and 0.24 m
(c) Sketch velocity profiles along a line of constant θ
(d) Check the volume flow rate calculated from the stream function by integrating the velocity profile along this line.

Solution: We will generate the stream function from the velocity field.

Governing Equations:

$$V_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} \qquad V_\theta = -\frac{\partial\psi}{\partial r} \qquad \text{(Definition of stream function)}$$

Assumptions: Incompressible flow
Flow is in the r-θ plane only

Integrating the θ-component of velocity yields the stream function: $\psi = -\int \frac{C}{r}\,dr + f(\theta) = -C\cdot\ln(r) + f(\theta)$

Now take the derivative of the stream function: $V_r = \frac{1}{r}\cdot\frac{df}{d\theta} = 0$ Therefore, $f(\theta) = C_1$

$$\psi = -C\cdot\ln(r) + C_1$$

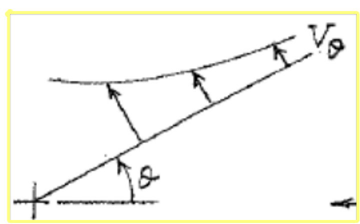
The volume flow rate per unit depth is: $Q = \psi(r_2) - \psi(r_1) = \left(-C\cdot\ln(r_2) + C_1\right) - \left(-C\cdot\ln(r_1) + C_1\right) = C\cdot\ln\left(\frac{r_1}{r_2}\right)$

Substituting in known values: $Q = 0.3\cdot\frac{\text{m}^2}{\text{s}} \times \ln\left(\frac{0.20}{0.24}\right)$

$$Q = -0.0547\cdot\frac{\text{m}^3}{\text{s}\cdot\text{m}}$$

Because Q<0, the flow is in the direction of e_θ

Along a line of constant θ, the velocity varies inversely with r:



From the velocity profile, $V_\theta = \frac{C}{r}$ Thus the flow rate is:

$$Q = \int_{r_1}^{r_2} V_\theta\,dr = \int_{r_1}^{r_2} \frac{C}{r}\,dr = C\cdot\ln\left(\frac{r_2}{r_1}\right)$$ Substituting known values: $Q = 0.3\cdot\frac{\text{m}^2}{\text{s}} \times \ln\left(\frac{0.24}{0.20}\right)$

$$Q = 0.0547\cdot\frac{\text{m}^3}{\text{s}\cdot\text{m}}$$

These two expressions are the same with the exception of the sign.

Problem 5.38

[Difficulty: 2]

5.38 Consider the flow field given by $\vec{V} = xy^2\hat{i} - \frac{1}{3}y^3\hat{j} + xy\hat{k}$. Determine (a) the number of dimensions of the flow, (b) if it is a possible incompressible flow, and (c) the acceleration of a fluid particle at point $(x, y, z) = (1, 2, 3)$.

Given: The velocity field provided above

Find: (a) the number of dimensions of the flow
(b) if this describes a possible incompressible flow
(c) the acceleration of a fluid particle at point (1,2,3)

Solution: We will check this flow field against the continuity equation, and then apply the definition of acceleration

Governing Equations:

$$\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)+\frac{\partial \rho}{\partial t}=0 \quad \text{(Continuity equation)}$$

$$\vec{a}_p=\frac{D\vec{V}}{Dt}=u\frac{\partial \vec{V}}{\partial x}+v\frac{\partial \vec{V}}{\partial y}+w\frac{\partial \vec{V}}{\partial z}+\frac{\partial \vec{V}}{\partial t} \quad \text{(Particle acceleration)}$$

Assumptions:
(1) Incompressible flow (ρ is constant)
(2) Two dimensional flow (velocity is not a function of z)
(3) Steady flow (velocity is not a function of t)

Based on assumption (2), we may state that: The flow is two dimensional.

Based on assumptions (1) and (3), the continuity equation reduces to: $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$

This is the criterion against which we will check the flow field.

$u = x\cdot y^2$

$v = -\frac{1}{3}\cdot y^3$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=y^2-y^2=0$$

This could be an incompressible flow field.

Based on assumptions (2) and (3), the acceleration reduces to: $\vec{a}_p=u\frac{\partial \vec{V}}{\partial x}+v\frac{\partial \vec{V}}{\partial y}$ and the partial derivatives of velocity are:

$\frac{\partial \vec{V}}{\partial x}=y^2\hat{i}+y\hat{k}$ and $\frac{\partial \vec{V}}{\partial y}=2xy\hat{i}-y^2\hat{j}+x\hat{k}$ Therefore the acceleration vector is equal to:

$$\vec{a}_p=xy^2\left(y^2\hat{i}+y\hat{k}\right)-\frac{1}{3}y^3\left(2xy\hat{i}-y^2\hat{j}+x\hat{k}\right)=\frac{1}{3}xy^4\hat{i}+\frac{1}{3}y^5\hat{j}+\frac{2}{3}xy^3\hat{k}$$

At point (1,2,3), the acceleration is:

$$\vec{a}_p=\left(\frac{1}{3}\times 1\times 2^4\right)\hat{i}+\left(\frac{1}{3}\times 2^5\right)\hat{j}+\left(\frac{2}{3}\times 1\times 2^3\right)\hat{k}=\frac{16}{3}\hat{i}+\frac{32}{3}\hat{j}+\frac{16}{3}\hat{k}$$

$$\vec{a}_p=\frac{16}{3}\hat{i}+\frac{32}{3}\hat{j}+\frac{16}{3}\hat{k}$$

Problem 5.39

[Difficulty: 3]

5.39 Consider the velocity field $\vec{V} = A(x^4 - 6x^2y^2 + y^4)\hat{i} + A(4xy^3 - 4x^3y)\hat{j}$ in the xy plane, where $A = 0.25\ \text{m}^{-3}\cdot\text{s}^{-1}$, and the coordinates are measured in meters. Is this a possible incompressible flow field? Calculate the acceleration of a fluid particle at point $(x, y) = (2, 1)$.

Given: Velocity field

Find: Whether flow is incompressible; Acceleration of particle at (2,1)

Solution:

Basic equations:

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$$

$$\vec{a}_p = \underbrace{\frac{D\vec{V}}{Dt}}_{\substack{\text{total}\\\text{acceleration}\\\text{of a particle}}} = \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\substack{\text{convective}\\\text{acceleration}}} + \underbrace{\frac{\partial \vec{V}}{\partial t}}_{\substack{\text{local}\\\text{acceleration}}}$$

$$u(x,y) = A\cdot\left(x^4 - 6\cdot x^2\cdot y^2 + y^4\right) \qquad v(x,y) = A\cdot\left(4\cdot x\cdot y^3 - 4\cdot x^3\cdot y\right)$$

For incompressible flow

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$$

Checking

$$\frac{\partial}{\partial x}\left[A\cdot\left(x^4 - 6\cdot x^2\cdot y^2 + y^4\right)\right] \rightarrow A\cdot\left(4\cdot x^3 - 12\cdot x\cdot y^2\right) \qquad \frac{\partial}{\partial y}\left[A\cdot\left(4\cdot x\cdot y^3 - 4\cdot x^3\cdot y\right)\right] \rightarrow -A\cdot\left(4\cdot x^3 - 12\cdot x\cdot y^2\right)$$

Hence

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$$

For this flow

$$a_x = u\cdot\frac{\partial}{\partial x}u + v\cdot\frac{\partial}{\partial y}u$$

$$a_x = A\cdot\left(x^4 - 6\cdot x^2\cdot y^2 + y^4\right)\cdot\frac{\partial}{\partial x}\left[A\cdot\left(x^4 - 6\cdot x^2\cdot y^2 + y^4\right)\right] + A\cdot\left(4\cdot x\cdot y^3 - 4\cdot x^3\cdot y\right)\cdot\frac{\partial}{\partial y}\left[A\cdot\left(x^4 - 6\cdot x^2\cdot y^2 + y^4\right)\right]$$

$$a_x = 4\cdot A^2\cdot x\cdot\left(x^2 + y^2\right)^3$$

$$a_y = u\cdot\frac{\partial}{\partial x}v + v\cdot\frac{\partial}{\partial y}v$$

$$a_y = A\cdot\left(x^4 - 6\cdot x^2\cdot y^2 + y^4\right)\cdot\frac{\partial}{\partial x}\left[A\cdot\left(4\cdot x\cdot y^3 - 4\cdot x^3\cdot y\right)\right] + A\cdot\left(4\cdot x\cdot y^3 - 4\cdot x^3\cdot y\right)\cdot\frac{\partial}{\partial y}\left[A\cdot\left(4\cdot x\cdot y^3 - 4\cdot x^3\cdot y\right)\right]$$

$$a_y = 4\cdot A^2\cdot y\cdot\left(x^2 + y^2\right)^3$$

Hence at (2,1)

$$a_x = 4\times\left(\frac{1}{4}\cdot\frac{1}{m^3\cdot s}\right)^2\times 2\cdot m\times\left[(2\cdot m)^2 + (1\cdot m)^2\right]^3 \qquad a_x = 62.5\,\frac{m}{s^2}$$

$$a_y = 4\times\left(\frac{1}{4}\cdot\frac{1}{m^3\cdot s}\right)^2\times 1\cdot m\times\left[(2\cdot m)^2 + (1\cdot m)^2\right]^3 \qquad a_y = 31.3\,\frac{m}{s^2} \qquad a = \sqrt{a_x^2 + a_y^2} \qquad a = 69.9\,\frac{m}{s^2}$$

Problem 5.40

[Difficulty: 2]

5.40 Consider the flow field given by $\vec{V} = ax^2y\hat{i} - by\hat{j} + cz^2\hat{k}$, where $a = 2\ \text{m}^{-2}\cdot\text{s}^{-1}$, $b = 2\ \text{s}^{-1}$, and $c = 1\ \text{m}^{-1}\cdot\text{s}^{-1}$. Determine (a) the number of dimensions of the flow, (b) if it is a possible incompressible flow, and (c) the acceleration of a fluid particle at point $(x, y, z) = (2, 1, 3)$.

Given: The velocity field provided above

Find: (a) the number of dimensions of the flow
(b) if this describes a possible incompressible flow
(c) the acceleration of a fluid particle at point (2,1,3)

Solution: We will check this flow field against the continuity equation, and then apply the definition of acceleration

Governing Equations:

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) + \frac{\partial \rho}{\partial t} = 0 \quad \text{(Continuity equation)}$$

$$\vec{a}_p = \frac{D\vec{V}}{Dt} = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} + \frac{\partial \vec{V}}{\partial t} \quad \text{(Particle acceleration)}$$

Assumptions: (1) Incompressible flow (ρ is constant)
(2) Steady flow (velocity is not a function of t)

Since the velocity is a function of x, y, and z, we may state that: The flow is three dimensional.

Based on assumptions (1) and (2), the continuity equation reduces to: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$

This is the criterion against which we will check the flow field.

$u = a\cdot x^2\cdot y$

$v = -b\cdot y$

$w = c\cdot z^2$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 2axy - b + 2cz \neq 0$$

This can not be incompressible.

Based on assumption (2), the acceleration reduces to: $\vec{a}_p = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}$ and the partial derivatives of velocity are:

$\frac{\partial \vec{V}}{\partial x} = 2axy\hat{i}$ $\quad \frac{\partial \vec{V}}{\partial y} = ax^2\hat{i} - b\hat{j}$ and $\frac{\partial \vec{V}}{\partial z} = 2cz\hat{k}$ Therefore the acceleration vector is equal to:

$$\vec{a}_p = ax^2y(2axy\hat{i}) - by(ax^2\hat{i} - b\hat{j}) + cz^2(2cz\hat{k}) = (2a^2x^3y^2 - abx^2y)\hat{i} + (b^2y)\hat{j} + (2c^2z^3)\hat{k}$$

At point (2,1,3):

$$\vec{a}_p = \left[2\times\left(\frac{2}{\text{m}^2\cdot\text{s}}\right)^2\times(2\ \text{m})^3\times(1\ \text{m})^2 - \frac{2}{\text{m}^2\cdot\text{s}}\times\frac{2}{\text{s}}\times(2\ \text{m})^2\times 1\ \text{m}\right]\hat{i} + \left[\left(\frac{2}{s}\right)^2\times 1\ \text{m}\right]\hat{j} + \left[2\times\left(\frac{1}{\text{m}\cdot\text{s}}\right)^2\times(3\ \text{m})^3\right]\hat{k}$$

$$= 48\hat{i} + 4\hat{j} + 54\hat{k}\ \frac{\text{m}}{\text{s}^2}$$

$$\vec{a}_p = 48\hat{i} + 4\hat{j} + 54\hat{k}\ \frac{\text{m}}{\text{s}^2}$$

Problem 5.41

[Difficulty: 3]

5.41 The x component of velocity in a steady, incompressible flow field in the xy plane is $u = A(x^5 - 10x^3y^2 + 5xy^4)$, where $A = 2\ \text{m}^{-4}\cdot\text{s}^{-1}$ and x is measured in meters. Find the simplest y component of velocity for this flow field. Evaluate the acceleration of a fluid particle at point $(x, y) = (1, 3)$.

Given: x component of velocity field

Find: Simplest y component for incompressible flow; Acceleration of particle at (1,3)

Solution:

Basic equations
$$u = \frac{\partial}{\partial y}\psi \qquad v = -\frac{\partial}{\partial x}\psi \qquad \underbrace{\vec{a}_p = \frac{D\vec{V}}{Dt}}_{\substack{\text{total}\\\text{acceleration}\\\text{of a particle}}} = \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\substack{\text{convective}\\\text{acceleration}}} + \underbrace{\frac{\partial \vec{V}}{\partial t}}_{\substack{\text{local}\\\text{acceleration}}}$$

We are given
$$u(x,y) = A\cdot\left(x^5 - 10\cdot x^3\cdot y^2 + 5\cdot x\cdot y^4\right)$$

Hence for incompressible flow
$$\psi(x,y) = \int u\, dy = \int A\cdot\left(x^5 - 10\cdot x^3\cdot y^2 + 5\cdot x\cdot y^4\right) dy = A\cdot\left(x^5\cdot y - \frac{10}{3}\cdot x^3\cdot y^3 + x\cdot y^5\right) + f(x)$$

$$v(x,y) = -\frac{\partial}{\partial x}\psi(x_y) = -\frac{\partial}{\partial x}\left[A\cdot\left(x^5\cdot y - \frac{10}{3}\cdot x^3\cdot y^3 + x\cdot y^5\right) + f(x)\right] = -A\cdot\left(5\cdot x^4\cdot y - 10\cdot x^2\cdot y^3 + y^5\right) + F(x)$$

Hence
$$v(x,y) = -A\cdot\left(5\cdot x^4\cdot y - 10\cdot x^2\cdot y^3 + y^5\right) + F(x)$$
where F(x) is an arbitrary function of x

The simplest is
$$v(x,y) = -A\cdot\left(5\cdot x^4\cdot y - 10\cdot x^2\cdot y^3 + y^5\right)$$

For this flow
$$a_x = u\cdot\frac{\partial}{\partial x}u + v\cdot\frac{\partial}{\partial y}u$$

$$a_x = A\cdot\left(x^5 - 10\cdot x^3\cdot y^2 + 5\cdot x\cdot y^4\right)\cdot\frac{\partial}{\partial x}\left[A\cdot\left(x^5 - 10\cdot x^3\cdot y^2 + 5\cdot x\cdot y^4\right)\right] - A\cdot\left(5\cdot x^4\cdot y - 10\cdot x^2\cdot y^3 + y^5\right)\cdot\frac{\partial}{\partial y}\left[A\cdot\left(x^5 - 10\cdot x^3\cdot y^2 + 5\cdot x\cdot y^4\right)\right]$$

$$a_x = 5\cdot A^2\cdot x\cdot\left(x^2 + y^2\right)^4$$

$$a_y = u\cdot\frac{\partial}{\partial x}v + v\cdot\frac{\partial}{\partial y}v$$

$$a_y = A\cdot\left(x^5 - 10\cdot x^3\cdot y^2 + 5\cdot x\cdot y^4\right)\cdot\frac{\partial}{\partial x}\left[-A\cdot\left(5\cdot x^4\cdot y - 10\cdot x^2\cdot y^3 + y^5\right)\right] - A\cdot\left(5\cdot x^4\cdot y - 10\cdot x^2\cdot y^3 + y^5\right)\cdot\frac{\partial}{\partial y}\left[-A\cdot\left(5\cdot x^4\cdot y - 10\cdot x^2\cdot y^3 + y^5\right)\right]$$

$$a_y = 5\cdot A^2\cdot y\cdot\left(x^2 + y^2\right)^4$$

Hence at (1,3)
$$a_x = 5\times\left(\frac{1}{2}\cdot\frac{1}{m^4\cdot s}\right)^2\times 1\cdot m\times\left[(1\cdot m)^2 + (3\cdot m)^2\right]^4 \qquad a_x = 1.25\times 10^4\,\frac{m}{s^2}$$

$$a_y = 5\times\left(\frac{1}{2}\cdot\frac{1}{m^4\cdot s}\right)^2\times 3\cdot m\times\left[(1\cdot m)^2 + (3\cdot m)^2\right]^4 \qquad a_y = 3.75\times 10^4\,\frac{m}{s^2} \qquad a = \sqrt{a_x^2 + a_y^2} \qquad a = 3.95\times 10^4\,\frac{m}{s^2}$$

Problem 5.42

[Difficulty: 2]

5.42 The velocity field within a laminar boundary layer is approximated by the expression

$$\vec{V} = \frac{AUy}{x^{1/2}}\hat{i} + \frac{AUy^2}{4x^{3/2}}\hat{j}$$

In this expression, $A = 141\ \mathrm{m}^{-1/2}$, and $U = 0.240$ m/s is the freestream velocity. Show that this velocity field represents a possible incompressible flow. Calculate the acceleration of a fluid particle at point $(x, y) = (0.5\ \mathrm{m}, 5\ \mathrm{mm})$. Determine the slope of the streamline through the point.

Given: The velocity field provided above

Find:
(a) if this describes a possible incompressible flow
(b) the acceleration of a fluid particle at point (x,y) = (0.5 m, 5 mm)
(c) the slope of the streamline through that point

Solution: We will check this flow field against the continuity equation, and then apply the definition of acceleration

Governing Equations:

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) + \frac{\partial \rho}{\partial t} = 0 \quad \text{(Continuity equation)}$$

$$\vec{a}_p = \frac{D\vec{V}}{Dt} = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} + \frac{\partial \vec{V}}{\partial t} \quad \text{(Particle acceleration)}$$

Assumptions:
(1) Incompressible flow (ρ is constant)
(2) Two-dimensional flow (velocity is not a function of z)
(3) Steady flow (velocity is not a function of t)

Based on the assumptions above, the continuity equation reduces to: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ This is the criterion against which we will check the flow field.

$u = \frac{A \cdot U \cdot y}{x^{\frac{1}{2}}}$ $\quad v = \frac{A \cdot U \cdot y^2}{4 \cdot x^{\frac{3}{2}}}$ $\quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{1}{2}\frac{AUy}{x^{\frac{3}{2}}} + 2\frac{AUy}{4x^{\frac{3}{2}}} = 0$ This represents a possible incompressible flow field.

Based on assumptions (2) and (3), the acceleration reduces to: $\vec{a}_p = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y}$ and the partial derivatives of velocity are:

$\frac{\partial \vec{V}}{\partial x} = -\frac{AUy}{2x^{3/2}}\hat{i} - \frac{3AUy^2}{8x^{5/2}}\hat{j}$ and $\frac{\partial \vec{V}}{\partial y} = \frac{AU}{x^{1/2}}\hat{i} + \frac{AUy}{2x^{3/2}}\hat{j}$ Therefore the acceleration vector is equal to:

$$\vec{a}_p = \frac{AUy}{x^{1/2}}\left(-\frac{AUy}{2x^{3/2}}\hat{i} - \frac{3AUy^2}{8x^{5/2}}\hat{j}\right) + \frac{AUy^2}{4x^{3/2}}\left(\frac{AU}{x^{1/2}}\hat{i} + \frac{AUy}{2x^{3/2}}\hat{j}\right) = -\frac{A^2U^2y^2}{4x^2}\hat{i} - \frac{A^2U^2y^3}{4x^3}\hat{j}$$

At (5 m, 5 mm):

$$\vec{a}_p = -\left[\frac{1}{4}\times\left(\frac{141}{\mathrm{m}^{1/2}}\right)^2 \times \left(0.240\frac{\mathrm{m}}{\mathrm{s}}\right)^2 \times \left(\frac{0.005}{0.5}\right)^2\right]\hat{i} - \left[\frac{1}{4}\times\left(\frac{141}{\mathrm{m}^{1/2}}\right)^2 \times \left(0.240\frac{\mathrm{m}}{\mathrm{s}}\right)^2 \times \left(\frac{0.005}{0.5}\right)^3\right]\hat{j}$$

$$\vec{a}_p = -2.86\left(10^{-2}\,\hat{i} + 10^{-4}\,\hat{j}\right)\frac{\text{m}}{\text{s}^2}$$

The slope of the streamline is given by: $\quad \text{slope} = \dfrac{v}{u} = \dfrac{A \cdot U \cdot y^2}{4 \cdot x^{\frac{3}{2}}} \cdot \dfrac{x^{\frac{1}{2}}}{A \cdot U \cdot y} = \dfrac{y}{4 \cdot x}$

Therefore, $\quad \text{slope} = \dfrac{0.005}{4 \times 0.5}$

$$\text{slope} = 2.50 \times 10^{-3}$$

Problem 5.43

[Difficulty: 2]

5.43 Wave flow of an incompressible fluid into a solid surface follows a sinusoidal pattern. Flow is two-dimensional with the x axis normal to the surface and y axis along the wall. The x component of the flow follows the pattern

$$u = Ax\sin\left(\frac{2\pi t}{T}\right)$$

Determine the y component of flow (v) and the convective and local components of the acceleration vector.

Given: X component of a 2-dimensional transient flow.

Find: Y component of flow and total acceleration.

Solution:

Governing Equations:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \qquad \text{(Continuity Equation for an Incompressible Fluid)}$$

$$\vec{a}_p=\frac{D\vec{V}}{Dt}=u\frac{\partial \vec{V}}{\partial x}+v\frac{\partial \vec{V}}{\partial y}+w\frac{\partial \vec{V}}{\partial z}+\frac{\partial \vec{V}}{\partial t} \qquad \text{(Material Derivative)}$$

Assumptions: Incompressible fluid
No motion along the wall ($x = 0$) limited to two dimensions ($w = 0$).

The given or available data is: $u = Ax\cdot\sin\left(\frac{2\pi t}{T}\right)$ $\qquad w = 0$

Simplify the continuity equation to find v: $\frac{\partial v}{\partial y}=-\frac{\partial u}{\partial x}=-A\cdot\sin\left(\frac{2\pi t}{T}\right)$

Integrate: $v = Ay\cdot\sin\left(\frac{2\pi t}{T}\right)+C$

Use the boundary condition of no flow at the origin to solve for the constant of integration

$$v = Ay\cdot\sin\left(\frac{2\pi t}{T}\right)$$

Give the velocity in vector form: $\vec{V}=A\cdot\sin\left(\frac{2\pi t}{T}\right)\times\left(x\hat{i}+y\hat{j}\right)$

Use the material derivative to find the acceleration. Start with the convective terms. $\vec{a}_{p,conv}=u\frac{\partial \vec{V}}{\partial x}+v\frac{\partial \vec{V}}{\partial y}$

$$\vec{a}_{p,conv}=u\frac{\partial \vec{V}}{\partial x}+v\frac{\partial \vec{V}}{\partial y}=Ax\cdot\sin\left(\frac{2\pi t}{T}\right)\times A\cdot\sin\left(\frac{2\pi t}{T}\right)\hat{i}+Ay\cdot\sin\left(\frac{2\pi t}{T}\right)\times A\cdot\sin\left(\frac{2\pi t}{T}\right)\hat{j}$$

$$=A^2\cdot\sin^2\left(\frac{2\pi t}{T}\right)\times\left(x\hat{i}+y\hat{j}\right)$$

$$\vec{a}_{p,conv}=A^2\cdot\sin^2\left(\frac{2\pi t}{T}\right)\times\left(x\hat{i}+y\hat{j}\right)$$

Finish the local term: $\vec{a}_{p,local}=\frac{\partial \vec{V}}{\partial t}=A\cdot\sin\left(\frac{2\pi t}{T}\right)\times\left(x\hat{i}+y\hat{j}\right)=\frac{2\pi}{T}A\cdot\cos\left(\frac{2\pi t}{T}\right)$

$$\vec{a}_{p,local}=\frac{2\pi A}{T}\cos\left(\frac{2\pi t}{T}\right)$$

Problem 5.44

[Difficulty: 2]

5.44 The y component of velocity in a two-dimensional, incompressible flow field is given by $v = -Axy$, where v is in m/s, x and y are in meters, and A is a dimensional constant. There is no velocity component or variation in the z direction. Determine the dimensions of the constant, A. Find the simplest x component of velocity in this flow field. Calculate the acceleration of a fluid particle at point $(x, y) = (1, 2)$.

Given: The 2-dimensional, incompressible velocity field provided above

Find:
(a) dimensions of the constant A
(b) simplest x-component of the velocity
(c) acceleration of a particle at (1,2)

Solution: We will check the dimensions against the function definition, check the flow field against the continuity equation, and then apply the definition of acceleration.

Governing Equations:

$$\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)+\frac{\partial \rho}{\partial t}=0 \quad \text{(Continuity equation)}$$

$$\vec{a}_p=\frac{D\vec{V}}{Dt}=u\frac{\partial \vec{V}}{\partial x}+v\frac{\partial \vec{V}}{\partial y}+w\frac{\partial \vec{V}}{\partial z}+\frac{\partial \vec{V}}{\partial t} \quad \text{(Particle acceleration)}$$

Assumptions:
(1) Incompressible flow (ρ is constant)
(2) Two-dimensional flow (velocity is not a function of z)
(3) Steady flow (velocity is not a function of t)

Since $v = -A\cdot x\cdot y$ it follows that $A = -\dfrac{v}{x\cdot y}$ and the dimensions of A are given by: $[A]=\left[\dfrac{v}{xy}\right]=\dfrac{L}{t}\cdot\dfrac{1}{L}\cdot\dfrac{1}{L}$

$$[A]=\frac{1}{Lt}$$

Based on the assumptions above, the continuity equation reduces to: $\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}=0$ Therefore: $\dfrac{\partial v}{\partial y}=-Ax=-\dfrac{\partial u}{\partial x}$

Integrating with respect to x will yield the x-component of velocity: $u=\displaystyle\int A\cdot x\,dx+f(y)=\frac{1}{2}\cdot A\cdot x^2+f(y)$

The simplest x-component of velocity is obtained for f(y) = 0:

$$u=\frac{1}{2}\cdot A\cdot x^2$$

Based on assumptions (2) and (3), the acceleration reduces to: $\vec{a}_p=u\dfrac{\partial \vec{V}}{\partial x}+v\dfrac{\partial \vec{V}}{\partial y}$ and the partial derivatives of velocity are:

$\dfrac{\partial \vec{V}}{\partial x}=Ax\hat{i}-Ay\hat{j}$ and $\dfrac{\partial \vec{V}}{\partial y}=-Ax\hat{j}$ Therefore the acceleration vector is equal to:

$$\vec{a}_p=\frac{1}{2}Ax^2\left(Ax\hat{i}-Ay\hat{j}\right)-Axy\left(-Ax\hat{j}\right)=\frac{1}{2}A^2x^3\hat{i}+\frac{1}{2}A^2x^2y\hat{j}$$

At (1 , 2):

$$\vec{a}_p=\left(\frac{1}{2}\times A^2\times 1^3\right)\hat{i}+\left(\frac{1}{2}\times A^2\times 1^2\times 2\right)\hat{j}$$

$$\vec{a}_p=A^2\left(\frac{1}{2}\hat{i}+\hat{j}\right)$$

Problem 5.45 [Difficulty: 2]

5.45 Consider the velocity field $\vec{V} = Ax/(x^2+y^2)\hat{i} + Ay/(x^2+y^2)\hat{j}$ in the xy plane, where $A = 10\ m^2/s$, and x and y are measured in meters. Is this an incompressible flow field? Derive an expression for the fluid acceleration. Evaluate the velocity and acceleration along the x axis, the y axis, and along a line defined by $y = x$. What can you conclude about this flow field?

Given: Velocity field

Find: Whether flow is incompressible; expression for acceleration; evaluate acceleration along axes and along $y = x$

Solution:

The given data is $\quad A = 10\cdot\frac{m^2}{s} \qquad u(x,y) = \frac{A\cdot x}{x^2+y^2} \qquad v(x,y) = \frac{A\cdot y}{x^2+y^2}$

For incompressible flow $\quad \frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$

Hence, checking $\quad \frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = -A\cdot\frac{\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2} + A\cdot\frac{\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2} = 0 \qquad$ Incompressible flow

The acceleration is given by

$$\vec{a}_p = \underset{\substack{\text{total}\\\text{acceleration}\\\text{of a particle}}}{\frac{D\vec{V}}{Dt}} = \underbrace{u\frac{\partial\vec{V}}{\partial x} + v\frac{\partial\vec{V}}{\partial y} + w\frac{\partial\vec{V}}{\partial z}}_{\substack{\text{convective}\\\text{acceleration}}} + \underset{\substack{\text{local}\\\text{acceleration}}}{\frac{\partial\vec{V}}{\partial t}}$$

For the present steady, 2D flow

$$a_x = u\cdot\frac{du}{dx} + v\cdot\frac{du}{dy} = \frac{A\cdot x}{x^2+y^2}\cdot\left[-\frac{A\cdot\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2}\right] + \frac{A\cdot y}{x^2+y^2}\cdot\left[-\frac{2\cdot A\cdot x\cdot y}{\left(x^2+y^2\right)^2}\right] \qquad a_x = -\frac{A^2\cdot x}{\left(x^2+y^2\right)^2}$$

$$a_y = u\cdot\frac{dv}{dx} + v\cdot\frac{dv}{dy} = \frac{A\cdot x}{x^2+y^2}\cdot\left[-\frac{2\cdot A\cdot x\cdot y}{\left(x^2+y^2\right)^2}\right] + \frac{A\cdot y}{x^2+y^2}\cdot\left[\frac{A\cdot\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2}\right] \qquad a_y = -\frac{A^2\cdot y}{\left(x^2+y^2\right)^2}$$

Along the x axis $\quad a_x = -\frac{A^2}{x^3} = -\frac{100}{x^3} \qquad a_y = 0$

Along the y axis $\quad a_x = 0 \qquad a_y = -\frac{A^2}{y^3} = -\frac{100}{y^3}$

Along the line $x = y$ $\quad a_x = -\frac{A^2\cdot x}{r^4} = -\frac{100\cdot x}{r^4} \qquad a_y = -\frac{A^2\cdot y}{r^4} = -\frac{100\cdot y}{r^4}$

where $\quad r = \sqrt{x^2+y^2}$

For this last case the acceleration along the line $x = y$ is $\quad a = \sqrt{{a_x}^2+{a_y}^2} = -\frac{A^2}{r^4}\cdot\sqrt{x^2+y^2} = -\frac{A^2}{r^3} = -\frac{100}{r^3} \quad a = -\frac{A^2}{r^3} = -\frac{100}{r^3}$

In each case the acceleration vector points towards the origin, proportional to 1/distance^3, so the flow field is a radial decelerating flow.

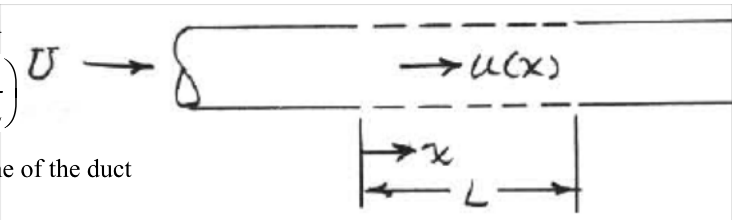
Problem 5.46

[Difficulty: 2]

5.46 An incompressible liquid with negligible viscosity flows steadily through a horizontal pipe of constant diameter. In a porous section of length $L = 0.3$ m, liquid is removed at a constant rate per unit length, so the uniform axial velocity in the pipe is $u(x) = U(1 - x/2L)$, where $U = 5$ m/s. Develop an expression for the acceleration of a fluid particle along the centerline of the porous section.

Given: Duct flow with incompressible, inviscid liquid

$$U = 5\cdot\frac{m}{s} \quad L = 0.3\cdot m \quad u(x) = U\cdot\left(1 - \frac{x}{2\cdot L}\right)$$



Find: Expression for acceleration along the centerline of the duct

Solution: We will apply the definition of acceleration to the velocity.

Governing Equation:

$$\vec{a}_p = \frac{D\vec{V}}{Dt} = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} + \frac{\partial \vec{V}}{\partial t} \quad \text{(Particle acceleration)}$$

Assumptions:

(1) Incompressible flow (ρ is constant)
(2) One-dimensional flow along centerline ($u = u(x)$ only)
(3) Steady flow (velocity is not a function of t)

Based on assumptions (2) and (3), the acceleration reduces to:

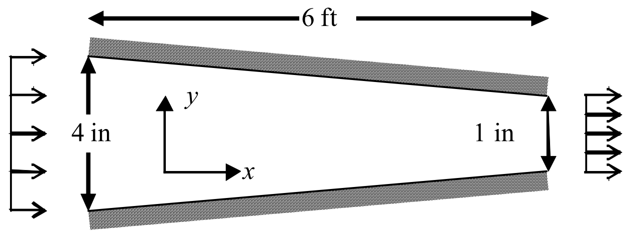
$$a_{px} = u\cdot\frac{\partial}{\partial x}u = \left[U\cdot\left(1 - \frac{x}{2\cdot L}\right)\right]\cdot\left(-\frac{U}{2\cdot L}\right) = -\frac{U^2}{2\cdot L}\cdot\left(1 - \frac{x}{2\cdot L}\right)$$

$$\boxed{a_{px} = -\frac{U^2}{2\cdot L}\cdot\left(1 - \frac{x}{2\cdot L}\right)}$$

Problem 5.47

[Difficulty: 4]

5.47 An incompressible liquid with negligible viscosity flows steadily through a horizontal pipe. The pipe diameter linearly varies from 4 in. to 1 in. over a length of 6 ft. Develop an expression for the acceleration of a fluid particle along the pipe centerline. Plot the centerline velocity and acceleration versus position along the pipe, if the inlet centerline velocity is 3 ft/s.



Given: Flow in a pipe with variable diameter

Find: Expression for particle acceleration; Plot of velocity and acceleration along centerline

Solution:

Basic equations:

$$\frac{\partial}{\partial t}\int_{CV}\rho\, d\mathcal{V} + \int_{CS}\rho\vec{V}\cdot d\vec{A} = 0$$

$$\vec{a}_p = \underbrace{\frac{D\vec{V}}{Dt}}_{\substack{\text{total}\\ \text{acceleration}\\ \text{of a particle}}} = \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\substack{\text{convective}\\ \text{acceleration}}} + \underbrace{\frac{\partial \vec{V}}{\partial t}}_{\substack{\text{local}\\ \text{acceleration}}}$$

Assumptions:

1) Incompressible flow
2) Uniform flow

Continuity reduces to $\sum_{CS}\vec{V}\cdot\vec{A} = 0$ and for the flow rate $Q = V\cdot A = V\cdot\frac{\pi\cdot D^2}{4}$

But $D = D_i + \frac{(D_o - D_i)}{L}\cdot x$ where D_i and D_o are the inlet and exit diameters, and x is distance along the pipe of length L: $D(0) = D_i$, $D(L) = D_o$.

Hence

$$V_i\cdot\frac{\pi\cdot D_i^{\,2}}{4} = V\cdot\frac{\pi\cdot\left[D_i + \frac{(D_o - D_i)}{L}\cdot x\right]^2}{4}$$

$$V = V_i\cdot\frac{D_i^{\,2}}{\left[D_i + \frac{(D_o - D_i)}{L}\cdot x\right]^2} = \frac{V_i}{\left[1 + \frac{\left(\frac{D_o}{D_i} - 1\right)}{L}\cdot x\right]^2} \qquad V(x) = \frac{V_i}{\left[1 + \frac{\left(\frac{D_o}{D_i} - 1\right)}{L}\cdot x\right]^2}$$

Some representative values are $V(0\cdot ft) = 3\cdot\frac{ft}{s}$ $\quad V\left(\frac{L}{2}\right) = 7.68\cdot\frac{ft}{s}$ $\quad V(L) = 48\cdot\frac{ft}{s}$

The acceleration is given by

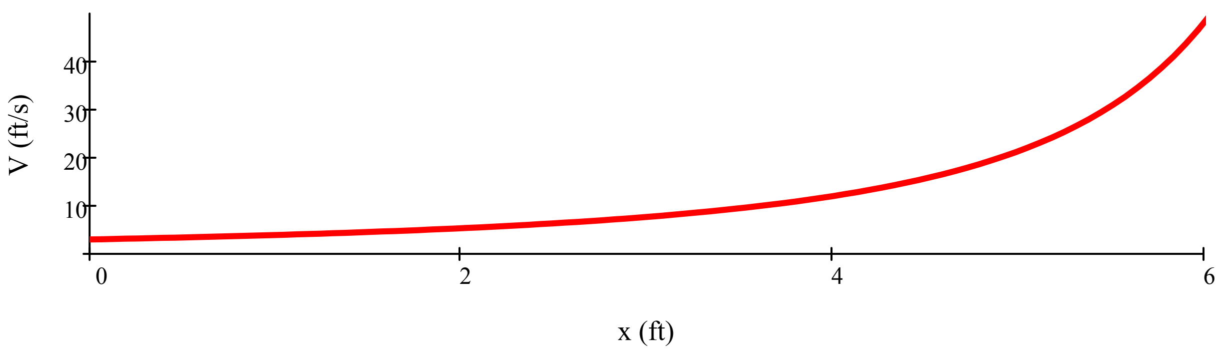
$$\vec{a}_p = \underbrace{\frac{D\vec{V}}{Dt}}_{\substack{\text{total}\\ \text{acceleration}\\ \text{of a particle}}} = \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\substack{\text{convective}\\ \text{acceleration}}} + \underbrace{\frac{\partial \vec{V}}{\partial t}}_{\substack{\text{local}\\ \text{acceleration}}}$$

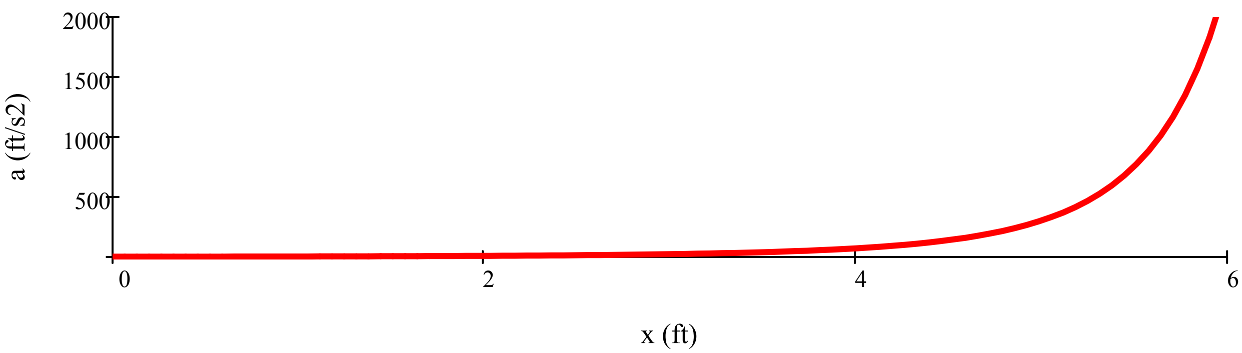
For this flow $a_x = V\cdot\frac{\partial}{\partial x}V$ $\quad a_x = \frac{V_i}{\left[1 + \frac{\left(\frac{D_o}{D_i} - 1\right)}{L}\cdot x\right]^2}\cdot\frac{\partial}{\partial x}\left[\frac{V_i}{\left[1 + \frac{\left(\frac{D_o}{D_i} - 1\right)}{L}\cdot x\right]^2}\right] = -\frac{2\cdot V_i^{\,2}\cdot\left(\frac{D_o}{D_i} - 1\right)}{L\cdot\left[\frac{x\cdot\left(\frac{D_o}{D_i} - 1\right)}{L} + 1\right]^5}$

$$a_x(x) = -\frac{2\cdot V_i^{\,2}\cdot\left(\frac{D_o}{D_i} - 1\right)}{L\cdot\left[\frac{x\cdot\left(\frac{D_o}{D_i} - 1\right)}{L} + 1\right]^5}$$

Some representative values are $a_x(0\cdot m) = 2.25\cdot\frac{ft}{s^2}$ $\quad a_x\left(\frac{L}{2}\right) = 23.6\cdot\frac{ft}{s^2}$ $\quad a_x(L) = 2.30\times 10^3\cdot\frac{ft}{s^2}$

The following plots can be done in *Excel*

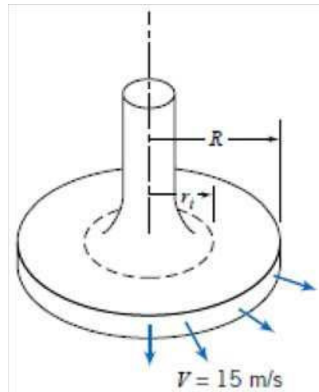




Problem 5.48

[Difficulty: 2]

5.48 Consider the low-speed flow of air between parallel disks as shown. Assume that the flow is incompressible and inviscid, and that the velocity is purely radial and uniform at any section. The flow speed is $V = 15$ m/s at $R = 75$ mm. Simplify the continuity equation to a form applicable to this flow field. Show that a general expression for the velocity field is $\vec{V} = V(R/r)\hat{e}_r$ for $r_i \le r \le R$. Calculate the acceleration of a fluid particle at the locations $r = r_i$ and $r = R$.



Given: Incompressible, inviscid flow of air between parallel disks

Find: (a) simplified version of continuity equation valid in this flow field
(b) show that the velocity is described by: $\vec{V} = V(R/r)\hat{e}_r$
(c) acceleration of a particle at $r = r_i$, $r = R$

Solution: We will apply the conservation of mass and the definition of acceleration to the velocity.

Governing Equations:

$$\frac{1}{r}\frac{\partial}{\partial r}(\rho r V_r) + \frac{1}{r}\frac{\partial \vec{V}}{\partial \theta}(\rho V_\theta) + \frac{\partial}{\partial z}(\rho V_z) + \frac{\partial \rho}{\partial t} = 0 \quad \text{(Continuity Equation)}$$

$$\vec{a}_p = \frac{D\vec{V}}{Dt} = (\vec{V}\cdot\nabla)\vec{V} + \frac{\partial \vec{V}}{\partial t} \quad \text{(Particle acceleration)}$$

Assumptions:
(1) Incompressible flow (ρ is constant)
(2) One-dimensional flow (velocity not a function of θ or z)
(3) Flow is only in the r-direction
(4) Steady flow (velocity is not a function of t)

Based on the above assumptions, the continuity equation reduces to: $\frac{1}{r}\cdot\frac{\partial}{\partial r}(r\cdot V_r) = 0$ or $r\cdot V_r = C$

Thus: $V_r = \frac{C}{r}$ should be the form of the solution. Now since at r = R: $R\cdot V = C$ it follows that: $V_r = \frac{R}{r}\cdot V$ or:

$$\vec{V} = V(R/r)\hat{e}_r$$

(Q.E.D.)

Based on assumptions (2) - (4), acceleration is radial only, and that acceleration is equal to: $a_{pr} = V_r\cdot\frac{\partial}{\partial r}V_r$

$$a_{pr} = \left(V\cdot\frac{R}{r}\right)\cdot\left(-V\cdot\frac{R}{r^2}\right) = -\frac{V^2}{R}\cdot\left(\frac{R}{r}\right)^3$$

Therefore, at r = ri: $a_{pr} = -\left(15\cdot\frac{m}{s}\right)^2 \times \frac{1}{0.075\cdot m} \times \left(\frac{75}{25}\right)^3$

$$a_{pr} = -8.1 \times 10^4 \frac{m}{s^2}$$

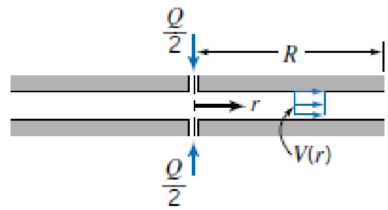
Therefore, at r = R: $a_{pr} = -\left(15\cdot\frac{m}{s}\right)^2 \times \frac{1}{0.075\cdot m} \times \left(\frac{75}{75}\right)^3$

$$a_{pr} = -3 \times 10^3 \frac{m}{s^2}$$

Problem 5.49

[Difficulty: 2]

5.49 Solve Problem 4.123 to show that the radial velocity in the narrow gap is $V_r = Q/2\pi rh$. Derive an expression for the acceleration of a fluid particle in the gap.



Given: Incompressible flow between parallel plates as shown

Find: (a) Show that the radial component of velocity is: $V_r = \dfrac{Q}{2\cdot\pi\cdot r\cdot h}$
(b) Acceleration in the gap

Solution: We will apply the conservation of mass and the definition of acceleration to the velocity.

Governing Equation:

$$\frac{1}{r}\frac{\partial}{\partial r}(\rho r V_r) + \frac{1}{r}\frac{\partial \vec{V}}{\partial \theta}(\rho V_\theta) + \frac{\partial}{\partial z}(\rho V_z) + \frac{\partial \rho}{\partial t} = 0 \quad \text{(Continuity Equation)}$$

$$\vec{a}_p = \frac{D\vec{V}}{Dt} = (\vec{V}\cdot\nabla)\vec{V} + \frac{\partial \vec{V}}{\partial t} \quad \text{(Particle acceleration)}$$

Assumptions:
(1) Incompressible flow (ρ is constant)
(2) One-dimensional flow (velocity not a function of θ or z)
(3) Flow is only in the r-direction
(4) Steady flow (velocity is not a function of t)

Based on the above assumptions, the continuity equation reduces to: $\frac{1}{r}\cdot\frac{\partial}{\partial r}(r\cdot V_r) = 0$ or $r\cdot V_r = C$

Thus: $V_r = \frac{C}{r}$ should be the form of the solution. Now since the volumetric flow rate is: $Q = 2\cdot\pi\cdot r\cdot h\cdot V_r$ it follows that:

$$V_r = \frac{Q}{2\cdot\pi\cdot r\cdot h}$$

(Q.E.D.)

Based on assumptions (2) - (4), acceleration is radial only, and that acceleration is equal to: $a_{pr} = V_r\cdot\frac{\partial}{\partial r}V_r$

$$a_{pr} = \frac{Q}{2\cdot\pi\cdot r\cdot h}\cdot\frac{-Q}{2\cdot\pi\cdot r^2\cdot h} = -\left(\frac{Q}{2\cdot\pi\cdot h}\right)^2\cdot\frac{1}{r^3}$$

Therefore, the particle acceleration is:

$$\vec{a}_p = -\left(\frac{Q}{2\pi h}\right)^2\frac{1}{r^3}\hat{e}_r$$

Problem 5.50

[Difficulty: 4]

5.50 As part of a pollution study, a model concentration c as a function of position x has been developed,

$$c(x) = A(e^{-x/2a} - e^{-x/a})$$

where $A = 3 \times 10^{-5}$ ppm (parts per million) and $a = 3$ ft. Plot this concentration from $x = 0$ to $x = 30$ ft. If a vehicle with a pollution sensor travels through the area at $u = U = 70$ ft/s, develop an expression for the measured concentration rate of change of c with time, and plot using the given data.

(a) At what location will the sensor indicate the most rapid rate of change?

(b) What is the value of this rate of change?

Given: Data on pollution concentration

Find: Plot of concentration; Plot of concentration over time for moving vehicle; Location and value of maximum rate change

Solution:

Basic equation:

$$\frac{D}{Dt} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} + \frac{\partial}{\partial t} \qquad \text{(Material Derivative)}$$

Assumption: Concentration of pollution is a function of x only
Sensor travels in x-direction only

For this case we have $\quad u = U \qquad v = 0 \qquad w = 0 \qquad c(x) = A\cdot\left(e^{-\frac{x}{2\cdot a}} - e^{-\frac{x}{a}}\right)$

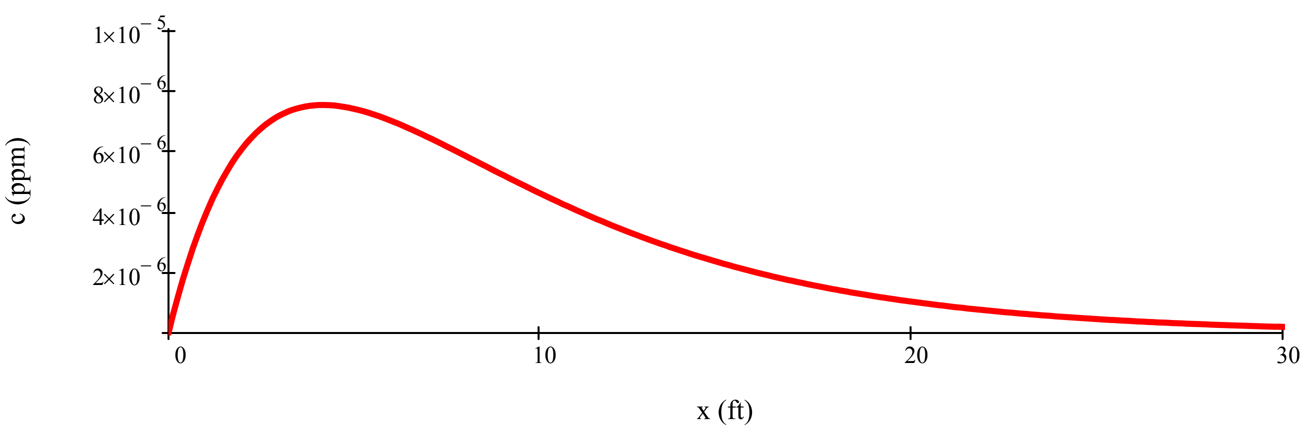
Hence

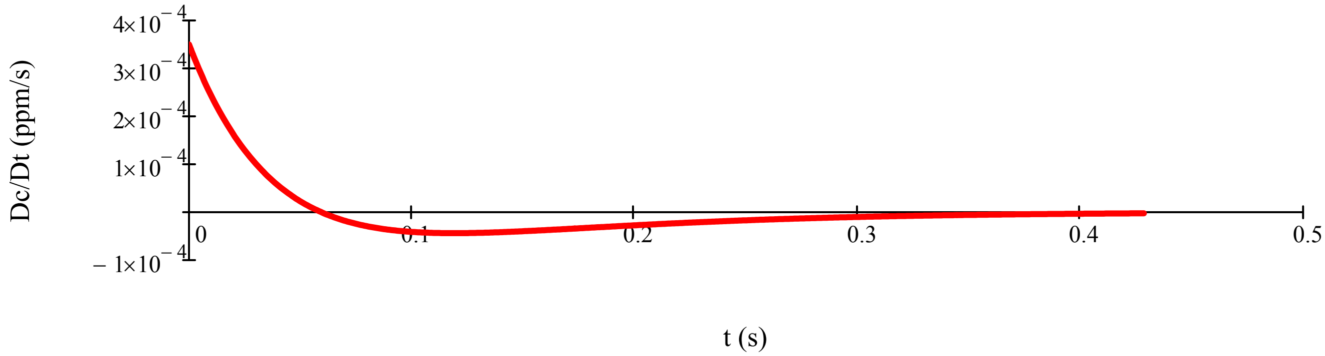
$$\frac{Dc}{Dt} = u\cdot\frac{dc}{dx} = U\cdot\frac{d}{dx}\left[A\cdot\left(e^{-\frac{x}{2\cdot a}} - e^{-\frac{x}{a}}\right)\right] = \frac{U\cdot A}{a}\cdot\left(e^{-\frac{x}{a}} - \frac{1}{2}\cdot e^{-\frac{x}{2\cdot a}}\right)$$

We need to convert this to a function of time. For this motion u = U so $\quad x = U\cdot t$

$$\frac{Dc}{Dt} = \frac{U\cdot A}{a}\cdot\left(e^{-\frac{U\cdot t}{a}} - \frac{1}{2}\cdot e^{-\frac{U\cdot t}{2\cdot a}}\right)$$

The following plots can be done in *Excel*





The magnitude of the rate of change is maximized when

$$\frac{d}{dx}\left(\frac{Dc}{Dt}\right) = \frac{d}{dx}\cdot\left[\frac{U\cdot A}{a}\cdot\left(e^{-\frac{x}{a}} - \frac{1}{2}\cdot e^{-\frac{x}{2\cdot a}}\right)\right] = 0$$

$$\frac{U\cdot A}{a^2}\cdot\left(\frac{1}{4}\cdot e^{-\frac{x}{2\cdot a}} - e^{-\frac{x}{a}}\right) = 0 \qquad \text{or} \qquad e^{\frac{x}{2\cdot a}} = 4$$

$$x_{max} = 2\cdot a\cdot \ln(4) = 2\times 3\cdot ft\times \ln(4) \qquad x_{max} = 8.32\cdot ft$$

$$t_{max} = \frac{x_{max}}{U} = 8.32\cdot ft\times\frac{s}{70\cdot ft} \qquad t_{max} = 0.119\cdot s$$

$$\frac{Dc_{max}}{Dt} = \frac{U\cdot A}{a}\cdot\left(e^{-\frac{x_{max}}{a}} - \frac{1}{2}\cdot e^{-\frac{x_{max}}{2\cdot a}}\right)$$

$$\frac{Dc_{max}}{Dt} = 70\cdot\frac{ft}{s}\times 3\times 10^{-5}\cdot ppm\times\frac{1}{3\cdot ft}\times\left(e^{-\frac{8.32}{3}} - \frac{1}{2}\times e^{-\frac{8.32}{2\cdot 3}}\right) \qquad \frac{Dc_{max}}{Dt} = -4.38\times 10^{-5}\cdot\frac{ppm}{s}$$

Note that there is another maximum rate, at t = 0 (x = 0)

$$\frac{Dc_{max}}{Dt} = 70\cdot\frac{ft}{s}\times 3\times 10^{-5}\cdot ppm\times\frac{1}{3\cdot ft}\cdot\left(1-\frac{1}{2}\right) \qquad \frac{Dc_{max}}{Dt} = 3.50\times 10^{-4}\cdot\frac{ppm}{s}$$

Problem 5.51

[Difficulty: 2]

5.51 After a rainfall the sediment concentration at a certain point in a river increases at the rate of 100 parts per million (ppm) per hour. In addition, the sediment concentration increases with distance downstream as a result of influx from tributary streams; this rate of increase is 50 ppm per mile. At this point the stream flows at 0.5 mph. A boat is used to survey the sediment concentration. The operator is amazed to find three different apparent rates of change of sediment concentration when the boat travels upstream, drifts with the current, or travels downstream. Explain physically why the different rates are observed. If the speed of the boat is 2.5 mph, compute the three rates of change.

Given: Sediment concentration fates in a river after a rainfall are:

$$\frac{\partial}{\partial t}c = 100\cdot\frac{\text{ppm}}{\text{hr}} \qquad \frac{\partial}{\partial x}c = 50\cdot\frac{\text{ppm}}{\text{mi}}$$

Stream speed is 0.5 mph, where a boat is used to survey the concentration.
The boat speed is 2.5 mph.

Find: (a) rates of change of sediment concentration observed when boat travels upstream, drifts with the current, or travels downstream.
(b) explain why the observed rates differ

Solution: We will apply the concept of substantial derivative

Governing Equation:

$$\frac{Dc}{Dt} = u\frac{\partial c}{\partial x} + v\frac{\partial c}{\partial y} + w\frac{\partial c}{\partial z} + \frac{\partial c}{\partial t} \qquad \text{(Substantial Derivative)}$$

Assumptions: (1) One-dimensional motion (velocity not a function of y or z)
(2) Steady flow (velocity is not a function of t)

Based on the above assumptions, the substantial derivative reduces to: $\frac{Dc}{Dt} = u\frac{\partial c}{\partial x} + \frac{\partial c}{\partial t}$

To obtain the rates of change from the boat, we set $u = u_B$

(i) For travel upstream, $u_B = u_s - V_b$ $\quad u_B = 0.5\cdot\text{mph} - 2.5\cdot\text{mph}$ $\quad u_B = -2\cdot\text{mph}$

$$D_{cup} = -2.0\cdot\frac{\text{mi}}{\text{hr}} \times 50\cdot\frac{10^{-6}}{\text{mi}} + 100\cdot\frac{10^{-6}}{\text{hr}}$$

$$D_{cup} = 0.00\cdot\frac{10^{-6}}{\text{hr}}$$

(ii) For drifting, $u_B = u_s$ $\quad u_B = 0.5\cdot\text{mph}$

$$D_{cdrift} = 0.5\cdot\frac{\text{mi}}{\text{hr}} \times 50\cdot\frac{10^{-6}}{\text{mi}} + 100\cdot\frac{10^{-6}}{\text{hr}}$$

$$D_{cdrift} = 125.0\cdot\frac{10^{-6}}{\text{hr}}$$

(iii) For travel downstream, $u_B = u_s + V_b$ $\quad u_B = 0.5\cdot\text{mph} + 2.5\cdot\text{mph}$ $\quad u_B = 3\cdot\text{mph}$

$$D_{cdown} = 3.0\cdot\frac{\text{mi}}{\text{hr}} \times 50\cdot\frac{10^{-6}}{\text{mi}} + 100\cdot\frac{10^{-6}}{\text{hr}}$$

$$D_{cdown} = 250\cdot\frac{10^{-6}}{\text{hr}}$$

Physically the observed rates of change differ because the observer is convected through the flow. The convective change may add to or subtract from the local rate of change.

Problem 5.52

[Difficulty: 2]

5.52 As an aircraft flies through a cold front, an onboard instrument indicates that ambient temperature drops at the rate of 0.7°F/min. Other instruments show an air speed of 400 knots and a 2500 ft/min rate of climb. The front is stationary and vertically uniform. Compute the rate of change of temperature with respect to horizontal distance through the cold front.

Given: Instruments on board an aircraft flying through a cold front show ambient temperature dropping at 0.7 °F/min, air speed of 400 knots and 2500 ft/min rate of climb.

Find: Rate of temperature change with respect to horizontal distance through cold front.

Solution: We will apply the concept of substantial derivative

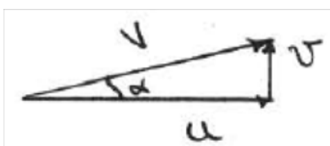
Governing Equation:

$$\frac{DT}{Dt} = u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} + \frac{\partial T}{\partial t} \qquad \text{(Substantial Derivative)}$$

Assumptions:
(1) Two-dimensional motion (velocity not a function of z)
(2) Steady flow (velocity is not a function of t)
(3) Temperature is constant in y direction

Based on the above assumptions, the substantial derivative reduces to: $\frac{DT}{Dt} = u\frac{\partial T}{\partial x}$

Finding the velocity components: $V = 400 \cdot \frac{\text{nmi}}{\text{hr}} \times \frac{6080 \cdot \text{ft}}{\text{nmi}} \times \frac{\text{hr}}{3600 \cdot \text{s}}$ $\quad V = 675.56 \cdot \frac{\text{ft}}{\text{s}}$ $\quad v = 2500 \cdot \frac{\text{ft}}{\text{min}} \times \frac{\text{min}}{60 \cdot \text{s}}$ $\quad v = 41.67 \cdot \frac{\text{ft}}{\text{s}}$



Therefore: $u = \sqrt{\left(675.56 \cdot \frac{\text{ft}}{\text{s}}\right)^2 - \left(41.67 \cdot \frac{\text{ft}}{\text{s}}\right)^2}$ $\quad u = 674.27 \cdot \frac{\text{ft}}{\text{s}}$

So the rate of change of temperature through the cold front is: $\delta T_x = \frac{-0.7 \cdot \Delta°\text{F}}{\text{min}} \times \frac{\text{s}}{674.27 \cdot \text{ft}} \times \frac{\text{min}}{60 \cdot \text{s}} \times \frac{5280 \cdot \text{ft}}{\text{mi}}$

$$\delta T_x = -0.0914 \cdot \frac{\Delta°\text{F}}{\text{mi}}$$

Problem 5.53

[Difficulty: 2]

5.53 An aircraft flies due north at 300 mph ground speed. Its rate of climb is 3000 ft/min. The vertical temperature gradient is $-3°F$ per 1000 ft of altitude. The ground temperature varies with position through a cold front, falling at the rate of 1°F per mile. Compute the rate of temperature change shown by a recorder on board the aircraft.

Given: Aircraft flying north with speed of 300 mph with respect to ground, 3000 ft/min vertical. Rate of temperature change is -3 deg F/1000 ft altitude. Ground temperature varied 1 deg F/mile.

Find: Rate of temperature change shown by on-board flight recorder

Solution: We will apply the concept of substantial derivative

Governing Equation:

$$\frac{DT}{Dt} = u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} + \frac{\partial T}{\partial t} \qquad \text{(Substantial Derivative)}$$

Assumptions:
(1) Two-dimensional motion (velocity not a function of z)
(2) Steady flow (velocity is not a function of t)

Based on the above assumptions, the substantial derivative reduces to: $$\frac{DT}{Dt} = u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}$$

Substituting numerical values: $$\delta T = \left(300 \cdot \frac{\text{mi}}{\text{hr}} \times \frac{-1 \cdot \Delta°\text{F}}{\text{mi}} \times \frac{\text{hr}}{60 \cdot \text{min}}\right) + \left(3000 \cdot \frac{\text{ft}}{\text{min}} \times \frac{-3 \cdot \Delta°\text{F}}{1000 \cdot \text{ft}}\right)$$

$$\delta T = -14 \cdot \frac{\Delta°\text{F}}{\text{min}}$$

Problem 5.54

[Difficulty: 4]

5.54 Wave flow of an incompressible fluid into a solid surface follows a sinusoidal pattern. Flow is axisymmetric about the z axis, which is normal to the surface. The z component of the flow follows the pattern

$$V_z = Az\sin\left(\frac{2\pi t}{T}\right)$$

Determine (a) the radial component of flow (V_r) and (b) the convective and local components of the acceleration vector.

Given: Z component of an axisymmetric transient flow.

Find: Radial component of flow and total acceleration.

Solution:

Governing Equations:

$$\frac{1}{r}\frac{\partial(rV_r)}{\partial r}+\frac{1}{r}\frac{\partial V_\theta}{\partial\theta}+\frac{\partial V_z}{\partial z}=0 \quad \text{(Continuity Equation for an Incompressible Fluid)}$$

$$a_{r,p}=V_r\frac{\partial V_r}{\partial r}+\frac{V_\theta}{r}\frac{\partial V_r}{\partial\theta}-\frac{V_\theta^2}{r}+V_z\frac{\partial V_r}{\partial z}+\frac{\partial V_r}{\partial t}$$

(Particle acceleration)

$$a_{z,p}=V_r\frac{\partial V_z}{\partial r}+\frac{V_\theta}{r}\frac{\partial V_z}{\partial\theta}+V_z\frac{\partial V_z}{\partial z}+\frac{\partial V_z}{\partial t}$$

Assumptions:
Incompressible fluid
No motion along the wall ($z = 0$) limited to two dimensions ($V_\theta = 0$ and all partials with respect to θ are zero).

The given or available data is: $V_Z = Az\cdot\sin\left(\frac{2\pi t}{T}\right)$ $\quad V_\theta = 0 \quad \frac{\partial(\;)}{\partial\theta}=0$

Simplify the continuity equation to find V_r: $\frac{1}{r}\frac{\partial(rV_r)}{\partial r}=-\frac{\partial V_z}{\partial z}\Rightarrow\frac{\partial(rV_r)}{\partial r}=r\times -A\cdot\sin\left(\frac{2\pi t}{T}\right)$

Solve using separation of variables: $rV_r=-\frac{r^2A}{2}\cdot\sin\left(\frac{2\pi t}{T}\right)+C$

Use the boundary condition of no flow at the origin to solve for the constant of integration

$$V_r=-\frac{rA}{2}\cdot\sin\left(\frac{2\pi t}{T}\right)$$

Find the convective terms of acceleration. $a_{r,conv}=V_r\frac{\partial V_r}{\partial r}+V_z\frac{\partial V_r}{\partial z}=-\frac{rA}{2}\sin\left(\frac{2\pi t}{T}\right)\times-\frac{A}{2}\sin\left(\frac{2\pi t}{T}\right)+Az\sin\left(\frac{2\pi t}{T}\right)\times 0$

$$a_{r,conv}=\frac{rA^2}{4}\sin^2\left(\frac{2\pi t}{T}\right)$$

$$a_{z,conv}=V_r\frac{\partial V_z}{\partial r}+V_z\frac{\partial V_z}{\partial z}=-\frac{rA}{2}\sin\left(\frac{2\pi t}{T}\right)\times 0+Az\cdot\sin\left(\frac{2\pi t}{T}\right)\times A\cdot\sin\left(\frac{2\pi t}{T}\right)$$

$$a_{z,conv}=zA^2\sin^2\left(\frac{2\pi t}{T}\right)$$

Find the local terms: $$a_{r,local} = \frac{\partial V_r}{\partial t} = -\frac{2\pi}{T} \times \frac{rA}{2} \cos\left(\frac{2\pi t}{T}\right)$$

$$a_{r,local} = \frac{-\pi r A}{T} \cos\left(\frac{2\pi t}{T}\right)$$

$$a_{z,local} = \frac{\partial V_z}{\partial t} = -\frac{2\pi}{T} \times Az \cdot \cos\left(\frac{2\pi t}{T}\right)$$

$$a_{z,local} = \frac{2\pi z A}{T} \cos\left(\frac{2\pi t}{T}\right)$$

Problem 5.55

[Difficulty: 3]

5.55 Expand $(\vec{V}\cdot\nabla)\vec{V}$ in rectangular coordinates by direct substitution of the velocity vector to obtain the convective acceleration of a fluid particle. Verify the results given in Eqs. 5.11.

Given: Definition of "del" operator

Find: an expression for the convective acceleration for a fluid particle.

Solution: We will directly substitute the velocity vector into the expression.

Governing Equation:

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$ ("del" operator in rectangular coordinates)

$$\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$$ (velocity vector)

Assumptions: None.

Directly substituting we get:

$$(\vec{V}\cdot\nabla)\vec{V} = (u\hat{i} + v\hat{j} + w\hat{k})\cdot\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(u\hat{i} + v\hat{j} + w\hat{k})$$

$$= \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)(u\hat{i} + v\hat{j} + w\hat{k})$$

$$= \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)\hat{i} + \left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)\hat{j} + \left(u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)\hat{k}$$

The components of this vector are the x-, y-, and z-components of the convective acceleration:

$$a_{xp} = \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) + \frac{\partial u}{\partial t}$$

$$a_{yp} = \left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) + \frac{\partial v}{\partial t}$$

$$a_{zp} = \left(u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) + \frac{\partial w}{\partial t}$$

Problem 5.56

[Difficulty: 3]

5.56 A steady, two-dimensional velocity field is given by $\vec{V} = Ax\hat{i} - Ay\hat{j}$, where $A = 1\ \mathrm{s}^{-1}$. Show that the streamlines for this flow are rectangular hyperbolas, $xy = C$. Obtain a general expression for the acceleration of a fluid particle in this velocity field. Calculate the acceleration of fluid particles at the points $(x, y) = (\frac{1}{2}, 2)$, $(1, 1)$, and $(2, \frac{1}{2})$, where x and y are measured in meters. Plot streamlines that correspond to $C = 0$, 1, and 2 m^2 and show the acceleration vectors on the streamline plot.

Given: Steady, two-dimensional velocity field represented above

Find:
(a) proof that streamlines are hyperbolas (xy = C)
(b) acceleration of a particle in this field
(c) acceleration of particles at (x,y) = (1/2m, 2m), (1m,1m), and (2m, 1/2m)
(d) plot streamlines corresponding to C = 0, 1, and 2 m^2 and show accelerations

Solution: We will apply the acceleration definition, and determine the streamline slope.

Governing Equations:
$$\vec{a}_p = \frac{D\vec{V}}{Dt} = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} + \frac{\partial \vec{V}}{\partial t} \quad \text{(Particle acceleration)}$$

Assumptions:
(1) Two-dimensional flow (velocity is not a function of z)
(2) Incompressible flow

Streamlines along the x-y plane are defined by $\frac{dy}{dx} = \frac{v}{u} = \frac{-A\cdot y}{A\cdot x}$ Thus: $\frac{dx}{x} + \frac{dy}{y} = 0$

After integrating: $\ln(x) + \ln(y) = \ln(C)$ which yields: $x\cdot y = C$ (Q.E.D.)

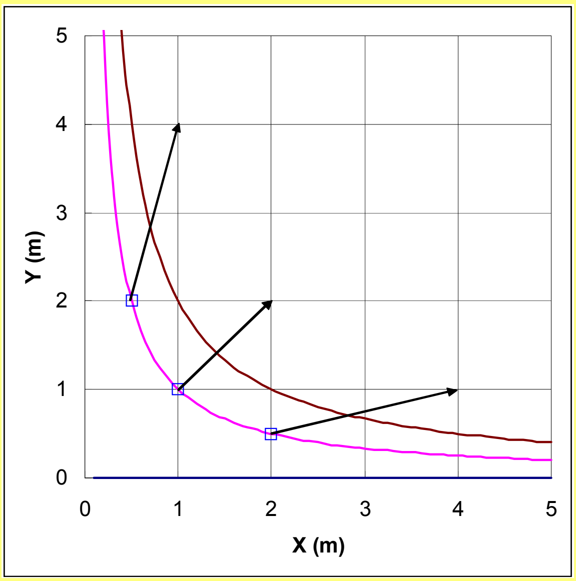
Based on the above assumptions the particle acceleration reduces to: $\vec{a}_p = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y}$ Substituting in the field:

$$\vec{a}_p = (Ax)A\hat{i} + (-Ay)(-A)\hat{j} = A^2\left(x\hat{i} + y\hat{j}\right)$$ which simplifies to $\vec{a}_p = A^2\left(x\hat{i} + y\hat{j}\right)$

At (x,y) = (0.5m, 2m) $\vec{a}_p = \left(0.5\hat{i} + 2\hat{j}\right)\frac{\mathrm{m}}{\mathrm{s}^2}$ At (x,y) = (1m, 1m) $\vec{a}_p = \left(\hat{i} + \hat{j}\right)\frac{\mathrm{m}}{\mathrm{s}^2}$ At (x,y) = (2m, 0.5m) $\vec{a}_p = \left(2\hat{i} + 0.5\hat{j}\right)\frac{\mathrm{m}}{\mathrm{s}^2}$

Here is the plot of the streamlines:

(When C = 0 the streamline is on the x- and y-axes.)



Problem 5.57

[Difficulty: 3]

5.57 A velocity field is represented by the expression $\vec{V} = (Ax - B)\hat{i} - Ay\hat{j}$, where $A = 0.2\ \mathrm{s}^{-1}$, $B = 0.6\ \mathrm{m \cdot s^{-1}}$, and the coordinates are expressed in meters. Obtain a general expression for the acceleration of a fluid particle in this velocity field. Calculate the acceleration of fluid particles at points $(x, y) = (0, \frac{4}{3})$, $(1, 2)$, and $(2, 4)$. Plot a few streamlines in the xy plane. Show the acceleration vectors on the streamline plot.

Given: Steady, two-dimensional velocity field represented above

Find:
(a) general acceleration of a particle in this field
(b) acceleration of particles at (x,y) = (0m, 4/3m), (1m,2m), and (2m, 4m)
(c) plot streamlines with acceleration vectors

Solution: We will apply the acceleration definition, and determine the streamline slope.

Governing Equations:

$$\vec{a}_p = \frac{D\vec{V}}{Dt} = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} + \frac{\partial \vec{V}}{\partial t} \quad \text{(Particle acceleration)}$$

Assumptions:
(1) Two-dimensional flow (velocity is not a function of z)
(2) Incompressible flow

Based on the above assumptions the particle acceleration reduces to: $\vec{a}_p = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y}$ Substituting in the field:

$$\vec{a}_p = (Ax - B)A\hat{i} + (-Ay)(-A)\hat{j} = \left(A^2x - AB\right)\hat{i} + \left(A^2y\right)\hat{j}$$

$$\vec{a}_p = \left(A^2x - AB\right)\hat{i} + \left(A^2y\right)\hat{j}$$

At (x,y) = (0m, 4/3m) $\vec{a}_p = \left(-0.12\hat{i} + 0.0533\hat{j}\right)\frac{\mathrm{m}}{\mathrm{s}^2}$

At (x,y) = (1m, 1m) $\vec{a}_p = \left(-0.08\hat{i} + 0.0800\hat{j}\right)\frac{\mathrm{m}}{\mathrm{s}^2}$

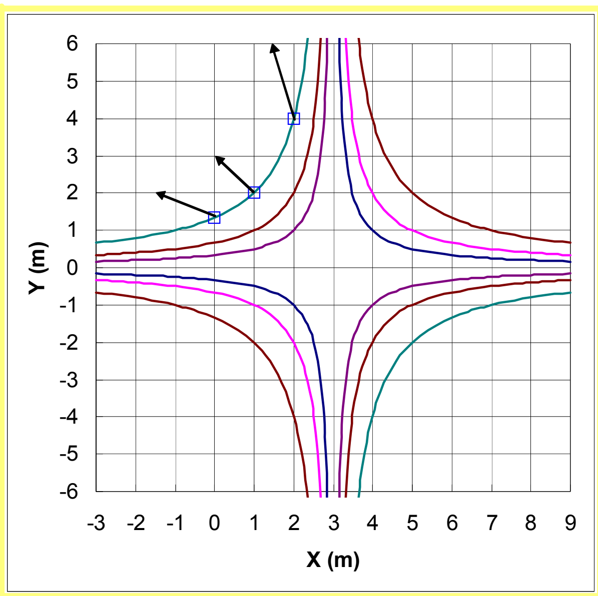
At (x,y) = (2m, 0.5m) $\vec{a}_p = \left(-0.04\hat{i} + 0.160\hat{j}\right)\frac{\mathrm{m}}{\mathrm{s}^2}$

Streamlines along the x-y plane are defined by $\frac{dy}{dx} = \frac{v}{u} = \frac{-A\cdot y}{A\cdot x - B}$ Thus: $\frac{dx}{A\cdot x - B} + \frac{dy}{A\cdot y} = 0$ After integrating:

$\frac{1}{A}\cdot\ln(A\cdot x - B) + \frac{1}{A}\cdot\ln(y) = \frac{1}{A}\cdot\ln(C)$ which yields:

Here is the plot of the streamlines:

$(A\cdot x - B)\cdot y = C$



Problem 5.58

[Difficulty: 3]

5.58 A velocity field is represented by the expression $\vec{V} = (Ax - B)\hat{i} + Cy\hat{j} + Dt\hat{k}$, where $A = 2\ \mathrm{s}^{-1}$, $B = 4\ \mathrm{m\cdot s^{-1}}$, $D = 5\ \mathrm{m\cdot s^{-2}}$, and the coordinates are measured in meters. Determine the proper value for C if the flow field is to be incompressible. Calculate the acceleration of a fluid particle located at point $(x, y) = (3, 2)$. Plot a few flow streamlines in the xy plane.

Given: Velocity field represented above

Find:
(a) the proper value for C if the flow field is incompressible
(b) acceleration of a particle at (x,y) = (3m,2m)
(c) sketch the streamlines in the x-y plane

Solution: We will check the velocity field against the continuity equation, apply the acceleration definition, and determine the streamline slope.

Governing Equations:

$$\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)+\frac{\partial \rho}{\partial t}=0 \qquad \text{(Continuity equation)}$$

$$\vec{a}_p=\frac{D\vec{V}}{Dt}=u\frac{\partial \vec{V}}{\partial x}+v\frac{\partial \vec{V}}{\partial y}+w\frac{\partial \vec{V}}{\partial z}+\frac{\partial \vec{V}}{\partial t} \qquad \text{(Particle acceleration)}$$

Assumptions:
(1) Two-dimensional flow (velocity is not a function of z)
(2) Incompressible flow

Based on the above assumptions the continuity equation reduces to: $\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$ This is the criterion to check the velocity.

The partial derivatives are: $\frac{\partial}{\partial x}u = A$ and $\frac{\partial}{\partial y}v = C$ Thus from continuity: $A + C = 0$ or $C = -A$ $\quad C = -2\cdot\mathrm{s}^{-1}$

Based on the above assumptions the particle acceleration reduces to: $\vec{a}_p = u\frac{\partial \vec{V}}{\partial x}+v\frac{\partial \vec{V}}{\partial y}+\frac{\partial \vec{V}}{\partial t}$ Substituting in the field:

$$\vec{a}_p=(Ax-B)A\hat{i}+(Cy)C\hat{j}+D\hat{k}=\left(A^2x-AB\right)\hat{i}+C^2y\hat{j}+D\hat{k}$$ At (x,y) = (3m, 2m)

$$\vec{a}_p=\left[\left(\frac{2}{\mathrm{s}}\right)^2\times 3\ \mathrm{m}-\frac{2}{\mathrm{s}}\times 4\frac{\mathrm{m}}{\mathrm{s}}\right]\hat{i}+\left(-\frac{2}{\mathrm{s}}\right)^2\times 2\ \mathrm{m}\hat{j}+5\frac{\mathrm{m}}{\mathrm{s}^2}\hat{k}$$

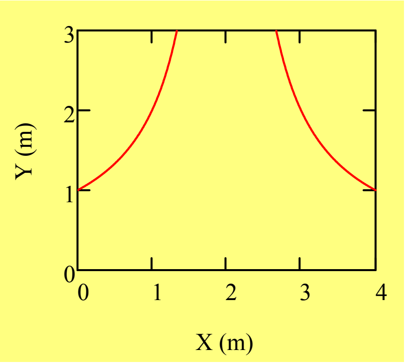
$$\vec{a}_p=\left(4\hat{i}+8\hat{j}+5\hat{k}\right)\frac{\mathrm{m}}{\mathrm{s}^2}$$

Streamlines along the x-y plane are defined by $\frac{dy}{dx} = \frac{v}{u} = \frac{C\cdot y}{A\cdot x - B}$ Thus: $-\frac{1}{A}\cdot\frac{dy}{y} = \frac{dx}{A\cdot x - B}$ or $\frac{dx}{x - \frac{B}{A}} + \frac{dy}{y} = 0$

Solving this ODE by integrating: $\ln\left(x - \frac{B}{A}\right) + \ln(y) = \mathrm{const}$

Therefore: $y\cdot\left(x - \frac{B}{A}\right) = \mathrm{constant}$

Here is a plot of the streamlines passing through (3, 2):



Problem 5.59

[Difficulty: 3]

5.59 A linear approximate velocity profile was used in Problem 5.10 to model a laminar incompressible boundary layer on a flat plate. For this profile, obtain expressions for the x and y components of acceleration of a fluid particle in the boundary layer. Locate the maximum magnitudes of the x and y accelerations. Compute the ratio of the maximum x magnitude to the maximum y magnitude for the flow conditions of Problem 5.10.

5.10 A crude approximation for the x component of velocity in an incompressible laminar boundary layer is a linear variation from $u = 0$ at the surface ($y = 0$) to the freestream velocity, U, at the boundary-layer edge ($y = \delta$). The equation for the profile is $u = Uy/\delta$, where $\delta = cx^{1/2}$ and c is a constant. Show that the simplest expression for the y component of velocity is $v = uy/4x$. Evaluate the maximum value of the ratio v/U, at a location where $x = 0.5$ m and $\delta = 5$ mm.

Given: Linear approximate profile for two-dimensional boundary layer

Find:
(a) x- and y-components of acceleration of a fluid particle
(b) locate the maximum values of acceleration
(c) compute ratio of maximum acceleration components

Solution: We will apply the acceleration definition.

Governing Equation:

$$\vec{a}_p = \frac{D\vec{V}}{Dt} = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} + \frac{\partial \vec{V}}{\partial t} \quad \text{(Particle acceleration)}$$

Assumptions:
(1) Two-dimensional flow (velocity is not a function of z)
(2) Incompressible flow
(3) Steady flow

Based on the above assumptions the particle acceleration reduces to: $\vec{a}_p = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y}$ The velocities and derivatives are:

$$u = \frac{U\cdot y}{\delta} \qquad v = \frac{u\cdot y}{4\cdot x} = \frac{U\cdot y^2}{4\cdot\delta\cdot x} \qquad \delta = c\cdot x^{\frac{1}{2}} \qquad \frac{\partial}{\partial x}u = \frac{\partial}{\partial \delta}\left(\frac{U\cdot y}{\delta}\right)\cdot\left(\frac{d}{dx}\delta\right) = -\frac{U\cdot y}{\delta^2}\cdot\frac{\delta}{2\cdot x} = -\frac{U\cdot y}{2\cdot\delta\cdot x} \qquad \frac{\partial}{\partial y}u = \frac{U}{\delta}$$

$$\frac{\partial}{\partial x}v = \frac{\partial}{\partial x}\left(\frac{U\cdot y^2}{4\cdot\delta\cdot x}\right) + \frac{\partial}{\partial \delta}\left(\frac{U\cdot y^2}{4\cdot\delta\cdot x}\right)\cdot\left(\frac{d}{dx}\delta\right) = -\frac{U\cdot y^2}{4\cdot\delta\cdot x^2} - \frac{U\cdot y^2}{4\cdot\delta^2\cdot x}\cdot\frac{\delta}{2\cdot x} = -\frac{3\cdot U\cdot y^2}{8\cdot\delta\cdot x^2} \qquad \frac{\partial}{\partial y}v = \frac{U\cdot y}{2\cdot\delta\cdot x}$$

So the accelerations are:

$$a_{px} = u\cdot\frac{\partial}{\partial x}u + v\cdot\frac{\partial}{\partial y}u = -\frac{U\cdot y}{\delta}\cdot\frac{U\cdot y}{2\cdot\delta\cdot x} + \frac{U\cdot y^2}{4\cdot\delta\cdot x}\cdot\frac{U}{\delta} = -\frac{U^2\cdot y^2}{4\cdot\delta\cdot x}$$

$$a_{px} = -\frac{U^2}{4\cdot x}\cdot\left(\frac{y}{\delta}\right)^2$$

$$a_{py} = u\cdot\frac{\partial}{\partial x}v + v\cdot\frac{\partial}{\partial y}v = -\frac{U\cdot y}{\delta}\cdot\frac{3\cdot U\cdot y^2}{8\cdot\delta\cdot x^2} + \frac{U\cdot y^2}{4\cdot\delta\cdot x}\cdot\frac{U\cdot y}{2\cdot\delta\cdot x} = -\frac{U\cdot y^3}{4\cdot\delta^2\cdot x^2}$$

$$a_{py} = -\frac{U^2}{4\cdot x}\cdot\left(\frac{y}{\delta}\right)^2\cdot\frac{y}{x}$$

The maximum values are when $y = \delta$:

$$a_{pxmax} = -\frac{U^2}{4\cdot x} \qquad a_{pymax} = -\frac{U^2}{4\cdot x}\cdot\frac{\delta}{x}$$

The ratio of the accelerations is:

$$\frac{a_{pymax}}{a_{pxmax}} = \frac{U^2}{4\cdot x}\cdot\frac{4\cdot x}{U^2}\cdot\frac{\delta}{x} = \frac{\delta}{x}$$

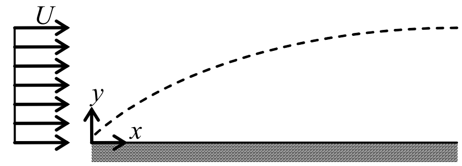
When $x = 0.5$ m and $\delta = 5$ mm: $\text{ratio} = \frac{0.5\cdot m}{0.005\cdot m}$

$$\text{ratio} = 100$$

Problem 5.60

[Difficulty: 4]

5.60 A parabolic approximate velocity profile was used in Problem 5.11 to model flow in a laminar incompressible boundary layer on a flat plate. For this profile, find the x component of acceleration, a_x, of a fluid particle within the boundary layer. Plot a_x at location $x = 0.8$ m, where $\delta = 1.2$ mm, for a flow with $U = 6$ m/s. Find the maximum value of a_x at this x location.



Given: Flow in boundary layer

Find: Expression for particle acceleration a_x; Plot acceleration and find maximum at x = 0.8 m

Solution:

Basic equations
$$\frac{u}{U} = 2\cdot\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2 \qquad \frac{v}{U} = \frac{\delta}{x}\cdot\left[\frac{1}{2}\cdot\left(\frac{y}{\delta}\right)^2 - \frac{1}{3}\cdot\left(\frac{y}{\delta}\right)^3\right] \qquad \delta = c\cdot\sqrt{x}$$

$$\underbrace{\vec{a}_p = \frac{D\vec{V}}{Dt}}_{\substack{\text{total}\\ \text{acceleration}\\ \text{of a particle}}} = \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\substack{\text{convective}\\ \text{acceleration}}} + \underbrace{\frac{\partial \vec{V}}{\partial t}}_{\substack{\text{local}\\ \text{acceleration}}}$$

We need to evaluate
$$a_x = u\cdot\frac{\partial}{\partial x}u + v\cdot\frac{\partial}{\partial y}u$$

First, substitute
$$\lambda(x,y) = \frac{y}{\delta(x)} \qquad \text{so} \qquad \frac{u}{U} = 2\cdot\lambda - \lambda^2 \qquad \frac{v}{U} = \frac{\delta}{x}\cdot\left(\frac{1}{2}\cdot\lambda^2 - \frac{1}{3}\cdot\lambda^3\right)$$

Then
$$\frac{\partial}{\partial x}u = \frac{du}{d\lambda}\cdot\frac{d\lambda}{dx} = U\cdot(2 - 2\cdot\lambda)\cdot\left(-\frac{y}{\delta^2}\right)\cdot\frac{d\delta}{dx} \qquad \frac{d\delta}{dx} = \frac{1}{2}\cdot c\cdot x^{-\frac{1}{2}}$$

$$\frac{\partial}{\partial x}u = U\cdot(2 - 2\cdot\lambda)\cdot\left(-\frac{\lambda}{\delta}\right)\cdot\frac{1}{2}\cdot c\cdot x^{-\frac{1}{2}} = U\cdot(2 - 2\cdot\lambda)\cdot\left(-\frac{\lambda}{c\cdot x^{\frac{1}{2}}}\right)\cdot\frac{1}{2}\cdot c\cdot x^{-\frac{1}{2}}$$

$$\frac{\partial}{\partial x}u = -U\cdot(2 - 2\cdot\lambda)\cdot\frac{\lambda}{2\cdot x} = -\frac{U\cdot\left(\lambda - \lambda^2\right)}{x}$$

$$\frac{\partial}{\partial y}u = U\cdot\left(\frac{2}{\delta} - 2\cdot\frac{y}{\delta^2}\right) = \frac{2\cdot U}{\delta}\cdot\left[\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2\right] = \frac{2\cdot U\cdot\left(\lambda - \lambda^2\right)}{y}$$

Hence
$$a_x = u\cdot\frac{\partial}{\partial x}u + v\cdot\frac{\partial}{\partial y}u = U\cdot\left(2\cdot\lambda - \lambda^2\right)\left[\frac{U\cdot\left(\lambda - \lambda^2\right)}{x}\right] + U\cdot\frac{\delta}{x}\cdot\left(\frac{1}{2}\cdot\lambda - \frac{1}{3}\cdot\lambda^3\right)\cdot\left[\frac{2\cdot U\cdot\left(\lambda - \lambda^2\right)}{y}\right]$$

Collecting terms
$$a_x = \frac{U^2}{x}\cdot\left(-\lambda^2 + \frac{4}{3}\cdot\lambda^3 - \frac{1}{3}\cdot\lambda^4\right) = \frac{U^2}{x}\cdot\left[-\left(\frac{y}{\delta}\right)^2 + \frac{4}{3}\cdot\left(\frac{y}{\delta}\right)^3 - \frac{1}{3}\cdot\left(\frac{y}{\delta}\right)^4\right]$$

To find the maximum
$$\frac{da_x}{d\lambda} = 0 = \frac{U^2}{x}\cdot\left(-2\cdot\lambda + 4\cdot\lambda^2 - \frac{4}{3}\cdot\lambda^3\right) \qquad \text{or} \qquad -1 + 2\cdot\lambda - \frac{2}{3}\cdot\lambda^2 = 0$$

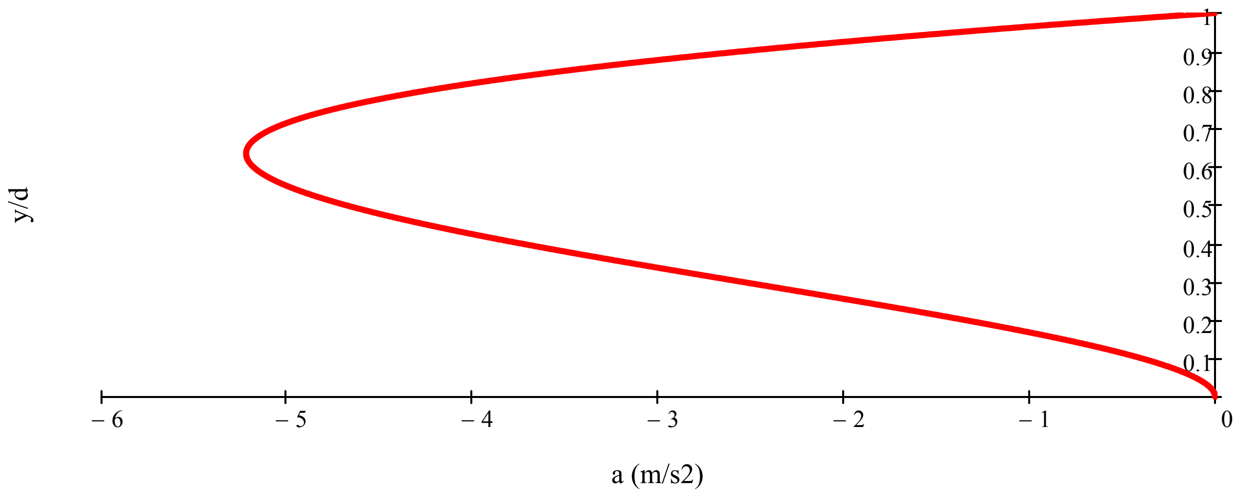
The solution of this quadratic ($\lambda < 1$) is
$$\lambda = \frac{3 - \sqrt{3}}{2} \qquad \lambda = 0.634 \qquad \frac{y}{\delta} = 0.634$$

At $\lambda = 0.634$

$$a_x = \frac{U^2}{x}\cdot\left(-0.634^2 + \frac{4}{3}\cdot 0.634^3 - \frac{1}{3}\cdot 0.634^4\right) = -0.116\cdot\frac{U^2}{x}$$

$$a_x = -0.116 \times \left(6\cdot\frac{m}{s}\right)^2 \times \frac{1}{0.8\cdot m} \qquad a_x = -5.22\,\frac{m}{s^2}$$

The following plot can be done in *Excel*



Problem 5.61

[Difficulty: 3]

5.61 Show that the velocity field of Problem 2.18 represents a possible incompressible flow field. Determine and plot the streamline passing through point $(x, y) = (2, 4)$ at $t = 1.5$ s. For the particle at the same point and time, show on the plot the velocity vector and the vectors representing the local, convective, and total accelerations.

2.18 Air flows downward toward an infinitely wide horizontal flat plate. The velocity field is given by $\vec{V} = (ax\hat{i} - ay\hat{j})(2 + \cos\omega t)$, where $a = 5\ \text{s}^{-1}$, $\omega = 2\pi\ \text{s}^{-1}$, x and y (measured in meters) are horizontal and vertically upward, respectively, and t is in s. Obtain an algebraic equation for a streamline at $t = 0$. Plot the streamline that passes through point $(x, y) = (3, 3)$ at this instant. Will the streamline change with time? Explain briefly. Show the velocity vector on your plot at the same point and time. Is the velocity vector tangent to the streamline? Explain.

Given: Steady, two-dimensional velocity field represented above

Find:
(a) prove velocity field represents a possible incompressible flow field
(b) expression for the streamline at t = 1.5 s
(c) plot of the streamline through (x,y) = (2m,4m) at that instant
(d) local velocity vector
(e) vectors representing local, convective, and total accelerations

Solution: We will apply the acceleration definition, and determine the streamline slope.

Governing Equations:

$$\vec{a}_p = \frac{D\vec{V}}{Dt} = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} + \frac{\partial \vec{V}}{\partial t} \quad \text{(Particle acceleration)}$$

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) + \frac{\partial \rho}{\partial t} = 0 \quad \text{(Continuity equation)}$$

Assumptions:
(1) Two-dimensional flow (velocity is not a function of z)
(2) Incompressible flow

Based on the two assumptions listed above, the continuity equation reduces to: $\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$

This is the criterion against which we will check all of the flow fields.

$u = (2 + \cos(\omega \cdot t)) \cdot ax$

$v = -(2 + \cos(\omega \cdot t)) \cdot ay$

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = (2 + \cos(\omega \cdot t)) \cdot (a - a) = 0$$

This could be an incompressible flow field.

Streamlines along the x-y plane are defined by $\frac{dy}{dx} = \frac{v}{u} = \frac{-a \cdot y \cdot (2 + \cos(\omega \cdot t))}{a \cdot x \cdot (2 + \cos(\omega \cdot t))} = -\frac{y}{x}$ Thus: $\frac{dx}{x} + \frac{dy}{y} = 0$ After integrating:

$\ln(x) + \ln(y) = \ln(C)$ which yields: $x \cdot y = C$ At (x,y) = (2m,4m) $C = 2 \cdot m \times 4 \cdot m$ $C = 8\,m^2$ $x \cdot y = 8 \cdot m^2$ (plot shown below)

At (x,y,t) = (2m,4m,1.5s) $\vec{V} = \frac{5}{\text{s}}\left(2\ \text{m}\hat{i} - 4\ \text{m}\hat{j}\right)\left(2 + \cos\left(\frac{2\pi}{\text{s}} \times 1.5\ \text{s}\right)\right)$

$$\vec{V} = \left(10\hat{i} - 20\hat{j}\right)\frac{\text{m}}{\text{s}}$$

Based on the above assumptions the particle acceleration reduces to: $\vec{a}_p = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + \frac{\partial \vec{V}}{\partial t}$

The local acceleration is: $\vec{a}_{p,local} = \frac{\partial \vec{V}}{\partial t} = a\omega\left(x\hat{i} - y\hat{j}\right)\left(-\sin(\omega t)\right) = \frac{5}{\text{s}} \times \frac{2\pi}{\text{s}}\left(2\ \text{m}\hat{i} - 4\ \text{m}\hat{j}\right)\left(-\sin\left(\frac{2\pi}{\text{s}} \times 1.5\ \text{s}\right)\right)$

$$\vec{a}_{p,local} = \left(0\hat{i} + 0\hat{j}\right)\frac{\text{m}}{\text{s}^2}$$

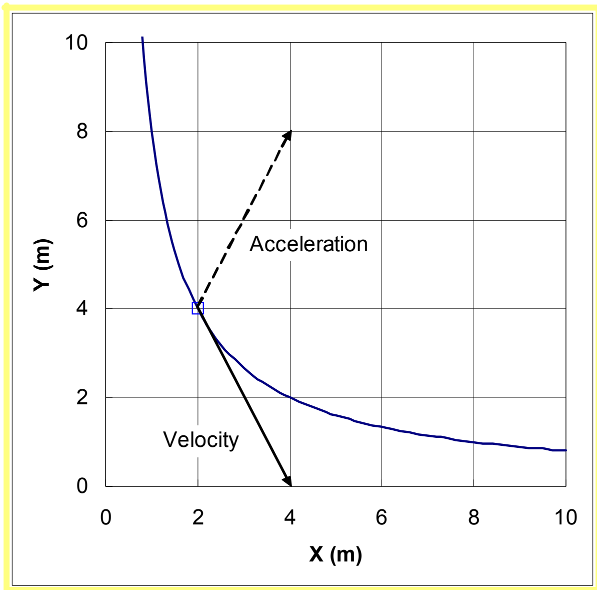
The convective acceleration is:

$$\vec{a}_{p,conv} = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} = a\left(u\hat{i} - v\hat{j}\right)\left(2 + \cos\omega t\right) = \frac{5}{\mathrm{s}} \times \left(10\frac{\mathrm{m}}{\mathrm{s}}\hat{i} + 20\frac{\mathrm{m}}{\mathrm{s}}\hat{j}\right)\left(2 + \cos\left(\frac{2\pi}{\mathrm{s}} \times 1.5\ \mathrm{s}\right)\right)$$

$$\vec{a}_{p,conv} = \left(50\hat{i} + 100\hat{j}\right)\frac{\mathrm{m}}{\mathrm{s}^2}$$

The total acceleration is the sum of the two acceleration terms: $\vec{a}_p = \left(50\hat{i} + 100\hat{j}\right)\frac{\mathrm{m}}{\mathrm{s}^2}$

Here is the plot of the streamline and the vectors:



Problem 5.62

[Difficulty: 3]

5.62 A sinusoidal approximate velocity profile was used in Problem 5.12 to model flow in a laminar incompressible boundary layer on a flat plate. For this profile, obtain an expression for the x and y components of acceleration of a fluid particle in the boundary layer. Plot a_x and a_y at location $x = 3$ ft, where $\delta = 0.04$ in., for a flow with $U = 20$ ft/s. Find the maxima of a_x and a_y at this x location.

5.12 A useful approximation for the x component of velocity in an incompressible laminar boundary layer is a sinusoidal variation from $u = 0$ at the surface ($y = 0$) to the freestream velocity, U, at the edge of the boundary layer ($y = \delta$). The equation for the profile is $u = U \sin(\pi y/2\delta)$, where $\delta = cx^{1/2}$ and c is a constant. Show that the simplest expression for the y component of velocity is

$$\frac{v}{U} = \frac{1}{\pi}\frac{\delta}{x}\left[\cos\left(\frac{\pi}{2}\frac{y}{\delta}\right) + \left(\frac{\pi}{2}\frac{y}{\delta}\right)\sin\left(\frac{\pi}{2}\frac{y}{\delta}\right) - 1\right]$$

Plot u/U and v/U versus y/δ, and find the location of the maximum value of the ratio v/U. Evaluate the ratio where $x = 0.5$ m and $\delta = 5$ mm.

Given: Sinusoidal profile for two-dimensional boundary layer

Find: (a) x- and y-components of acceleration of a fluid particle
(b) plot components as functions of y/δ for U = 20 ft/s, x = 3 ft, δ = 0.04 in
(c) maximum values of acceleration at this x location

Solution: We will apply the acceleration definition.

Governing Equation: (Particle acceleration)

Assumptions:
(1) Two-dimensional flow (velocity is not a function of z)
(2) Incompressible flow
(3) Steady flow

Based on the above assumptions the particle acceleration reduces to: $\vec{a}_p = u\dfrac{\partial \vec{V}}{\partial x} + v\dfrac{\partial \vec{V}}{\partial y}$ To make this easier, define η:

$$\eta = \frac{\pi}{2}\cdot\frac{y}{\delta} = \eta(x,y) \qquad \frac{\partial}{\partial y}\eta = \frac{\pi}{2\cdot\delta} \qquad \delta = c\cdot x^{\frac{1}{2}} \qquad \frac{d}{dx}\delta = \frac{1}{2}\cdot c\cdot x^{-\frac{1}{2}} = \frac{\delta}{2\cdot x}$$

$$\frac{\partial}{\partial x}\eta = \left(\frac{\partial}{\partial \delta}\eta\right)\cdot\left(\frac{d}{dx}\delta\right) = -\frac{\pi\cdot y}{2\cdot\delta^2}\cdot\frac{\delta}{2\cdot x} = -\frac{\pi}{4\cdot x}\cdot\frac{y}{\delta}$$

The velocities and derivatives are: $u = U\cdot\sin(\eta)$

$$\frac{\partial}{\partial x}u = \frac{\partial}{\partial \eta}u\cdot\frac{\partial}{\partial x}\eta = U\cdot\cos(\eta)\cdot-\frac{\pi}{4\cdot x}\cdot\frac{y}{\delta} = -\frac{U}{2\cdot x}\cdot\eta\cdot\cos(\eta) \qquad \text{(Eqn. 1)}$$

$$\frac{\partial}{\partial y}u = \frac{\partial}{\partial \eta}u\cdot\frac{\partial}{\partial y}\eta = U\cdot\cos(\eta)\cdot\frac{\pi}{2\cdot\delta} = \frac{U\cdot\pi}{2\cdot\delta}\cdot\cos(\eta) \qquad \text{(Eqn. 2)}$$

$$v = \frac{U}{\pi}\cdot\frac{\delta}{x}\cdot(\cos(\eta) + \eta\cdot\sin(\eta) - 1)$$

We find the derivatives of v using product and chain rules:

$$\frac{\partial}{\partial x}v = \frac{U}{\pi}\cdot\left[\left(\frac{1}{x}\cdot\frac{\delta}{2\cdot x} - \frac{\delta}{x^2}\right)\cdot(\cos(\eta) + \eta\cdot\sin(\eta) - 1) + \frac{\delta}{x}\cdot(-\sin(\eta) + \sin(\eta) + \eta\cdot\cos(\eta))\cdot-\frac{\pi}{4\cdot x}\cdot\frac{y}{\delta}\right]$$

Simplifying this expression:

$$\frac{\partial}{\partial x}v = -\frac{U\cdot\delta}{2\cdot\pi\cdot x^2}\cdot\left[(\cos(\eta) + \eta\cdot\sin(\eta) - 1) + \eta^2\cdot\cos(\eta)\right] \qquad \text{(Eqn. 3)}$$

$$\frac{\partial}{\partial y}v = \frac{\partial}{\partial \eta}v \cdot \frac{\partial}{\partial y}\eta = \frac{U}{\pi}\cdot\frac{\delta}{x}\cdot(-\sin(\eta) + \sin(\eta) + \eta\cdot\cos(\eta))\cdot\frac{\pi}{2\cdot\delta} = \frac{U}{2\cdot x}\cdot\eta\cdot\cos(\eta) \qquad \text{(Eqn. 4)}$$

So the accelerations are:

$$a_{px} = u\cdot\frac{\partial}{\partial x}u + v\cdot\frac{\partial}{\partial y}u = U\cdot\sin(\eta)\cdot -\frac{U}{2\cdot x}\cdot\eta\cdot\cos(\eta) + \frac{U}{\pi}\cdot\frac{\delta}{x}\cdot(\cos(\eta) + \eta\cdot\sin(\eta) - 1)\cdot\frac{U\cdot\pi}{2\cdot\delta}\cdot\cos(\eta)$$

Simplifying this expression:

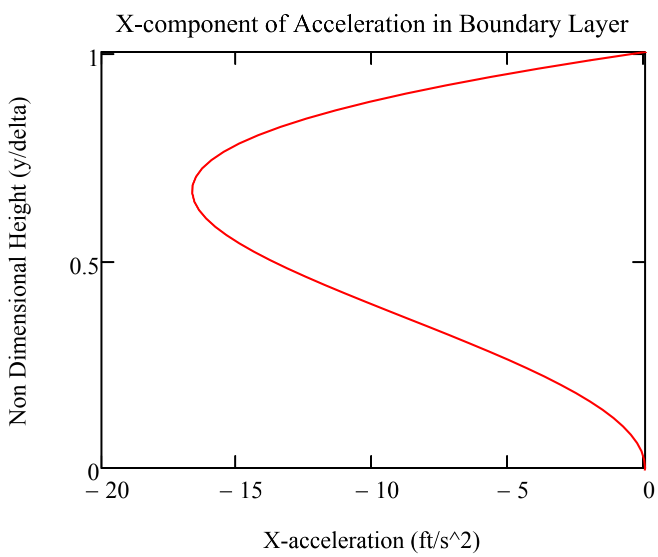
$$a_{px} = \frac{U^2}{2\cdot x}\cdot\cos(\eta)\cdot(\cos(\eta) - 1)$$

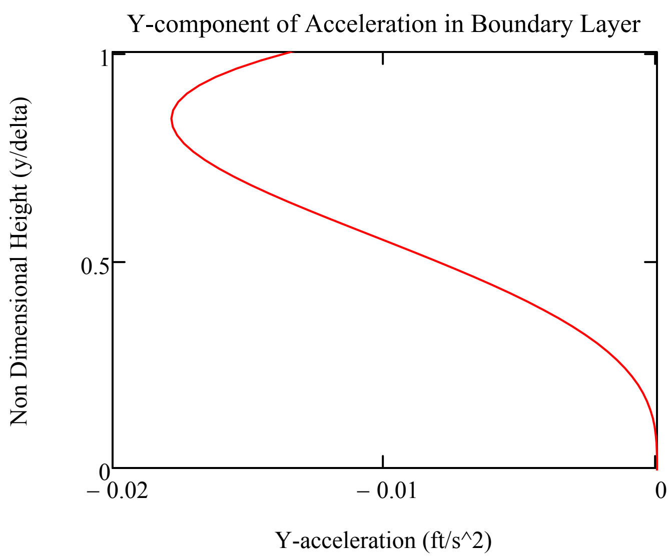
$$a_{py} = u\cdot\frac{\partial}{\partial x}v + v\cdot\frac{\partial}{\partial y}v = U\cdot\sin(\eta)\cdot -\frac{U\cdot\delta}{2\cdot\pi\cdot x^2}\cdot\left(\cos(\eta) + \eta\cdot\sin(\eta) - 1 + \eta^2\cdot\cos(\eta)\right) + \frac{U}{\pi}\cdot\frac{\delta}{x}\cdot(\cos(\eta) + \eta\cdot\sin(\eta) - 1)\cdot\frac{U}{2\cdot x}\cdot\eta\cdot\cos(\eta)$$

Simplifying this expression:

$$a_{py} = \frac{U^2\cdot\delta}{2\cdot\pi\cdot x^2}\cdot\left[\eta\cdot\cos(\eta)\cdot(\cos(\eta) + \eta\cdot\sin(\eta) - 1) - \sin(\eta)\cdot\left[\left(1 + \eta^2\right)\cdot\cos(\eta) + \eta\cdot\sin(\eta) - 1\right]\right]$$

Here are the plots of the acceleration components:





The maximum values and their locations may be found using Excel or Mathcad:

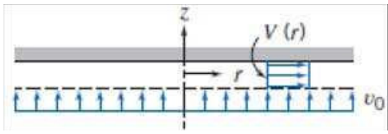
$$a_{pxmax} = -16.7\cdot\frac{ft}{s^2} \qquad a_{pymax} = -0.0178\cdot\frac{ft}{s^2}$$

$$\frac{y}{\delta} = 0.667 \qquad \frac{y}{\delta} = 0.839$$

Problem 5.63

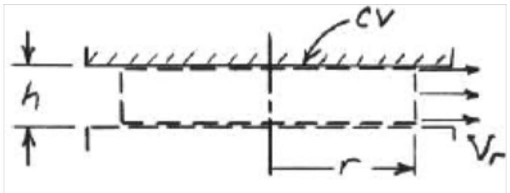
[Difficulty: 3]

5.63 Air flows into the narrow gap, of height h, between closely spaced parallel disks through a porous surface as shown. Use a control volume, with outer surface located at position r, to show that the uniform velocity in the r direction is $V = v_0 r/2h$. Find an expression for the velocity component in the z direction ($v_0 \ll V$). Evaluate the components of acceleration for a fluid particle in the gap.



Given: Flow between parallel disks through porous surface

Find: (a) show that $V_r = v_o r/2h$
(b) expression for the z-component of velocity ($v_o << V$)
(c) expression for acceleration of fluid particle in the gap



Solution: We will apply the continuity equation to the control volume shown:

Governing Equations:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho dV + \int_{CS} \rho \vec{V} \cdot d\vec{A} \qquad \text{(Continuity)}$$

$$\vec{a}_p = \frac{D\vec{V}}{Dt} = \left(\vec{V} \cdot \nabla\right)\vec{V} + \frac{\partial \vec{V}}{\partial t} \qquad \text{(Particle Accleration)}$$

Assumptions:
(1) Steady flow
(2) Incompressible flow
(3) Uniform flow at every section
(4) Velocity in θ-direction is zero

Based on the above assumptions the continuity equation reduces to: $0 = -\rho \cdot v_0 \cdot \pi \cdot r^2 + \rho \cdot V_r \cdot 2 \cdot \pi \cdot r \cdot h$ Solving for Vr: $V_r = v_0 \cdot \frac{r}{2 \cdot h}$

We apply the differential form of continuity to find V_z: $\frac{1}{r} \cdot \frac{\partial}{\partial r}\left(r \cdot V_r\right) + \frac{\partial}{\partial z} V_z = 0$ $\quad \frac{1}{r} \cdot \frac{\partial}{\partial r}\left(r \cdot V_r\right) = \frac{v_0}{h} = -\frac{\partial}{\partial z} V_z$ Therefore:

$V_z = -\int \frac{v_0}{h}\, dz + f(r) = -v_0 \cdot \frac{z}{h} + f(r)$ Now at z = 0: $V_z = v_0$ Therefore we can solve for f(r): $v_0 = -v_0 \cdot \frac{0}{h} + f(r)$ $\quad f(r) = v_0$

So we find that the z-component of velocity is: $V_z = v_0 \cdot \left(1 - \frac{z}{h}\right)$

Based on the above assumptions the particle acceleration reduces to: $\vec{a}_p = V_r \frac{\partial \vec{V}}{\partial r} + V_z \frac{\partial \vec{V}}{\partial z}$

$\frac{\partial}{\partial r} V_r = \frac{v_0}{2 \cdot h} \quad \frac{\partial}{\partial z} V_r = 0 \quad \frac{\partial}{\partial r} V_z = 0 \quad \frac{\partial}{\partial z} V_z = -\frac{v_0}{h}$

So the accelerations are:

$$a_{pr} = V_r \cdot \frac{\partial}{\partial r} V_r + V_z \cdot \frac{\partial}{\partial z} V_r = v_0 \cdot \frac{r}{2 \cdot h} \cdot \frac{v_0}{2 \cdot h} + v_0 \cdot \left(1 - \frac{z}{h}\right) \cdot 0 \qquad a_{pr} = \frac{v_0^2 \cdot r}{4 \cdot h^2}$$

$$a_{pz} = V_r \cdot \frac{\partial}{\partial r} V_z + V_z \cdot \frac{\partial}{\partial z} V_z = v_0 \cdot \frac{r}{2 \cdot h} \cdot 0 + v_0 \cdot \left(1 - \frac{z}{h}\right) \cdot -\frac{v_0}{h} \qquad a_{pz} = \frac{v_0^2}{h} \cdot \left(\frac{z}{h} - 1\right)$$

Problem 5.64

[Difficulty: 3]

5.64 The velocity field for steady inviscid flow from left to right over a circular cylinder, of radius R, is given by

$$\vec{V} = U\cos\theta\left[1-\left(\frac{R}{r}\right)^2\right]\hat{e}_r - U\sin\theta\left[1+\left(\frac{R}{r}\right)^2\right]\hat{e}_\theta$$

Obtain expressions for the acceleration of a fluid particle moving along the stagnation streamline ($\theta = \pi$) and for the acceleration along the cylinder surface ($r = R$). Plot a_r as a function of r/R for $\theta = \pi$, and as a function of θ for $r = R$; plot a_θ as a function of θ for $r = R$. Comment on the plots. Determine the locations at which these accelerations reach maximum and minimum values.

Given: Steady inviscid flow over a circular cylinder of radius R

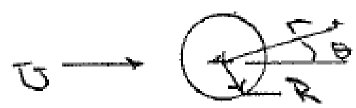
Find:
(a) Expression for acceleration of particle moving along $\theta = \pi$
(b) Expression for accleeration of particle moving along $r = R$
(c) Locations at which accelerations in r- and θ- directions reach maximum and minimum values
(d) Plot a_r as a function of R/r for $\theta = \pi$ and as a function of θ for r = R
(e) Plot a_θ as a function of θ for r = R

Solution: We will apply the particle acceleration definition to the velocity field

Governing Equation:

$$\vec{a}_p = \frac{D\vec{V}}{Dt} = \left(\vec{V}\cdot\nabla\right)\vec{V} + \frac{\partial\vec{V}}{\partial t} \qquad \text{(Particle Accleration)}$$

Assumptions:
(1) Steady flow
(2) Inviscid flow
(3) No flow in z-direction, velocity is not a function of z



Based on the above assumptions the particle acceleration reduces to: $\vec{a}_p = V_r\dfrac{\partial\vec{V}}{\partial r} + \dfrac{V_\theta}{r}\dfrac{\partial\vec{V}}{\partial\theta}$ and the components are:

$$a_{pr} = V_r\cdot\frac{\partial}{\partial r}V_r + \frac{V_\theta}{r}\cdot\frac{\partial}{\partial\theta}V_r - \frac{V_\theta^{\,2}}{r} \qquad a_{p\theta} = V_r\cdot\frac{\partial}{\partial r}V_\theta + \frac{V_\theta}{r}\cdot\frac{\partial}{\partial\theta}V_\theta + \frac{V_r\cdot V_\theta}{r}$$

When $\theta = \pi$: $\quad V_r = -U\cdot\left[1-\left(\dfrac{R}{r}\right)^2\right] \quad V_\theta = 0 \quad \dfrac{\partial}{\partial r}V_r = -U\cdot -2\cdot -\dfrac{R^2}{r^3} = -2\cdot U\cdot\dfrac{R^2}{r^3} \quad \dfrac{\partial}{\partial\theta}V_r = 0 \quad \dfrac{\partial}{\partial r}V_\theta = 0 \quad \dfrac{\partial}{\partial\theta}V_\theta = 0$

So the accelerations are:

$$a_{pr} = -U\cdot\left[1-\left(\frac{R}{r}\right)^2\right]\cdot -2\cdot U\cdot\frac{R^2}{r^3} = \frac{2\cdot U^2}{R}\cdot\left(\frac{R}{r}\right)^3\cdot\left[1-\left(\frac{R}{r}\right)^2\right]$$

$$a_{pr} = \frac{2\cdot U^2}{R}\cdot\left(\frac{R}{r}\right)^3\cdot\left[1-\left(\frac{R}{r}\right)^2\right]$$

$$a_{p\theta} = 0$$

To find the maximum acceleration, we take the derivative of the accleration and set it to zero: Let $\eta = \dfrac{R}{r}$

$$\frac{d}{d\eta}a_{pr} = \frac{2\cdot U^2}{R}\cdot\left[3\cdot\eta^2\cdot\left(1-\eta^2\right) - \eta^3\cdot 2\cdot\eta\right] = \frac{2\cdot U^2}{R}\left(-5\cdot\eta^4 + 3\eta^2\right) = 0 \quad \text{Therefore:} \quad \eta = \sqrt{\frac{3}{5}} \quad \text{or} \quad r = 1.291\cdot R$$

The maximum acceleration would then be:

$$a_{prmax} = \frac{2\cdot U^2}{R}\cdot\left(\frac{1}{1.291}\right)^3\cdot\left[1-\left(\frac{1}{1.291}\right)^2\right] \qquad a_{prmax} = 0.372\cdot\frac{U^2}{R}$$

When r = R: $V_r = 0$ $V_\theta = -2\cdot U\cdot\sin(\theta)$ $\frac{\partial}{\partial r}V_r = 0$ $\frac{\partial}{\partial\theta}V_r = 0$ $\frac{\partial}{\partial r}V_\theta = 0$ $\frac{\partial}{\partial\theta}V_\theta = -2\cdot U\cdot\cos(\theta)$

So the accelerations are: $a_{pr} = -\frac{(-2\cdot U\cdot\cos(\theta))^2}{R} = -\frac{4\cdot U^2}{R}\cdot(\sin(\theta))^2$

$$a_{pr} = -\frac{4\cdot U^2}{R}\cdot(\sin(\theta))^2$$

$a_{p\theta} = \frac{-2\cdot U\cdot\sin(\theta)}{R}\cdot -2\cdot U\cdot\cos(\theta) = \frac{4\cdot U^2}{R}\cdot\sin(\theta)\cdot\cos(\theta)$

$$a_{p\theta} = \frac{4\cdot U^2}{R}\cdot\sin(\theta)\cdot\cos(\theta)$$

Radial acceleration is minimum at $\theta = 180\cdot deg$

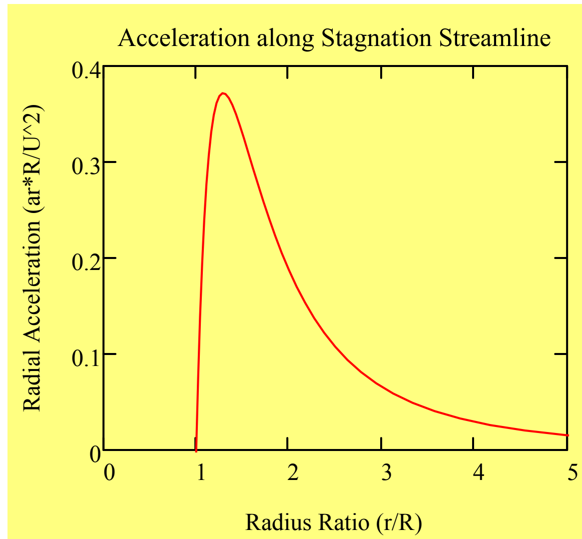
Accelerations at this angle are: $a_{rmin} = -4\cdot\frac{U^2}{R}$ $a_\theta = 0$

Azimuthal acceleration is maximum at $\theta = 45\cdot deg$

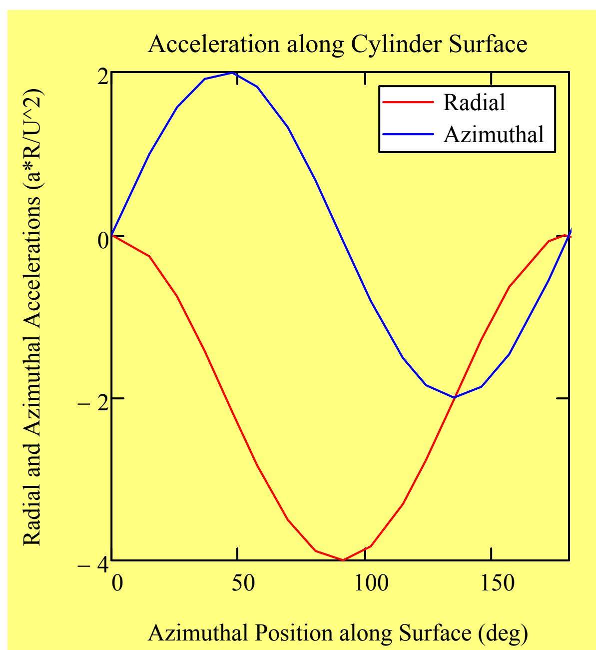
Accelerations at this angle are: $a_r = -2\cdot\frac{U^2}{R}$ $a_{\theta max} = 2\cdot\frac{U^2}{R}$

Azimuthal acceleration is minimum at $\theta = 135\cdot deg$

Accelerations at this angle are: $a_r = -2\cdot\frac{U^2}{R}$ $a_{\theta min} = -2\cdot\frac{U^2}{R}$



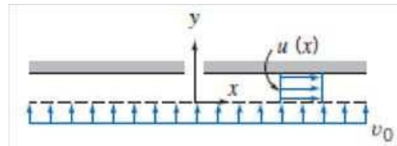
The plots of acceleration along the stagnation streamline and the cylinder surface are shown here. In all cases the accelerations have been normalized by U^2/R



Problem 5.65

[Difficulty: 3]

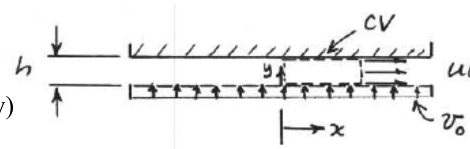
5.65 Air flows into the narrow gap, of height h, between closely spaced parallel plates through a porous surface as shown. Use a control volume, with outer surface located at position x, to show that the uniform velocity in the x direction is $u = v_0 x/h$. Find an expression for the velocity component in the y direction. Evaluate the acceleration of a fluid particle in the gap.



Given: Air flow through porous surface into narrow gap

Find:
(a) show that $u(x) = v_0 x/h$
(b) expression for the y-component of velocity
(c) expression for acceleration of fluid particle in the gap

Solution: We will apply the continuity equation to the control volume shown:



Governing Equations:

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A} \qquad \text{(Continuity)}$$

$$\vec{a}_p = \frac{D\vec{V}}{Dt} = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} + \frac{\partial \vec{V}}{\partial t} \qquad \text{(Particle Accleration)}$$

Assumptions:
(1) Steady flow
(2) Incompressible flow
(3) Uniform flow at every section

Based on the above assumptions the continuity equation reduces to: $0 = -x\cdot w\cdot v_0 + h\cdot w\cdot u(x)$ Solving for u: $u(x) = v_0\cdot\frac{x}{h}$

We apply the differential form of continuity to find v: $\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$ $\quad \frac{\partial}{\partial x}u = \frac{v_0}{h} = -\frac{\partial}{\partial y}v$ Therefore the y-velocity v is:

$v = -\int \frac{v_0}{h}\,dy + f(x) = -v_0\cdot\frac{y}{h} + f(x)$ Now at y = 0: $v = v_0$ Therefore we can solve for f(x): $v_0 = -v_0\cdot\frac{0}{h} + f(x)$ $\quad f(x) = v_0$

So we find that the y-component of velocity is: $v = v_0\cdot\left(1 - \frac{y}{h}\right)$

Based on the above assumptions the particle acceleration reduces to: $\vec{a}_p = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y}$

$\frac{\partial}{\partial x}u = \frac{v_0}{h} \qquad \frac{\partial}{\partial y}u = 0 \qquad \frac{\partial}{\partial x}v = 0 \qquad \frac{\partial}{\partial y}v = -\frac{v_0}{h}$

So the accelerations are:

$$a_{px} = u\cdot\frac{\partial}{\partial x}u + v\cdot\frac{\partial}{\partial y}u = v_0\cdot\frac{x}{h}\cdot\frac{v_0}{h} + v_0\cdot\left(1 - \frac{y}{h}\right)\cdot 0 \qquad a_{px} = \frac{v_0^{\,2}\cdot x}{h^2}$$

$$a_{py} = u\cdot\frac{\partial}{\partial x}v + v\cdot\frac{\partial}{\partial y}v = v_0\cdot\frac{x}{h}\cdot 0 + v_0\cdot\left(1 - \frac{y}{h}\right)\cdot -\frac{v_0}{h} \qquad a_{py} = \frac{v_0^{\,2}}{h}\cdot\left(\frac{y}{h} - 1\right)$$

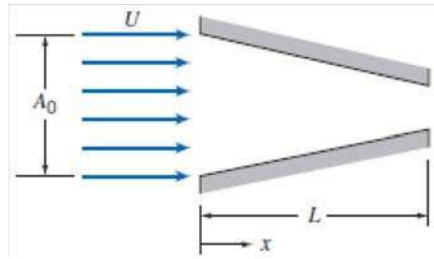
The acceleration vector would be:

$$\vec{a}_p = \frac{v_0^2}{h}\left[\frac{x}{h}\hat{i} + \left(\frac{y}{h} - 1\right)\hat{j}\right]$$

Problem 5.66

[Difficulty: 3]

5.66 Consider the incompressible flow of a fluid through a nozzle as shown. The area of the nozzle is given by $A = A_0(1 - bx)$ and the inlet velocity varies according to $U = U_0(0.5 + 0.5\cos\omega t)$ where $A_0 = 5\ \text{ft}^2$, $L = 20$ ft, $b = 0.02\ \text{ft}^{-1}$, $\omega = 0.16$ rad/s, and $U_0 = 20$ ft/s. Find and plot the acceleration on the centerline, with time as a parameter.



Given: Velocity field and nozzle geometry

Find: Acceleration along centerline; plot

Solution:

Assumption: Incompressible flow

The given data is $A_0 = 5\cdot\text{ft}^2$ $\quad L = 20\cdot\text{ft}$ $\quad b = 0.2\cdot\text{ft}^{-1}$ $\quad U_0 = 20\cdot\frac{\text{ft}}{\text{s}}$ $\quad \omega = 0.16\cdot\frac{\text{rad}}{\text{s}}$ $\quad A(x) = A_0\cdot(1 - b\cdot x)$

The velocity on the centerline is obtained from continuity $\quad u(x)\cdot A(x) = U_0\cdot A_o$

so

$$u(x,t) = \frac{A_0}{A(x)}\cdot U_0\cdot(0.5 + 0.5\cdot\cos(\omega\cdot t)) = \frac{U_0}{(1 - b\cdot x)}\cdot(0.5 + 0.5\cdot\cos(\omega\cdot t))$$

The acceleration is given by

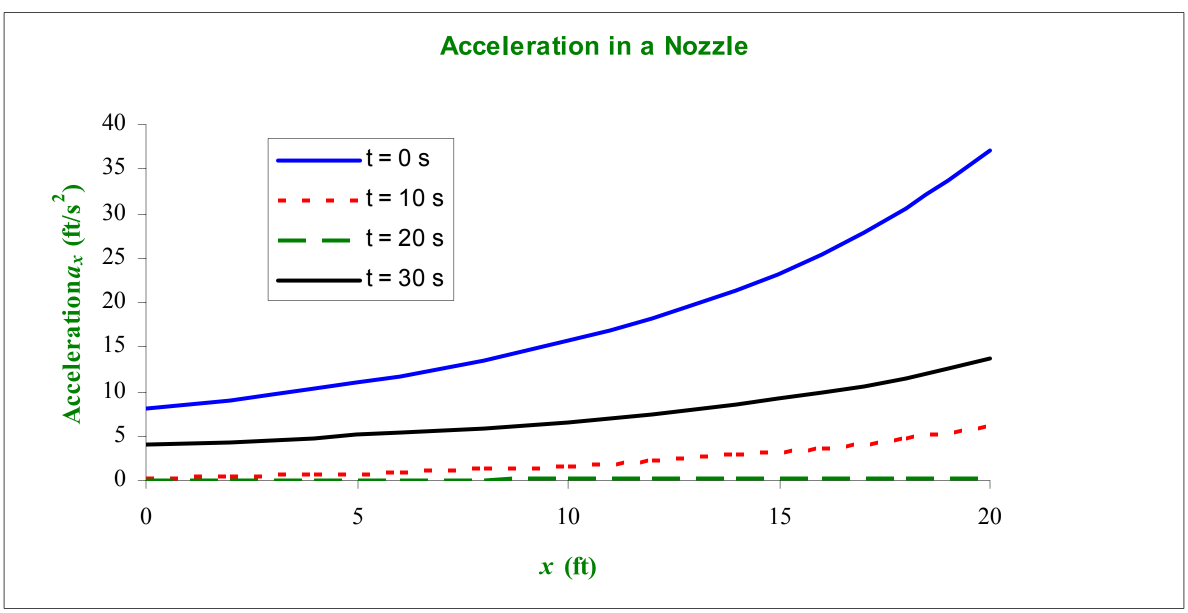
$$\vec{a}_p = \underbrace{\frac{D\vec{V}}{Dt}}_{\substack{\text{total}\\\text{acceleration}\\\text{of a particle}}} = \underbrace{u\frac{\partial\vec{V}}{\partial x} + v\frac{\partial\vec{V}}{\partial y} + w\frac{\partial\vec{V}}{\partial z}}_{\substack{\text{convective}\\\text{acceleration}}} + \underbrace{\frac{\partial\vec{V}}{\partial t}}_{\substack{\text{local}\\\text{acceleration}}}$$

For the present 1D flow

$$a_x = \frac{\partial}{\partial t}u + u\cdot\frac{\partial}{\partial x}u = -\frac{0.5\cdot U_0\cdot\omega\cdot\sin(\omega\cdot t)}{1 - b\cdot x} + \frac{U_0}{(1 - b\cdot x)}\cdot(0.5 + 0.5\cdot\cos(\omega\cdot t))\cdot\left[\frac{U_0\cdot b\cdot(0.5\cdot\cos(\omega\cdot t) + 0.5)}{(1 - b\cdot x)^2}\right]$$

$$a_x = \frac{U_0}{(1 - b\cdot x)}\cdot\left[-(0.5\cdot\omega\cdot\sin(\omega\cdot t)) + (0.5 + 0.5\cdot\cos(\omega\cdot t))\cdot\left[\frac{U_0\cdot b\cdot(0.5\cdot\cos(\omega\cdot t) + 0.5)}{(1 - b\cdot x)^2}\right]\right]$$

The plot is shown here:



Problem 5.67

[Difficulty: 4]

5.67 Consider again the steady, two-dimensional velocity field of Problem 5.56. Obtain expressions for the particle coordinates, $x_p = f_1(t)$ and $y_p = f_2(t)$, as functions of time and the initial particle position, (x_0, y_0) at $t = 0$. Determine the time required for a particle to travel from initial position, $(x_0, y_0) = (\frac{1}{2}, 2)$ to positions $(x, y) = (1, 1)$ and $(2, \frac{1}{2})$. Compare the particle accelerations determined by differentiating $f_1(t)$ and $f_2(t)$ with those obtained in Problem 5.56.

5.56 A steady, two-dimensional velocity field is given by $\vec{V} = Ax\hat{i} - Ay\hat{j}$, where $A = 1\,\mathrm{s}^{-1}$. Show that the streamlines for this flow are rectangular hyperbolas, $xy = C$. Obtain a general expression for the acceleration of a fluid particle in this velocity field. Calculate the acceleration of fluid particles at the points $(x, y) = (\frac{1}{2}, 2)$, $(1, 1)$, and $(2, \frac{1}{2})$, where x and y are measured in meters. Plot streamlines that correspond to $C = 0$, 1, and 2 m^2 and show the acceleration vectors on the streamline plot.

Given: Steady, two-dimensional velocity field of Problem 5.56

Find: (a) expressions for particle coordinates, $x_p = f_1(t)$ and $y_p = f_2(t)$
(b) Time requires for particle to travel from (0.5, 2) to (1, 1) and (2, 0.5)
(c) compare acceleration determined from $f_1(t)$ and $f_2(t)$ to those found in Problem 5.56

Solution: We will apply the particle acceleration definition to the velocity field

Governing Equation:

$$\vec{a}_p = \frac{D\vec{V}}{Dt} = \left(\vec{V}\cdot\nabla\right)\vec{V} + \frac{\partial \vec{V}}{\partial t} \qquad \text{(Particle Accleration)}$$

Assumptions:
(1) Incompressible flow
(2) Two-dimensional flow
(3) Steady flow

For the given flow, $u = A\cdot x$ and $v = -A\cdot y$ Thus $u_p = \frac{d}{dt}f_1 = A\cdot x_p = A\cdot f_1$ or $\frac{df_1}{f_1} = A\cdot dt$ Integrating from x_0 to f_1 yields:

$$\int_{x_0}^{f_1} \frac{1}{f_1}\,df_1 = \ln\left(\frac{f_1}{x_0}\right) = A\cdot t$$ Solving for f_1 yields: $f_1(t) = x_0\cdot e^{A\cdot t}$

Similarly, we can find: $v_p = \frac{d}{dt}f_2 = -A\cdot y_p = -A\cdot f_2$ or $\frac{df_2}{f_2} = -A\cdot dt$ Integrating from y_0 to f_2 yields: $f_2(t) = y_0\cdot e^{-A\cdot t}$

In this problem, $x_0 = \frac{1}{2}\cdot m$ and $y_0 = 2\cdot m$ Knowing the final position, we can solve for the time required.

To reach (1, 1):

$x = 1.0\cdot m$ $\quad t = \ln(2)\times 1s \quad t = 0.693\,s$ $\qquad y = 1.0\cdot m \quad t = -\ln\left(\frac{1}{2}\right)\times 1s \quad t = 0.693\,s$ $\qquad t = 0.693\,s$

To reach (2, 0.5):

$x = 2.0\cdot m$ $\quad t = \ln(4)\times 1s \quad t = 1.386\,s$ $\qquad y = 0.5\cdot m \quad t = -\ln\left(\frac{1}{4}\right)\times 1s \quad t = 1.386\,s$ $\qquad t = 1.386\,s$

The acceleration components are:

$$a_x = \frac{d^2}{dt^2}f_1 = x_0\cdot A^2\cdot e^{A\cdot t} = A^2\cdot f_1(t) \qquad a_y = \frac{d^2}{dt^2}f_2 = x_0\cdot(-A)^2\cdot e^{-A\cdot t} = A^2\cdot f_2(t)$$

At (x, y) = (1, 1): $\vec{a}_p = \left(\hat{i} + \hat{j}\right)\frac{m}{s^2}$

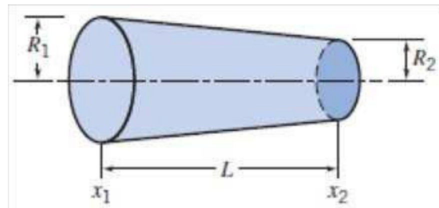
At (x, y) = (2, 0.5): $\vec{a}_p = \left(2\hat{i} + 0.5\hat{j}\right)\frac{m}{s^2}$

These results are identical to the accelerations calculated in Problem 5.56.

Problem 5.68

[Difficulty: 3]

5.68 Consider the one-dimensional, incompressible flow through the circular channel shown. The velocity at section ① is given by $U = U_0 + U_1 \sin \omega t$, where $U_0 = 20$ m/s, $U_1 = 2$ m/s, and $\omega = 0.3$ rad/s. The channel dimensions are $L = 1$ m, $R_1 = 0.2$ m, and $R_2 = 0.1$ m. Determine the particle acceleration at the channel exit. Plot the results as a function of time over a complete cycle. On the same plot, show the acceleration at the channel exit if the channel is constant area, rather than convergent, and explain the difference between the curves.



Given: One-dimensional, incompressible flow through circular channel.

Find:
(a) the acceleration of a particle at the channel exit
(b) plot as a function of time for a compleye cycle.
(c) plot acceleration if channel is constant area
(d) explain difference between the two acceleration cases

Solution: We will apply the particle acceleration definition to the velocity field

Governing Equations:

$$\vec{a}_p = \frac{D\vec{V}}{Dt} = \left(\vec{V} \cdot \nabla\right)\vec{V} + \frac{\partial \vec{V}}{\partial t} \qquad \text{(Particle Accleration)}$$

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A} \qquad \text{(Continuity equation)}$$

Assumptions:
(1) Incompressible flow
(2) One-dimensional flow

Based on the above assumptions the continuity equation can provide the velocity at any location: $u = U \cdot \frac{A_1}{A} = \left(\frac{R_1}{r}\right)^2$

Now based on the geometry of the channel we can write $r = R_1 - \left(R_1 - R_2\right) \cdot \frac{x}{L} = R_1 - \Delta R \cdot \frac{x}{L}$ Therefore the flow speed is:

$$u = U \cdot \frac{R_1^{\,2}}{\left(R_1 - \Delta R \cdot \frac{x}{L}\right)^2} = \frac{\left(U_0 + U_1 \cdot \sin(\omega \cdot t)\right)}{\left[1 - \frac{\Delta R}{R_1} \cdot \left(\frac{x}{L}\right)\right]^2}$$

Based on the above assumptions the particle acceleration reduces to:

$$\vec{a}_p = \left(u \frac{\partial u}{\partial x} + \frac{\partial u}{\partial t}\right)\hat{i}$$

Substituting the velocity and derivatives into this expression we can get the acceleration in the x-direction:

$$a_x = \frac{\left(U_0 + U_1 \cdot \sin(\omega \cdot t)\right)}{\left[1 - \frac{\Delta R}{R_1} \cdot \left(\frac{x}{L}\right)\right]^2} \cdot \frac{\left(U_0 + U_1 \cdot \sin(\omega \cdot t)\right)}{\left[1 - \frac{\Delta R}{R_1} \cdot \left(\frac{x}{L}\right)\right]^3} \cdot (-2) \cdot \left(-\frac{\Delta R}{R_1 \cdot L}\right) + \frac{\omega \cdot U_1 \cdot \cos(\omega \cdot t)}{\left[1 - \frac{\Delta R}{R_1} \cdot \left(\frac{x}{L}\right)\right]^2}$$

When we simplify this expression we get:

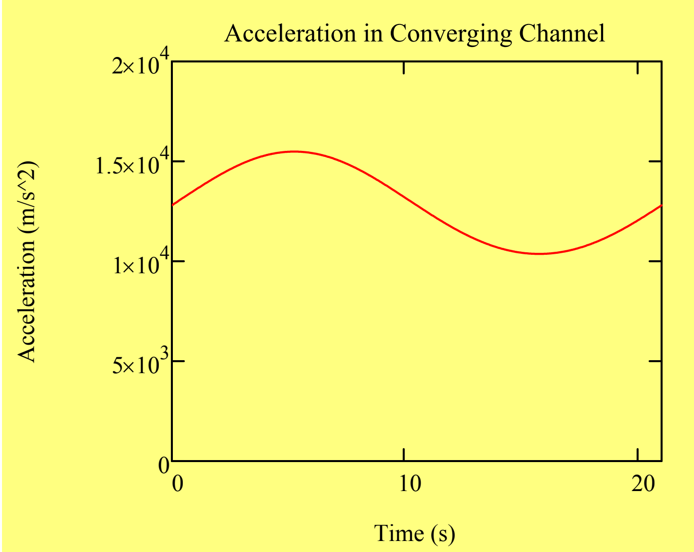
$$a_x = \frac{2 \cdot \Delta R}{R_1 \cdot L} \cdot \frac{\left(U_0 + U_1 \cdot \sin(\omega \cdot t)\right)^2}{\left[1 - \frac{\Delta R}{R_1} \cdot \left(\frac{x}{L}\right)\right]^5} + \frac{\omega \cdot U_1 \cdot \cos(\omega \cdot t)}{\left[1 - \frac{\Delta R}{R_1} \cdot \left(\frac{x}{L}\right)\right]^2}$$

Now we substitute the given values into this expression we get:

$$a_x = 2 \times 0.1 \cdot m \times \frac{1}{0.2 \cdot m} \times \frac{1}{1 \cdot m} \times (20 + 2 \cdot \sin(\omega \cdot t))^2 \cdot \frac{m^2}{s^2} \times \frac{1}{\left(1 - \frac{0.1 \cdot m}{0.2 \cdot m} \times 1\right)^5} + 0.3 \cdot \frac{rad}{s} \times 2 \cdot \frac{m}{s} \times \cos(\omega \cdot t) \times \frac{1}{\left(1 - \frac{0.1 \cdot m}{0.2 \cdot m} \times 1\right)^2}$$

$$a_x = \left[32 \cdot \left(20 + 2 \cdot \sin\left(0.3 \cdot \frac{rad}{s} \cdot t\right)\right)^2 + 2.4 \cdot \cos\left(0.3 \cdot \frac{rad}{s} \cdot t\right)\right] \cdot \frac{m}{s^2}$$

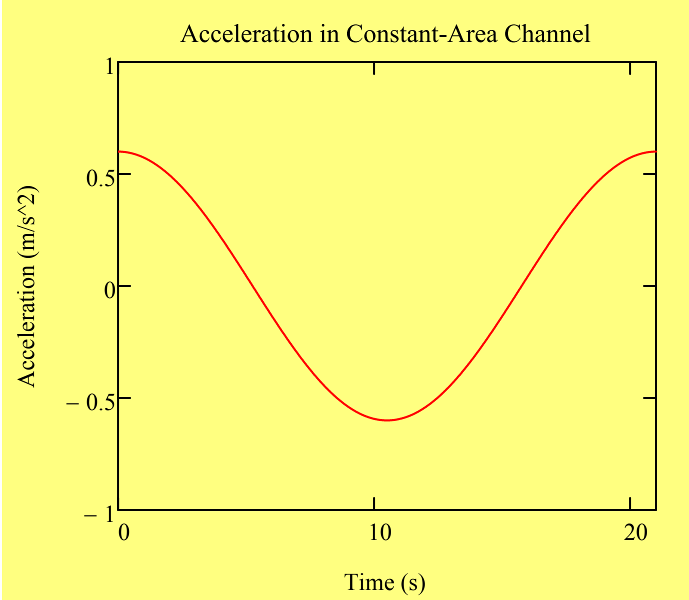
Here is a plot of the acceleration versus time.



For a constant area channel, ΔR = 0 and the acceleration becomes:

$$a_x = \left(0.6 \cdot \cos\left(0.3 \cdot \frac{rad}{s} \cdot t\right)\right) \cdot \frac{m}{s^2}$$

The plot of that acceleration is shown below.
The acceleration is so much larger for the converging channel than in the constant area channel because the convective acceleration is generated by the converging channel - the constant area channel has only local acceleration.



Problem 5.69

[Difficulty: 2]

5.69 Which, if any, of the following flow fields are irrotational?

(a) $u = 2x^2 + y^2 - x^2y;\ v = x^3 + x(y^2 - 2y)$

(b) $u = 2xy - x^2 + y;\ v = 2xy - y^2 + x^2$

(c) $u = xt + 2y;\ v = xt^2 - yt$

(d) $u = (x + 2y)xt;\ v = -(2x + y)yt$

Given: Velocity components

Find: Which flow fields are irrotational

Solution:

For a 2D field, the irrotationality the test is $\frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u = 0$

a) $u(x,y,t) = 2\cdot x^2 + y^2 \quad v(x,y,t) = x^3 + x\cdot\left(y^2 - 2\cdot y\right) \quad \frac{\partial}{\partial x}v(x,y,t) = 3\cdot x^2 + y^2 - 2\cdot y \quad \frac{\partial}{\partial y}u(x,y,t) = 2\cdot y$

Hence $\frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u \neq 0$

Not irrotational

b) $u(x,y,t) = 2\cdot x\cdot y - x^2 + y \quad v(x,y,t) = 2\cdot x\cdot y - y^2 + x^2 \quad \frac{\partial}{\partial x}v(x,y,t) = 2\cdot x + 2\cdot y \quad \frac{\partial}{\partial y}u(x,y,t) = 2\cdot x + 1$

Hence $\frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u \neq 0$

Not irrotational

c) $u(x,y,t) = x\cdot t + 2\cdot y \quad v(x,y,t) = x\cdot t^2 - y\cdot t \quad \frac{\partial}{\partial x}v(x,y,t) = t^2 \quad \frac{\partial}{\partial y}u(x,y,t) = 2$

Hence $\frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u \neq 0$

Not irrotational

d) $u(x,y,t) = (x + 2\cdot y)\cdot x\cdot t \quad v(x,y,t) = -(2\cdot x + y)\cdot y\cdot t \quad \frac{\partial}{\partial x}v(x,y,t) = -2\cdot t\cdot y \quad \frac{\partial}{\partial y}u(x,y,t) = 2\cdot t\cdot x$

Hence $\frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u \neq 0$

Not irrotational

Problem 5.70

[Difficulty: 4]

5.70 Expand $(\vec{V}\cdot\nabla)\vec{V}$ in cylindrical coordinates by direct substitution of the velocity vector to obtain the convective acceleration of a fluid particle. (Recall the hint in footnote 1 on page 178.) Verify the results given in Eqs. 5.12.

Given: Definition of "del" operator in cylindrical coordinates, velocity vector

Find: (a) An expression for $(\vec{V}\cdot\vec{\nabla})\vec{V}$ in cylindrical coordinates.
(b) Show result is identical to Equations 5.12.

Solution: We will apply the velocity field to the del operator and simplify.

Governing Equations:

$$\vec{\nabla} = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{k}\frac{\partial}{\partial z} \qquad \text{(Definition of "del" operator)}$$

$$\vec{V} = V_r\hat{e}_r + V_\theta\hat{e}_\theta + V_z\hat{k} \qquad \text{(Velocity flow field)}$$

$$\frac{\partial \hat{e}_r}{\partial \theta} = \hat{e}_\theta \qquad \frac{\partial \hat{e}_\theta}{\partial \theta} = -\hat{e}_r \qquad \text{(Hints from footnote)}$$

Substituting $(\vec{V}\cdot\vec{\nabla})\vec{V}$ using the governing equations yields:

$$(\vec{V}\cdot\vec{\nabla})\vec{V} = \left[\left(V_r\hat{e}_r + V_\theta\hat{e}_\theta + V_z\hat{k}\right)\cdot\left(\hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{k}\frac{\partial}{\partial z}\right)\right]\left(V_r\hat{e}_r + V_\theta\hat{e}_\theta + V_z\hat{k}\right)$$

$$= \left(V_r\frac{\partial}{\partial r} + \frac{V_\theta}{r}\frac{\partial}{\partial\theta} + V_z\frac{\partial}{\partial z}\right)\left(V_r\hat{e}_r + V_\theta\hat{e}_\theta + V_z\hat{k}\right)$$

$$= V_r\frac{\partial}{\partial r}\left(V_r\hat{e}_r + V_\theta\hat{e}_\theta + V_z\hat{k}\right) + \frac{V_\theta}{r}\frac{\partial}{\partial\theta}\left(V_r\hat{e}_r + V_\theta\hat{e}_\theta + V_z\hat{k}\right) + V_z\frac{\partial}{\partial z}\left(V_r\hat{e}_r + V_\theta\hat{e}_\theta + V_z\hat{k}\right)$$

$$= V_r\frac{\partial}{\partial r}V_r\hat{e}_r + V_r\frac{\partial}{\partial r}V_\theta\hat{e}_\theta + V_r\frac{\partial}{\partial r}V_z\hat{k} + \frac{V_\theta}{r}\frac{\partial}{\partial\theta}(V_r\hat{e}_r) + \frac{V_\theta}{r}\frac{\partial}{\partial\theta}(V_\theta\hat{e}_\theta) + \frac{V_\theta}{r}\frac{\partial}{\partial\theta}V_z\hat{k} + V_z\frac{\partial}{\partial z}V_r\hat{e}_r$$
$$+ V_z\frac{\partial}{\partial z}V_\theta\hat{e}_\theta + V_z\frac{\partial}{\partial z}V_z\hat{k}$$

Applying the product rule to isolate derivatives of the unit vectors:

$$(\vec{V}\cdot\vec{\nabla})\vec{V} = V_r\frac{\partial V_r}{\partial r}\hat{e}_r + V_r\frac{\partial V_\theta}{\partial r}\hat{e}_\theta + V_r\frac{\partial V_z}{\partial r}\hat{k} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial\theta}\hat{e}_r + \frac{V_\theta}{r}\frac{\partial \hat{e}_r}{\partial\theta}V_r + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial\theta}\hat{e}_\theta + \frac{V_\theta}{r}\frac{\partial \hat{e}_\theta}{\partial\theta}V_\theta + \frac{V_\theta}{r}\frac{\partial V_z}{\partial\theta}\hat{k}$$
$$+ V_z\frac{\partial V_r}{\partial z}\hat{e}_r + V_z\frac{\partial V_\theta}{\partial z}\hat{e}_\theta + V_z\frac{\partial V_z}{\partial z}\hat{k}$$

Collecting terms:

$$(\vec{V}\cdot\vec{\nabla})\vec{V} = \left(V_r\frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial\theta} - \frac{V_\theta^2}{r} + V_z\frac{\partial V_r}{\partial z}\right)\hat{e}_r + \left(V_r\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial\theta} + \frac{V_rV_\theta}{r} + V_z\frac{\partial V_\theta}{\partial z}\right)\hat{e}_\theta$$
$$+ \left(V_r\frac{\partial V_z}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_z}{\partial\theta} + V_z\frac{\partial V_z}{\partial z}\right)\hat{k}$$

The three terms in parentheses are the three components of convective acceleration given in Equations 5.12.

Problem 5.71

[Difficulty: 4]

5.71 Consider again the sinusoidal velocity profile used to model the x component of velocity for a boundary layer in Problem 5.12. Neglect the vertical component of velocity. Evaluate the circulation around the contour bounded by $x = 0.4$ m, $x = 0.6$ m, $y = 0$, and $y = 8$ mm. What would be the results of this evaluation if it were performed 0.2 m further downstream? Assume $U = 0.5$ m/s.

Given: Sinusoidal approximation to boundary-layer velocity profile:

$$u = U \cdot \sin\left(\frac{\pi}{2} \cdot \frac{y}{\delta}\right) \quad \text{where} \quad \delta = 5 \cdot \text{mm at} \quad x = 0.5 \cdot \text{m}$$

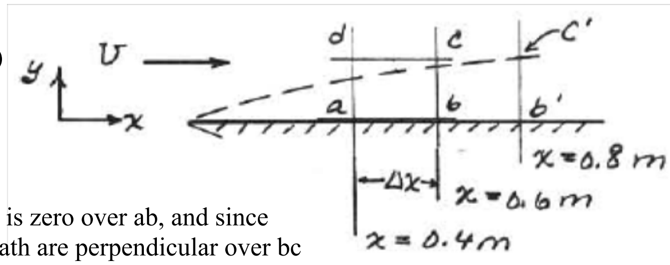
Neglect the vertical component of velocity. $U = 0.5 \cdot \frac{\text{m}}{\text{s}}$

Find: (a) Circulation about a contour bounded by x = 0.4 m, x = 0.6 m, y = 0, and y = 8 mm.
(b) Result if evaluated Δx = 0.2 m further downstream

Solution: We will apply the definition of circulation to the given velocity field.

Governing Equation:

$$\Gamma = \oint \vec{V} \cdot d\vec{s} \qquad \text{(Definition of circulation)}$$



From the definition of circulation we break up the integral:

$$\Gamma = \int \vec{V} \cdot d\vec{s} + \int \vec{V} \cdot d\vec{s} + \int \vec{V} \cdot d\vec{s} + \int_{da} \vec{V} \cdot d\vec{s}$$

Since the velocity is zero over ab, and since the velocity and path are perpendicular over bc and da:

$$\Gamma = \int_{x_c}^{x_d} U\,dx = -U \cdot (x_c - x_d) \qquad \Gamma = -0.5 \cdot \frac{\text{m}}{\text{s}} \times (0.6 \cdot \text{m} - 0.4 \cdot \text{m})$$

$$\boxed{\Gamma = -0.1 \frac{\text{m}^2}{\text{s}}}$$

At the downstream location, since $\delta = c \cdot x^{\frac{1}{2}}$ $\quad \delta' = \delta \cdot \left(\frac{x}{x'}\right)^{\frac{1}{2}}$ $\quad \delta' = 5 \cdot \text{mm} \cdot \left(\frac{0.8 \cdot \text{m}}{0.5 \cdot \text{m}}\right)^{\frac{1}{2}}$ $\quad \delta' = 6.325 \cdot \text{mm}$

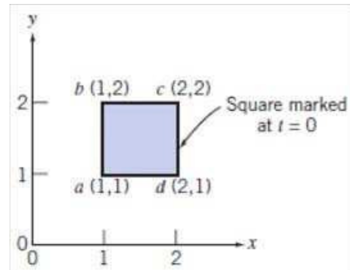
Now since the boundary layer is less than 8 mm thick at point c', the integral along c'c will be the same as that along cd.

$$\boxed{\Gamma_{bb'c'c} = \Gamma_{abcd}}$$

Problem 5.72

[Difficulty: 2]

5.72 Consider the velocity field for flow in a rectangular "corner," $\vec{V} = Ax\hat{i} - Ay\hat{j}$, with $A = 0.3\,\mathrm{s}^{-1}$, as in Example 5.8. Evaluate the circulation about the unit square of Example 5.8.



Given: Velocity field for flow in a rectangular corner as in Example 5.8.

Find: Circulation about the unit square shown above.

Solution: We will apply the definition of circulation to the given velocity field.

Governing Equation:

$$\Gamma = \oint \vec{V} \cdot d\vec{s} \qquad \text{(Definition of circulation)}$$

From the definition of circulation we break up the integral: $\Gamma = \int_{ab} \vec{V} \cdot d\vec{s} + \int_{bc} \vec{V} \cdot d\vec{s} + \int_{cd} \vec{V} \cdot d\vec{s} + \int_{da} \vec{V} \cdot d\vec{s}$

The integrand is equal to: $\vec{V} \cdot d\vec{s} = \left(Ax\hat{i} - Ay\hat{j}\right) \cdot \left(dx\hat{i} + dy\hat{j}\right) = Axdx - Aydy$ Therefore, the circulation is equal to:

$$\Gamma = \int_{x_a}^{x_d} A \cdot x\, dx + \int_{y_d}^{y_c} -A \cdot y\, dy + \int_{x_c}^{x_b} A \cdot x\, dx + \int_{y_b}^{y_a} -A \cdot y\, dy = \frac{A}{2} \cdot \left[\left(x_d^{\,2} - x_a^{\,2}\right) - \left(y_c^{\,2} - y_d^{\,2}\right) + \left(x_b^{\,2} - x_c^{\,2}\right) - \left(y_a^{\,2} - y_b^{\,2}\right)\right]$$

$$\Gamma = \frac{1}{2} \times 0.3 \cdot \frac{1}{s} \times \left[\left(2^2 - 1^2\right) - \left(2^2 - 1^2\right) + \left(1^2 - 2^2\right) - \left(1^2 - 2^2\right)\right] \cdot m^2$$

$$\boxed{\Gamma = 0\,\frac{m^2}{s}}$$

This result is to be expected since the flow is irrotational and by Stokes' theorem, the circulation is equal to the curl of the velocity over the bounded area (Eqn. 5.18).

Problem 5.73

[Difficulty: 3]

5.73 A flow is represented by the velocity field $\vec{V} = (x^7 - 21x^5y^2 + 35x^3y^4 - 7xy^6)\hat{i} + (7x^6y - 35x^4y^3 + 21x^2y^5 - y^7)\hat{j}$. Determine if the field is (a) a possible incompressible flow and (b) irrotational.

Given: Flow field

Find: If the flow is incompressible and irrotational

Solution:

Basic equations: Incompressibility $\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$ Irrotationality $\frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u = 0$

a) $u(x,y) = x^7 - 21\cdot x^5\cdot y^2 + 35\cdot x^3\cdot y^4 - 7\cdot x\cdot y^6$ $v(x,y) = 7\cdot x^6\cdot y - 35\cdot x^4\cdot y^3 + 21\cdot x^2\cdot y^5 - y^7$

$\frac{\partial}{\partial x}u(x,y) = 7\cdot x^6 - 105\cdot x^4\cdot y^2 + 105\cdot x^2\cdot y^4 - 7\cdot y^6$ $\frac{\partial}{\partial y}v(x,y) = 7\cdot x^6 - 105\cdot x^4\cdot y^2 + 105\cdot x^2\cdot y^4 - 7\cdot y^6$

Hence $\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v \neq 0$ COMPRESSIBLE

b) $u(x,y) = x^7 - 21\cdot x^5\cdot y^2 + 35\cdot x^3\cdot y^4 - 7\cdot x\cdot y^6$ $v(x,y) = 7\cdot x^6\cdot y - 35\cdot x^4\cdot y^3 + 21\cdot x^2\cdot y^5 - y^7$

$\frac{\partial}{\partial x}v(x,y) = 42\cdot x^5\cdot y - 140\cdot x^3\cdot y^3 + 42\cdot x\cdot y^5$ $-\frac{\partial}{\partial y}u(x,y) = 42\cdot x^5\cdot y - 140\cdot x^3\cdot y^3 + 42\cdot x\cdot y^5$

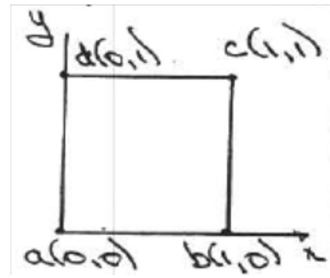
Hence $\frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u \neq 0$ ROTATIONAL

Note that if we define $v(x,y) = -\left(7\cdot x^6\cdot y - 35\cdot x^4\cdot y^3 + 21\cdot x^2\cdot y^5 - y^7\right)$ then the flow is incompressible and irrotational!

Problem 5.74

[Difficulty: 2]

5.74 Consider the two-dimensional flow field in which $u = Ax^2$ and $v = Bxy$, where $A = 1/2\ \text{ft}^{-1}\cdot\text{s}^{-1}$, $B = -1\ \text{ft}^{-1}\cdot\text{s}^{-1}$, and the coordinates are measured in feet. Show that the velocity field represents a possible incompressible flow. Determine the rotation at point $(x, y) = (1, 1)$. Evaluate the circulation about the "curve" bounded by $y = 0$, $x = 1$, $y = 1$, and $x = 0$.



Given: Two-dimensional flow field

Find:
(a) show that the velocity field represents a possible incompressible flow
(b) Rotation at (x, y) = (1, 1)
(c) Circulation about the unit square shown above

Solution: We will apply the definition of circulation to the given velocity field.

Governing Equations:

$$\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)+\frac{\partial \rho}{\partial t}=0 \quad \text{(Continuity equation)}$$

$$\vec{\omega}=\frac{1}{2}\nabla\times\vec{V} \quad \text{(Definition of rotation)}$$

$$\Gamma=\oint \vec{V}\cdot d\vec{s} \quad \text{(Definition of circulation)}$$

Assumptions:
(1) Steady flow
(2) Incompressible flow
(3) Two dimensional flow (velocity is not a function of z)

Based on the assumptions listed above, the continuity equation reduces to: $\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$

This is the criterion against which we will check the flow field.

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 2A\cdot x + B\cdot x = 2\times\frac{1}{2\cdot \text{ft}\cdot\text{s}}\cdot x + \frac{-1}{\text{ft}\cdot\text{s}}\cdot x = 0$$

This could be an incompressible flow field.

From the definition of rotation:
$$\vec{\omega}=\frac{1}{2}\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ Ax^2 & Bxy & 0\end{vmatrix}=\frac{1}{2}By\hat{k}$$
At (x, y) = (1, 1)
$$\vec{\omega}=-0.5\hat{k}\,\frac{\text{rad}}{\text{s}}$$

From the definition of circulation we break up the integral: $\Gamma=\int_{ab}\vec{V}\cdot d\vec{s}+\int_{bc}\vec{V}\cdot d\vec{s}+\int_{cd}\vec{V}\cdot d\vec{s}+\int_{da}\vec{V}\cdot d\vec{s}$

The integrand is equal to: $\vec{V}\cdot d\vec{s}=\left(Ax^2\hat{i}+Bxy\hat{j}\right)\cdot\left(dx\hat{i}+dy\hat{j}\right)=Ax^2dx+Bxydy$ Therefore, the circulation is equal to:

$$\Gamma=\int_{x_a}^{x_b}A\cdot x^2\,dx+\int_{y_b}^{y_c}B\cdot x\cdot y\,dy+\int_{x_c}^{x_d}A\cdot x^2\,dx+\int_{y_d}^{y_a}B\cdot x\cdot y\,dy$$
Evaluating the integrals:

$$\Gamma=\frac{A}{3}\cdot\left(x_b^3-x_a^3+x_d^3-x_c^3\right)+\frac{B}{2}\left[x_c\cdot\left(y_c^2-y_b^2\right)+x_a\cdot\left(y_a^2-y_d^2\right)\right]$$
Since $x_a = x_d = 0$ and $x_b = x_c$ we can simplify:

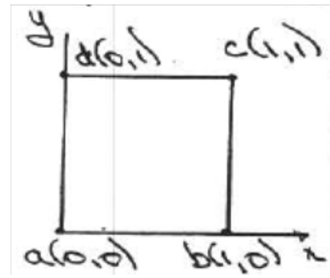
$$\Gamma=\frac{B}{2}\cdot x_c\cdot\left(y_c^2-y_b^2\right)$$
Substituting given values:
$$\Gamma=\frac{1}{2}\times\left(-\frac{1}{\text{ft}\cdot\text{s}}\right)\times 1\cdot\text{ft}\times\left(1^2-0^2\right)\cdot\text{ft}^2$$

$$\Gamma=-0.500\cdot\frac{\text{ft}^2}{\text{s}}$$

Problem 5.75

[Difficulty: 2]

5.75 Consider the two-dimensional flow field in which $u = Axy$ and $v = By^2$, where $A = 1\,\mathrm{m}^{-1}\cdot\mathrm{s}^{-1}$, $B = -\frac{1}{2}\mathrm{m}^{-1}\cdot\mathrm{s}^{-1}$, and the coordinates are measured in meters. Show that the velocity field represents a possible incompressible flow. Determine the rotation at point $(x, y) = (1, 1)$. Evaluate the circulation about the "curve" bounded by $y = 0$, $x = 1$, $y = 1$, and $x = 0$.



Given: Two-dimensional flow field

Find:
(a) show that the velocity field represents a possible incompressible flow
(b) Rotation at (x, y) = (1, 1)
(c) Circulation about the unit square shown above

Solution: We will apply the definition of circulation to the given velocity field.

Governing Equations:

$$\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)+\frac{\partial \rho}{\partial t}=0 \quad \text{(Continuity equation)}$$

$$\vec{\omega}=\frac{1}{2}\nabla\times\vec{V} \quad \text{(Definition of rotation)}$$

$$\Gamma=\oint \vec{V}\cdot d\vec{s} \quad \text{(Definition of circulation)}$$

Assumptions:
(1) Steady flow
(2) Incompressible flow
(3) Two dimensional flow (velocity is not a function of z)

Based on the assumptions listed above, the continuity equation reduces to: $\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$

This is the criterion against which we will check the flow field.

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = A\cdot y + 2\cdot B\cdot y = \frac{1}{\mathrm{m}\cdot\mathrm{s}}\cdot y + 2\times\frac{-1}{2\cdot\mathrm{m}\cdot\mathrm{s}}\cdot y = 0$$

This could be an incompressible flow field.

From the definition of rotation: $\vec{\omega}=\frac{1}{2}\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ Axy & By^2 & 0\end{vmatrix}=-\frac{1}{2}Ax\hat{k}$ At (x, y) = (1, 1) $\vec{\omega}=-0.5\hat{k}\,\frac{\mathrm{rad}}{\mathrm{s}}$

From the definition of circulation we break up the integral: $\Gamma=\int_{ab}\vec{V}\cdot d\vec{s}+\int_{bc}\vec{V}\cdot d\vec{s}+\int_{cd}\vec{V}\cdot d\vec{s}+\int_{da}\vec{V}\cdot d\vec{s}$

The integrand is equal to: $\vec{V}\cdot d\vec{s}=\left(Axy\hat{i}+By^2\hat{j}\right)\cdot\left(dx\hat{i}+dy\hat{j}\right)=Axydx+By^2dy$ Therefore, the circulation is equal to:

$$\Gamma=\int_{x_a}^{x_b}A\cdot x\cdot y\,dx+\int_{y_b}^{y_c}B\cdot y^2\,dy+\int_{x_c}^{x_d}A\cdot x\cdot y\,dx+\int_{y_d}^{y_a}B\cdot y^2\,dy=\frac{A}{2}\cdot\left(x_b^{\,2}-x_a^{\,2}\right)\left(y_a-y_c\right)+\frac{B}{3}\left(y_c^{\,3}-y_b^{\,3}+y_a^{\,3}-y_d^{\,3}\right)$$

Since $y_a = y_d = 0$ and $y_b = y_c$ we can simplify: $\Gamma=-\frac{A}{2}\cdot\left(x_b^{\,2}-x_a^{\,2}\right)\cdot -y_c$ Substituting given values:

$$\Gamma=\frac{1}{2}\times\frac{1}{\mathrm{m}\cdot\mathrm{s}}\times\left(1^2-0^2\right)\cdot\mathrm{m}^2\times -1\cdot\mathrm{m} \qquad \Gamma=-0.5\,\frac{\mathrm{m}^2}{\mathrm{s}}$$

Problem 5.76

[Difficulty: 3]

*5.76 Consider a flow field represented by the stream function $\psi = 3x^5y - 10x^3y^3 + 3xy^5$. Is this a possible two-dimensional incompressible flow? Is the flow irrotational?

Given: Stream function

Find: If the flow is incompressible and irrotational

Solution:

Basic equations: Incompressibility $\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$ Irrotationality $\frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u = 0$

Note: The fact that ψ exists means the flow is incompressible, but we check anyway

$$\psi(x,y) = 3\cdot x^5\cdot y - 10\cdot x^3\cdot y^3 + 3\cdot x\cdot y^5$$

Hence $u(x,y) = \frac{\partial}{\partial y}\psi(x,y) = 3\cdot x^5 - 30\cdot x^3\cdot y^2 + 15\cdot x\cdot y^4$ $v(x,y) = -\frac{\partial}{\partial x}\psi(x,y) = 30\cdot x^2\cdot y^3 - 15\cdot x^4\cdot y - 3\cdot y^5$

For incompressibility

$$\frac{\partial}{\partial x}u(x,y) = 15\cdot x^4 - 90\cdot x^2\cdot y^2 + 15\cdot y^4 \qquad \frac{\partial}{\partial y}v(x,y) = 90\cdot x^2\cdot y^2 - 15\cdot x^4 - 15\cdot y^4$$

Hence $\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$ INCOMPRESSIBLE

For irrotationality

$$\frac{\partial}{\partial x}v(x,y) = 60\cdot x\cdot y^3 - 60\cdot x^3\cdot y \qquad -\frac{\partial}{\partial y}u(x,y) = 60\cdot x^3\cdot y - 60\cdot x\cdot y^3$$

Hence $\frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u = 0$ IRROTATIONAL

Problem 5.77

[Difficulty: 3]

*5.77 Consider the flow field represented by the stream function $\psi = x^6 - 15x^4y^2 + 15x^2y^4 - y^6$. Is this a possible two-dimensional, incompressible flow? Is the flow irrotational?

Given: Stream function

Find: If the flow is incompressible and irrotational

Solution:

Basic equations: Incompressibility $\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$ Irrotationality $\frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u = 0$

Note: The fact that ψ exists means the flow is incompressible, but we check anyway

$$\psi(x,y) = x^6 - 15\cdot x^4\cdot y^2 + 15\cdot x^2\cdot y^4 - y^6$$

Hence $u(x,y) = \frac{\partial}{\partial y}\psi(x,y) = 60\cdot x^2\cdot y^3 - 30\cdot x^4\cdot y - 6\cdot y^5$ $v(x,y) = -\frac{\partial}{\partial x}\psi(x,y) = 60\cdot x^3\cdot y^2 - 6\cdot x^5 - 30\cdot x\cdot y^4$

For incompressibility

$$\frac{\partial}{\partial x}u(x,y) = 120\cdot x\cdot y^3 - 120\cdot x^3\cdot y \qquad \frac{\partial}{\partial y}v(x,y) = 120\cdot x^3\cdot y - 120\cdot x\cdot y^3$$

Hence $\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$ INCOMPRESSIBLE

For irrotationality

$$\frac{\partial}{\partial x}v(x,y) = 180\cdot x^2\cdot y^2 - 30\cdot x^4 - 30\cdot y^4 \qquad -\frac{\partial}{\partial y}u(x,y) = 30\cdot x^4 - 180\cdot x^2\cdot y^2 + 30\cdot y^4$$

Hence $\frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u = 0$ IRROTATIONAL

Problem 5.78

[Difficulty: 2]

***5.78** Consider a velocity field for motion parallel to the x axis with constant shear. The shear rate is $du/dy = A$, where $A = 0.1\ \text{s}^{-1}$. Obtain an expression for the velocity field, $\vec{V}$. Calculate the rate of rotation. Evaluate the stream function for this flow field.

Given: Velocity field for motion in the x-direction with constant shear

Find: (a) Expression for the velocity field
(b) Rate of rotation
(c) Stream function

Solution: We will apply the definition of circulation to the given velocity field.

Governing Equations:

$$\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)+\frac{\partial \rho}{\partial t}=0 \quad \text{(Continuity equation)}$$

$$\vec{\omega}=\frac{1}{2}\nabla\times\vec{V} \quad \text{(Definition of rotation)}$$

Assumptions: (1) Steady flow
(2) Incompressible flow

The x-component of velocity is: $u = \int A\,dy + f(x) = Ay + f(x)$ Since flow is parallel to the x-axis: $\vec{V}=\left[Ay+f(x)\right]\hat{i}$

From the definition of rotation: $$\vec{\omega}=\frac{1}{2}\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ Ay+f(x) & 0 & 0\end{vmatrix}=-\frac{1}{2}A\hat{k}$$ $\vec{\omega}=-0.05\hat{k}\,\dfrac{\text{rad}}{\text{s}}$

From the definition of the stream function $\psi = \int u\,dy + g(x) = \int (A\cdot y + f(x))\,dy + g(x) = \frac{1}{2}\cdot A\cdot y^2 + f(x)\cdot y + g(x)$

$v = -\frac{\partial}{\partial x}\psi = -\frac{d}{dx}f(x)\cdot y - \frac{d}{dx}g(x) = 0$ Therefore, the derivatives of both f and g are zero, and thus f and g are constants:

$$\psi = \frac{1}{2}\cdot A\cdot y^2 + c_1\cdot y + c_2$$

Problem 5.79

[Difficulty: 2]

***5.79** Consider a flow field represented by the stream function $\psi = -A/2(x^2 + y^2)$, where A = constant. Is this a possible two-dimensional incompressible flow? Is the flow irrotational?

Given: The stream function

Find: Whether or not the flow is incompressible; whether or not the flow is irrotational

Solution:

The stream function is $$\psi(x,y) = -\frac{A}{2\cdot\pi\cdot\left(x^2 + y^2\right)}$$

The velocity components are $$u(x,y) = \frac{\partial}{\partial y}\psi(x,y) = \frac{A\cdot y}{\pi\cdot\left(x^2 + y^2\right)^2} \qquad v(x,y) = -\frac{\partial}{\partial x}\psi(x,y) = -\frac{A\cdot x}{\pi\cdot\left(x^2 + y^2\right)^2}$$

Because a stream function exists, the flow is: Incompressible

Alternatively, we can check with $$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$$

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = -\frac{4\cdot A\cdot x\cdot y}{\pi\left(x^2 + y^2\right)^3} + \frac{4\cdot A\cdot x\cdot y}{\pi\left(x^2 + y^2\right)^3} = 0$$ Incompressible

For a 2D field, the irrotionality the test is $$\frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u = 0$$

$$\frac{\partial}{\partial x}v(x,y) - \frac{\partial}{\partial y}u(x,y) = \frac{4\cdot A\cdot x^2}{\pi\cdot\left(x^2 + y^2\right)^3} - \frac{2\cdot A}{\pi\cdot\left(x^2 + y^2\right)^2} + \frac{4\cdot A\cdot y^2}{\pi\cdot\left(x^2 + y^2\right)^3} = \frac{2\cdot A}{\pi\cdot\left(x^2 + y^2\right)^2}$$ Not irrotational

Problem 5.80

[Difficulty: 2]

***5.80** Consider the flow field represented by the stream function $\psi = Axy + Ay^2$, where $A = 1\ \mathrm{s}^{-1}$. Show that this represents a possible incompressible flow field. Evaluate the rotation of the flow. Plot a few streamlines in the upper half plane.

Given: Flow field represented by a stream function.

Find:
(a) Show that this represents an incompressible velocity field
(b) the rotation of the flow
(c) Plot several streamlines in the upper half plane

Solution: We will apply the definition of rotation to the given velocity field.

Governing Equation:

$$\vec{\omega} = \frac{1}{2}\nabla \times \vec{V} \qquad \text{(Definition of rotation)}$$

Assumptions:
(1) Steady flow
(2) Incompressible flow

From the definition of the stream function: $u = \frac{\partial}{\partial y}\psi = A \cdot x + 2 \cdot A \cdot y \qquad v = -\frac{\partial}{\partial x}\psi = -A \cdot y$ Applying the continuity equation:

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = A - A = 0$$

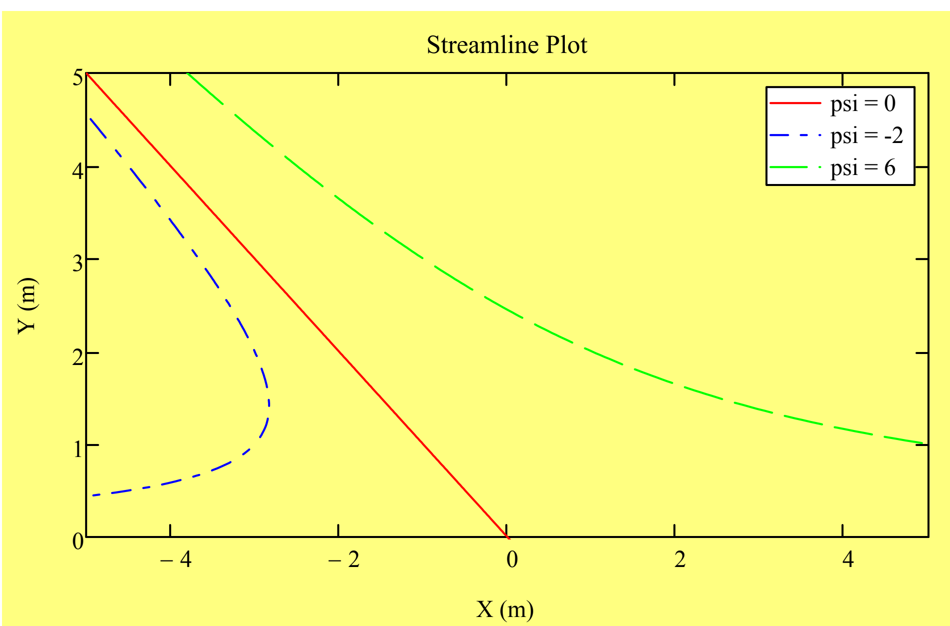
This could be an incompressible flow field

From the definition of rotation:

$$\vec{\omega} = \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A(x+2y) & -Ay & 0 \end{vmatrix} = \frac{1}{2}(-2A)\hat{k} = -A\hat{k}$$

$$\vec{\omega} = -A\hat{k}$$

The streamlines are curves where the stream function is constant, i.e., $\psi = \text{constant}$ Here is a plot of streamlines:



Problem 5.81

[Difficulty: 3]

*5.81 A flow field is represented by the stream function $\psi = x^2 - y^2$. Find the corresponding velocity field. Show that this flow field is irrotational. Plot several streamlines and illustrate the velocity field.

Given: Flow field represented by a stream function.

Find: (a) Expression for the velocity field
(b) Show that flow field is irrotational
(c) Plot several streamlines and illustrate velocity field

Solution: We will apply the definition of circulation to the given velocity field.

Governing Equation: $\vec{\omega} = \frac{1}{2}\nabla \times \vec{V}$ (Definition of rotation)

Assumptions: (1) Steady flow
(2) Incompressible flow

From the definition of the stream function: $u = \frac{\partial}{\partial y}\psi = -2 \cdot y$ $\quad v = -\frac{\partial}{\partial x}\psi = -2 \cdot x$ In vector notation: $\vec{V} = -2y\hat{i} - 2x\hat{j}$

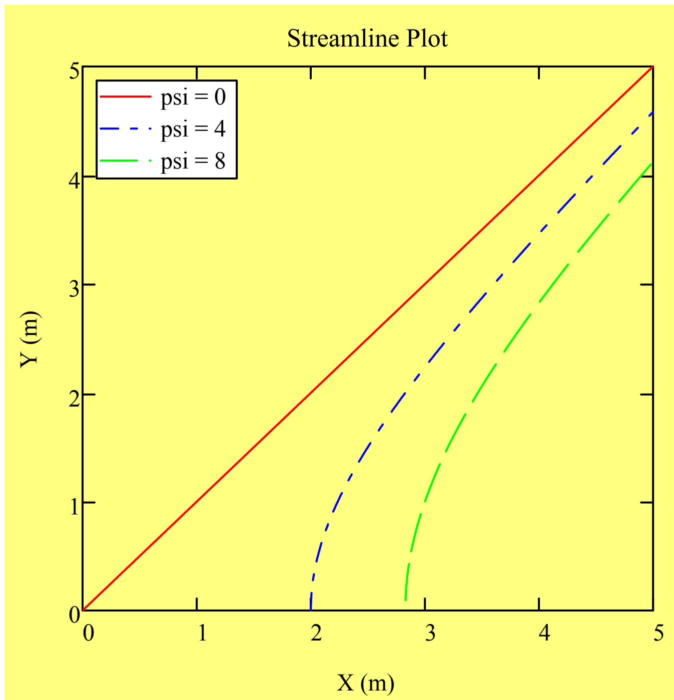
From the definition of rotation:

$$\vec{\omega} = \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -2y & -2x & 0 \end{vmatrix} = \frac{1}{2}(-2+2)\hat{k} = \vec{0}$$

$\vec{\omega} = \vec{0}$

Flow is irrotational

The streamlines are curves where the stream function is constant, i.e., $\psi = \text{constant}$ Here is a plot of streamlines:

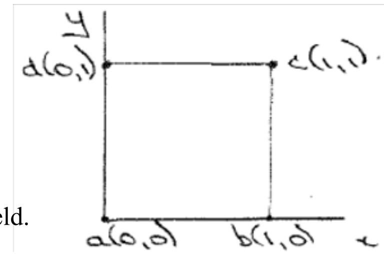


Problem 5.82

[Difficulty: 2]

*5.82 Consider the velocity field given by $\vec{V} = Ax^2\hat{i} + Bxy\hat{j}$, where $A = 1\ \mathrm{ft^{-1} \cdot s^{-1}}$, $B = -2\ \mathrm{ft^{-1} \cdot s^{-1}}$, and the coordinates are measured in feet.
(a) Determine the fluid rotation.
(b) Evaluate the circulation about the "curve" bounded by $y = 0$, $x = 1$, $y = 1$, and $x = 0$.
(c) Obtain an expression for the stream function.
(d) Plot several streamlines in the first quadrant.

Given: Flow field represented by a velocity function.



Find:
(a) Fluid rotation
(b) Circulation about the curve shown
(c) Stream function
(d) Plot several streamlines in first quadrant

Solution: We will apply the definition of rotation and circulation to the given velocity field.

Governing Equation:

$$\vec{\omega} = \frac{1}{2}\nabla \times \vec{V} \qquad \text{(Definition of rotation)}$$

$$\Gamma = \oint \vec{V} \cdot d\vec{s} \qquad \text{(Definition of circulation)}$$

Assumption: Steady flow

From the definition of rotation:
$$\vec{\omega} = \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ Ax^2 & Bxy & 0 \end{vmatrix} = \frac{1}{2}(By)\hat{k}$$

$$\vec{\omega} = -y\hat{k}\,\frac{\text{rad}}{\text{ft} \cdot \text{s}}$$

From the definition of circulation we break up the integral: $\Gamma = \int_{ab} \vec{V} \cdot d\vec{s} + \int_{bc} \vec{V} \cdot d\vec{s} + \int_{cd} \vec{V} \cdot d\vec{s} + \int_{da} \vec{V} \cdot d\vec{s}$

The integrand is equal to: $\vec{V} \cdot d\vec{s} = \left(Ax^2\hat{i} + Bxy\hat{j}\right) \cdot \left(dx\hat{i} + dy\hat{j}\right) = Ax^2dx + Bxydy$ Therefore, the circulation is equal to:

$$\Gamma = \int_{x_a}^{x_b} A \cdot x^2\, dx + \int_{y_b}^{y_c} B \cdot x \cdot y\, dy + \int_{x_c}^{x_d} A \cdot x^2\, dx + \int_{y_d}^{y_a} B \cdot x \cdot y\, dy$$ Evaluating the integral:

$$\Gamma = \frac{A}{3} \cdot \left(x_b^{\ 3} - x_a^{\ 3} + x_d^{\ 3} - x_c^{\ 3}\right) + \frac{B}{2}\left[x_c \cdot \left(y_c^{\ 2} - y_b^{\ 2}\right) + x_a \cdot \left(y_a^{\ 2} - y_d^{\ 2}\right)\right]$$ Since $x_a = x_d = 0$ and $x_b = x_c$ we can simplify:

$$\Gamma = \frac{B}{2} \cdot x_c \cdot \left(y_c^{\ 2} - y_b^{\ 2}\right)$$ Substituting given values: $$\Gamma = \frac{1}{2} \times -\frac{2}{\text{ft} \cdot \text{s}} \times 1 \cdot \text{ft} \times \left(1^2 - 0^2\right) \cdot \text{ft}^2$$

$$\Gamma = -1.000 \cdot \frac{\text{ft}^2}{\text{s}}$$

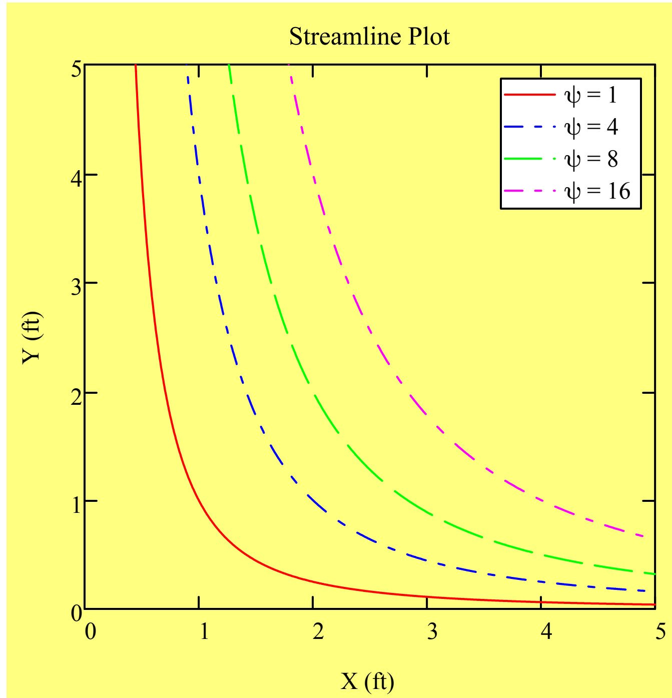
From the definition of the stream function: $u = \dfrac{\partial}{\partial y}\psi \qquad \psi = \displaystyle\int u\, dy + f(x) = \int A \cdot x^2\, dy + f(x) = A \cdot x^2 \cdot y + f(x)$

In addition, $v = -\dfrac{\partial}{\partial x}\psi \qquad \psi = -\displaystyle\int v\, dx + g(y) = -\int B \cdot x \cdot y\, dx + g(y) = -\frac{B}{2} \cdot x^2 \cdot y + g(y)$ Comparing the two stream functions:

$\dfrac{1}{\text{ft} \cdot \text{s}} \cdot x^2 \cdot y + f(x) = \dfrac{1}{\text{ft} \cdot \text{s}} \cdot x^2 \cdot y + g(y)$ Thus, $f = g = \text{constant}$ Taking $f(x) = 0$:

$$\psi = A \cdot x^2 \cdot y$$

The streamlines are curves where the stream function is constant, i.e., $\psi = \text{constant}$ Here is a plot of streamlines:

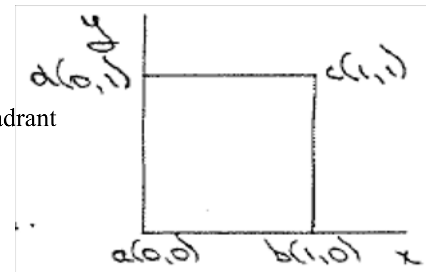


Problem 5.83

[Difficulty: 2]

***5.83** Consider the flow represented by the velocity field $\vec{V} = (Ay + B)\hat{i} + Ax\hat{j}$, where A = 10 s^{-1}, B = 10 ft/s, and the coordinates are measured in feet.
(a) Obtain an expression for the stream function.
(b) Plot several streamlines (including the stagnation streamline) in the first quadrant.
(c) Evaluate the circulation about the "curve" bounded by $y = 0$, $x = 1$, $y = 1$, and $x = 0$.

Given: Flow field represented by a velocity function.



Find:
(a) An expression for the stream function
(b) Circulation about the curve shown
(c) Plot several streamlines (including the stagnation streamline) in first quadrant

Solution: We will apply the definition of circulation to the given velocity field.

Governing Equation:
$$\Gamma = \oint \vec{V} \cdot d\vec{s} \qquad \text{(Definition of circulation)}$$

Assumptions: Steady flow

From the definition of the stream function: $u = \frac{\partial}{\partial y}\psi \qquad \psi = \int u\,dy + f(x) = \int (A\cdot y + B)\,dy + f(x) = \frac{A}{2}\cdot y^2 + B\cdot y + f(x)$

In addition, $v = -\frac{\partial}{\partial x}\psi \qquad \psi = -\int v\,dx + g(y) = -\int A\cdot x\,dx + g(y) = -\frac{A}{2}\cdot x^2 + g(y)$ Comparing the two stream functions:

$\frac{A}{2}\cdot y^2 + B\cdot y + f(x) = -\frac{A}{2}\cdot x^2 + g(y)$ Thus, $f(x) = -\frac{A}{2}\cdot x^2 + C$ Taking C = 0:

$$\psi = \frac{A}{2}\cdot\left(y^2 - x^2\right) + B\cdot y$$

From the definition of circulation we break up the integral: $\Gamma = \int_{ab}\vec{V}\cdot d\vec{s} + \int_{bc}\vec{V}\cdot d\vec{s} + \int_{cd}\vec{V}\cdot d\vec{s} + \int_{da}\vec{V}\cdot d\vec{s}$

The integrand is equal to: $\vec{V}\cdot d\vec{s} = \left((Ay+B)\hat{i} + Ax\hat{j}\right)\cdot\left(dx\hat{i} + dy\hat{j}\right) = (Ay+B)dx + Axdy$ Therefore, the circulation is:

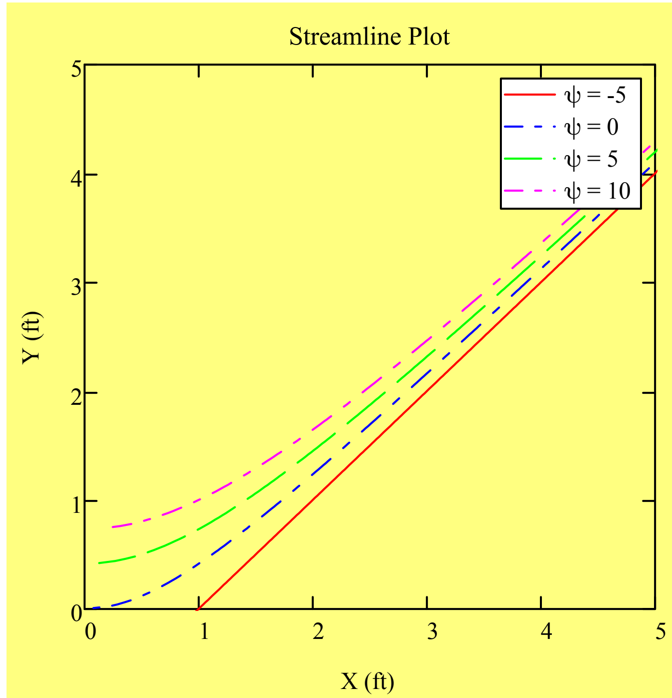
$$\Gamma = \int_{x_a}^{x_b}(A\cdot y + B)\,dx + \int_{y_b}^{y_c}A\cdot x\,dy + \int_{x_c}^{x_d}(A\cdot y + B)\,dx + \int_{y_d}^{y_a}A\cdot x\,dy$$ Evaluating the integral:

$$\Gamma = \left(A\cdot y_a + B\right)\cdot\left(x_b - x_a\right) + A\cdot x_b\cdot\left(y_c - y_b\right) + \left(A\cdot y_c + B\right)\cdot\left(x_d - x_c\right) + A\cdot x_d\cdot\left(y_a - y_d\right)$$ Substituting known values:

$$\Gamma = \left(\frac{10}{s}\times 0\cdot ft + 10\cdot\frac{ft}{s}\right)\times(1-0)\cdot ft + \frac{10}{s}\times 1\cdot ft\times(1-0)\cdot ft + \left(\frac{10}{s}\times 1\cdot ft + 10\cdot\frac{ft}{s}\right)\times(0-1)\cdot ft + \frac{10}{s}\times 0\cdot ft\times(1-0)\cdot ft$$

$$\Gamma = 0\cdot\frac{ft^2}{s}$$

The streamlines are curves where the stream function is constant, i.e., $\psi = \text{constant}$ Here is a plot of streamlines:



The stagnation streamline is the one running through the point where the velocity vanishes:

$$A \cdot y_{stag} + B = 0 \qquad y_{stag} = -\frac{B}{A} = -1 \cdot ft$$

$$A \cdot x_{stag} = 0 \qquad x_{stag} = 0$$

Plugging this information in to find the stream function at the stagnation point yields:

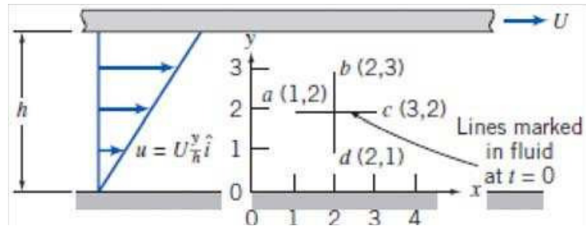
$$\psi_{stag} = \frac{10}{2 \cdot s} \cdot \left[(-1 \cdot ft)^2 - (0 \cdot ft)^2 \right] + 10 \cdot \frac{ft}{s} \cdot -1 \cdot ft$$

$$\psi_{stag} = -5 \cdot \frac{ft^2}{s}$$

Problem 5.84

[Difficulty: 3]

5.84 Consider again the viscometric flow of Example 5.7. Evaluate the average rate of rotation of a pair of perpendicular line segments oriented at ±45° from the x axis. Show that this is the same as in the example.

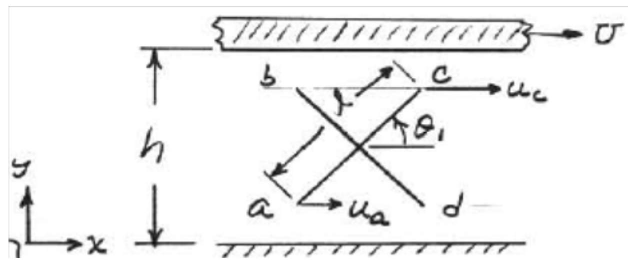


Given: Viscometric flow of Example 5.7, V = U(y/h)i, where U = 4 mm/s and h = 4 mm

Find: (a) Average rate of rotation of two line segments at +/- 45 degrees
(b) Show that this is the same as in the example

Solution: We will apply the definition of rotation to the given velocity field.

Governing Equation: $\vec{\omega} = \frac{1}{2}\nabla \times \vec{V}$ (Definition of rotation)

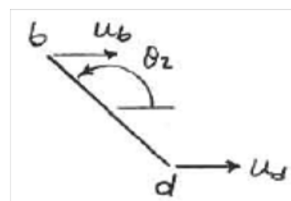


Assumptions:
(1) Steady flow
(2) Incompressible flow

Considering the lines shown: $u_c = u_a + \frac{\partial}{\partial y}u \cdot (l \cdot \sin(\theta_1))$

$-\omega_{ac} = \frac{(u_c - u_a)\cdot \sin(\theta_1)}{l}$ since the component normal to l is $u \cdot \sin(\theta_1)$

$-\omega_{ac} = \frac{\partial}{\partial y}u \cdot \frac{(l \cdot \sin(\theta_1))\cdot \sin(\theta_1)}{l} = \frac{\partial}{\partial y}u \cdot (\sin(\theta_1))^2 = \frac{U}{h}\cdot(\sin(\theta_1))^2$ Now consider this sketch:



$u_b = u_d + \frac{\partial}{\partial y}u \cdot (l \cdot \sin(\theta_2))$ $\quad -\omega_{bd} = \frac{(u_d - u_b)\cdot \sin(\theta_2)}{l}$ since the component normal to l is $u \cdot \sin(\theta_2)$

$-\omega_{bd} = \frac{\partial}{\partial y}u \cdot \frac{(l \cdot \sin(\theta_2))\cdot \sin(\theta_2)}{l} = \frac{\partial}{\partial y}u \cdot (\sin(\theta_2))^2 = \frac{U}{h}\cdot(\sin(\theta_2))^2$ Now we sum these terms:

$\omega = \frac{1}{2}\cdot(\omega_{ac} + \omega_{bd}) = -\frac{1}{2}\cdot\frac{U}{h}\cdot\left[(\sin(\theta_1))^2 + (\sin(\theta_2))^2\right]$ When $\theta_1 = 45\cdot\text{deg}$ and $\theta_2 = 135\cdot\text{deg}$

$$\omega = -\frac{1}{2}\cdot\frac{U}{h}\cdot\left[\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2\right]$$

$$\omega = -\frac{1}{2}\cdot\frac{U}{h}$$

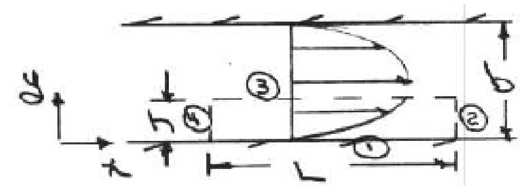
Substituting for U and h: $\omega = -\frac{1}{2} \times 4\cdot\frac{mm}{s} \times \frac{1}{4\cdot mm}$

$$\omega = -0.5\frac{1}{s}$$

Problem 5.85

[Difficulty: 3]

5.85 Consider the pressure-driven flow between stationary parallel plates separated by distance b. Coordinate y is measured from the bottom plate. The velocity field is given by $u = U(y/b)[1-(y/b)]$. Obtain an expression for the circulation about a closed contour of height h and length L. Evaluate when $h = b/2$ and when $h = b$. Show that the same result is obtained from the area integral of the Stokes Theorem (Eq. 5.18).



Given: Velocity field for pressure-driven flow between stationary parallel plates

Find:
(a) Expression for circulation about a closed contour of height h and length L
(b) Evaluate part (a) for h = b/2 and h = b
(c) Show that the same result is obtained from area integral of Stokes Theorem (Eq. 5.14)

Solution: We will apply the definition of circulation to the given velocity field.

Governing Equations:

$$\Gamma = \oint \vec{V} \cdot d\vec{s} \qquad \text{(Definition of circulation)}$$

$$\int_A \left(\nabla \times \vec{V}\right) dA = \oint \vec{V} \cdot d\vec{s} \qquad \text{(Stokes Theorem)}$$

Assumptions: (1) Steady flow

From the definition of circulation we break up the integral: $\Gamma = \int_1 \vec{V} \cdot d\vec{s} + \int_2 \vec{V} \cdot d\vec{s} + \int_3 \vec{V} \cdot d\vec{s} + \int_4 \vec{V} \cdot d\vec{s}$

The integrand is equal to: $\vec{V} \cdot d\vec{s} = U \dfrac{y}{b}\left(1 - \dfrac{y}{b}\right)\hat{i} \cdot \left(dx\hat{i} + dy\hat{j}\right) = U \dfrac{y}{b}\left(1 - \dfrac{y}{b}\right)dx$ Therefore, the circulation is equal to:

$$\Gamma = \int_0^L U \cdot \frac{0}{b} \cdot \left(1 - \frac{0}{b}\right) dx + \int_L^0 U \cdot \frac{h}{b} \cdot \left(1 - \frac{h}{b}\right) dx = -U \cdot L \cdot \frac{h}{b} \cdot \left(1 - \frac{h}{b}\right)$$

$$\Gamma = -U \cdot L \cdot \frac{h}{b} \cdot \left(1 - \frac{h}{b}\right)$$

For h = b/2: $\Gamma = -U \cdot L \cdot \dfrac{1}{b} \cdot \dfrac{b}{2} \cdot \left(1 - \dfrac{1}{b} \cdot \dfrac{b}{2}\right)$ $\quad \Gamma = -\dfrac{U \cdot L}{4}$

For h = b: $\Gamma = -U \cdot L \cdot 1 \cdot (1 - 1)$ $\quad \Gamma = 0$

From Stokes Theorem: $\Gamma = \int_A \left(\nabla \times \vec{V}\right) dA = \int_A \left(\dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}\right) dA = -U \int_A \left(\dfrac{1}{b} - \dfrac{2y}{b}\right) dA$ We define dA = L dy:

$$\Gamma = -U \cdot L \cdot \int_0^h \left(\frac{1}{b} - \frac{2}{b} \cdot y\right) dy = -U \cdot L \cdot \left(\frac{h}{b} - \frac{h^2}{b}\right) = -U \cdot L \cdot \frac{h}{b} \cdot \left(1 - \frac{h}{b}\right)$$

$$\Gamma = -U \cdot L \cdot \frac{h}{b} \cdot \left(1 - \frac{h}{b}\right)$$

Problem 5.86

[Difficulty: 3]

*5.86 The velocity field near the core of a tornado can be approximated as

$$\vec{V} = -\frac{q}{2\pi r}\hat{e}_r + \frac{K}{2\pi r}\hat{e}_\theta$$

Is this an irrotational flow field? Obtain the stream function for this flow.

Given: Velocity field approximation for the core of a tornado

Find: (a) Whether or not this is an irrotational flow
(b) Stream function for the flow

Solution: We will apply the definition of rotation to the given velocity field.

Governing Equation:

$$\vec{\omega} = \frac{1}{2}\nabla \times \vec{V} \qquad \text{(Definition of rotation)}$$

$$\vec{\nabla} = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{k}\frac{\partial}{\partial z} \qquad \text{(Definition of "del" operator)}$$

$$\frac{\partial \hat{e}_r}{\partial \theta} = \hat{e}_\theta \qquad \frac{\partial \hat{e}_\theta}{\partial \theta} = -\hat{e}_r \qquad \text{(Hints from text)}$$

Assumptions: (1) Steady flow
(2) Two-dimensional flow (no z velocity, velocity is not a function of θ or z)

From the definition of rotation: $\vec{\omega} = \frac{1}{2}\left(\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(V_r \hat{e}_r + V_\theta \hat{e}_\theta\right)$ Employing assumption (2) yields:

$$\vec{\omega} = \frac{1}{2}\left(\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta}\right) \times \left(V_r \hat{e}_r + V_\theta \hat{e}_\theta\right) = \frac{1}{2}\left[\hat{e}_r \times \left(\hat{e}_r \frac{\partial V_r}{\partial r} + \hat{e}_\theta \frac{\partial V_\theta}{\partial r}\right) + \hat{e}_\theta \frac{1}{r} \times \frac{\partial}{\partial \theta}\left(V_r \hat{e}_r + V_\theta \hat{e}_\theta\right)\right]$$ From product rule:

$$\vec{\omega} = \frac{1}{2}\left[\left(\hat{e}_r \times \hat{e}_r\right)\frac{\partial V_r}{\partial r} + \left(\hat{e}_r \times \hat{e}_\theta\right)\frac{\partial V_\theta}{\partial r} + \hat{e}_\theta \frac{1}{r} \times \left(\hat{e}_r \frac{\partial V_r}{\partial \theta} + V_r \frac{\partial \hat{e}_r}{\partial \theta} + \hat{e}_\theta \frac{\partial V_\theta}{\partial \theta} + V_\theta \frac{\partial \hat{e}_\theta}{\partial \theta}\right)\right]$$ Using the hints from the text:

$$\vec{\omega} = \frac{1}{2}\left[\left(\hat{e}_r \times \hat{e}_r\right)\frac{\partial V_r}{\partial r} + \left(\hat{e}_r \times \hat{e}_\theta\right)\left(\frac{\partial V_\theta}{\partial r} - \frac{1}{r}\frac{\partial V_r}{\partial \theta} + \frac{V_\theta}{r}\right) + \left(\hat{e}_\theta \times \hat{e}_\theta\right)\left(\frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{V_r}{r}\right)\right] = \frac{1}{2}\left(\frac{\partial V_\theta}{\partial r} - \frac{1}{r}\frac{\partial V_r}{\partial \theta} + \frac{V_\theta}{r}\right)\hat{k}$$

Since V is only a function of r: $\vec{\omega} = \frac{1}{2}\left(\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\right)\hat{k} = \frac{1}{2}\left(-\frac{K}{2\pi r^2} + \frac{K}{2\pi r^2}\right)\hat{k} = \vec{0}$

Flow is irrotational.

To build the stream function: $V_r = \frac{1}{r}\cdot\frac{\partial}{\partial \theta}\psi \qquad \psi = \int r\cdot V_r\, d\theta + f(r) = -\int \frac{q}{2\cdot\pi}\, d\theta + f(r) = -\frac{q\cdot\theta}{2\cdot\pi} + f(r)$

$V_\theta = -\frac{\partial}{\partial r}\psi \qquad \psi = -\int V_\theta\, dr + g(\theta) = -\int \frac{K}{2\cdot\pi\cdot r}\, dr + g(\theta) = -\frac{K}{2\cdot\pi}\cdot\ln(r) + g(\theta)$ Comparing these two expressions:

$$-\frac{q\cdot\theta}{2\cdot\pi} + f(r) = -\frac{K}{2\cdot\pi}\cdot\ln(r) + g(\theta) \qquad f(r) = -\frac{K}{2\cdot\pi}\cdot\ln(r)$$

$$\psi = -\frac{K}{2\cdot\pi}\cdot\ln(r) - \frac{q\cdot\theta}{2\cdot\pi}$$

Problem 5.87 [Difficulty: 2]

5.87 The velocity profile for fully developed flow in a circular tube is $V_z = V_{max}[1 - (r/R)^2]$. Evaluate the rates of linear and angular deformation for this flow. Obtain an expression for the vorticity vector, $\vec{\zeta}$.

Given: Velocity field for fully-developed flow in a circular tube

Find: (a) Rates of linear and angjular deformation for this flow
(b) Expression for the vorticity vector

Solution: We will apply the definition of vorticity to the given velocity field.

Governing Equation: $\vec{\zeta} = \nabla \times \vec{V}$ (Definition of vorticity)

Assumptions: (1) Steady flow

The volume dilation rate of the flow is: $\nabla \cdot \vec{V} = \frac{1}{r}\frac{\partial(rV_r)}{\partial r} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z} = 0$

Rates of linear deformation in all three directions is zero.

The angular deformations are: r-θ plane: $\frac{1}{r}\cdot\frac{\partial}{\partial r}\left(\frac{V_\theta}{r}\right) + \frac{\partial}{\partial \theta}V_r = 0$

θ-z plane: $\frac{\partial}{\partial z}V_\theta + \frac{1}{r}\cdot\frac{\partial}{\partial \theta}V_z = 0$

z-r plane: $\frac{\partial}{\partial z}V_r + \frac{\partial}{\partial r}V_z = -V_{max}\cdot\frac{2\cdot r}{R^2}$

$$\text{angdef} = -V_{max}\cdot\frac{2\cdot r}{R^2}$$

The vorticity in cylindrical coordinates is: $\vec{\zeta} = \nabla \times \vec{V} = \left(\frac{1}{r}\frac{\partial V_z}{\partial \theta} - \frac{\partial V_\theta}{\partial z}\right)\hat{e}_r + \left(\frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r}\right)\hat{e}_\theta + \left(\frac{1}{r}\frac{\partial r V_\theta}{\partial r} - \frac{1}{r}\frac{\partial V_r}{\partial \theta}\right)\hat{k}$

$$\vec{\zeta} = -V_{max}\frac{2r}{R^2}\hat{e}_\theta$$

Problem 5.88

[Difficulty: 3]

5.88 Consider the pressure-driven flow between stationary parallel plates separated by distance $2b$. Coordinate y is measured from the channel centerline. The velocity field is given by $u = u_{max}[1-(y/b)^2]$. Evaluate the rates of linear and angular deformation. Obtain an expression for the vorticity vector, $\vec{\zeta}$. Find the location where the vorticity is a maximum.

Given: Velocity field for pressure-driven flow between stationary parallel plates

Find:
(a) Rates of linear and angjular deformation for this flow
(b) Expression for the vorticity vector
(c) Location of maximum vorticity

Solution: We will apply the definition of vorticity to the given velocity field.

Governing Equation: $\vec{\zeta} = \nabla \times \vec{V}$ (Definition of vorticity)

Assumptions: (1) Steady flow

The volume dilation rate of the flow is: $$\nabla \cdot \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Rates of linear deformation in all three directions is zero.

The angular deformations are:

x-y plane: $\frac{\partial}{\partial x}v + \frac{\partial}{\partial y}u = -u_{max} \cdot \frac{2 \cdot y}{b^2}$

y-z plane: $\frac{\partial}{\partial y}w + \frac{\partial}{\partial z}v = 0$

z-x plane: $\frac{\partial}{\partial z}u + \frac{\partial}{\partial x}w = 0$

$$\text{angdef} = -u_{max} \cdot \frac{2 \cdot y}{b^2}$$

The vorticity is: $$\vec{\zeta} = \nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u_{max}\left[1-\left(\frac{y}{b}\right)^2\right] & 0 & 0 \end{vmatrix} = u_{max}\frac{2y}{b^2}\hat{k}$$

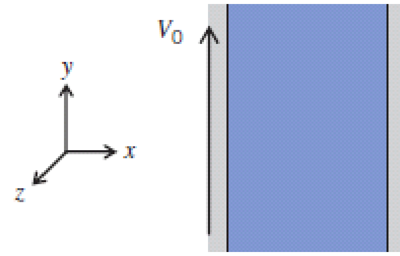
$$\vec{\zeta} = u_{max}\frac{2y}{b^2}\hat{k}$$

The vorticity is a maximum at y=b and y=-b

Problem 5.89

[Difficulty: 2]

5.89 Consider a steady, laminar, fully developed, incompressible flow between two infinite plates, as shown. The flow is due to the motion of the left plate as well a pressure gradient that is applied in the y direction. Given the conditions that $\vec{V} \neq \vec{V}(z)$, $w = 0$, and that gravity points in the negative y direction, prove that $u = 0$ and that the pressure gradient in the y direction must be constant.



Given: Flow between infinite plates

Find: Prove that $u = 0$, dP/dy = constant

Solution:

Governing Equations:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \qquad \text{(Continuity Equation)}$$

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=\rho g_x-\frac{\partial P}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=\rho g_y-\frac{\partial P}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right) \qquad \text{(Navier-Stokes Equations)}$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=\rho g_z-\frac{\partial P}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)$$

Assumptions:
Incompressible fluid
No motion along the wall ($x = 0$) limited to two dimensions ($w = 0$).

Prove that u = 0:

Given that $\vec{V} \neq \vec{V}(z)$ this means that $\frac{\partial u}{\partial z}=\frac{\partial v}{\partial z}=\frac{\partial w}{\partial z}=0$

Also given that the flow is fully developed which means that $\vec{V} \neq \vec{V}(y)$ so that $\frac{\partial u}{\partial y}=\frac{\partial v}{\partial y}=\frac{\partial w}{\partial y}=0$

And steady flow implies that $\vec{V} \neq \vec{V}(t)$

The continuity equation becomes $\frac{\partial u}{\partial x}=0$, but because $u \neq u(y,z,t)$ then $u = u(x)$ meaning that the partial derivative here becomes an ordinary derivative: $\frac{du}{dx}=0$

Integrating the ordinary derivative gives: $u = \text{constant}$

By the no-slip boundary condition u = 0 at the surface of either plate meaning the constant must be zero.

Hence: $u = 0$

Prove that $\frac{\partial P}{\partial y} = \text{constant}$ **:**

Due to the fact that u = 0, and gravity is in the negative y-direction the x-component of the Navier-Stokes Equation becomes:

$$\frac{\partial P}{\partial x} = 0 \quad \text{hence } P \neq P(x)$$

Due to the fact that w = 0, and gravity is in the negative y-direction the z-component of the Navier-Stokes Equation becomes:

$$\frac{\partial P}{\partial z} = 0 \quad \text{hence } P \neq P(z)$$

The y-component of the Navier-Stokes Equation reduces to:

$$0 = -\frac{\partial P}{\partial y} - \rho g + \mu\left(\frac{\partial^2 v}{\partial x^2}\right)$$

So then

$$\frac{\partial P}{\partial y} = -\rho g + \mu\left(\frac{\partial^2 v}{\partial x^2}\right) \quad [1]$$

It has been shown that $P \neq P(x,z)$ and because the flow is steady $P \neq P(t)$ meaning that $P = P(y)$. This means that the left hand side of [1] can only be a function of y or a constant. Additionally, by the fully developed, steady flow, and $\vec{V} \neq \vec{V}(z)$ conditions it is shown that $v = v(x)$. For this reason the right hand side of [1] can only be a function or x or a constant.

Mathematically speaking it is impossible for: $f(y) = g(x)$ so each side of [1] must be a constant.

Hence, $\frac{\partial P}{\partial y} = \text{constant}$

Problem 5.90

[Difficulty: 3]

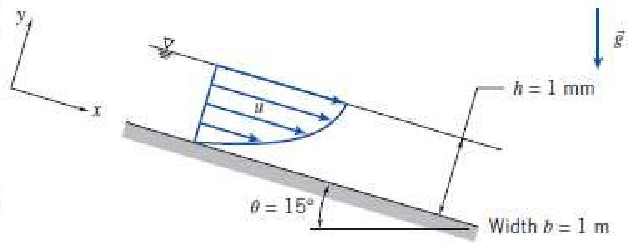
5.90 Assume the liquid film in Example 5.9 is not isothermal, but instead has the following distribution:

$$T(y) = T_0 + (T_w - T_0)\left(1 - \frac{y}{h}\right)$$

where T_0 and T_w are, respectively, the ambient temperature and the wall temperature. The fluid viscosity decreases with increasing temperature and is assumed to be described by

$$\mu = \frac{\mu_0}{1 + a(T - T_0)}$$

with $a > 0$. In a manner similar to Example 5.9, derive an expression for the velocity profile.



Given: temperature profile and temperature-dependent viscosity expression

Find: Velocity Profile

Solution:

Governing Equations:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad \text{(Continuity Equation)}$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho g_x - \frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = \rho g_y - \frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \qquad \text{(Navier-Stokes Equations)}$$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = \rho g_z - \frac{\partial P}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

Assumptions: Incompressible fluid

Similar to the Example 5.9, the x-component momentum equation can be simplified to

$$\frac{d\tau_{yx}}{dy} = -\rho g \sin\theta \qquad (1)$$

Integrating once, one has

$$\tau_{yx} = -\rho g y \sin\theta + C_1 \qquad (2)$$

Using the boundary condition: $\tau_{yx}(y = h) = 0$

$$c_1 = \rho g h \sin\theta \qquad (3)$$

Substituting c_1 into eq. (2),

$$\tau_{yx} = \mu\frac{du}{dy} = \rho g(h - y)\sin\theta \qquad (4)$$

Here, the fluid viscosity depends on the temperature,

$$\mu = \frac{\mu_0}{1 + a(T_w - T_0)(1 - y/h)} \qquad (5)$$

Substituting equation (5) into equation (4), we have

$$\frac{du}{dy} = \frac{\rho g h (1 - y/h)\sin\theta}{\mu_0}(1 + a(T_w - T_0)(1 - y/h)) \qquad (6)$$

Integrating equation (6) once

$$u = \frac{\rho g h \sin\theta}{\mu_0}(y(1 - \frac{y}{2h}) + a(T_w - T_0)y(1 - \frac{y}{h} + \frac{y^2}{3h^2})) + C_2 \qquad (7)$$

At y=0, u=0: c_2=0.

Substituting c_2=0 into eq. (7), one obtains

$$u = \frac{\rho g h \sin\theta}{\mu_0}(y(1 - \frac{y}{2h}) + a(T_w - T_0)y(1 - \frac{y}{h} + \frac{y^2}{3h^2})) \qquad (8)$$

When *a*=0, eq. (8) can be simplified to

$u = \frac{\rho g h \sin\theta}{\mu_0} y(1 - \frac{y}{2h})$, and it is exactly the same velocity profile in Example 5.9.

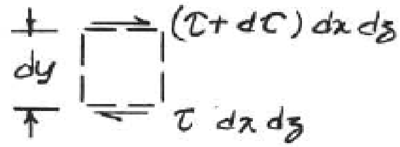
Problem 5.91

[Difficulty: 2]

5.91 The x component of velocity in a laminar boundary layer in water is approximated as $u = U\sin(\pi y/2\delta)$, where $U = 3$ m/s and $\delta = 2$ mm. The y component of velocity is much smaller than u. Obtain an expression for the net shear force per unit volume in the x direction on a fluid element. Calculate its maximum value for this flow.

Given: Sinusoidal approximation for velocity profile in laminar boundary layer

Find: (a) Express shear force per unit volume in the x-direction
(b) Maximum value at these conditions



Solution: We will evaluate a differential volume of fluid in this flow field

Assumptions: (1) Steady flow

The differential of shear force would be: $dF_{shear} = (\tau + d\tau)\cdot dx\cdot dz - \tau\cdot dx\cdot dz = d\tau\cdot dx\cdot dz$ and $\dfrac{dF_{sx}}{dV} = \dfrac{d\tau}{dy} = \dfrac{d}{dy}\left(\mu\cdot\dfrac{du}{dy}\right) = \mu\cdot\dfrac{d^2}{dy^2}u$

From the given profile: $\dfrac{d}{dy}u = \dfrac{\pi\cdot U}{2\cdot\delta}\cdot\cos\left(\dfrac{\pi\cdot y}{2\cdot\delta}\right)$ and $\dfrac{d^2}{dy^2}u = -U\cdot\left(\dfrac{\pi}{2\cdot\delta}\right)^2\cdot\sin\left(\dfrac{\pi\cdot y}{2\cdot\delta}\right)$ Thus, $\dfrac{dF_{sx}}{dV} = -\mu\cdot U\cdot\left(\dfrac{\pi}{2\cdot\delta}\right)^2\cdot\sin\left(\dfrac{\pi\cdot y}{2\cdot\delta}\right)$

The maximum magnitue for this shear force is when $y = \delta$:

$$F_{vmax} = \frac{dF_{sxmax}}{dV} = -\mu\cdot U\cdot\left(\frac{\pi}{2\cdot\delta}\right)^2$$

For water: $\mu = 0.001\cdot\dfrac{N\cdot s}{m^2}$ $\quad U = 3\cdot\dfrac{m}{s}$ $\quad \delta = 2\cdot mm$ Substituting these values:

$$F_{vmax} = -0.001\cdot\frac{N\cdot s}{m^2}\times 3\cdot\frac{m}{s}\times\left(\frac{\pi}{2}\times\frac{1}{0.002\cdot m}\right)^2$$

$$F_{vmax} = -1.851\cdot\frac{kN}{m^3}$$

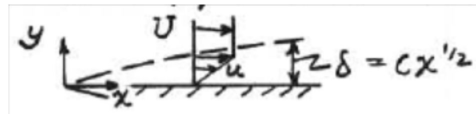
Problem 5.92

[Difficulty: 3]

5.92 A linear velocity profile was used to model flow in a laminar incompressible boundary layer in Problem 5.10. Express the rotation of a fluid particle. Locate the maximum rate of rotation. Express the rate of angular deformation for a fluid particle. Locate the maximum rate of angular deformation. Express the rates of linear deformation for a fluid particle. Locate the maximum rates of linear deformation. Express the shear force per unit volume in the x direction. Locate the maximum shear force per unit volume; interpret this result.

5.10 A crude approximation for the x component of velocity in an incompressible laminar boundary layer is a linear variation from $u = 0$ at the surface ($y = 0$) to the freestream velocity, U, at the boundary-layer edge ($y = \delta$). The equation for the profile is $u = Uy/\delta$, where $\delta = cx^{1/2}$ and c is a constant. Show that the simplest expression for the y component of velocity is $v = uy/4x$. Evaluate the maximum value of the ratio v/U, at a location where $x = 0.5$ m and $\delta = 5$ mm.

Given: Linear approximation for velocity profile in laminar boundary layer



Find:
(a) Express rotation, find maximum
(b) Express angular deformation, locate maximum
(c) Express linear deformation, locate maximum
(d) Express shear force per unit volume, locate maximum

Solution: We will apply the definition of rotation to the given velocity field.

Governing Equation: $$\vec{\omega} = \frac{1}{2}\nabla \times \vec{V}$$ (Definition of rotation)

Assumptions: (1) Steady flow

The rotation is: $$\vec{\omega} = \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ U\frac{y}{\delta} & \frac{U}{4}\frac{y}{x}\frac{y}{\delta} & 0 \end{vmatrix} = \frac{1}{2}\left[\frac{\partial}{\partial x}\left(\frac{U}{4}\frac{y}{x}\frac{y}{\delta}\right) - \frac{\partial}{\partial y}\left(U\frac{y}{\delta}\right)\right]\hat{k}$$ Computing the partial derivatives:

$$\omega_z = -\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{3}{2}\cdot\frac{U\cdot y^2}{c\cdot x^{\frac{5}{2}}} - \frac{1}{2}\cdot\frac{U}{c\cdot x^{\frac{1}{2}}} = -\frac{1}{2}\cdot\left(\frac{3}{8}\cdot\frac{U\cdot y^2}{c\cdot x^{\frac{5}{2}}} + \frac{U}{c\cdot x^{\frac{1}{2}}}\right) = -\frac{U}{2\cdot c\cdot x^{\frac{1}{2}}}\cdot\left[1 + \frac{3}{8}\cdot\left(\frac{y}{x}\right)^2\right]$$

$$\omega_z = -\frac{U}{2\cdot\delta}\cdot\left[1 + \frac{3}{8}\cdot\left(\frac{y}{x}\right)^2\right]$$

Maximum value at $y = \delta$

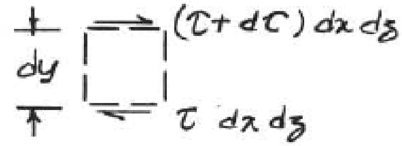
The angular deformation is: $$\frac{\partial}{\partial x}v + \frac{\partial}{\partial y}u = \frac{\partial}{\partial x}\left(\frac{U}{4}\cdot\frac{y^2}{c\cdot x^{\frac{3}{2}}}\right) + \frac{\partial}{\partial y}\left(U\cdot\frac{y}{c\cdot x^{\frac{1}{2}}}\right) = \frac{U}{c}\left(-\frac{1}{4}\cdot\frac{3}{2}\cdot\frac{y^2}{x^{\frac{5}{2}}} + \frac{1}{x^{\frac{1}{2}}}\right) = \frac{U}{c\cdot x^{\frac{1}{2}}}\cdot\left[1 - \frac{3}{8}\cdot\left(\frac{y}{x}\right)^2\right]$$

$$\text{angdef} = \frac{U}{\delta}\cdot\left[1 - \frac{3}{8}\cdot\left(\frac{y}{x}\right)^2\right]$$ Maximum value at $y = 0$

Linear deformation: $$\frac{\partial}{\partial x}u = \frac{\partial}{\partial x}\left(U\cdot\frac{y}{c\cdot x^{\frac{1}{2}}}\right) = -\frac{1}{2}\cdot\frac{U\cdot y}{c\cdot x^{\frac{3}{2}}}$$ $$\frac{\partial}{\partial x}u = -\frac{U}{2\delta}\cdot\frac{y}{x}$$ Maximum value at $y = \delta$

$$\frac{\partial}{\partial y}v = \frac{\partial}{\partial y}\left(\frac{U}{4}\cdot\frac{y^2}{c\cdot x^{\frac{3}{2}}}\right) = \frac{2}{4}\cdot\frac{U\cdot y}{c\cdot x^{\frac{3}{2}}}$$ $$\frac{\partial}{\partial y}v = \frac{U}{2\delta}\cdot\frac{y}{x}$$ Maximum value at $y = \delta$

The shear stress is $\tau_{yx} = \mu \cdot \left(\frac{\partial}{\partial x} v + \frac{\partial}{\partial y} u \right) = \frac{\mu \cdot U}{\delta} \cdot \left[1 - \frac{3}{8} \cdot \left(\frac{y}{x} \right)^2 \right]$



The net shear force on a fluid element is dτ dx dz: $d\tau = \frac{\partial}{\partial y} \tau \cdot dy = \frac{\mu \cdot U}{\delta} \cdot \left(-\frac{3}{8} \cdot \frac{2 \cdot y}{x^2} \right) \cdot dy = -\frac{3 \cdot \mu \cdot U \cdot y}{4 \cdot \delta \cdot x^2} \cdot dy$

Therefore the shear stress per unit volume is: $\frac{d}{dV} F = -\frac{3 \cdot \mu \cdot U}{4 \cdot \delta \cdot x} \cdot \frac{y}{x}$ Maximum value at $y = \delta$

Problem 5.93

[Difficulty: 2]

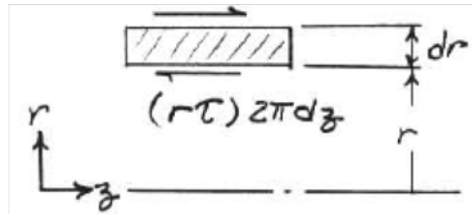
5.93 Problem 4.35 gave the velocity profile for fully developed laminar flow in a circular tube as $u = u_{max}[1 - (r/R)^2]$. Obtain an expression for the shear force per unit volume in the x direction for this flow. Evaluate its maximum value for the conditions of Problem 4.35.

Given: Velocity profile for fully developed laminar flow in a tube

Find: (a) Express shear force per unit volume in the x-direction
(b) Maximum value at these conditions

Solution: We will evaluate a differential volume of fluid in this flow field

Assumptions: (1) Steady flow



The differential of shear force would be: $dF_{shear} = (\tau + d\tau)\cdot 2\cdot\pi\cdot r\cdot dz\cdot dr - \tau\cdot 2\cdot\pi\cdot r\cdot dz\cdot dr = 2\cdot\pi\cdot r\cdot d\tau\cdot dz\cdot dr$

and in cylindrical coordinates: $$\frac{dF_{sz}}{dV} = \frac{1}{2\cdot\pi\cdot r\cdot dr\cdot dz}\cdot\frac{d}{dr}(r\cdot\tau)\cdot 2\cdot\pi\cdot dr\cdot dz = \frac{1}{r}\frac{d}{dr}\left(r\cdot\mu\cdot\frac{d}{dr}u\right)$$

From the given profile: $\frac{d}{dr}u = -u_{max}\cdot\frac{2\cdot r}{R^2}$ Therefore: $\frac{dF_{sz}}{dV} = -2\cdot\frac{\mu\cdot u_{max}}{r}\cdot\frac{d}{dr}\left(\frac{r^2}{R^2}\right) = -2\cdot\frac{\mu\cdot u_{max}}{r}\cdot\frac{2\cdot r}{R^2} = -4\cdot\frac{\mu\cdot u_{max}}{R^2}$

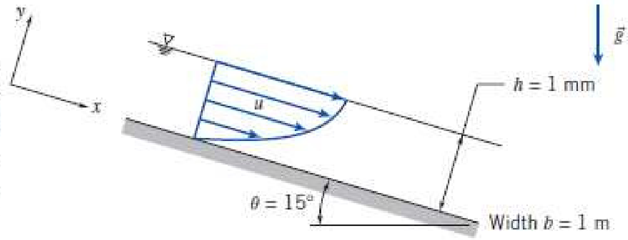
$$F_{vmax} = \frac{dF_{szmax}}{dV} = -4\cdot\frac{\mu\cdot u_{max}}{R^2}$$

For water: $\mu = 2.1\times 10^{-5}\cdot\frac{lbf\cdot s}{ft^2}$ $u_{max} = 10\cdot\frac{ft}{s}$ $R = 3\cdot in$ Substituting these values:

$$F_{vmax} = -4\times 2.1\times 10^{-5}\cdot\frac{lbf\cdot s}{ft^2}\times 10\cdot\frac{ft}{s}\times\frac{1}{(3\cdot in)^2}\times\left(\frac{12\cdot in}{ft}\right)^2$$ $$F_{vmax} = -0.0134\cdot\frac{lbf}{ft^3}$$

Problem 5.94 [Difficulty: 3]

5.94 Assume the liquid film in Example 5.9 is horizontal (i.e., $\theta = 0°$) and that the flow is driven by a constant shear stress on the top surface $(y = h)$, $\tau_{yx} = C$. Assume that the liquid film is thin enough and flat and that the flow is fully developed with zero net flow rate (flow rate $Q = 0$). Determine the velocity profile $u(y)$ and the pressure gradient dp/dx.



Given: Horizontal, fully developed flow

Find: Velocity Profile and pressure gradient

Solution:

Governing Equations:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \qquad \text{(Continuity Equation)}$$

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=\rho g_x-\frac{\partial P}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=\rho g_y-\frac{\partial P}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right) \qquad \text{(Navier-Stokes Equations)}$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=\rho g_z-\frac{\partial P}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)$$

Assumptions:
(1) Incompressible fluid
(2) Zero net flow rate

For fully developed flow

$$\frac{d^2u}{dy^2}=\frac{1}{\mu}\frac{dp}{dx} \qquad (1)$$

The general solution for equation (1) is

$$u=\frac{y^2}{2\mu}\frac{dp}{dx}+C_1y+C_2 \qquad (2)$$

where C_1 and C_2 are constants.

Apply the boundary conditions

$$u=0 \text{ at } y=0$$

$$\mu\frac{du}{dy}=C \text{ at } y=h$$

Then, we can get $C_1=\frac{1}{\mu}(C-h\frac{dp}{dx})$ and $C_2=0$

$$u=-\frac{h^2}{\mu}\frac{dp}{dx}(y'-\frac{1}{2}y'^2)+\frac{Ch}{\mu}y', \text{ where } y'=\frac{y}{h}$$

The net flow or flow rate is zero:

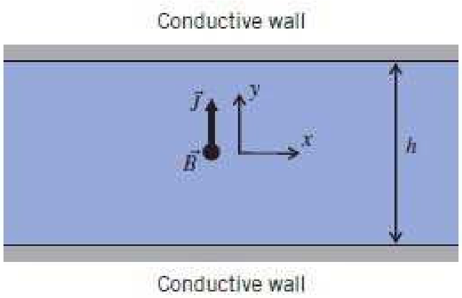
$$0 = -\frac{h^2}{\mu}\frac{dp}{dx}\int_0^1 (y' - \frac{1}{2}y'^2)dy + \frac{Ch}{\mu}\int_0^1 y' dy$$

Thus, $\frac{dp}{dx} = \frac{3}{2}\frac{C}{h}$

Problem 5.95

[Difficulty: 3]

5.95 Consider a planar microchannel of width h, as shown (it is actually very long in the x direction and open at both ends). A Cartesian coordinate system with its origin positioned at the center of the microchannel is used in the study. The microchannel is filled with a weakly conductive solution. When an electric current is applied across the two conductive walls, the current density, $\vec{J}$, transmitted through the solution is parallel to the y axis. The entire device is placed in a constant magnetic field, $\vec{B}$, which is pointed outward from the plane (the z direction), as shown. Interaction between the current density and the magnetic field induces a *Lorentz force* of density $\vec{J} \times \vec{B}$. Assume that the conductive solution is incompressible, and since the sample volume is very small in lab-on-a-chip applications, the gravitational body force is neglected. Under steady state, the flow driven by the Lorentz force is described by the continuity (Eq. 5.1a) and Navier–Stokes equations (Eqs. 5.27), except the x, y, and z components of the latter have extra Lorentz force components on the right. Assuming that the flow is fully developed and the velocity field $\vec{V}$ is a function of y only, find the three components of velocity.



Given: N-S equations and simplification assumptions

Find: Fluid Velocity

Solution:

Governing Equations:

$\nabla \cdot \mathbf{u} = 0$ (Continuity Equation)

$\rho \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{J} \times \mathbf{B}$ (Momentum Equation)

Assumptions:

(1) Incompressible fluid
(2) Two dimensional, fully developed flow driven by Lorentz force
(2) Zero pressure gradient

Write the 2D continuity and momentum equations in Cartesian coordinates:

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \qquad (1)$$

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + JB \qquad (2)$$

$$\rho(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y})=-\frac{\partial p}{\partial y}+\mu(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}) \qquad (3)$$

Simplify the above equations:

$$v=0\rightarrow\frac{\partial u}{\partial x}=0\rightarrow u=u(y)$$

Using the assumption of zero pressure gradient, equation (3) vanishes, and equation (2) can be simplified as

$$0=\mu\frac{d^2u}{dy^2}+JB \qquad (4)$$

General solution for equation (4) is

$$u=-\frac{1}{2}\frac{JB}{\mu}y^2+C_1y+C_2 \qquad (5)$$

Apply the no slip boundary conditions into equation (5), we get

$$\begin{cases} u(-\frac{h}{2})=0=-\frac{1}{2}\frac{JB}{\mu}\frac{h^2}{4}-C_1\frac{h}{2}+C_2 \\ u(\frac{h}{2})=0=-\frac{1}{2}\frac{JB}{\mu}\frac{h^2}{4}+C_1\frac{h}{2}+C_2 \end{cases}$$

Therefore, C_1=0 and $C_2=\frac{JB}{8\mu}h^2$

The fluid velocity is given as

$$u(y)=\frac{JB}{8\mu}(h^2-4y^2)$$

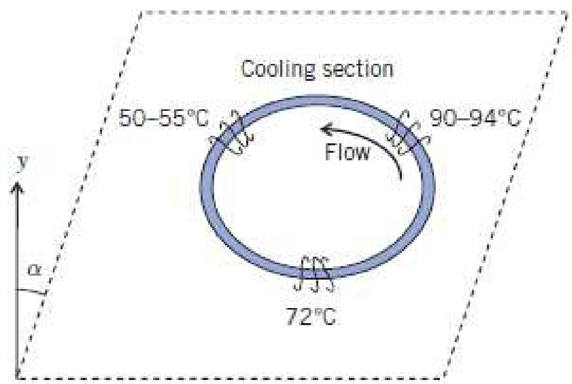
Problem 5.96

[Difficulty: 2]

5.96 The common thermal polymerase chain reaction (PCR) process requires the cycling of reagents through three distinct temperatures for denaturation (90–94°C), annealing (50–55°C), and extension (72°C). In continuous-flow PCR reactors, the temperatures of the three thermal zones are maintained as fixed while the reagents are cycled continuously through these zones. These temperature variations induce significant variations in the fluid density, which under appropriate conditions can be used to generate fluid motion. The figure depicts a thermosiphon-based PCR device (Chen et al., 2004, *Analytical Chemistry*, 76, 3707–3715). The closed loop is filled with PCR reagents. The plan of the loop is inclined at an angle α with respect to the vertical. The loop is surrounded by three heaters and coolers that maintain different temperatures.

(a) Explain why the fluid automatically circulates in the closed loop along the counterclockwise direction.

(b) What is the effect of the angle α on the fluid velocity?



Given: Temperature-dependent fluid density and the Navier-Stokes equations

Find: Explanation for the buoyancy-driven flow; effect of angle on fluid velocity

Solution:

Governing Equations:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \qquad \text{(Continuity Equation)}$$

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=\rho g_x-\frac{\partial P}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=\rho g_y-\frac{\partial P}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right) \qquad \text{(Navier-Stokes Equations)}$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=\rho g_z-\frac{\partial P}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)$$

Assumption: Incompressible fluid

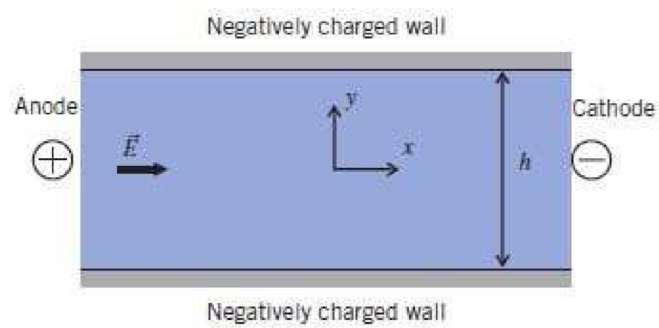
(1) The first term in the right-hand-side of the momentum equations (5.27a)-(5.27c) represents the gravitational body force, which is proportional to the local fluid density. The fluid density in the region at temperature 72^{o}C is higher than that in the region at temperature 90-94 ^{o}C, and meanwhile is lower than that in the region at temperature 50-55 ^{o}C. Thus, the net gravitational force induces counter-clockwise fluid circulation within the loop.

(2) Since the fluid circulation is driven by buoyancy force which is proportional to $g\times\cos\alpha$ where g is the gravitational acceleration, one can control the flow rate in the loop by adjusting the inclination angle α. When the angle $\alpha=90^{o}$, there is no fluid motion. When $\alpha=0$, the flow rate is the maximum.

Problem 5.97

[Difficulty: 3]

5.97 Electro-osmotic flow (EOF) is the motion of liquid induced by an applied electric field across a charged capillary tube or microchannel. Assume the channel wall is negatively charged, a thin layer called the electric double layer (EDL) forms in the vicinity of the channel wall in which the number of positive ions is much larger than that of the negative ions. The net positively charged ions in the EDL then drag the electrolyte solution along with them and cause the fluid to flow toward the cathode. The thickness of the EDL is typically on the order of 10 nm. When the channel dimensions are much larger than the thickness of EDL, there is a slip velocity, $y-\frac{\varepsilon\zeta}{\mu}\vec{E}$, on the channel wall, where ε is the fluid permittivity, ζ is the negative surface electric potential, $\vec{E}$ is the electric field intensity, and μ is the fluid dynamic viscosity. Consider a microchannel formed by two parallel plates. The walls of the channel have a negative surface electric potential of ζ. The microchannel is filled with an electrolyte solution, and the microchannel ends are subjected to an electric potential difference that gives rise to a uniform electric field strength of E along the x direction. The pressure gradient in the channel is zero. Derive the velocity of the steady, fully developed electro-osmotic flow. Compare the velocity profile of the EOF to that of pressure-driven flow. Calculate the EOF velocity using $\varepsilon = 7.08\times10^{-10}$ C·V^{-1}m^{-1}, $\zeta=-0.1$ V, $\mu=10^{-3}$ Pa·s, and $E = 1000$ V/m.



Given: N-S equations

Find: Fluid velocity

Solution:

Governing Equations:

$\nabla\cdot\mathbf{u}=0$ (Continuity equation)

$\rho\mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p+\mu\nabla^2\mathbf{u}$ (Momentum equation)

Assumptions:
(1) Two-dimensional fully developed flow
(2) Zero pressure gradient

(1) Write the continuity and momentum equations in Cartesian form:

$$\begin{cases}\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}=0 & (1)\\ \rho(u\dfrac{\partial u}{\partial x}+v\dfrac{\partial u}{\partial y})=-\dfrac{\partial p}{\partial x}+\mu(\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}) & (2)\\ \rho(u\dfrac{\partial v}{\partial x}+v\dfrac{\partial v}{\partial y})=-\dfrac{\partial p}{\partial y}+\mu(\dfrac{\partial^2 v}{\partial x^2}+\dfrac{\partial^2 v}{\partial y^2}) & (3)\end{cases}$$

Simplify the above equations:

$$v = 0 \rightarrow \frac{\partial u}{\partial x} = 0 \rightarrow u = u(y)$$

Using the assumption of zero pressure gradient, equation (3) vanishes, and equation (2) can be simplified as

$$0 = \mu \frac{d^2 u}{dy^2} \qquad (4)$$

General solution for equation (4) is given as

$$u = C_1 y + C_2 \qquad (5)$$

Apply the boundary condition into equation (5), we get

$$\begin{cases} u(-\frac{h}{2}) = -\frac{\varepsilon\zeta}{\mu}E = -C_1\frac{h}{2} + C_2 \\ u(\frac{h}{2}) = -\frac{\varepsilon\zeta}{\mu}E = C_1\frac{h}{2} + C_2 \end{cases}$$

Therefore, C_1=0 and $C_2 = -\frac{\varepsilon\zeta}{\mu}E$

The fluid velocity is given as

$$u(y) = -\frac{\varepsilon\zeta}{\mu}E \qquad (6)$$

(2) Pressure-driven flow has a parabolic flow velocity profile; while EOF has a plug velocity profile and it is independent of the channel size.

(3) Substituting $\varepsilon=7.08\times10^{-10}$ $CV^{-1}m^{-1}$, $\zeta=-0.1V$, $\mu=10^{-3}$ Pa.s, and E=1000 V/m into equation (6), one obtains

$$u(y) = -\frac{7.08\times10^{-10}\,\mathrm{C}\cdot V^{-1}\cdot m^{-1}\times -0.1V}{10^{-3}\,\mathrm{Pa\cdot s}}\times 1000\,\mathrm{V/m}$$
$$= 70.8\times10^{-6}\,\frac{C\cdot V}{Pa\cdot s\cdot m^2} = 70.8\times10^{-6}\,\frac{N}{Pa\cdot s\cdot m} = 70.8\times10^{-6}\,m/s$$

Problem 5.98

[Difficulty: 3]

5.98 A tank contains water (20°C) at an initial depth $y_0 = 1$ m. The tank diameter is $D = 250$ mm, and a tube of diameter $d = 3$ mm and length $L = 4$ m is attached to the bottom of the tank. For laminar flow a reasonable model for the water level over time is

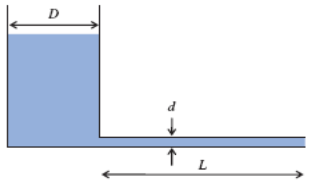
$$\frac{dy}{dt} = -\frac{d^4 \rho g}{32 D^2 \mu L} y \qquad y(0) = y_0$$

Using Euler methods with time steps of 12 min and 6 min:

(a) Estimate the water depth after 120 min, and compute the errors compared to the exact solution

$$y_{\text{exact}}(t) = y_0 e^{-\frac{d^4 \rho g}{32 D^2 \mu L} t}$$

(b) Plot the Euler and exact results.



$d =$ 3 mm
$D =$ 250 mm
$y_0 =$ 1 m
$L =$ 4 m

$\rho =$ 999 kg/m^3
$\mu =$ 0.001 N s/m^2

$h =$ 12 min $\qquad$ $h =$ 6 min

$$\frac{dy}{dt} = -\frac{d^4 \rho g}{32 D^2 \mu L} y \qquad y(0) = y_0$$

$$y_{\text{Exact}}(t) = y_0 e^{-\frac{d^4 \rho g}{32 D^2 \mu L} t}$$

$$y_{n+1} = y_n + h k y_n \qquad k = -\frac{d^4 \rho g}{32 D^2 \mu L} \qquad t_{n+1} = t_n + h$$

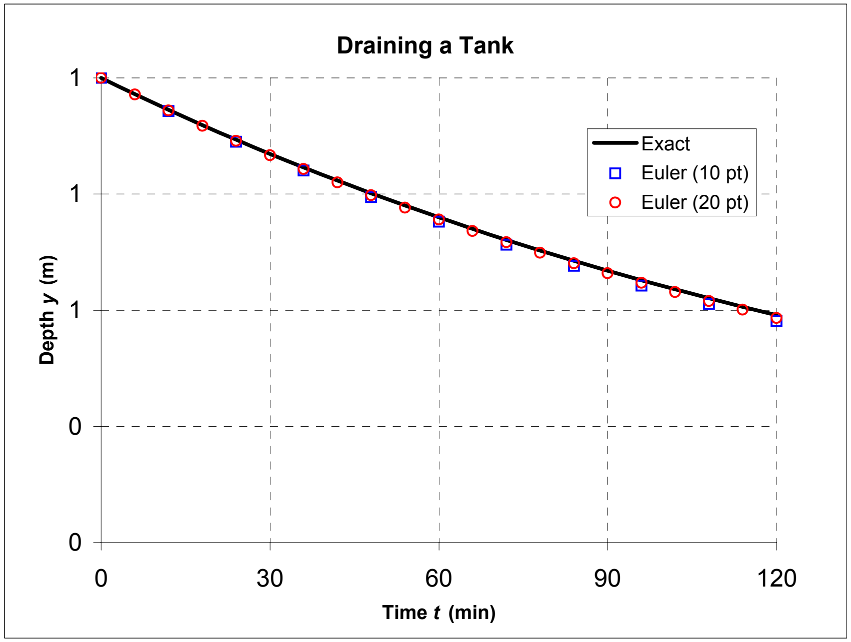
$k =$ 0.000099 s^{-1}

n	t_n (min)	y_n (m)
0	0	1
1	12	0.929
2	24	0.862
3	36	0.801
4	48	0.743
5	60	0.690
6	72	0.641
7	84	0.595
8	96	0.553
9	108	0.513
10	120	0.477

Error: 3%

n	t_n (min)	y_n (m)	y_{Exact} (m)
0	0.0	1	1
1	6.0	0.964	0.965
2	12.0	0.930	0.931
3	18.0	0.897	0.898
4	24.0	0.865	0.867
5	30.0	0.834	0.836
6	36.0	0.804	0.807
7	42.0	0.775	0.779
8	48.0	0.748	0.751
9	54.0	0.721	0.725
10	60.0	0.695	0.700
11	66.0	0.670	0.675
12	72.0	0.646	0.651
13	78.0	0.623	0.629
14	84.0	0.601	0.606
15	90.0	0.579	0.585
16	96.0	0.559	0.565
17	102.0	0.539	0.545
18	108.0	0.520	0.526
19	114.0	0.501	0.507
20	120.0	0.483	0.489

Error: 1%



5.99 Use the Euler method to solve and plot

$$\frac{dy}{dx} = \cos(x) \qquad y(0) = 0$$

from $x = 0$ to $x = \pi/2$, using step sizes of $\pi/48$, $\pi/96$, and $\pi/144$. Also plot the exact solution,

$$y(x) = \sin(x)$$

and compute the errors at $x = \pi/2$ for the three Euler method solutions.

$$\boxed{y_{n+1} = y_n + \Delta x \cos(x_n) \qquad x_{n+1} = x_n + \Delta x}$$

Δx
0.06545

n	x	y
0	0.000	0.000
1	0.065	0.065
2	0.131	0.131
3	0.196	0.196
4	0.262	0.260
5	0.327	0.323
6	0.393	0.385
7	0.458	0.446
8	0.524	0.504
9	0.589	0.561
10	0.654	0.615
11	0.720	0.667
12	0.785	0.716
13	0.851	0.763
14	0.916	0.806
15	0.982	0.846
16	1.047	0.882
17	1.113	0.915
18	1.178	0.944
19	1.244	0.969
20	1.309	0.990
21	1.374	1.007
22	1.440	1.020
23	1.505	1.028
24	1.571	1.032

Error 3.24%

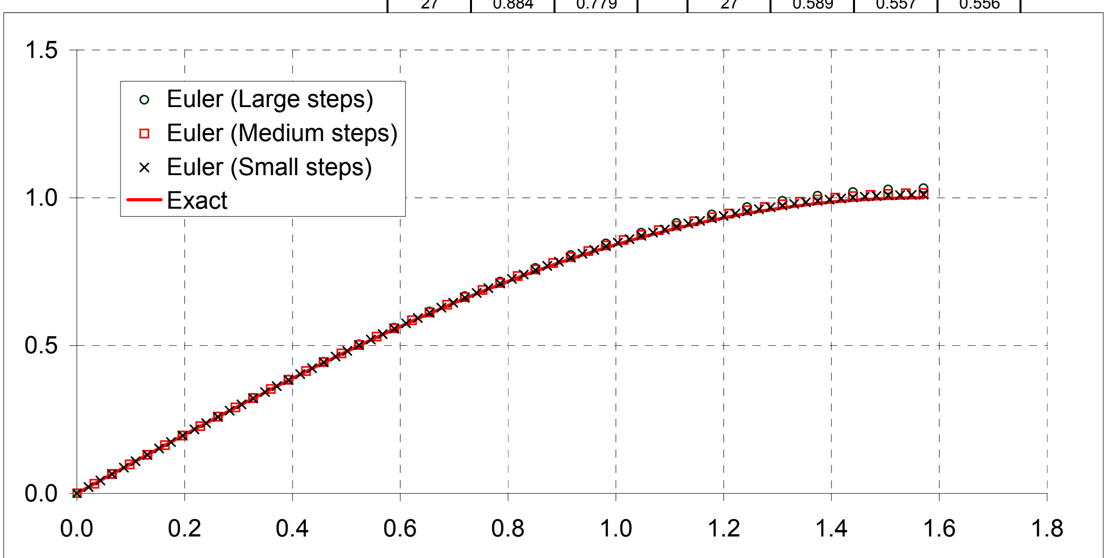
Δx
0.032725

n	x	y
0	0.000	0.000
1	0.033	0.033
2	0.065	0.065
3	0.098	0.098
4	0.131	0.131
5	0.164	0.163
6	0.196	0.195
7	0.229	0.227
8	0.262	0.259
9	0.295	0.291
10	0.327	0.322
11	0.360	0.353
12	0.393	0.384
13	0.425	0.414
14	0.458	0.444
15	0.491	0.473
16	0.524	0.502
17	0.556	0.530
18	0.589	0.558
19	0.622	0.585
20	0.654	0.612
21	0.687	0.638
22	0.720	0.663
23	0.753	0.688
24	0.785	0.712
25	0.818	0.735
26	0.851	0.757
27	0.884	0.779

Error 1.63%

Δx
0.021817

n	x	y	y_{Exact}
0	0.000	0.000	0.000
1	0.022	0.022	0.022
2	0.044	0.044	0.044
3	0.065	0.065	0.065
4	0.087	0.087	0.087
5	0.109	0.109	0.109
6	0.131	0.131	0.131
7	0.153	0.152	0.152
8	0.175	0.174	0.174
9	0.196	0.195	0.195
10	0.218	0.217	0.216
11	0.240	0.238	0.238
12	0.262	0.259	0.259
13	0.284	0.280	0.280
14	0.305	0.301	0.301
15	0.327	0.322	0.321
16	0.349	0.343	0.342
17	0.371	0.363	0.362
18	0.393	0.383	0.383
19	0.415	0.404	0.403
20	0.436	0.424	0.423
21	0.458	0.443	0.442
22	0.480	0.463	0.462
23	0.502	0.482	0.481
24	0.524	0.501	0.500
25	0.545	0.520	0.519
26	0.567	0.539	0.537
27	0.589	0.557	0.556
55	1.200	0.939	0.932
56	1.222	0.947	0.940
57	1.244	0.954	0.947
58	1.265	0.961	0.954
59	1.287	0.968	0.960
60	1.309	0.974	0.966
61	1.331	0.980	0.971
62	1.353	0.985	0.976
63	1.374	0.990	0.981
64	1.396	0.994	0.985
65	1.418	0.998	0.988
66	1.440	1.001	0.991
67	1.462	1.004	0.994
68	1.484	1.006	0.996
69	1.505	1.008	0.998
70	1.527	1.009	0.999
71	1.549	1.010	1.000
72	1.571	1.011	1.000



Error 1.09%

Problem 5.100

[Difficulty: 3]

*5.100 Use *Excel* to generate the solution of Eq. 5.31 for $m = 1$ shown in Fig. 5.18. To do so, you need to learn how to perform linear algebra in *Excel*. For example, for $N = 4$ you will end up with the matrix equation of Eq. 5.37. To solve this equation for the u values, you will have to compute the inverse of the 4×4 matrix, and then multiply this inverse into the 4×1 matrix on the right of the equation. In *Excel*, to do *array operations*, you must use the following rules: Pre-select the cells that will contain the result; use the appropriate *Excel array function* (look at *Excel's* Help for details); press Ctrl+Shift+Enter, not just Enter. For example, to invert the 4×4 matrix you would: Pre-select a blank 4×4 array that will contain the inverse matrix; type = *minverse*([array containing matrix to be inverted]); press Ctrl+Shift+Enter. To multiply a 4×4 matrix into a 4×1 matrix you would: Pre-select a blank 4×1 array that will contain the result; type = *mmult*([array containing 4×4 matrix], [array containing 4×1 matrix]); press Ctrl+Shift+Enter.

$$\frac{du}{dx} + u^m = 0; \qquad 0 \le x \le 1; \qquad u(0) = 1$$

N = **4**

Δx = **0.333**

	Eq. 5.34 (LHS)				**(RHS)**
	1.000	0.000	0.000	0.000	1
	-1.000	1.333	0.000	0.000	0
	0.000	-1.000	1.333	0.000	0
	0.000	0.000	-1.000	1.333	0

x	**Inverse Matrix**				**Result**	**Exact**	**Error**
0.000	1.000	0.000	0.000	0.000	1.000	1.000	0.000
0.333	0.750	0.750	0.000	0.000	0.750	0.717	0.000
0.667	0.563	0.563	0.750	0.000	0.563	0.513	0.001
1.000	0.422	0.422	0.563	0.750	0.422	0.368	0.001
							0.040

N = **8**
Δx = **0.143**

Eq. 5.34 (LHS) / **(RHS)**

								(RHS)
1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	1
-1.000	1.143	0.000	0.000	0.000	0.000	0.000	0.000	0
0.000	-1.000	1.143	0.000	0.000	0.000	0.000	0.000	0
0.000	0.000	-1.000	1.143	0.000	0.000	0.000	0.000	0
0.000	0.000	0.000	-1.000	1.143	0.000	0.000	0.000	0
0.000	0.000	0.000	0.000	-1.000	1.143	0.000	0.000	0
0.000	0.000	0.000	0.000	0.000	-1.000	1.143	0.000	0
0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.143	0

Inverse Matrix

x	1	2	3	4	5	6	7	8	Result	Exact	Error
0.000	1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	1.000	1.000	0.000
0.143	0.875	0.875	0.000	0.000	0.000	0.000	0.000	0.000	0.875	0.867	0.000
0.286	0.766	0.766	0.875	0.000	0.000	0.000	0.000	0.000	0.766	0.751	0.000
0.429	0.670	0.670	0.766	0.875	0.000	0.000	0.000	0.000	0.670	0.651	0.000
0.571	0.586	0.586	0.670	0.766	0.875	0.000	0.000	0.000	0.586	0.565	0.000
0.714	0.513	0.513	0.586	0.670	0.766	0.875	0.000	0.000	0.513	0.490	0.000
0.857	0.449	0.449	0.513	0.586	0.670	0.766	0.875	0.000	0.449	0.424	0.000
1.000	0.393	0.393	0.449	0.513	0.586	0.670	0.766	0.875	0.393	0.368	0.000
											0.019

N = **16**
Δx = **0.067**

Eq. 5.34 (LHS)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	(RHS)
1	1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	1
2	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0
3	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0
4	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0
5	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0
6	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0
7	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0
8	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0
9	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0
10	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0
11	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0
12	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0
13	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0
14	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0
15	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0
16	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0

x	**Inverse Matrix**																**Result**	**Exact**	**Error**
0.000	1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	1.000	1.000	0.000
0.067	0.938	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.938	0.936	0.000
0.133	0.879	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.879	0.875	0.000
0.200	0.824	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.824	0.819	0.000
0.267	0.772	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.772	0.766	0.000
0.333	0.724	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.724	0.717	0.000
0.400	0.679	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.679	0.670	0.000
0.467	0.637	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.637	0.627	0.000
0.533	0.597	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.597	0.587	0.000
0.600	0.559	0.559	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.559	0.549	0.000
0.667	0.524	0.524	0.559	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.524	0.513	0.000
0.733	0.492	0.492	0.524	0.559	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.492	0.480	0.000
0.800	0.461	0.461	0.492	0.524	0.559	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.461	0.449	0.000
0.867	0.432	0.432	0.461	0.492	0.524	0.559	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.432	0.420	0.000
0.933	0.405	0.405	0.432	0.461	0.492	0.524	0.559	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.405	0.393	0.000
1.000	0.380	0.380	0.405	0.432	0.461	0.492	0.524	0.559	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.380	0.368	0.000
																			0.009

N	Δx	**Error**
4	0.333	0.040
8	0.143	0.019
16	0.067	0.009

Problem 5.101

[Difficulty: 3]

***5.101** Following the steps to convert the differential equation Eq. 5.31 (for $m = 1$) into a difference equation (for example, Eq. 5.37 for $N = 4$), solve

$$\frac{du}{dx} + u = 2\cos(2x) \qquad 0 \le x \le 1 \quad u(0) = 0$$

for $N = 4$, 8, and 16 and compare to the exact solution

$$u_{\text{exact}} = \frac{2}{5}\cos(2x) + \frac{4}{5}\sin(2x) - \frac{2}{5}e^{-x}$$

Hints: Follow the rules for *Excel* array operations as described in Problem 5.100. Only the right side of the difference equations will change, compared to the solution method of Eq. 5.31 (for example, only the right side of Eq. 5.37 needs modifying).

New Eq. 5.37: $$-u_{i-1} + (1 + \Delta x)u_i = 2\Delta x \cdot \cos(2x_i)$$

N = **4**

Δx = **0.333**

	Eq. 5.34 (LHS)				(RHS)
	1.000	0.000	0.000	0.000	0
	-1.000	1.333	0.000	0.000	0.52392
	0.000	-1.000	1.333	0.000	0.15683
	0.000	0.000	-1.000	1.333	-0.2774

x	Inverse Matrix				Result	Exact	Error
0.000	1.000	0.000	0.000	0.000	0.000	0.000	0.000
0.333	0.750	0.750	0.000	0.000	0.393	0.522	0.004
0.667	0.563	0.563	0.750	0.000	0.412	0.666	0.016
1.000	0.422	0.422	0.563	0.750	0.101	0.414	0.024
							0.212

N = **8**

Δx = **0.143**

Eq. 5.34 (LHS)								(RHS)
1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0
-1.000	1.143	0.000	0.000	0.000	0.000	0.000	0.000	0.27413
0.000	-1.000	1.143	0.000	0.000	0.000	0.000	0.000	0.24032
0.000	0.000	-1.000	1.143	0.000	0.000	0.000	0.000	0.18703
0.000	0.000	0.000	-1.000	1.143	0.000	0.000	0.000	0.11857
0.000	0.000	0.000	0.000	-1.000	1.143	0.000	0.000	0.0405
0.000	0.000	0.000	0.000	0.000	-1.000	1.143	0.000	-0.0409
0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.143	-0.1189

x	Inverse Matrix 1	2	3	4	5	6	7	8	Result	Exact	Error
0.000	1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.143	0.875	0.875	0.000	0.000	0.000	0.000	0.000	0.000	0.240	0.263	0.000
0.286	0.766	0.766	0.875	0.000	0.000	0.000	0.000	0.000	0.420	0.469	0.000
0.429	0.670	0.670	0.766	0.875	0.000	0.000	0.000	0.000	0.531	0.606	0.001
0.571	0.586	0.586	0.670	0.766	0.875	0.000	0.000	0.000	0.569	0.668	0.001
0.714	0.513	0.513	0.586	0.670	0.766	0.875	0.000	0.000	0.533	0.653	0.002
0.857	0.449	0.449	0.513	0.586	0.670	0.766	0.875	0.000	0.431	0.565	0.002
1.000	0.393	0.393	0.449	0.513	0.586	0.670	0.766	0.875	0.273	0.414	0.002
											0.094

N = **16**

Δx = **0.067** **Eq. 5.34 (LHS)**

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	(RHS)
1	1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0
2	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.13215
3	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.12862
4	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.12281
5	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.11482
6	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.10478
7	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.09289
8	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.07935
9	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.06441
10	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.04831
11	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.03137
12	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.01386
13	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	-0.0039
14	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	-0.0216
15	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	-0.0389
16	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	-0.0555

x	**Inverse Matrix**																**Result**	**Exact**	**Error**
0.000	1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.067	0.938	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.124	0.129	0.000
0.133	0.879	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.237	0.247	0.000
0.200	0.824	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.337	0.352	0.000
0.267	0.772	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.424	0.445	0.000
0.333	0.724	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.495	0.522	0.000
0.400	0.679	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.552	0.584	0.000
0.467	0.637	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.591	0.630	0.000
0.533	0.597	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.615	0.659	0.000
0.600	0.559	0.559	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.622	0.671	0.000
0.667	0.524	0.524	0.559	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.612	0.666	0.000
0.733	0.492	0.492	0.524	0.559	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.587	0.645	0.000
0.800	0.461	0.461	0.492	0.524	0.559	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.547	0.608	0.000
0.867	0.432	0.432	0.461	0.492	0.524	0.559	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.492	0.557	0.000
0.933	0.405	0.405	0.432	0.461	0.492	0.524	0.559	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.425	0.491	0.000
1.000	0.380	0.380	0.405	0.432	0.461	0.492	0.524	0.559	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.346	0.414	0.000
																			0.044

N	Δx	**Error**
4	0.333	0.212
8	0.143	0.094
16	0.067	0.044

Problem 5.102

[Difficulty: 3]

***5.102** Following the steps to convert the differential equation Eq. 5.31 (for $m = 1$) into a difference equation (for example, Eq. 5.37 for $N = 4$), solve

$$\frac{du}{dx} + u = 2x^2 + x \qquad 0 \le x \le 1 \quad u(0) = 3$$

for $N = 4$, 8, and 16 and compare to the exact solution

$$u_{\text{exact}} = 2x^2 - 3x + 3$$

Hint: Follow the hints provided in Problem 5.101.

New Eq. 5.37: $-u_{i-1} + (1+\Delta x)u_i = \Delta x \cdot (2x_i^2 + x_i)$

N = **4**

Δx = **0.333**

Eq. 5.34 (LHS)				**(RHS)**
1.000	0.000	0.000	0.000	3
-1.000	1.333	0.000	0.000	0.18519
0.000	-1.000	1.333	0.000	0.51852
0.000	0.000	-1.000	1.333	1

x	**Inverse Matrix**				**Result**	**Exact**	**Error**
0.000	1.000	0.000	0.000	0.000	3.000	3.000	0.000
0.333	0.750	0.750	0.000	0.000	2.389	2.222	0.007
0.667	0.563	0.563	0.750	0.000	2.181	1.889	0.021
1.000	0.422	0.422	0.563	0.750	2.385	2.000	0.037
							0.256

N = **8**

Δx = **0.143**

Eq. 5.34 (LHS)								**(RHS)**
1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	3
-1.000	1.143	0.000	0.000	0.000	0.000	0.000	0.000	0.02624
0.000	-1.000	1.143	0.000	0.000	0.000	0.000	0.000	0.06414
0.000	0.000	-1.000	1.143	0.000	0.000	0.000	0.000	0.1137
0.000	0.000	0.000	-1.000	1.143	0.000	0.000	0.000	0.17493
0.000	0.000	0.000	0.000	-1.000	1.143	0.000	0.000	0.24781
0.000	0.000	0.000	0.000	0.000	-1.000	1.143	0.000	0.33236
0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.143	0.42857

	Inverse Matrix										
x	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**Result**	**Exact**	**Error**
0.000	1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	3.000	3.000	0.000
0.143	0.875	0.875	0.000	0.000	0.000	0.000	0.000	0.000	2.648	2.612	0.000
0.286	0.766	0.766	0.875	0.000	0.000	0.000	0.000	0.000	2.373	2.306	0.001
0.429	0.670	0.670	0.766	0.875	0.000	0.000	0.000	0.000	2.176	2.082	0.001
0.571	0.586	0.586	0.670	0.766	0.875	0.000	0.000	0.000	2.057	1.939	0.002
0.714	0.513	0.513	0.586	0.670	0.766	0.875	0.000	0.000	2.017	1.878	0.002
0.857	0.449	0.449	0.513	0.586	0.670	0.766	0.875	0.000	2.055	1.898	0.003
1.000	0.393	0.393	0.449	0.513	0.586	0.670	0.766	0.875	2.174	2.000	0.004
											0.113

N = **16**

Δx = **0.067** **Eq. 5.34 (LHS)**

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**(RHS)**
1	1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	3
2	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.00504
3	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.01126
4	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.01867
5	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.02726
6	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.03704
7	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.048
8	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.06015
9	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.07348
10	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.088
11	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.1037
12	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.12059
13	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.13867
14	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.15793
15	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.17837
16	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.2

x	**Inverse Matrix**																**Result**	**Exact**	**Error**
0.000	1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	3.000	3.000	0.000
0.067	0.938	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	2.817	2.809	0.000
0.133	0.879	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	2.652	2.636	0.000
0.200	0.824	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	2.503	2.480	0.000
0.267	0.772	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	2.373	2.342	0.000
0.333	0.724	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	2.259	2.222	0.000
0.400	0.679	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	2.163	2.120	0.000
0.467	0.637	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	2.084	2.036	0.000
0.533	0.597	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	2.023	1.969	0.000
0.600	0.559	0.559	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	1.979	1.920	0.000
0.667	0.524	0.524	0.559	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	1.952	1.889	0.000
0.733	0.492	0.492	0.524	0.559	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	1.943	1.876	0.000
0.800	0.461	0.461	0.492	0.524	0.559	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	1.952	1.880	0.000
0.867	0.432	0.432	0.461	0.492	0.524	0.559	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	1.978	1.902	0.000
0.933	0.405	0.405	0.432	0.461	0.492	0.524	0.559	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	2.022	1.942	0.000
1.000	0.380	0.380	0.405	0.432	0.461	0.492	0.524	0.559	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	2.083	2.000	0.000
																			0.054

N	**Δ*x***	**Error**
4	0.333	0.256
8	0.143	0.113
16	0.067	0.054

Problem 5.103

[Difficulty: 3]

*5.103 A 50-mm cube of mass $M = 3$ kg is sliding across an oiled surface. The oil viscosity is $\mu = 0.45$ N·s/m^2, and the thickness of the oil between the cube and surface is $\delta = 0.2$ mm. If the initial speed of the block is $u_0 = 1$ m/s, use the numerical method that was applied to the linear form of Eq. 5.31 to predict the cube motion for the first second of motion. Use $N = 4$, 8, and 16 and compare to the exact solution

$$u_{exact} = u_0 e^{-(A\mu/M\delta)t}$$

where A is the area of contact. *Hint:* Follow the hints provided in Problem 5.101.

Equation of motion: $$M\frac{du}{dt} = -\mu\frac{du}{dy}A = -\mu A\frac{u}{\delta}$$

$$\frac{du}{dt} + \left(\frac{\mu A}{M\delta}\right)u = 0$$

$$\frac{du}{dt} + k \cdot u = 0$$

New Eq. 5.37: $$-u_{i-1} + (1 + k\cdot\Delta t)u_i = 0$$

A = 0.0025 m^2
δ = 0.5 mm
μ = 0.45 N.s/m^2
M = 3 kg
k = 0.75 s^{-1}

N = **4**
Δt = **0.333**

	Eq. 5.34 (LHS)				**(RHS)**
	1.000	0.000	0.000	0.000	1
	-1.000	1.250	0.000	0.000	0
	0.000	-1.000	1.250	0.000	0
	0.000	0.000	-1.000	1.250	0

t	**Inverse Matrix**				**Result**	**Exact**	**Error**
0.000	1.000	0.000	0.000	0.000	1.000	1.000	0.0E+00
0.333	0.800	0.800	0.000	0.000	0.800	0.779	1.1E-04
0.667	0.640	0.640	0.800	0.000	0.640	0.607	2.8E-04
1.000	0.512	0.512	0.640	0.800	0.512	0.472	3.9E-04
							0.028

N = **8**
Δt = **0.143**

	Eq. 5.34 (LHS)								**(RHS)**
	1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	1
	-1.000	1.107	0.000	0.000	0.000	0.000	0.000	0.000	0
	0.000	-1.000	1.107	0.000	0.000	0.000	0.000	0.000	0
	0.000	0.000	-1.000	1.107	0.000	0.000	0.000	0.000	0
	0.000	0.000	0.000	-1.000	1.107	0.000	0.000	0.000	0
	0.000	0.000	0.000	0.000	-1.000	1.107	0.000	0.000	0
	0.000	0.000	0.000	0.000	0.000	-1.000	1.107	0.000	0
	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.107	0

	Inverse Matrix										
t	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**Result**	**Exact**	**Error**
0.000	1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	1.000	1.000	0.0E+00
0.143	0.903	0.903	0.000	0.000	0.000	0.000	0.000	0.000	0.903	0.898	2.9E-06
0.286	0.816	0.816	0.903	0.000	0.000	0.000	0.000	0.000	0.816	0.807	9.5E-06
0.429	0.737	0.737	0.816	0.903	0.000	0.000	0.000	0.000	0.737	0.725	1.7E-05
0.571	0.666	0.666	0.737	0.816	0.903	0.000	0.000	0.000	0.666	0.651	2.5E-05
0.714	0.601	0.601	0.666	0.737	0.816	0.903	0.000	0.000	0.601	0.585	3.2E-05
0.857	0.543	0.543	0.601	0.666	0.737	0.816	0.903	0.000	0.543	0.526	3.7E-05
1.000	0.490	0.490	0.543	0.601	0.666	0.737	0.816	0.903	0.490	0.472	4.1E-05
											0.013

N = **16**

Δt = **0.067** **Eq. 5.34 (LHS)**

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**(RHS)**
1	1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	1
2	-1.000	1.050	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0
3	0.000	-1.000	1.050	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0
4	0.000	0.000	-1.000	1.050	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0
5	0.000	0.000	0.000	-1.000	1.050	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0
6	0.000	0.000	0.000	0.000	-1.000	1.050	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0
7	0.000	0.000	0.000	0.000	0.000	-1.000	1.050	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0
8	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.050	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0
9	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.050	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0
10	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.050	0.000	0.000	0.000	0.000	0.000	0.000	0
11	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.050	0.000	0.000	0.000	0.000	0.000	0
12	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.050	0.000	0.000	0.000	0.000	0
13	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.050	0.000	0.000	0.000	0
14	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.050	0.000	0.000	0
15	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.050	0.000	0
16	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.050	0

t	Inverse Matrix																Result	Exact	Error
0.000	1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	1.000	1.000	0.0E+00
0.067	0.952	0.952	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.952	0.951	8.3E-08
0.133	0.907	0.907	0.952	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.907	0.905	3.0E-07
0.200	0.864	0.864	0.907	0.952	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.864	0.861	6.1E-07
0.267	0.823	0.823	0.864	0.907	0.952	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.823	0.819	9.9E-07
0.333	0.784	0.784	0.823	0.864	0.907	0.952	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.784	0.779	1.4E-06
0.400	0.746	0.746	0.784	0.823	0.864	0.907	0.952	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.746	0.741	1.8E-06
0.467	0.711	0.711	0.746	0.784	0.823	0.864	0.907	0.952	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.711	0.705	2.2E-06
0.533	0.677	0.677	0.711	0.746	0.784	0.823	0.864	0.907	0.952	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.677	0.670	2.7E-06
0.600	0.645	0.645	0.677	0.711	0.746	0.784	0.823	0.864	0.907	0.952	0.000	0.000	0.000	0.000	0.000	0.000	0.645	0.638	3.0E-06
0.667	0.614	0.614	0.645	0.677	0.711	0.746	0.784	0.823	0.864	0.907	0.952	0.000	0.000	0.000	0.000	0.000	0.614	0.607	3.4E-06
0.733	0.585	0.585	0.614	0.645	0.677	0.711	0.746	0.784	0.823	0.864	0.907	0.952	0.000	0.000	0.000	0.000	0.585	0.577	3.7E-06
0.800	0.557	0.557	0.585	0.614	0.645	0.677	0.711	0.746	0.784	0.823	0.864	0.907	0.952	0.000	0.000	0.000	0.557	0.549	4.0E-06
0.867	0.530	0.530	0.557	0.585	0.614	0.645	0.677	0.711	0.746	0.784	0.823	0.864	0.907	0.952	0.000	0.000	0.530	0.522	4.3E-06
0.933	0.505	0.505	0.530	0.557	0.585	0.614	0.645	0.677	0.711	0.746	0.784	0.823	0.864	0.907	0.952	0.000	0.505	0.497	4.5E-06
1.000	0.481	0.481	0.505	0.530	0.557	0.585	0.614	0.645	0.677	0.711	0.746	0.784	0.823	0.864	0.907	0.952	0.481	0.472	4.7E-06
																			0.006

N	Δ*t*	Error
4	0.333	0.028
8	0.143	0.013
16	0.067	0.006

Problem 5.104

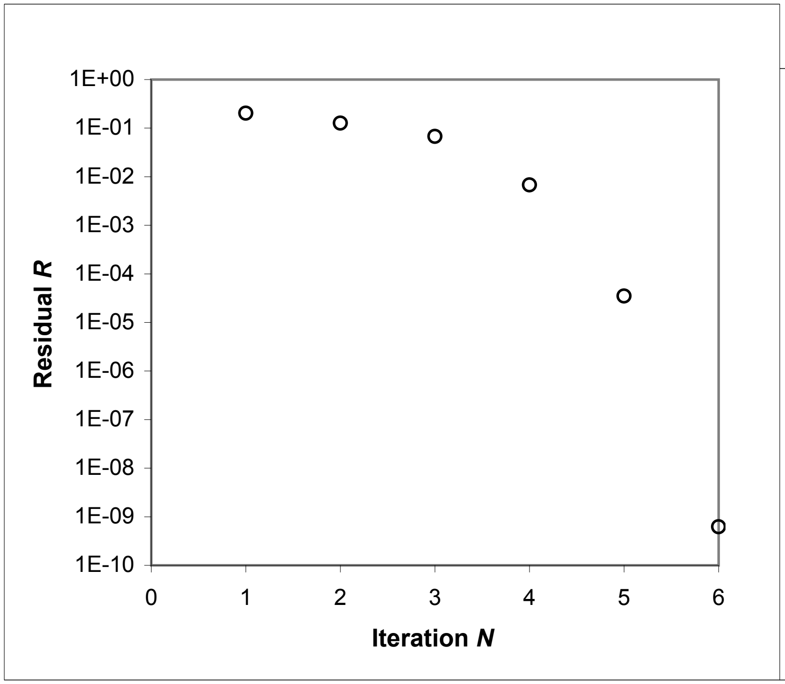
[Difficulty: 3]

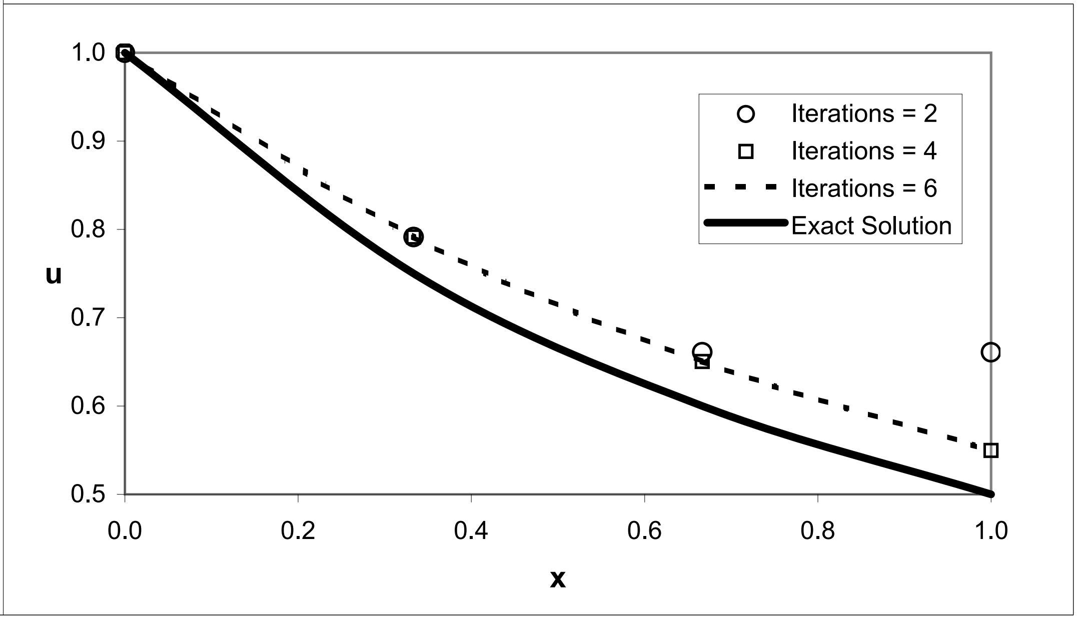
***5.104** Use *Excel* to generate the solutions of Eq. 5.31 for $m = 2$, as shown in Fig. 5.21.

$$u_i = \frac{u_{g_{i-1}} + \Delta x\, u_{g_i}^2}{1 + 2\Delta x\, u_{g_i}}$$

$\Delta x =$ 0.333

	x				
Iteration	0.000	0.333	0.667	1.000	**Residuals**
0	1.000	1.000	1.000	1.000	
1	1.000	0.800	0.800	0.800	0.204
2	1.000	0.791	0.661	0.661	0.127
3	1.000	0.791	0.650	0.560	0.068
4	1.000	0.791	0.650	0.550	0.007
5	1.000	0.791	0.650	0.550	0.000
6	1.000	0.791	0.650	0.550	0.000
Exact	**1.000**	**0.750**	**0.600**	**0.500**	





Problem 5.105

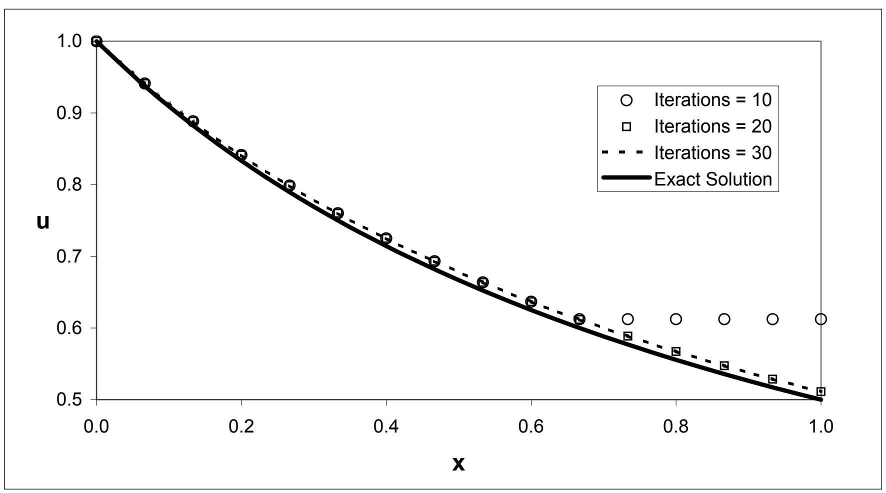
[Difficulty: 3]

***5.105** Use *Excel* to generate the solutions of Eq. 5.31 for $m = 2$, as shown in Fig. 5.21, except use 16 points and as many iterations as necessary to obtain reasonable convergence.

$$u_i = \frac{u_{g_{i-1}} + \Delta x\, u_{g_i}^2}{1 + 2\Delta x\, u_{g_i}}$$

$\Delta x =$ 0.0667

Iteration	*x* = 0.000	0.067	0.133	0.200	0.267	0.333	0.400	0.467	0.533	0.600	0.667	0.733	0.800	0.867	0.933	1.000
0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1	1.000	0.941	0.941	0.941	0.941	0.941	0.941	0.941	0.941	0.941	0.941	0.941	0.941	0.941	0.941	0.941
2	1.000	0.941	0.889	0.889	0.889	0.889	0.889	0.889	0.889	0.889	0.889	0.889	0.889	0.889	0.889	0.889
3	1.000	0.941	0.888	0.842	0.842	0.842	0.842	0.842	0.842	0.842	0.842	0.842	0.842	0.842	0.842	0.842
4	1.000	0.941	0.888	0.841	0.799	0.799	0.799	0.799	0.799	0.799	0.799	0.799	0.799	0.799	0.799	0.799
5	1.000	0.941	0.888	0.841	0.799	0.761	0.761	0.761	0.761	0.761	0.761	0.761	0.761	0.761	0.761	0.761
6	1.000	0.941	0.888	0.841	0.799	0.760	0.726	0.726	0.726	0.726	0.726	0.726	0.726	0.726	0.726	0.726
7	1.000	0.941	0.888	0.841	0.799	0.760	0.725	0.694	0.694	0.694	0.694	0.694	0.694	0.694	0.694	0.694
8	1.000	0.941	0.888	0.841	0.799	0.760	0.725	0.693	0.664	0.664	0.664	0.664	0.664	0.664	0.664	0.664
9	1.000	0.941	0.888	0.841	0.799	0.760	0.725	0.693	0.664	0.637	0.637	0.637	0.637	0.637	0.637	0.637
10	**1.000**	**0.941**	**0.888**	**0.841**	**0.799**	**0.760**	**0.725**	**0.693**	**0.664**	**0.637**	**0.612**	**0.612**	**0.612**	**0.612**	**0.612**	**0.612**
11	1.000	0.941	0.888	0.841	0.799	0.760	0.725	0.693	0.664	0.637	0.612	0.589	0.589	0.589	0.589	0.589
12	1.000	0.941	0.888	0.841	0.799	0.760	0.725	0.693	0.664	0.637	0.612	0.589	0.568	0.568	0.568	0.568
13	1.000	0.941	0.888	0.841	0.799	0.760	0.725	0.693	0.664	0.637	0.612	0.589	0.567	0.548	0.548	0.548
14	1.000	0.941	0.888	0.841	0.799	0.760	0.725	0.693	0.664	0.637	0.612	0.589	0.567	0.547	0.529	0.529
15	1.000	0.941	0.888	0.841	0.799	0.760	0.725	0.693	0.664	0.637	0.612	0.589	0.567	0.547	0.529	0.512
16	1.000	0.941	0.888	0.841	0.799	0.760	0.725	0.693	0.664	0.637	0.612	0.589	0.567	0.547	0.529	0.511
17	1.000	0.941	0.888	0.841	0.799	0.760	0.725	0.693	0.664	0.637	0.612	0.589	0.567	0.547	0.529	0.511
18	1.000	0.941	0.888	0.841	0.799	0.760	0.725	0.693	0.664	0.637	0.612	0.589	0.567	0.547	0.529	0.511
19	1.000	0.941	0.888	0.841	0.799	0.760	0.725	0.693	0.664	0.637	0.612	0.589	0.567	0.547	0.529	0.511
20	**1.000**	**0.941**	**0.888**	**0.841**	**0.799**	**0.760**	**0.725**	**0.693**	**0.664**	**0.637**	**0.612**	**0.589**	**0.567**	**0.547**	**0.529**	**0.511**
21	1.000	0.941	0.888	0.841	0.799	0.760	0.725	0.693	0.664	0.637	0.612	0.589	0.567	0.547	0.529	0.511
22	1.000	0.941	0.888	0.841	0.799	0.760	0.725	0.693	0.664	0.637	0.612	0.589	0.567	0.547	0.529	0.511
23	1.000	0.941	0.888	0.841	0.799	0.760	0.725	0.693	0.664	0.637	0.612	0.589	0.567	0.547	0.529	0.511
24	1.000	0.941	0.888	0.841	0.799	0.760	0.725	0.693	0.664	0.637	0.612	0.589	0.567	0.547	0.529	0.511
25	1.000	0.941	0.888	0.841	0.799	0.760	0.725	0.693	0.664	0.637	0.612	0.589	0.567	0.547	0.529	0.511
26	1.000	0.941	0.888	0.841	0.799	0.760	0.725	0.693	0.664	0.637	0.612	0.589	0.567	0.547	0.529	0.511
27	1.000	0.941	0.888	0.841	0.799	0.760	0.725	0.693	0.664	0.637	0.612	0.589	0.567	0.547	0.529	0.511
28	1.000	0.941	0.888	0.841	0.799	0.760	0.725	0.693	0.664	0.637	0.612	0.589	0.567	0.547	0.529	0.511
29	1.000	0.941	0.888	0.841	0.799	0.760	0.725	0.693	0.664	0.637	0.612	0.589	0.567	0.547	0.529	0.511
30	**1.000**	**0.941**	**0.888**	**0.841**	**0.799**	**0.760**	**0.725**	**0.693**	**0.664**	**0.637**	**0.612**	**0.589**	**0.567**	**0.547**	**0.529**	**0.511**
Exact	**1.000**	**0.938**	**0.882**	**0.833**	**0.789**	**0.750**	**0.714**	**0.682**	**0.652**	**0.625**	**0.600**	**0.577**	**0.556**	**0.536**	**0.517**	**0.500**



Problem 5.106

[Difficulty: 3]

***5.106** Use *Excel* to generate the solutions of Eq. 5.31 for $m = -1$, with $u(0) = 3$, using 4 and 16 points over the interval from $x = 0$ to $x = 3$, with sufficient iterations, and compare to the exact solution

$$u_{\text{exact}} = \sqrt{9 - 2x}$$

To do so, follow the steps described in "Dealing with Non-linearity" section.

$$\Delta u_i = u_i - u_{g_i}$$

$$\frac{1}{u_i} = \frac{1}{u_{g_i} + \Delta u_i} \approx \frac{1}{u_{g_i}}\left(1 - \frac{\Delta u_i}{u_{g_i}}\right)$$

$$\frac{u_i - u_{i-1}}{\Delta x} + \frac{1}{u_i} = 0$$

$$\frac{u_i - u_{i-1}}{\Delta x} + \frac{1}{u_{g_i}}\left(1 - \frac{u_i - u_{g_i}}{u_{g_i}}\right) = 0$$

$$\frac{u_i - u_{i-1}}{\Delta x} + \frac{1}{u_{g_i}}\left(2 - \frac{u_i}{u_{g_i}}\right) = 0$$

$$u_i\left(1 - \frac{\Delta x}{u_{g_i}^2}\right) = u_{i-1} - \frac{2\Delta x}{u_{g_i}}$$

$$u_i = \frac{u_{i-1} - \dfrac{2\Delta x}{u_{g_i}}}{1 - \dfrac{\Delta x}{u_{g_i}^2}}$$

$\Delta x =$ 1.500

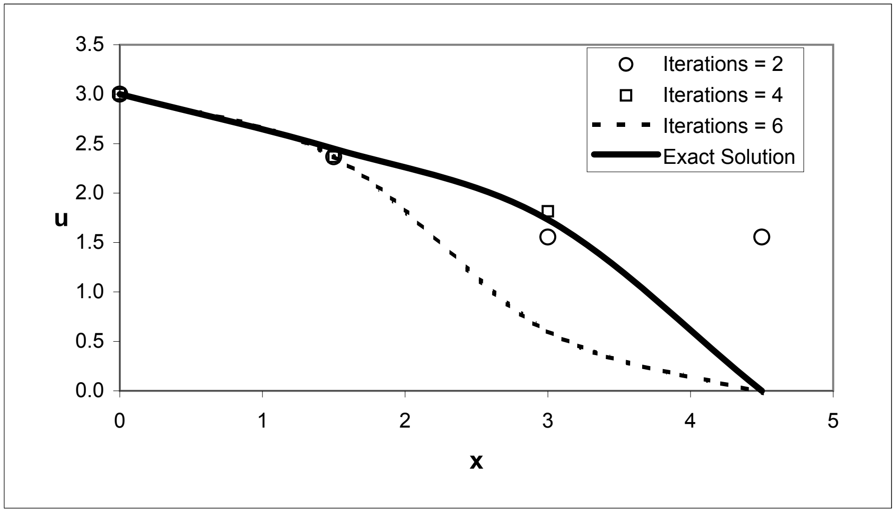
	x			
Iteration	0.000	1.500	3.000	4.500
0	3.000	3.000	3.000	3.000
1	3.000	2.400	2.400	2.400
2	3.000	2.366	1.555	1.555
3	3.000	2.366	1.151	-0.986
4	3.000	2.366	1.816	-7.737
5	3.000	2.366	1.310	2.260
6	3.000	2.366	0.601	-0.025
Exact	**3.000**	**2.449**	**1.732**	**0.000**

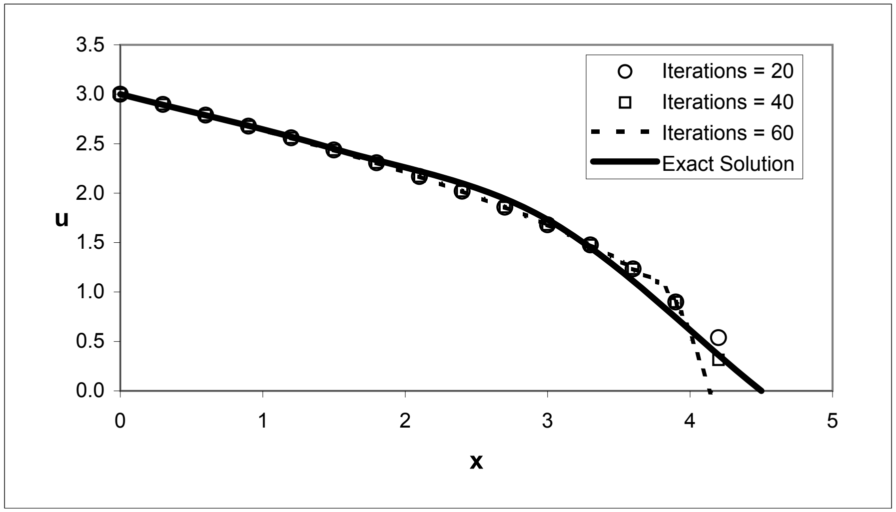
$\Delta x =$ 0.300

	x															
Iteration	0.000	0.300	0.600	0.900	1.200	1.500	1.800	2.100	2.400	2.700	3.000	3.300	3.600	3.900	4.200	4.500
0	3.000	3.000	3.000	3.000	3.000	3.000	3.000	3.000	3.000	3.000	3.000	3.000	3.000	3.000	3.000	3.000
1	3.000	2.897	2.897	2.897	2.897	2.897	2.897	2.897	2.897	2.897	2.897	2.897	2.897	2.897	2.897	2.897
2	3.000	2.896	2.789	2.789	2.789	2.789	2.789	2.789	2.789	2.789	2.789	2.789	2.789	2.789	2.789	2.789
3	3.000	2.896	2.789	2.677	2.677	2.677	2.677	2.677	2.677	2.677	2.677	2.677	2.677	2.677	2.677	2.677
4	3.000	2.896	2.789	2.677	2.560	2.560	2.560	2.560	2.560	2.560	2.560	2.560	2.560	2.560	2.560	2.560
5	3.000	2.896	2.789	2.677	2.560	2.438	2.438	2.438	2.438	2.438	2.438	2.438	2.438	2.438	2.438	2.438
6	3.000	2.896	2.789	2.677	2.560	2.436	2.308	2.308	2.308	2.308	2.308	2.308	2.308	2.308	2.308	2.308
7	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.170	2.170	2.170	2.170	2.170	2.170	2.170	2.170	2.170
8	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.023	2.023	2.023	2.023	2.023	2.023	2.023	2.023
9	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.862	1.862	1.862	1.862	1.862	1.862	1.862
10	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.686	1.686	1.686	1.686	1.686	1.686
11	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.487	1.487	1.487	1.487	1.487
12	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.254	1.254	1.254	1.254
13	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.958	0.958	0.958
14	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.901	0.493	0.493
15	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	1.349	3.091
16	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	0.544	1.192
17	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	14.403	0.051
18	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	0.859	-0.024
19	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	0.338	-0.051

20	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	0.538	-0.105
21	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	5.953	-0.239
22	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	0.805	-1.998
23	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	0.286	1.195
24	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	0.450	-0.273
25	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	0.900	-0.876
26	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	0.369	2.601
27	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	0.605	0.145
28	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	-0.517	0.266
29	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	-17.059	0.858
30	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	0.935	-29.971
31	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	0.392	0.955
32	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	0.663	-0.352
33	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	-0.020	-1.662
34	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	-0.041	0.383
35	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	-0.088	1.534
36	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	-0.204	-0.549
37	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	-0.621	198.629
38	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	8.435	-0.624
39	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	0.831	41.087
40	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	0.313	0.817
41	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	0.494	-0.765
42	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	1.379	2.623
43	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	0.551	1.203
44	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	-16.722	0.066
45	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	0.936	0.377
46	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	0.392	0.591
47	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	0.664	-4.391
48	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	-0.014	0.813
49	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	-0.029	-1.376
50	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	-0.061	0.483
51	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	-0.135	4.578
52	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	-0.347	-0.270
53	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	-1.765	-0.603
54	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	1.371	-4.389
55	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	0.549	1.532
56	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	-40.363	0.180
57	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	0.914	5.316
58	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	0.379	0.810
59	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	0.627	-0.668
60	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	-0.243	4.652
Exact	**3.000**	**2.898**	**2.793**	**2.683**	**2.569**	**2.449**	**2.324**	**2.191**	**2.049**	**1.897**	**1.732**	**1.549**	**1.342**	**1.095**	**0.775**	**0.000**

Here are graphs comparing the numerical and exact solutions.





Problem 5.107

[Difficulty: 3]

***5.107** An environmental engineer drops a pollution measuring probe with a mass of 0.3 slugs into a fast moving river (the speed of the water is $U = 25$ ft/s). The equation of motion for your speed u is

$$M\frac{du}{dt} = k(U-u)^2$$

where $k = 0.02$ lbf·s²/ft² is a constant indicating the drag of the water. Use *Excel* to generate and plot the probe speed versus time (for the first 10 s) using the same approach as the solutions of Eq. 5.31 for $m = 2$, as shown in Fig 5.21, except use 16 points and as many iterations as necessary to obtain reasonable convergence. Compare your results to the exact solution

$$u_{\text{exact}} = \frac{kU^2t}{M + kUt}$$

Hint: Use a substitution for $(U - u)$ so that the equation of motion looks similar to Eq. 5.31.

$$M\frac{du}{dt} = k(U-u)^2$$

$$v = U - u$$

$$dv = -du$$

$$-M\frac{dv}{dt} = kv^2$$

$$\frac{dv}{dt} + \frac{k}{M}v^2 = 0$$

$$v_i^2 \approx 2v_{g_i}v_i - v_{g_i}^2$$

$$\frac{v_i - v_{i-1}}{\Delta t} + \frac{k}{M}\left(2v_{g_i}v_i - v_{g_i}^2\right) = 0$$

$$v_i = \frac{v_{g_{i-1}} + \frac{k}{M}\Delta t\, v_{g_i}^2}{1 + 2\frac{k}{M}\Delta t\, v_{g_i}}$$

$\Delta t =$ 1.000

$k =$ 0.02 lbf.s²/ft²

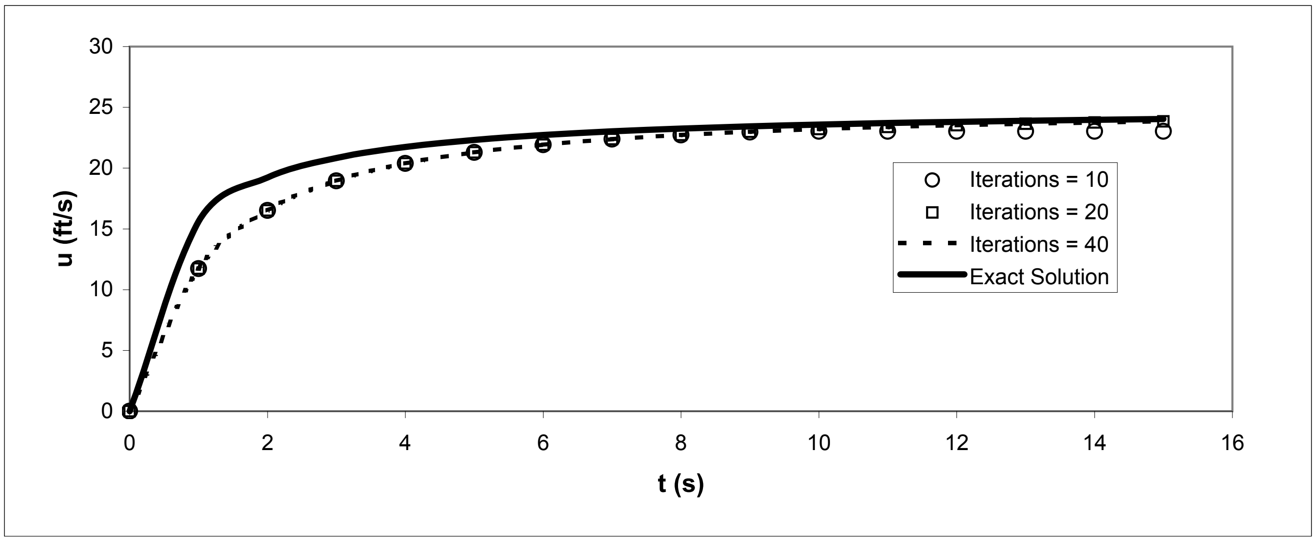
$M =$ 0.3 slug

	t															
Iteration	0.000	1.000	2.000	3.000	4.000	5.000	6.000	7.000	8.000	9.000	10.000	11.000	12.000	13.000	14.000	15.000
0	25.000	25.000	25.000	25.000	25.000	25.000	25.000	25.000	25.000	25.000	25.000	25.000	25.000	25.000	25.000	25.000
1	25.000	15.385	15.385	15.385	15.385	15.385	15.385	15.385	15.385	15.385	15.385	15.385	15.385	15.385	15.385	15.385
2	25.000	13.365	10.213	10.213	10.213	10.213	10.213	10.213	10.213	10.213	10.213	10.213	10.213	10.213	10.213	10.213
3	25.000	13.267	8.603	7.269	7.269	7.269	7.269	7.269	7.269	7.269	7.269	7.269	7.269	7.269	7.269	7.269
4	25.000	13.267	8.477	6.158	5.480	5.480	5.480	5.480	5.480	5.480	5.480	5.480	5.480	5.480	5.480	5.480
5	25.000	13.267	8.476	6.043	4.715	4.323	4.323	4.323	4.323	4.323	4.323	4.323	4.323	4.323	4.323	4.323
6	25.000	13.267	8.476	6.042	4.621	3.781	3.533	3.533	3.533	3.533	3.533	3.533	3.533	3.533	3.533	3.533
7	25.000	13.267	8.476	6.042	4.620	3.706	3.136	2.967	2.967	2.967	2.967	2.967	2.967	2.967	2.967	2.967
8	25.000	13.267	8.476	6.042	4.620	3.705	3.075	2.668	2.547	2.547	2.547	2.547	2.547	2.547	2.547	2.547
9	25.000	13.267	8.476	6.042	4.620	3.705	3.075	2.618	2.314	2.224	2.224	2.224	2.224	2.224	2.224	2.224
10	25.000	13.267	8.476	6.042	4.620	3.705	3.075	2.618	2.274	2.039	1.970	1.970	1.970	1.970	1.970	1.970
11	25.000	13.267	8.476	6.042	4.620	3.705	3.075	2.618	2.273	2.006	1.820	1.765	1.765	1.765	1.765	1.765
12	25.000	13.267	8.476	6.042	4.620	3.705	3.075	2.618	2.273	2.005	1.792	1.641	1.597	1.597	1.597	1.597
13	25.000	13.267	8.476	6.042	4.620	3.705	3.075	2.618	2.273	2.005	1.791	1.617	1.493	1.457	1.457	1.457
14	25.000	13.267	8.476	6.042	4.620	3.705	3.075	2.618	2.273	2.005	1.791	1.617	1.473	1.369	1.338	1.338
15	25.000	13.267	8.476	6.042	4.620	3.705	3.075	2.618	2.273	2.005	1.791	1.617	1.472	1.351	1.263	1.237
16	25.000	13.267	8.476	6.042	4.620	3.705	3.075	2.618	2.273	2.005	1.791	1.617	1.472	1.351	1.247	1.172
17	25.000	13.267	8.476	6.042	4.620	3.705	3.075	2.618	2.273	2.005	1.791	1.617	1.472	1.351	1.247	1.158
18	25.000	13.267	8.476	6.042	4.620	3.705	3.075	2.618	2.273	2.005	1.791	1.617	1.472	1.351	1.247	1.158
19	25.000	13.267	8.476	6.042	4.620	3.705	3.075	2.618	2.273	2.005	1.791	1.617	1.472	1.351	1.247	1.158
20	25.000	13.267	8.476	6.042	4.620	3.705	3.075	2.618	2.273	2.005	1.791	1.617	1.472	1.351	1.247	1.158
21	25.000	13.267	8.476	6.042	4.620	3.705	3.075	2.618	2.273	2.005	1.791	1.617	1.472	1.351	1.247	1.158
22	25.000	13.267	8.476	6.042	4.620	3.705	3.075	2.618	2.273	2.005	1.791	1.617	1.472	1.351	1.247	1.158
23	25.000	13.267	8.476	6.042	4.620	3.705	3.075	2.618	2.273	2.005	1.791	1.617	1.472	1.351	1.247	1.158
24	25.000	13.267	8.476	6.042	4.620	3.705	3.075	2.618	2.273	2.005	1.791	1.617	1.472	1.351	1.247	1.158
25	25.000	13.267	8.476	6.042	4.620	3.705	3.075	2.618	2.273	2.005	1.791	1.617	1.472	1.351	1.247	1.158
26	25.000	13.267	8.476	6.042	4.620	3.705	3.075	2.618	2.273	2.005	1.791	1.617	1.472	1.351	1.247	1.158
27	25.000	13.267	8.476	6.042	4.620	3.705	3.075	2.618	2.273	2.005	1.791	1.617	1.472	1.351	1.247	1.158
28	25.000	13.267	8.476	6.042	4.620	3.705	3.075	2.618	2.273	2.005	1.791	1.617	1.472	1.351	1.247	1.158

29	25.000	13.267	8.476	6.042	4.620	3.705	3.075	2.618	2.273	2.005	1.791	1.617	1.472	1.351	1.247	1.158
30	25.000	13.267	8.476	6.042	4.620	3.705	3.075	2.618	2.273	2.005	1.791	1.617	1.472	1.351	1.247	1.158
31	25.000	13.267	8.476	6.042	4.620	3.705	3.075	2.618	2.273	2.005	1.791	1.617	1.472	1.351	1.247	1.158
32	25.000	13.267	8.476	6.042	4.620	3.705	3.075	2.618	2.273	2.005	1.791	1.617	1.472	1.351	1.247	1.158
33	25.000	13.267	8.476	6.042	4.620	3.705	3.075	2.618	2.273	2.005	1.791	1.617	1.472	1.351	1.247	1.158
34	25.000	13.267	8.476	6.042	4.620	3.705	3.075	2.618	2.273	2.005	1.791	1.617	1.472	1.351	1.247	1.158
35	25.000	13.267	8.476	6.042	4.620	3.705	3.075	2.618	2.273	2.005	1.791	1.617	1.472	1.351	1.247	1.158
36	25.000	13.267	8.476	6.042	4.620	3.705	3.075	2.618	2.273	2.005	1.791	1.617	1.472	1.351	1.247	1.158
37	25.000	13.267	8.476	6.042	4.620	3.705	3.075	2.618	2.273	2.005	1.791	1.617	1.472	1.351	1.247	1.158
38	25.000	13.267	8.476	6.042	4.620	3.705	3.075	2.618	2.273	2.005	1.791	1.617	1.472	1.351	1.247	1.158
39	25.000	13.267	8.476	6.042	4.620	3.705	3.075	2.618	2.273	2.005	1.791	1.617	1.472	1.351	1.247	1.158
40	25.000	13.267	8.476	6.042	4.620	3.705	3.075	2.618	2.273	2.005	1.791	1.617	1.472	1.351	1.247	1.158

Above values are for v! To get u we compute u = U - v

Iteration																
10	0.000	11.733	16.524	18.958	20.380	21.295	21.925	22.382	22.726	22.961	23.030	23.030	23.030	23.030	23.030	23.030
20	0.000	11.733	16.524	18.958	20.380	21.295	21.925	22.382	22.727	22.995	23.209	23.383	23.528	23.649	23.753	23.842
40	0.000	11.733	16.524	18.958	20.380	21.295	21.925	22.382	22.727	22.995	23.209	23.383	23.528	23.649	23.753	23.842
Exact	**0.000**	**15.625**	**19.231**	**20.833**	**21.739**	**22.321**	**22.727**	**23.026**	**23.256**	**23.438**	**23.585**	**23.707**	**23.810**	**23.897**	**23.973**	**24.038**



Problem 6.1

[Difficulty: 2]

6.1 Consider the flow field with velocity given by $\vec{V} = [A(y^2 - x^2) - Bx]\hat{i} + [2Axy + By]\hat{j}$; $A = 1\ \text{ft}^{-1}\cdot\text{s}^{-1}$, $B = 1\ \text{ft}^{-1}\cdot\text{s}^{-1}$; the coordinates are measured in feet. The density is 2 slug/ft^3, and gravity acts in the negative y direction. Calculate the acceleration of a fluid particle and the pressure gradient at point $(x, y) = (1, 1)$.

Given: Velocity field

Find: Acceleration of particle and pressure gradient at (1,1)

Solution:

NOTE: Units of B are s^{-1} not $ft^{-1}s^{-1}$

Basic equations

$$\vec{a}_p = \underset{\substack{\text{total}\\\text{acceleration}\\\text{of a particle}}}{\frac{D\vec{V}}{Dt}} = \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\substack{\text{convective}\\\text{acceleration}}} + \underset{\substack{\text{local}\\\text{acceleration}}}{\frac{\partial \vec{V}}{\partial t}} \qquad \rho\frac{D\vec{V}}{Dt} = \rho\vec{g} - \nabla p$$

For this flow

$$u(x,y) = A\cdot\left(y^2 - x^2\right) - B\cdot x \qquad v(x,y) = 2\cdot A\cdot x\cdot y + B\cdot y$$

$$a_x = u\cdot\frac{\partial}{\partial x}u + v\cdot\frac{\partial}{\partial y}u = \left[A\cdot\left(y^2 - x^2\right) - B\cdot x\right]\cdot\frac{\partial}{\partial x}\left[A\cdot\left(y^2 - x^2\right) - B\cdot x\right] + (2\cdot A\cdot x\cdot y + B\cdot y)\cdot\frac{\partial}{\partial y}\left[A\cdot\left(y^2 - x^2\right) - B\cdot x\right]$$

$$a_x = (B + 2\cdot A\cdot x)\cdot\left(A\cdot x^2 + B\cdot x + A\cdot y^2\right)$$

$$a_y = u\cdot\frac{\partial}{\partial x}v + v\cdot\frac{\partial}{\partial y}v = \left[A\cdot\left(y^2 - x^2\right) - B\cdot x\right]\cdot\frac{\partial}{\partial x}(2\cdot A\cdot x\cdot y + B\cdot y) + (2\cdot A\cdot x\cdot y + B\cdot y)\cdot\frac{\partial}{\partial y}(2\cdot A\cdot x\cdot y + B\cdot y)$$

$$a_y = (B + 2\cdot A\cdot x)\cdot(B\cdot y + 2\cdot A\cdot x\cdot y) - 2\cdot A\cdot y\cdot\left[B\cdot x + A\cdot\left(x^2 - y^2\right)\right]$$

Hence at (1,1)

$$a_x = (1 + 2\cdot 1\cdot 1)\cdot\frac{1}{s} \times \left(1\cdot 1^2 + 1\cdot 1 + 1\cdot 1^2\right)\cdot\frac{ft}{s} \qquad a_x = 9\cdot\frac{ft}{s^2}$$

$$a_y = (1 + 2\cdot 1\cdot 1)\cdot\frac{1}{s} \times (1\cdot 1 + 2\cdot 1\cdot 1\cdot 1)\cdot\frac{ft}{s} - 2\cdot 1\cdot 1\cdot\frac{1}{s} \times \left[1\cdot 1 + 1\cdot\left(1^2 - 1^2\right)\right]\cdot\frac{ft}{s} \qquad a_y = 7\cdot\frac{ft}{s^2}$$

$$a = \sqrt{a_x^2 + a_y^2} \qquad \theta = \operatorname{atan}\left(\frac{a_y}{a_x}\right) \qquad a = 11.4\cdot\frac{ft}{s^2} \qquad \theta = 37.9\cdot\deg$$

For the pressure gradient

$$\frac{\partial}{\partial x}p = \rho\cdot g_x - \rho\cdot a_x = -2\cdot\frac{slug}{ft^3} \times 9\cdot\frac{ft}{s^2} \times \frac{lbf\cdot s^2}{slug\cdot ft} \qquad \frac{\partial}{\partial x}p = -18\cdot\frac{\frac{lbf}{ft^2}}{ft} = -0.125\cdot\frac{psi}{ft}$$

$$\frac{\partial}{\partial y}p = \rho\cdot g_y - \rho\cdot a_y = 2\cdot\frac{slug}{ft^3} \times (-32.2 - 7)\cdot\frac{ft}{s^2} \times \frac{lbf\cdot s^2}{slug\cdot ft} \qquad \frac{\partial}{\partial y}p = -78.4\cdot\frac{\frac{lbf}{ft^2}}{ft} = -0.544\cdot\frac{psi}{ft}$$

Problem 6.2

[Difficulty: 2]

6.2 An incompressible frictionless flow field is given by $\vec{V} = (Ax + By)\hat{i} + (Bx - Ay)\hat{j}$, where $A = 2\,s^{-1}$ and $B = 2\,s^{-1}$, and the coordinates are measured in meters. Find the magnitude and direction of the acceleration of a fluid particle at point $(x, y) = (2, 2)$. Find the pressure gradient at the same point, if $\vec{g} = -g\hat{j}$ and the fluid is water.

Given: Velocity field

Find: Acceleration of particle and pressure gradient at (2,2)

Solution:

Basic equations

$$\vec{a}_p = \underset{\substack{\text{total}\\\text{acceleration}\\\text{of a particle}}}{\frac{D\vec{V}}{Dt}} = \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\substack{\text{convective}\\\text{acceleration}}} + \underset{\substack{\text{local}\\\text{acceleration}}}{\frac{\partial \vec{V}}{\partial t}} \qquad \rho\frac{D\vec{V}}{Dt} = \rho\vec{g} - \nabla p$$

Given data $\quad A = 1\cdot\frac{1}{s} \quad B = 3\cdot\frac{1}{s} \quad x = 2\cdot m \quad y = 2\cdot m \quad \rho = 999\cdot\frac{kg}{m^3}$

For this flow $\quad u(x,y) = A\cdot x + B\cdot y \qquad v(x,y) = B\cdot x - A\cdot y$

$$a_x = u\cdot\frac{\partial}{\partial x}u + v\cdot\frac{\partial}{\partial y}u = (A\cdot x + B\cdot y)\cdot\frac{\partial}{\partial x}(A\cdot x + B\cdot y) + (B\cdot x - A\cdot y)\cdot\frac{\partial}{\partial y}(A\cdot x + B\cdot y) \qquad a_x = \left(A^2 + B^2\right)\cdot x$$

$$a_y = u\cdot\frac{\partial}{\partial x}v + v\cdot\frac{\partial}{\partial y}v = (A\cdot x + B\cdot y)\cdot\frac{\partial}{\partial x}(B\cdot x - A\cdot y) + (B\cdot x - A\cdot y)\cdot\frac{\partial}{\partial y}(B\cdot x - A\cdot y) \qquad a_y = \left(A^2 + B^2\right)\cdot y$$

Hence at (2,2) $\quad a_x = (1 + 9)\frac{1}{s} \times 2\cdot m \quad a_x = 20\frac{m}{s} \quad a_y = (1 + 9)\frac{1}{s} \times 2\cdot m \quad a_y = 20\frac{m}{s}$

$$a = \sqrt{a_x^{\,2} + a_y^{\,2}} \qquad \theta = \operatorname{atan}\left(\frac{a_y}{a_x}\right) \qquad a = 28.28\frac{m}{s} \qquad \theta = 45\cdot deg$$

For the pressure gradient

$$\frac{\partial}{\partial x}p = \rho\cdot g_x - \rho\cdot a_x = -999\cdot\frac{kg}{m^3} \times 20\cdot\frac{m}{s^2} \times \frac{N\cdot s^2}{kg\cdot m} \qquad \frac{\partial}{\partial x}p = -20000\frac{Pa}{m} = -20.0\cdot\frac{kPa}{m}$$

$$\frac{\partial}{\partial y}p = -\rho\cdot g_y - \rho\cdot a_y = 999\cdot\frac{kg}{m^3} \times (-9.81 - 20)\cdot\frac{m}{s^2} \times \frac{N\cdot s^2}{kg\cdot m} \qquad \frac{\partial}{\partial y}p = -29800\frac{Pa}{m} = -29.8\cdot\frac{kPa}{m}$$

Problem 6.3

[Difficulty: 2]

6.3 A horizontal flow of water is described by the velocity field $\vec{V} = (-Ax + Bt)\hat{i} + (Ay + Bt)\hat{j}$, where $A = 1\ \mathrm{s}^{-1}$ and $B = 2\ \mathrm{m/s^2}$, x and y are in meters, and t is in seconds. Find expressions for the local acceleration, the convective acceleration, and the total acceleration. Evaluate these at point (1, 2) at $t = 5$ seconds. Evaluate ∇p at the same point and time.

Given: Velocity field

Find: Acceleration of particle and pressure gradient at (1,2)

Solution:

Basic equations

$$\underbrace{\vec{a}_p = \frac{D\vec{V}}{Dt}}_{\text{total acceleration of a particle}} = \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\text{convective acceleration}} + \underbrace{\frac{\partial \vec{V}}{\partial t}}_{\text{local acceleration}} \qquad \rho\frac{D\vec{V}}{Dt} = \rho\vec{g} - \nabla p$$

Given data

$$A = 1\cdot\frac{1}{\mathrm{s}} \qquad B = 2\cdot\frac{\mathrm{m}}{\mathrm{s}^2} \qquad x = 1\cdot\mathrm{m} \qquad y = 2\cdot\mathrm{m} \qquad t = 5\cdot\mathrm{s} \qquad \rho = 999\cdot\frac{\mathrm{kg}}{\mathrm{m}^3}$$

$$u(x,y,t) = -A\cdot x + B\cdot t \qquad v(x,y,t) = A\cdot y + B\cdot t$$

The acceleration components and values are

$$a_{xt}(x,y,t) = \frac{\partial}{\partial t}u(x,y,t) = B \qquad a_{xt}(x,y,t) = B \qquad a_{xt}(x,y,t) = 2\,\frac{\mathrm{m}}{\mathrm{s}^2}$$

$$a_{xc}(x,y,t) = u(x,y,t)\cdot\frac{\partial}{\partial x}u(x,y,t) + v(x,y,t)\cdot\frac{\partial}{\partial y}u(x,y,t) \qquad a_{xc}(x,y,t) = A^2\cdot x - A\cdot B\cdot t \qquad a_{xc}(x,y,t) = -9\,\frac{\mathrm{m}}{\mathrm{s}^2}$$

$$a_{yt}(x,y,t) = \frac{\partial}{\partial t}v(x,y,t) \qquad a_{yt}(x,y,t) = B \qquad a_{yt}(x,y,t) = 2\,\frac{\mathrm{m}}{\mathrm{s}^2}$$

$$a_{yc}(x,y,t) = u(x,y,t)\cdot\frac{\partial}{\partial x}v(x,y,t) + v(x,y,t)\cdot\frac{\partial}{\partial y}v(x,y,t) \qquad a_{yc}(x,y,t) = y\cdot A^2 + B\cdot t\cdot A \qquad a_{yc}(x,y,t) = 12\,\frac{\mathrm{m}}{\mathrm{s}^2}$$

$$a_x(x,y,t) = a_{xt}(x,y,t) + a_{xc}(x,y,t) \qquad a_x(x,y,t) = x\cdot A^2 - B\cdot t\cdot A + B \qquad a_x(x,y,t) = -7\,\frac{\mathrm{m}}{\mathrm{s}^2}$$

$$a_y(x,y,t) = a_{yt}(x,y,t) + a_{yc}(x,y,t) \qquad a_y(x,y,t) = y\cdot A^2 + B\cdot t\cdot A + B \qquad a_y(x,y,t) = 14\,\frac{\mathrm{m}}{\mathrm{s}^2}$$

For overall acceleration

$$a(x,y,t) = \sqrt{a_x(x,y,t)^2 + a_y(x,y,t)^2} \qquad a(x,y,t) = \sqrt{\left(x\cdot A^2 - B\cdot t\cdot A + B\right)^2 + \left(y\cdot A^2 + B\cdot t\cdot A + B\right)^2} \qquad a(x,y,t) = 15.7\,\frac{m}{s^2}$$

$$\theta = \operatorname{atan}\left(\frac{a_y(x,y,t)}{a_x(x,y,t)}\right) \qquad \theta = -63.4\cdot deg$$

For the pressure gradient we need $\quad -\rho\cdot a_x(x,y,t) = 6.99\cdot\frac{kPa}{m} \quad -\rho\cdot a_y(x,y,t) = -13.99\cdot\frac{kPa}{m} \quad -\rho\cdot g = -9.80\cdot\frac{kPa}{m}$

Hence for the pressure gradient

$$\frac{\partial}{\partial x}p = \rho\cdot g_x - \rho\cdot a_x = 999\cdot\frac{kg}{m^3}\times 7\cdot\frac{m}{s^2}\times\frac{N\cdot s^2}{kg\cdot m} \qquad \frac{\partial}{\partial x}p = 6990\cdot\frac{Pa}{m} = 6.99\cdot\frac{kPa}{m}$$

$$\frac{\partial}{\partial y}p = -\rho\cdot g_y - \rho\cdot a_y = 999\cdot\frac{kg}{m^3}\times(-9.81 - 14)\cdot\frac{m}{s^2}\times\frac{N\cdot s^2}{kg\cdot m} \qquad \frac{\partial}{\partial y}p = -23800\cdot\frac{Pa}{m} = -23.8\cdot\frac{kPa}{m}$$

Problem 6.4 [Difficulty: 2]

6.4 A velocity field in a fluid with density of 1000 kg/m³ is given by $\vec{V} = (-Ax + By)t\hat{i} + (Ay + Bx)t\hat{j}$, where $A = 2\ \mathrm{s}^{-2}$ and $B = 1\,\mathrm{s}^{-2}$, x and y are in meters, and t is in seconds. Body forces are negligible. Evaluate ∇p at point $(x, y) = (1, 1)$ at $t = 1$ s.

Given: Velocity field

Find: Pressure gradient at (1,1) at 1 s

Solution:

Basic equations

$$\vec{a}_p = \underset{\substack{\text{total}\\\text{acceleration}\\\text{of a particle}}}{\frac{D\vec{V}}{Dt}} = \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\substack{\text{convective}\\\text{acceleration}}} + \underset{\substack{\text{local}\\\text{acceleration}}}{\frac{\partial \vec{V}}{\partial t}} \qquad \rho\frac{D\vec{V}}{Dt} = \rho\vec{g} - \nabla p$$

Given data $A = 2\cdot\frac{1}{s^2}$ $\quad B = 1\cdot\frac{1}{s^2}$ $\quad x = 1\cdot m$ $\quad y = 1\cdot m$ $\quad t = 1\cdot s$ $\quad \rho = 1000\cdot\frac{kg}{m^3}$

$u(x,y,t) = (-A\cdot x + B\cdot y)\cdot t \qquad v(x,y,t) = (A\cdot y + B\cdot x)\cdot t$

The acceleration components and values are

$$a_{xt}(x,y,t) = \frac{\partial}{\partial t}u(x,y,t) = B\cdot y - A\cdot x \qquad a_{xt}(x,y,t) = B\cdot y - A\cdot x \qquad a_{xt}(x,y,t) = -1\,\frac{m}{s^2}$$

$$a_{xc}(x,y,t) = u(x,y,t)\cdot\frac{\partial}{\partial x}u(x,y,t) + v(x,y,t)\cdot\frac{\partial}{\partial y}u(x,y,t) \qquad a_{xc}(x,y,t) = t^2\cdot x\cdot\left(A^2 + B^2\right) \qquad a_{xc}(x,y,t) = 5\,\frac{m}{s^2}$$

$$a_{yt}(x,y,t) = \frac{\partial}{\partial t}v(x,y,t) \qquad a_{yt}(x,y,t) = A\cdot y + B\cdot x \qquad a_{yt}(x,y,t) = 3\,\frac{m}{s^2}$$

$$a_{yc}(x,y,t) = u(x,y,t)\cdot\frac{\partial}{\partial x}v(x,y,t) + v(x,y,t)\cdot\frac{\partial}{\partial y}v(x,y,t) \qquad a_{yc}(x,y,t) = t^2\cdot y\cdot\left(A^2 + B^2\right) \qquad a_{yc}(x,y,t) = 5\,\frac{m}{s^2}$$

$$a_x(x,y,t) = a_{xt}(x,y,t) + a_{xc}(x,y,t) \qquad a_x(x,y,t) = x\cdot A^2\cdot t^2 - x\cdot A + x\cdot B^2\cdot t^2 + y\cdot B \qquad a_x(x,y,t) = 4\,\frac{m}{s^2}$$

$$a_y(x,y,t) = a_{yt}(x,y,t) + a_{yc}(x,y,t) \qquad a_y(x,y,t) = y\cdot A^2\cdot t^2 + y\cdot A + y\cdot B^2\cdot t^2 + x\cdot B \qquad a_y(x,y,t) = 8\,\frac{m}{s^2}$$

Hence for the pressure gradient

$$\frac{\partial}{\partial x}p = -\rho\cdot a_x = -1000\cdot\frac{kg}{m^3}\times 4\cdot\frac{m}{s^2}\times\frac{N\cdot s^2}{kg\cdot m} \qquad \frac{\partial}{\partial x}p = -4000\cdot\frac{Pa}{m} = -4\cdot\frac{kPa}{m}$$

$$\frac{\partial}{\partial y}p = -\rho\cdot a_y = -1000\cdot\frac{kg}{m^3}\times 8\cdot\frac{m}{s^2}\times\frac{N\cdot s^2}{kg\cdot m} \qquad \frac{\partial}{\partial y}p = -8000\cdot\frac{Pa}{m} = -8\cdot\frac{kPa}{m}$$

Problem 6.5

[Difficulty: 2]

6.5 Consider the flow field with velocity given by $\vec{V} = [A(x^2 - y^2) - 3Bx]\hat{i} - [2Axy - 3By]\hat{j}$, where $A = 1\ \text{ft}^{-1}\cdot\text{s}^{-1}$, $B = 1\ \text{s}^{-1}$, and the coordinates are measured in feet. The density is 2 slug/ft^3 and gravity acts in the negative y direction. Determine the acceleration of a fluid particle and the pressure gradient at point $(x, y) = (1, 1)$.

Given: Velocity field

Find: Acceleration of particle and pressure gradient at (1,1)

Solution:

Basic equations

$$\underset{\substack{\text{total}\\\text{acceleration}\\\text{of a particle}}}{\vec{a}_p = \frac{D\vec{V}}{Dt}} = \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\substack{\text{convective}\\\text{acceleration}}} + \underset{\substack{\text{local}\\\text{acceleration}}}{\frac{\partial \vec{V}}{\partial t}} \qquad \rho\frac{D\vec{V}}{Dt} = \rho\vec{g} - \nabla p$$

For this flow $\quad u(x,y) = A\cdot\left(x^2 - y^2\right) - 3\cdot B\cdot x \qquad v(x,y) = -2\cdot A\cdot x\cdot y + 3\cdot B\cdot y$

$$a_x = u\cdot\frac{\partial}{\partial x}u + v\cdot\frac{\partial}{\partial y}u = \left[A\cdot\left(x^2 - y^2\right) - 3\cdot B\cdot x\right]\cdot\frac{\partial}{\partial x}\left[A\cdot\left(x^2 - y^2\right) - 3\cdot B\cdot x\right] + (-2\cdot A\cdot x\cdot y + 3\cdot B\cdot y)\cdot\frac{\partial}{\partial y}\left[A\cdot\left(x^2 - y^2\right) - 3\cdot B\cdot x\right]$$

$$a_x = (2\cdot A\cdot x - 3\cdot B)\cdot\left(A\cdot x^2 - 3\cdot B\cdot x + A\cdot y^2\right)$$

$$a_y = u\cdot\frac{\partial}{\partial x}v + v\cdot\frac{\partial}{\partial y}v = \left[A\cdot\left(x^2 - y^2\right) - 3\cdot B\cdot x\right]\cdot\frac{\partial}{\partial x}(-2\cdot A\cdot x\cdot y + 3\cdot B\cdot y) + (-2\cdot A\cdot x\cdot y + 3\cdot B\cdot y)\cdot\frac{\partial}{\partial y}(-2\cdot A\cdot x\cdot y + 3\cdot B\cdot y)$$

$$a_y = (3\cdot B\cdot y - 2\cdot A\cdot x\cdot y)\cdot(3\cdot B - 2\cdot A\cdot x) - 2\cdot A\cdot y\cdot\left[A\cdot\left(x^2 - y^2\right) - 3\cdot B\cdot x\right]$$

Hence at (1,1)

$$a_x = (2\cdot 1\cdot 1 - 3\cdot 1)\cdot\frac{1}{s} \times \left(1\cdot 1^2 - 3\cdot 1\cdot 1 + 1\cdot 1^2\right)\cdot\frac{ft}{s} \qquad a_x = 1\cdot\frac{ft}{s^2}$$

$$a_y = (3\cdot 1\cdot 1 - 2\cdot 1\cdot 1\cdot 1)\cdot\frac{1}{s} \times (3\cdot 1 - 2\cdot 1\cdot 1)\cdot\frac{ft}{s} - 2\cdot 1\cdot 1\cdot\frac{1}{s} \times \left[1\cdot\left(1^2 - 1^2\right) - 3\cdot 1\cdot 1\right]\cdot\frac{ft}{s} \qquad a_y = 7\cdot\frac{ft}{s^2}$$

$$a = \sqrt{{a_x}^2 + {a_y}^2} \qquad \theta = \operatorname{atan}\left(\frac{a_y}{a_x}\right) \qquad a = 7.1\cdot\frac{ft}{s^2} \qquad \theta = 81.9\cdot\text{deg}$$

For the pressure gradient

$$\frac{\partial}{\partial x}p = \rho\cdot g_x - \rho\cdot a_x = -2\cdot\frac{slug}{ft^3} \times 1\cdot\frac{ft}{s^2} \times \frac{lbf\cdot s^2}{slug\cdot ft} \qquad \frac{\partial}{\partial x}p = -2\cdot\frac{\frac{lbf}{ft^2}}{ft} = -0.0139\cdot\frac{psi}{ft}$$

$$\frac{\partial}{\partial y}p = \rho\cdot g_y - \rho\cdot a_y = 2\cdot\frac{slug}{ft^3} \times (-32.2 - 7)\cdot\frac{ft}{s^2} \times \frac{lbf\cdot s^2}{slug\cdot ft} \qquad \frac{\partial}{\partial y}p = -78.4\cdot\frac{\frac{lbf}{ft^2}}{ft} = -0.544\cdot\frac{psi}{ft}$$

Problem 6.6

[Difficulty: 2]

6.6 The x component of velocity in an incompressible flow field is given by $u = Ax$, where $A = 2\ \mathrm{s}^{-1}$ and the coordinates are measured in meters. The pressure at point $(x, y) = (0, 0)$ is $p_0 = 190$ kPa (gage). The density is $\rho = 1.50\ \mathrm{kg/m^3}$ and the z axis is vertical. Evaluate the simplest possible y component of velocity. Calculate the fluid acceleration and determine the pressure gradient at point $(x, y) = (2, 1)$. Find the pressure distribution along the positive x axis.

Given: Velocity field

Find: Simplest y component of velocity; Acceleration of particle and pressure gradient at (2,1); pressure on x axis

Solution:

Basic equations
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \underbrace{\vec{a}_p = \frac{D\vec{V}}{Dt}}_{\text{total acceleration of a particle}} = \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\text{convective acceleration}} + \underbrace{\frac{\partial \vec{V}}{\partial t}}_{\text{local acceleration}} \qquad \rho\frac{D\vec{V}}{Dt} = \rho\vec{g} - \nabla p$$

For this flow $\quad u(x,y) = A\cdot x \qquad \frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0 \quad$ so $\quad v(x,y) = -\int \frac{\partial}{\partial x}u\,dy = -\int A\,dy = -A\cdot y + c$

Hence $\quad v(x,y) = -A\cdot y \quad$ is the simplest y component of velocity

For acceleration
$$a_x = u\cdot\frac{\partial}{\partial x}u + v\cdot\frac{\partial}{\partial y}u = A\cdot x\cdot\frac{\partial}{\partial x}(A\cdot x) + (-A\cdot y)\cdot\frac{\partial}{\partial y}(A\cdot x) = A^2\cdot x \qquad a_x = A^2\cdot x$$

$$a_y = u\cdot\frac{\partial}{\partial x}v + v\cdot\frac{\partial}{\partial y}v = A\cdot x\cdot\frac{\partial}{\partial x}(-A\cdot y) + (-A\cdot y)\cdot\frac{\partial}{\partial y}(-A\cdot y) \qquad a_y = A^2\cdot y$$

Hence at (2,1) $\quad a_x = \left(\frac{2}{s}\right)^2 \times 2\cdot m \qquad a_y = \left(\frac{2}{s}\right)^2 \times 1\cdot m \qquad a_x = 8\frac{m}{s^2} \qquad a_y = 4\frac{m}{s^2}$

$$a = \sqrt{a_x^2 + a_y^2} \qquad \theta = \mathrm{atan}\left(\frac{a_y}{a_x}\right) \qquad a = 8.94\frac{m}{s^2} \qquad \theta = 26.6\cdot deg$$

For the pressure gradient

$$\frac{\partial}{\partial x}p = \rho\cdot g_x - \rho\cdot a_x = -1.50\cdot\frac{kg}{m^3} \times 8\cdot\frac{m}{s^2} \times \frac{N\cdot s^2}{kg\cdot m} \qquad \frac{\partial}{\partial x}p = -12\cdot\frac{Pa}{m}$$

$$\frac{\partial}{\partial y}p = \rho\cdot g_y - \rho\cdot a_y = -1.50\cdot\frac{kg}{m^3} \times 4\cdot\frac{m}{s^2} \times \frac{N\cdot s^2}{kg\cdot m} \qquad \frac{\partial}{\partial y}p = -6\cdot\frac{Pa}{m}$$

$$\frac{\partial}{\partial z}p = \rho\cdot g_z - \rho\cdot a_z = 1.50 \times \frac{kg}{m^3} \times (-9.81)\cdot\frac{m}{s^2} \times \frac{N\cdot s^2}{kg\cdot m} \qquad \frac{\partial}{\partial y}p = -14.7\cdot\frac{Pa}{m}$$

For the pressure on the x axis $\quad dp = \frac{\partial}{\partial x}p \qquad p - p_0 = \int_0^x \left(\rho\cdot g_x - \rho\cdot a_x\right)dx = \int_0^x \left(-\rho\cdot A^2\cdot x\right)dx = -\frac{1}{2}\cdot\rho\cdot A^2\cdot x^2$

$$p(x) = p_0 - \frac{1}{2}\cdot\rho\cdot A^2\cdot x^2 \qquad p(x) = 190\cdot kPa - \frac{1}{2}\cdot 1.5\cdot\frac{kg}{m^3} \times \left(\frac{2}{s}\right)^2 \times \frac{N\cdot s^2}{kg\cdot m} \times x^2 \qquad p(x) = 190 - \frac{3}{1000}\cdot x^2 \quad \text{(p in kPa, x in m)}$$

Problem 6.7

[Difficulty: 3]

6.7 Consider the flow field with velocity given by $\vec{V} = Ax\sin(2\pi\omega t)\hat{i} - Ay\sin(2\pi\omega t)\hat{j}$, where $A = 2\text{ s}^{-1}$ and $\omega = 1\text{ s}^{-1}$. The fluid density is 2 kg/m³. Find expressions for the local acceleration, the convective acceleration, and the total acceleration. Evaluate these at point (1, 1) at $t = 0$, 0.5, and 1 seconds. Evaluate ∇p at the same point and times.

Given: Velocity field

Find: Expressions for local, convective and total acceleration; evaluate at several points; evaluate pressure gradient

Solution:

The given data is $A = 2\cdot\frac{1}{s}$ $\quad \omega = 1\cdot\frac{1}{s}$ $\quad \rho = 2\cdot\frac{kg}{m^3}$ $\quad u = A\cdot x\cdot\sin(2\cdot\pi\cdot\omega\cdot t)$ $\quad v = -A\cdot y\cdot\sin(2\cdot\pi\cdot\omega\cdot t)$

Check for incompressible flow $\quad \frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$

Hence $\quad \frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = A\cdot\sin(2\cdot\pi\cdot\omega\cdot t) - A\cdot\sin(2\cdot\pi\cdot\omega\cdot t) = 0$ $\quad$ Incompressible flow

The governing equation for acceleration is

$$\vec{a}_p = \underbrace{\frac{D\vec{V}}{Dt}}_{\substack{\text{total}\\ \text{acceleration}\\ \text{of a particle}}} = \underbrace{u\frac{\partial\vec{V}}{\partial x} + v\frac{\partial\vec{V}}{\partial y} + w\frac{\partial\vec{V}}{\partial z}}_{\substack{\text{convective}\\ \text{acceleration}}} + \underbrace{\frac{\partial\vec{V}}{\partial t}}_{\substack{\text{local}\\ \text{acceleration}}}$$

The local acceleration is then

x - component $\quad \frac{\partial}{\partial t}u = 2\cdot\pi\cdot A\cdot\omega\cdot x\cdot\cos(2\cdot\pi\cdot\omega\cdot t)$

y - component $\quad \frac{\partial}{\partial t}v = -2\cdot\pi\cdot A\cdot\omega\cdot y\cdot\cos(2\cdot\pi\cdot\omega\cdot t)$

For the present steady, 2D flow, the convective acceleration is

x - component $\quad u\cdot\frac{\partial}{\partial x}u + v\cdot\frac{\partial}{\partial y}u = A\cdot x\cdot\sin(2\cdot\pi\cdot\omega\cdot t)\cdot(A\cdot\sin(2\cdot\pi\cdot\omega\cdot t)) + (-A\cdot y\cdot\sin(2\cdot\pi\cdot\omega\cdot t))\cdot 0 = A^2\cdot x\cdot\sin(2\cdot\pi\cdot\omega\cdot t)^2$

y - component $\quad u\cdot\frac{\partial}{\partial x}v + v\cdot\frac{\partial}{\partial y}v = A\cdot x\cdot\sin(2\cdot\pi\cdot\omega\cdot t)\cdot 0 + (-A\cdot y\cdot\sin(2\cdot\pi\cdot\omega\cdot t))\cdot(-A\cdot\sin(2\cdot\pi\cdot\omega\cdot t)) = A^2\cdot y\cdot\sin(2\cdot\pi\cdot\omega\cdot t)^2$

The total acceleration is then

x - component $\quad \frac{\partial}{\partial t}u + u\cdot\frac{\partial}{\partial x}u + v\cdot\frac{\partial}{\partial y}u = 2\cdot\pi\cdot A\cdot\omega\cdot x\cdot\cos(2\cdot\pi\cdot\omega\cdot t) + A^2\cdot x\cdot\sin(2\cdot\pi\cdot\omega\cdot t)^2$

y - component $\quad \frac{\partial}{\partial t}v + u\cdot\frac{\partial}{\partial x}v + v\cdot\frac{\partial}{\partial y}v = -2\cdot\pi\cdot A\cdot\omega\cdot y\cdot\cos(2\cdot\pi\cdot\omega\cdot t) + A^2\cdot y\cdot\sin(2\cdot\pi\cdot\omega\cdot t)^2$

Evaluating at point (1,1) at

$t = 0\cdot s$	Local	$12.6\cdot\frac{m}{s^2}$	and	$-12.6\cdot\frac{m}{s^2}$	Convective	$0\cdot\frac{m}{s^2}$	and	$0\cdot\frac{m}{s^2}$
	Total	$12.6\cdot\frac{m}{s^2}$	and	$-12.6\cdot\frac{m}{s^2}$				
$t = 0.5\cdot s$	Local	$-12.6\cdot\frac{m}{s^2}$	and	$12.6\cdot\frac{m}{s^2}$	Convective	$0\cdot\frac{m}{s^2}$	and	$0\cdot\frac{m}{s^2}$
	Total	$-12.6\cdot\frac{m}{s^2}$	and	$12.6\cdot\frac{m}{s^2}$				
$t = 1\cdot s$	Local	$12.6\cdot\frac{m}{s^2}$	and	$-12.6\cdot\frac{m}{s^2}$	Convective	$0\cdot\frac{m}{s^2}$	and	$0\cdot\frac{m}{s^2}$
	Total	$12.6\cdot\frac{m}{s^2}$	and	$-12.6\cdot\frac{m}{s^2}$				

The governing equation (assuming inviscid flow) for computing the pressure gradient is $\rho\frac{D\vec{V}}{Dt} = \rho\vec{g} - \nabla p$

Hence, the components of pressure gradient (neglecting gravity) are

$$\frac{\partial}{\partial x}p = -\rho\cdot\frac{Du}{Dt} \qquad \frac{\partial}{\partial x}p = -\rho\cdot\left(2\cdot\pi\cdot A\cdot\omega\cdot x\cdot\cos(2\cdot\pi\cdot\omega\cdot t) + A^2\cdot x\cdot\sin(2\cdot\pi\cdot\omega\cdot t)^2\right)$$

$$\frac{\partial}{\partial y}p = -\rho\cdot\frac{Dv}{Dt} \qquad \frac{\partial}{\partial x}p = -\rho\cdot\left(-2\cdot\pi\cdot A\cdot\omega\cdot y\cdot\cos(2\cdot\pi\cdot\omega\cdot t) + A^2\cdot y\cdot\sin(2\cdot\pi\cdot\omega\cdot t)^2\right)$$

Evaluated at (1,1) and time	$t = 0\cdot s$	*x* comp.	$-25.1\cdot\frac{Pa}{m}$	*y* comp.	$25.1\cdot\frac{Pa}{m}$
	$t = 0.5\cdot s$	*x* comp.	$25.1\cdot\frac{Pa}{m}$	*y* comp.	$-25.1\cdot\frac{Pa}{m}$
	$t = 1\cdot s$	*x* comp.	$-25.1\cdot\frac{Pa}{m}$	*y* comp.	$25.1\cdot\frac{Pa}{m}$

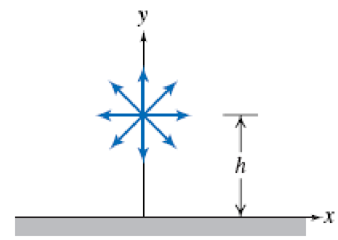
Problem 6.8

[Difficulty: 3]

6.8 The velocity field for a plane source located distance $h = 1$ m above an infinite wall aligned along the x axis is given by

$$\vec{V} = \frac{q}{2\pi[x^2 + (y-h)^2]}[x\hat{i} + (y-h)\hat{j}] + \frac{q}{2\pi[x^2 + (y+h)^2]}[x\hat{i} + (y+h)\hat{j}]$$

where $q = 2$ m^3/s/m. The fluid density is 1000 kg/m^3 and body forces are negligible. Derive expressions for the velocity and acceleration of a fluid particle that moves along the wall, and plot from $x = 0$ to $x = +10h$. Verify that the velocity and acceleration normal to the wall are zero. Plot the pressure gradient $\partial p/\partial x$ along the wall. Is the pressure gradient along the wall adverse (does it oppose fluid motion) or not?



Given: Velocity field

Find: Expressions for velocity and acceleration along wall; plot; verify vertical components are zero; plot pressure gradient

Solution:

The given data is $\quad q = 2\cdot\frac{\frac{m^3}{s}}{m} \qquad h = 1\cdot m \qquad \rho = 1000\cdot\frac{kg}{m^3}$

$$u = \frac{q\cdot x}{2\cdot\pi\left[x^2 + (y-h)^2\right]} + \frac{q\cdot x}{2\cdot\pi\left[x^2 + (y+h)^2\right]} \qquad v = \frac{q\cdot(y-h)}{2\cdot\pi\left[x^2 + (y-h)^2\right]} + \frac{q\cdot(y+h)}{2\cdot\pi\left[x^2 + (y+h)^2\right]}$$

The governing equation for acceleration is

$$\vec{a}_p = \underset{\substack{\text{total}\\\text{acceleration}\\\text{of a particle}}}{\frac{D\vec{V}}{Dt}} = \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\substack{\text{convective}\\\text{acceleration}}} + \underset{\substack{\text{local}\\\text{acceleration}}}{\frac{\partial \vec{V}}{\partial t}}$$

For steady, 2D flow this reduces to (after considerable math!)

x - component
$$a_x = u\cdot\frac{\partial}{\partial x}u + v\cdot\frac{\partial}{\partial y}u = -\frac{q^2\cdot x\cdot\left[\left(x^2+y^2\right)^2 - h^2\cdot\left(h^2 - 4\cdot y^2\right)\right]}{\left[x^2 + (y+h)^2\right]^2\cdot\left[x^2 + (y-h)^2\right]^2\cdot\pi^2}$$

y - component
$$a_y = u\cdot\frac{\partial}{\partial x}v + v\cdot\frac{\partial}{\partial y}v = -\frac{q^2\cdot y\cdot\left[\left(x^2+y^2\right)^2 - h^2\cdot\left(h^2 + 4\cdot x^2\right)\right]}{\pi^2\cdot\left[x^2 + (y+h)^2\right]^2\cdot\left[x^2 + (y-h)^2\right]^2}$$

For motion along the wall $\qquad y = 0\cdot m$

$$u = \frac{q\cdot x}{\pi\cdot\left(x^2 + h^2\right)} \qquad v = 0 \quad \text{(No normal velocity)} \qquad a_x = -\frac{q^2\cdot x\cdot\left(x^2 - h^2\right)}{\pi^2\cdot\left(x^2 + h^2\right)^3} \qquad a_y = 0 \quad \text{(No normal acceleration)}$$

The governing equation (assuming inviscid flow) for computing the pressure gradient is

$$\rho \frac{D\vec{V}}{Dt} = \rho \vec{g} - \nabla p$$

Hence, the component of pressure gradient (neglecting gravity) along the wall is

$$\frac{\partial}{\partial x} p = -\rho \cdot \frac{Du}{Dt} \qquad \frac{\partial}{\partial x} p = \frac{\rho \cdot q^2 \cdot x \cdot \left(x^2 - h^2\right)}{\pi^2 \cdot \left(x^2 + h^2\right)^3}$$

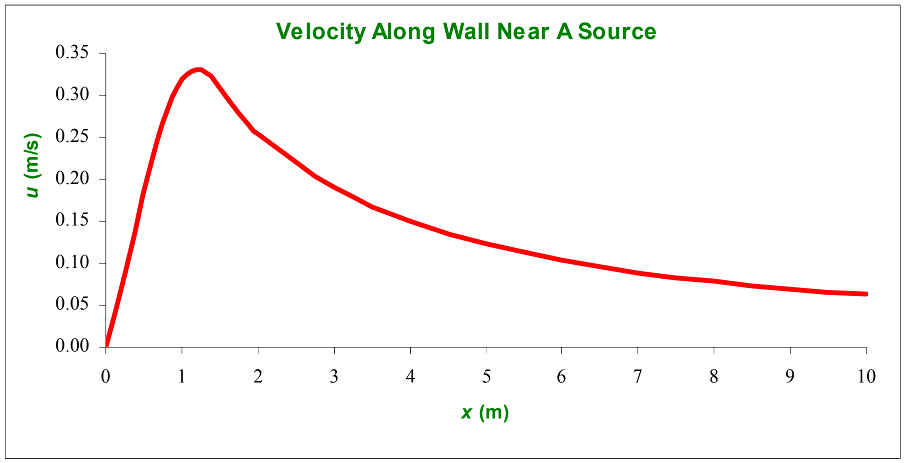
The plots of velocity, acceleration, and pressure gradient are shown below, done in *Excel*. From the plots it is clear that the fluid experiences an adverse pressure gradient from the origin to $x = 1$ m, then a negative one promoting fluid acceleration. If flow separates, it will likely be in the region $x = 0$ to $x = h$.

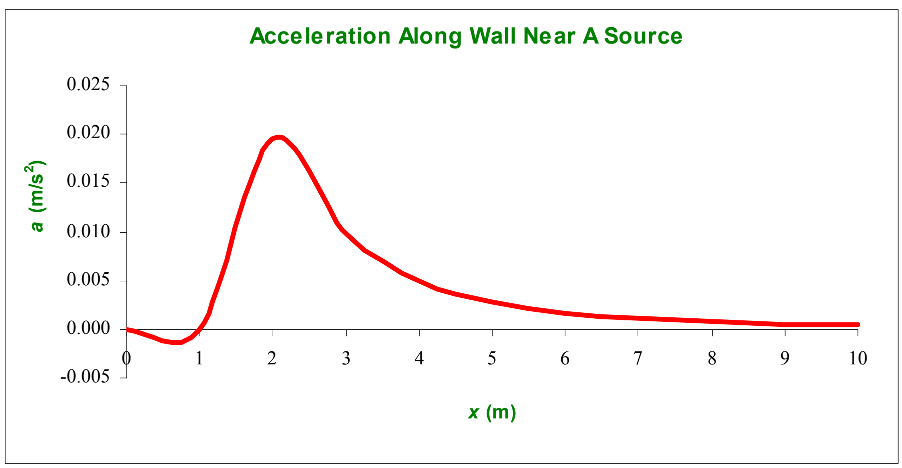
$q =$ 2 $m^3/s/m$

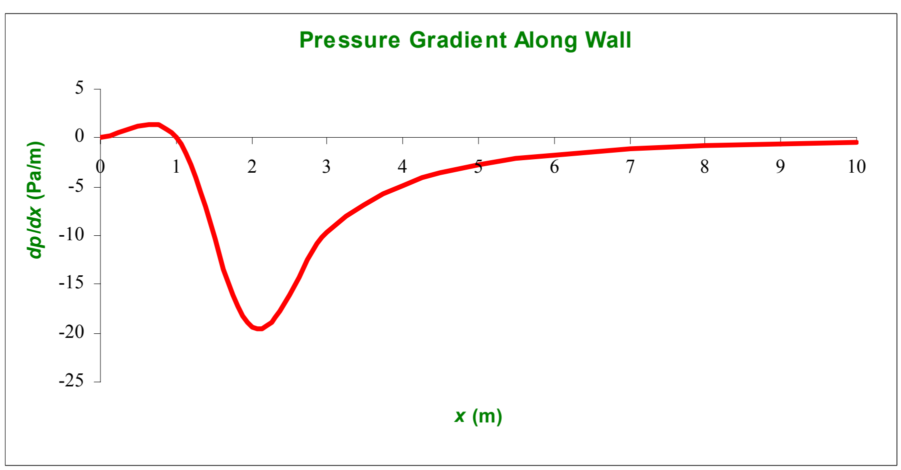
$h =$ 1 m

$\angle =$ 1000 kg/m^3

x (m)	u (m/s)	a (m/s^2)	dp/dx (Pa/m)
0.0	0.00	0.00000	0.00
1.0	0.32	0.00000	0.00
2.0	0.25	0.01945	-19.45
3.0	0.19	0.00973	-9.73
4.0	0.15	0.00495	-4.95
5.0	0.12	0.00277	-2.77
6.0	0.10	0.00168	-1.68
7.0	0.09	0.00109	-1.09
8.0	0.08	0.00074	-0.74
9.0	0.07	0.00053	-0.53
10.0	0.06	0.00039	-0.39







Problem 6.9

[Difficulty: 2]

6.9 The velocity distribution in a two-dimensional steady flow field in the xy plane is $\vec{V} = (Ax - B)\hat{i} + (C - Ay)\hat{j}$, where $A = 2\ \text{s}^{-1}$, $B = 5\ \text{m}\cdot\text{s}^{-1}$, and $C = 3\ \text{m}\cdot\text{s}^{-1}$; the coordinates are measured in meters, and the body force distribution is $\vec{g} = -g\hat{k}$. Does the velocity field represent the flow of an incompressible fluid? Find the stagnation point of the flow field. Obtain an expression for the pressure gradient in the flow field. Evaluate the difference in pressure between point $(x, y) = (1, 3)$ and the origin, if the density is 1.2 kg/m³.

Solution:

(a) Apply the continuity equation, $\frac{\partial \rho}{\partial t} + \nabla \cdot \rho\vec{V} = 0$, for the given conditions. If ρ constant, then

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 = \frac{\partial}{\partial x}(2x-5) + \frac{\partial}{\partial y}(3-2y) = 2-2 = 0 \checkmark$$

∴ velocity field represents an incompressible flow

(b) At the stagnation point, $\vec{V} = 0$. For $\vec{V} = 0$, then

$$u = 2x - 5 = 0 \quad \text{and} \quad v = (3-2y) = 0$$

Thus stagnation point is at $(x,y) = \left(\frac{5}{2}, \frac{3}{2}\right)$

(c) Euler's equation, $\rho\vec{g} - \nabla p = \rho\frac{D\vec{V}}{Dt}$, can be used to obtain an expression for the pressure gradient

$$\nabla p = \rho\vec{g} - \rho\frac{D\vec{V}}{Dt} = \rho\vec{g} - \rho\left[\underbrace{\cancel{\frac{\partial \vec{V}}{\partial t}}}_{\text{steady}} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + \underbrace{\cancel{w\frac{\partial \vec{V}}{\partial z}}}_{w=0}\right].$$

$$\nabla p = \rho\left[\vec{g} - u\frac{\partial \vec{V}}{\partial x} - v\frac{\partial \vec{V}}{\partial y}\right] = \rho\left[-g\hat{k} - (2x-5)\,2\hat{i} - (3-2y)(-2\hat{j})\right]$$

$$\nabla p = -\rho\left[(4x-10)\hat{i} + (4y-6)\hat{j} + g\hat{k}\right]$$

∇p

(d) Since $p = p(x,y,z)$ we can write

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz = -\rho(4x-10)dx - \rho(4y-6)dy - \rho g\,dz$$

We can integrate to obtain Δp between any two points in the field if, and only if, the integral of the right hand side is independent of the path of integration. This is true for the present case.

$$\therefore p_{1,3} - p_{0,0} = -\rho\left\{\int_0^1 (4x-10)dx + \int_0^3 (4y-6)dy\right\} = -\rho\left\{\left[2x^2 - 10x\right]_0^1 + \left[2y^2 - 6y\right]_0^3\right\}$$

$$= -\rho\{-8 - 0\} = 8\rho$$

$$p_{1,3} - p_{0,0} = 8\ \frac{\text{m}^2}{\text{s}^2} \cdot 1.2\frac{\text{kg}}{\text{m}^3} \cdot \frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}} = 9.6\ \text{N/m}^2$$

Δp

Problem 6.10

[Difficulty: 2]

6.10 In a two-dimensional frictionless, incompressible ($\rho = 1500\ \text{kg/m}^3$) flow, the velocity field in meters per second is given by $\vec{V} = (Ax + By)\hat{i} + (Bx - Ay)\hat{j}$; the coordinates are measured in meters, and $A = 4\ \text{s}^{-1}$ and $B = 2\ \text{s}^{-1}$. The pressure is $p_0 = 200$ kPa at point $(x, y) = (0, 0)$. Obtain an expression for the pressure field, $p(x, y)$ in terms of p_0, A, and B, and evaluate at point $(x, y) = (2, 2)$.

Given: Velocity field

Find: Expression for pressure field; evaluate at (2,2)

Solution:

Basic equations

$$\vec{a}_p = \underset{\substack{\text{total}\\\text{acceleration}\\\text{of a particle}}}{\frac{D\vec{V}}{Dt}} = \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\substack{\text{convective}\\\text{acceleration}}} + \underset{\substack{\text{local}\\\text{acceleration}}}{\frac{\partial \vec{V}}{\partial t}} \qquad \rho\frac{D\vec{V}}{Dt} = \rho\vec{g} - \nabla p$$

Given data $\quad A = 4\cdot\frac{1}{s} \quad B = 2\cdot\frac{1}{s} \quad x = 2\cdot m \quad y = 2\cdot m \quad \rho = 1500\cdot\frac{kg}{m^3} \quad p_0 = 200\cdot kPa$

For this flow $\quad u(x,y) = A\cdot x + B\cdot y \qquad v(x,y) = B\cdot x - A\cdot y$

Note that $\quad \frac{\partial}{\partial x}u(x,y) + \frac{\partial}{\partial y}v(x,y) = 0 \qquad \frac{\partial}{\partial x}v(x,y) - \frac{\partial}{\partial y}u(x,y) = 0$

Then $\quad a_x(x,y) = u(x,y)\cdot\frac{\partial}{\partial x}u(x,y) + v(x,y)\cdot\frac{\partial}{\partial y}u(x,y) \qquad a_x(x,y) = x\cdot\left(A^2 + B^2\right) \qquad a_x(x,y) = 40\,\frac{m}{s^2}$

$$a_y(x,y) = u(x,y)\cdot\frac{\partial}{\partial x}v(x,y) + v(x,y)\cdot\frac{\partial}{\partial y}v(x,y) \qquad a_y(x,y) = y\cdot\left(A^2 + B^2\right) \qquad a_y(x,y) = 40\,\frac{m}{s^2}$$

The momentum equation becomes $\quad \frac{\partial}{\partial x}p = -\rho\cdot a_x \quad \frac{\partial}{\partial y}p = -\rho\cdot a_y \quad$ and $\quad p = dx\cdot\frac{\partial}{\partial x}p + dy\cdot\frac{\partial}{\partial y}p$

Integrating

$$p(x,y) = p_0 - \rho\cdot\int_0^x a_x(x,y)\,dx - \rho\cdot\int_0^y a_y(x,y)\,dy$$

$$p(x,y) = p_0 - \frac{\rho\cdot\left(A^2 + B^2\right)\cdot y^2}{2} - \frac{\rho\cdot\left(A^2 + B^2\right)\cdot x^2}{2}$$

$$p(x,y) = 80\cdot kPa$$

Problem 6.11

[Difficulty: 2]

6.11 An incompressible liquid with a density of 1250 kg/m^3 and negligible viscosity flows steadily through a horizontal pipe of constant diameter. In a porous section of length L = 5 m, liquid is removed at a constant rate per unit length so that the uniform axial velocity in the pipe is $u(x) = U(1 - x/L)$, where U = 15 m/s. Develop expressions for and plot the pressure gradient along the centerline. Evaluate the outlet pressure if the pressure at the inlet to the porous section is 100 kPa (gage).

Given: Velocity field

Find: Expression for pressure gradient; plot; evaluate pressure at outlet

Solution:

Basic equations

$$\vec{a}_p = \underbrace{\frac{D\vec{V}}{Dt}}_{\substack{\text{total}\\\text{acceleration}\\\text{of a particle}}} = \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\substack{\text{convective}\\\text{acceleration}}} + \underbrace{\frac{\partial \vec{V}}{\partial t}}_{\substack{\text{local}\\\text{acceleration}}} \qquad \rho\frac{D\vec{V}}{Dt} = \rho\vec{g} - \nabla p$$

Given data $\quad U = 15\cdot\frac{m}{s} \quad L = 5\cdot m \quad p_{in} = 100\cdot kPa \quad \rho = 1250\cdot\frac{kg}{m^3}$

Here $\quad u(x) = U\cdot\left(1 - \frac{x}{L}\right) \quad u(0) = 15\,\frac{m}{s} \quad u(L) = 0\,\frac{m}{s}$

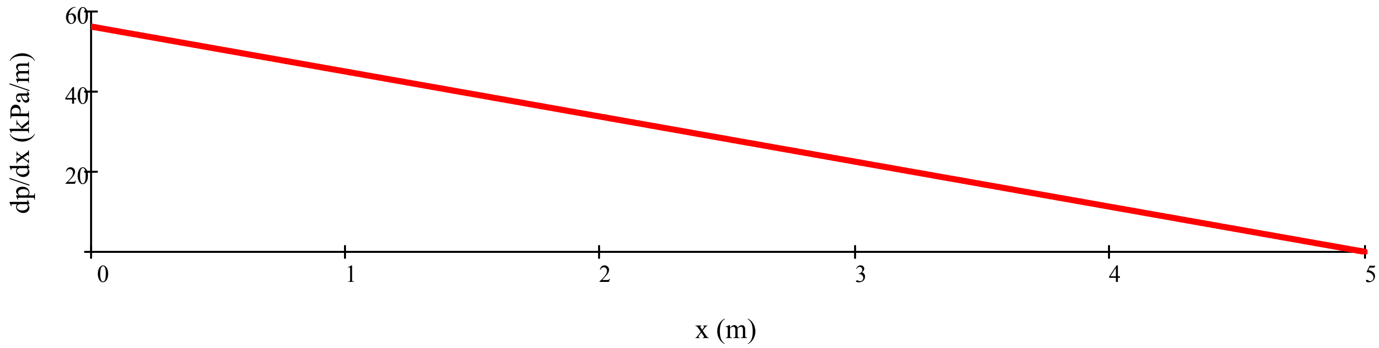
The x momentum becomes $\quad \rho\cdot u\cdot\frac{d}{dx}u = \rho\cdot a_a = -\frac{d}{dx}p$

Hence $\quad a_x(x) = u(x)\cdot\frac{\partial}{\partial x}u(x) \quad a_x(x) = \frac{U^2\cdot\left(\frac{x}{L} - 1\right)}{L}$

The pressure gradient is then $\quad \frac{dp}{dx} = -\rho\cdot\frac{U^2}{L}\cdot\left(\frac{x}{L} - 1\right)$

Integrating momentum $\quad p(x) = p_{in} - \rho\cdot\int_0^x a_x(x)\,dx \quad p(x) = p_{in} - \frac{U^2\cdot\rho\cdot x\cdot(x - 2\cdot L)}{2\cdot L^2}$

Hence $\quad p(L) = \frac{\rho\cdot U^2}{2} + p_{in} \quad p(L) = 241\cdot kPa$



Problem 6.12

[Difficulty: 2]

6.12 An incompressible liquid with a density of 900 kg/m^3 and negligible viscosity flows steadily through a horizontal pipe of constant diameter. In a porous section of length $L = 2$ m, liquid is removed at a variable rate along the length so that the uniform axial velocity in the pipe is $u(x) = Ue^{-x/L}$, where $U = 20$ m/s. Develop expressions for and plot the acceleration of a fluid particle along the centerline of the porous section and the pressure gradient along the centerline. Evaluate the outlet pressure if the pressure at the inlet to the porous section is 50 kPa (gage).

Given: Velocity field

Find: Expression for acceleration and pressure gradient; plot; evaluate pressure at outlet

Solution:

Basic equations

$$\vec{a}_p = \underset{\substack{\text{total}\\\text{acceleration}\\\text{of a particle}}}{\frac{D\vec{V}}{Dt}} = \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\substack{\text{convective}\\\text{acceleration}}} + \underset{\substack{\text{local}\\\text{acceleration}}}{\frac{\partial \vec{V}}{\partial t}} \qquad \rho\frac{D\vec{V}}{Dt} = \rho\vec{g} - \nabla p$$

Given data $\quad U = 20\cdot\frac{m}{s} \quad L = 2\cdot m \quad p_{in} = 50\cdot kPa \quad \rho = 900\cdot\frac{kg}{m^3}$

Here $\quad u(x) = U\cdot e^{-\frac{x}{L}} \quad u(0) = 20\frac{m}{s} \quad u(L) = 7.36\frac{m}{s}$

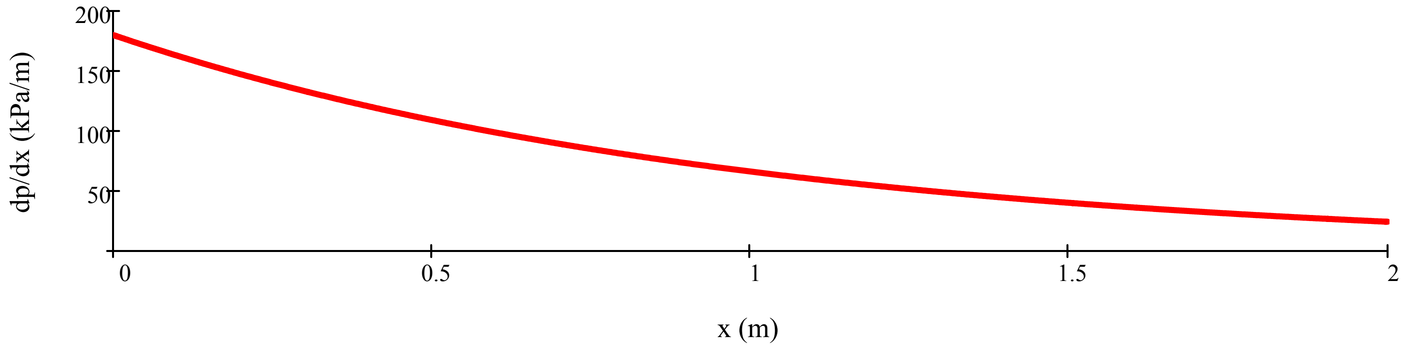
The x component of acceleration is then $\quad a_x(x) = u(x)\cdot\frac{\partial}{\partial x}u(x) \qquad a_x(x) = -\frac{U^2\cdot e^{-\frac{2\cdot x}{L}}}{L}$

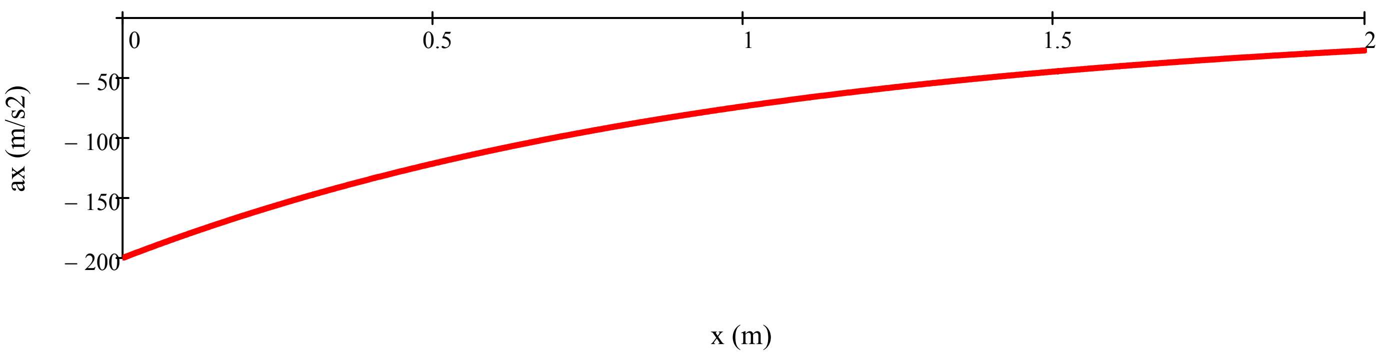
The x momentum becomes $\quad \rho\cdot u\cdot\frac{d}{dx}u = \rho\cdot a_a = -\frac{d}{dx}p$

The pressure gradient is then $\quad \frac{dp}{dx} = \frac{\rho}{L}\cdot U^2\cdot e^{-\frac{2\cdot x}{L}}$

Integrating momentum $\quad p(x) = p_{in} - \rho\cdot\int_0^x a_x(x)\,dx \qquad p(x) = p_{in} - \frac{U^2\cdot\rho\cdot\left(e^{-\frac{2\cdot x}{L}} - 1\right)}{2}$

Hence $\quad p(L) = p_{in} - \frac{U^2\cdot\rho\cdot\left(e^{-2} - 1\right)}{2} \qquad p(L) = 206\cdot kPa$


200
150
100
50
dp/dx (kPa/m)
0
0.5
1
1.5
2
x (m)


0
0.5
1
1.5
2
− 50
− 100
− 150
− 200
ax (m/s2)
x (m)

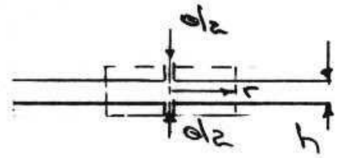
Problem 6.13

[Difficulty: 2]

6.13 For the flow of Problem 4.123 show that the uniform radial velocity is $V_r = Q/2\pi rh$. Obtain expressions for the r component of acceleration of a fluid particle in the gap and for the pressure variation as a function of radial distance from the central holes.

Solution:

Apply the conservation of mass to a CV with outer edge at r.



Basic equation: $0 = \cancel{\frac{\partial}{\partial t}\int_{CV} \rho \, d\forall}^{\,0(1)} + \int_{CS} \rho \vec{V} \cdot d\vec{A}$

Assumptions: (1) steady flow
(2) incompressible flow
(3) uniform flow at each section

Then

$$0 = \int_{CS} \vec{V} \cdot d\vec{A} = -2 \times \frac{Q}{2} + V_r 2\pi r h$$

and $V_r = \dfrac{Q}{2\pi r h}$ ← V_r

From Eq. 6.4a

$$g_r - \frac{1}{\rho}\frac{\partial p}{\partial r} = a_r = \frac{\partial V_r}{\partial t} + V_r \frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} + V_z \frac{\partial V_r}{\partial z} - \frac{V_\theta^2}{r}$$

Since $V_r = V_r(r)$ and $V_\theta = 0$, then

$$a_r = V_r \frac{\partial V_r}{\partial r} = \frac{Q}{2\pi r h}\left[\frac{Q}{2\pi h}\left(-\frac{1}{r^2}\right)\right] = -\left(\frac{Q}{2\pi r h}\right)^2 \frac{1}{r}$$

$a_r = -\dfrac{V_r^2}{r}$ ← a_r

Since $g_r = 0$, then

$$-\frac{1}{\rho}\frac{\partial p}{\partial r} = a_r$$

$\dfrac{\partial p}{\partial r} = -\rho a_r = \rho \dfrac{V_r^2}{r}$ ← $\dfrac{\partial p}{\partial r}$

Problem 6.14

[Difficulty: 3]

6.14 The velocity field for a plane vortex sink is given by $\vec{V} = (-q/2\pi r)\hat{e}_r + (K/2\pi r)\hat{e}_\theta$, where $q = 2$ $\text{m}^3/\text{s/m}$ and $K = 1$ $\text{m}^3/\text{s/m}$. The fluid density is 1000 kg/m^3. Find the acceleration at (1, 0), (1, $\pi/2$), and (2, 0). Evaluate ∇p under the same conditions.

Given: Velocity field

Find: The acceleration at several points; evaluate pressure gradient

Solution:

The given data is $\quad q = 2 \cdot \dfrac{\frac{m^3}{s}}{m} \qquad K = 1 \cdot \dfrac{\frac{m^3}{s}}{m} \qquad \rho = 1000 \cdot \dfrac{kg}{m^3} \qquad V_r = -\dfrac{q}{2 \cdot \pi \cdot r} \qquad V_\theta = \dfrac{K}{2 \cdot \pi \cdot r}$

The governing equations for this 2D flow are

$$\rho a_r = \rho \left(\frac{\partial V_r}{\partial t} + V_r \frac{\partial V_r}{\partial r} + \frac{V_\theta}{r} \frac{\partial V_r}{\partial \theta} + V_z \frac{\partial V_r}{\partial z} - \frac{V_\theta^2}{r} \right) = \rho g_r - \frac{\partial p}{\partial r} \qquad (6.3a)$$

$$\rho a_\theta = \rho \left(\frac{\partial V_\theta}{\partial t} + V_r \frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r} \frac{\partial V_\theta}{\partial \theta} + V_z \frac{\partial V_\theta}{\partial z} + \frac{V_r V_\theta}{r} \right) = \rho g_\theta - \frac{1}{r} \frac{\partial p}{\partial \theta} \qquad (6.3b)$$

The total acceleration for this steady flow is then

r - component $\quad a_r = V_r \cdot \dfrac{\partial}{\partial r} V_r + \dfrac{V_\theta}{r} \cdot \dfrac{\partial}{\partial \theta} V_r - \dfrac{V_\theta^2}{r} \qquad a_r = -\dfrac{q^2 + K^2}{4 \cdot \pi^2 \cdot r^3}$

θ - component $\quad a_\theta = V_r \cdot \dfrac{\partial}{\partial r} V_\theta + \dfrac{V_\theta}{r} \cdot \dfrac{\partial}{\partial \theta} V_\theta + \dfrac{V_r \cdot V_\theta}{r} \qquad a_\theta = 0$

Evaluating at point (1,0) $\quad a_r = -0.127 \dfrac{m}{s^2} \qquad a_\theta = 0$

Evaluating at point (1,π/2) $\quad a_r = -0.127 \dfrac{m}{s^2} \qquad a_\theta = 0$

Evaluating at point (2,0) $\quad a_r = -0.0158 \dfrac{m}{s^2} \qquad a_\theta = 0$

From Eq. 6.3, pressure gradient is $\quad \dfrac{\partial}{\partial r} p = -\rho \cdot a_r \qquad \dfrac{\partial}{\partial r} p = \dfrac{\rho \cdot \left(q^2 + K^2\right)}{4 \cdot \pi^2 \cdot r^3}$

$\dfrac{1}{r} \cdot \dfrac{\partial}{\partial \theta} p = -\rho \cdot a_\theta \qquad \dfrac{1}{r} \cdot \dfrac{\partial}{\partial \theta} p = 0$

Evaluating at point (1,0) $\quad \dfrac{\partial}{\partial r} p = 127 \cdot \dfrac{Pa}{m} \qquad \dfrac{1}{r} \cdot \dfrac{\partial}{\partial \theta} p = 0$

Evaluating at point (1,π/2) $\quad \dfrac{\partial}{\partial r} p = 127 \cdot \dfrac{Pa}{m} \qquad \dfrac{1}{r} \cdot \dfrac{\partial}{\partial \theta} p = 0$

Evaluating at point (2,0) $\quad \dfrac{\partial}{\partial r} p = 15.8 \cdot \dfrac{Pa}{m} \qquad \dfrac{1}{r} \cdot \dfrac{\partial}{\partial \theta} p = 0$

Problem 6.15

[Difficulty: 3]

6.15 An incompressible, inviscid fluid flows into a horizontal round tube through its porous wall. The tube is closed at the left end and the flow discharges from the tube to the atmosphere at the right end. For simplicity, consider the x component of velocity in the tube uniform across any cross section. The density of the fluid is ρ, the tube diameter and length are D and L, respectively, and the uniform inflow velocity is v_0. The flow is steady. Obtain an algebraic expression for the x component of acceleration of a fluid particle located at position x, in terms of v_0, x, and D. Find an expression for the pressure gradient, $\partial p/\partial x$, at position x. Integrate to obtain an expression for the gage pressure at $x = 0$.

Solution: Apply conservation of mass using the CV shown.

Basic equations: $0 = \overbrace{\frac{\partial}{\partial t}\int_{CV} \rho \, dV}^{=0(1)} + \int_{CS} \rho \vec{V} \cdot d\vec{A}$

$$a_{px} = u\frac{\partial u}{\partial x} + \overbrace{v\frac{\partial u}{\partial y}}^{\approx 0(5)} + \overbrace{w\frac{\partial u}{\partial z}}^{\approx 0(5)} + \overbrace{\frac{\partial u}{\partial t}}^{=0(1)} \; ; \quad -\frac{\partial p}{\partial x} + \rho g_x = \rho a_{px}$$

Assumptions: (1) Steady flow (2) Incompressible flow (3) Uniform flow at each cross-section (4) Horizontal; $g_x = 0$ (5) $v \approx 0$ in channel ($w \approx 0$ too) (6) Inviscid flow

Then

$$\int \vec{V} \cdot d\vec{A} = \left\{-\left|v_0 \pi D x\right|\right\} + \left\{+\left|u \frac{\pi D^2}{4}\right|\right\} = 0 \quad \text{or} \quad u(x) = 4 v_0 \frac{x}{D}$$

and

$$a_{px} = 4 v_0 \frac{x}{D}\left(4 v_0 \frac{1}{D}\right) = 16 v_0^2 \frac{x}{D^2}$$ a_{px}

From the Euler equation,

$$-\frac{\partial p}{\partial x} = \rho a_{px} \quad \text{so} \quad \frac{\partial p}{\partial x} = -\rho a_{px} = -16 \rho v_0^2 \frac{x}{D^2}$$ $\frac{\partial p}{\partial x}$

Since $v \approx w \approx 0$, then $p(x)$ and $dp = \frac{\partial p}{\partial x} dx$. Integrating

$$\int_0^L dp = p_L - p(0) = \int_0^L -16 \rho v_0^2 \frac{x}{D^2}\, dx = -\frac{16 \rho v_0^2}{D^2} \frac{x^2}{2}\Big]_0^L = -\frac{8 \rho v_0^2 L^2}{D^2}$$

Thus, since $p_L = p_{atm}$, the gage pressure at $x = 0$ is

$$p(0) = 8 \rho v_0^2 \left(\frac{L}{D}\right)^2$$ $p(0)$

Problem 6.16

[Difficulty: 3]

6.16 An incompressible liquid with negligible viscosity and density $\rho = 1.75$ slug/ft^3 flows steadily through a horizontal pipe. The pipe cross-section area linearly varies from 15 in^2 to 2.5 in^2 over a length of 10 feet. Develop an expression for and plot the pressure gradient and pressure versus position along the pipe, if the inlet centerline velocity is 5 ft/s and inlet pressure is 35 psi. What is the exit pressure? *Hint:* Use relation

$$u\frac{\partial u}{\partial x} = \frac{1}{2}\frac{\partial}{\partial x}(u^2)$$

Given: Flow in a pipe with variable area

Find: Expression for pressure gradient and pressure; Plot them; exit pressure

Solution:

Assumptions: 1) Incompressible flow 2) Flow profile remains unchanged so centerline velocity can represent average velocity

Basic equations $\quad Q = V \cdot A \qquad \vec{a}_p = \underbrace{\frac{D\vec{V}}{Dt}}_{\substack{\text{total}\\ \text{acceleration}\\ \text{of a particle}}} = \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\substack{\text{convective}\\ \text{acceleration}}} + \underbrace{\frac{\partial \vec{V}}{\partial t}}_{\substack{\text{local}\\ \text{acceleration}}} \qquad \rho\frac{D\vec{V}}{Dt} = \rho\vec{g} - \nabla p$

Given data $\quad \rho = 1.75\cdot\frac{\text{slug}}{\text{ft}^3} \qquad p_i = 35\cdot\text{psi} \qquad A_i = 15\cdot\text{in}^2 \qquad A_e = 2.5\cdot\text{in}^2 \qquad L = 10\cdot\text{ft} \qquad u_i = 5\cdot\frac{\text{ft}}{\text{s}}$

For this 1D flow $\quad Q = u_i\cdot A_i = u\cdot A \qquad A = A_i - \frac{(A_i - A_e)}{L}\cdot x \qquad$ so $\qquad u(x) = u_i\cdot\frac{A_i}{A} = u_i\cdot\frac{A_i}{A_i - \left[\frac{(A_i - A_e)}{L}\cdot x\right]}$

$$a_x = u\cdot\frac{\partial}{\partial x}u + v\cdot\frac{\partial}{\partial y}u = u_i\cdot\frac{A_i}{A_i - \left[\frac{(A_i - A_e)}{L}\cdot x\right]}\cdot\frac{\partial}{\partial x}\left[u_i\cdot\frac{A_i}{A_i - \left[\frac{(A_i - A_e)}{L}\cdot x\right]}\right] = \frac{A_i^2\cdot L^2\cdot u_i^2\cdot(A_e - A_i)}{(A_i\cdot L + A_e\cdot x - A_i\cdot x)^3}$$

For the pressure $\quad \frac{\partial}{\partial x}p = -\rho\cdot a_x - \rho\cdot g_x = -\frac{\rho\cdot A_i^2\cdot L^2\cdot u_i^2\cdot(A_e - A_i)}{(A_i\cdot L + A_e\cdot x - A_i\cdot x)^3}$

and $\quad dp = \frac{\partial}{\partial x}p\cdot dx \qquad p - p_i = \int_0^x \frac{\partial}{\partial x}p\,dx = \int_0^x -\frac{\rho\cdot A_i^2\cdot L^2\cdot u_i^2\cdot(A_e - A_i)}{(A_i\cdot L + A_e\cdot x - A_i\cdot x)^3}\,dx$

This is a tricky integral, so instead consider the following: $\quad \frac{\partial}{\partial x}p = -\rho\cdot a_x = -\rho\cdot u\cdot\frac{\partial}{\partial x}u = -\frac{1}{2}\cdot\rho\cdot\frac{\partial}{\partial x}(u^2)$

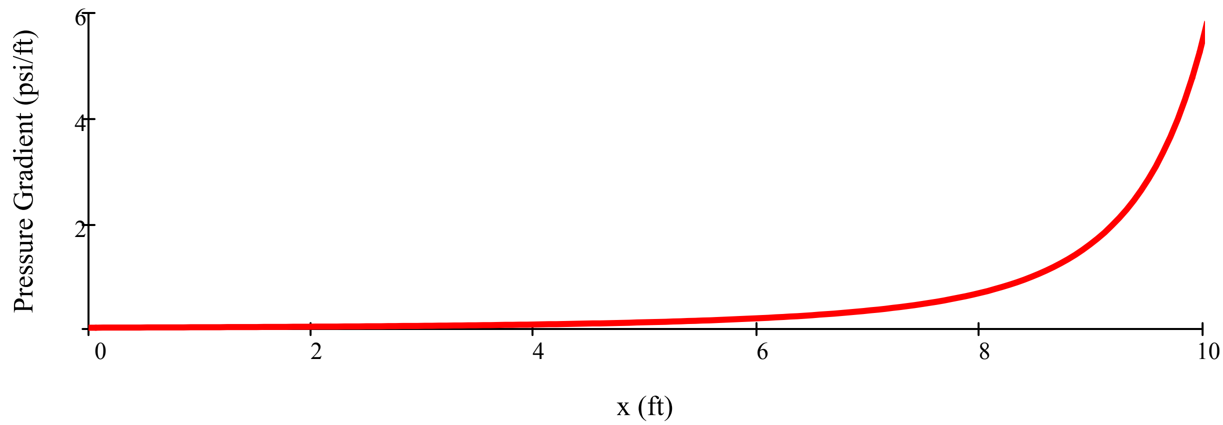
Hence $\quad p - p_i = \int_0^x \frac{\partial}{\partial x}p\,dx = -\frac{\rho}{2}\cdot\int_0^x \frac{\partial}{\partial x}(u^2)\,dx = \frac{\rho}{2}\cdot\left(u(x=0)^2 - u(x)^2\right)$

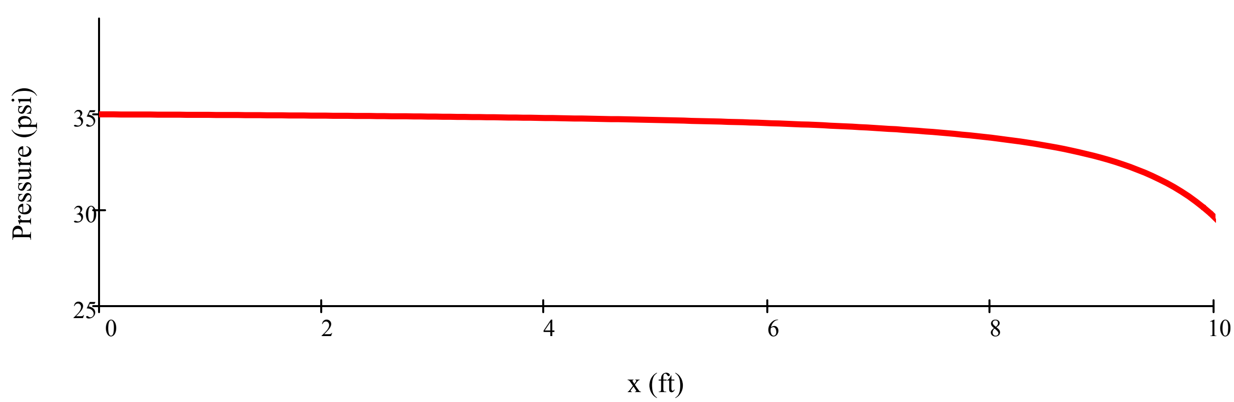
$$p(x) = p_i + \frac{\rho}{2} \cdot \left(u_i^2 - u(x)^2\right)$$ which we recognise as the Bernoulli equation!

Hence $$p(x) = p_i + \frac{\rho \cdot u_i^2}{2} \cdot \left[1 - \left[\frac{A_i}{A_i - \left[\frac{(A_i - A_e)}{L} \cdot x\right]}\right]^2\right]$$

At the exit $p(L) = 29.7\ \text{psi}$

The following plots can be done in *Excel*





Problem 6.17

[Difficulty: 3]

6.17 An incompressible liquid with negligible viscosity and density $\rho = 1250\ \text{kg/m}^3$ flows steadily through a 5-m-long convergent-divergent section of pipe for which the area varies as

$$A(x) = A_0(1 + e^{-x/a} - e^{-x/2a})$$

where $A_0 = 0.25\ \text{m}^2$ and $a = 1.5$ m. Plot the area for the first 5 m. Develop an expression for and plot the pressure gradient and pressure versus position along the pipe, for the first 5 m, if the inlet centerline velocity is 10 m/s and inlet pressure is 300 kPa. *Hint:* Use relation

$$u\frac{\partial u}{\partial x} = \frac{1}{2}\frac{\partial}{\partial x}(u^2)$$

Given: Flow in a pipe with variable area

Find: Expression for pressure gradient and pressure; Plot them

Solution:

Assumptions: 1) Incompressible flow 2) Flow profile remains unchanged so centerline velocity can represent average velocity

Basic equations

$$Q = V\cdot A \qquad \underbrace{\vec{a}_p = \frac{D\vec{V}}{Dt}}_{\substack{\text{total}\\\text{acceleration}\\\text{of a particle}}} = \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\text{convective acceleration}} + \underbrace{\frac{\partial \vec{V}}{\partial t}}_{\substack{\text{local}\\\text{acceleration}}} \qquad \rho\frac{D\vec{V}}{Dt} = \rho\vec{g} - \nabla p$$

Given data $\quad \rho = 1250\cdot\frac{\text{kg}}{\text{m}^3} \quad A_0 = 0.25\cdot\text{m}^2 \quad a = 1.5\cdot\text{m} \quad L = 5\cdot\text{m} \quad u_0 = 10\cdot\frac{\text{m}}{\text{s}} \quad p_0 = 300\cdot\text{kPa}$

For this 1D flow $\quad Q = u_0\cdot A_0 = u\cdot A \qquad A(x) = A_0\cdot\left(1 + e^{-\frac{x}{a}} - e^{-\frac{x}{2\cdot a}}\right)$

so

$$u(x) = u_0\cdot\frac{A_0}{A} = \frac{u_0}{\left(1 + e^{-\frac{x}{a}} - e^{-\frac{x}{2\cdot a}}\right)}$$

$$a_x = u\cdot\frac{\partial}{\partial x}u + v\cdot\frac{\partial}{\partial y}u = \frac{u_0}{\left(1 + e^{-\frac{x}{a}} - e^{-\frac{x}{2\cdot a}}\right)}\cdot\frac{\partial}{\partial x}\left[\frac{u_0}{\left(1 + e^{-\frac{x}{a}} - e^{-\frac{x}{2\cdot a}}\right)}\right] = \frac{u_0^{\,2}\cdot e^{-\frac{x}{2\cdot a}}\cdot\left(2\cdot e^{-\frac{x}{2\cdot a}} - 1\right)}{2\cdot a\cdot\left(e^{-\frac{x}{a}} - e^{-\frac{x}{2\cdot a}} + 1\right)^3}$$

For the pressure

$$\frac{\partial}{\partial x}p = -\rho\cdot a_x - \rho\cdot g_x = -\frac{\rho\cdot u_0^{\,2}\cdot e^{-\frac{x}{2\cdot a}}\cdot\left(2\cdot e^{-\frac{x}{2\cdot a}} - 1\right)}{2\cdot a\cdot\left(e^{-\frac{x}{a}} - e^{-\frac{x}{2\cdot a}} + 1\right)^3}$$

and $\quad dp = \frac{\partial}{\partial x}p\cdot dx \qquad p - p_i = \int_0^x \frac{\partial}{\partial x}p\,dx = \int_0^x -\frac{\rho\cdot u_0^{\,2}\cdot e^{-\frac{x}{2\cdot a}}\cdot\left(2\cdot e^{-\frac{x}{2\cdot a}} - 1\right)}{2\cdot a\cdot\left(e^{-\frac{x}{a}} - e^{-\frac{x}{2\cdot a}} + 1\right)^3}\,dx$

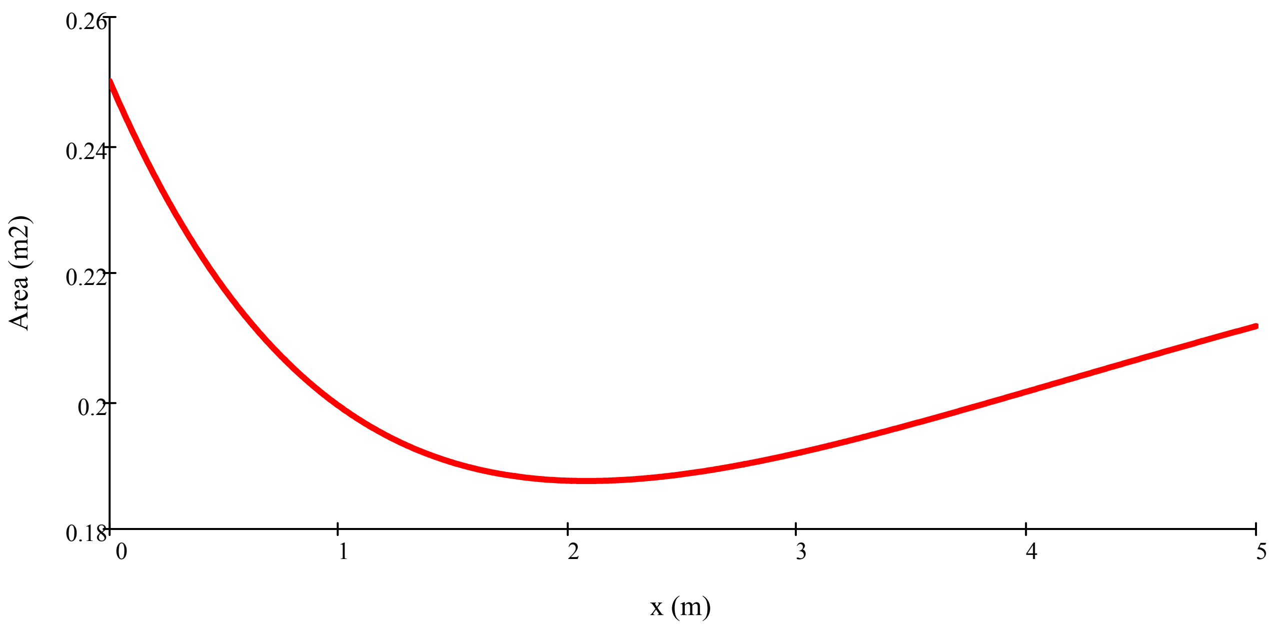
This is a tricky integral, so instead consider the following: $\quad \frac{\partial}{\partial x}p = -\rho\cdot a_x = -\rho\cdot u\cdot\frac{\partial}{\partial x}u = -\frac{1}{2}\cdot\rho\cdot\frac{\partial}{\partial x}\left(u^2\right)$

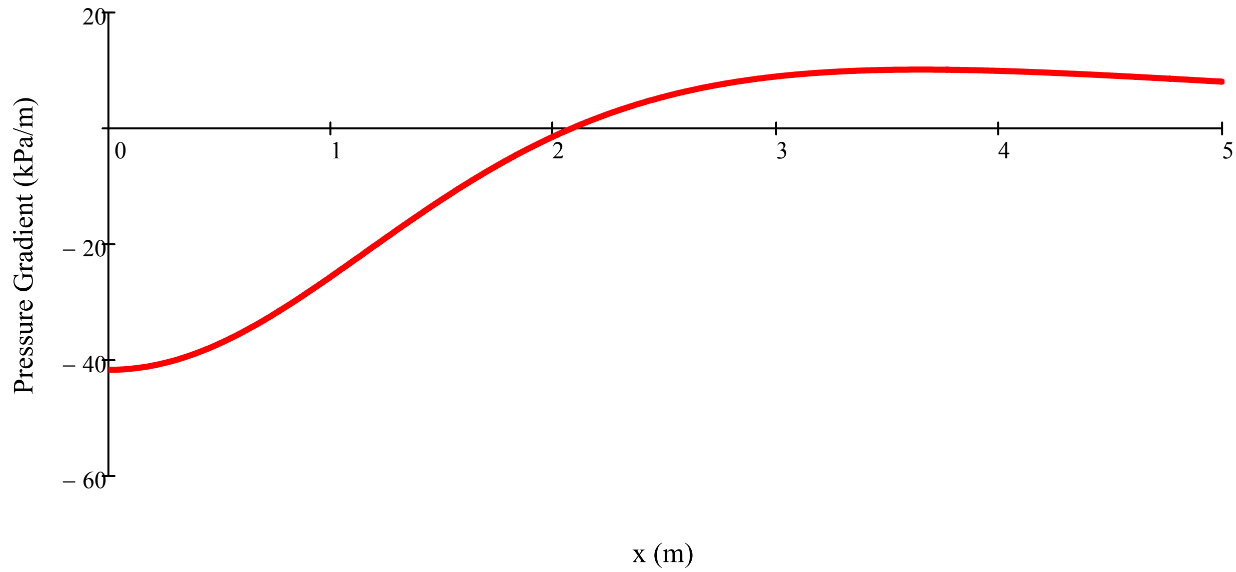
Hence $\quad p - p_i = \int_0^x \frac{\partial}{\partial x}p\,dx = -\frac{\rho}{2}\cdot\int_0^x \frac{\partial}{\partial x}\left(u^2\right)dx = \frac{\rho}{2}\cdot\left(u(x=0)^2 - u(x)^2\right)$

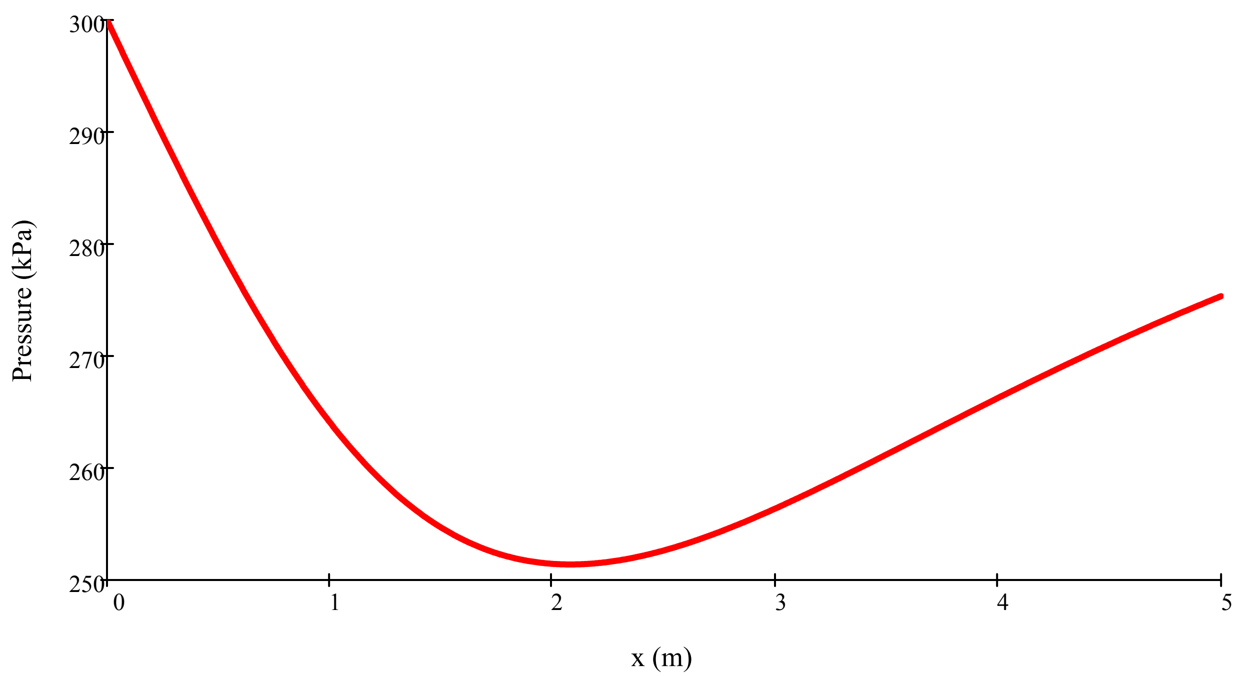
$p(x) = p_0 + \frac{\rho}{2}\cdot\left(u_0^{\,2} - u(x)^2\right)$ $\quad$ which we recognise as the Bernoulli equation!

$$p(x) = p_0 + \frac{\rho\cdot u_0^{\,2}}{2}\cdot\left[1 - \left[\frac{1}{\left(1 + e^{-\frac{x}{a}} - e^{-\frac{x}{2\cdot a}}\right)}\right]^2\right]$$

The following plots can be done in *Excel*




Pressure Gradient (kPa/m)
20
−20
−40
−60
0
1
2
3
4
5
x (m)


Pressure (kPa)
300
290
280
270
260
250
0
1
2
3
4
5
x (m)

Problem 6.18

[Difficulty: 3]

6.18 A nozzle for an incompressible, inviscid fluid of density $\rho = 1000\ \text{kg/m}^3$ consists of a converging section of pipe. At the inlet the diameter is $D_i = 100$ mm, and at the outlet the diameter is $D_o = 20$ mm. The nozzle length is $L = 500$ mm, and the diameter decreases linearly with distance x along the nozzle. Derive and plot the acceleration of a fluid particle, assuming uniform flow at each section, if the speed at the inlet is $V_i = 1$ m/s. Plot the pressure gradient through the nozzle, and find its maximum absolute value. If the pressure gradient must be no greater than 5 MPa/m in absolute value, how long would the nozzle have to be?

Given: Nozzle geometry

Find: Acceleration of fluid particle; Plot; Plot pressure gradient; find L such that pressure gradient < 5 MPa/m in absolute value

Solution:

The given data is $D_i = 0.1 \cdot m$ $\quad D_o = 0.02 \cdot m$ $\quad L = 0.5 \cdot m$ $\quad V_i = 1 \cdot \frac{m}{s}$ $\quad \rho = 1000 \cdot \frac{kg}{m^3}$

For a linear decrease in diameter $\quad D(x) = D_i + \frac{D_o - D_i}{L} \cdot x$

From continuity $\quad Q = V \cdot A = V \cdot \frac{\pi}{4} \cdot D^2 = V_i \cdot \frac{\pi}{4} \cdot D_i^2 \qquad Q = 0.00785 \frac{m^3}{s}$

Hence $\quad V(x) \cdot \frac{\pi}{4} \cdot D(x)^2 = Q \qquad V(x) = \frac{4 \cdot Q}{\pi \cdot \left(D_i + \frac{D_o - D_i}{L} \cdot x\right)^2}$

or $\quad V(x) = \frac{V_i}{\left(1 + \frac{D_o - D_i}{L \cdot D_i} \cdot x\right)^2}$

The governing equation for this flow is

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho g_x - \frac{\partial p}{\partial x} \qquad (6.2a)$$

or, for steady 1D flow, in the notation of the problem

$$a_x = V \cdot \frac{d}{dx} V = \frac{V_i}{\left(1 + \frac{D_o - D_i}{L \cdot D_i} \cdot x\right)^2} \cdot \frac{d}{dx} \frac{V_i}{\left(1 + \frac{D_o - D_i}{L \cdot D_i} \cdot x\right)^2} \qquad a_x(x) = -\frac{2 \cdot V_i^2 \cdot \left(D_o - D_i\right)}{D_i \cdot L \cdot \left[1 + \frac{\left(D_o - D_i\right)}{D_i \cdot L} \cdot x\right]^5}$$

This is plotted in the associated *Excel* workbook

From Eq. 6.2a, pressure gradient is

$$\frac{\partial}{\partial x} p = -\rho \cdot a_x \qquad \frac{\partial}{\partial x} p = \frac{2 \cdot \rho \cdot V_i^2 \cdot \left(D_o - D_i\right)}{D_i \cdot L \cdot \left[1 + \frac{\left(D_o - D_i\right)}{D_i \cdot L} \cdot x\right]^5}$$

This is also plotted in the associated *Excel* workbook. Note that the pressure gradient is always negative: separation is unlikely to occur in the nozzle

At the inlet $\quad \frac{\partial}{\partial x}p = -3.2\cdot\frac{kPa}{m}$ $\quad$ At the exit $\quad \frac{\partial}{\partial x}p = -10\cdot\frac{MPa}{m}$

To find the length *L* for which the absolute pressure gradient is no more than 5 MPa/m, we need to solve

$$\left|\frac{\partial}{\partial x}p\right| \le 5\cdot\frac{MPa}{m} = \frac{2\cdot\rho\cdot V_i^{2}\cdot\left(D_o - D_i\right)}{D_i\cdot L\cdot\left[1 + \frac{\left(D_o - D_i\right)}{D_i\cdot L}\cdot x\right]^{5}}$$

with *x* = *L* m (the largest pressure gradient is at the outlet)

Hence $\quad L \ge \frac{2\cdot\rho\cdot V_i^{2}\cdot\left(D_o - D_i\right)}{D_i\cdot\left(\frac{D_o}{D_i}\right)^{5}\cdot\left|\frac{\partial}{\partial x}p\right|}$ $\quad L \ge 1\cdot m$

This result is also obtained using *Goal Seek* in the *Excel* workbook

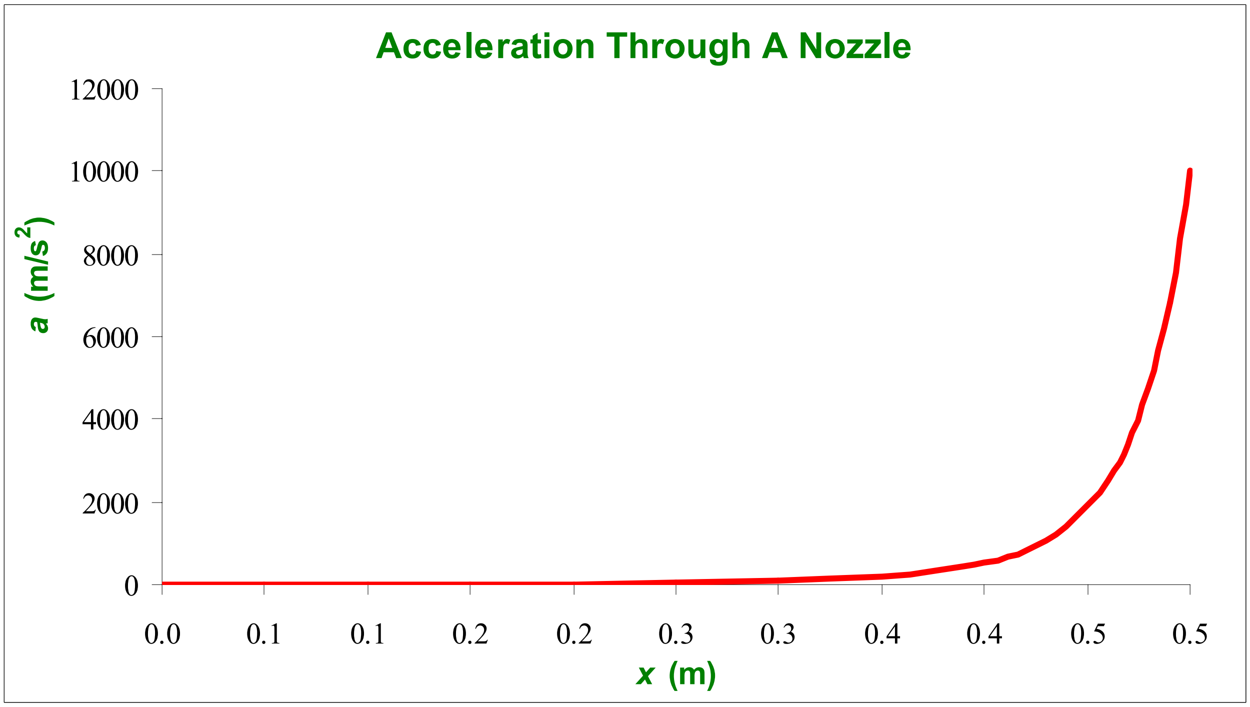
From Excel

***x* (m)**	***a* (m/s^2)**	***dp/dx* (kPa/m)**
0.000	3.20	-3.20
0.050	4.86	-4.86
0.100	7.65	-7.65
0.150	12.6	-12.6
0.200	22.0	-22.0
0.250	41.2	-41.2
0.300	84.2	-84.2
0.350	194	-194
0.400	529	-529
0.420	843	-843
0.440	1408	-1408
0.460	2495	-2495
0.470	3411	-3411
0.480	4761	-4761
0.490	6806	-6806
0.500	10000	-10000

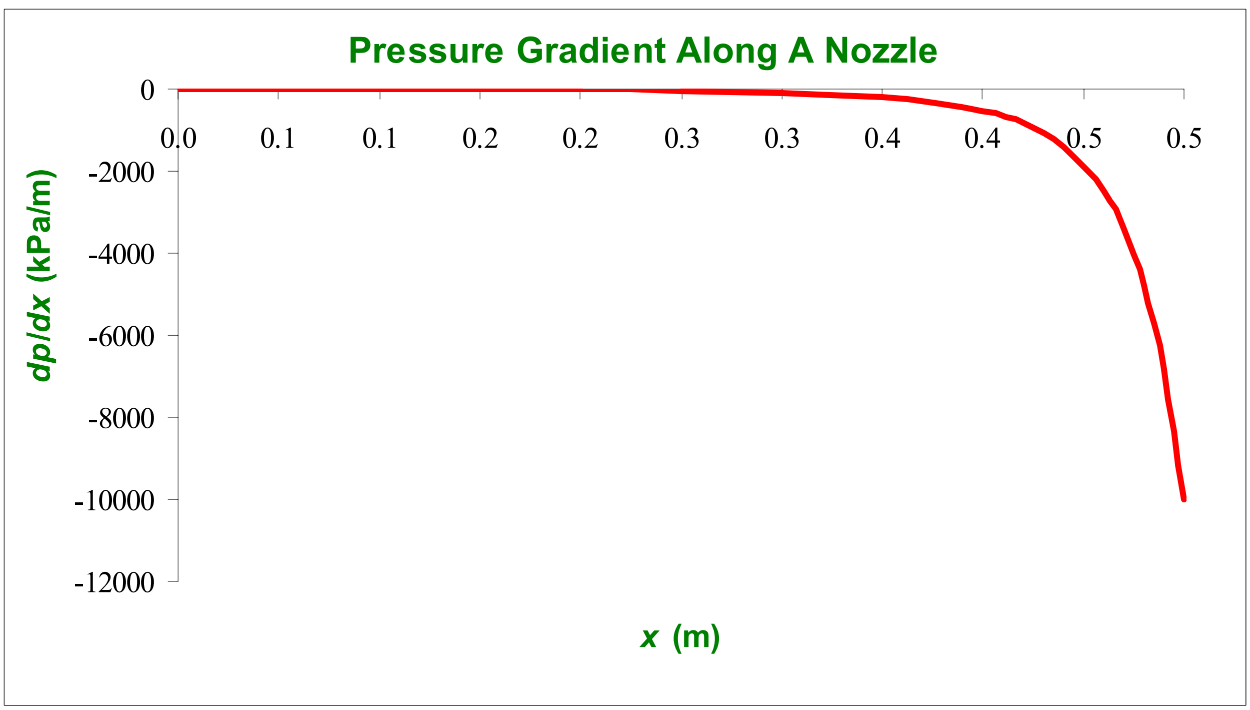
For the length *L* required for the pressure gradient to be less than 5 MPa/m (abs) use *Goal Seek*

L = 1.00 m

***x* (m)**	***dp/dx* (kPa/m)**
1.00	-5000



Acceleration Through A Nozzle
12000
10000
8000
6000
4000
2000
0
a (m/s2)
0.0
0.1
0.1
0.2
0.2
0.3
0.3
0.4
0.4
0.5
0.5
x (m)



Pressure Gradient Along A Nozzle
0
-2000
-4000
-6000
-8000
-10000
-12000
dp/dx (kPa/m)
0.0
0.1
0.1
0.2
0.2
0.3
0.3
0.4
0.4
0.5
0.5
x (m)

Problem 6.19

[Difficulty: 3]

6.19 A diffuser for an incompressible, inviscid fluid of density $\rho = 1000$ kg/m^3 consists of a diverging section of pipe. At the inlet the diameter is $D_i = 0.25$ m, and at the outlet the diameter is $D_o = 0.75$ m. The diffuser length is $L = 1$ m, and the diameter increases linearly with distance x along the diffuser. Derive and plot the acceleration of a fluid particle, assuming uniform flow at each section, if the speed at the inlet is $V_i = 5$ m/s. Plot the pressure gradient through the diffuser, and find its maximum value. If the pressure gradient must be no greater than 25 kPa/m, how long would the diffuser have to be?

Given: Diffuser geometry

Find: Acceleration of a fluid particle; plot it; plot pressure gradient; find *L* such that pressure gradient is less than 25 kPa/m

Solution:

The given data is $D_i = 0.25\cdot m$ $\quad D_o = 0.75\cdot m$ $\quad L = 1\cdot m$ $\quad V_i = 5\cdot\frac{m}{s}$ $\quad \rho = 1000\cdot\frac{kg}{m^3}$

For a linear increase in diameter $\quad D(x) = D_i + \frac{D_o - D_i}{L}\cdot x$

From continuity $\quad Q = V\cdot A = V\cdot\frac{\pi}{4}\cdot D^2 = V_i\cdot\frac{\pi}{4}\cdot D_i^2 \qquad Q = 0.245\frac{m^3}{s}$

Hence $\quad V(x)\cdot\frac{\pi}{4}\cdot D(x)^2 = Q \qquad V(x) = \frac{4\cdot Q}{\pi\cdot\left(D_i + \frac{D_o - D_i}{L}\cdot x\right)^2}$ $\quad$ or $\quad V(x) = \frac{V_i}{\left(1 + \frac{D_o - D_i}{L\cdot D_i}\cdot x\right)^2}$

The governing equation for this flow is

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho g_x - \frac{\partial p}{\partial x} \qquad (6.2a)$$

or, for steady 1D flow, in the notation of the problem

$$a_x = V\cdot\frac{d}{dx}V = \frac{V_i}{\left(1 + \frac{D_o - D_i}{L\cdot D_i}\cdot x\right)^2}\cdot\frac{d}{dx}\frac{V_i}{\left(1 + \frac{D_o - D_i}{L\cdot D_i}\cdot x\right)^2}$$

Hence

$$a_x(x) = -\frac{2\cdot V_i^2\cdot\left(D_o - D_i\right)}{D_i\cdot L\cdot\left[1 + \frac{\left(D_o - D_i\right)}{D_i\cdot L}\cdot x\right]^5}$$

This can be plotted in *Excel* (see below)

From Eq. 6.2a, pressure gradient is $\quad \frac{\partial}{\partial x}p = -\rho\cdot a_x \qquad \frac{\partial}{\partial x}p = \frac{2\cdot\rho\cdot V_i^2\cdot\left(D_o - D_i\right)}{D_i\cdot L\cdot\left[1 + \frac{\left(D_o - D_i\right)}{D_i\cdot L}\cdot x\right]^5}$

This can also plotted in *Excel*. Note that the pressure gradient is adverse: separation is likely to occur in the diffuser, and occur near the entrance

At the inlet $\frac{\partial}{\partial x}p = 100 \cdot \frac{kPa}{m}$ At the exit $\frac{\partial}{\partial x}p = 412 \cdot \frac{Pa}{m}$

To find the length *L* for which the pressure gradient is no more than 25 kPa/m, we need to solve

$$\frac{\partial}{\partial x}p \le 25 \cdot \frac{kPa}{m} = \frac{2 \cdot \rho \cdot V_i^2 \cdot \left(D_o - D_i\right)}{D_i \cdot L \cdot \left[1 + \frac{\left(D_o - D_i\right)}{D_i \cdot L} \cdot x\right]^5}$$

with $x = 0$ m (the largest pressure gradient is at the inlet)

Hence $$L \ge \frac{2 \cdot \rho \cdot V_i^2 \cdot \left(D_o - D_i\right)}{D_i \cdot \frac{\partial}{\partial x}p}$$ $L \ge 4 \cdot m$

This result is also obtained using *Goal Seek* in *Excel.*

In *Excel*:

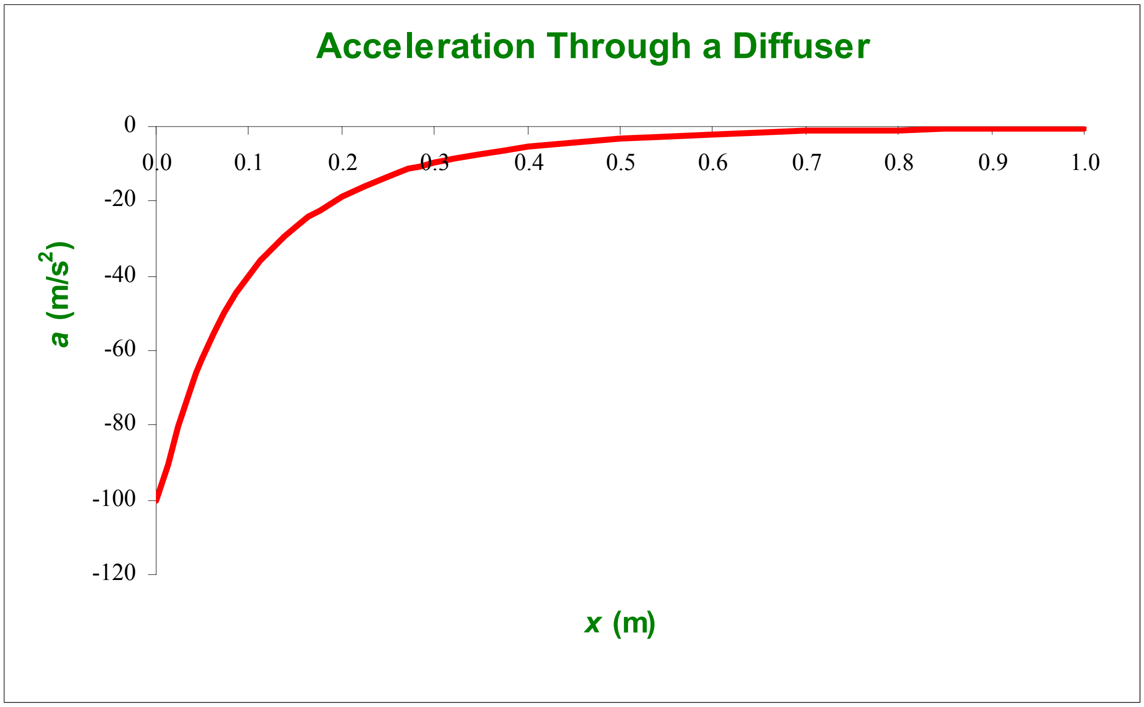
D_i =	0.25	m
D_o =	0.75	m
L =	1	m
V_i =	5	m/s
(=	1000	kg/m^3

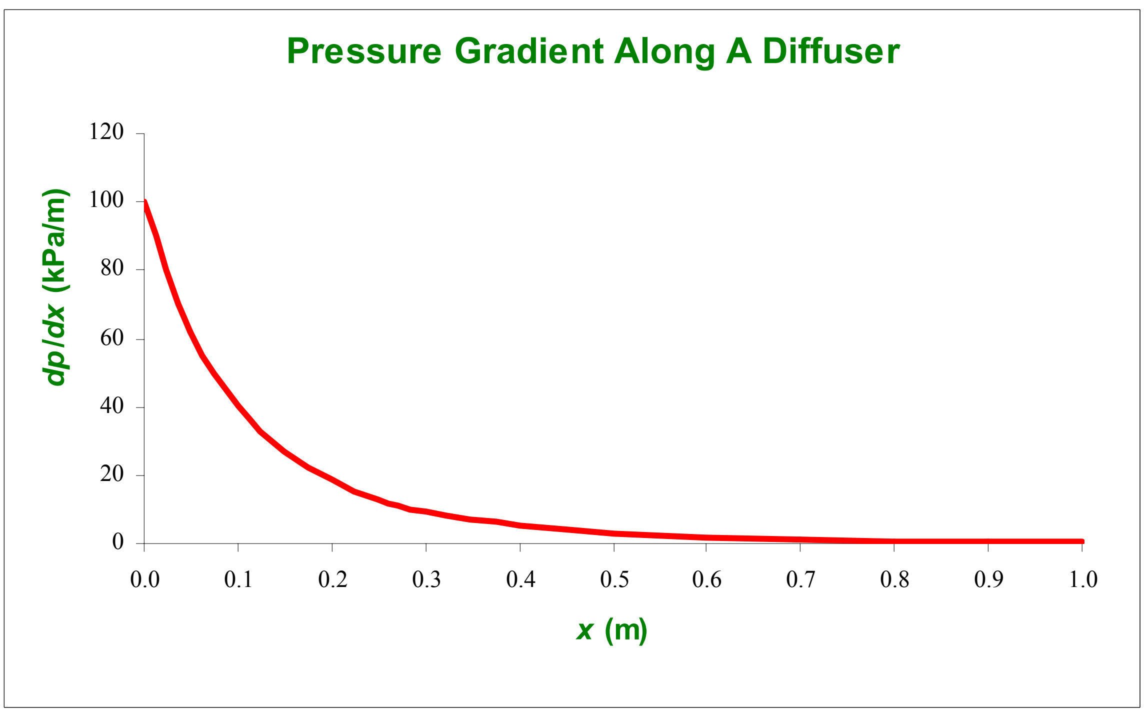
x (m)	*a* (m/s^2)	*dp* /*dx* (kPa/m)
0.00	-100	100
0.05	-62.1	62.1
0.10	-40.2	40.2
0.15	-26.9	26.93
0.20	-18.59	18.59
0.25	-13.17	13.17
0.30	-9.54	9.54
0.40	-5.29	5.29
0.50	-3.125	3.125
0.60	-1.940	1.940
0.70	-1.256	1.256
0.80	-0.842	0.842
0.90	-0.581	0.581
1.00	-0.412	0.412

For the length *L* required for the pressure gradient to be less than 25 kPa/m use *Goal Seek*

L = 4.00 m

x (m)	*dp* /*dx* (kPa/m)
0.0	25.0

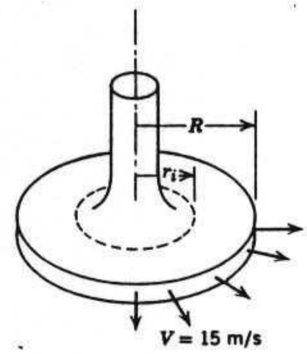

Acceleration Through a Diffuser
0
-20
-40
-60
-80
-100
-120
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
a (m/s²)
x (m)


Pressure Gradient Along A Diffuser
120
100
80
60
40
20
0
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
dp/dx (kPa/m)
x (m)

Problem 6.20

[Difficulty: 4]

6.20 Consider the flow of Problem 5.48. Evaluate the magnitude and direction of the net pressure force that acts on the upper plate between r_i and R, if $r_i = R/2$.

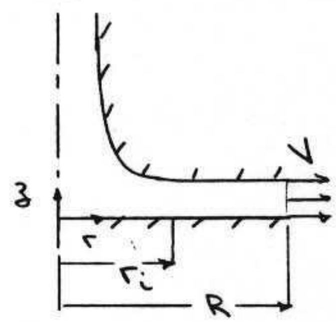


Solution:

Basic equations: $\rho \vec{g} - \nabla p = \rho \frac{D\vec{V}}{Dt}$ $\qquad \vec{F} = -\int p\, d\vec{A}$

Assumptions:
(1) incompressible flow
(2) steady flow
(3) frictionless flow
(4) uniform flow at each section.

To determine the pressure distribution $p(r)$, apply Eulers equation in the r direction



$$-\frac{\partial p}{\partial r} + \cancel{\rho g_r} = \rho a_r = \rho V_r \frac{\partial V_r}{\partial r}$$

$$\frac{\partial p}{\partial r} = -\rho V_r \frac{\partial V_r}{\partial r} = -\rho V \frac{R}{r} \frac{\partial}{\partial r}\left(V\frac{R}{r}\right) = \rho V \frac{R}{r} \frac{VR}{r^2}$$

$$\frac{dp}{dr} = \rho V^2 \frac{R^2}{r^3}$$

$$dp = \rho V^2 \frac{R^2}{r^3}\, dr$$

Integrating we obtain

$$p - p_{atm} = \int_{p_{atm}}^{p} dp = \rho V^2 R^2 \int_R^r r^{-3}\, dr = \rho V^2 R^2 \left[-\frac{1}{2r^2}\right]_R^r = \frac{1}{2}\rho V^2 R^2 \left[\frac{1}{R^2} - \frac{1}{r^2}\right]$$

Then

$$F_z = \int (p - p_{atm})\, dA = \int_{R/2}^{R} \frac{1}{2}\rho V^2 R^2 \left[\frac{1}{R^2} - \frac{1}{r^2}\right] 2\pi r\, dr = \rho V^2 R^2 \pi \left[\frac{r^2}{2R^2} - \ln r\right]_{R/2}^{R}$$

$$= \rho V^2 R^2 \pi \left[\frac{1}{2R^2}\left(R^2 - \frac{R^2}{4}\right) - \ln \frac{R}{R/2}\right] = \rho V^2 R^2 \pi \left[0.375 - \ln 2\right] = -0.318\pi \rho V^2 R^2$$

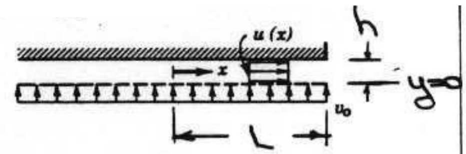
$$= -0.318\pi \times 1.23 \frac{kg}{m^3} \times (15)^2 \frac{m^2}{s^2} \times (0.075)^2 m^2 \times \frac{N \cdot s^2}{kg \cdot m}$$

$F_z = -1.56\text{N}$ $\quad$ ($F_z < 0$, so force acts down) $\qquad F_z$

Problem 6.21

[Difficulty: 4]

6.21 Consider again the flow field of Problem 5.65. Assume the flow is incompressible with $\rho = 1.23\ \text{kg/m}^3$ and friction is negligible. Further assume the vertical air flow velocity is $v_0 = 15$ mm/s, the half-width of the cavity is $L = 22$ mm, and its height is $h = 1.2$ mm. Calculate the pressure gradient at $(x, y) = (L, h)$. Obtain an equation for the flow streamlines in the cavity.



Solution:

Eulers equation, $\rho\vec{g} - \nabla p = \rho \frac{D\vec{V}}{Dt}$, can be used to determine the pressure gradient for incompressible frictionless flow.

We need first to determine the velocity field. With $u = v_0 x/h$, for 2-D, incompressible flow we can use the continuity equation to determine v.

Since $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$, then $\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -\frac{\partial}{\partial x}\left(\frac{v_0 x}{h}\right) = -\frac{v_0}{h}$

Then
$$v = \int \frac{\partial v}{\partial y}\,dy + f(x) = -\frac{v_0}{h}y + f(x)$$

But $v = v_0$ at $y = 0$ and hence $f(x) = v_0$ and $v = v_0\left(1 - \frac{y}{h}\right)$

Then
$$\nabla p = \rho\vec{g} - \rho\frac{D\vec{V}}{Dt} = \rho\left[\vec{g} - u\frac{\partial \vec{V}}{\partial x} - v\frac{\partial \vec{V}}{\partial y}\right] = \rho\left[-g\hat{j} - \frac{v_0 x}{h}\left(\frac{v_0}{h}\hat{i}\right) - v_0\left(1-\frac{y}{h}\right)\left(-\frac{v_0}{h}\hat{j}\right)\right]$$

$$\nabla p = \rho\left[-g\hat{j} - \frac{v_0^2 x}{h^2}\hat{i} - \frac{v_0^2}{h}\left(1 - \frac{y}{h}\right)\hat{j}\right]$$

At the point $(x,y) = (L,h)$
$$\nabla p = \rho\left[-\frac{v_0^2 L}{h^2}\hat{i} - g\hat{j}\right]$$

$$= 1.23\,\frac{\text{kg}}{\text{m}^3}\left[-(15)^2\,\frac{\text{mm}^2}{\text{s}^2}\times 0.022\,\text{m}\times\frac{1}{(1.2)^2\,\text{mm}^2}\hat{i} - 9.81\,\frac{\text{m}}{\text{s}^2}\hat{j}\right]\cdot\frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}}$$

$$\nabla p\Big]_{L,h} = -4.23\,\hat{i} - 12.1\,\hat{j}\ \ \text{N/m}^3$$

∇p

(b) The slope of the streamlines is given by $\frac{dy}{dx} = \frac{v}{u}$

$\therefore \frac{dy}{dx} = \frac{v_0(1 - y/h)}{\frac{v_0 x}{h}}$ and separating variables, we can write

$\frac{d\left(\frac{y}{h}\right)}{(1 - y/h)} = \frac{d\left(\frac{x}{h}\right)}{x/h}$ Then integrating we obtain

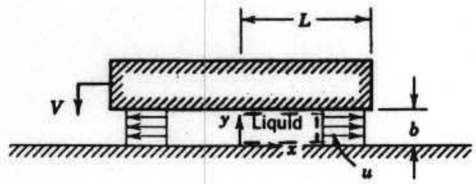
$$-\ln(1 - y/h) = \ln\frac{x}{h} - \ln c$$

or
$$\frac{x}{h}\left(1 - \frac{y}{h}\right) = \text{constant}$$

Problem 6.22

[Difficulty: 3]

6.22 A liquid layer separates two plane surfaces as shown. The lower surface is stationary; the upper surface moves downward at constant speed V. The moving surface has width w, perpendicular to the plane of the diagram, and $w \gg L$. The incompressible liquid layer, of density ρ, is squeezed from between the surfaces. Assume the flow is uniform at any cross section and neglect viscosity as a first approximation. Use a suitably chosen control volume to show that $u = Vx/b$ within the gap, where $b = b_0 - Vt$. Obtain an algebraic expression for the acceleration of a fluid particle located at x. Determine the pressure gradient, $\partial p/\partial x$, in the liquid layer. Find the pressure distribution, $p(x)$. Obtain an expression for the net pressure force that acts on the upper (moving) flat surface.



Solution:

Basic equations: $0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$

$-\nabla p + \rho \vec{g} = \rho \frac{D\vec{V}}{Dt}$ $\qquad \vec{F} = -\int p \, d\vec{A}$

(a) For the deformable CV shown

$$0 = \frac{\partial}{\partial t}\int_0^y \rho w x \, dy + \rho u w y = \rho w x \frac{dy}{dt} + \rho u w y$$

But $dy/dt = -V$ and hence $u = \frac{Vx}{y}$

If $y = b_0$ at $t=0$, then $y = b = b_0 - Vt$ at any time t

$$\therefore \; u = \frac{Vx}{b} \qquad u(t)$$

(b) $$a_x = \frac{Du}{Dt} = u\frac{\partial u}{\partial x} + \cancel{v\frac{\partial u}{\partial y}}^{\,0(1)} + \cancel{w\frac{\partial u}{\partial z}}^{\,0(1)} + \frac{\partial u}{\partial t}$$

Assumptions: (1) $u \neq u(y)$, $w = 0$

$$a_x = \frac{Vx}{b}\left(\frac{V}{b}\right) + \frac{\partial u}{\partial b}\frac{\partial b}{\partial t} = \frac{V^2 x}{b^2} + \left(-\frac{Vx}{b^2}\right)(-V) = \frac{2V^2 x}{b^2} \qquad a_x$$

(c) From Euler's equation in the x direction with $g_x = 0$

$$\frac{\partial p}{\partial x} = -\rho a_x = -\frac{2\rho V^2 x}{b^2} \qquad \frac{\partial p}{\partial x}$$

(d) $$p - p_{atm} = \int_L^x \frac{\partial p}{\partial x} dx = \int_L^x -\frac{2\rho V^2}{b^2} x \, dx = -\frac{\rho V^2 x^2}{b^2}\Big]_L^x = \frac{\rho V^2 L^2}{b^2}\left[1 - \left(\frac{x}{L}\right)^2\right] \qquad p(x)$$

(e) $$F_y = \int_A (p - p_{atm}) dA = 2\int_0^L \frac{\rho V^2 L^2}{b^2}\left[1 - \left(\frac{x}{L}\right)^2\right] w \, dx$$

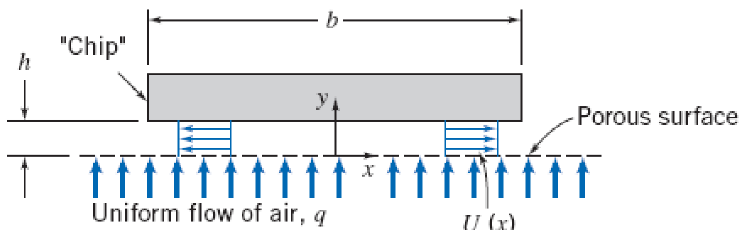
$$= 2\int_0^1 \frac{\rho V^2 L^3}{b^2}\left[1 - \left(\frac{x}{L}\right)^2\right] w \, d\left(\frac{x}{L}\right) = \frac{2\rho V^2 L^3 w}{b^2}\left[\left(\frac{x}{L}\right) - \frac{1}{3}\left(\frac{x}{L}\right)^3\right]_0^1$$

$$F_y = \frac{4\rho V^2 L^3 w}{3b^2} \qquad \text{(upward, since } F_y > 0\text{)} \qquad F_y$$

Problem 6.23

[Difficulty: 4]

6.23 A rectangular computer chip floats on a thin layer of air, $h = 0.5$ mm thick, above a porous surface. The chip width is $b = 40$ mm, as shown. Its length, L, is very long in the direction perpendicular to the diagram. There is no flow in the z direction. Assume flow in the x direction in the gap under the chip is uniform. Flow is incompressible, and frictional effects may be neglected. Use a suitably chosen control volume to show that $U(x) = qx/h$ in the gap. Find a general expression for the (2D) acceleration of a fluid particle in the gap in terms of q, h, x, and y. Obtain an expression for the pressure gradient $\partial p/\partial x$. Assuming atmospheric pressure on the chip upper surface, find an expression for the net pressure force on the chip; is it directed upward or downward? Explain. Find the required flow rate q ($m^3/s/m^2$) and the maximum velocity, if the mass per unit length of the chip is 0.005 kg/m. Plot the pressure distribution as part of your explanation of the direction of the net force.



Given: Rectangular chip flow

Find: Velocity field; acceleration; pressure gradient; net force; required flow rate; plot pressure

Solution:

Basic equations
$$\sum_{CS} \left(\vec{V}\cdot\vec{A}\right) = 0 \qquad \frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$$

$$\vec{a}_p = \underbrace{\frac{D\vec{V}}{Dt}}_{\substack{\text{total}\\\text{acceleration}\\\text{of a particle}}} = \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\substack{\text{convective}\\\text{acceleration}}} + \underbrace{\frac{\partial \vec{V}}{\partial t}}_{\substack{\text{local}\\\text{acceleration}}} \qquad \rho\frac{D\vec{V}}{Dt} = \rho\vec{g} - \nabla p$$

The given data is $\rho = 1.23\cdot\frac{kg}{m^3}$ $\quad p_{atm} = 101\cdot kPa$ $\quad h = 0.5\cdot mm$ $\quad b = 40\cdot mm$ $\quad M_{length} = 0.005\cdot\frac{kg}{m}$

Assuming a CV that is from the centerline to any point x, and noting that q is inflow per unit area, continuity leads to

$$q\cdot x\cdot L = U\cdot h\cdot L \qquad \text{or} \qquad u(x) = U(x) = q\cdot\frac{x}{h}$$

For acceleration we will need the vertical velocity v; we can use

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0 \qquad \text{or} \qquad \frac{\partial}{\partial y}v = -\frac{\partial}{\partial x}u = -\frac{du}{dx} = -\frac{d}{dx}\left(q\cdot\frac{x}{h}\right) = -\frac{q}{h}$$

Hence
$$v(y = y) - v(y = 0) = -\int_0^y \frac{q}{h}\,dy = -q\cdot\frac{y}{h}$$

But
$$v(y = 0) = q \qquad \text{so} \qquad v(y) = q\cdot\left(1 - \frac{y}{h}\right)$$

For the x acceleration
$$a_x = u\cdot\frac{\partial}{\partial x}u + v\cdot\frac{\partial}{\partial y}u \qquad a_x = q\cdot\frac{x}{h}\cdot\left(\frac{q}{h}\right) + q\cdot\left(1 - \frac{y}{h}\right)\cdot(0) \qquad a_x = \frac{q^2}{h^2}\cdot x$$

For the y acceleration $$a_y = u\cdot\frac{\partial}{\partial x}v + v\cdot\frac{\partial}{\partial y}v \qquad a_y = q\cdot\frac{x}{h}\cdot(0) + q\cdot\left(1-\frac{y}{h}\right)\cdot\left(-\frac{q}{h}\right) \qquad a_x = \frac{q^2}{h}\cdot\left(\frac{y}{h}-1\right)$$

For the pressure gradient we use x and y momentum (Euler equation) $$\rho\cdot\frac{Du}{Dx} = \rho\cdot\left(u\cdot\frac{\partial}{\partial x}u + v\cdot\frac{\partial}{\partial y}u\right) = \rho\cdot a_x = \frac{\partial}{\partial x}p$$

Hence $$\frac{\partial}{\partial x}p = -\rho\cdot\frac{q^2}{h^2}\cdot x$$

Also $$\rho\cdot\frac{Dv}{Dx} = \rho\cdot\left(u\cdot\frac{\partial}{\partial x}v + v\cdot\frac{\partial}{\partial y}v\right) = \rho\cdot a_y = -\frac{\partial}{\partial y}p \qquad \frac{\partial}{\partial y}p = \rho\cdot\frac{q^2}{h}\cdot\left(1-\frac{y}{h}\right)$$

For the pressure distribution, integrating from the outside edge (x = b/2) to any point x

$$p(x=x) - p\left(x=\frac{b}{2}\right) = p(x) - p_{atm} = \int_{\frac{b}{2}}^{x}\frac{\partial}{\partial x}p\,dx = \int_{\frac{b}{2}}^{x}-\rho\cdot\frac{q^2}{h^2}\cdot x\,dx = -\rho\cdot\frac{q^2}{2\cdot h^2}\cdot x^2 + \rho\cdot\frac{q^2}{8\cdot h^2}\cdot b^2$$

$$p(x) = p_{atm} + \rho\cdot\frac{q^2\cdot b^2}{8\cdot h^2}\cdot\left[1-4\cdot\left(\frac{x}{b}\right)^2\right]$$

For the net force we need to integrate this ... actually the gage pressure, as this pressure is opposed on the outer surface by p_{atm}

$$p_g(x) = \frac{\rho\cdot q^2\cdot b^2}{8\cdot h^2}\cdot\left[1-4\cdot\left(\frac{x}{b}\right)^2\right]$$

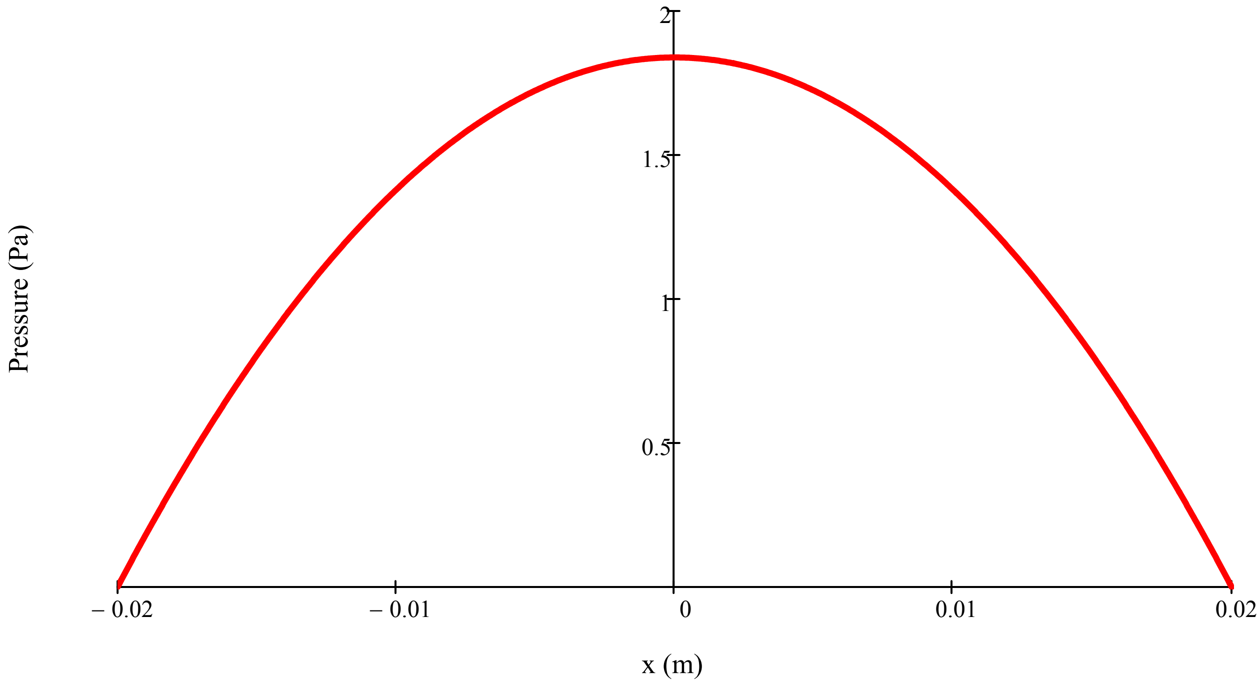
$$F_{net} = 2\cdot L\cdot\int_0^{\frac{b}{2}}p_g(x)\,dx = 2\cdot L\cdot\int_0^{\frac{b}{2}}\frac{\rho\cdot q^2\cdot b^2}{8\cdot h^2}\cdot\left[1-4\cdot\left(\frac{x}{b}\right)^2\right]dx = \frac{\rho\cdot q^2\cdot b^2\cdot L}{4\cdot h^2}\cdot\left(\frac{b}{2}-\frac{1}{3}\cdot\frac{b}{2}\right) \qquad F_{net} = \frac{\rho\cdot q^2\cdot b^3\cdot L}{12\cdot h^2}$$

The weight of the chip must balance this force $$M\cdot g = M_{length}\cdot L\cdot g = F_{net} = \frac{\rho\cdot q^2\cdot b^3\cdot L}{12\cdot h^2} \quad \text{or} \quad M_{length}\cdot g = \frac{\rho\cdot q^2\cdot b^3}{12\cdot h^2}$$

Solving for q for the given mass/length $$q = \sqrt{\frac{12\cdot h^2\cdot g\cdot M_{length}}{\rho\cdot b^3}} \qquad q = 0.0432\cdot\frac{\frac{m^3}{s}}{m^2}$$

The maximum speed $$U_{max} = u\left(x=\frac{b}{2}\right) = q\cdot\frac{\frac{b}{2}}{h} \qquad U_{max} = \frac{b\cdot q}{2\cdot h} \qquad U_{max} = 1.73\,\frac{m}{s}$$

The following plot can be done in *Excel* $$p_g(x) = \frac{\rho\cdot q^2\cdot b^2}{8\cdot h^2}\cdot\left[1-4\cdot\left(\frac{x}{b}\right)^2\right]$$

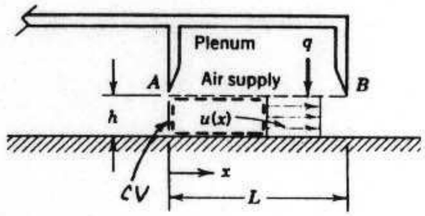


The net force is such that the chip is floating on air due to a Bernoulli effect: the speed is maximum at the edges and zero at the center; pressure has the opposite trend - pressure is minimum (p_{atm}) at the edges and maximum at the center.

Problem 6.24

[Difficulty: 3]

6.24 Heavy weights can be moved with relative ease on air cushions by using a load pallet as shown. Air is supplied from the plenum through porous surface AB. It enters the gap vertically at uniform speed, q. Once in the gap, all air flows in the positive x direction (there is no flow across the plane at $x = 0$). Assume air flow in the gap is incompressible and uniform at each cross section, with speed $u(x)$, as shown in the enlarged view. Although the gap is narrow ($h \ll L$), neglect frictional effects as a first approximation. Use a suitably chosen control volume to show that $u(x) = qx/h$ in the gap. Calculate the acceleration of a fluid particle in the gap. Evaluate the pressure gradient, $\partial p/\partial x$, and sketch the pressure distribution within the gap. Be sure to indicate the pressure at $x = L$.



Solution: Choose a CV in the gap, from 0 to x, as shown.

Basic equations: $0 = \cancelto{0(1)}{\frac{\partial}{\partial t}}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$

$$a_x = u\frac{\partial u}{\partial x} + \cancelto{0(3)}{v\frac{\partial u}{\partial y}} + \cancelto{0(4)}{w\frac{\partial u}{\partial z}} + \cancelto{0(1)}{\frac{\partial u}{\partial t}} \qquad -\frac{\partial p}{\partial x} + \cancelto{0(5)}{\rho g_x} = \rho a_{p_x}$$

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Uniform flow at each section
(4) No variation with z
(5) Horizontal, so $g_x = 0$

From continuity,

$$0 = \{-|\rho q w x|\} + \{+|\rho u(x) w h|\} \quad \text{so} \quad u(x) = q\frac{x}{h}$$

$u(x)$

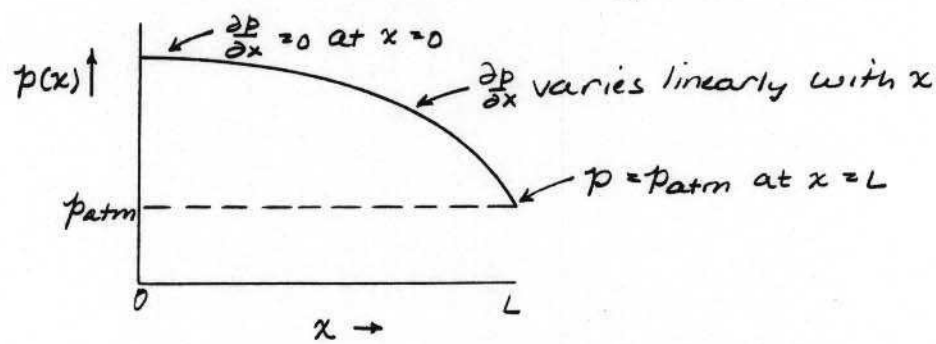
The acceleration is $a_{p_x} = \left(q\frac{x}{h}\right)\left(q\frac{1}{h}\right) = \frac{q^2 x}{h^2}$

a_{px}

The pressure gradient is $\frac{\partial p}{\partial x} = -\rho a_{p_x} = -\frac{\rho q^2 x}{h^2}$

$\frac{\partial p}{\partial x}$

Sketching:



Sketch

Problem 6.25

[Difficulty: 4]

6.25 A velocity field is given by $\vec{V} = [Ax^3 + Bxy^2]\hat{i} + [Ay^3 + Bx^2y]\hat{j}$; $A = 0.2\ \mathrm{m^{-2} \cdot s^{-1}}$, B is a constant, and the coordinates are measured in meters. Determine the value and units for B if this velocity field is to represent an incompressible flow. Calculate the acceleration of a fluid particle at point $(x, y) = (2, 1)$. Evaluate the component of particle acceleration normal to the velocity vector at this point.

Given: Velocity field

Find: Constant B for incompressible flow; Acceleration of particle at (2,1); acceleration normal to velocity at (2,1)

Solution:

Basic equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \vec{a}_p = \underbrace{\frac{D\vec{V}}{Dt}}_{\substack{\text{total} \\ \text{acceleration} \\ \text{of a particle}}} = \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\substack{\text{convective} \\ \text{acceleration}}} + \underbrace{\frac{\partial \vec{V}}{\partial t}}_{\substack{\text{local} \\ \text{acceleration}}}$$

For this flow

$$u(x,y) = A\cdot x^3 + B\cdot x\cdot y^2 \qquad v(x,y) = A\cdot y^3 + B\cdot x^2\cdot y$$

$$\frac{\partial}{\partial x}u(x,y) + \frac{\partial}{\partial y}v(x,y) = \frac{\partial}{\partial x}\left(A\cdot x^3 + B\cdot x\cdot y^2\right) + \frac{\partial}{\partial y}\left(A\cdot y^3 + B\cdot x^2\cdot y\right) = 0$$

$$\frac{\partial}{\partial x}u(x,y) + \frac{\partial}{\partial y}v(x,y) = (3\cdot A + B)\cdot\left(x^2 + y^2\right) = 0$$

Hence $B = -3\cdot A$ $\qquad B = -0.6\frac{1}{m^2\cdot s}$

We can write

$$u(x,y) = A\cdot x^3 - 3\cdot A\cdot x\cdot y^2 \qquad v(x,y) = A\cdot y^3 - 3\cdot A\cdot x^2\cdot y$$

Hence for a_x

$$a_x = u\cdot\frac{\partial}{\partial x}u + v\cdot\frac{\partial}{\partial y}u = \left(A\cdot x^3 - 3\cdot A\cdot x\cdot y^2\right)\cdot\frac{\partial}{\partial x}\left(A\cdot x^3 - 3\cdot A\cdot x\cdot y^2\right) + \left(A\cdot y^3 - 3\cdot A\cdot x^2\cdot y\right)\cdot\frac{\partial}{\partial y}\left(A\cdot x^3 - 3\cdot A\cdot x\cdot y^2\right)$$

$$a_x = 3\cdot A^2\cdot x\cdot\left(x^2 + y^2\right)^2$$

For a_y

$$a_y = u\cdot\frac{\partial}{\partial x}v + v\cdot\frac{\partial}{\partial y}v = \left(A\cdot x^3 - 3\cdot A\cdot x\cdot y^2\right)\cdot\frac{\partial}{\partial x}\left(A\cdot y^3 - 3\cdot A\cdot x^2\cdot y\right) + \left(A\cdot y^3 - 3\cdot A\cdot x^2\cdot y\right)\cdot\frac{\partial}{\partial y}\left(A\cdot y^3 - 3\cdot A\cdot x^2\cdot y\right)$$

$$a_y = 3\cdot A^2\cdot y\cdot\left(x^2 + y^2\right)^2$$

Hence at (2,1)

$$a_x = 3\cdot\left(\frac{0.2}{m^2\cdot s}\right)^2 \times 2\cdot m \times \left[(2\cdot m)^2 + (1\cdot m)^2\right]^2 \qquad a_x = 6.00\cdot\frac{m}{s^2}$$

$$a_y = 3\cdot\left(\frac{0.2}{m^2\cdot s}\right)^2 \times 1\cdot m \times \left[(2\cdot m)^2 + (1\cdot m)^2\right]^2 \qquad a_y = 3.00\cdot\frac{m}{s^2}$$

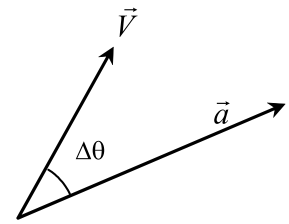
$$a = \sqrt{a_x^{\ 2} + a_y^{\ 2}} \qquad a = 6.71\frac{m}{s^2}$$

We need to find the component of acceleration normal to the velocity vector

At (2,1) the velocity vector is at angle

$$\theta_{vel} = \text{atan}\left(\frac{v}{u}\right) = \text{atan}\left(\frac{A \cdot y^3 - 3 \cdot A \cdot x^2 \cdot y}{A \cdot x^3 - 3 \cdot A \cdot x \cdot y^2}\right)$$

$$\theta_{vel} = \text{atan}\left(\frac{1^3 - 3 \cdot 2^2 \cdot 1}{2^3 - 3 \cdot 2 \cdot 1^2}\right) \qquad \theta_{vel} = -79.7 \cdot \text{deg}$$



At (1,2) the acceleration vector is at angle $\quad \theta_{accel} = \text{atan}\left(\frac{a_y}{a_x}\right) \qquad \theta_{accel} = \text{atan}\left(\frac{1}{2}\right) \qquad \theta_{accel} = 26.6 \cdot \text{deg}$

Hence the angle between the acceleration and velocity vectors is $\quad \Delta\theta = \theta_{accel} - \theta_{vel} \qquad \Delta\theta = 106 \cdot \text{deg}$

The component of acceleration normal to the velocity is then $\quad a_n = a \cdot \sin(\Delta\theta) = 6.71 \cdot \frac{m}{s^2} \cdot \sin(106 \cdot \text{deg}) \qquad a_n = 6.45 \cdot \frac{m}{s^2}$

Problem 6.26

6.26 The y component of velocity in a two-dimensional incompressible flow field is given by $v = -Axy$, where v is in m/s, the coordinates are measured in meters, and $A = 1\ \mathrm{m^{-1} \cdot s^{-1}}$. There is no velocity component or variation in the z direction. Calculate the acceleration of a fluid particle at point $(x, y) = (1, 2)$. Estimate the radius of curvature of the streamline passing through this point. Plot the streamline and show both the velocity vector and the acceleration vector on the plot. (Assume the simplest form of the x component of velocity.)

Solution:

For 2-D incompressible flow $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$, so $\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$.

$$u = \int \frac{\partial u}{\partial x}\,dx + f(y) = \int -\frac{\partial v}{\partial y}\,dx + f(y) = -\int(-Ax)\,dx + f(y) = \frac{Ax^2}{2} + f(y)$$

Choose the simplest solution, $f(y) = 0$, so $u = \frac{Ax^2}{2}$. Hence

$$\vec{V} = \frac{Ax^2}{2}\hat{i} - Axy\hat{j} = A\left(\frac{x^2}{2}\hat{i} - xy\hat{j}\right)$$

The acceleration of a fluid particle is

$$\vec{a}_p = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} = \frac{Ax^2}{2}(Ax\hat{i} - Ay\hat{j}) - Axy(-Ax\hat{j})$$

$$\vec{a}_p = \frac{A^2x^3}{2}\hat{i} + \frac{A^2x^2y}{2}\hat{j} = \frac{A^2}{2}(x^3\hat{i} + x^2y\hat{j})$$

At the point (1,2)

$$\vec{a}_p = \frac{1}{2}\times(1)\frac{1}{\mathrm{m^2 s^2}}\left[(1)^3\,\mathrm{m^3}\,\hat{i} + (1)^2(2)\,\mathrm{m^3}\,\hat{j}\right] = 0.5\hat{i} + \hat{j}\ \mathrm{m/s^2} \qquad \vec{a}(1,2)$$

$$\vec{V} = \frac{1}{\mathrm{m \cdot s}}\left[\frac{1}{2}(1)^2\mathrm{m^2}\,\hat{i} - (1)(2)\,\mathrm{m^2}\,\hat{j}\right] = 0.5\hat{i} - 2\hat{j}\ \mathrm{m/s}$$

The unit vector tangent to the streamline is

$$\hat{e}_t = \frac{\vec{V}}{|\vec{V}|} = \frac{0.5\hat{i} - 2\hat{j}}{[(0.5)^2 + (-2)^2]^{1/2}} = 0.243\hat{i} - 0.970\hat{j}$$

The unit vector normal to the streamline is

$$\vec{e}_n = \hat{e}_t \times \hat{k} = (0.243\hat{i} - 0.970\hat{j}) \times \hat{k} = -0.970\hat{i} - 0.243\hat{j}$$

The normal component of acceleration is

$$a_n = -\frac{V^2}{R} = \vec{a}\cdot\hat{e}_n = (0.5\hat{i} + \hat{j})\cdot(-0.970\hat{i} - 0.243\hat{j})$$

$$-\frac{V^2}{R} = -0.728\ \mathrm{m/s^2}$$

$$R = \frac{V^2}{0.728} = \frac{4.25\ \mathrm{m^2/s^2}}{0.728\ \mathrm{m/s^2}} = 5.84\ \mathrm{m} \qquad R(1,2)$$

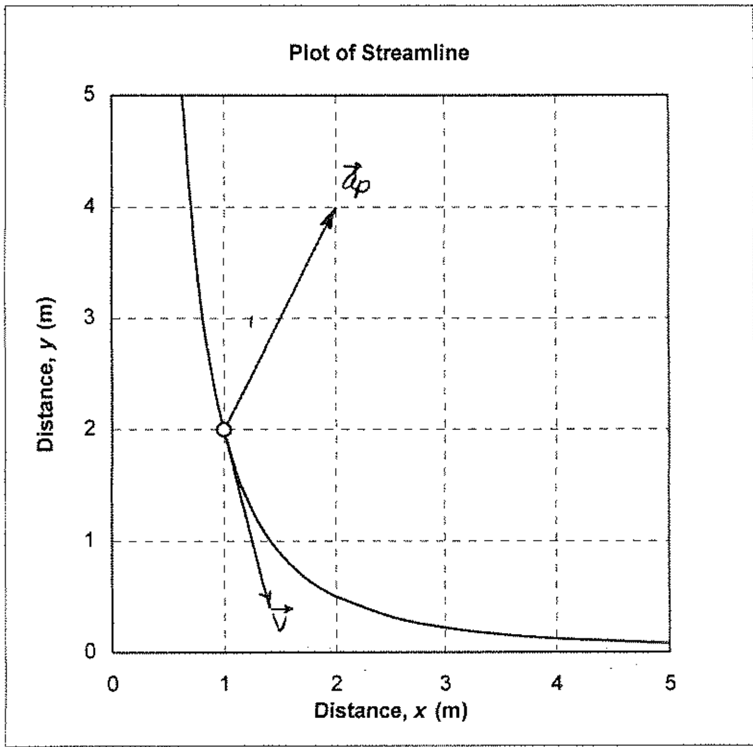
The slope of the streamlines is given by

$$\left.\frac{dy}{dx}\right)_{s\ell} = \frac{v}{u} = \frac{-Axy}{Ax^2/2} = -\frac{2y}{x}$$

Thus $\frac{dy}{y} + \frac{2dx}{x} = 0$ and $\ln y + \ln x^2 = \ln c$ or

$$x^2 y = c$$

The equation of the streamline through (1,2) is $x^2 y = 2$



Problem 6.27

[Difficulty: 5]

6.27 Consider the velocity field $\vec{V} = A[x^4 - 6x^2y^2 + y^4]\hat{i} + B[x^3y - xy^3]\hat{j}$; $A = 2\ \mathrm{m^{-3}\cdot s^{-1}}$, B is a constant, and the coordinates are measured in meters. Find B for this to be an incompressible flow. Obtain the equation of the streamline through point $(x, y) = (1, 2)$. Derive an algebraic expression for the acceleration of a fluid particle. Estimate the radius of curvature of the streamline at $(x, y) = (1, 2)$.

Given: Velocity field

Find: Constant B for incompressible flow; Equation for streamline through (1,2); Acceleration of particle; streamline curvature

Solution:

Basic equations
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$
$$\vec{a}_p = \underbrace{\frac{D\vec{V}}{Dt}}_{\substack{\text{total}\\\text{acceleration}\\\text{of a particle}}} = \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\substack{\text{convective}\\\text{acceleration}}} + \underbrace{\frac{\partial \vec{V}}{\partial t}}_{\substack{\text{local}\\\text{acceleration}}}$$

For this flow
$$u(x,y) = A\cdot\left(x^4 - 6\cdot x^2\cdot y^2 + y^4\right) \qquad v(x,y) = B\cdot\left(x^3\cdot y - x\cdot y^3\right)$$
$$\frac{\partial}{\partial x}u(x,y) + \frac{\partial}{\partial y}v(x,y) = \frac{\partial}{\partial x}\left[A\cdot\left(x^4 - 6\cdot x^2\cdot y^2 + y^4\right)\right] + \frac{\partial}{\partial y}\left[B\cdot\left(x^3\cdot y - x\cdot y^3\right)\right] = 0$$
$$\frac{\partial}{\partial x}u(x,y) + \frac{\partial}{\partial y}v(x,y) = B\cdot\left(x^3 - 3\cdot x\cdot y^2\right) + A\cdot\left(4\cdot x^3 - 12\cdot x\cdot y^2\right) = (4\cdot A + B)\cdot x\cdot\left(x^2 - 3\cdot y^2\right) = 0$$

Hence $\quad B = -4\cdot A \qquad\qquad B = -8\,\frac{1}{m^3\cdot s}$

Hence for a_x
$$a_x = u\cdot\frac{\partial}{\partial x}u + v\cdot\frac{\partial}{\partial y}u = A\cdot\left(x^4 - 6\cdot x^2\cdot y^2 + y^4\right)\cdot\frac{\partial}{\partial x}\left[A\cdot\left(x^4 - 6\cdot x^2\cdot y^2 + y^4\right)\right] + \left[-4\cdot A\cdot\left(x^3\cdot y - x\cdot y^3\right)\right]\cdot\frac{\partial}{\partial y}\left[A\cdot\left(x^4 - 6\cdot x^2\cdot y^2 + y^4\right)\right]$$
$$a_x = 4\cdot A^2\cdot x\cdot\left(x^2 + y^2\right)^3$$

For a_y
$$a_y = u\cdot\frac{\partial}{\partial x}v + v\cdot\frac{\partial}{\partial y}v = A\cdot\left(x^4 - 6\cdot x^2\cdot y^2 + y^4\right)\cdot\frac{\partial}{\partial x}\left[-4\cdot A\cdot\left(x^3\cdot y - x\cdot y^3\right)\right] + \left[-4\cdot A\cdot\left(x^3\cdot y - x\cdot y^3\right)\right]\cdot\frac{\partial}{\partial y}\left[-4\cdot A\cdot\left(x^3\cdot y - x\cdot y^3\right)\right]$$
$$a_y = 4\cdot A^2\cdot y\cdot\left(x^2 + y^2\right)^3$$

For a streamline $\quad \frac{dy}{dx} = \frac{v}{u} \quad$ so $\quad \frac{dy}{dx} = \frac{-4\cdot A\cdot\left(x^3\cdot y - x\cdot y^3\right)}{A\cdot\left(x^4 - 6\cdot x^2\cdot y^2 + y^4\right)} = -\frac{4\cdot\left(x^3\cdot y - x\cdot y^3\right)}{\left(x^4 - 6\cdot x^2\cdot y^2 + y^4\right)}$

Let $\quad u = \frac{y}{x} \qquad \frac{du}{dx} = \frac{d\left(\frac{y}{x}\right)}{dx} = \frac{1}{x}\cdot\frac{dy}{dx} + y\cdot\frac{d\left(\frac{1}{x}\right)}{dx} = \frac{1}{x}\cdot\frac{dy}{dx} - \frac{y}{x^2} \quad$ so $\quad \frac{dy}{dx} = x\cdot\frac{du}{dx} + u$

Hence $$\frac{dy}{dx} = x\cdot\frac{du}{dx} + u = -\frac{4\cdot\left(x^3\cdot y - x\cdot y^3\right)}{\left(x^4 - 6\cdot x^2\cdot y^2 + y^4\right)} = -\frac{4\cdot\left(1 - u^2\right)}{\left(\frac{1}{u} - 6\cdot u + \right.}u + \frac{4\cdot\left(1 - u^2\right)}{\left(\frac{1}{u} - 6\cdot u + u^3\right)}$$

$$x\cdot\frac{du}{dx} = -\left[u + \frac{4\cdot\left(1 - u^2\right)}{\left(\frac{1}{u} - 6\cdot u + u^3\right)}\right] = -\frac{u\cdot\left(u^4 - 10\cdot u^2 + 5\right)}{u^4 - 6\cdot u^2 + 1}$$

Separating variables $$\frac{dx}{x} = -\frac{u^4 - 6\cdot u^2 + 1}{u\cdot\left(u^4 - 10\cdot u^2 + 5\right)}\cdot du \qquad \ln(x) = -\frac{1}{5}\cdot\ln\left(u^5 - 10\cdot u^3 + 5\cdot u\right) + C$$

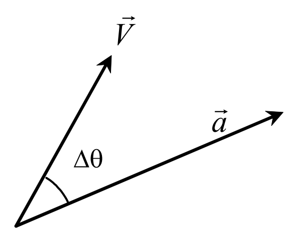
$$\left(u^5 - 10\cdot u^3 + 5\cdot u\right)\cdot x^5 = c \qquad y^5 - 10\cdot y^3\cdot x^2 + 5\cdot y\cdot x^4 = \text{const}$$

For the streamline through (1,2) $$y^5 - 10\cdot y^3\cdot x^2 + 5\cdot y\cdot x^4 = -38$$

Note that it would be MUCH easier to use the stream function method here!

To find the radius of curvature we use $a_n = -\frac{V^2}{R}$ or $|R| = \frac{V^2}{a_n}$

We need to find the component of acceleration normal to the velocity vector



At (1,2) the velocity vector is at angle $$\theta_{vel} = \text{atan}\left(\frac{v}{u}\right) = \text{atan}\left[-\frac{4\cdot\left(x^3\cdot y - x\cdot y^3\right)}{\left(x^4 - 6\cdot x^2\cdot y^2 + y^4\right)}\right]$$

$$\theta_{vel} = \text{atan}\left[-\frac{4\cdot(2-8)}{1 - 24 + 16}\right] \qquad \theta_{vel} = -73.7\cdot\text{deg}$$

At (1,2) the acceleration vector is at angle

$$\theta_{accel} = \text{atan}\left(\frac{a_y}{a_x}\right) = \text{atan}\left[\frac{4\cdot A^2\cdot y\cdot\left(x^2 + y^2\right)^3}{4\cdot A^2\cdot x\cdot\left(x^2 + y^2\right)^3}\right] = \text{atan}\left(\frac{y}{x}\right) \qquad \theta_{accel} = \text{atan}\left(\frac{2}{1}\right) \qquad \theta_{accel} = 63.4\cdot\text{deg}$$

Hence the angle between the acceleration and velocity vectors is $\Delta\theta = \theta_{accel} - \theta_{vel}$ $\quad \Delta\theta = 137\cdot\text{deg}$

The component of acceleration normal to the velocity is then $a_n = a\cdot\sin(\Delta\theta)$ where $a = \sqrt{a_x^2 + a_y^2}$

At (1,2) $$a_x = 4\cdot A^2\cdot x\cdot\left(x^2 + y^2\right)^3 = 500\cdot m^7 \times A^2 = 500\cdot m^7 \times \left(\frac{2}{m^3\cdot s}\right)^2 = 2000\cdot\frac{m}{s^2} \qquad a_y = 4\cdot A^2\cdot y\cdot\left(x^2 + y^2\right)^3 = 4000\cdot\frac{m}{s^2}$$

$$a = \sqrt{2000^2 + 4000^2}\cdot\frac{m}{s^2} \qquad a = 4472\frac{m}{s^2} \qquad a_n = a\cdot\sin(\Delta\theta) \qquad a_n = 3040\frac{m}{s^2}$$

$$u = A\cdot\left(x^4 - 6\cdot x^2\cdot y^2 + y^4\right) = -14\cdot\frac{m}{s} \qquad v = B\cdot\left(x^3\cdot y - x\cdot y^3\right) = 48\cdot\frac{m}{s} \qquad V = \sqrt{u^2 + v^2} = 50\cdot\frac{m}{s}$$

Then $$|R| = \frac{V^2}{a_n} \qquad R = \left(50\cdot\frac{m}{s}\right)^2 \times \frac{1}{3040}\cdot\frac{s^2}{m} \qquad R = 0.822\ m$$

Problem 6.28

[Difficulty: 2]

6.28 The velocity field for a plane doublet is given in Table 6.2. Find an expression for the pressure gradient at any point (r, θ).

Given: Velocity field for doublet

Find: Expression for pressure gradient

Solution:

Basic equations

$$\rho a_r = \rho\left(\frac{\partial V_r}{\partial t} + V_r\frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} + V_z\frac{\partial V_r}{\partial z} - \frac{V_\theta^2}{r}\right) = \rho g_r - \frac{\partial p}{\partial r}$$

$$\rho a_\theta = \rho\left(\frac{\partial V_\theta}{\partial t} + V_r\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + V_z\frac{\partial V_\theta}{\partial z} + \frac{V_r V_\theta}{r}\right) = \rho g_\theta - \frac{1}{r}\frac{\partial p}{\partial \theta}$$

For this flow $\quad V_r(r,\theta) = -\frac{\Lambda}{r^2}\cdot\cos(\theta) \qquad V_\theta(r,\theta) = -\frac{\Lambda}{r^2}\cdot\sin(\theta) \qquad V_z = 0$

Hence for r momentum $\quad \rho\cdot g_r - \frac{\partial}{\partial r}p = \rho\cdot\left(V_r\cdot\frac{\partial}{\partial r}V_r + \frac{V_\theta}{r}\cdot\frac{\partial}{\partial \theta}V_r - \frac{V_\theta^2}{r}\right)$

Ignoring gravity

$$\frac{\partial}{\partial r}p = -\rho\cdot\left[\left(-\frac{\Lambda}{r^2}\cdot\cos(\theta)\right)\cdot\frac{\partial}{\partial r}\left(-\frac{\Lambda}{r^2}\cdot\cos(\theta)\right) + \frac{\left(-\frac{\Lambda}{r^2}\cdot\sin(\theta)\right)}{r}\cdot\frac{\partial}{\partial \theta}\left(-\frac{\Lambda}{r^2}\cdot\cos(\theta)\right) - \frac{\left(-\frac{\Lambda}{r^2}\cdot\sin(\theta)\right)^2}{r}\right] \qquad \frac{\partial}{\partial r}p = \frac{2\cdot\Lambda^2\cdot\rho}{r^5}$$

For θ momentum $\quad \rho\cdot g_\theta - \frac{1}{r}\cdot\frac{\partial}{\partial \theta}p = \rho\cdot\left(V_r\cdot\frac{\partial}{\partial r}V_\theta + \frac{V_\theta}{r}\cdot\frac{\partial}{\partial \theta}V_\theta + \frac{V_r\cdot V_\theta}{r}\right)$

Ignoring gravity

$$\frac{\partial}{\partial \theta}p = -r\cdot\rho\cdot\left[\left(-\frac{\Lambda}{r^2}\cdot\cos(\theta)\right)\cdot\frac{\partial}{\partial r}\left(-\frac{\Lambda}{r^2}\cdot\sin(\theta)\right) + \frac{\left(-\frac{\Lambda}{r^2}\cdot\sin(\theta)\right)}{r}\cdot\frac{\partial}{\partial \theta}\left(-\frac{\Lambda}{r^2}\cdot\sin(\theta)\right) + \frac{\left(-\frac{\Lambda}{r^2}\cdot\sin(\theta)\right)\cdot\left(-\frac{\Lambda}{r^2}\cdot\cos(\theta)\right)}{r}\right] \qquad \frac{\partial}{\partial \theta}p = 0$$

The pressure gradient is purely radial

Problem 6.29

[Diffculty: 4]

6.29 Air flow over a stationary circular cylinder of radius a is modeled as a steady, frictionless, and incompressible flow from right to left, given by the velocity field

$$\vec{V} = U\left[\left(\frac{a}{r}\right)^2 - 1\right]\cos\theta\,\hat{e}_r + U\left[\left(\frac{a}{r}\right)^2 + 1\right]\sin\theta\,\hat{e}_\theta$$

Consider flow along the streamline forming the cylinder surface, $r = a$. Express the components of the pressure gradient in terms of angle θ. Obtain an expression for the variation of pressure (gage) on the surface of the cylinder. For $U = 75$ m/s and $a = 150$ mm, plot the pressure distribution (gage) and explain, and find the minimum pressure. Plot the speed V as a function of r along the radial line $\theta = \pi/2$ for $r > a$ (that is, directly above the cylinder), and explain.

Given: Velocity field for flow over a cylinder

Find: Expression for pressure gradient; pressure variation; minimum pressure; plot velocity

Solution:

Basic equations

$$\rho a_r = \rho\left(\frac{\partial V_r}{\partial t} + V_r\frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} + V_z\frac{\partial V_r}{\partial z} - \frac{V_\theta^2}{r}\right) = \rho g_r - \frac{\partial p}{\partial r}$$

$$\rho a_\theta = \rho\left(\frac{\partial V_\theta}{\partial t} + V_r\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + V_z\frac{\partial V_\theta}{\partial z} + \frac{V_r V_\theta}{r}\right) = \rho g_\theta - \frac{1}{r}\frac{\partial p}{\partial \theta}$$

Given data $\quad \rho = 1.23\cdot\frac{kg}{m^3} \qquad a = 150\cdot mm \qquad U = 75\cdot\frac{m}{s}$

For this flow $\quad V_r = U\cdot\left[\left(\frac{a}{r}\right)^2 - 1\right]\cdot\cos(\theta) \qquad V_\theta = U\cdot\left[\left(\frac{a}{r}\right)^2 + 1\right]\cdot\sin(\theta)$

On the surface r = a $\quad V_r = 0 \qquad V_\theta = 2\cdot U\cdot\sin(\theta)$

Hence on the surface:

For r momentum

$$\rho\cdot\left(V_r\cdot\frac{\partial}{\partial r}V_r + \frac{V_\theta}{r}\cdot\frac{\partial}{\partial\theta}V_r - \frac{V_\theta^2}{r}\right) = \rho\cdot\left(-\frac{V_\theta^2}{a}\right) = -\frac{\partial}{\partial r}p \qquad \frac{\partial}{\partial r}p = \rho\cdot\frac{V_\theta^2}{a} = \rho\cdot 4\cdot U^2\cdot\sin(\theta)^2$$

For θ momentum

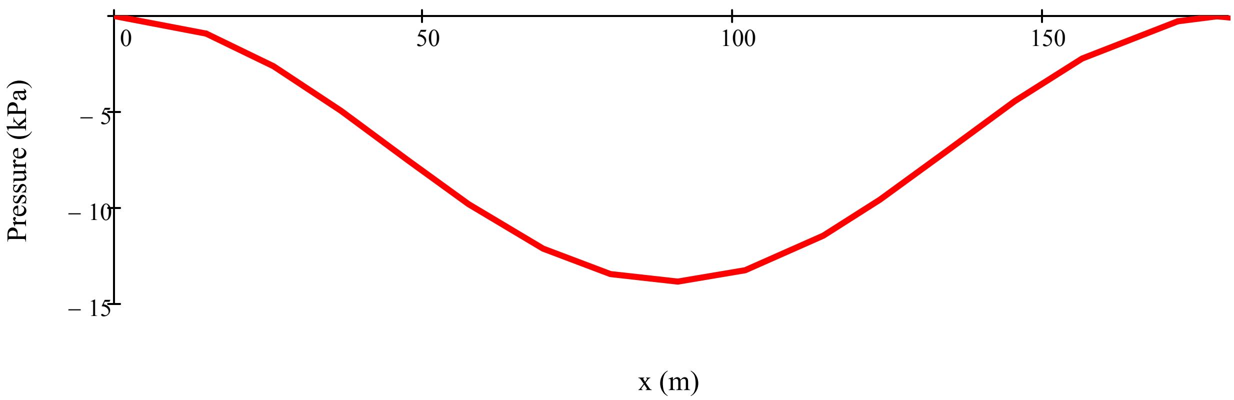
$$\rho\cdot\left(V_r\cdot\frac{\partial}{\partial r}V_\theta + \frac{V_\theta}{r}\cdot\frac{\partial}{\partial\theta}V_\theta + \frac{V_r\cdot V_\theta}{r}\right) = \rho\cdot\left(\frac{V_\theta}{a}\cdot\frac{\partial}{\partial\theta}V_\theta\right) = \rho\cdot\frac{2\cdot U\cdot\sin(\theta)}{a}\cdot 2\cdot U\cdot\cos(\theta) = -\frac{1}{r}\cdot\frac{\partial}{\partial\theta}p$$

$$\frac{\partial}{\partial\theta}p = -\frac{4\cdot\rho\cdot U^2}{a}\cdot\sin(\theta)\cdot\cos(\theta) = -\frac{2\cdot\rho\cdot U^2}{a}\cdot\sin(2\cdot\theta)$$

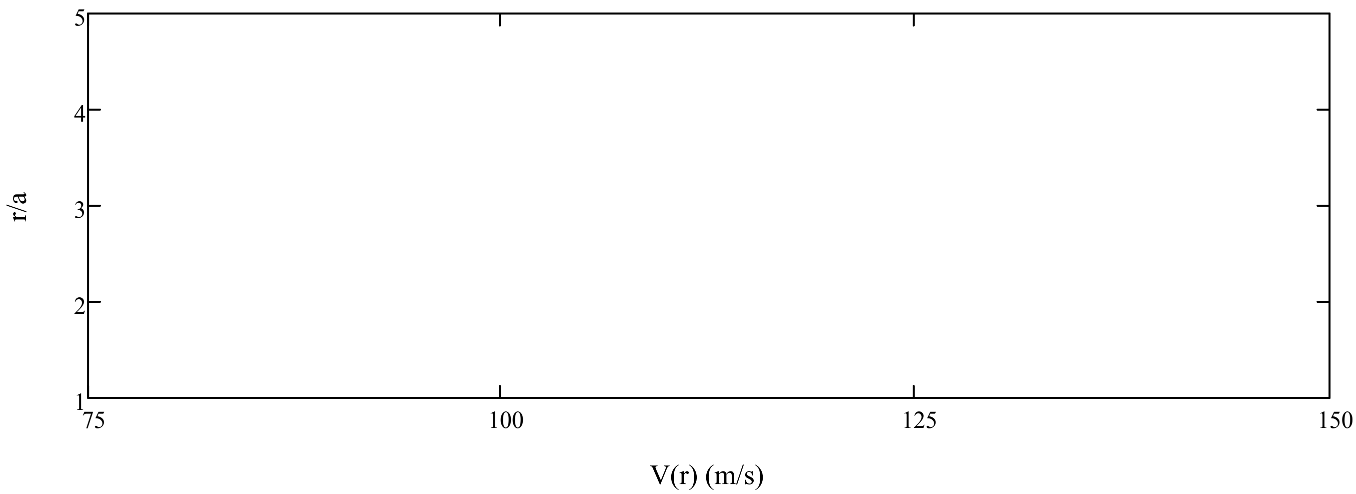
For the pressure distribution we integrate from θ = 0 to θ = θ, assuming $p(0) = p_{atm}$ (a stagnation point)

$$p(\theta) - p_{atm} = \int_0^\theta \frac{\partial}{\partial\theta}p\,d\theta = \int_0^\theta -\frac{4\cdot\rho\cdot U^2}{a}\cdot\sin(\theta)\cdot\cos(\theta)\,d\theta$$

$$p(\theta) = -4 \cdot \rho \cdot U^2 \int_0^\theta \sin(\theta) \cdot \cos(\theta)\, d\theta \qquad p(\theta) = -2 \cdot U^2 \cdot \rho \cdot \sin(\theta)^2 \qquad \text{Minimum p:} \qquad p\left(\frac{\pi}{2}\right) = -13.8 \cdot \text{kPa}$$



For the velocity as a function of radial position at $\theta = \pi/2$ $\qquad V_r = 0 \qquad$ so $\qquad V = V_\theta \qquad V_\theta(r) = U \cdot \left[\left(\frac{a}{r}\right)^2 + 1\right]$



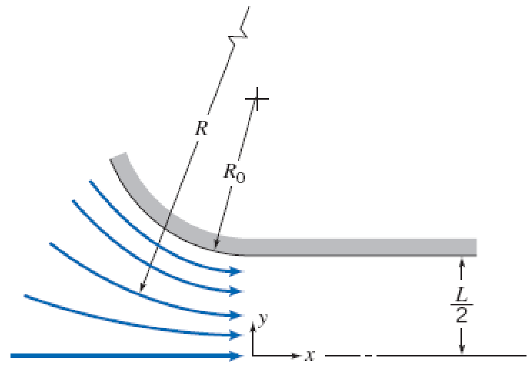
The velocity falls off to V = U as directly above the cylinder we have uniform horizontal as the effect of the cylinder decreases

$$V_\theta(100 \cdot a) = 75\,\frac{m}{s}$$

Problem 6.30

[Difficulty: 2]

6.30 To model the velocity distribution in the curved inlet section of a water channel, the radius of curvature of the streamlines is expressed as $R = LR_0/2y$. As an approximation, assume the water speed along each streamline is $V = 10$ m/s. Find an expression for and plot the pressure distribution from $y = 0$ to the tunnel wall at $y = L/2$, if the centerline pressure (gage) is 50 kPa, $L = 75$ mm, and $R_0 = 0.2$ m. Find the value of V for which the wall static pressure becomes 35 kPa.



Given: Flow in a curved section

Find: Expression for pressure distribution; plot; V for wall static pressure of 35 kPa

Solution:

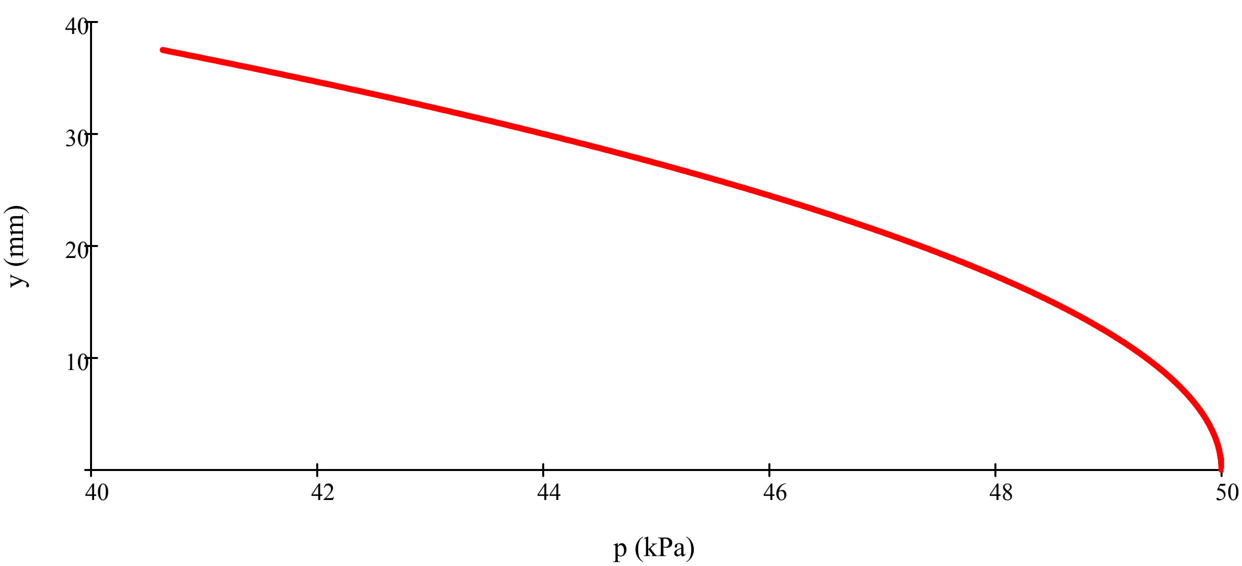
Basic equation $\quad \frac{\partial}{\partial n} p = \rho \cdot \frac{V^2}{R}$

Assumptions: Steady; frictionless; no body force; constant speed along streamline

Given data $\quad \rho = 999 \cdot \frac{kg}{m^3} \quad V = 10 \cdot \frac{m}{s} \quad L = 75 \cdot mm \quad R_0 = 0.2 \cdot m \quad p_c = 50 \cdot kPa$

At the inlet section $\quad p = p(y) \quad$ hence $\quad \frac{\partial}{\partial n} p = -\frac{dp}{dy} = \rho \cdot \frac{V^2}{R} = \rho \cdot V^2 \cdot \frac{2 \cdot y}{L \cdot R_0} \quad dp = -\rho \cdot V^2 \cdot \frac{2 \cdot y}{L \cdot R_0} \cdot dy$

Integrating from y = 0 to y = y $\quad p(y) = p_c - \frac{\rho \cdot V^2}{R_0 \cdot L} \cdot y^2 \quad (1) \quad p(0) = 50 \cdot kPa \quad p\left(\frac{L}{2}\right) = 40.6 \cdot kPa$

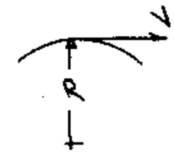


For a new wall pressure $\quad p_{wall} = 35 \cdot kPa \quad$ solving Eq 1 for V gives $\quad V = \sqrt{\frac{4 \cdot R_0 \cdot \left(p_c - p_{wall}\right)}{\rho \cdot L}} \quad V = 12.7 \frac{m}{s}$

Problem 6.31

[Difficulty: 2]

6.31 Air at 20 psia and 100°F flows around a smooth corner at the inlet to a diffuser. The air speed is 150 ft/s, and the radius of curvature of the streamlines is 3 in. Determine the magnitude of the centripetal acceleration experienced by a fluid particle rounding the corner. Express your answer in gs. Evaluate the pressure gradient, $\partial p/\partial r$.



Solution:

Basic equations: $\rho\vec{g} - \nabla p = \rho \frac{D\vec{V}}{Dt}$ ...... (1)

$\frac{D\vec{V}}{Dt} = \vec{a}_p$ ... (2) $\qquad p = \rho R T$ .... (3)

Assumptions: (1) ρ = constant
(2) frictionless flow
(3) $\vec{g} = -g\hat{k}$

Writing the r component of equation (1)

$$\cancel{g_r}^{0} - \frac{1}{\rho}\frac{\partial p}{\partial r} = a_r = \cancel{\frac{\partial V_r}{\partial t}}^{0} + \cancel{V_r}^{0}\frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\cancel{\frac{\partial V_r}{\partial \theta}}^{0} + \cancel{V_z}^{0}\frac{\partial V_r}{\partial z} - \frac{V_\theta^2}{r}$$

$$a_r = -\frac{V_\theta^2}{r} \qquad \frac{a_r}{g} = -\frac{V_\theta^2}{rg} = -(150)^2\frac{ft^2}{s^2} \times \frac{1}{3\,in} \times \frac{12\,in}{ft} \times \frac{s^2}{32.2\,ft}$$

$$\frac{a_r}{g} = -2800\ G's \qquad \frac{a_r}{g}$$

Also

$$\frac{\partial p}{\partial r} = \rho\frac{V_\theta^2}{r}$$

where $\rho = \frac{p}{RT} = 20\,\frac{lbf}{in^2} \times \frac{lbm\,°R}{53.3\,ft\text{-}lbf} \times \frac{1}{560°R} \times \frac{144\,in^2}{ft^2} \times \frac{slug}{32.2\,lbm}$

$$\rho = 0.003\ slug/ft^3$$

$$\frac{\partial p}{\partial r} = \rho\frac{V_\theta^2}{r} = 0.003\frac{slug}{ft^3} \times (150)^2\frac{ft^2}{s^2} \times \frac{1}{3\,in} \times \frac{12\,in}{ft} \times \frac{lbf\text{-}s^2}{ft\text{-}slug}$$

$$\frac{\partial p}{\partial r} = 270\ \frac{lbf/ft^2}{ft} \qquad \frac{\partial p}{\partial r}$$

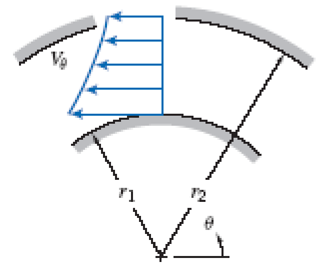
Problem 6.32

[Difficulty: 3]

6.32 Repeat Example 6.1, but with the somewhat more realistic assumption that the flow is similar to a free vortex (irrotational) profile, $V_\theta = c/r$ (where c is a constant), as shown in Fig. P6.32. In doing so, prove that the flow rate is given by $Q = k\sqrt{\Delta p}$, where k is

$$k = w \ln\left(\frac{r_2}{r_1}\right)\sqrt{\frac{2r_2^2 r_1^2}{\rho(r_2^2 - r_1^2)}}$$

and w is the depth of the bend.



Given: Velocity field for free vortex flow in elbow

Find: Similar solution to Example 6.1; find k (above)

Solution:

Basic equation $\quad \frac{\partial}{\partial r}p = \frac{\rho \cdot V^2}{r} \quad$ with $\quad V = V_\theta = \frac{c}{r}$

Assumptions: 1) Frictionless 2) Incompressible 3) free vortex

For this flow $\quad p \neq p(\theta) \quad$ so $\quad \frac{\partial}{\partial r}p = \frac{d}{dr}p = \frac{\rho \cdot V^2}{r} = \frac{\rho \cdot c^2}{r^3}$

Hence
$$\Delta p = p_2 - p_1 = \int_{r_1}^{r_2} \frac{\rho \cdot c^2}{r^3}\,dr = \frac{\rho \cdot c^2}{2}\cdot\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right) = \frac{\rho \cdot c^2 \cdot \left(r_2^2 - r_1^2\right)}{2 \cdot r_1^2 \cdot r_2^2} \qquad (1)$$

Next we obtain c in terms of Q

$$Q = \int \vec{V}\,d\vec{A} = \int_{r_1}^{r_2} V \cdot w\,dr = \int_{r_1}^{r_2} \frac{w \cdot c}{r}\,dr = w \cdot c \cdot \ln\left(\frac{r_2}{r_1}\right)$$

Hence
$$c = \frac{Q}{w \cdot \ln\left(\frac{r_2}{r_1}\right)}$$

Using this in Eq 1
$$\Delta p = p_2 - p_1 = \frac{\rho \cdot c^2 \cdot \left(r_2^2 - r_1^2\right)}{2 \cdot r_1^2 \cdot r_2^2} = \frac{\rho \cdot Q^2 \cdot \left(r_2^2 - r_1^2\right)}{2 \cdot w^2 \cdot \ln\left(\frac{r_2}{r_1}\right)^2 \cdot r_1^2 \cdot r_2^2}$$

Solving for Q
$$Q = w \cdot \ln\left(\frac{r_2}{r_1}\right)\cdot\sqrt{\frac{2 \cdot r_1^2 \cdot r_2^2}{\rho \cdot \left(r_2^2 - r_1^2\right)}}\cdot\sqrt{\Delta p} \qquad k = w \cdot \ln\left(\frac{r_2}{r_1}\right)\cdot\sqrt{\frac{2 \cdot r_1^2 \cdot r_2^2}{\rho \cdot \left(r_2^2 - r_1^2\right)}}$$

6.33 The velocity field in a two-dimensional, steady, inviscid flow field in the horizontal xy plane is given by $\vec{V} = (Ax + B)\hat{i} - Ay\hat{j}$, where $A = 1\ \text{s}^{-1}$ and $B = 2$ m/s; x and y are measured in meters. Show that streamlines for this flow are given by $(x + B/A)y = \text{constant}$. Plot streamlines passing through points $(x, y) = (1, 1)$, $(1, 2)$, and $(2, 2)$. At point $(x, y) = (1, 2)$, evaluate and plot the acceleration vector and the velocity vector. Find the component of acceleration along the streamline at the same point; express it as a vector. Evaluate the pressure gradient along the streamline at the same point if the fluid is air. What statement, if any, can you make about the relative value of the pressure at points (1, 1) and (2, 2)?

Solution:

The slope of a streamline is $\left.\frac{dy}{dx}\right)_{s.l} = \frac{v}{u} = \frac{-Ay}{Ax+B} = \frac{-y}{x+B/A}$

Then $\frac{dy}{y} + \frac{dx}{x+B/A} = 0$ and $\ln y + \ln(x + B/A) = \ln c$.

and $(x + B/A)y = \text{constant}$ ← Streamlines

For (1,1) $(x+2)y = 3$
(1,2) $(x+2)y = 6$
(2,2). $(x+2)y = 8$

These streamlines are shown in the plot at the end of the problem solution

The particle acceleration $\vec{a}_p = \frac{D\vec{V}}{Dt} = \cancel{\frac{\partial \vec{V}}{\partial t}}^{0(1)} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + \cancel{w\frac{\partial \vec{V}}{\partial z}}^{0(2)}$.

Assumptions: (1) steady flow (given)
(2) 2-D (given) $\vec{V} \neq \vec{V}(z)$.

$\vec{a}_p = (Ax+B)\frac{\partial}{\partial x}\left[(Ax+B)\hat{i} - Ay\hat{j}\right] - Ay\frac{\partial}{\partial y}\left[(Ax+B)\hat{i} - Ay\hat{j}\right]$

$\vec{a}_p = (Ax+B)A\hat{i} - Ay(-A\hat{j}) = A(Ax+B)\hat{i} + A^2y\hat{j}$

At point (1,2).

$\vec{a}_p = \frac{1}{s}\left(\frac{1}{s} \times 1\text{m} + 2\frac{\text{m}}{s}\right)\hat{i} + \frac{1}{s^2} \times 2\text{m}\,\hat{j} = 3\hat{i} + 2\hat{j}\ \text{m/s}^2$ ← $\vec{a}_{(1,2)}$

$\vec{V} = \left(\frac{1}{s} \times 1\text{m} + 2\frac{\text{m}}{s}\right)\hat{i} - \frac{1}{s}\,2\text{m}\,\hat{j} = 3\hat{i} - 2\hat{j}\ \text{m/s}$ ← $\vec{V}_{(1,2)}$

$\vec{V}$ and $\vec{a}$ are shown on the streamline plot.

(b) The component of $\vec{a}_p$ along (tangent to) the streamline is given by $a_t = \vec{a}_p \cdot \hat{e}_t$ where $\hat{e}_t = \frac{\vec{V}}{|\vec{V}|}$

Thus $\hat{e}_t = \frac{3\hat{i} - 2\hat{j}}{\left[3^2 + (-2)^2\right]^{1/2}} = 0.832\hat{i} - 0.555\hat{j}$

and

$a_t = \vec{a}_p \cdot \hat{e}_t = (3\hat{i}+2\hat{j})\,m/s^2 \cdot (0.832\hat{i} - 0.555\hat{j}) = 1.39\ m/s^2$

$\vec{a}_t = 1.39\ \hat{e}_t = 1.16\hat{i} - 0.771\hat{j}\ m/s^2$ $\quad \vec{a}_t(1,2)$

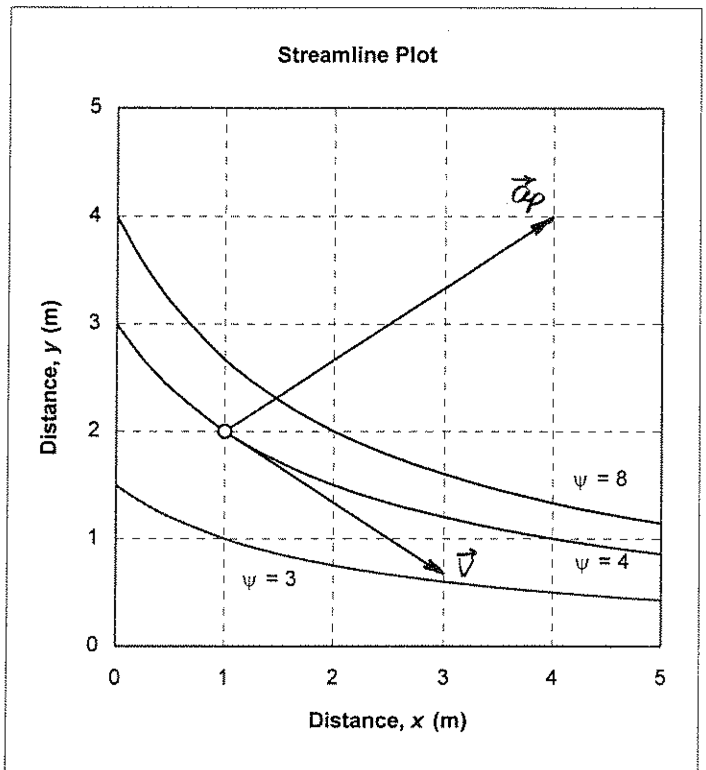
For frictionless flow, Euler's equation along a streamline (neglecting gravity, i.e. assuming flow in horizontal plane) is

$$\frac{\partial p}{\partial s} = -\rho V \frac{\partial V}{\partial s} = -\rho a_t = -1.23\ \frac{kg}{m^3} \times 1.39\ \frac{m}{s^2} \times \frac{N s^2}{kg \cdot m}$$

$$\frac{\partial p}{\partial s} = -1.71\ N/m^2/m \quad \frac{\partial p}{\partial s}(1,2)$$

Looking at the streamline we would expect $p(2,2)$ to be less than $p(1,1)$ due to streamline curvature; Euler's equation normal to a streamline says

$$\partial p / \partial n = \frac{\rho V^2}{R}$$



Problem 6.34

[Difficulty: 2]

6.34 Using the analyses of Example 6.1 and Problem 6.32, plot the discrepancy (percent) between the flow rates obtained from assuming uniform flow and the free vortex (irrotational) profile as a function of inner radius r_1.

Given: Flow rates in elbow for uniform flow and free vortes

Find: Plot discrepancy

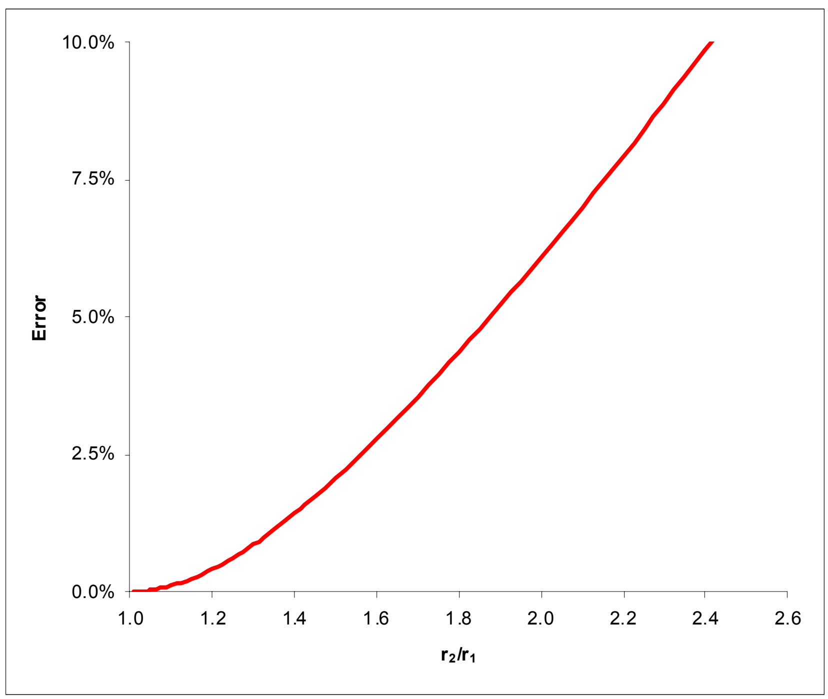
Solution:

For Example 6.1 $\quad Q_{Uniform} = V \cdot A = w \cdot (r_2 - r_1) \cdot \sqrt{\dfrac{1}{\rho \cdot \ln\left(\dfrac{r_2}{r_1}\right)}} \cdot \sqrt{\Delta p}$ $\quad$ or $\quad \dfrac{Q_{Uniform} \cdot \sqrt{\rho}}{w \cdot r_1 \cdot \sqrt{\Delta p}} = \dfrac{\left(\dfrac{r_2}{r_1} - 1\right)}{\sqrt{\ln\left(\dfrac{r_2}{r_1}\right)}}$ $\quad$ (1)

For Problem 6.32 $\quad Q = w \cdot \ln\left(\dfrac{r_2}{r_1}\right) \cdot \sqrt{\dfrac{2 \cdot r_1^{\,2} \cdot r_2^{\,2}}{\rho \cdot \left(r_2^{\,2} - r_1^{\,2}\right)}} \cdot \sqrt{\Delta p}$ $\quad$ or $\quad \dfrac{Q \cdot \sqrt{\rho}}{w \cdot r_1 \cdot \sqrt{\Delta p}} = \left(\dfrac{r_2}{r_1}\right) \cdot \ln\left(\dfrac{r_2}{r_1}\right) \cdot \sqrt{\dfrac{2}{\left[\left(\dfrac{r_2}{r_1}\right)^2 - 1\right]}}$ $\quad$ (2)

It is convenient to plot these as functions of r_2/r_1

r_2/r_1	Eq. 1	Eq. 2	Error
1.01	0.100	0.100	0.0%
1.05	0.226	0.226	0.0%
1.10	0.324	0.324	0.1%
1.15	0.401	0.400	0.2%
1.20	0.468	0.466	0.4%
1.25	0.529	0.526	0.6%
1.30	0.586	0.581	0.9%
1.35	0.639	0.632	1.1%
1.40	0.690	0.680	1.4%
1.45	0.738	0.726	1.7%
1.50	0.785	0.769	2.1%
1.55	0.831	0.811	2.4%
1.60	0.875	0.851	2.8%
1.65	0.919	0.890	3.2%
1.70	0.961	0.928	3.6%
1.75	1.003	0.964	4.0%
1.80	1.043	1.000	4.4%
1.85	1.084	1.034	4.8%
1.90	1.123	1.068	5.2%
1.95	1.162	1.100	5.7%
2.00	1.201	1.132	6.1%
2.05	1.239	1.163	6.6%
2.10	1.277	1.193	7.0%
2.15	1.314	1.223	7.5%
2.20	1.351	1.252	8.0%
2.25	1.388	1.280	8.4%
2.30	1.424	1.308	8.9%
2.35	1.460	1.335	9.4%
2.40	1.496	1.362	9.9%
2.45	1.532	1.388	10.3%
2.50	1.567	1.414	10.8%



Problem 6.35

6.35 The x component of velocity in a two-dimensional, incompressible flow field is given by $u = Ax^2$; the coordinates are measured in feet and $A = 1\ \text{ft}^{-1}\cdot\text{s}^{-1}$. There is no velocity component or variation in the z direction. Calculate the acceleration of a fluid particle at point $(x, y) = (1, 2)$. Estimate the radius of curvature of the streamline passing through this point. Plot the streamline and show both the velocity vector and the acceleration vector on the plot. (Assume the simplest form of the y component of velocity.)

Solution:

For 2-D incompressible flow $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$, so $\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}$

$$v = \int \frac{\partial v}{\partial y}dy + f(x) = \int -\frac{\partial u}{\partial x}dy + f(x) = -\int 2Ax\,dy + f(x) = -2Axy + f(x).$$

Choose the simplest solution, $f(x) = 0$, so $v = -2Axy$. Hence

$$\vec{V} = Ax^2\hat{i} - 2Axy\hat{j} = A\left[x^2\hat{i} - 2xy\hat{j}\right]$$

The acceleration of a fluid particle is

$$\vec{a}_p = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} = Ax^2\left[A(2x\hat{i} - 2y\hat{j})\right] - 2Axy\left[-2Ax\hat{j}\right]$$

$$\vec{a}_p = 2A^2x^3\hat{i} + 2A^2x^2y\hat{j} = 2A^2x^2\left[x\hat{i} + y\hat{j}\right]$$

At the point (1,2)

$$\vec{a}_p = 2\times\frac{(1)^2}{\text{ft}^2\text{s}^2}\times(1)^2\text{m}^2\left[1\text{m}\,\hat{i} + 2\text{m}\,\hat{j}\right] = 2\hat{i} + 4\hat{j}\ \text{ft/s}^2 \qquad \vec{a}_{(1,2)}$$

$$\vec{V} = \frac{1}{\text{ft}\cdot\text{s}}\left[(1)^2\text{m}^2\hat{i} - 2(1\text{m})(2\text{m})\hat{j}\right] = \hat{i} - 4\hat{j}\ \text{ft/s}$$

The unit vector tangent to the streamline is

$$\hat{e}_t = \frac{\vec{V}}{|\vec{V}|} = \frac{\hat{i} - 4\hat{j}}{\left[(1)^2 + (-4)^2\right]^{1/2}} = 0.243\hat{i} - 0.970\hat{j}$$

The unit vector normal to the streamline is

$$\hat{e}_n = \hat{e}_t \times \hat{k} = (0.243\hat{i} - 0.970\hat{j})\times\hat{k} = -0.970\hat{i} - 0.243\hat{j}$$

The normal component of acceleration is

$$a_n = -\frac{V^2}{R} = \vec{a}\cdot\hat{e}_n = (2\hat{i} + 4\hat{j})\cdot(-0.970\hat{i} - 0.243\hat{j})$$

$$-\frac{V^2}{R} = -2.91\ \text{ft/s}^2$$

$$R = \frac{V^2}{2.91} = \frac{17\ \text{ft}^2/\text{s}^2}{2.91\ \text{ft/s}^2} = 5.84\ \text{ft} \qquad R$$

The slope of the streamline is given by

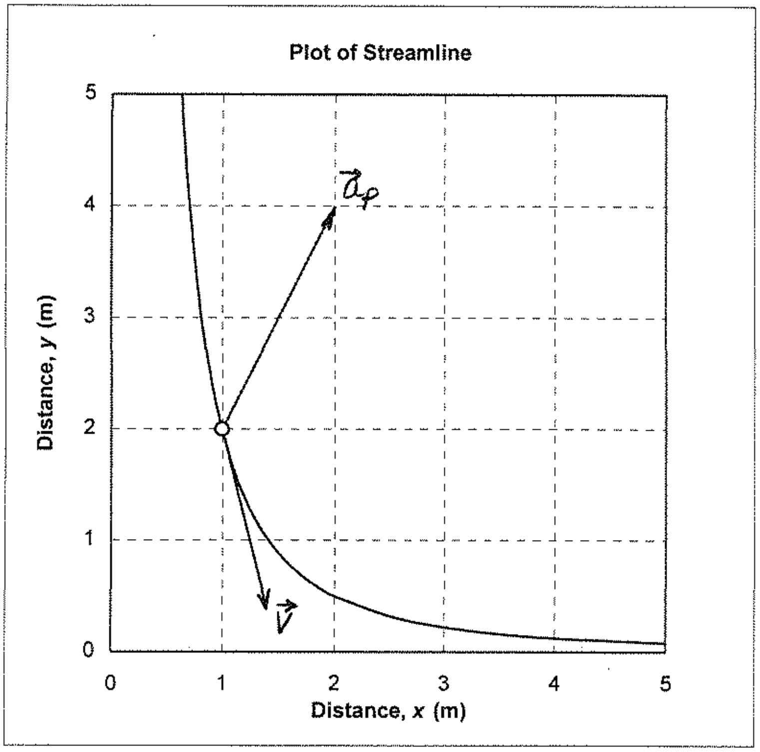
$$\left.\frac{dy}{dx}\right)_{s.l} = \frac{v}{u} = \frac{-2Axy}{Ax^2} = \frac{-2y}{x}$$

Problem 6.35

Thus $\frac{dy}{y} + \frac{2\,dx}{x} = 0$ and $\ln y + \ln x^2 = \ln c$ or

$$x^2 y = c$$

The equation of the streamline through (1,2) is $x^2 y = 2$.



Problem 6.36

[Difficulty: 4]

6.36 The x component of velocity in a two-dimensional incompressible flow field is given by $u = -\Lambda(x^2 - y^2)/(x^2 + y^2)^2$, where u is in m/s, the coordinates are measured in meters, and $\Lambda = 2\ \mathrm{m^3 \cdot s^{-1}}$. Show that the simplest form of the y component of velocity is given by $v = -2\Lambda xy/(x^2 + y^2)^2$. There is no velocity component or variation in the z direction. Calculate the acceleration of fluid particles at points $(x, y) = (0, 1)$, $(0, 2)$, and $(0, 3)$. Estimate the radius of curvature of the streamlines passing through these points. What does the relation among the three points and their radii of curvature suggest to you about the flow field? Verify this by plotting these streamlines. [*Hint*: You will need to use an integrating factor.]

Given: x component of velocity field

Find: y component of velocity field; acceleration at several points; estimate radius of curvature; plot streamlines

Solution:

The given data is $\quad \Lambda = 2 \cdot \dfrac{m^3}{s} \qquad u = -\dfrac{\Lambda \cdot (x^2 - y^2)}{(x^2 + y^2)^2}$

The basic equation (continuity) is $\quad \dfrac{\partial}{\partial x}u + \dfrac{\partial}{\partial y}v = 0$

The basic equation for acceleration is

$$\vec{a}_p = \underbrace{\frac{D\vec{V}}{Dt}}_{\substack{\text{total}\\\text{acceleration}\\\text{of a particle}}} = \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\substack{\text{convective}\\\text{acceleration}}} + \underbrace{\frac{\partial \vec{V}}{\partial t}}_{\substack{\text{local}\\\text{acceleration}}}$$

Hence $\quad v = -\displaystyle\int \frac{du}{dx}\,dy = -\int \frac{2\cdot\Lambda\cdot x\cdot(x^2 - 3\cdot y^2)}{(x^2 + y^2)^3}\,dy$

Integrating (using an integrating factor) $\quad v = -\dfrac{2\cdot\Lambda\cdot x\cdot y}{(x^2 + y^2)^2}$

Alternatively, we could check that the given velocities u and v satisfy continuity

$$u = -\frac{\Lambda\cdot(x^2 - y^2)}{(x^2 + y^2)^2} \qquad \frac{\partial}{\partial x}u = \frac{2\cdot\Lambda\cdot x\cdot(x^2 - 3\cdot y^2)}{(x^2 + y^2)^3} \qquad v = -\frac{2\cdot\Lambda\cdot x\cdot y}{(x^2 + y^2)^2} \qquad \frac{\partial}{\partial y}v = -\frac{2\cdot\Lambda\cdot x\cdot(x^2 - 3\cdot y^2)}{(x^2 + y^2)^3}$$

so $\quad \dfrac{\partial}{\partial x}u + \dfrac{\partial}{\partial y}v = 0$

For steady, 2D flow the acceleration components reduce to (after considerable math!):

x - component

$$a_x = u\cdot\frac{\partial}{\partial x}u + v\cdot\frac{\partial}{\partial y}u$$

$$a_x = \left[-\frac{\Lambda\cdot\left(x^2 - y^2\right)}{\left(x^2 + y^2\right)^2}\right]\cdot\left[\frac{2\cdot\Lambda\cdot x\cdot\left(x^2 - 3\cdot y^2\right)}{\left(x^2 + y^2\right)^3}\right] + \left[-\frac{2\cdot\Lambda\cdot x\cdot y}{\left(x^2 + y^2\right)^2}\right]\cdot\left[\frac{2\cdot\Lambda\cdot y\cdot\left(3\cdot x^2 - y^2\right)}{\left(x^2 + y^2\right)^3}\right] \qquad a_x = -\frac{2\cdot\Lambda^2\cdot x}{\left(x^2 + y^2\right)^3}$$

y - component

$$a_y = u\cdot\frac{\partial}{\partial x}v + v\cdot\frac{\partial}{\partial y}v$$

$$a_y = \left[-\frac{\Lambda\cdot\left(x^2 - y^2\right)}{\left(x^2 + y^2\right)^2}\right]\cdot\left[\frac{2\cdot\Lambda\cdot y\cdot\left(3\cdot x^2 - y^2\right)}{\left(x^2 + y^2\right)^3}\right] + \left[-\frac{2\cdot\Lambda\cdot x\cdot y}{\left(x^2 + y^2\right)^2}\right]\cdot\left[\frac{2\cdot\Lambda\cdot y\cdot\left(3\cdot y^2 - x^2\right)}{\left(x^2 + y^2\right)^3}\right] \qquad a_y = -\frac{2\cdot\Lambda^2\cdot y}{\left(x^2 + y^2\right)^3}$$

Evaluating at point (0,1) $\quad u = 2\cdot\frac{m}{s} \quad v = 0\cdot\frac{m}{s} \quad a_x = 0\cdot\frac{m}{s^2} \quad a_y = -8\cdot\frac{m}{s^2}$

Evaluating at point (0,2) $\quad u = 0.5\cdot\frac{m}{s} \quad v = 0\cdot\frac{m}{s} \quad a_x = 0\cdot\frac{m}{s^2} \quad a_y = -0.25\cdot\frac{m}{s^2}$

Evaluating at point (0,3) $\quad u = 0.222\cdot\frac{m}{s} \quad v = 0\cdot\frac{m}{s} \quad a_x = 0\cdot\frac{m}{s^2} \quad a_y = -0.0333\cdot\frac{m}{s^2}$

The instantaneous radius of curvature is obtained from $\quad a_{radial} = -a_y = -\frac{u^2}{r} \quad$ or $\quad r = -\frac{u^2}{a_y}$

For the three points $\quad y = 1\,m \quad r = \frac{\left(2\cdot\frac{m}{s}\right)^2}{8\cdot\frac{m}{s^2}} \quad r = 0.5\,m$

$y = 2\,m \quad r = \frac{\left(0.5\cdot\frac{m}{s}\right)^2}{0.25\cdot\frac{m}{s^2}} \quad r = 1\,m$

$y = 3\,m \quad r = \frac{\left(0.2222\cdot\frac{m}{s}\right)^2}{0.03333\cdot\frac{m}{s^2}} \quad r = 1.5\cdot m$

The radius of curvature in each case is 1/2 of the vertical distance from the origin. The streamlines form circles tangent to the *x* axis

The streamlines are given by
$$\frac{dy}{dx} = \frac{v}{u} = \frac{-\dfrac{2\cdot\Lambda\cdot x\cdot y}{\left(x^2+y^2\right)^2}}{-\dfrac{\Lambda\cdot\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2}} = \frac{2\cdot x\cdot y}{\left(x^2-y^2\right)}$$

so
$$-2\cdot x\cdot y\cdot dx + \left(x^2-y^2\right)\cdot dy = 0$$

This is an inexact integral, so an integrating factor is needed

First we try
$$R = \frac{1}{-2\cdot x\cdot y}\cdot\left[\frac{d}{dx}\left(x^2-y^2\right) - \frac{d}{dy}(-2\cdot x\cdot y)\right] = -\frac{2}{y}$$

Then the integrating factor is
$$F = e^{\int -\frac{2}{y}\,dy} = \frac{1}{y^2}$$

The equation becomes an exact integral
$$-2\cdot\frac{x}{y}\cdot dx + \frac{\left(x^2-y^2\right)}{y^2}\cdot dy = 0$$

So
$$u = \int -2\cdot\frac{x}{y}\,dx = -\frac{x^2}{y} + f(y) \quad \text{and} \quad u = \int \frac{\left(x^2-y^2\right)}{y^2}\,dy = -\frac{x^2}{y} - y + g(x)$$

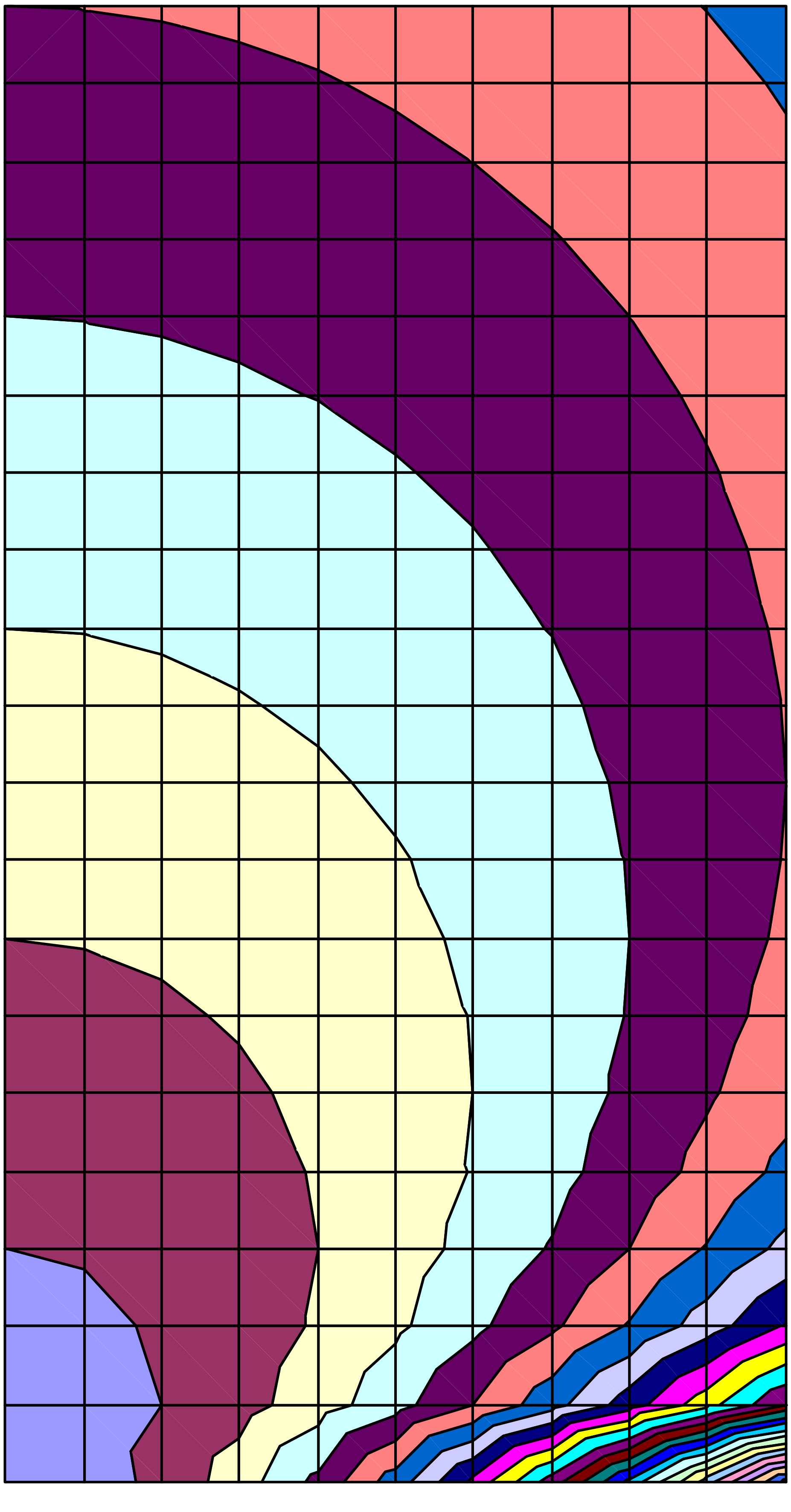
Comparing solutions
$$\psi = \frac{x^2}{y} + y \qquad (1) \qquad \text{or} \qquad x^2 + y^2 = \psi\cdot y = \mathrm{const}\cdot y$$

These form circles that are tangential to the *x* axis, as can be shown in *Excel*:

The stream function can be evaluated using Eq 1

	y values																				
x values	**0.10**	**0.25**	**0.50**	**0.75**	**1.00**	**1.25**	**1.50**	**1.75**	**2.00**	**2.25**	**2.50**	**2.75**	**3.00**	**3.25**	**3.50**	**3.75**	**4.00**	**4.25**	**4.50**	**4.75**	**5.00**
2.50	62.6	25.3	13.0	9.08	7.25	6.25	5.67	5.32	5.13	5.03	5.00	5.02	5.08	5.17	5.29	5.42	5.56	5.72	5.89	6.07	6.25
2.25	50.7	20.5	10.6	7.50	6.06	5.30	4.88	4.64	4.53	4.50	4.53	4.59	4.69	4.81	4.95	5.10	5.27	5.44	5.63	5.82	6.01
2.00	40.1	16.3	8.50	6.08	5.00	4.45	4.17	4.04	4.00	4.03	4.10	4.20	4.33	4.48	4.64	4.82	5.00	5.19	5.39	5.59	5.80
1.75	30.7	12.5	6.63	4.83	4.06	3.70	3.54	3.50	3.53	3.61	3.73	3.86	4.02	4.19	4.38	4.57	4.77	4.97	5.18	5.39	5.61
1.50	22.6	9.25	5.00	3.75	3.25	3.05	3.00	3.04	3.13	3.25	3.40	3.57	3.75	3.94	4.14	4.35	4.56	4.78	5.00	5.22	5.45
1.25	15.7	6.50	3.63	2.83	2.56	2.50	2.54	2.64	2.78	2.94	3.13	3.32	3.52	3.73	3.95	4.17	4.39	4.62	4.85	5.08	5.31
1.00	10.1	4.25	2.50	2.08	2.00	2.05	2.17	2.32	2.50	2.69	2.90	3.11	3.33	3.56	3.79	4.02	4.25	4.49	4.72	4.96	5.20
0.75	5.73	2.50	1.63	1.50	1.56	1.70	1.88	2.07	2.28	2.50	2.73	2.95	3.19	3.42	3.66	3.90	4.14	4.38	4.63	4.87	5.11
0.50	2.60	1.25	1.00	1.08	1.25	1.45	1.67	1.89	2.13	2.36	2.60	2.84	3.08	3.33	3.57	3.82	4.06	4.31	4.56	4.80	5.05
0.25	0.73	0.50	0.63	0.83	1.06	1.30	1.54	1.79	2.03	2.28	2.53	2.77	3.02	3.27	3.52	3.77	4.02	4.26	4.51	4.76	5.01
0.00	0.10	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.25	2.50	2.75	3.00	3.25	3.50	3.75	4.00	4.25	4.50	4.75	5.00

See next page for plot:



6.37 The x component of velocity in a two-dimensional, incompressible flow field is given by $u = Axy$; the coordinates are measured in meters and $A = 2\ \text{m}^{-1}\cdot\text{s}^{-1}$. There is no velocity component or variation in the z direction. Calculate the acceleration of a fluid particle at point $(x, y) = (2, 1)$. Estimate the radius of curvature of the streamline passing through this point. Plot the streamline and show both the velocity vector and the acceleration vector on the plot. (Assume the simplest form of the y component of velocity.)

Solution: For two-d. incompressible flow, $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$, so

$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -Ay$; Integrating, $v = -\frac{1}{2}Ay^2$; $\vec{V} = Axy\hat{\imath} - \frac{1}{2}Ay^2\hat{\jmath}$.

The acceleration is

$$a_{p_x} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = (Axy)(Ay) + (-\tfrac{1}{2}Ay^2)(Ax) = \tfrac{1}{2}A^2xy^2$$

$$a_{p_y} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = (Axy)(0) + (-\tfrac{1}{2}Ay^2)(-Ay) = \tfrac{1}{2}A^2y^3$$

$$\vec{a}_p = \tfrac{1}{2}A^2xy^2\hat{\imath} + \tfrac{1}{2}A^2y^3\hat{\jmath};\ \text{at } (2,1)\ \ \vec{a}_p = 4\hat{\imath} + 2\hat{\jmath}\ (\text{ft/s}^2)$$ $\vec{a}_p$

Note $a_n = \frac{V^2}{R}$, so $R = \frac{V^2}{a_n}$, where a_n is acceleration normal to $\vec{V}$

At $(2,1)$, $\vec{V} = 4\hat{\imath} - 1\hat{\jmath}$ ft/s, so $V^2 = (4)^2 + (1)^2 = 17\ \text{ft}^2/\text{s}^2$

To find a_n, dot $\vec{a}_p$ with $\hat{e}_n$, the unit normal vector. To find $\hat{e}_n$, set

$$\hat{e}_n = -\frac{v}{V}\hat{\imath} + \frac{u}{V}\hat{\jmath} = \frac{1}{\sqrt{17}}\hat{\imath} + \frac{4}{\sqrt{17}}\hat{\jmath}$$

$$a_n = \hat{e}_n \cdot \hat{a}_p = \frac{4}{\sqrt{17}} + \frac{8}{\sqrt{17}} = \frac{12}{\sqrt{17}} = 2.91\ \text{ft/s}^2$$

Substituting

$$R = \frac{V^2}{a_n} = 17\frac{\text{ft}^2}{\text{s}^2} \times \frac{\text{s}^2}{2.91\ \text{ft}} = 5.84\ \text{ft}$$ R

The streamline is $\frac{dx}{u} = \frac{dy}{v} = \frac{dx}{Axy} = \frac{dy}{-\frac{1}{2}Ay^2}$ or $\frac{dx}{x} + 2\frac{dy}{y} = 0$

Integrating, $\ln x + 2\ln y = \ln c$ or $xy^2 = c$

For $(x,y) = (2,1)$, then $c = 2\ \text{ft}^3$.

The plot and streamlines are on the following page.

Components of Velocity and Acceleration:

Input Parameters:

$A =$ 2 $ft^{-1}s^{-1}$

Calculated Values:

$C =$ 2 ft^3

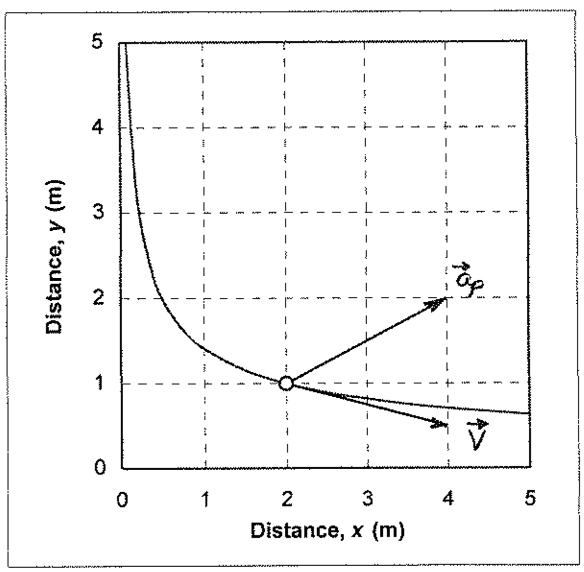
Coord. x	Coord. y	Velocity, V_x	Velocity, V_y	Velocity, V	Accel., a_x	Accel., a_y	Accel., a	Normal Accel., a_n
0.08	5.00							
0.2	3.16							
0.4	2.24							
0.5	2.00	2.00	-4.00	4.47	2.00	16.0	16.1	8.94
0.6	1.83							
0.8	1.58							
1.0	1.41	2.83	-2.00	3.46	2.83	5.66	6.32	6.25
1.5	1.15	3.46	-1.33	3.71	3.46	3.08	4.63	4.12
2.0	1.00	4.00	-1.00	4.12	4.00	2.00	4.47	2.91
2.5	0.89	4.47	-0.80	4.54	4.47	1.43	4.70	2.20
3.0	0.82	4.90	-0.67	4.94	4.90	1.09	5.02	1.74
3.5	0.76	5.29	-0.57	5.32	5.29	0.86	5.36	1.43
4.0	0.71	5.66	-0.50	5.68	5.66	0.71	5.70	1.20
4.5	0.67	6.00	-0.44	6.02	6.00	0.59	6.03	1.03
5.0	0.63	6.32	-0.40	6.34	6.32	0.51	6.34	0.90

Acceleration:

2	1
4	2

Velocity:

2	1
4	0.5



Problem 6.38

[Difficulty: 1]

6.38 Water flows at a speed of 25 ft/s. Calculate the dynamic pressure of this flow. Express your answer in inches of mercury.

Given: Water at speed 25 ft/s

Find: Dynamic pressure in in. Hg

Solution:

Basic equations

$$p_{dynamic} = \frac{1}{2}\cdot\rho\cdot V^2 \qquad\qquad p = \rho_{Hg}\cdot g\cdot\Delta h = SG_{Hg}\cdot\rho\cdot g\cdot\Delta h$$

Hence

$$\Delta h = \frac{\rho\cdot V^2}{2\cdot SG_{Hg}\cdot\rho\cdot g} = \frac{V^2}{2\cdot SG_{Hg}\cdot g}$$

$$\Delta h = \frac{1}{2}\times\left(25\cdot\frac{ft}{s}\right)^2\times\frac{1}{13.6}\times\frac{s^2}{32.2\cdot ft}\times\frac{12\cdot in}{1\cdot ft} \qquad \Delta h = 8.56\cdot in$$

Problem 6.39

[Difficulty: 1]

6.39 Calculate the dynamic pressure that corresponds to a speed of 100 km/hr in standard air. Express your answer in millimeters of water.

Given: Air speed of 100 km/hr

Find: Dynamic pressure in mm water

Solution:

Basic equations $p_{dynamic} = \frac{1}{2}\cdot\rho_{air}\cdot V^2$ $\qquad p = \rho_w\cdot g\cdot\Delta h$

Hence $\Delta h = \frac{\rho_{air}}{\rho_w}\cdot\frac{V^2}{2\cdot g}$

$$\Delta h = \frac{1.23\cdot\frac{kg}{m^3}}{999\cdot\frac{kg}{m^3}} \times \frac{1}{2} \times \left(100\cdot\frac{km}{hr}\right)^2 \times \left(\frac{1000\cdot m}{1\cdot km}\right)^2 \times \left(\frac{1\cdot hr}{3600\cdot s}\right)^2 \times \frac{s^2}{9.81\cdot m} \qquad \Delta h = 48.4\cdot mm$$

Problem 6.40

[Difficulty: 2]

6.40 Plot the speed of air versus the dynamic pressure (in millimeters of mercury), up to a dynamic pressure of 250 mm Hg.

Given: Air speed

Find: Plot dynamic pressure in mm Hg

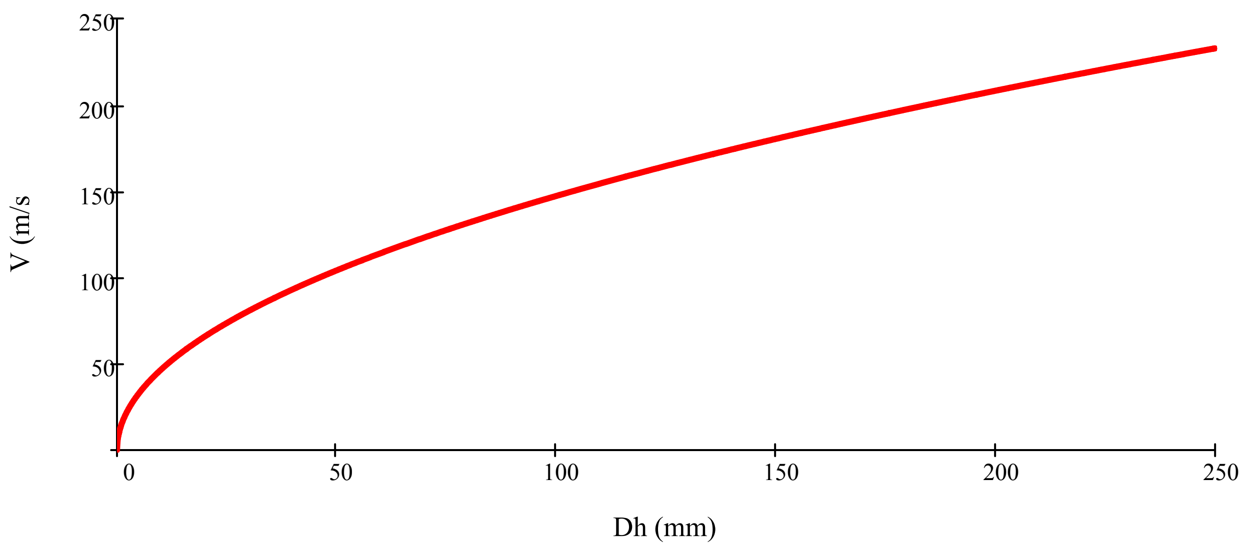
Solution:

Basic equations $p_{dynamic} = \frac{1}{2} \cdot \rho_{air} \cdot V^2$ $\quad p = \rho_{Hg} \cdot g \cdot \Delta h = SG_{Hg} \cdot \rho_w \cdot g \cdot \Delta h$

Available data $\rho_w = 999 \cdot \frac{kg}{m^3}$ $\quad \rho_{air} = 1.23 \cdot \frac{kg}{m^3}$ $\quad SG_{Hg} = 13.6$

Hence $\frac{1}{2} \cdot \rho_{air} \cdot V^2 = SG_{Hg} \cdot \rho_w \cdot g \cdot \Delta h$

Solving for V $V(\Delta h) = \sqrt{\frac{2 \cdot SG_{Hg} \cdot \rho_w \cdot g \cdot \Delta h}{\rho_{air}}}$



Problem 6.41

[Difficulty: 2]

6.41 You present your open hand out of the window of an automobile perpendicular to the airflow. Assuming for simplicity that the air pressure on the entire front surface is stagnation pressure (with respect to automobile coordinates), with atmospheric pressure on the rear surface, estimate the net force on your hand when driving at (a) 30 mph and (b) 60 mph. Do these results roughly correspond with your experience? Do the simplifications tend to make the calculated force an over- or underestimate?

Given: Velocity of automobile

Find: Estimates of aerodynamic force on hand

Solution:

The basic equation is the Bernoulli equation (in coordinates attached to the vehicle)

$$p_{atm} + \frac{1}{2}\cdot\rho\cdot V^2 = p_{stag}$$

where V is the free stream velocity

For air $\rho = 0.00238\cdot\frac{\text{slug}}{\text{ft}^3}$

We need an estimate of the area of a typical hand. Personal inspection indicates that a good approximation is a square of sides 9 cm and 17 cm

$$A = 9\cdot\text{cm} \times 17\cdot\text{cm} \qquad A = 153\cdot\text{cm}^2$$

Hence, for p_{stag} on the front side of the hand, and p_{atm} on the rear, by assumption,

$$F = \left(p_{stag} - p_{atm}\right)\cdot A = \frac{1}{2}\cdot\rho\cdot V^2\cdot A$$

(a) $V = 30\cdot\text{mph}$

$$F = \frac{1}{2}\cdot\rho\cdot V^2\cdot A = \frac{1}{2} \times 0.00238\cdot\frac{\text{slug}}{\text{ft}^3} \times \left(30\cdot\text{mph}\cdot\frac{22\cdot\frac{\text{ft}}{\text{s}}}{15\cdot\text{mph}}\right)^2 \times 153\cdot\text{cm}^2 \times \left(\frac{\frac{1}{12}\cdot\text{ft}}{2.54\cdot\text{cm}}\right)^2 \qquad F = 0.379\cdot\text{lbf}$$

(b) $V = 60\cdot\text{mph}$

$$F = \frac{1}{2}\cdot\rho\cdot V^2\cdot A = \frac{1}{2} \times 0.00238\cdot\frac{\text{slug}}{\text{ft}^3} \times \left(60\cdot\text{mph}\cdot\frac{22\cdot\frac{\text{ft}}{\text{s}}}{15\cdot\text{mph}}\right)^2 \times 153\cdot\text{cm}^2 \times \left(\frac{\frac{1}{12}\cdot\text{ft}}{2.54\cdot\text{cm}}\right)^2 \qquad F = 1.52\cdot\text{lbf}$$

These values pretty much agree with experience. However, they overestimate a bit as the entire front of the hand is not at stagnation pressure - there is flow around the had - so the pressure is less than stagnation over most of the surface.

Problem 6.42

[Difficulty: 2]

6.42 A jet of air from a nozzle is blown at right angles against a wall in which two pressure taps are located. A manometer connected to the tap directly in front of the jet shows a head of 25 mm of mercury above atmospheric. Determine the approximate speed of the air leaving the nozzle if it is at $-10°C$ and 200 kPa. At the second tap a manometer indicates a head of 5 mm of mercury above atmospheric; what is the approximate speed of the air there?

Given: Air jet hitting wall generating pressures

Find: Speed of air at two locations

Solution:

Basic equations
$$\frac{p}{\rho_{air}} + \frac{V^2}{2} + g \cdot z = \text{const} \qquad \rho_{air} = \frac{p}{R_{air} \cdot T} \qquad \Delta p = \rho_{Hg} \cdot g \cdot \Delta h = SG_{Hg} \cdot \rho \cdot g \cdot \Delta h$$

Assumptions: 1) Incompressible flow 2) Inviscid 3) Steady 4) Along a streamline

Available data
$$R = 287 \cdot \frac{J}{kg \cdot K} \qquad T = -10\,°C \qquad \rho = 999 \cdot \frac{kg}{m^3} \qquad p = 200 \cdot kPa \qquad SG_{Hg} = 13.6$$

For the air
$$\rho_{air} = \frac{p}{R \cdot T} \qquad \rho_{air} = 2.65\,\frac{kg}{m^3}$$

Hence, applying Bernoulli between the jet and where it hits the wall directly

$$\frac{p_{atm}}{\rho_{air}} + \frac{V_j^2}{2} = \frac{p_{wall}}{\rho_{air}} \qquad p_{wall} = \frac{\rho_{air} \cdot V_j^2}{2} \qquad \text{(working in gage pressures)}$$

Hence
$$p_{wall} = SG_{Hg} \cdot \rho \cdot g \cdot \Delta h = \frac{\rho_{air} \cdot V_j^2}{2} \qquad \text{so} \qquad V_j = \sqrt{\frac{2 \cdot SG_{Hg} \cdot \rho \cdot g \cdot \Delta h}{\rho_{air}}}$$

$$\Delta h = 25 \cdot mm \qquad \text{hence} \qquad V_j = \sqrt{2 \times 13.6 \times 999 \cdot \frac{kg}{m^3} \times \frac{1}{2.65} \cdot \frac{m^3}{kg} \times 9.81 \cdot \frac{m}{s^2} \times 25 \cdot mm \times \frac{1 \cdot m}{1000 \cdot mm}} \qquad V_j = 50.1\,\frac{m}{s}$$

Repeating the analysis for the second point

$$\Delta h = 5 \cdot mm \qquad \frac{p_{atm}}{\rho_{air}} + \frac{V_j^2}{2} = \frac{p_{wall}}{\rho_{air}} + \frac{V^2}{2} \qquad V = \sqrt{V_j^2 - \frac{2 \cdot p_{wall}}{\rho_{air}}} = \sqrt{V_j^2 - \frac{2 \cdot SG_{Hg} \cdot \rho \cdot g \cdot \Delta h}{\rho_{air}}}$$

Hence
$$V = \sqrt{\left(50.1 \cdot \frac{m}{s}\right)^2 - 2 \times 13.6 \times 999 \cdot \frac{kg}{m^3} \times \frac{1}{2.65} \cdot \frac{m^3}{kg} \times 9.81 \cdot \frac{m}{s^2} \times 5 \cdot mm \times \frac{1 \cdot m}{1000 \cdot mm}} \qquad V = 44.8\,\frac{m}{s}$$

Problem 6.43

[Difficulty: 2]

6.43 A pitot-static tube is used to measure the speed of air at standard conditions at a point in a flow. To ensure that the flow may be assumed incompressible for calculations of engineering accuracy, the speed is to be maintained at 100 m/s or less. Determine the manometer deflection, in millimeters of water, that corresponds to the maximum desirable speed.

Solution: Manometer reads $p_0 - p$ in mm of H_2O.

Basic equations: $\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$ for flow

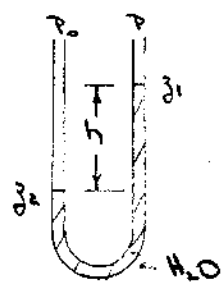
$\frac{dp}{dz} = -\rho g$ for manometer

Assumptions: (1) steady flow
(2) incompressible flow
(3) flow along a streamline
(4) frictionless deceleration to p_0
(5) ρ = constant for manometer

From the Bernoulli equation

$$\frac{p_0}{\rho} = \frac{p}{\rho} + \frac{V^2}{2}$$

$$p_0 - p = \rho \frac{V^2}{2}$$



For the manometer, $dp = -\rho g\, dz$

$$p_0 - p = \int_p^{p_0} dp = -\rho g (z_2 - z_1) = \rho g h$$

Then,

$$\rho_{H_2O}\, g h = \rho_{air} \frac{V^2}{2}$$

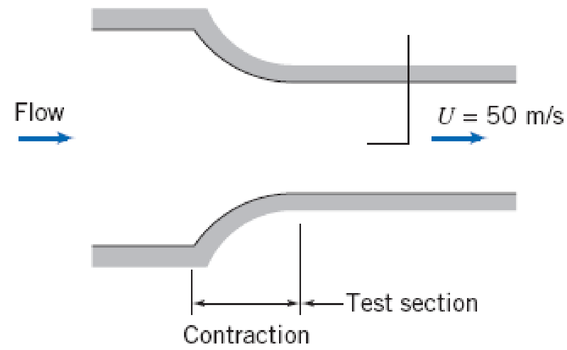
and

$$h = \frac{\rho_{air}}{\rho_{H_2O}} \frac{V^2}{2g} = \frac{1.23}{999} \times (100)^2 \frac{m^2}{s^2} \times \frac{1}{2} \times \frac{s^2}{9.81\,m} \times \frac{10^3\,mm}{m} = 62.8\ mm \longleftarrow h$$

Problem 6.44

[Difficulty: 2]

6.44 The inlet contraction and test section of a laboratory wind tunnel are shown. The air speed in the test section is $U = 50$ m/s. A total-head tube pointed upstream indicates that the stagnation pressure on the test section centerline is 10 mm of water below atmospheric. The laboratory is maintained at atmospheric pressure and a temperature of −5°C. Evaluate the dynamic pressure on the centerline of the wind tunnel test section. Compute the static pressure at the same point. Qualitatively compare the static pressure at the tunnel wall with that at the centerline. Explain why the two may not be identical.



Given: Wind tunnel with inlet section

Find: Dynamic and static pressures on centerline; compare with Speed of air at two locations

Solution:

Basic equations $\quad p_{dyn} = \frac{1}{2}\cdot\rho_{air}\cdot U^2 \quad p_0 = p_s + p_{dyn} \quad \rho_{air} = \frac{p}{R_{air}\cdot T} \quad \Delta p = \rho_w\cdot g\cdot \Delta h$

Assumptions: 1) Incompressible flow 2) Inviscid 3) Steady 4) Along a streamline

Available data $\quad T = -5\,°C \quad U = 5(R = 287\cdot\frac{J}{kg\cdot K} \quad p_{atm} = 101\cdot kPa \quad h_0 = -10\cdot mm \quad \rho_w = 999\cdot\frac{kg}{m^3}$

For air $\quad \rho_{air} = \frac{p_{atm}}{R\cdot T} \quad \rho_{air} = 1.31\,\frac{kg}{m^3}$

$$p_{dyn} = \frac{1}{2}\cdot\rho_{air}\cdot U^2 \qquad p_{dyn} = 1.64\cdot kPa$$

Also $\quad p_0 = \rho_w\cdot g\cdot h_0 \quad p_0 = -98.0\,Pa \quad$ (gage)

and $\quad p_0 = p_s + p_{dyn} \quad$ so $\quad p_s = p_0 - p_{dyn} \quad p_s = -1.738\,kPa$ (gage) $\quad h_s = \frac{p_s}{\rho_w\cdot g} \quad h_s = -177\,mm$

Streamlines in the test section are straight so $\quad \frac{\partial}{\partial n}p = 0 \quad$ and $\quad p_w = p_{centerline}$

In the curved section $\quad \frac{\partial}{\partial n}p = \rho_{air}\cdot\frac{V^2}{R} \quad$ so $\quad p_w < p_{centerline}$

Problem 6.45

[Difficulty: 2]

6.45 Maintenance work on high-pressure hydraulic systems requires special precautions. A small leak can result in a high-speed jet of hydraulic fluid that can penetrate the skin and cause serious injury (therefore troubleshooters are cautioned to use a piece of paper or cardboard, *not a finger*, to search for leaks). Calculate and plot the jet speed of a leak versus system pressure, for pressures up to 40 MPa (gage). Explain how a high-speed jet of hydraulic fluid can cause injury.

Solution:

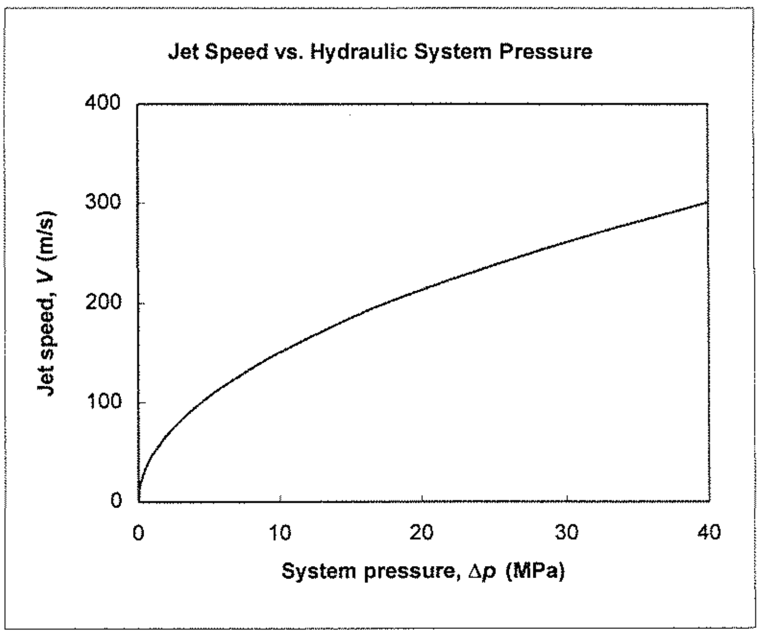
Basic equation: $\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$

Assumptions: (1) steady flow
(2) incompressible flow
(3) frictionless flow
(4) flow along a streamline.

The Bernoulli equation gives

$$V = \left[\frac{2(p_0 - p_{atm})}{\rho}\right]^{1/2}$$

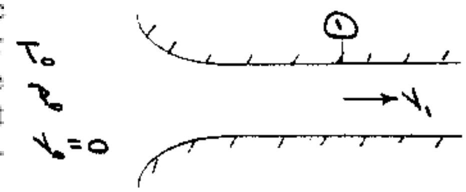
From Table A.2 (Appendix A) for lubricating oil SG = 0.88



The high stagnation pressure ruptures the skin causing the jet to penetrate the tissue.

Problem 6.46

6.46 An open-circuit wind tunnel draws in air from the atmosphere through a well-contoured nozzle. In the test section, where the flow is straight and nearly uniform, a static pressure tap is drilled into the tunnel wall. A manometer connected to the tap shows that static pressure within the tunnel is 45 mm of water below atmospheric. Assume that the air is incompressible, and at 25°C, 100 kPa (abs). Calculate the air speed in the wind-tunnel test section.



Solution:

Basic equations: $\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$

Assumptions: (1) steady flow
(2) incompressible flow
(3) frictionless flow
(4) flow along a streamline
(5) air behaves as an ideal gas
(6) stagnation pressure = p_{atm}

From the Bernoulli equation, $\frac{p_0}{\rho} = \frac{p_1}{\rho} + \frac{V_1^2}{2}$

$$p_0 - p_1 = p_{atm} - p_1 = \frac{1}{2}\rho V_1^2$$

$$V_1 = \left[\frac{2(p_{atm} - p_1)}{\rho}\right]^{1/2}$$

From the manometer reading, $p_{atm} - p_1 = \rho_{H_2O}\, g h$ Then

$$V_1 = \left[\frac{2\rho_{H_2O}\, g h}{\rho}\right]^{1/2}$$

From the ideal gas equation of state

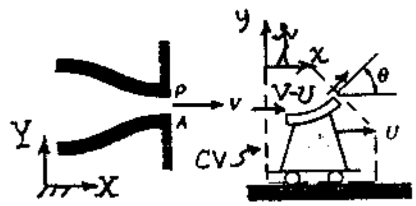
$$\rho = \frac{p}{RT} = 100 \times 10^3 \frac{N}{m^2} \times \frac{kg \cdot K}{287\, N \cdot m} \times \frac{1}{298\, K} = 1.17\, kg/m^3$$

$$V_1 = \left[\frac{2\rho_{H_2O}}{\rho} gh\right]^{1/2} = \left[2 \times \frac{999}{1.17} \times 9.81 \frac{m}{s^2} \times 0.045\, m\right]^{1/2} = 27.5\, m/s \longleftarrow V_1$$

Problem 6.47

[Difficulty: 4]

6.47 The wheeled cart shown in Problem 4.128 rolls with negligible resistance. The cart is to accelerate to the right. The jet speed is $V = 40$ m/s. The jet area remains constant at $A = 25$ mm^2. Neglect viscous forces between the water and vane. When the cart attains speed $U = 15$ m/s, calculate the stagnation pressure of the water leaving the nozzle with respect to a fixed observer, the stagnation pressure of the water jet leaving the nozzle with respect to an observer on the vane, the absolute velocity of the jet leaving the vane with respect to a fixed observer, and the stagnation pressure of the jet leaving the vane with respect to a fixed observer. How would viscous forces affect the latter stagnation pressure, i.e., would viscous forces increase, decrease, or leave unchanged this stagnation pressure? Justify your answer.



<u>Solution:</u> Stagnation pressure is $p_0 = p + \frac{1}{2}\rho V^2$ or $p_0 - p = \frac{1}{2}\rho V^2$

At jet, $p_{0j} = \frac{1}{2}\rho V^2 = \frac{1}{2} \times 999 \frac{kg}{m^3} \times (40)^2 \frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m} = 799$ kPa (gage) $\quad p_{0j}$

At cart, $p_{0,rel} = \frac{1}{2}\rho (V-U)^2 = \frac{1}{2} \times 999 \frac{kg}{m^3} \times (40-15)^2 \frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m} = 312$ kPa (gage) $\quad p_{0,rel}$

Leaving vane, $\vec{V}_{abs} = U\hat{i} + (V-U)(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$

$$\vec{V}_{abs} = [U + (V-U)\cos\theta]\hat{i} + (V-U)\sin\theta\,\hat{j}$$

$$= \left[15\,\frac{m}{s} + (40-15)\frac{m}{s} \times \left(-\frac{1}{2}\right)\right]\hat{i} + (40-15)\,\frac{m}{s} \times 0.866\,\hat{j}$$

$$\vec{V}_{abs} = 2.5\,\hat{i} + 21.7\,\hat{j} \ \text{m/s}$$ $\vec{V}_{abs}$

The magnitude $|\vec{V}_{abs}| = \left[(2.5)^2 + (21.7)^2\right]^{\frac{1}{2}}$ m/s $= 21.8$ m/s

Leaving vane, $p_0 = \frac{1}{2}\rho |\vec{V}_{abs}|^2$, relative to a fixed observer. Thus

$$p_{0,fixed} = \frac{1}{2} \times 999 \frac{kg}{m^3} \times (21.8)^2 \frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m} = 237 \text{ kPa (gage)}$$ $p_{0,fixed}$

{The corresponding absolute pressures are 900, 413, and 338 kPa (abs).}

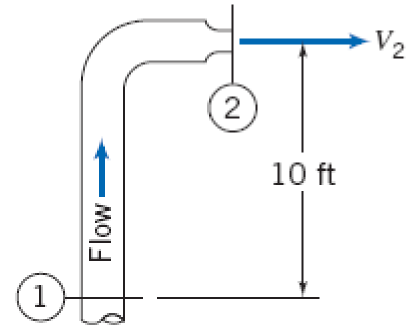
Discussion: Viscous forces would slow the jet speed relative to the vane. The jet would enter the vane with relative speed $(V - U)$; it would leave the vane with speed $\alpha\,(V - U)$, where $\alpha < 1$.

Friction would reduce both components of relative velocity leaving the vane. The absolute velocity of the jet leaving the vane, as seen by a fixed observer, would decrease. Thus the stagnation pressure of the flow leaving the vane, relative to a fixed observer, would decrease.

Problem 6.48

[Difficulty: 2]

6.48 Water flows steadily up the vertical 1-in.-diameter pipe and out the nozzle, which is 0.5 in. in diameter, discharging to atmospheric pressure. The stream velocity at the nozzle exit must be 30 ft/s. Calculate the minimum gage pressure required at section (1). If the device were inverted, what would be the required minimum pressure at section (1) to maintain the nozzle exit velocity at 30 ft/s?



Given: Flow in pipe/nozzle device

Find: Gage pressure needed for flow rate; repeat for inverted

Solution:

Basic equations $\quad Q = V \cdot A \qquad \dfrac{p}{\rho} + \dfrac{V^2}{2} + g \cdot z = \text{const}$

Assumptions: 1) Incompressible flow 2) Inviscid 3) Steady 4) Along a streamline

Available data $\quad D_1 = 1 \cdot in \qquad D_2 = 0.5 \cdot in \qquad V_2 = 30 \cdot \dfrac{ft}{s} \qquad z_2 = 10 \cdot ft \qquad \rho = 1.94 \cdot \dfrac{slug}{ft^3}$

From continuity $\quad Q = V_1 \cdot A_1 = V_2 \cdot A_2 \qquad V_1 = V_2 \cdot \dfrac{A_2}{A_1} \quad$ or $\quad V_1 = V_2 \cdot \left(\dfrac{D_2}{D_1}\right)^2 \qquad V_1 = 7.50 \dfrac{ft}{s}$

Hence, applying Bernoulli between locations 1 and 2

$$\frac{p_1}{\rho} + \frac{V_1^{\ 2}}{2} + 0 = \frac{p_2}{\rho} + \frac{V_2^{\ 2}}{2} + g \cdot z_2 = 0 + \frac{V_2^{\ 2}}{2} + g \cdot z_2 \text{ working in gage pressures}$$

Solving for p_1 (gage) $\quad p_1 = \rho \cdot \left(\dfrac{V_2^{\ 2} - V_1^{\ 2}}{2} + g \cdot z_2\right) \qquad p_1 = 10.0 \cdot psi$

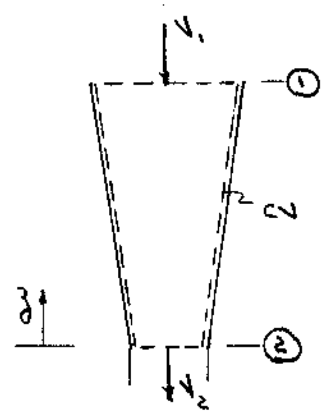
When it is inverted $\quad z_2 = -10 \cdot ft$

$$p_2 = \rho \cdot \left(\frac{V_2^{\ 2} - V_1^{\ 2}}{2} + g \cdot z_2\right) \qquad p_2 = 1.35 \cdot psi$$

Problem 6.49

[Difficulty: 2]

6.49 Water flows in a circular duct. At one section the diameter is 0.3 m, the static pressure is 260 kPa (gage), the velocity is 3 m/s, and the elevation is 10 m above ground level. At a section downstream at ground level, the duct diameter is 0.15 m. Find the gage pressure at the downstream section if frictional effects may be neglected.



Solution: Apply continuity to cv shown to determine V_2; the Bernoulli equation is then applied along a streamline from ① to ② to determine p_2.

Basic equations:

$$0 = \cancel{\frac{\partial}{\partial t}}^{\,0(1)} \int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + g z_2$$

Assumptions:
(1) steady flow
(2) incompressible flow
(3) frictionless flow
(4) flow along a streamline
(5) uniform flow at sections ① and ②

From the continuity equation

$$0 = -|\rho V_1 A_1| + |\rho V_2 A_2|$$

Then,

$$V_2 = \frac{A_1}{A_2} V_1 = \left(\frac{D_1}{D_2}\right)^2 V_1 = \left(\frac{0.3}{0.15}\right)^2 \times 3 \frac{m}{s} = 12 \ m/s$$

From the Bernoulli equation,

$$p_2 = p_1 + \frac{\rho}{2}(V_1^2 - V_2^2) + \rho g (z_1 - z_2)$$

$$= 260 \frac{kN}{m^2} + \frac{1}{2} \times 999 \frac{kg}{m^3} \times \left[(3)^2 - (12)^2\right] \frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m} + 999 \frac{kg}{m^3} \times 9.81 \frac{m}{s^2} \times 10 m \times \frac{N \cdot s^2}{kg \cdot m}$$

$$p_2 = 291 \ kN/m^2 = 291 \ kPa \ \text{(gage)}.$$

p_2

Problem 6.50

[Difficulty: 2]

6.50 You are on a date. Your date runs out of gas unexpectedly. You come to your own rescue by siphoning gas from another car. The height difference for the siphon is about 1 ft. The hose diameter is 0.5 in. What is your gasoline flow rate?

Given: Siphoning of gasoline

Find: Flow rate

Solution:

Basic equation $$\frac{p}{\rho_{gas}} + \frac{V^2}{2} + g \cdot z = \text{const}$$

Assumptions: 1) Incompressible flow 2) Inviscid 3) Steady 4) Along a streamline

Hence, applying Bernoulli between the gas tank free surface and the siphon exit

$$\frac{p_{atm}}{\rho_{gas}} = \frac{p_{atm}}{\rho_{gas}} + \frac{V^2}{2} - g \cdot h$$

where we assume the tank free surface is slowly changing so V_{tank} <<, and h is the difference in levels

Hence $$V = \sqrt{2 \cdot g \cdot h}$$

The flow rate is then $$Q = V \cdot A = \frac{\pi \cdot D^2}{4} \cdot \sqrt{2 \cdot g \cdot h}$$

$$Q = \frac{\pi}{4} \times (.5 \cdot \text{in})^2 \times \frac{1 \cdot \text{ft}^2}{144 \cdot \text{in}^2} \times \sqrt{2 \times 32.2 \frac{\text{ft}}{\text{s}^2} \times 1 \cdot \text{ft}} \qquad Q = 0.0109 \cdot \frac{\text{ft}^3}{\text{s}} \qquad Q = 4.91 \cdot \frac{\text{gal}}{\text{min}}$$

Problem 6.51

[Difficulty: 2]

6.51 You (a young person of legal drinking age) are making homemade beer. As part of the process you have to siphon the wort (the fermenting beer with sediment at the bottom) into a clean tank using a 5-mm ID tubing. Being a young engineer, you're curious about the flow you can produce. Find an expression for and plot the flow rate Q (liters per minute) versus the differential in height h (millimeters) between the wort free surface and the location of the hose exit. Find the value of h for which $Q = 2$ L/min.

Given: Siphoning of wort

Find: Flow rate; plot; height for a flow of 2 L/min

Solution:

Basic equation $$\frac{p}{\rho_{wort}} + \frac{V^2}{2} + g\cdot z = \text{const}$$

Assumptions: 1) Incompressible flow 2) Inviscid 3) Steady 4) Along a streamline

Hence, applying Bernoulli between the open surface of the full tank and tube exit to atmosphere

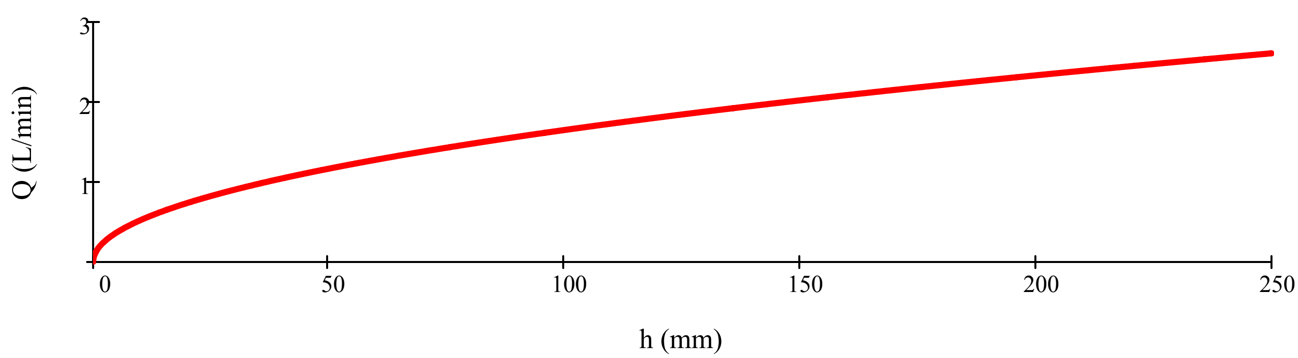
$$\frac{p_{atm}}{\rho_{gas}} = \frac{p_{atm}}{\rho_{gas}} + \frac{V^2}{2} - g\cdot h$$

where we assume the tank free surface is slowly changing so V_{tank} <<, and h is the difference in levels

Hence $$V = \sqrt{2\cdot g\cdot h}$$

The flow rate is then $$Q = V\cdot A = \frac{\pi\cdot D^2}{4}\cdot\sqrt{2\cdot g\cdot h}$$

For $D = 5\cdot mm$



For a flow rate of $Q = 2\cdot\frac{L}{min}$ $\quad Q = 3.33\times 10^{-5}\frac{m^3}{s}$ $\quad$ solving for h $\quad h = \frac{8\cdot Q^2}{\pi^2\cdot D^4\cdot g}$ $\quad h = 147\cdot mm$

Problem 6.52

[Difficulty: 2]

6.52 A tank at a pressure of 50 kPa (gage) gets a pinhole rupture and benzene shoots into the air. Ignoring losses, to what height will the benzene rise?

Given: Ruptured pipe

Find: Height benzene rises from tank

Solution:

Basic equation
$$\frac{p}{\rho_{ben}} + \frac{V^2}{2} + g \cdot z = \text{const}$$

Assumptions: 1) Incompressible flow 2) Inviscid 3) Steady 4) Along a streamline

Available data $\rho = 999 \cdot \frac{kg}{m^3}$ $p_{ben} = 50 \cdot kPa$ (gage) From Table A.2 $SG_{ben} = 0.879$

Hence, applying Bernoulli between the pipe and the rise height of the benzene

$$\frac{p_{ben}}{\rho_{ben}} = \frac{p_{atm}}{\rho_{ben}} + g \cdot h$$ where we assume V_{pipe} <<, and h is the rise height

Hence $$h = \frac{p_{ben}}{SG_{ben} \cdot \rho \cdot g}$$ where p_{ben} is now the gage pressure

$$h = 5.81\, m$$

Problem 6.53

[Difficulty: 2]

6.53 A can of Coke (you're not sure if it's diet or regular) has a small pinhole leak in it. The Coke sprays vertically into the air to a height of 0.5 m. What is the pressure inside the can of Coke? (Estimate for both kinds of Coke.)

Given: Ruptured Coke can

Find: Pressure in can

Solution:

Basic equation $$\frac{p}{\rho_{Coke}} + \frac{V^2}{2} + g \cdot z = \text{const}$$

Assumptions: 1) Incompressible flow 2) Inviscid 3) Steady 4) Along a streamline

Available data $\rho_w = 999 \cdot \frac{kg}{m^3}$ $\quad h = 0.5 \cdot m$ $\quad$ From a web search $\quad SG_{DietCoke} = 1$ $\quad SG_{RegularCoke} = 1.11$

Hence, applying Bernoulli between the coke can and the rise height of the coke

$$\frac{p_{can}}{\rho_{Coke}} = \frac{p_{atm}}{\rho_{Coke}} + g \cdot h$$ where we assume V_{Coke} <<, and h is the rise height

Hence $\quad p_{Coke} = \rho_{Coke} \cdot g \cdot h = SG_{Coke} \cdot \rho_w \cdot g \cdot h$ $\quad$ where p_{Coke} is now the gage pressure

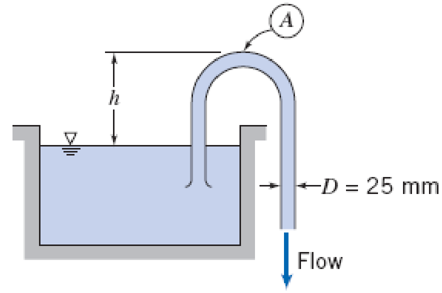
Hence $\quad p_{Diet} = SG_{DietCoke} \cdot \rho_w \cdot g \cdot h$ $\quad p_{Diet} = 4.90 \cdot kPa$ $\quad$ (gage)

and $\quad p_{Regular} = SG_{RegularCoke} \cdot \rho_w \cdot g \cdot h$ $\quad p_{Regular} = 5.44 \cdot kPa$ $\quad$ (gage)

Problem 6.54

[Difficulty: 3]

6.54 The water flow rate through the siphon is 5 L/s, its temperature is 20°C, and the pipe diameter is 25 mm. Compute the maximum allowable height, h, so that the pressure at point A is above the vapor pressure of the water. (Assume the flow is frictionless.)



Given: Flow rate through siphon

Find: Maximum height h to avoid cavitation

Solution:

Basic equation $\quad \dfrac{p}{\rho} + \dfrac{V^2}{2} + g\cdot z = \text{const} \qquad Q = V\cdot A$

Assumptions: 1) Incompressible flow 2) Inviscid 3) Steady 4) Along a streamline

Available data $\quad Q = 5\cdot\dfrac{L}{s} \qquad Q = 5\times 10^{-3}\dfrac{m^3}{s} \qquad D = 25\cdot mm \qquad \rho = 999\cdot\dfrac{kg}{m^3} \qquad p_{atm} = 101\cdot kPa$

From continuity $\quad V = \dfrac{Q}{A} = \dfrac{4\cdot Q}{\pi\cdot D^2} \qquad V = \dfrac{4}{\pi}\times 0.005\dfrac{m^3}{s}\times\left(\dfrac{1}{.025\cdot m}\right)^2 \qquad V = 10.2\dfrac{m}{s}$

Hence, applying Bernoulli between the free surface and point A

$$\frac{p_{atm}}{\rho} = \frac{p_A}{\rho} + g\cdot h + \frac{V^2}{2} \qquad \text{where we assume } V_{Surface} <<$$

Hence $\quad p_A = p_{atm} - \rho\cdot g\cdot h - \rho\cdot\dfrac{V^2}{2}$

From the steam tables, at 20°C the vapor pressure is $\quad p_v = 2.358\cdot kPa$

This is the lowest permissible value of p_A

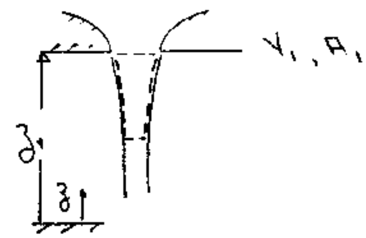
Hence $\quad p_A = p_v = p_{atm} - \rho\cdot g\cdot h - \rho\cdot\dfrac{V^2}{2} \quad$ or $\quad h = \dfrac{p_{atm} - p_v}{\rho\cdot g} - \dfrac{V^2}{2\cdot g}$

Hence $\quad h = (101 - 2.358)\times 10^3\cdot\dfrac{N}{m^2}\times\dfrac{1}{999}\cdot\dfrac{m^3}{kg}\times\dfrac{s^2}{9.81\cdot m}\times\dfrac{kg\cdot m}{N\cdot s^2} - \dfrac{1}{2}\times\left(10.2\dfrac{m}{s}\right)^2\times\dfrac{s^2}{9.81\cdot m} \qquad h = 4.76\, m$

Problem 6.55

[Difficulty: 2]

6.55 A stream of liquid moving at low speed leaves a nozzle pointed directly downward. The velocity may be considered uniform across the nozzle exit and the effects of friction may be ignored. At the nozzle exit, located at elevation z_0, the jet velocity and area are V_0 and A_0, respectively. Determine the variation of jet area with elevation.



Solution:

Basic equations: $$\frac{p_1}{\rho} + \frac{V_1^2}{2} + g z_1 = \frac{p}{\rho} + \frac{V^2}{2} + g z$$

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$

Assumptions:
(1) steady flow
(2) incompressible flow
(3) frictionless flow
(4) flow along a streamline
(5) $p = p_1 = p_{atm}$
(6) uniform flow at a section

From the Bernoulli equation
$$V^2 = V_1^2 + 2g(z_1 - z)$$

From the continuity equation
$$0 = \int_{CS} \rho \vec{V} \cdot d\vec{A} = -\{|\rho V_1 A_1|\} + \{|\rho V A|\}$$

and
$$V_1 A_1 = V A \quad \text{or} \quad V = V_1 \frac{A_1}{A}$$

Thus
$$V_1^2 \left(\frac{A_1}{A}\right)^2 = V_1^2 + 2g(z_1 - z)$$

Solving for A,
$$A = A_1 \sqrt{\frac{1}{1 + \frac{2g(z_1 - z)}{V_1^2}}}$$

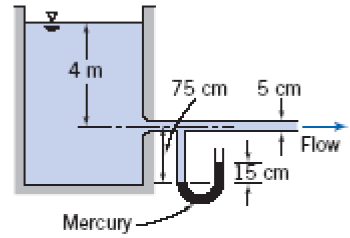
$A(z)$

{Note: jet area decreases as z decreases, owing to the higher velocity}

Problem 6.56

[Difficulty: 2]

6.56 Water flows from a very large tank through a 5-cm-diameter tube. The dark liquid in the manometer is mercury. Estimate the velocity in the pipe and the rate of discharge from the tank. (Assume the flow is frictionless.)



Given: Flow through tank-pipe system

Find: Velocity in pipe; Rate of discharge

Solution:

Basic equations $\frac{p}{\rho} + \frac{V^2}{2} + g \cdot z = \text{const}$ $\qquad \Delta p = \rho \cdot g \cdot \Delta h$ $\qquad Q = V \cdot A$

Assumptions: 1) Incompressible flow 2) Inviscid 3) Steady 4) Along a streamline

Hence, applying Bernoulli between the free surface and the manometer location

$$\frac{p_{atm}}{\rho} = \frac{p}{\rho} - g \cdot H + \frac{V^2}{2}$$

where we assume $V_{Surface}$ <<, and H = 4 m

Hence $\quad p = p_{atm} + \rho \cdot g \cdot H - \rho \cdot \frac{V^2}{2}$

For the manometer $\quad p - p_{atm} = SG_{Hg} \cdot \rho \cdot g \cdot h_2 - \rho \cdot g \cdot h_1$

Note that we have water on one side and mercury on the other of the manometer

Combining equations $\quad \rho \cdot g \cdot H - \rho \cdot \frac{V^2}{2} = SG_{Hg} \cdot \rho \cdot g \cdot h_2 - \rho \cdot g \cdot h_1$ $\quad$ or $\quad V = \sqrt{2 \cdot g \cdot \left(H - SG_{Hg} \cdot h_2 + h_2\right)}$

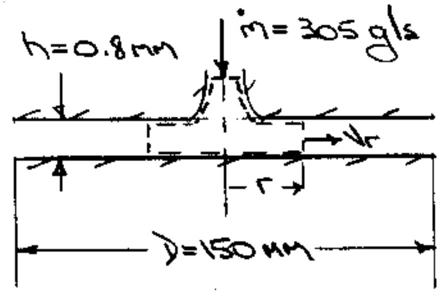
Hence $\quad V = \sqrt{2 \times 9.81 \cdot \frac{m}{s^2} \times (4 - 13.6 \times 0.15 + 0.75) \cdot m}$ $\qquad V = 7.29 \frac{m}{s}$

The flow rate is $\quad Q = V \cdot \frac{\pi \cdot D^2}{4}$ $\qquad Q = \frac{\pi}{4} \times 7.29 \cdot \frac{m}{s} \times (0.05 \cdot m)^2$ $\qquad Q = 0.0143 \frac{m^3}{s}$

Problem 6.57

[Difficulty: 2]

6.57 In a laboratory experiment, water flows radially outward at moderate speed through the space between circular plane parallel disks. The perimeter of the disks is open to the atmosphere. The disks have diameter $D = 150$ mm and the spacing between the disks is $h = 0.8$ mm. The measured mass flow rate of water is $\dot{m} = 305$ g/s. Assuming frictionless flow in the space between the disks, estimate the theoretical static pressure between the disks at radius $r = 50$ mm. In the laboratory situation, where *some* friction is present, would the pressure measured at the same location be above or below the theoretical value? Why?



Solution:

Basic equations: $0 = \frac{\partial}{\partial t}\int_{CV}\rho\, d\forall + \int_{CS}\rho\vec{V}\cdot d\vec{A}$

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2$$

Assumptions:
(1) steady flow
(2) incompressible flow
(3) flow along a streamline
(4) neglect friction
(5) uniform flow at each section

Apply continuity to the CV shown

$$0 = \{-\dot{m}\} + \{\rho V_r 2\pi r h\} \quad \text{so} \quad V = \frac{\dot{m}}{2\pi \rho r h}$$

$$V_1 = V_{r=50mm} = \frac{1}{2\pi}\times 0.305\frac{kg}{s}\times\frac{m^3}{999\,kg}\times\frac{1}{0.050\,m}\times\frac{1}{8\times10^{-4}\,m} = 1.21\ m/s$$

$$V_2 = V_{r=R} = \frac{1}{2\pi}\times 0.305\frac{kg}{s}\times\frac{m^3}{999\,kg}\times\frac{1}{0.075\,m}\times\frac{1}{8\times10^{-4}\,m} = 0.810\ m/s$$

From the Bernoulli equation

$$p_1 - p_2 = p_{r=50m} - p_{atm} = \frac{1}{2}\rho V_2^2 - \frac{1}{2}\rho V_1^2 = \frac{\rho}{2}(V_2^2 - V_1^2)$$

$$p_{r=50mm} = \frac{1}{2}\times 999\frac{kg}{m^3}\left[(0.810)^2 - (1.21)^2\right]\frac{m^2}{s^2}\times\frac{N\cdot s^2}{kg\cdot m}$$

$$p_{r=50mm} = -404\ N/m^2\ (gage) \longleftarrow p_{r=50mm}$$

Friction would cause a pressure drop in the flow direction. Since the discharge pressure is fixed at p_{atm}, the measured pressure would be greater than the theoretical value.

Problem 6.58

[Difficulty: 3]

6.58 Consider frictionless, incompressible flow of air over the wing of an airplane flying at 200 km/hr. The air approaching the wing is at 65 kPa and −10°C. At a certain point in the flow, the pressure is 60 kPa. Calculate the speed of the air relative to the wing at this point and the absolute air speed.

Given: Air flow over a wing

Find: Air speed relative to wing at a point; absolute air speed

Solution:

Basic equation $\quad \frac{p}{\rho} + \frac{V^2}{2} + g \cdot z = \text{const} \qquad p = \rho \cdot R \cdot T$

Assumptions: 1) Incompressible flow 2) Inviscid 3) Steady 4) Along a streamline

Available data $\quad T = -10\,°C \qquad p_1 = 65 \cdot kPa \qquad V_1 = 200 \cdot \frac{km}{hr} \qquad p_2 = 60 \cdot kPa \qquad R = 286.9 \cdot \frac{N \cdot m}{kg \cdot K}$

For air $\quad \rho = \frac{p_1}{R \cdot T} \qquad \rho = (65) \times 10^3 \cdot \frac{N}{m^2} \times \frac{kg \cdot K}{286.9 \cdot N \cdot m} \times \frac{1}{(-10 + 273) \cdot K} \qquad \rho = 0.861 \, \frac{kg}{m^3}$

Hence, applying Bernoulli between the upstream point (1) and the point on the wing (2)

$$\frac{p_1}{\rho} + \frac{{V_1}^2}{2} = \frac{p_2}{\rho} + \frac{{V_2}^2}{2} \qquad \text{where we ignore gravity effects}$$

Hence $\quad V_2 = \sqrt{{V_1}^2 + 2 \cdot \frac{(p_1 - p_2)}{\rho}}$

Then $\quad V_2 = \sqrt{\left(200 \cdot \frac{km}{hr}\right)^2 \times \left(\frac{1000 \cdot m}{1 \cdot km}\right)^2 \times \left(\frac{1 \cdot hr}{3600 \cdot s}\right)^2 + 2 \times \frac{m^3}{0.861 \cdot kg} \times (65 - 60) \times 10^3 \cdot \frac{N}{m^2} \times \frac{kg \cdot m}{N \cdot s^2}} \qquad V_2 = 121 \, \frac{m}{s}$

NOTE: At this speed, significant density changes will occur, so this result is not very realistic

The absolute velocity is

$$V_{2abs} = V_2 - V_1 \qquad V_{2abs} = 65.7 \, \frac{m}{s}$$

Problem 6.59

[Difficulty: 3]

6.59 A speedboat on hydrofoils is moving at 20 m/s in a freshwater lake. Each hydrofoil is 3 m below the surface. Assuming, as an approximation, frictionless, incompressible flow, find the stagnation pressure (gage) at the front of each hydrofoil. At one point on a hydrofoil, the pressure is −75 kPa (gage). Calculate the speed of the water relative to the hydrofoil at this point and the absolute water speed.

Given: Water flow over a hydrofoil

Find: Stagnation pressure; water speed relative to airfoil at a point; absolute value

Solution:

Basic equations $\frac{p}{\rho} + \frac{V^2}{2} + g \cdot z = \text{const}$ $\quad \Delta p = \rho \cdot g \cdot h$

Assumptions: 1) Incompressible flow 2) Inviscid 3) Steady 4) Along a streamline

Available data $V_1 = 20 \cdot \frac{m}{s}$ $\quad \rho = 999 \cdot \frac{kg}{m^3}$ $\quad h = 3 \cdot m$ $\quad p_2 = -75 \cdot kPa$ (gage)

Using coordinates fixed to the hydrofoil, the pressure at depth h is $p_1 = \rho \cdot g \cdot h$ $\quad p_1 = 29.4 \cdot kPa$

Applying Bernoulli between the upstream (1) and the stagnation point (at the front of the hydrofoil)

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} = \frac{p_0}{\rho} \qquad \text{or} \qquad p_0 = p_1 + \frac{1}{2} \cdot \rho \cdot V_1^2 \qquad p_0 = 229 \cdot kPa$$

Applying Bernoulli between the upstream point (1) and the point on the hydrofoil (2)

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} = \frac{p_2}{\rho} + \frac{V_2^2}{2}$$

Hence $$V_2 = \sqrt{V_1^2 + 2 \cdot \frac{(p_1 - p_2)}{\rho}} \qquad V_2 = 24.7 \frac{m}{s}$$

This is the speed of the water relative to the hydrofoil; in absolute coordinates $V_{2abs} = V_2 + V_1$ $\quad V_{2abs} = 44.7 \frac{m}{s}$

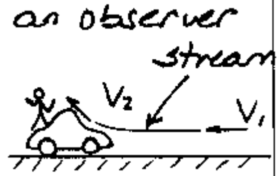
Problem 6.60

[Difficulty: 3]

6.60 A mercury barometer is carried in a car on a day when there is no wind. The temperature is 20°C and the corrected barometer height is 761 mm of mercury. One window is open slightly as the car travels at 105 km/hr. The barometer reading in the moving car is 5 mm lower than when the car is stationary. Explain what is happening. Calculate the local speed of the air flowing past the window, *relative to* the automobile.

Solution: (a) Air speed relative to car is higher than in the freestream, thus lowering the pressure at window.

(b) Apply the Bernoulli equation in frame seen by an observer on the car:



Basic equation: $\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2$

Assumptions: (1) Steady flow (seen by observer on car)
(2) Incompressible flow
(3) Neglect friction
(4) Flow along a streamline
(5) Neglect Δz

Then $V_2^2 = \left[V_1^2 + 2\left(\frac{p_1 - p_2}{\rho}\right)\right]$ or $V_2 = \left[V_1^2 + \frac{2(p_1 - p_2)}{\rho}\right]^{1/2}$ (1)

From fluid statics

$$p_1 - p_2 = \rho g (h_1 - h_2) = SG \rho_{H_2O}\, g\, \Delta h$$

$$= 13.6 \times 1000 \frac{kg}{m^3} \times 9.81 \frac{m}{s^2} \times 0.005\, m \times \frac{N \cdot s^2}{kg \cdot m}$$

$$p_1 - p_2 = 667\ N/m^2$$

and from ideal gas

$$\rho = \frac{p}{RT} = 13.6 \times 1000 \frac{kg}{m^3} \times 9.81 \frac{m}{s^2} \times 0.761\, m \times \frac{kg \cdot K}{287\ N \cdot m} \times \frac{1}{(273+20)\,K} \times \frac{N \cdot s^2}{kg \cdot m}$$

$$\rho = 1.21\ kg/m^3$$

Substituting into Eq. 1

$$V_2 = \left[\left(105 \frac{km}{hr} \times 1000 \frac{m}{km} \times \frac{hr}{3600\, s}\right)^2 + 2 \times 667 \frac{N}{m^2} \times \frac{m^2}{1.21\ kg} \times \frac{kg \cdot m}{N \cdot s^2}\right]^{1/2}$$

$V_2 = 44.2\ m/s$ (159 km/hr) relative to car

V_2

Problem 6.61

[Difficulty: 2]

6.61 A fire nozzle is coupled to the end of a hose with inside diameter $D = 3$ in. The nozzle is contoured smoothly and has outlet diameter $d = 1$ in. The design inlet pressure for the nozzle is $p_1 = 100$ psi (gage). Evaluate the maximum flow rate the nozzle could deliver.

Given: Flow through fire nozzle

Find: Maximum flow rate

Solution:

Basic equation $\quad \frac{p}{\rho} + \frac{V^2}{2} + g \cdot z = \text{const} \qquad Q = V \cdot A$

Assumptions: 1) Incompressible flow 2) Inviscid 3) Steady 4) Along a streamline

Hence, applying Bernoulli between the inlet (1) and exit (2)

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} = \frac{p_2}{\rho} + \frac{V_2^2}{2} \qquad \text{where we ignore gravity effects}$$

But we have $\quad Q = V_1 \cdot A_1 = V_1 \cdot \frac{\pi \cdot D^2}{4} = V_2 \cdot A_2 = \frac{\pi \cdot d^2}{4} \qquad \text{so} \qquad V_1 = V_2 \cdot \left(\frac{d}{D}\right)^2$

$$V_2^2 - V_2^2 \cdot \left(\frac{d}{D}\right)^4 = \frac{2 \cdot (p_2 - p_1)}{\rho}$$

Hence $\quad V_2 = \sqrt{\frac{2 \cdot (p_1 - p_2)}{\rho \cdot \left[1 - \left(\frac{d}{D}\right)^4\right]}}$

Then $\quad V_2 = \sqrt{2 \times \frac{ft^3}{1.94 \cdot slug} \times (100 - 0) \cdot \frac{lbf}{in^2} \times \left(\frac{12 \cdot in}{1 \cdot ft}\right)^2 \times \frac{1}{1 - \left(\frac{1}{3}\right)^3} \times \frac{slug \cdot ft}{lbf \cdot s^2}} \qquad V_2 = 124 \cdot \frac{ft}{s}$

$$Q = V_2 \cdot \frac{\pi \cdot d^2}{4} \qquad Q = \frac{\pi}{4} \times 124 \cdot \frac{ft}{s} \times \left(\frac{1}{12} \cdot ft\right)^2 \qquad Q = 0.676 \cdot \frac{ft^3}{s} \qquad Q = 304 \cdot \frac{gal}{min}$$

Problem 6.62

[Difficulty: 2]

6.62 A racing car travels at 235 mph along a straightaway. The team engineer wishes to locate an air inlet on the body of the car to obtain cooling air for the driver's suit. The plan is to place the inlet at a location where the air speed is 60 mph along the surface of the car. Calculate the static pressure at the proposed inlet location. Express the pressure rise above ambient as a fraction of the freestream dynamic pressure.

Given: Race car on straightaway

Find: Air inlet where speed is 60 mph; static pressure; pressure rise

Solution:

Basic equation $$\frac{p}{\rho} + \frac{V^2}{2} + g \cdot z = \text{const}$$

Assumptions: 1) Incompressible flow 2) Inviscid 3) Steady 4) Along a streamline 5) Standard atmosphere

Available data $p_{atm} = 101 \cdot kPa$ $\quad \rho = 0.002377 \frac{slug}{ft^3}$ $\quad V_1 = 235 \cdot mph$ $\quad V_2 = 60 \cdot mph$

Between location 1 (the upstream flow at 235 mph with respect to the car), and point 2 (on the car where V = 60 mph), Bernoulli becomes

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} = \frac{p_{atm}}{\rho} + \frac{V_1^2}{2} = \frac{p_2}{\rho} + \frac{V_2^2}{2}$$

Hence $$p_2 = p_{atm} + \frac{1}{2} \cdot \rho \cdot V_1^2 \cdot \left[1 - \left(\frac{V_2}{V_1}\right)^2\right] \qquad p_2 = 15.6\,psi$$

Note that the pressure rise is $$\Delta p = \frac{1}{2} \cdot \rho \cdot V_1^2 \cdot \left[1 - \left(\frac{V_2}{V_1}\right)^2\right] \qquad \Delta p = 0.917 \cdot psi$$

The freestream dynamic pressure is $$q = \frac{1}{2} \cdot \rho \cdot V_1^2 \qquad q = 0.980 \cdot psi$$

Then $$\frac{\Delta p}{q} = 93.5 \cdot \%$$

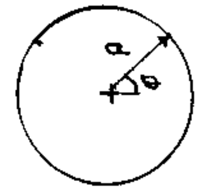
Note that at this speed the flow is borderline compressible!

Problem 6.63

[Difficulty: 2]

6.63 Steady, frictionless, and incompressible flow from left to right over a stationary circular cylinder, of radius a, is represented by the velocity field

$$\vec{V} = U\left[1 - \left(\frac{a}{r}\right)^2\right]\cos\theta\,\hat{e}_r - U\left[1 + \left(\frac{a}{r}\right)^2\right]\sin\theta\,\hat{e}_\theta$$



Obtain an expression for the pressure distribution along the streamline forming the cylinder surface, $r = a$. Determine the locations where the static pressure on the cylinder is equal to the freestream static pressure.

Solution:

Basic equation: $\dfrac{p}{\rho} + \dfrac{V^2}{2} + gz = \text{constant}$

Assumptions: (1) steady flow (given).
(2) incompressible flow (given)
(3) frictionless flow (given)
(4) flow along a streamline.

Along the cylinder surface $r = a$ and $\vec{V} = -2U\sin\theta\,\hat{e}_\theta$

Applying the Bernoulli equation along the streamline $r = a$,

$$\frac{p}{\rho} + \frac{V^2}{2} = \frac{p_\infty}{\rho} + \frac{U^2}{2}$$

$$p = p_\infty + \tfrac{1}{2}\rho\left(U^2 - V^2\right) = p_\infty + \tfrac{1}{2}\rho\left(U^2 - 4U^2\sin^2\theta\right)$$

$$p = p_\infty + \tfrac{1}{2}\rho U^2\left(1 - 4\sin^2\theta\right) \qquad p$$

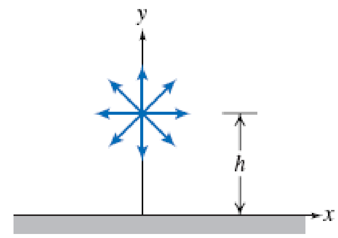
For $p = p_\infty$, $1 - 4\sin^2\theta = 0$ and $\sin\theta = \pm 0.5$

$$\therefore \theta = 30^\circ, 150^\circ, 210^\circ, 330^\circ \qquad \theta$$

Problem 6.64

[Difficulty: 3]

6.64 The velocity field for a plane source at a distance h above an infinite wall aligned along the x axis was given in Problem 6.8. Using the data from that problem, plot the pressure distribution along the wall from $x = -10h$ to $x = +10h$ (assume the pressure at infinity is atmospheric). Find the net force on the wall if the pressure on the lower surface is atmospheric. Does the force tend to pull the wall towards the source, or push it away?



Given: Velocity field

Find: Pressure distribution along wall; plot distribution; net force on wall

Solution:

The given data is $\quad q = 2\cdot\dfrac{\frac{m^3}{s}}{m} \qquad h = 1\cdot m \qquad \rho = 1000\cdot\dfrac{kg}{m^3}$

$$u = \frac{q\cdot x}{2\cdot\pi\left[x^2 + (y-h)^2\right]} + \frac{q\cdot x}{2\cdot\pi\left[x^2 + (y+h)^2\right]} \qquad v = \frac{q\cdot(y-h)}{2\cdot\pi\left[x^2 + (y-h)^2\right]} + \frac{q\cdot(y+h)}{2\cdot\pi\left[x^2 + (y+h)^2\right]}$$

The governing equation is the Bernoulli equation

$$\frac{p}{\rho} + \frac{1}{2}\cdot V^2 + g\cdot z = \text{const} \qquad \text{where} \qquad V = \sqrt{u^2 + v^2}$$

Apply this to point arbitrary point (x,0) on the wall and at infinity (neglecting gravity)

At $\quad |x| \rightarrow 0 \qquad u \rightarrow 0 \qquad v \rightarrow 0 \qquad V \rightarrow 0$

At point (x,0) $\quad u = \dfrac{q\cdot x}{\pi\cdot\left(x^2 + h^2\right)} \qquad v = 0 \qquad V = \dfrac{q\cdot x}{\pi\cdot\left(x^2 + h^2\right)}$

Hence the Bernoulli equation becomes $\quad \dfrac{p_{atm}}{\rho} = \dfrac{p}{\rho} + \dfrac{1}{2}\cdot\left[\dfrac{q\cdot x}{\pi\cdot\left(x^2 + h^2\right)}\right]^2$

or (with pressure expressed as gage pressure) $\quad p(x) = -\dfrac{\rho}{2}\cdot\left[\dfrac{q\cdot x}{\pi\cdot\left(x^2 + h^2\right)}\right]^2$

(Alternatively, the pressure distribution could have been obtained from Problem 6.8, where the momentum equation was used to find the pressure gradient $\dfrac{\partial}{\partial x}p = \dfrac{\rho\cdot q^2\cdot x\cdot\left(x^2 - h^2\right)}{\pi^2\cdot\left(x^2 + h^2\right)^3}$ along the wall. Integration of this with respect to x leads to the same result for $p(x)$)

The plot of pressure can be done in *Excel* (see below). From the plot it is clear that the wall experiences a negative gage pressure on the upper surface (and zero gage pressure on the lower), so the net force on the wall is upwards, towards the source

The force per width on the wall is given by $\quad F = \int_{-10\cdot h}^{10\cdot h} \left(p_{upper} - p_{lower}\right) dx \qquad F = -\frac{\rho\cdot q^2}{2\cdot\pi^2}\cdot\int_{-10\cdot h}^{10\cdot h} \frac{x^2}{\left(x^2+h^2\right)^2}\, dx$

The integral is $\quad \int \frac{x^2}{\left(x^2+h^2\right)^2}\, dx = \frac{\operatorname{atan}\left(\frac{x}{h}\right)}{2\cdot h} - \frac{x}{2\cdot h^2 + 2\cdot x^2}$

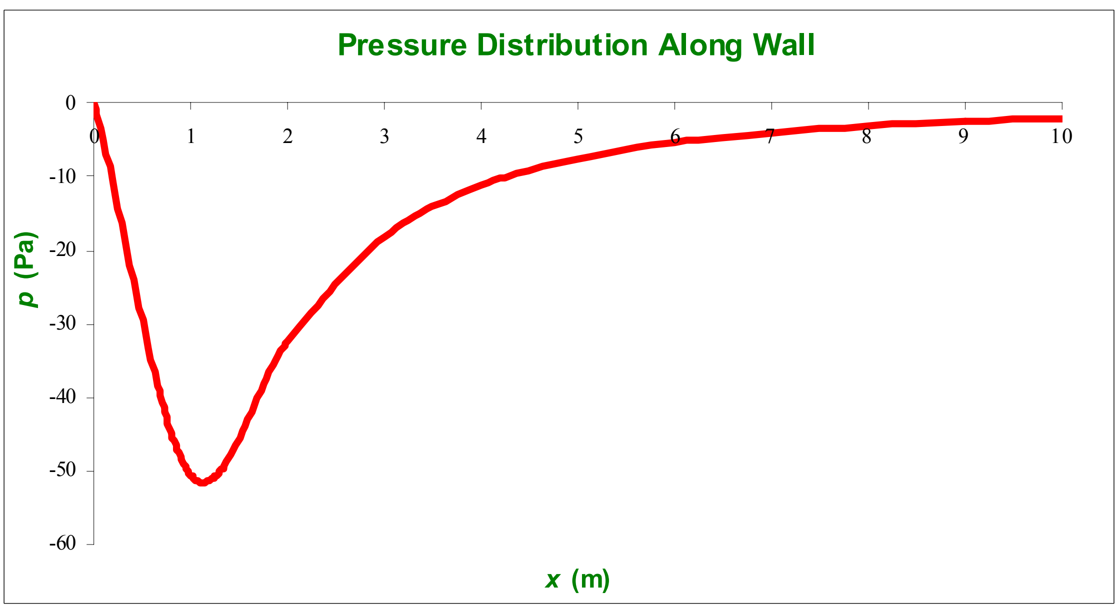
so $\quad F = -\frac{\rho\cdot q^2}{2\cdot\pi^2\cdot h}\cdot\left(-\frac{10}{101} + \operatorname{atan}(10)\right)$

$$F = -\frac{1}{2\cdot\pi^2} \times 1000\cdot\frac{kg}{m^3} \times \left(2\cdot\frac{m^2}{s}\right)^2 \times \frac{1}{1\cdot m} \times \left(-\frac{10}{101} + \operatorname{atan}(10)\right) \times \frac{N\cdot s^2}{kg\cdot m} \qquad F = -278\cdot\frac{N}{m}$$

In *Excel*:

q = 2 $m^3/s/m$
h = 1 m
ℵ= 1000 kg/m^3

x (m)	p (Pa)
0.0	0.00
1.0	-50.66
2.0	-32.42
3.0	-18.24
4.0	-11.22
5.0	-7.49
6.0	-5.33
7.0	-3.97
8.0	-3.07
9.0	-2.44
10.0	-1.99



Problem 6.65

[Difficulty: 3]

6.65 The velocity field for a plane doublet is given in Table 6.2. If $\Lambda = 3\ \mathrm{m^3 \cdot s^{-1}}$, the fluid density is $\rho = 1.5\ \mathrm{kg/m^3}$, and the pressure at infinity is 100 kPa, plot the pressure along the x axis from $x = -2.0$ m to -0.5 m and $x = 0.5$ m to 2.0 m.

Given: Velocity field for plane doublet

Find: Pressure distribution along x axis; plot distribution

Solution:

The governing equation is the Bernoulli equation $\quad \dfrac{p}{\rho} + \dfrac{1}{2}\cdot V^2 + g\cdot z = \text{const} \quad$ where $\quad V = \sqrt{u^2 + v^2}$

The given data is $\quad \Lambda = 3\cdot\dfrac{m^3}{s} \quad \rho = 1000\cdot\dfrac{kg}{m^3} \quad p_0 = 100\cdot kPa$

From Table 6.1 $\quad V_r = -\dfrac{\Lambda}{r^2}\cdot\cos(\theta) \quad V_\theta = -\dfrac{\Lambda}{r^2}\cdot\sin(\theta)$

where V_r and V_θ are the velocity components in cylindrical coordinates (r,θ). For points along the x axis, $r = x$, $\theta = 0$, $V_r = u$ and $V_\theta = v = 0$

$$u = -\frac{\Lambda}{x^2} \qquad v = 0$$

so (neglecting gravity) $\quad \dfrac{p}{\rho} + \dfrac{1}{2}\cdot u^2 = \text{const}$

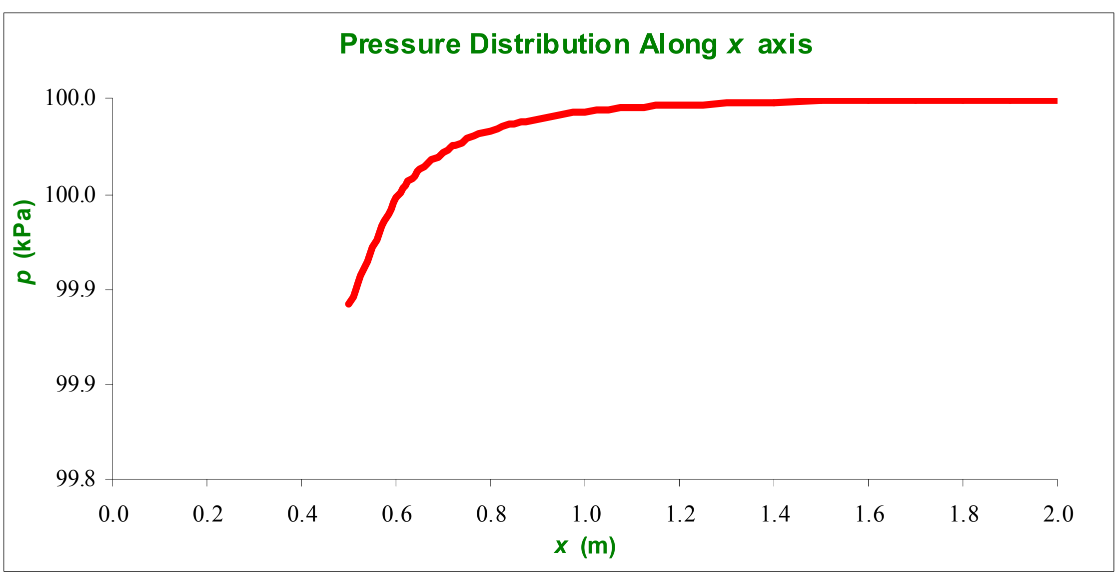
Apply this to point arbitrary point $(x,0)$ on the x axis and at infinity

At $\quad |x| \to 0 \quad u \to 0 \quad p \to p_0 \qquad$ At point $(x,0)$ $\quad u = -\dfrac{\Lambda}{x^2}$

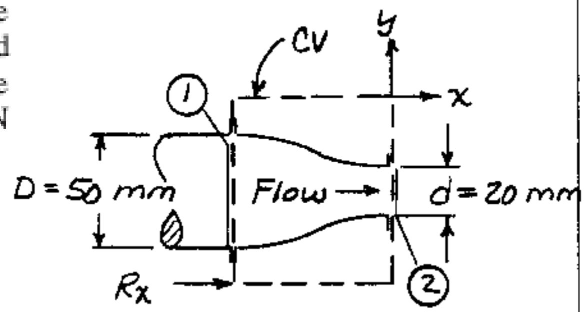
Hence the Bernoulli equation becomes $\quad \dfrac{p_0}{\rho} = \dfrac{p}{\rho} + \dfrac{\Lambda^2}{2\cdot x^4} \quad$ or $\quad p(x) = p_0 - \dfrac{\rho\cdot\Lambda^2}{2\cdot x^4}$

The plot of pressure can be done in *Excel*:

x (m)	p (Pa)
0.5	99.892
0.6	99.948
0.7	99.972
0.8	99.984
0.9	99.990
1.0	99.993
1.1	99.995
1.2	99.997
1.3	99.998
1.4	99.998
1.5	99.999
1.6	99.999
1.7	99.999
1.8	99.999
1.9	99.999
2.0	100.000



6.66 A smoothly contoured nozzle, with outlet diameter $d = 20$ mm, is coupled to a straight pipe by means of flanges. Water flows in the pipe, of diameter $D = 50$ mm, and the nozzle discharges to the atmosphere. For steady flow and neglecting the effects of viscosity, find the volume flow rate in the pipe corresponding to a calculated axial force of 45.5 N needed to keep the nozzle attached to the pipe.



Solution: Apply continuity, x momentum, and Bernoulli.

Basic equation:
$$0 = \cancelto{=0(1)}{\frac{\partial}{\partial t}}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A}$$

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + g\cancel{z_1} = \cancelto{=0(7)}{\frac{p_2}{\rho}} + \frac{V_2^2}{2} + g\cancel{z_2}$$

$$F_{Sx} + \cancelto{=0(6)}{F_{Bx}} = \cancelto{=0(1)}{\frac{\partial}{\partial t}}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$$

Assumptions: (1) Steady flow
(2) Uniform flow at each section
(3) Flow along a streamline
(4) Incompressible flow
(5) No friction
(6) Horizontal, $F_{Bx} = 0$, $z_1 = z_2$
(7) Use gage pressures

Then
$$0 = \{-V_1A_1\} + \{+V_2A_2\} \;;\quad V_2 = V_1\frac{A_1}{A_2} = V_1\left(\frac{D}{d}\right)^2 \;;\quad Q = V_1A_1 = V_2A_2$$

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} = \frac{V_2^2}{2} \;;\quad p_1 = \rho\left(\frac{V_2^2}{2} - \frac{V_1^2}{2}\right) = \frac{\rho V_1^2}{2}\left[\left(\frac{V_2}{V_1}\right)^2 - 1\right] = \frac{\rho V_1^2}{2}\left[\left(\frac{D}{d}\right)^4 - 1\right]$$

$$R_x + p_1A_1 - \cancelto{=0(7)}{p_2}A_2 = u_1\{-|\rho V_1A_1|\} + u_2\{+|\rho V_2A_2|\} = \rho V_1A_1(V_2 - V_1)$$

$$u_1 = V_1 \qquad u_2 = V_2$$

$$R_x + A_1\frac{\rho V_1^2}{2}\left[\left(\frac{D}{d}\right)^4 - 1\right] = \rho V_1^2A_1\left(\frac{V_2}{V_1} - 1\right) = \rho V_1^2A\left[\left(\frac{D}{d}\right)^2 - 1\right]$$

Thus
$$V_1^2 = \frac{-2R_x}{\rho A_1}\frac{1}{\left(\frac{D}{d}\right)^4 - 2\left(\frac{D}{d}\right)^2 + 1} \quad\text{so}\quad V_1 = \sqrt{\frac{-2R_x}{\rho A_1}}\frac{1}{\left(\frac{D}{d}\right)^2 - 1}$$

$$V_1 = \left[-2\times -45.5\,\text{N}\times\frac{\text{m}^3}{999\,\text{kg}}\times\frac{4}{\pi(0.050)^2\text{m}^2}\times\frac{\text{kg}\cdot\text{m}}{\text{N}\cdot\text{s}^2}\right]^{\frac{1}{2}}\frac{1}{\left(\frac{50}{20}\right)^2 - 1} = 1.30\ \text{m/s}$$

Finally,

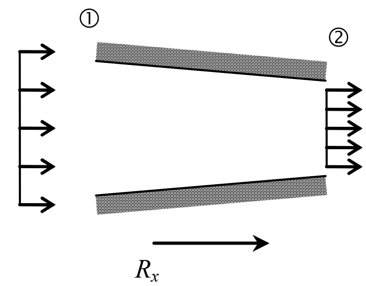
$$Q = V_1A_1 = 1.30\,\frac{\text{m}}{\text{s}}\times\frac{\pi}{4}(0.050)^2\text{m}^2 = 2.55\times10^{-3}\,\text{m}^3/\text{s}$$ Q

{Note: It is necessary to recognize that $R_x < 0$ for a nozzle, see Example Problem 4.7.}

Problem 6.67

[Difficulty: 3]

6.67 A fire nozzle is coupled to the end of a hose with inside diameter $D = 75$ mm. The nozzle is smoothly contoured and its outlet diameter is $d = 25$ mm. The nozzle is designed to operate at an inlet water pressure of 700 kPa (gage). Determine the design flow rate of the nozzle. (Express your answer in L/s.) Evaluate the axial force required to hold the nozzle in place. Indicate whether the hose coupling is in tension or compression.



Given: Flow through fire nozzle

Find: Maximum flow rate

Solution:

Basic equation $\quad \frac{p}{\rho} + \frac{V^2}{2} + g\cdot z = \text{const} \qquad Q = V\cdot A \qquad F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\, d\forall + \int_{CS} u\,\rho\,\vec{V}\cdot d\vec{A}$

Assumptions: 1) Incompressible flow 2) Inviscid 3) Steady 4) Along a streamline

Hence, applying Bernoulli between the inlet (1) and exit (2)

$$\frac{p_1}{\rho} + \frac{V_1^{\,2}}{2} = \frac{p_2}{\rho} + \frac{V_2^{\,2}}{2} \qquad \text{where we ignore gravity effects}$$

But we have $\quad Q = V_1\cdot A_1 = V_1\cdot\frac{\pi\cdot D^2}{4} = V_2\cdot\frac{\pi\cdot d^2}{4} \qquad$ so $\qquad V_1 = V_2\cdot\left(\frac{d}{D}\right)^2$

Hence in Bernoulli $\quad V_2^{\,2} - V_2^{\,2}\cdot\left(\frac{d}{D}\right)^4 = \frac{2\cdot(p_2 - p_1)}{\rho} \qquad$ or $\qquad V_2 = \sqrt{\frac{2\cdot(p_1 - p_2)}{\rho\cdot\left[1 - \left(\frac{d}{D}\right)^4\right]}}$

$$V_2 = \sqrt{2 \times \frac{m^3}{1000\cdot kg} \times (700 - 0) \times 10^3\cdot\frac{N}{m^2} \times \frac{1}{1 - \left(\frac{25}{75}\right)^4} \times \frac{kg\cdot m}{N\cdot s^2}} \qquad V_2 = 37.6\,\frac{m}{s}$$

Then $\quad Q = V_2\cdot\frac{\pi\cdot d^2}{4} \qquad Q = \frac{\pi}{4} \times 37.6\cdot\frac{m}{s} \times (0.025\cdot m)^2 \qquad Q = 0.0185\cdot\frac{m^3}{s} \qquad Q = 18.5\cdot\frac{L}{s}$

From x momentum $\quad R_x + p_1\cdot A_1 = u_1\cdot(-\rho\cdot V_1\cdot A_1) + u_2\cdot(\rho\cdot V_2\cdot A_2) \qquad$ using gage pressures

Hence $\quad R_x = -p_1\cdot\frac{\pi\cdot D^2}{4} + \rho\cdot Q\cdot(V_2 - V_1) = -p_1\cdot\frac{\pi\cdot D^2}{4} + \rho\cdot Q\cdot V_2\cdot\left[1 - \left(\frac{d}{D}\right)^2\right]$

$$R_x = -700 \times 10^3\cdot\frac{N}{m^2} \times \frac{\pi}{4}\cdot(0.075\cdot m)^2 + 1000\cdot\frac{kg}{m^3} \times 0.0185\cdot\frac{m^3}{s} \times 37.6\cdot\frac{m}{s} \times \left[1 - \left(\frac{25}{75}\right)^3\right] \times \frac{N\cdot s^2}{kg\cdot m} \qquad R_x = -2423\,N$$

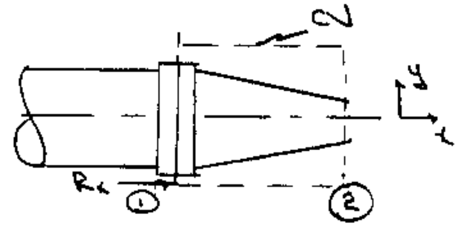
This is the force of the nozzle on the fluid; hence the force of the fluid on the nozzle is 2400 N to the right; the nozzle is in tension

Problem 6.68

[Difficulty: 3]

6.68 Water flows steadily through a 3.25-in.-diameter pipe and discharges through a 1.25-in.-diameter nozzle to atmospheric pressure. The flow rate is 24.5 gpm. Calculate the minimum static pressure required in the pipe to produce this flow rate. Evaluate the axial force of the nozzle assembly on the pipe flange.

Solution:



Apply the Bernoulli equation along the central streamline between sections ① and ②

$$\frac{p_1}{\rho}+\frac{V_1^2}{2}+\cancel{gz_1}=\frac{p_2}{\rho}+\frac{V_2^2}{2}+\cancel{gz_2}$$

Assumptions: (1) steady flow (2) incompressible flow (3) frictionless flow (4) flow along a streamline. (5) $\Delta z=0$ (6) uniform flow at each section

Then $p_1=p_2+\frac{\rho}{2}(V_2^2-V_1^2)=p_2+\frac{\rho V_2^2}{2}\left[1-\left(\frac{V_1}{V_2}\right)^2\right]$

$p_2=p_{atm}$ and from continuity, $A_2V_2=A_1V_1$

$$\therefore\ p_{1g}=\frac{\rho}{2}V_2^2\left[1-\left(\frac{A_2}{A_1}\right)^2\right]=\frac{\rho V_2^2}{2}\left[1-\left(\frac{D_2}{D_1}\right)^4\right]$$

$$V_2=\frac{Q}{A}=\frac{4Q}{\pi d^2}=\frac{4}{\pi}\times 24.5\,\frac{gal}{min}\times\frac{ft^3}{7.48\,gal}\times\frac{min}{60\,s}\times\frac{1}{(1.25)^2\,in^2}\times\frac{144\,in^2}{ft^2}$$

$V_2=6.41$ ft/s and

$$p_{1g}=\frac{1}{2}\times 1.94\,\frac{slug}{ft^3}\times(6.41)^2\frac{ft^2}{s^2}\times\frac{lbf\cdot s^2}{ft\cdot slug}\left[1-\left(\frac{1.25}{3.25}\right)^4\right]=39.0\ \text{psfgage}$$ p_1

(b) Apply the x momentum equation to the CV

$$F_{Sx}+F_{Bx}=\cancel{\frac{\partial}{\partial t}\int_{CV}u\rho\,d\forall}^{\,0}+\int_{CS}u\rho\vec{V}\cdot d\vec{A}$$

$$R_x+p_{1g}A_1=u_1\{-\dot{m}\}+u_2\{\dot{m}\}=-V_1\dot{m}+V_2\dot{m}$$

$$R_x=-p_{1g}A_1+\dot{m}(V_2-V_1)=-p_{1g}A_1+\rho QV_2\left(1-\frac{V_1}{V_2}\right)$$

$$=-39\,\frac{lbf}{ft^2}\times\frac{\pi}{4}\left(\frac{3.25}{12}\right)^2ft^2+1.94\,\frac{slug}{ft^3}\times 24.5\,\frac{gal}{min}\times\frac{ft^3}{7.48\,gal}\times\frac{min}{60\,s}\times 6.41\,\frac{ft}{s}\left[1-\left(\frac{1.25}{3.25}\right)^2\right]\frac{lbf\cdot s^2}{slug\cdot ft}$$

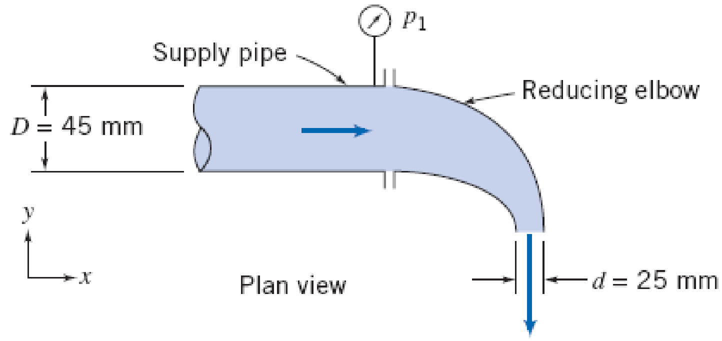
$R_x=-2.25+0.58=-1.67$ lbf

Force of nozzle on flange $K_x=-R_x=1.67$ lbf K_x

Problem 6.69

[Difficulty: 3]

6.69 Water flows steadily through the reducing elbow shown. The elbow is smooth and short, and the flow accelerates; so the effect of friction is small. The volume flow rate is $Q = 2.5$ L/s. The elbow is in a horizontal plane. Estimate the gage pressure at section (1). Calculate the x component of the force exerted by the reducing elbow on the supply pipe.



Given: Flow through reducing elbow

Find: Gage pressure at location 1; x component of force

Solution:

Basic equations: $\frac{p}{\rho} + \frac{V^2}{2} + g \cdot z = \text{const}$ $\quad F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\,dV\!\!\!\!- + \int_{CS} u\,\rho\,\vec{V}\cdot d\vec{A}$ $\quad Q = V \cdot A$

Assumptions: 1) Incompressible flow 2) Inviscid 3) Steady 4) Along a streamline 5) Ignore elevation change 6) $p_2 = p_{atm}$

Available data: $Q = 2.5 \cdot \frac{L}{s}$ $\quad Q = 2.5 \times 10^{-3} \frac{m^3}{s}$ $\quad D = 45 \cdot mm$ $\quad d = 25 \cdot mm$ $\quad \rho = 999 \cdot \frac{kg}{m^3}$

From contnuity $\quad V_1 = \frac{Q}{\left(\frac{\pi \cdot D^2}{4}\right)}$ $\quad V_1 = 1.57 \frac{m}{s}$ $\quad V_2 = \frac{Q}{\left(\frac{\pi \cdot d^2}{4}\right)}$ $\quad V_2 = 5.09 \frac{m}{s}$

Hence, applying Bernoulli between the inlet (1) and exit (2) $\quad \frac{p_1}{\rho} + \frac{V_1^2}{2} = \frac{p_2}{\rho} + \frac{V_2^2}{2}$

or, in gage pressures $\quad p_{1g} = \frac{\rho}{2} \cdot \left(V_2^2 - V_1^2\right)$ $\quad p_{1g} = 11.7 \cdot kPa$

From x-momentum $\quad R_x + p_{1g} \cdot A_1 = u_1 \cdot \left(-m_{rate}\right) + u_2 \cdot \left(m_{rate}\right) = -m_{rate} \cdot V_1 = -\rho \cdot Q \cdot V_1$ $\quad$ because $\quad u_1 = V_1 \quad u_2 = 0$

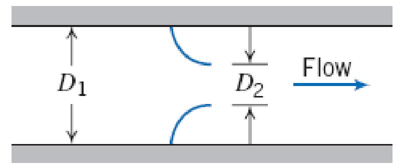
$$R_x = -p_{1g} \cdot \frac{\pi \cdot D^2}{4} - \rho \cdot Q \cdot V_1 \qquad R_x = -22.6\,N$$

The force on the supply pipe is then $\quad K_x = -R_x \quad K_x = 22.6\,N$ on the pipe to the right

Problem 6.70

[Difficulty: 3]

6.70 A flow nozzle is a device for measuring the flow rate in a pipe. This particular nozzle is to be used to measure low-speed air flow for which compressibility may be neglected. During operation, the pressures p_1 and p_2 are recorded, as well as upstream temperature, T_1. Find the mass flow rate in terms of $\Delta p = p_2 - p_1$ and T_1, the gas constant for air, and device diameters D_1 and D_2. Assume the flow is frictionless. Will the actual flow be more or less than this predicted flow? Why?



Given: Flow nozzle

Find: Mass flow rate in terms of Δp, T_1 and D_1 and D_2

Solution:

Basic equation $$\frac{p}{\rho} + \frac{V^2}{2} + g \cdot z = \text{const} \qquad Q = V \cdot A$$

Assumptions: 1) Incompressible flow 2) Inviscid 3) Steady 4) Along a streamline

Hence, applying Bernoulli between the inlet (1) and exit (2)

$$\frac{p_1}{\rho} + \frac{V_1^{\,2}}{2} = \frac{p_2}{\rho} + \frac{V_2^{\,2}}{2} \qquad \text{where we ignore gravity effects}$$

But we have $$Q = V_1 \cdot A_1 = V_1 \cdot \frac{\pi \cdot D_1^{\,2}}{4} = V_2 \cdot \frac{\pi \cdot D_2^{\,2}}{4} \qquad \text{so} \qquad V_1 = V_2 \cdot \left(\frac{D_2}{D_1}\right)^2$$

Note that we assume the flow at D_2 is at the same pressure as the entire section 2; this will be true if there is turbulent mixing

Hence $$V_2^{\,2} - V_2^{\,2} \cdot \left(\frac{D_2}{D_1}\right)^4 = \frac{2 \cdot (p_2 - p_1)}{\rho} \qquad \text{or} \qquad V_2 = \sqrt{\frac{2 \cdot (p_1 - p_2)}{\rho \cdot \left[1 - \left(\frac{D_2}{D_1}\right)^4\right]}}$$

Then the mass flow rate is $$m_{flow} = \rho \cdot V_2 \cdot A_2 = \rho \cdot \frac{\pi \cdot D_2^{\,2}}{4} \cdot \sqrt{\frac{2 \cdot (p_1 - p_2)}{\rho \cdot \left[1 - \left(\frac{D_2}{D_1}\right)^4\right]}} = \frac{\pi \cdot D_2^{\,2}}{2 \cdot \sqrt{2}} \cdot \sqrt{\frac{\Delta p \cdot \rho}{\left[1 - \left(\frac{D_2}{D_1}\right)^4\right]}}$$

Using $$p = \rho \cdot R \cdot T \qquad m_{flow} = \frac{\pi \cdot D_2^{\,2}}{2 \cdot \sqrt{2}} \cdot \sqrt{\frac{\Delta p \cdot p_1}{R \cdot T_1 \cdot \left[1 - \left(\frac{D_2}{D_1}\right)^4\right]}}$$

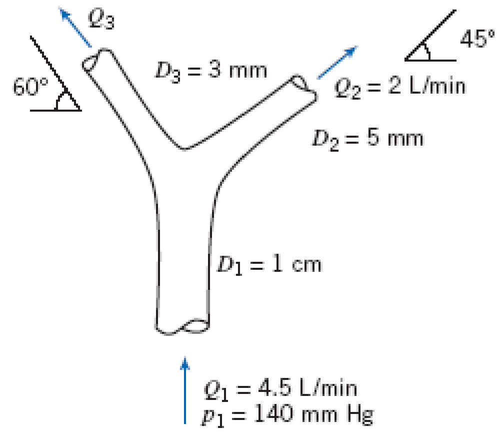
For a flow nozzle $$m_{flow} = k \cdot \sqrt{\Delta p} \quad \text{where} \qquad k = \frac{\pi \cdot D_2^{\,2}}{2 \cdot \sqrt{2}} \cdot \sqrt{\frac{p_1}{R \cdot T_1 \cdot \left[1 - \left(\frac{D_2}{D_1}\right)^4\right]}}$$

We can expect the actual flow will be less because there is actually significant loss in the device. Also the flow will experience a vena contracta so that the minimum diameter is actually smaller than D_2. We will discuss this device in Chapter 8.

Problem 6.71

[Difficulty: 4]

6.71 The branching of a blood vessel is shown. Blood at a pressure of 140 mm Hg flows in the main vessel at 4.5 L/min. Estimate the blood pressure in each branch, assuming that blood vessels behave as rigid tubes, that we have frictionless flow, and that the vessel lies in the horizontal plane. What is the force generated at the branch by the blood? You may approximate blood to have a density of 1060 kg/m³.



Given: Flow through branching blood vessel

Find: Blood pressure in each branch; force at branch

Solution:

Basic equations $\quad \dfrac{p}{\rho} + \dfrac{V^2}{2} + g\cdot z = \text{const} \qquad \sum_{CV} Q = 0 \qquad Q = V\cdot A \qquad \Delta p = \rho\cdot g\cdot \Delta h$

$$F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\, d\forall + \int_{CS} u\,\rho\,\vec{V}\cdot d\vec{A}$$

$$F_y = F_{S_y} + F_{B_y} = \frac{\partial}{\partial t}\int_{CV} v\,\rho\, d\forall + \int_{CS} v\,\rho\,\vec{V}\cdot d\vec{A}$$

Assumptions: 1) Incompressible flow 2) Inviscid 3) Steady 4) Along a streamline

Given data $\quad Q_1 = 4.5\cdot\dfrac{L}{min} \qquad Q_2 = 2\cdot\dfrac{L}{min} \qquad D_1 = 10\cdot mm \qquad D_2 = 5\cdot mm \qquad D_3 = 3\cdot mm$

$SG_{Hg} = 13.6 \qquad \rho = 999\cdot\dfrac{kg}{m^3} \qquad \rho_b = 1060\cdot\dfrac{kg}{m^3} \qquad h_1 = 140\cdot mm$ (pressure in in. Hg)

For Q_3 we have $\quad \sum_{CV} Q = -Q_1 + Q_2 + Q_3 = 0 \quad$ so $\quad Q_3 = Q_1 - Q_2 \qquad Q_3 = 2.50\cdot\dfrac{L}{min}$

We will need each velocity $\quad V_1 = \dfrac{Q_1}{A_1} \qquad V_1 = \dfrac{4\cdot Q_1}{\pi\cdot D_1{}^2} \qquad V_1 = 0.955\,\dfrac{m}{s}$

Similarly $\quad V_2 = \dfrac{4\cdot Q_2}{\pi\cdot D_2{}^2} \qquad V_2 = 1.70\,\dfrac{m}{s} \qquad V_3 = \dfrac{4\cdot Q_3}{\pi\cdot D_3{}^2} \qquad V_3 = 5.89\,\dfrac{m}{s}$

Hence, applying Bernoulli between the inlet (1) and exit (2)

$$\frac{p_1}{\rho} + \frac{V_1{}^2}{2} = \frac{p_2}{\rho} + \frac{V_2{}^2}{2}$$ where we ignore gravity effects

$$p_2 = p_1 + \frac{\rho}{2}\cdot\left(V_1^{\ 2} - V_2^{\ 2}\right) \qquad \text{and} \qquad p_1 = SG_{Hg}\cdot\rho\cdot g\cdot h_1 \qquad p_1 = 18.7\cdot kPa$$

Hence $\quad p_2 = p_1 + \frac{\rho_b}{2}\cdot\left(V_1^{\ 2} - V_2^{\ 2}\right) \qquad p_2 = 17.6\cdot kPa$

In mm Hg $\quad h_2 = \frac{p_2}{SG_{Hg}\cdot\rho\cdot g} \qquad h_2 = 132\cdot mm$

Similarly for exit (3) $\quad p_3 = p_1 + \frac{\rho}{2}\cdot\left(V_1^{\ 2} - V_3^{\ 2}\right) \qquad p_3 = 1.75\cdot kPa$

In mm Hg $\quad h_3 = \frac{p_3}{SG_{Hg}\cdot\rho\cdot g} \qquad h_3 = 13.2\cdot mm$

Note that all pressures are gage.

For x momentum $\quad R_x + p_3\cdot A_3\cdot\cos(60\cdot deg) - p_2\cdot A_2\cdot\cos(45\cdot deg) = u_3\cdot(\rho\cdot Q_3) + u_2\cdot(\rho\cdot Q_2)$

$$R_x = p_2\cdot A_2\cdot\cos(45\cdot deg) - p_3\cdot A_3\cdot\cos(60\cdot deg) + \rho\cdot\left(Q_2\cdot V_2\cdot\cos(45\cdot deg) - Q_3\cdot V_3\cdot\cos(60\cdot deg)\right)$$

$$R_x = p_2\cdot\frac{\pi\cdot D_2^{\ 2}}{4}\cdot\cos(45\cdot deg) - p_3\cdot\frac{\pi\cdot D_3^{\ 2}}{4}\cdot\cos(60\cdot deg) + \rho\cdot\left(Q_2\cdot V_2\cdot\cos(45\cdot deg) - Q_3\cdot V_3\cdot\cos(60\cdot deg)\right) \qquad R_x = 0.156\,N$$

For y momentum $\quad R_y - p_3\cdot A_3\cdot\sin(60\cdot deg) - p_2\cdot A_2\cdot\sin(45\cdot deg) + p_1\cdot A_1 = v_3\cdot(\rho\cdot Q_3) + v_2\cdot(\rho\cdot Q_2)$

$$R_y = p_2\cdot A_2\cdot\sin(45\cdot deg) + p_3\cdot A_3\cdot\sin(60\cdot deg) - p_1\cdot A_1 + \rho\cdot\left(Q_2\cdot V_2\cdot\sin(45\cdot deg) + Q_3\cdot V_3\cdot\sin(60\cdot deg)\right)$$

$$R_y = p_2\cdot\frac{\pi\cdot D_2^{\ 2}}{4}\cdot\sin(45\cdot deg) + p_3\cdot\frac{\pi\cdot D_3^{\ 2}}{4}\cdot\sin(60\cdot deg) - p_1\cdot\frac{\pi\cdot D_1^{\ 2}}{4} + \rho\cdot\left(Q_2\cdot V_2\cdot\sin(45\cdot deg) + Q_3\cdot V_3\cdot\sin(60\cdot deg)\right)$$

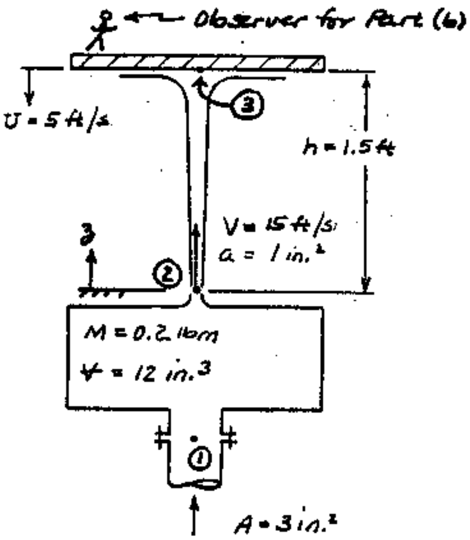
$$R_y = -0.957\,N$$

Problem 6.72

[Difficulty: 3] Part 1/2

6.72 An object, with a flat horizontal lower surface, moves downward into the jet of the spray system of Problem 4.81 with speed $U = 5$ ft/s. Determine the minimum supply pressure needed to produce the jet leaving the spray system at $V = 15$ ft/s. Calculate the maximum pressure exerted by the liquid jet on the flat object at the instant when the object is $h = 1.5$ ft above the jet exit. Estimate the force of the water jet on the flat object.

Solution:



(a)

The minimum pressure occurs when friction is neglected, and so we apply the Bernoulli equation

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + \cancel{g z_1}^{(5)} = \frac{p_2}{\rho} + \frac{V_2^2}{2} + \cancel{g z_2}^{(5)}$$

Assume: (1) steady flow
(2) incompressible flow
(3) no friction
(4) flow along a streamline
(5) neglect $z_2 - z_1$
(6) $p_2 = p_{atm}$
(7) uniform flow at ① & ②

Then

$$p_{1g} = p_1 - p_{atm} = \frac{\rho}{2}(V_2^2 - V_1^2) = \frac{\rho V_2^2}{2}\left[1 - \left(\frac{V_1}{V_2}\right)^2\right]$$

From continuity, $A_1 V_1 = A_2 V_2$, and $\frac{V_1}{V_2} = \frac{A_2}{A_1} = \frac{a}{A}$. Then,

$$p_{1g} = \frac{\rho V_2^2}{2}\left[1 - \left(\frac{a}{A}\right)^2\right] = \frac{1}{2} \times 1.94 \frac{\text{slug}}{\text{ft}^3} \times (15)^2 \frac{\text{ft}^2}{\text{s}^2}\left[1 - \left(\frac{1}{3}\right)^2\right] \times \frac{\text{lbf}\cdot\text{s}^2}{\text{ft}\cdot\text{slug}} \times \frac{\text{ft}^2}{144\,\text{in.}^2} = 1.35\ \text{psig} \quad p_{1g}$$

Frictional effects would cause this value to be higher.

(b) The maximum pressure of the jet on the object is the stagnation pressure

$$p_0 = p + \frac{1}{2}\rho V_r^2$$

where V_r is the velocity of the impinging jet relative to the object.

At $z = 1.5$ ft, the jet velocity, V_4, in the absence of the object can be calculated from

$$\frac{p_2}{\rho} + \frac{V_2^2}{2} + g z_2 = \cancel{\frac{p_4}{\rho}} + \frac{V_4^2}{2} + g z_4$$

$$V_4 = \left[V_2^2 - 2g(z_4 - z_2)\right]^{1/2} = \left[(15)^2 \frac{\text{ft}^2}{\text{s}^2} - 2 \times 32.2 \frac{\text{ft}}{\text{s}^2}(1.5)\,\text{ft}\right]^{1/2} = 11.3\ \text{ft/s}$$

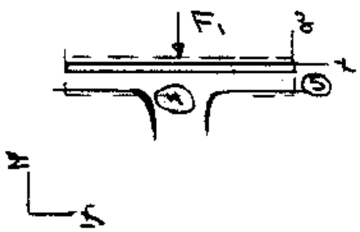
Then

$$V_{rel} = V_4 - (-U) = (11.3 + 5)\ \text{ft/s} = 16.3\ \text{ft/s}$$

and

$$p_0 - p_{atm} = p_{0g} = \frac{1}{2}\rho V_r^2 = \frac{1}{2} \times 1.94 \frac{\text{slug}}{\text{ft}^3} \times (16.3)^2 \frac{\text{ft}^2}{\text{s}^2} \times \frac{\text{lbf}\cdot\text{s}^2}{\text{ft}\cdot\text{slug}} \times \frac{\text{ft}^2}{144\,\text{in.}^2} = 1.79\ \text{psig} \quad p_{0g}$$

(c) To determine the force of the water on the object we apply the z component of the momentum equation to the CV shown



$$F_{S_z} + \cancel{F_{B_z}}^{\,\approx 0(9)} = \cancel{\frac{\partial}{\partial t}\int_{CV} w_{xyz}\,\rho\, d\forall}^{\,\approx 0(8)} + \int_{CS} w_{xyz}\left(\rho \vec{V}_{xyz}\cdot d\vec{A}\right)$$

Assumptions: (8) neglect $\frac{\partial}{\partial t}\int_{CV}$

(9) neglect body forces

(10) uniform radial flow at ⑤

(11) uniform vertical flow at ④ with $z_4 = 1.5$ ft

Then $-F_1 = -w_{4xyz}\left|\rho V_{4xyz} A_4\right|$

where F_1 is applied force necessary to maintain motion of plate at constant speed U

$$V_{4xyz} = V_4 - (-U) = V_4 + U$$

$$w_{4xyz} = V_{4xyz} = V_4 + U$$

$$\therefore \quad F_1 = \rho\,(V_4 + U)^2 A_4$$

From continuity $\quad A_2V_2 = A_4V_4$

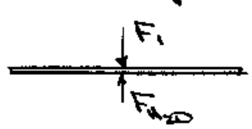
$$\text{and } A_4 = \frac{V_2}{V_4}A_2 = \frac{15}{11.3}\times 1\,\text{in}^2 = 1.33\ \text{in}^2$$

Then

$$F_1 = \rho\,(V_4+U)^2 A_4 = 1.94\,\frac{\text{slug}}{\text{ft}^3}\,(11.3+5)^2\,\frac{\text{ft}^2}{\text{s}^2}\times 1.33\,\text{in}^2\times\frac{\text{ft}^2}{144\,\text{in}^2}\times\frac{\text{lbf}\cdot\text{s}^2}{\text{ft}\cdot\text{slug}}$$

$F_1 = 4.76$ lbf (in the direction shown)

Since the plate is moving at constant speed, then



$$\sum \vec{F}_{plate} = m\bar{a} = 0 \quad \text{and}$$

neglecting the weight of the plate then

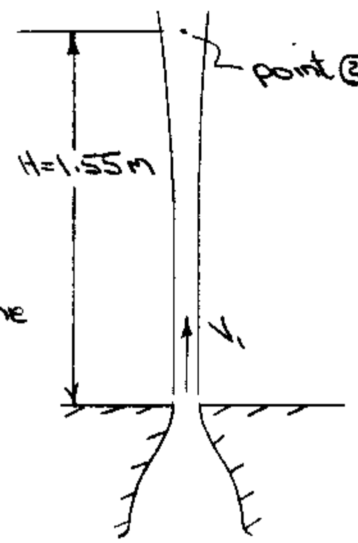
$$F_{H_2O} = F_1 = 4.76\ \text{lbf}$$

$$\vec{F}_{H_2O} = 4.76\,\hat{k}\ \text{lbf}$$

Problem 6.73

[Difficulty: 3]

6.73 A water jet is directed upward from a well-designed nozzle of area $A_1 = 600\ \text{mm}^2$; the exit jet speed is $V_1 = 6.3$ m/s. The flow is steady and the liquid stream does not break up. Point ② is located $H = 1.55$ m above the nozzle exit plane. Determine the velocity in the undisturbed jet at point ②. Calculate the pressure that would be sensed by a stagnation tube located there. Evaluate the force that would be exerted on a flat plate placed normal to the stream at point ②. Sketch the pressure distribution on the plate.



Solution: Apply Bernoulli and then y-momentum equation

Basis eq.: $\cancel{\frac{p_1}{\rho}} + \frac{V_1^2}{2} + gz_1 = \cancel{\frac{p_2}{\rho}} + \frac{V_2^2}{2} + gz_2$

Assumptions: (1) steady flow
(2) incompressible flow
(3) frictionless flow
(4) flow along a streamline
(5) $p_1 = p_2 = p_{atm}$

Then

$$V_2 = \left[V_1^2 + 2g(z_1 - z_2)\right]^{1/2}$$

$$V_2 = \left[(6.3)^2\,\frac{m^2}{s^2} + 2\times 9.81\,\frac{m}{s^2}\,(-1.55\,m)\right]^{1/2}$$

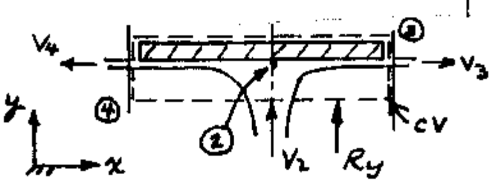
$V_2 = 3.05$ m/s ← V_2

By definition, $p_{0_2} = p_2 + \frac{1}{2}\rho V_2^2 = p_{atm} + \frac{1}{2}\rho V_2^2$, so

$$p_{0_2\,gage} = \frac{1}{2}\times 999\,\frac{kg}{m^3}\times(3.05)^2\,\frac{m^2}{s^2}\times\frac{N\cdot s^2}{kg\cdot m} = 4.65\,kPa\,(g)$$ ← p_{0_2}

Apply y-momentum equation to CV surrounding plate

Basic eq.: $F_{S_y} + \underbrace{\cancel{F_{B_y}}}_{\approx 0(6)} = \underbrace{\cancel{\frac{\partial}{\partial t}}}_{=0(1)}\int_{CV} v\rho\, dV + \int_{CS} v\,\rho\vec{V}\cdot d\vec{A}$



Assumptions: (6) neglect mass in CV
(7) V_2 enters CV uniformly
(8) $v_3 = v_4 = 0$

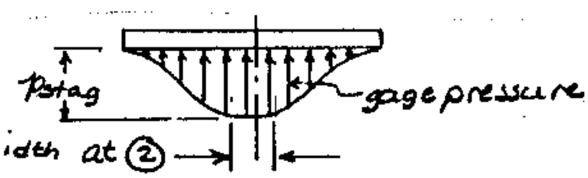
Then

$$R_y = v_2\{-\rho V_1 A_1\} + \underbrace{\cancel{v_3}}_{=0(8)}\{\dot{m}_3\} + \underbrace{\cancel{v_4}}_{=0(8)}\{\dot{m}_4\} = -\rho V_1 A_1 V_2 \quad \text{and}$$

$$K_y = -R_y = \rho V_1 A_1 V_2 = 999\,\frac{kg}{m^3}\times 6.3\,\frac{m}{s}\times 600\,mm^2\times 3.05\,\frac{m}{s}\times\frac{m^2}{10^6\,mm^2}\times\frac{N\cdot s^2}{kg\cdot m}$$

$K_y = 11.5$ N (force up) ← K_y

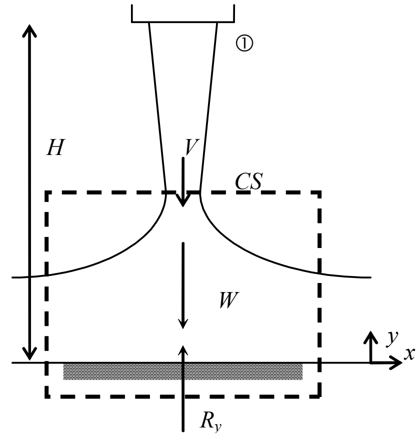
The pressure distribution on the plate is as shown.



Problem 6.74

[Difficulty: 3]

6.74 Water flows out of a kitchen faucet of 1.25 cm diameter at the rate of 0.1 L/s. The bottom of the sink is 45 cm below the faucet outlet. Will the cross-sectional area of the fluid stream increase, decrease, or remain constant between the faucet outlet and the bottom of the sink? Explain briefly. Obtain an expression for the stream cross section as a function of distance y above the sink bottom. If a plate is held directly under the faucet, how will the force required to hold the plate in a horizontal position vary with height above the sink? Explain briefly.



Given: Flow through kitchen faucet

Find: Area variation with height; force to hold plate as function of height

Solution:

Basic equation
$$\frac{p}{\rho} + \frac{V^2}{2} + g \cdot z = \text{const} \qquad Q = V \cdot A \qquad F_y = F_{S_y} + F_{B_y} = \frac{\partial}{\partial t}\int_{CV} v\,\rho\,dV + \int_{CS} v\,\rho\,\vec{V}\cdot d\vec{A}$$

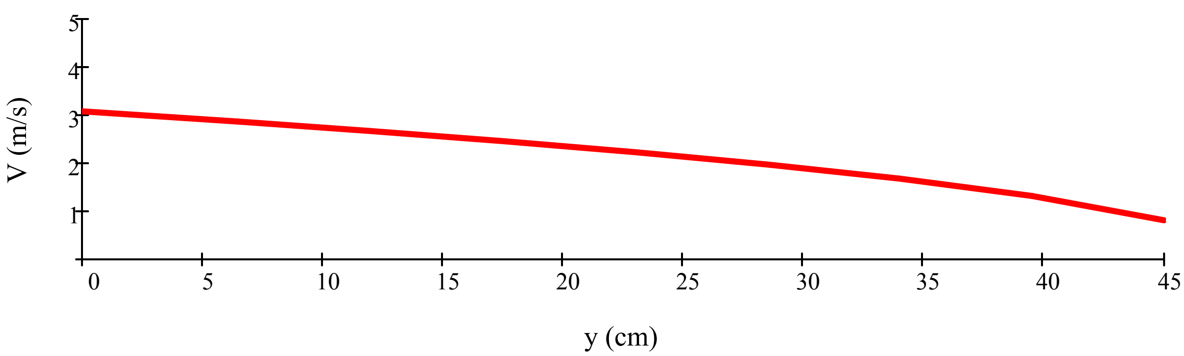
Assumptions: 1) Incompressible flow 2) Inviscid 3) Steady 4) Along a streamline

Hence, applying Bernoulli between the faucet (1) and any height y

$$\frac{V_1^2}{2} + g \cdot H = \frac{V^2}{2} + g \cdot y$$ where we assume the water is at p_{atm}

Hence $$V(y) = \sqrt{V_1^2 + 2 \cdot g \cdot (H - y)}$$

The problem doesn't require a plot, but it looks like $V_1 = 0.815\,\frac{m}{s}$ $\qquad V(0 \cdot m) = 3.08\,\frac{m}{s}$

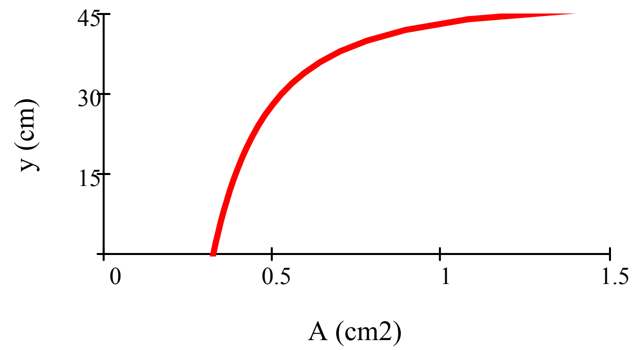


The speed increases as y decreases because the fluid particles "trade" potential energy for kinetic, just as a falling solid particle does!

But we have $$Q = V_1 \cdot A_1 = V_1 \cdot \frac{\pi \cdot D^2}{4} = V \cdot A$$

Hence $$A = \frac{V_1 \cdot A_1}{V} \qquad A(y) = \frac{\pi \cdot D_1^2 \cdot V_1}{4 \cdot \sqrt{V_1^2 + 2 \cdot g \cdot (H - y)}}$$

The problem doesn't require a plot, but it looks like



$$A(H) = 1.23 \cdot cm^2$$

$$A(0) = 0.325 \cdot cm^2$$

The area decreases as the speed increases. If the stream falls far enough the flow will change to turbulent.

For the CV above $\quad R_y - W = u_{in} \cdot \left(-\rho \cdot V_{in} \cdot A_{in}\right) = -V \cdot (-\rho \cdot Q)$

$$R_y = W + \rho \cdot V^2 \cdot A = W + \rho \cdot Q \cdot \sqrt{{V_1}^2 + 2 \cdot g \cdot (H - y)}$$

Hence R_y increases in the same way as V as the height y varies; the maximum force is when y = H

$$R_{ymax} = W + \rho \cdot Q \cdot \sqrt{{V_1}^2 + 2 \cdot g \cdot H}$$

Problem 6.75

[Difficulty: 4]

6.75 An old magic trick uses an empty thread spool and a playing card. The playing card is placed against the bottom of the spool. Contrary to intuition, when one blows downward through the central hole in the spool, the card is not blown away. Instead it is "sucked" up against the spool. Explain.

Open-Ended Problem Statement: An old magic trick uses an empty thread spool and a playing card. The playing card is placed against the bottom of the spool. Contrary to intuition, when one blows downward through the central hole in the spool, the card is not blown away. Instead it is ''sucked'' up against the spool. Explain.

Discussion: The secret to this "parlor trick" lies in the velocity distribution, and hence the pressure distribution, that exists between the spool and the playing cards.

Neglect viscous effects for the purposes of discussion. Consider the space between the end of the spool and the playing card as a pair of parallel disks. Air from the hole in the spool enters the annular space surrounding the hole, and then flows radially outward between the parallel disks. For a given flow rate of air the edge of the hole is the cross-section of minimum flow area and therefore the location of maximum air speed.

After entering the space between the parallel disks, air flows radially outward. The flow area becomes larger as the radius increases. Thus the air slows and its pressure increases. The largest flow area, slowest air speed, and highest pressure between the disks occur at the outer periphery of the spool where the air is discharged from an annular area.

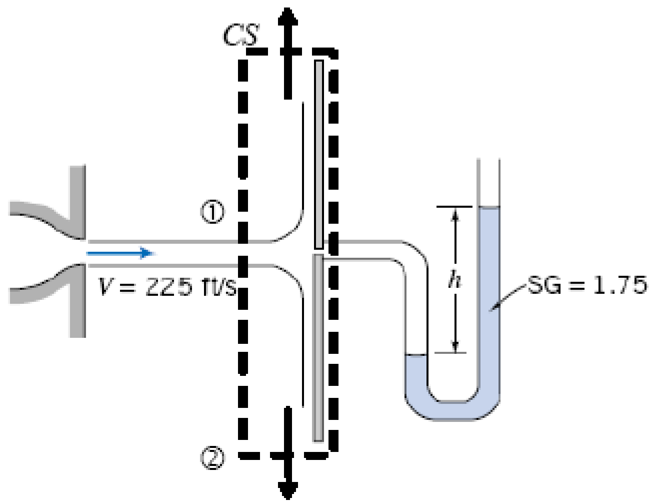
The air leaving the annular space between the disk and card must be at atmospheric pressure. This is the location of the highest pressure in the space between the parallel disks. Therefore pressure at smaller radii between the disks must be lower, and hence the pressure between the disks is sub-atmospheric. Pressure above the card is less than atmospheric pressure; pressure beneath the card is atmospheric. Each portion of the card experiences a pressure difference acting upward. This causes a net pressure force to act upward on the whole card. The upward pressure force acting on the card tends to keep it from blowing off the spool when air is introduced through the central hole in the spool.

Viscous effects are present in the narrow space between the disk and card. However, they only reduce the pressure rise as the air flows outward, they do not dominate the flow behavior.

Problem 6.76

[Difficulty: 4]

6.76 A horizontal axisymmetric jet of air with 0.4 in. diameter strikes a stationary vertical disk of 7.5 in. diameter. The jet speed is 225 ft/s at the nozzle exit. A manometer is connected to the center of the disk. Calculate (a) the deflection, if the manometer liquid has SG = 1.75, (b) the force exerted by the jet on the disk, and (c) the force exerted on the disk if it is assumed that the stagnation pressure acts on the entire forward surface of the disk. Sketch the streamline pattern and plot the distribution of pressure on the face of the disk.



Given: Air jet striking disk

Find: Manometer deflection; Force to hold disk; Force assuming p_0 on entire disk; plot pressure distribution

Solution:

Basic equations: Hydrostatic pressure, Bernoulli, and momentum flux in x direction

$$\Delta p = SG\cdot\rho\cdot g\cdot\Delta h \qquad \frac{p}{\rho} + \frac{V^2}{2} + g\cdot z = \text{constant} \qquad F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\, dV + \int_{CS} u\,\rho\,\vec{V}\cdot d\vec{A}$$

Assumptions: 1) Steady flow 2) Incompressible 3) No friction 4) Flow along streamline 5) Uniform flow 6) Horizontal flow ($g_x = 0$)

Applying Bernoulli between jet exit and stagnation point

$$\frac{p_{atm}}{\rho_{air}} + \frac{V^2}{2} = \frac{p_0}{\rho_{air}} + 0 \qquad p_0 - p_{atm} = \frac{1}{2}\cdot\rho_{air}\cdot V^2$$

But from hydrostatics $\quad p_0 - p_{atm} = SG\cdot\rho\cdot g\cdot\Delta h \quad$ so

$$\Delta h = \frac{\frac{1}{2}\cdot\rho_{air}\cdot V^2}{SG\cdot\rho\cdot g} = \frac{\rho_{air}\cdot V^2}{2\cdot SG\cdot\rho\cdot g}$$

$$\Delta h = 0.002377\cdot\frac{slug}{ft^3}\times\left(225\cdot\frac{ft}{s}\right)^2\times\frac{1}{2\cdot 1.75}\times\frac{ft^3}{1.94\cdot slug}\times\frac{s^2}{32.2\cdot ft} \qquad \Delta h = 0.55\cdot ft \qquad \Delta h = 6.60\cdot in$$

For x momentum

$$R_x = V\cdot\left(-\rho_{air}\cdot A\cdot V\right) = -\rho_{air}\cdot V^2\cdot\frac{\pi\cdot d^2}{4}$$

$$R_x = -0.002377\cdot\frac{slug}{ft^3}\times\left(225\cdot\frac{ft}{s}\right)^2\times\frac{\pi\cdot\left(\frac{0.4}{12}\cdot ft\right)^2}{4}\times\frac{lbf\cdot s^2}{slug\cdot ft} \qquad R_x = -0.105\cdot lbf$$

The force of the jet on the plate is then $\quad F = -R_x \qquad F = 0.105\cdot lbf$

The stagnation pressure is $\quad p_0 = p_{atm} + \frac{1}{2}\cdot\rho_{air}\cdot V^2$

The force on the plate, assuming stagnation pressure on the front face, is

$$F = (p_0 - p)\cdot A = \frac{1}{2}\cdot\rho_{air}\cdot V^2\cdot\frac{\pi\cdot D^2}{4}$$

$$F = \frac{\pi}{8} \times 0.002377\cdot\frac{slug}{ft^3} \times \left(225\cdot\frac{ft}{s}\right)^2 \times \left(\frac{7.5}{12}\cdot ft\right)^2 \times \frac{lbf\cdot s^2}{slug\cdot ft} \qquad F = 18.5\cdot lbf$$

Obviously this is a huge overestimate!

For the pressure distribution on the disk, we use Bernoulli between the disk outside edge any radius r for radial flow

$$\frac{p_{atm}}{\rho_{air}} + \frac{1}{2}\cdot v_{edge}^2 = \frac{p}{\rho_{air}} + \frac{1}{2}\cdot v^2$$

We need to obtain the speed v as a function of radius. If we assume the flow remains constant thickness h, then

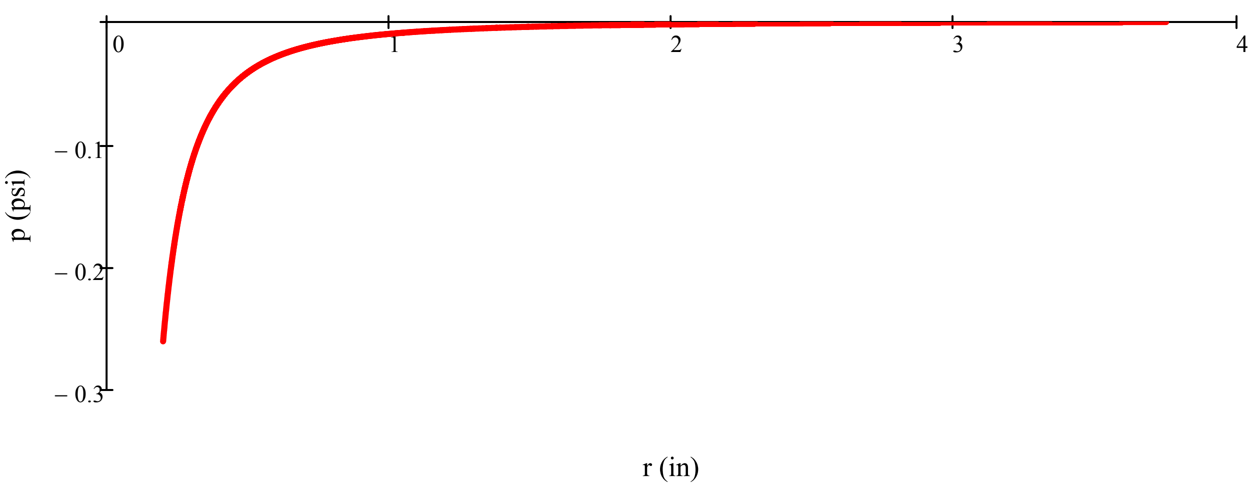
$$Q = v\cdot 2\cdot\pi\cdot r\cdot h = V\cdot\frac{\pi\cdot d^2}{4} \qquad\qquad v(r) = V\cdot\frac{d^2}{8\cdot h\cdot r}$$

We need an estimate for h. As an approximation, we assume that h = d (this assumption will change the scale of p(r) but not the basic shape)

Hence $\quad v(r) = V\cdot\dfrac{d}{8\cdot r}$

Using this in Bernoulli $\quad p(r) = p_{atm} + \dfrac{1}{2}\cdot\rho_{air}\cdot\left(v_{edge}^2 - v(r)^2\right) = p_{atm} + \dfrac{\rho_{air}\cdot V^2\cdot d^2}{128}\cdot\left(\dfrac{4}{D^2} - \dfrac{1}{r^2}\right)$

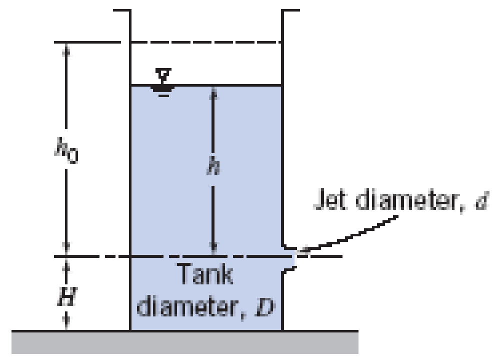
Expressed as a gage pressure $\quad p(r) = \dfrac{\rho_{air}\cdot V^2\cdot d^2}{128}\cdot\left(\dfrac{4}{D^2} - \dfrac{1}{r^2}\right)$



Problem 6.77

[Difficulty: 4] Part 1/2

6.77 The tank, of diameter D, has a well-rounded nozzle with diameter d. At $t = 0$, the water level is at height h_0. Develop an expression for dimensionless water height, h/h_0, at any later time. For $D/d = 10$, plot h/h_0 as a function of time with h_0 as a parameter for $0.1 \le h_0 \le 1$ m. For $h_0 = 1$ m, plot h/h_0 as a function of time with D/d as a parameter for $2 \le D/d \le 10$.



Solution:

Apply the Bernoulli equation along a streamline between the surface and the jet

Basic equation: $\cancel{\frac{p_s}{\rho}} + \frac{V_t^2}{2} + g z_s = \cancel{\frac{p_j}{\rho}} + \frac{V_j^2}{2} + g z_j$

Assumptions: (1) quasi-steady flow, ie neglect acceleration in tank.
(2) incompressible flow
(3) neglect frictional effects
(4) flow along a streamline
(5) $p_t = p_j = p_{atm}$.

From continuity, $V_t A_t = V_j A_j$ or $V_j = V_t \frac{A_t}{A_j} = V_t \left(\frac{D}{d}\right)^2$

Solving,

$$\frac{V_t^2}{2} - \frac{V_j^2}{2} = \frac{V_t^2}{2}\left[1 - \left(\frac{V_j}{V_t}\right)^2\right] = g(z_j - z_s) = g[H - (H+h)] = -gh$$

Then

$$V_t = \left[\frac{2gh}{(V_j/V_t)^2 - 1}\right]^{1/2} = \left[\frac{2gh}{(A_t/A_j)^2 - 1}\right]^{1/2} = \left[\frac{2gh}{(D/d)^4 - 1}\right]^{1/2} = -\frac{dh}{dt}$$

Separating variables,

$$\frac{dh}{h^{1/2}} = -\left[\frac{2g}{(D/d)^4 - 1}\right]^{1/2} dt$$

Integrating,

$$2h^{1/2} = -\left[\frac{2g}{(D/d)^4 - 1}\right]^{1/2} t + c$$

At $t = 0$, $h = h_0$, so $c = 2h_0^{1/2}$ and

$$h = \left\{ h_0^{1/2} - \frac{1}{2}\left[\frac{2g}{(D/d)^4 - 1}\right]^{1/2} t \right\}^2$$

Nondimensionalize (divide by h_0) to obtain

$$\frac{h}{h_0} = \left\{1 - \sqrt{\frac{g}{2h_0\{(D/d)^4-1\}}}\,t\right\}^2 \qquad h/h_0$$

Draining of a cylindrical liquid tank:

Plot of h/h_0 vs. t for $0.1 < h_0 < 1$ m

Input Data: D = 50 mm, d = 5 mm

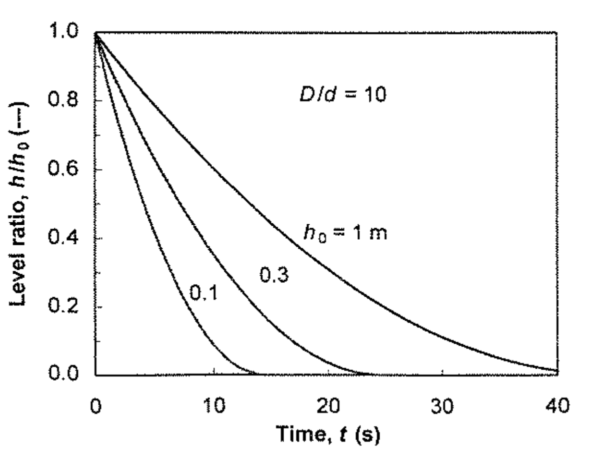
h_0 (m) =	0.1	0.3	1
Time, t (s)	**h/h_0 (---)**	**h/h_0 (---)**	**h/h_0 (---)**
0	1.00	1.00	1.00
2	0.739	0.845	0.913
4	0.518	0.703	0.831
6	0.336	0.574	0.752
8	0.193	0.458	0.677
10	0.090	0.355	0.606
12	0.025	0.265	0.539
14	0.000	0.188	0.476
16		0.125	0.417
18		0.074	0.362
20		0.037	0.310
22		0.012	0.263
24		0.001	0.219
26			0.180
28			0.144
30			0.113
32			0.085
34			0.061
36			0.041
38			0.025
40			0.013
45			0.000

Plot of h/h_0 vs. t for $10 < D/d < 2$

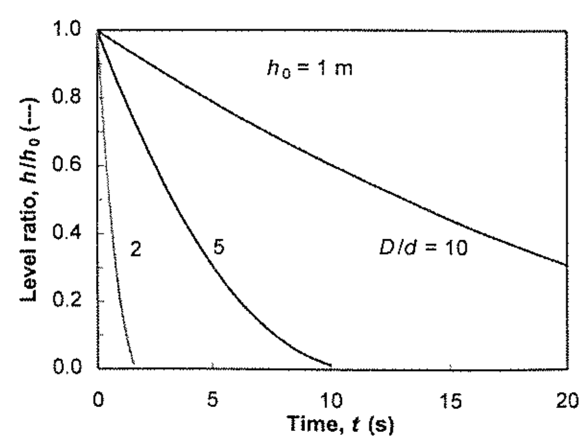
h_0 = 1 m

D/d (---) =	2	5	10
Time, t (s)	**h/h_0 (---)**	**h/h_0 (---)**	**h/h_0 (---)**
0	1.00	1.00	1.00
0.5	0.523	0.913	0.978
1	0.199	0.831	0.956
1.5	0.029	0.752	0.935
1.6	0.013	0.737	0.930
3		0.539	0.872
4		0.417	0.831
5		0.310	0.791
6		0.219	0.752
7		0.144	0.714
8		0.085	0.677
9		0.041	0.641
10		0.013	0.606
12			0.539
14			0.476
16			0.417
18			0.362
20			0.310

Level Ratio vs. Time for Tank Draining



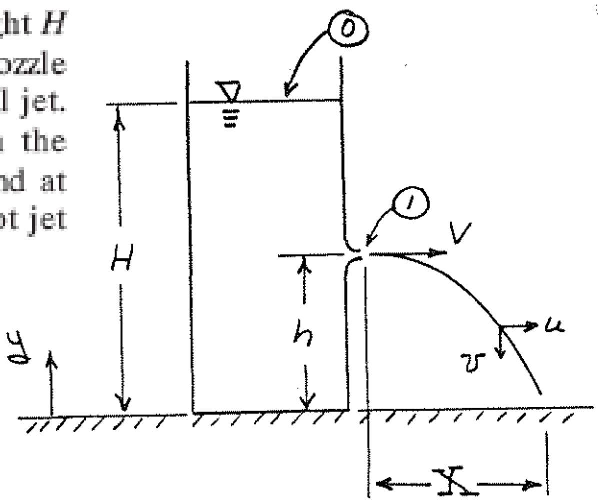
Level Ratio vs. Time for Tank Draining



Problem 6.78

[Difficulty: 4] Part 1/2

6.78 The water level in a large tank is maintained at height H above the surrounding level terrain. A rounded nozzle placed in the side of the tank discharges a horizontal jet. Neglecting friction, determine the height h at which the orifice should be placed so the water strikes the ground at the maximum horizontal distance X from the tank. Plot jet speed V and distance X as functions of h $(0 < h < H)$.



Apply Bernoulli equation between tank surface and jet.

Basic equation: $\cancel{\frac{p_0}{\rho}} + \cancel{\frac{V_0^2}{2}}^{\approx 0} + g y_0 = \cancel{\frac{p_1}{\rho}} + \frac{V_1^2}{2} + g y_1$

Assumptions: (1) steady flow (2) incompressible flow (3) flow along streamline (4) no friction

Then

$$gH = \frac{V^2}{2} + gh \quad \text{or} \quad V = \sqrt{2g(H-h)} \qquad (1)$$

Assume no air resistance in the stream. Then u = constant, and $X = ut = \sqrt{2g(H-h)}\,t$ — — — — — — — (2)

The only force acting on the stream is gravity

$$\sum F_y = -mg = m a_y = m\frac{dv}{dt} \; ; \text{ thus } \frac{dv}{dt} = -g$$

Integrating we obtain $v = \cancel{v_0}^{0} - gt$ and

$$y = y_0 + \cancel{v_0 t}^{0} - \frac{1}{2}gt^2$$

Solving for t, $t = \left[\frac{2(y_0 - y)}{g}\right]^{1/2}$

The time of flight is then $t = \sqrt{\frac{2y_0}{g}} = \sqrt{\frac{2h}{g}}$

Substituting into Eq. 2

$$X = \sqrt{2g(H-h)}\sqrt{\frac{2h}{g}} = 2\sqrt{h(H-h)} \qquad (3)$$

X will be maximized when $h(H-h)$ is maximized, or when

$$\frac{d}{dh}\left[h(H-h)\right] = 0 = (H-h) + h(-1) = H - 2h \quad \text{or} \quad h = H/2 \longleftarrow h$$

The corresponding range is

$$X = 2\sqrt{\frac{H}{2}\cdot\frac{H}{2}} = H$$

See the next page for plots

Problem 6.78

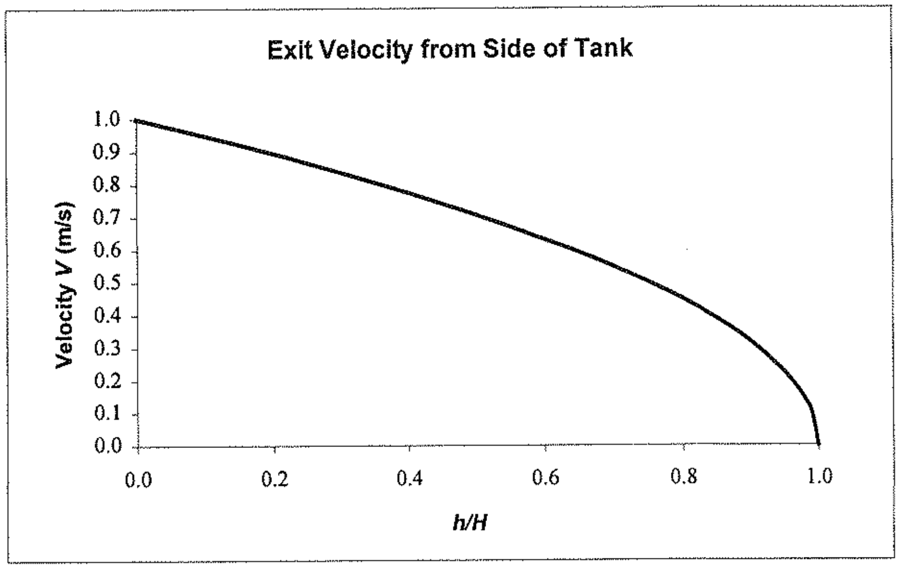
[Difficulty: 4] Part 2/2

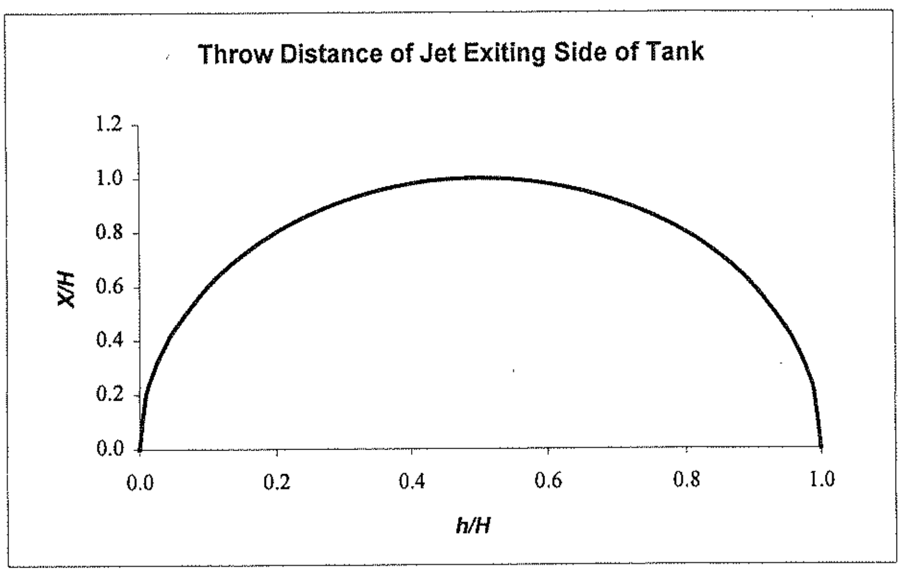
From Eq.1, $\frac{V}{\sqrt{2gH}} = \sqrt{1-\frac{h}{H}}$

From Eq.2, $\frac{X}{H} = 2\sqrt{\frac{h}{H}\left(1-\frac{h}{H}\right)}$

Exit velocity and throw distance from orifice in side of tank, versus height *h/H*

h/H	$V/(2gH)^{1/2}$	*X/H*
0.00	1.00	0.000
0.01	0.995	0.199
0.02	0.990	0.280
0.03	0.985	0.341
0.04	0.980	0.392
0.05	0.975	0.436
0.10	0.949	0.600
0.15	0.922	0.714
0.20	0.894	0.800
0.25	0.866	0.866
0.30	0.837	0.917
0.35	0.806	0.954
0.40	0.775	0.980
0.45	0.742	0.995
0.50	0.707	1.000
0.55	0.671	0.995
0.60	0.632	0.980
0.65	0.592	0.954
0.70	0.548	0.917
0.75	0.500	0.866
0.80	0.447	0.800
0.85	0.387	0.714
0.90	0.316	0.600
0.95	0.224	0.436
0.96	0.200	0.392
0.97	0.173	0.341
0.98	0.141	0.280
0.99	0.100	0.199
1.00	0.00	0.00

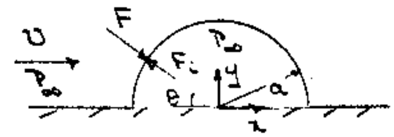




Problem 6.79

[Difficulty: 4]

6.79 The flow over a Quonset hut may be approximated by the velocity distribution of Problem 6.63 with $0 \le \theta \le \pi$. During a storm the wind speed reaches 100 km/hr; the outside temperature is 5°C. A barometer inside the hut reads 720 mm of mercury; pressure p_∞ is also 720 mm Hg. The hut has a diameter of 6 m and a length of 18 m. Determine the net force tending to lift the hut off its foundation.



Solution:

Basic equations: $\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{const}$ $F = \int p\,dA$

Assumptions: (1) steady flow
(2) incompressible flow
(3) frictionless flow
(4) flow along a streamline

Along the top half of the cylinder, $r = a$ and $\vec{V} = -2U\sin\theta\,\hat{i}_\theta$, $0 \le \theta \le \pi$

Applying the Bernoulli equation along the streamline $(r = a)$

$$\frac{p}{\rho} + \frac{V^2}{2} = \frac{p_\infty}{\rho} + \frac{V_\infty^2}{2}$$

$$p - p_\infty = \frac{\rho}{2}(V_\infty^2 - V^2) = \frac{\rho}{2}(U^2 - 4U^2\sin^2\theta) = \frac{\rho U^2}{2}(1 - 4\sin^2\theta)$$

$$F_{R_y} = \int_A (p_\infty - p)\,dA\,\sin\theta = \int_0^\pi (p_\infty - p)\sin\theta\,L a\,d\theta$$

$$= \int_0^\pi \frac{\rho U^2}{2}(4\sin^2\theta - 1)\sin\theta\,L a\,d\theta = \frac{\rho U^2}{2} a L \left\{ 4\left[\frac{\cos^3\theta}{3} - \cos\theta\right]_0^\pi + \cos\theta\Big]_0^\pi \right\}$$

$$= \frac{\rho U^2}{2} a L \left\{ 4\left[\left(-\frac{1}{3} + 1\right) - \left(\frac{1}{3} - 1\right)\right] + (-1 - 1) \right\}$$

$$F_{R_y} = \frac{\rho U^2}{2} a L \left(\frac{10}{3}\right) = \frac{5}{3}\rho U^2 a L$$

From the ideal gas equation of state

$$\rho = \frac{p}{RT} = 720\,\text{mm Hg} \times \frac{\text{atm}}{760\,\text{mm}} \times 1.01 \times 10^5 \frac{\text{N}}{\text{m}^2\cdot\text{atm}} \times \frac{\text{kg}\cdot\text{K}}{287\,\text{N}\cdot\text{m}} \times \frac{1}{278\,\text{K}} = 1.20\,\text{kg/m}^3$$

$$F_{R_y} = \frac{5}{3}\rho U^2 a L = \frac{5}{3} \times 1.20\frac{\text{kg}}{\text{m}^3} \times (10^5)^2 \frac{\text{m}^2}{\text{hr}^2} \times \frac{\text{hr}^2}{(3600)^2\text{s}^2} \times 3\,\text{m} \times 18\,\text{m} \times \frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}}$$

$$F_{R_y} = 83.3\,\text{kN}$$ F_{R_y}

Comment: The actual pressure distribution over the rear portion of the hut is not modelled well by ideal flow. The force calculated here is lower than the actual force.

Problem 6.80

[Difficulty: 3]

6.80 Many recreation facilities use inflatable "bubble" structures. A tennis bubble to enclose four courts is shaped roughly as a circular semicylinder with a diameter of 50 ft and a length of 50 ft. The blowers used to inflate the structure can maintain the air pressure inside the bubble at 0.75 in. of water above ambient pressure. The bubble is subjected to a wind that blows at 35 mph in a direction perpendicular to the axis of the semicylindrical shape. Using polar coordinates, with angle θ measured from the ground on the upwind side of the structure, the resulting pressure distribution may be expressed as

$$\frac{p - p_\infty}{\frac{1}{2}\rho V_\infty^2} = 1 - 4\sin^2\theta$$

where p is the pressure at the surface, p_∞ the atmospheric pressure, and V_w the wind speed. Determine the net vertical force exerted on the structure.

Given: Air flow over "bubble" structure

Find: Net vertical force

Solution: The net force is given by $\vec{F} = \int p\, d\vec{A}$ also $\Delta p = \rho\cdot g\cdot \Delta h$

Available data $L = 50\cdot ft$ $R = 25ft$ $V = 35\cdot mph$ $\Delta h = 0.75\cdot in$ $\rho = 1.94\cdot\frac{slug}{ft^3}$ $\rho_{air} = 0.00238\cdot\frac{slug}{ft^3}$

The internal pressure is $\Delta p = \rho\cdot g\cdot\Delta h$ $\Delta p = 187\,Pa$

Through symmetry only the vertical component of force is no-zero $F_V = \int_0^\pi (p_i - p)\cdot\sin(\theta)\cdot R\cdot L\, d\theta$

where pi is the internal pressure and p the external $p_i = p_{atm} + \Delta p$ $p = p_{atm} - \frac{1}{2}\cdot\rho_{air}\cdot V^2\cdot\left(1 - 4\cdot\sin(\theta)^2\right)$

Hence

$$F_V = \int_0^\pi \left[\Delta p - \frac{1}{2}\cdot\rho_{air}\cdot V^2\cdot\left(1 - 4\cdot\sin(\theta)^2\right)\right]\cdot\sin(\theta)\cdot R\cdot L\, d\theta$$

$$F_V = R\cdot L\cdot\Delta p\cdot\int_0^\pi \sin(\theta)\, d\theta - R\cdot L\cdot\frac{1}{2}\cdot\rho_{air}\cdot V^2\cdot\int_0^\pi \left(1 - 4\cdot\sin(\theta)^2\right)\cdot\sin(\theta)\, d\theta$$

But $\int \left(\sin(\theta) - 4\cdot\sin(\theta)^3\right) d\theta = -\cos(\theta) + 4\cdot\left(\cos(\theta) - \frac{1}{3}\cdot\cos(\theta)^3\right)$ so $\int_0^\pi \left(\sin(\theta) - 4\cdot\sin(\theta)^3\right) d\theta = -\frac{10}{3}$

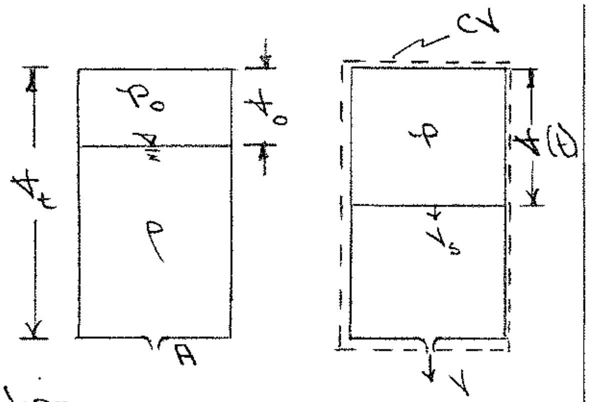
$\int \sin(\theta)\, d\theta = -\cos(\theta)$ so $\int_0^\pi \sin(\theta)\, d\theta = 2$

Combining results $F_V = R\cdot L\cdot\left(2\cdot\Delta p + \frac{5}{3}\cdot\rho_{air}\cdot V^2\right)$ $F_V = 2.28\times 10^4\cdot lbf$ $F_V = 22.8\cdot kip$

Problem 6.81

6.81 High-pressure air forces a stream of water from a tiny, rounded orifice, of area A, in a tank. The pressure is high enough that gravity may be neglected. The air expands slowly, so that the expansion may be considered isothermal. The initial volume of air in the tank is $\forall_0$. At later instants the volume of air is $\forall(t)$; the total volume of the tank is $\forall_t$. Obtain an algebraic expression for the mass flow rate of water leaving the tank. Find an algebraic expression for the rate of change in mass of the water inside the tank. Develop an ordinary differential equation and solve for the water mass in the tank at any instant. If $\forall_0 = 5\ \text{m}^3$, $\forall_t = 10\ \text{m}^3$, $A = 25\ \text{mm}^2$, and $p_0 = 1$ MPa, plot the water mass in the tank versus time for the first forty minutes.

Solution:



Basic equations: $\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{const}$

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$

Assumptions: (1) quasi steady flow $V_s \ll V$

(2) frictionless

(3) incompressible

(4) flow along a streamline

(5) uniform flow at outlet.

(6) neglect gravity

(7) $p \gg p_{atm}$ $\therefore p_{abs} \approx p_{gage}$

Apply Bernoulli equation between liquid surface and orifice

$$V_j = \left[\frac{2(p - p_{atm})}{\rho}\right]^{1/2} \approx \sqrt{\frac{2p}{\rho}}$$

$$\dot{m} = \rho A V_j = \rho A \sqrt{\frac{2p}{\rho}} = \sqrt{2p\rho}\, A \qquad \dot{m}$$

Rate of change of mass in tank is $\frac{dM}{dt} = \frac{\partial}{\partial t}\int \rho\, d\forall$

$$\frac{dM}{dt} = \rho_w \frac{d\forall_w}{dt} = -\rho_w \frac{d\forall_{air}}{dt} \quad (\forall_t = \forall_{air} + \forall_w) \qquad \frac{dM}{dt}$$

For isothermal flow, $\frac{p}{\rho} = RT = \text{constant} = \frac{p_0}{\rho_0}$

where ρ is the air density and $\rho = M_{air}/\forall_{air}$

thus

$$p\forall = p_0 \forall_0 \quad \text{or} \quad p = p_0 \frac{\forall_0}{\forall}$$

From continuity

$$0 = \rho_w \frac{d\forall_w}{dt} + \dot{m}$$

and

$$0 = -\rho_w \frac{d\forall_{air}}{dt} + \sqrt{2p\rho_w}\, A$$

$$\frac{d\forall}{dt} = \sqrt{\frac{2p}{\rho_w}} = \sqrt{\frac{2p_0\forall_0}{\rho_w \forall}}$$

Separating variables, $\quad V^{1/2}\, dV = \sqrt{\dfrac{2 p_0 V_0}{\rho}}\, A\, dt$

Integrating $\quad \dfrac{2}{3} V^{3/2} \Big]_{V_0}^{V} = \sqrt{\dfrac{2 p_0 V_0}{\rho_w}}\, At$

$$\frac{2}{3}\left(V^{3/2} - V_0^{3/2}\right) = \frac{2 V_0^{3/2}}{3}\left[\left(\frac{V}{V_0}\right)^{3/2} - 1\right] = \sqrt{\frac{2 p_0 V_0}{\rho_w}}\, At$$

Then

$$\left(\frac{V}{V_0}\right)^{3/2} = \left[1 + \frac{3}{2 V_0^{3/2}} \sqrt{\frac{2 p_0 V_0}{\rho_w}}\, At\right]$$

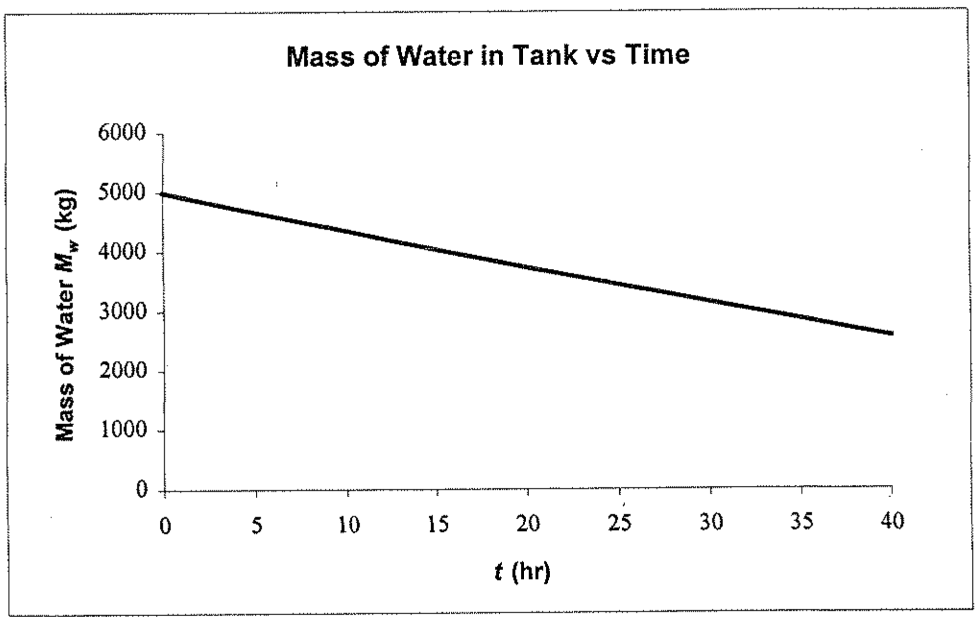
$$\frac{V}{V_0} = \left[1 + 1.5\sqrt{\frac{2 p_0}{\rho_w}}\, \frac{At}{V_0}\right]^{2/3}$$

But $\quad M_w = \rho_w (V_t - V) = \rho V_0 \left\{\dfrac{V_t}{V_0} - \dfrac{V}{V_0}\right\}$

$$\therefore M_w = \rho_w V_0 \left\{\frac{V_t}{V_0} - \left[1 + 1.5\sqrt{\frac{2 p_0}{\rho_w}}\, \frac{At}{V_0}\right]^{2/3}\right\}$$

M_w

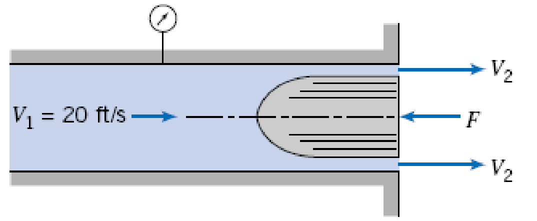
t (s)	M_w (kg)
0	4995
2	4862
4	4730
6	4600
8	4472
10	4345
12	4220
14	4096
16	3973
18	3851
20	3731
22	3612
24	3494
26	3377
28	3260
30	3145
32	3031
34	2918
36	2806
38	2695
40	2584



Problem 6.82

[Difficulty: 4]

6.82 Water flows at low speed through a circular tube with inside diameter of 2 in. A smoothly contoured body of 1.5 in. diameter is held in the end of the tube where the water discharges to atmosphere. Neglect frictional effects and assume uniform velocity profiles at each section. Determine the pressure measured by the gage and the force required to hold the body.



Given: Water flow out of tube

Find: Pressure indicated by gage; force to hold body in place

Solution:

Basic equations: Bernoulli, and momentum flux in x direction

$$\frac{p}{\rho} + \frac{V^2}{2} + g \cdot z = \text{constant} \qquad Q = V \cdot A \qquad F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\,dV\!\!\!\!- + \int_{CS} u\,\rho\,\vec{V}\cdot d\vec{A}$$

Assumptions: 1) Steady flow 2) Incompressible 3) No friction 4) Flow along streamline 5) Uniform flow 6) Horizontal flow ($g_x = 0$)

Applying Bernoulli between jet exit and stagnation point

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} = \frac{p_2}{\rho} + \frac{V_2^2}{2} = \frac{V_2^2}{2} \qquad \text{where we work in gage pressure}$$

$$p_1 = \frac{\rho}{2}\cdot\left(V_2^2 - V_1^2\right)$$

But from continuity $Q = V_1 \cdot A_1 = V_2 \cdot A_2$ $\qquad V_2 = V_1 \cdot \frac{A_1}{A_2} = V_1 \cdot \frac{D^2}{D^2 - d^2}$ $\qquad$ where D = 2 in and d = 1.5 in

$$V_2 = 20 \cdot \frac{ft}{s} \cdot \left(\frac{2^2}{2^2 - 1.5^2}\right) \qquad V_2 = 45.7 \cdot \frac{ft}{s}$$

Hence

$$p_1 = \frac{1}{2} \times 1.94 \cdot \frac{slug}{ft^3} \times \left(45.7^2 - 20^2\right)\cdot\left(\frac{ft}{s}\right)^2 \times \frac{lbf \cdot s^2}{slug \cdot ft} \qquad p_1 = 1638 \cdot \frac{lbf}{ft^2} \qquad p_1 = 11.4 \cdot psi \qquad \text{(gage)}$$

The x mometum is $\quad -F + p_1 \cdot A_1 - p_2 \cdot A_2 = u_1 \cdot \left(-\rho \cdot V_1 \cdot A_1\right) + u_2 \cdot \left(\rho \cdot V_2 \cdot A_2\right)$

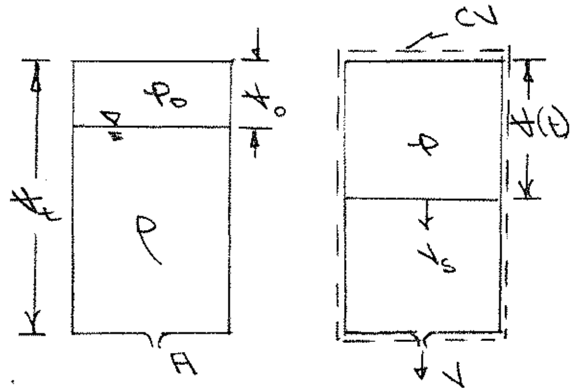
$$F = p_1 \cdot A_1 + \rho \cdot \left(V_1^2 \cdot A_1 - V_2^2 \cdot A_2\right) \qquad \text{using gage pressures}$$

$$F = 11.4 \cdot \frac{lbf}{in^2} \times \frac{\pi \cdot (2 \cdot in)^2}{4} + 1.94 \cdot \frac{slug}{ft^3} \times \left[\left(20 \cdot \frac{ft}{s}\right)^2 \times \frac{\pi \cdot (2 \cdot in)^2}{4} - \left(45.7 \cdot \frac{ft}{s}\right)^2 \times \frac{\pi \cdot \left[(2 \cdot in)^2 - (1.5 \cdot in)^2\right]}{4}\right] \times \left(\frac{1 \cdot ft}{12 \cdot in}\right)^2 \times \frac{lbf \cdot s^2}{slug \cdot ft}$$

$$F = 14.1 \cdot lbf \qquad \text{in the direction shown}$$

Problem 6.83

6.83 Repeat Problem 6.81 assuming the air expands so rapidly that the expansion may be treated as adiabatic.



Solution:

Basic equations: $\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{const}$

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$

Assumptions:
(1) quasi steady flow $V_s \ll V$
(2) frictionless
(3) incompressible
(4) flow along a streamline
(5) uniform flow at outlet
(6) neglect gravity
(7) $p \Rightarrow p_{atm}$ $\therefore$ $p_{abs} = p_{gage}$

Apply Bernoulli equation between liquid surface and orifice

$$V_j = \left[\frac{2(p - p_{atm})}{\rho}\right]^{1/2} = \sqrt{\frac{2p}{\rho}}$$

$$\dot{m} = \rho A V_j = \rho A \sqrt{\frac{2p}{\rho}} = \sqrt{2p\rho}\, A \quad \dot{m}$$

Rate of change of mass in tank is $\frac{dM}{dt} = \frac{\partial}{\partial t}\int \rho \, d\forall$

$$\frac{dM}{dt} = \rho_w \frac{d\forall_w}{dt} = -\rho_w \frac{d\forall_{air}}{dt} \quad (\forall_t = \forall_{air} + \forall_w) \quad \frac{dM}{dt}$$

For adiabatic expansion of air $p/\rho^k = \text{constant}$

Since mass of air is constant, $p_0 \forall_0^k = p \forall^k$

From continuity, $-\rho_w \frac{d\forall_{air}}{dt} + \sqrt{2p\rho_w}\, A = 0$

$$\frac{d\forall_{air}}{dt} = \frac{A\sqrt{2}}{\sqrt{\rho_w}} p^{1/2} = A\sqrt{\frac{2}{\rho_w}}\left[\frac{p_0 \forall_0^k}{\forall^k}\right]^{1/2} = A\sqrt{\frac{2 p_0 \forall_0^k}{\rho_w}}\, \forall^{-k/2}$$

$$\forall^{k/2}\, d\forall = A\sqrt{\frac{2 p_0 \forall_0^k}{\rho_w}}\, dt = c\, dt \quad \text{where } c = A\sqrt{\frac{2 p_0 \forall_0^k}{\rho_w}}$$

Integrating

$$\frac{2}{(k+2)} \forall^{\frac{k}{2}+1}\Big]_{\forall_0}^{\forall} = ct$$

$$V^{\left(\frac{k+2}{2}\right)} - V_0^{\left(\frac{k+2}{2}\right)} = \frac{k+2}{2} c t$$

$$\left(\frac{V}{V_0}\right)^{\left(\frac{k+2}{2}\right)} = 1 + \frac{(k+2)}{2} c t \cdot \frac{1}{V_0^{\frac{k+2}{2}}}$$

$$= 1 + \left(\frac{k+2}{2}\right) A \left[\frac{2 p_0 V_0^{k}}{\rho_w}\right]^{1/2} \cdot \left[\frac{1}{V_0^{(k+2)}}\right]^{1/2} t$$

$$\left(\frac{V}{V_0}\right)^{\frac{k+2}{2}} = 1 + \left(\frac{k+2}{2}\right) A \left[\frac{2 p_0}{\rho_w V_0^2}\right]^{1/2} t = 1 + \left(\frac{k+2}{2}\right) \frac{A}{V_0} \left[\frac{2 p_0}{\rho_w}\right]^{1/2} t$$

$$\frac{V}{V_0} = \left[1 + \frac{A}{V_0}\sqrt{\frac{2 p_0}{\rho_w}} \left(\frac{k+2}{2}\right) t\right]^{2/(k+2)}$$

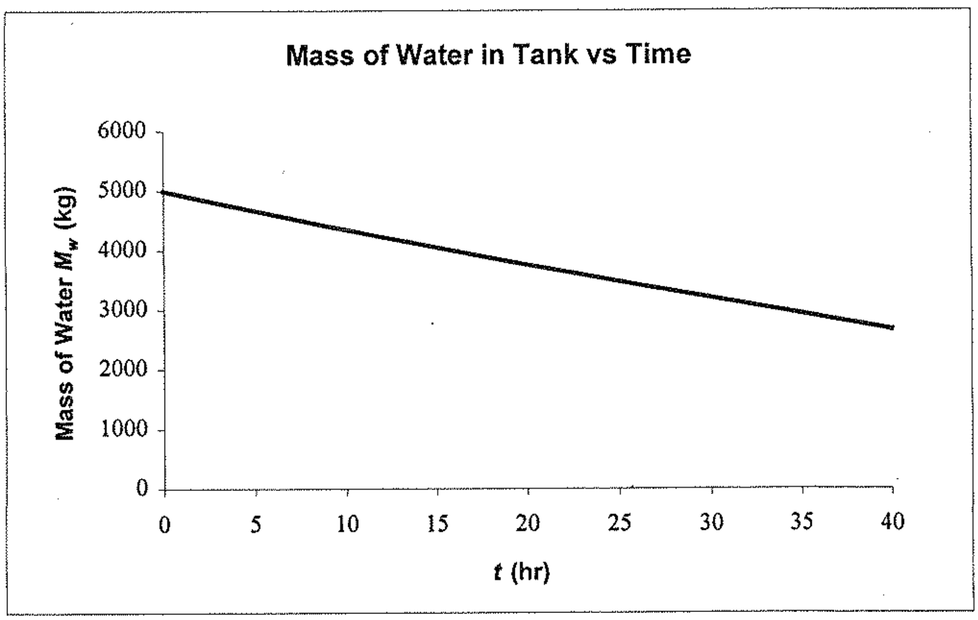
$$M_w = \rho_w (V_t - V) = \rho V_0 \left\{\frac{V_t}{V_0} - \frac{V}{V_0}\right\}$$

$$M_w = \rho_w V_0 \left\{\frac{V_t}{V_0} - \left[1 + \frac{A}{V_0}\sqrt{\frac{2 p_0}{\rho_w}} \left(\frac{k+2}{2}\right) t\right]^{2/(k+2)}\right\}$$

$$M_w = \rho_w V_0 \left\{\frac{V_t}{V_0} - \left[1 + 1.70 \sqrt{\frac{2 p_0}{\rho_w}} \frac{A t}{V_0}\right]^{0.588}\right\}$$

M_w

t (s)	M_w (kg)
0	4995
2	4862
4	4732
6	4603
8	4477
10	4353
12	4231
14	4110
16	3991
18	3874
20	3759
22	3645
24	3532
26	3420
28	3310
30	3202
32	3094
34	2988
36	2882
38	2778
40	2675



Problem 6.84

[Difficulty: 5]

6.84 Describe the pressure distribution on the exterior of a multistory building in a steady wind. Identify the locations of the maximum and minimum pressures on the outside of the building. Discuss the effect of these pressures on infiltration of outside air into the building.

Open-Ended Problem Statement: Describe the pressure distribution on the exterior of a multistory building in a steady wind. Identify the locations of the maximum and minimum pressures on the outside of the building. Discuss the effect of these pressures on infiltration of outside air into the building.

Discussion: A multi-story building acts as a bluff-body obstruction in a thick atmospheric boundary layer. The boundary-layer velocity profile causes the air speed near the top of the building to be highest and that toward the ground to be lower.

Obstruction of air flow by the building causes regions of stagnation pressure on upwind surfaces. The stagnation pressure is highest where the air speed is highest. Therefore the maximum surface pressure occurs near the roof on the upwind side of the building. Minimum pressure on the upwind surface of the building occurs near the ground where the air speed is lowest.

The minimum pressure on the entire building will likely be in the low-speed, low-pressure wake region on the downwind side of the building.

Static pressure inside the building will tend to be an average of all the surface pressures that act on the outside of the building. It is never possible to seal all openings completely. Therefore air will tend to infiltrate into the building in regions where the outside surface pressure is above the interior pressure, and will tend to pass out of the building in regions where the outside surface pressure is below the interior pressure. Thus generally air will tend to move through the building from the upper floors toward the lower floors, and from the upwind side to the downwind side.

Problem 6.85

[Difficulty: 5]

6.85 Imagine a garden hose with a stream of water flowing out through a nozzle. Explain why the end of the hose may be unstable when held a half meter or so from the nozzle end.

Open-Ended Problem Statement: Imagine a garden hose with a stream of water flowing out through a nozzle. Explain why the end of the hose may be unstable when held a half meter or so from the nozzle end.

Discussion: Water flowing out of the nozzle tends to exert a thrust force on the end of the hose. The thrust force is aligned with the flow from the nozzle and is directed toward the hose.

Any misalignment of the hose will lead to a tendency for the thrust force to bend the hose further. This will quickly become unstable, with the result that the free end of the hose will "flail" about, spraying water from the nozzle in all directions.

This instability phenomenon can be demonstrated easily in the backyard. However, it will tend to do least damage when the person demonstrating it is wearing a bathing suit!

Problem 6.86

[Difficulty: 5]

6.86 An aspirator provides suction by using a stream of water flowing through a venturi. Analyze the shape and dimensions of such a device. Comment on any limitations on its use.

Open-Ended Problem Statement: An aspirator provides suction by using a stream of water flowing through a venturi. Analyze the shape and dimensions of such a device. Comment on any limitations on its use.

Discussion: The basic shape of the aspirator channel should be a converging nozzle section to reduce pressure followed by a diverging diffuser section to promote pressure recovery. The basic shape is that of a venturi flow meter.

If the diffuser exhausts to atmosphere, the exit pressure will be atmospheric. The pressure rise in the diffuser will cause the pressure at the diffuser inlet (venturi throat) to be below atmospheric.

A small tube can be brought in from the side of the throat to aspirate another liquid or gas into the throat as a result of the reduced pressure there.

The following comments can be made about limitations on the aspirator:

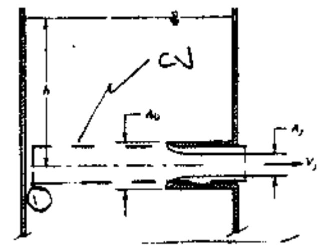
1. It is desirable to minimize the area of the aspirator tube compared to the flow area of the venturi throat. This minimizes the disturbance of the main flow through the venturi and promotes the best possible pressure recovery in the diffuser.
2. It is desirable to avoid cavitation in the throat of the venturi. Cavitation alters the effective shape of the flow channel and destroys the pressure recovery in the diffuser. To avoid cavitation, the reduced pressure must always be above the vapor pressure of the driver liquid.
3. It is desirable to limit the flow rate of gas into the venturi throat. A large amount of gas can alter the flow pattern and adversely affect pressure recovery in the diffuser.

The best combination of specific dimensions could be determined experimentally by a systematic study of aspirator performance. A good starting point probably would be to use dimensions similar to those of a commercially available venturi flow meter.

Problem 6.87

[Difficulty: 5]

6.87 A tank with a *reentrant* orifice called a *Borda mouthpiece* is shown. The fluid is inviscid and incompressible. The reentrant orifice essentially eliminates flow along the tank walls, so the pressure there is nearly hydrostatic. Calculate the *contraction coefficient*, $C_c = A_j/A_0$. *Hint:* Equate the unbalanced hydrostatic pressure force and momentum flux from the jet.



Solution:

Apply the x-component of the momentum equation to the CV shown

$$F_{S_x} + \cancel{F_{B_x}} = \cancel{\frac{\partial}{\partial t}\int_{CV} u\rho\, dV}^{\,0(1)} + \int_{CS} u\rho\vec{V}\cdot d\vec{A}$$

Assumptions: (1) steady flow
(2) uniform flow at jet exit.
(3) hydrostatic pressure variation across CS ①, $V_1 \approx 0$
(4) x momentum flux across horizontal portion of CS is negligible.
(5) ρ = constant

Then

$$\int_{A_0} p\, dA_1 = \dot{m}V_j = \rho V_j A_j V_j = \rho A_j V_j^2$$

$$\bar{p}_1 A_0 = \rho g h A_0 = \rho A_j V_j^2$$

$$\therefore \frac{A_0}{A_j} = \frac{V_j^2}{gh}$$

Apply the Bernoulli equation along the central streamline from ① to the jet exit.

$$\frac{p_1}{\rho} + \cancel{\frac{V_1^2}{2}}^{\,0(3)} + g\cancel{z_1} = \cancel{\frac{p_j}{\rho}}^{\,0(gage)} + \frac{V_j^2}{2} + g\cancel{z_j}$$

Assumptions: (6) frictionless flow

$$p_1 = \rho g h = \rho\frac{V_j^2}{2}$$

$$\therefore \frac{V_j^2}{2} = gh$$

and

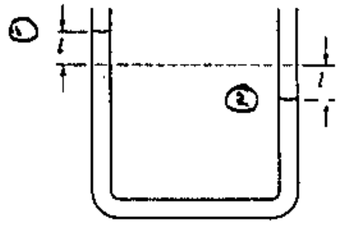
$$\frac{A_0}{A_j} = \frac{V_j^2}{gh} = 2$$

$$\therefore C_c = \frac{A_j}{A_0} = \frac{1}{2} \longleftarrow C_c$$

Problem 6.88

[Difficulty: 3]

6.88 Apply the unsteady Bernoulli equation to the U-tube manometer of constant diameter shown. Assume that the manometer is initially deflected and then released. Obtain a differential equation for l as a function of time.



Solution

Basic equation: $\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2 + \int_1^2 \frac{\partial V_s}{\partial t}\,ds$

Assumptions: (1) incompressible flow
(2) frictionless flow
(3) flow along a streamline

Since $p_1 = p_2 = p_{atm}$ and $V_1^2 = V_2^2$, then

$$g(z_1 - z_2) = \int_1^2 \frac{\partial V_s}{\partial t}\,ds$$

Let L = total length of column
l = deflection

Then $ds = dL$

$$V_s = V = \frac{dl}{dt}$$

$$\therefore\ 2gl = \int_1^2 \frac{\partial V}{\partial t}\,dL = \frac{\partial V}{\partial t}\int_1^2 dL = L\frac{\partial V}{\partial t}$$

Since $V = -\frac{dl}{dt}$

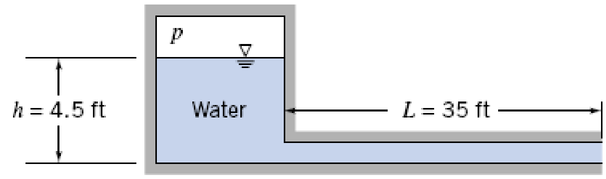
$$2gl = L\frac{\partial V}{\partial t} = -L\frac{d^2 l}{dt^2}$$

Finally $\frac{d^2 l}{dt^2} + \frac{2g}{L}\,l = 0$

Problem 6.89

[Difficulty: 3]

6.89 Compressed air is used to accelerate water from a tube. Neglect the velocity in the reservoir and assume the flow in the tube is uniform at any section. At a particular instant, it is known that $V = 6$ ft/s and $dV/dt = 7.5$ ft/s^2. The cross-sectional area of the tube is $A = 32$ in.^{2}. Determine the pressure in the tank at this instant.



Given: Unsteady water flow out of tube

Find: Pressure in the tank

Solution:

Basic equation: Unsteady Bernoulli

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2 + \int_1^2 \frac{\partial V}{\partial t} ds$$

Assumptions: 1) Unsteady flow 2) Incompressible 3) No friction 4) Flow along streamline 5) Uniform flow 6) Horizontal flow (g_x = (

Applying unsteady Bernoulli between reservoir and tube exit

$$\frac{p}{\rho} + g \cdot h = \frac{V^2}{2} + \int_1^2 \frac{\partial}{\partial t} V\, ds = \frac{V^2}{2} + \frac{dV}{dt} \cdot \int_1^2 1\, ds$$

where we work in gage pressure

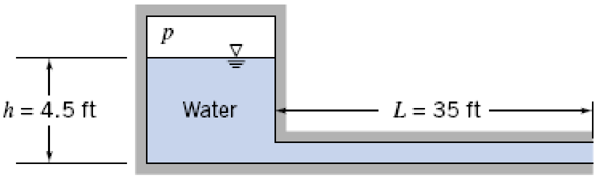
Hence $$p = \rho \cdot \left(\frac{V^2}{2} - g \cdot h + \frac{dV}{dt} \cdot L \right)$$

Hence $$p = 1.94 \cdot \frac{\text{slug}}{\text{ft}^3} \times \left(\frac{6^2}{2} - 32.2 \times 4.5 + 7.5 \times 35 \right) \cdot \left(\frac{\text{ft}}{\text{s}} \right)^2 \times \frac{\text{lbf} \cdot \text{s}^2}{\text{slug} \cdot \text{ft}} \qquad p = 263 \cdot \frac{\text{lbf}}{\text{ft}^2} \qquad p = 1.83 \cdot \text{psi}$$ (gage)

Problem 6.90

[Difficulty: 3]

6.90 If the water in the pipe in Problem 6.89 is initially at rest and the air pressure is 3 psig, what will be the initial acceleration of the water in the pipe?



Given: Unsteady water flow out of tube

Find: Initial acceleration

Solution:

Basic equation: Unsteady Bernoulli

$$\frac{p_1}{\rho}+\frac{V_1^2}{2}+gz_1=\frac{p_2}{\rho}+\frac{V_2^2}{2}+gz_2+\int_1^2\frac{\partial V}{\partial t}ds$$

Assumptions: 1) Unsteady flow 2) Incompressible 3) No friction 4) Flow along streamline 5) Uniform flow 6) Horizontal flow (g_x = (

Applying unsteady Bernoulli between reservoir and tube exit

$$\frac{p}{\rho}+g\cdot h=\int_1^2\frac{\partial}{\partial t}V\,ds=\frac{dV}{dt}\cdot\int_1^2 1\,ds=a_x\cdot L$$

where we work in gage pressure

Hence $$a_x=\frac{1}{L}\cdot\left(\frac{p}{\rho}+g\cdot h\right)$$

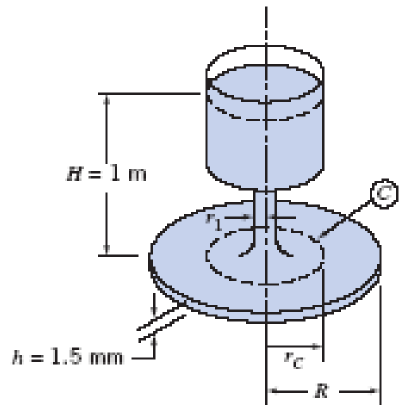
Hence $$a_x=\frac{1}{35\cdot ft}\times\left[3\cdot\frac{lbf}{in^2}\times\left(\frac{12\cdot in}{1\cdot ft}\right)^2\times\frac{ft^3}{1.94\cdot slug}\times\frac{slug\cdot ft}{s^2\cdot lbf}+32.2\cdot\frac{ft}{s^2}\times 4.5\cdot ft\right]\qquad a_x=10.5\cdot\frac{ft}{s^2}$$

Note that we obtain the same result if we treat the water in the pipe as a single body at rest with gage pressure p + ρgh at the left end!

Problem 6.91

[Difficulty: 4]

6.91 Consider the reservoir and disk flow system with the reservoir level maintained constant. Flow between the disks is started from rest at $t = 0$. Evaluate the rate of change of volume flow rate at $t = 0$, if $r_1 = 50$ mm.



Solution:

Apply the unsteady Bernoulli equation from the surface to the exit.

$$\cancel{\frac{p_s}{\rho}} + \cancel{\frac{V_s^2}{2}} + g z_s = \cancel{\frac{p_e}{\rho}} + \frac{V_e^2}{2} + \cancel{g z_e} + \int_1^2 \frac{\partial V_s}{\partial t}\, ds$$

$$gH = \frac{V_e^2}{2} + \int_1^2 \frac{\partial V_s}{\partial t}\, ds.$$

Assumptions: (1) frictionless flow
(2) incompressible flow
(3) flow along a streamline.

For uniform flow at any section between the plates, for $r \geq r_1$, the volume flow rate is given by

$$Q = \int \vec{V} \cdot d\vec{A} = V_r\, 2\pi r h \quad \text{and} \quad V_r = \frac{Q}{2\pi r h}$$

At the exit $V_e = Q/2\pi R h$

Assume that the rate of change of fluid velocity in the reservoir (out to $r = r_1$) is negligible. Then

$$\int_1^2 \frac{\partial V_s}{\partial t}\, ds = \frac{\partial}{\partial t}\int_1^2 V_r\, dr = \frac{\partial}{\partial t}\int_1^2 \frac{Q}{2\pi h}\frac{dr}{r} = \frac{\ln R/r_1}{2\pi h}\frac{dQ}{dt}$$

Then substituting into the unsteady Bernoulli equation, we obtain

$$gH = \frac{Q^2}{8\pi^2 R^2 h^2} + \frac{\ln R/r_1}{2\pi h}\frac{dQ}{dt}$$

At $t = 0$, $Q = 0$ and

$$\frac{dQ}{dt} = \frac{2\pi h g H}{\ln R/r_1}$$

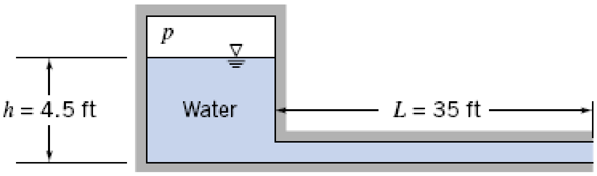
$$= 2\pi \times 0.0015\,\text{m} \times 9.81\,\frac{\text{m}}{\text{s}^2} \times 1\,\text{m} \times \frac{1}{\ln \frac{300}{50}}$$

$$\frac{dQ}{dt} = 0.0516\ \text{m}^3/\text{s}/\text{s} \longleftarrow \left.\frac{dQ}{dt}\right|_{t=0}$$

Problem 6.92

[Difficulty: 4]

6.92 If the water in the pipe of Problem 6.89 is initially at rest, and the air pressure is maintained at 1.5 psig, derive a differential equation for the velocity V in the pipe as a function of time, integrate, and plot V versus t for $t = 0$ to 5 s.



Given: Unsteady water flow out of tube

Find: Differential equation for velocity; Integrate; Plot v versus time

Solution:

Basic equation: Unsteady Bernoulli

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2 + \int_1^2 \frac{\partial V}{\partial t} ds$$

Assumptions: 1) Unsteady flow 2) Incompressible 3) No friction 4) Flow along streamline 5) Uniform flow 6) Horizontal flow (g_x = (

Applying unsteady Bernoulli between reservoir and tube exit

$$\frac{p}{\rho} + g\cdot h = \frac{V^2}{2} + \int_1^2 \frac{\partial}{\partial t} V\, ds = \frac{V^2}{2} + \frac{dV}{dt}\cdot\int_1^2 1\, ds = \frac{V^2}{2} + \frac{dV}{dt}\cdot L$$

where we work in gage pressure

Hence

$$\frac{dV}{dt} + \frac{V^2}{2\cdot L} = \frac{1}{L}\cdot\left(\frac{p}{\rho} + g\cdot h\right)$$

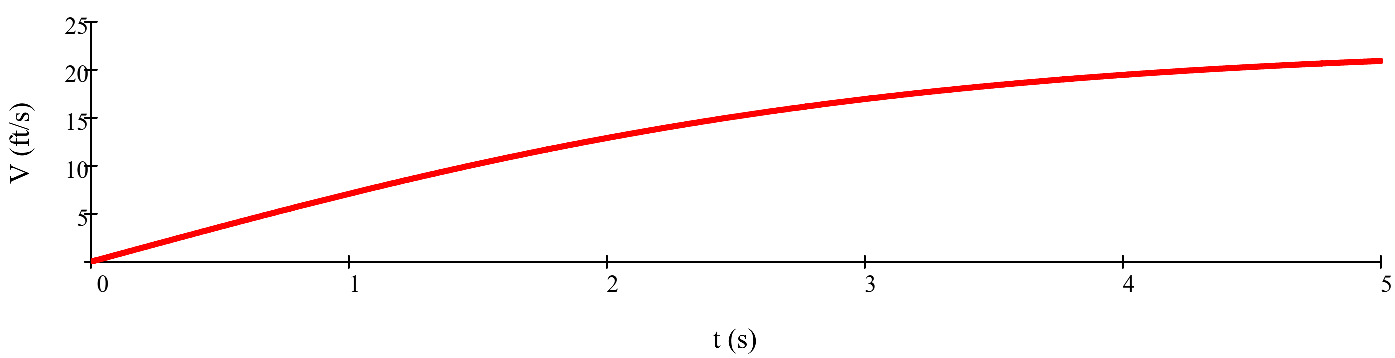
is the differential equation for the flow

Separating variables

$$\frac{L\cdot dV}{\frac{p}{\rho} + g\cdot h - \frac{V^2}{2}} = dt$$

Integrating and using limits V(0) = 0 and V(t) = V

$$V(t) = \sqrt{2\cdot\left(\frac{p}{\rho} + g\cdot h\right)}\cdot\tanh\left(\sqrt{\frac{\frac{p}{\rho} + g\cdot h}{2\cdot L^2}}\cdot t\right)$$



This graph is suitable for plotting in *Excel*

For large times

$$V = \sqrt{2\cdot\left(\frac{p}{\rho} + g\cdot h\right)} \qquad V = 22.6\cdot\frac{ft}{s}$$

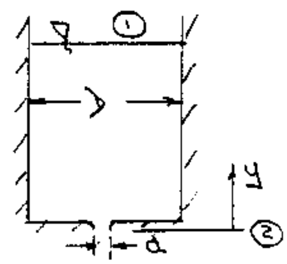
Problem 6.93

[Difficulty: 5] Part 1/2

6.93 Consider the tank of Problem 4.46. Using the Bernoulli equation for unsteady flow along a streamline, evaluate the minimum diameter ratio, D/d, required to justify the assumption that flow from the tank is quasi-steady.

Solution:

For incompressible, frictionless flow along a streamline, the unsteady Bernoulli equation is



$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gy_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gy_2 + \int_1^2 \frac{\partial V_s}{\partial t}\,dy$$

$p_1 = p_2 = p_{atm}$, $y_2 = 0$

From continuity $V_1A_1 = V_2A_2 = V_jA_j$

$$\therefore \quad \frac{1}{2}V_j^2\left(\frac{A_j}{A_1}\right)^2 + gy_1 = \frac{1}{2}V_j^2 + \int_1^2 \frac{\partial V_s}{\partial t}\,dy$$

or,

$$gy_1 = \frac{1}{2}V_j^2\left[1-\left(\frac{A_j}{A_1}\right)^2\right] + \int_1^2 \frac{\partial V_s}{\partial t}\,dy$$

If we assume quasi-steady flow, we say that $\int_1^2 \frac{\partial V_s}{\partial t}\,dy$ is negligible and hence $\frac{2gy}{V_j^2[1-AR^2]} = 1$ where $AR = \frac{A_j}{A_1}$

Now, $\int_1^2 \frac{\partial V_s}{\partial t} \approx y\frac{dV_s}{dt} = y\frac{dV_1}{dt} = y\frac{d}{dt}\left(V_j\frac{A_j}{A_1}\right) = y\frac{A_j}{A_1}\frac{dV_j}{dt}$

Thus for the assumption to be reasonable we must have

$$\left|y\frac{A_j}{A_1}\frac{dV_j}{dt}\right| \ll gy \qquad \text{or} \qquad \left|\frac{A_j}{A_1}\frac{dV_j}{dt}\right| \ll g$$

Under the assumption of quasi-steady flow

$$V_j = \left[2gy\frac{1}{(1-AR^2)}\right]^{1/2} \qquad \text{where } AR = A_j/A_1$$

then,

$$\frac{dV_j}{dt} = \sqrt{\frac{2g}{(1-AR^2)}}\,\frac{1}{2\sqrt{y}}\,\frac{dy}{dt} = \frac{dy}{dt}\sqrt{\frac{g}{2y(1-AR^2)}}$$

Since $\frac{dy}{dt} = -V_1 = -V_j\frac{A_j}{A_1}$, then

$$\frac{dV_j}{dt} = -V_j\frac{A_j}{A_1}\sqrt{\frac{g}{2y(1-AR^2)}} = -\frac{A_j}{A_1}\sqrt{\frac{V_j^2(1-AR^2)}{2gy}}\;\frac{g}{(1-AR^2)}$$

and

$$\frac{dV_j}{dt} = -\frac{A_j}{A_1}\frac{g}{(1-AR^2)}$$

For $\left|\frac{A_j}{A_1}\frac{dV_j}{dt}\right| \ll g$, then $\left(\frac{A_j}{A_1}\right)^2\frac{1}{(1-AR^2)} \ll 1$

If we take $\left(\frac{A_j}{A_1}\right)^2\frac{1}{(1-AR^2)} \approx 0.01$

then,

$$\left(\frac{A_j}{A_1}\right)^2 = 0.01(1-AR^2) = 0.01\left[1-\left(\frac{A_j}{A_1}\right)^2\right]$$

and

$$1.01\left(\frac{A_j}{A_1}\right)^2 = 0.01$$

$$\frac{A_j}{A_1} = 0.0995$$

or

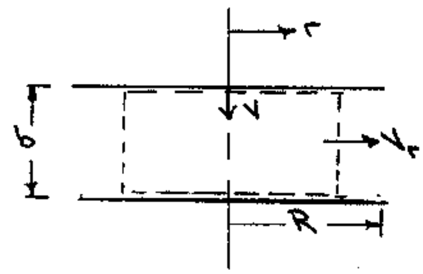
$$\frac{D_j}{D_1} = \left(\frac{A_j}{A_1}\right)^{1/2} = 0.32 \longleftarrow$$

In problem 4.48, $D_j/D_1 = d/D = .04$ and hence the assumption of quasi-steady flow is valid.

Problem 6.94

[Difficulty: 5]

6.94 Two circular disks, of radius R, are separated by distance b. The upper disk moves toward the lower one at constant speed V. The space between the disks is filled with a frictionless, incompressible fluid, which is squeezed out as the disks come together. Assume that, at any radial section, the velocity is uniform across the gap width b. However, note that b is a function of time. The pressure surrounding the disks is atmospheric. Determine the gage pressure at $r = 0$.



Solution.

Basic equation: $\frac{p_1}{\rho}+\frac{V_1^2}{2}+gz_1 = \frac{p_2}{\rho}+\frac{V_2^2}{2}+gz_2+\int_1^2 \frac{\partial V_s}{\partial t}\,ds$

$$0=\frac{\partial}{\partial t}\int_{CV}\rho\, d\forall + \int_{CS}\rho \vec{V}\cdot d\vec{A}$$

Assumptions: (1) incompressible flow
(2) frictionless flow
(3) flow along a streamline
(4) uniform radial flow at any r
(5) neglect elevation changes.

Then,

$$0 = \frac{\partial}{\partial t}\int_{CV}\rho\, d\forall + \int_{CS}\rho\vec{V}\cdot d\vec{A} = \frac{\partial}{\partial t}(\rho\pi r^2 b)+\rho V_r 2\pi r b$$

$$= \rho\pi r^2\frac{\partial b}{\partial t}+\rho V_r 2\pi r b. \quad \text{But } \frac{\partial b}{\partial t}=-V$$

$$\therefore 0 = -\rho\pi r^2 V+\rho V_r 2\pi r b \quad \text{and} \quad V_r = V\frac{r}{2b}$$

Applying the Bernoulli equation between point ① $(r=r)$ and point ② $(r=R)$

$$p_1-p_2=\frac{\rho}{2}\left[V_2^2-V_1^2\right]+\int_r^R\rho\frac{\partial V_r}{\partial t}\,dr \quad \text{Now, } \frac{\partial V_r}{\partial t}=\frac{\partial}{\partial t}\left(V\frac{r}{2b}\right)=\frac{rV}{2}\left(-\frac{1}{b^2}\frac{db}{dt}\right)=\frac{V^2 r}{2b^2}$$

$$=\frac{\rho}{2}\left[\left(\frac{VR}{2b}\right)^2-\left(\frac{Vr}{2b}\right)^2\right]+\int_r^R \rho\frac{V^2 r}{2b^2}\,dr$$

$$=\frac{\rho V^2}{8b^2}\left[R^2-r^2\right]+\frac{\rho V^2}{4b^2}r^2\Big]_r^R = \frac{\rho V^2}{8b^2}\left[R^2-r^2\right]+\frac{\rho V^2}{4b^2}\left[R^2-r^2\right]$$

$$p_1-p_{atm}=\frac{3}{8}\frac{\rho V^2}{b^2}\left[R^2-r^2\right]=\frac{3}{8}\rho\frac{V^2R^2}{b^2}\left[1-\left(\frac{r}{R}\right)^2\right]$$

When $r=0$ $\quad p_1=p_0$

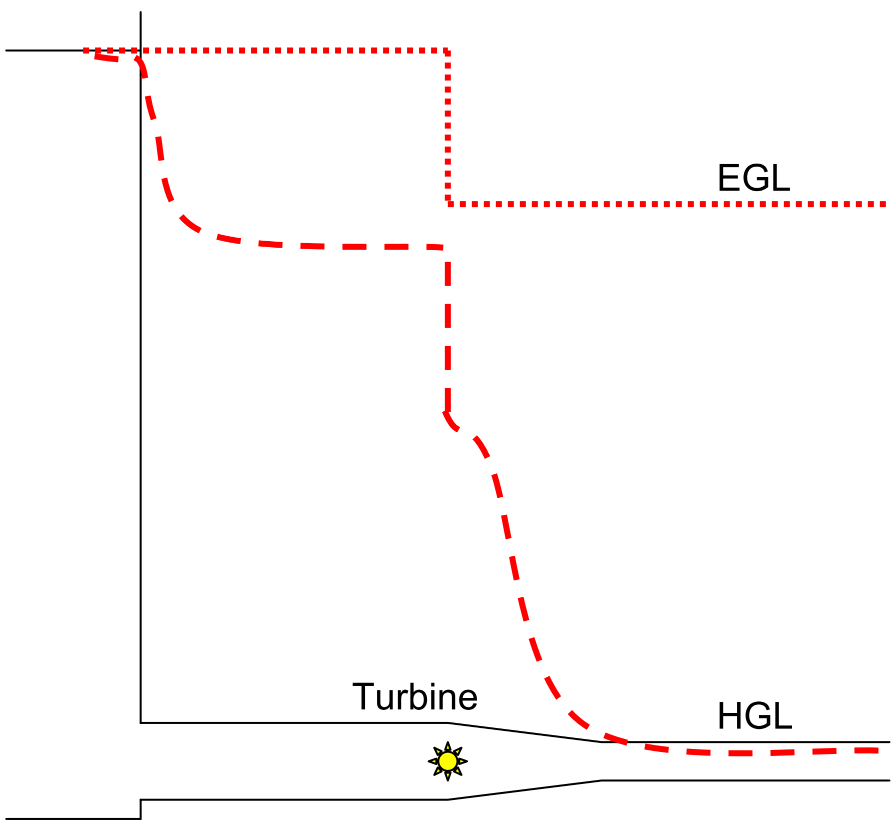
$$\therefore p_0-p_{atm}=\frac{3}{8}\rho\frac{V^2R^2}{b^2}$$

Problem 6.95

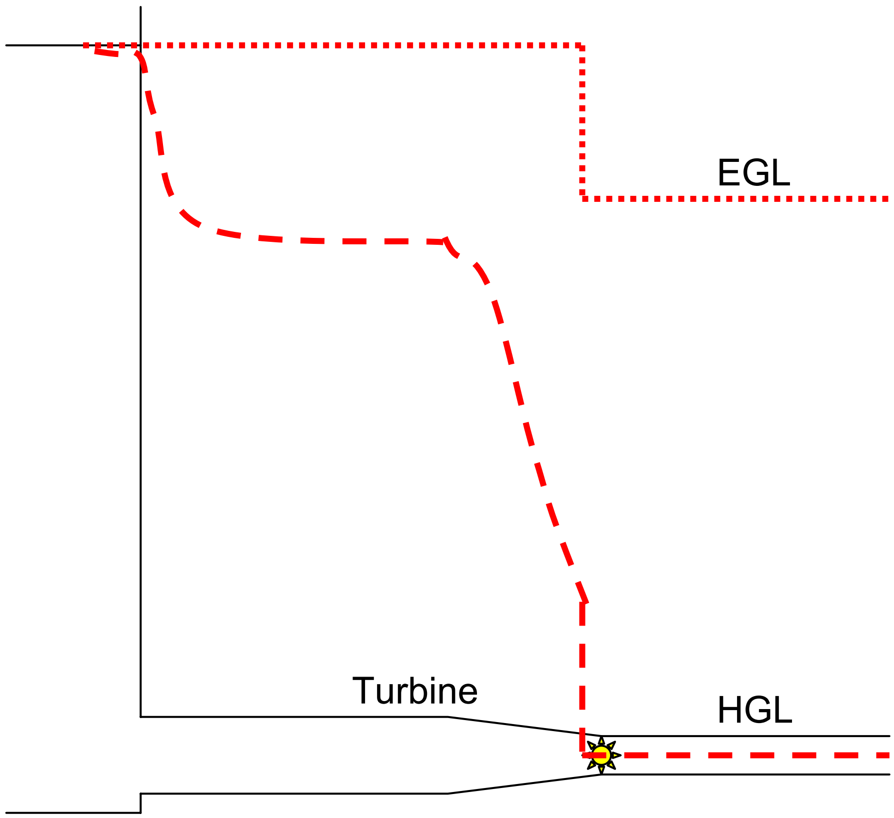
[Difficulty: 2]

6.95 Carefully sketch the energy grade lines (EGL) and hydraulic grade lines (HGL) for the system shown in Fig. 6.6 if the pipe is horizontal (i.e., the outlet is at the base of the reservoir), and a water turbine (extracting energy) is located at point (2), or at point (3). In Chapter 8 we will investigate the effects of friction on internal flows. Can you anticipate and sketch the effect of friction on the EGL and HGL for the two cases?

(a) Note that the effect of friction would be that the EGL would tend to drop: suddenly at the contraction, gradually in the large pipe, more steeply in the small pipe. The HGL would then "hang" below the HGL in a manner similar to that shown.



(b) Note that the effect of friction would be that the EGL would tend to drop: suddenly at the contraction, gradually in the large pipe, more steeply in the small pipe. The HGL would then "hang" below the HGL in a manner similar to that shown.

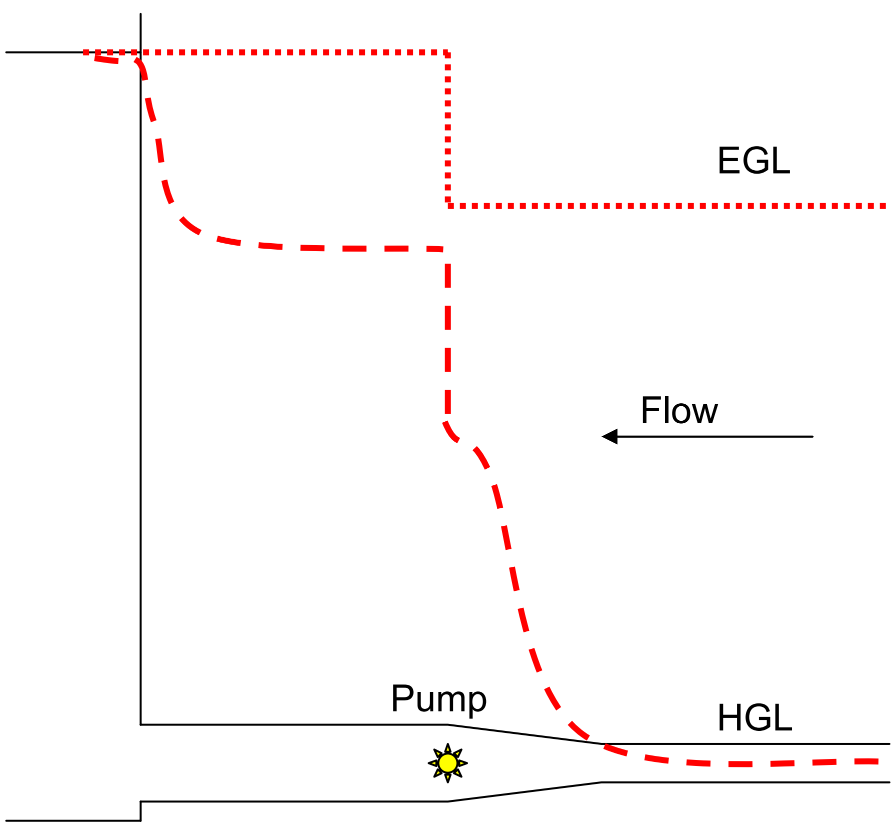


Problem 6.96

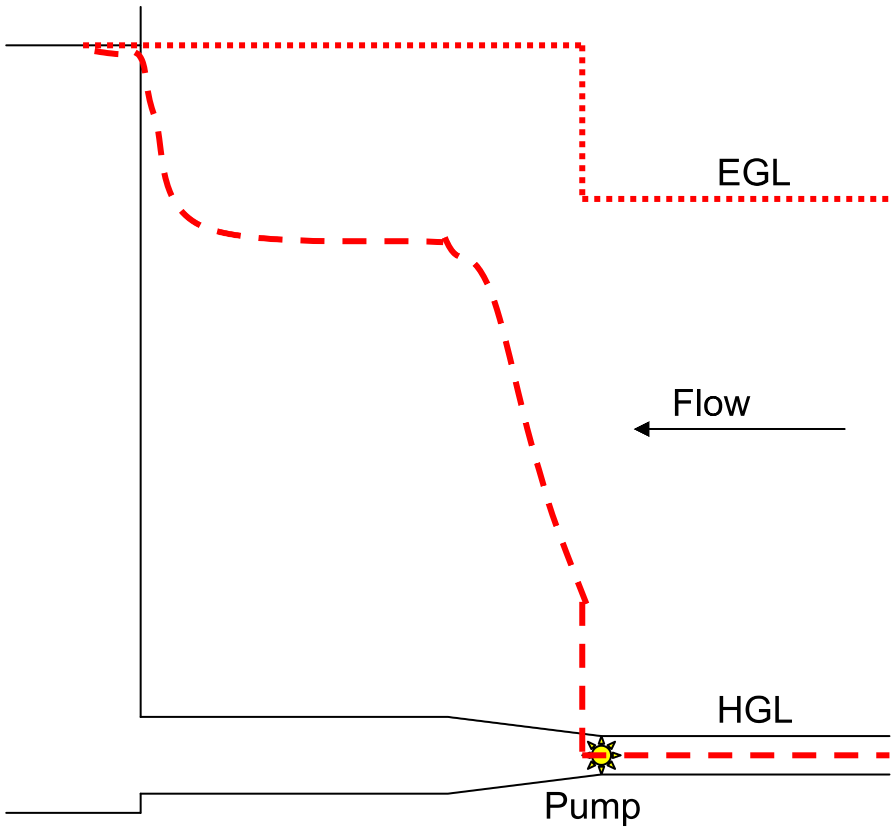
[Difficulty: 2]

6.96 Carefully sketch the energy grade lines (EGL) and hydraulic grade lines (HGL) for the system shown in Fig. 6.6 if a pump (adding energy to the fluid) is located at point (2), or at point (3), such that flow is *into* the reservoir. In Chapter 8 we will investigate the effects of friction on internal flows. Can you anticipate and sketch the effect of friction on the EGL and HGL for the two cases?

(a) Note that the effect of friction would be that the EGL would tend to drop from right to left: steeply in the small pipe, gradually in the large pipe, and suddenly at the expansion. The HGL would then "hang" below the HGL in a manner similar to that shown.



(b) Note that the effect of friction would be that the EGL would tend to drop from right to left: steeply in the small pipe, gradually in the large pipe, and suddenly at the expansion. The HGL would then “hang” below the HGL in a manner similar to that shown.



Problem 6.97

[Difficulty: 2]

6.97 Determine whether the Bernoulli equation can be applied between different radii for the vortex flow fields (a) $\vec{V} = \omega r \hat{e}_\theta$ and (b) $\vec{V} = \hat{e}_\theta K/2\pi r$.

Solution: Since $V_r = 0$, the streamlines are concentric circles.

In order for it to be possible to apply the Bernoulli equation between different radii, it is necessary that the flow be irrotational.

Basic equation: $\vec{\omega} = \frac{1}{2}\nabla \times \vec{V}$

Flow (1)

$$\nabla \times \vec{V}_1 = \left(\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{k}\frac{\partial}{\partial z}\right) \times \omega r \hat{e}_\theta$$

$$= \hat{e}_r \times \hat{e}_\theta \frac{\partial}{\partial r}(\omega r) + \hat{e}_r \times \omega r \cancelto{0}{\frac{\partial \hat{e}_\theta}{\partial r}} + \cancelto{0}{\hat{e}_\theta \times \hat{e}_\theta} \frac{1}{r}\frac{\partial(\omega r)}{\partial \theta} + \hat{e}_\theta \times \frac{\omega r}{r}\frac{\partial \hat{e}_\theta}{\partial \theta}$$

$$= \hat{k}\omega + \hat{e}_\theta \times \omega(-\hat{e}_r)$$

$$\nabla \times \vec{V}_1 = 2\omega \hat{k}$$

$\therefore$ Flow (1) is rotational and Bernoulli equation cannot be applied between different radii.

Flow (2)

$$\nabla \times \vec{V}_2 = \left(\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{k}\frac{\partial}{\partial z}\right) \times \frac{K}{2\pi r}\hat{e}_\theta$$

$$= \hat{e}_r \times \hat{e}_\theta \frac{\partial}{\partial r}\left(\frac{K}{2\pi r}\right) + \hat{e}_r \times \left(\frac{K}{2\pi r}\right)\cancelto{0}{\frac{\partial \hat{e}_\theta}{\partial r}} + \cancelto{0}{\hat{e}_\theta \times \hat{e}_\theta}\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{K}{2\pi r}\right) + \hat{e}_\theta \times \frac{1}{r}\left(\frac{K}{2\pi r}\right)\frac{\partial \hat{e}_\theta}{\partial \theta}$$

$$= -\hat{k}\frac{K}{2\pi r^2} + \hat{e}_\theta \frac{K}{2\pi r^2} \times (-\hat{e}_r)$$

$$= -\hat{k}\frac{K}{2\pi r^2} + \hat{k}\frac{K}{2\pi r^2}$$

$$\nabla \times \vec{V}_2 = 0$$

Since the flow field is irrotational, Bernoulli equation can be applied between different radii if the flow is also incompressible and frictionless.

Problem 6.98

[Difficulty: 3]

6.98 Consider a two-dimensional fluid flow: $u = ax + by$ and $v = cx + dy$, where a, b, c and d are constant. If the flow is incompressible and irrotational, find the relationships among a, b, c, and d. Find the stream function and velocity potential function of this flow.

Given: 2D incompressible, inviscid flow field

Find: Relationships among constants; stream function; velocity potential

Solution:

Basic equations Incompressible $u = \frac{\partial}{\partial y}\psi$ $\quad v = -\frac{\partial}{\partial x}\psi$ Irrotational $u = -\frac{\partial}{\partial x}\phi$ $\quad v = -\frac{\partial}{\partial y}\phi$

$$u(x,y) = a\cdot x + b\cdot y \qquad v(x,y) = c\cdot x + d\cdot y$$

Check incompressibility $\frac{\partial}{\partial x}u(x,y) + \frac{\partial}{\partial y}v(x,y) = a + d$ Hence must have $d = -a$

Check irrotational $\frac{\partial}{\partial x}v(x,y) - \frac{\partial}{\partial y}u(x,y) = c - b$ Hence must have $c = b$

Hence for the streamfunction $\psi(x,y) = \int u(x,y)\,dy = a\cdot x\cdot y + \frac{1}{2}\cdot b\cdot y^2 + f(x)$

$$\psi(x,y) = -\int v(x,y)\,dx = -\frac{1}{2}\cdot c\cdot x^2 - d\cdot x\cdot y + g(y)$$

Comparing $\psi(x,y) = \frac{1}{2}\cdot b\cdot y^2 - \frac{1}{2}\cdot c\cdot x^2 + a\cdot x\cdot y$ $\qquad \psi(x,y) = a\cdot x\cdot y + \frac{1}{2}\cdot b\cdot\left(y^2 - x^2\right)$

Check $u(x,y) = \frac{\partial}{\partial y}\psi(x,y) = a\cdot x + b\cdot y$ $\qquad v(x,y) = -\frac{\partial}{\partial x}\psi(x,y) = b\cdot x - a\cdot y$

Hence for the velocity potential $\phi(x,y) = -\int u(x,y)\,dx = -\frac{1}{2}\cdot a\cdot x^2 - b\cdot x\cdot y + f(y)$

$$\psi(x,y) = -\int v(x,y)\,dy = -c\cdot x\cdot y - \frac{1}{2}\cdot d\cdot y^2 + g(x)$$

Comparing $\phi(x,y) = -\frac{1}{2}\cdot a\cdot x^2 - \frac{1}{2}\cdot d\cdot y^2 - b\cdot x\cdot y$ $\qquad \phi(x,y) = -b\cdot x\cdot y - \frac{1}{2}\cdot a\cdot\left(x^2 - y^2\right)$

Check $u(x,y) = -\frac{\partial}{\partial x}\phi(x,y) = a\cdot x + b\cdot y$ $\qquad v(x,y) = -\frac{\partial}{\partial y}\phi(x,y) = b\cdot x - a\cdot y$

Problem 6.99

[Difficulty: 3]

6.99 Consider the flow represented by the stream function $\psi = Ax^2y$, where A is a dimensional constant equal to 2.5 $m^{-1}\cdot s^{-1}$. The density is 1200 kg/m³. Is the flow rotational? Can the pressure difference between points $(x, y) = (1, 4)$ and $(2, 1)$ be evaluated? If so, calculate it, and if not, explain why.

Given: Stream function

Find: If the flow is irrotational; Pressure difference between points (1,4) and (2,1)

Solution:

Basic equations: Incompressibility because ψ exists $u = \frac{\partial}{\partial y}\psi$ $v = -\frac{\partial}{\partial x}\psi$ Irrotationality $\frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u = 0$

$$\psi(x,y) = A\cdot x^2\cdot y$$

$$u(x,y) = \frac{\partial}{\partial y}\psi(x,y) = \frac{\partial}{\partial y}\left(A\cdot x^2\cdot y\right) \qquad u(x,y) = A\cdot x^2$$

$$v(x,y) = -\frac{\partial}{\partial x}\psi(x,y) = -\frac{\partial}{\partial x}\left(A\cdot x^2\cdot y\right) \qquad v(x,y) = -2\cdot A\cdot x\cdot y$$

Hence $\frac{\partial}{\partial x}v(x,y) - \frac{\partial}{\partial y}u(x,y) = -2\cdot A\cdot y$ $\quad \frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u \neq 0$ so flow is NOT IRROTATIONAL

Since flow is rotational, we must be on same streamline to be able to use Bernoulli

At point (1,4) $\psi(1,4) = 4\,A$ and at point (2,1) $\psi(2,1) = 4\,A$

Hence these points are on same streamline so Bernoulli can be used. The velocity at a point is $V(x,y) = \sqrt{u(x,y)^2 + v(x,y)^2}$

Hence at (1,4)
$$V_1 = \sqrt{\left[\frac{2.5}{m\cdot s}\times(1\cdot m)^2\right]^2 + \left(-2\times\frac{2.5}{m\cdot s}\times 1\cdot m\times 4\cdot m\right)^2} \qquad V_1 = 20.2\frac{m}{s}$$

Hence at (2,1)
$$V_2 = \sqrt{\left[\frac{2.5}{m\cdot s}\times(2\cdot m)^2\right]^2 + \left(-2\times\frac{2.5}{m\cdot s}\times 2\cdot m\times 1\cdot m\right)^2} \qquad V_2 = 14.1\frac{m}{s}$$

Using Bernoulli
$$\frac{p_1}{\rho} + \frac{1}{2}\cdot V_1^2 = \frac{p_2}{\rho} + \frac{1}{2}\cdot V_2^2 \qquad \Delta p = \frac{\rho}{2}\cdot\left(V_2^2 - V_1^2\right)$$

$$\Delta p = \frac{1}{2}\times 1200\cdot\frac{kg}{m^3}\times\left(14.1^2 - 20.2^2\right)\cdot\left(\frac{m}{s}\right)^2\times\frac{N\cdot s^2}{kg\cdot m} \qquad \Delta p = -126\cdot kPa$$

Problem 6.100

[Difficulty: 2]

6.100 The velocity field for a two-dimensional flow is $\vec{V} = (Ax - By)t\hat{i} - (Bx + Ay)t\hat{j}$, where $A = 1\ \text{s}^{-2}$ $B = 2\ \text{s}^{-2}$, t is in seconds, and the coordinates are measured in meters. Is this a possible incompressible flow? Is the flow steady or unsteady? Show that the flow is irrotational and derive an expression for the velocity potential.

Solution: For incompressible flow, $\nabla \cdot \vec{V} = 0$

For given flow $\nabla \cdot \vec{V} = \frac{\partial}{\partial x}(Ax - By)t - \frac{\partial}{\partial y}(Bx + Ay)t = At - At = 0$

$\therefore$ velocity field represents a possible incompressible flow

The flow is unsteady since $\vec{V} = \vec{V}(x, y, t)$

The rotation is given by $\vec{\omega} = \frac{1}{2}\nabla \times \vec{V} = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k}$

$$\vec{\omega} = \frac{1}{2}\left[\frac{\partial}{\partial x} - (Bx + Ay)t - \frac{\partial}{\partial y}(Ax - By)t\right] = -Bt + Bt = 0$$

$\vec{\omega} = 0$, so flow is irrotational

From the definition of the velocity potential, $\vec{V} = -\nabla\phi$

$u = -\frac{\partial\phi}{\partial x}$ and $\phi = \int -u\,dx + f(y,t) = \int -(Ax - By)t\,dx + f(y,t)$

$$\phi = \left(-A\frac{x^2}{2} + Bxy\right)t + f(y,t)$$

$v = -\frac{\partial\phi}{\partial y}$ and $\phi = \int -v\,dy + g(x,t) = \int (Bx + Ay)t\,dy + g(x,t)$

$$\phi = \left(Bxy + A\frac{y^2}{2}\right)t + g(x,t)$$

Comparing the two expressions for ϕ we conclude

$$f(y,t) = \frac{A}{2}y^2 t \quad \text{and} \quad g(x,t) = -\frac{A}{2}x^2 t$$

Hence,

$$\phi = \left\{\frac{A}{2}(y^2 - x^2) + Bxy\right\}t \qquad \phi$$

Problem 6.101

[Difficulty: 3]

6.101 Using Table 6.2, find the stream function and velocity potential for a plane source, of strength q, near a 90° corner. The source is equidistant h from each of the two infinite planes that make up the corner. Find the velocity distribution along one of the planes, assuming $p = p_0$ at infinity. By choosing suitable values for q and h, plot the streamlines and lines of constant velocity potential. (*Hint*: Use the *Excel* workbook of Example 6.10.)

Given: Data from Table 6.2

Find: Stream function and velocity potential for a source in a corner; plot; velocity along one plane

Solution:

From Table 6.2, for a source at the origin $\quad \psi(r,\theta) = \frac{q}{2\cdot\pi}\cdot\theta \qquad \phi(r,\theta) = -\frac{q}{2\cdot\pi}\cdot\ln(r)$

Expressed in Cartesian coordinates $\quad \psi(x,y) = \frac{q}{2\cdot\pi}\cdot\operatorname{atan}\left(\frac{y}{x}\right) \qquad \phi(x,y) = -\frac{q}{4\cdot\pi}\cdot\ln\left(x^2+y^2\right)$

To build flow in a corner, we need image sources at three locations so that there is symmetry about both axes. We need sources at (h,h), (h,- h), (- h,h), and (- h,- h)

Hence the composite stream function and velocity potential are

$$\psi(x,y) = \frac{q}{2\cdot\pi}\cdot\left(\operatorname{atan}\left(\frac{y-h}{x-h}\right) + \operatorname{atan}\left(\frac{y+h}{x-h}\right) + \operatorname{atan}\left(\frac{y+h}{x+h}\right) + \operatorname{atan}\left(\frac{y-h}{x+h}\right)\right)$$

$$\phi(x,y) = -\frac{q}{4\cdot\pi}\cdot\ln\left[\left[(x-h)^2+(y-h)^2\right]\cdot\left[(x-h)^2+(y+h)^2\right]\right] - \frac{q}{4\cdot\pi}\cdot\left[(x+h)^2+(y+h)^2\right]\cdot\left[(x+h)^2+(y-h)^2\right]$$

By a similar reasoning the horizontal velocity is given by

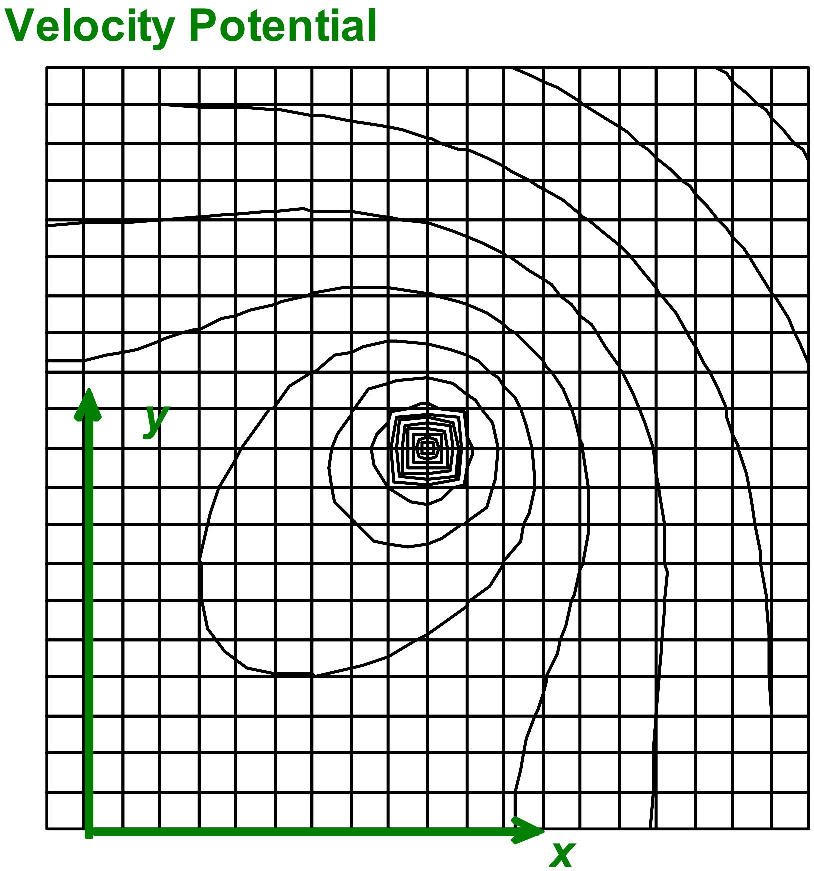
$$u = \frac{q\cdot(x-h)}{2\cdot\pi\left[(x-h)^2+(y-h)^2\right]} + \frac{q\cdot(x-h)}{2\cdot\pi\left[(x-h)^2+(y+h)^2\right]} + \frac{q\cdot(x+h)}{2\cdot\pi\left[(x+h)^2+(y+h)^2\right]} + \frac{q\cdot(x+h)}{2\cdot\pi\left[(x+h)^2+(y+h)^2\right]}$$

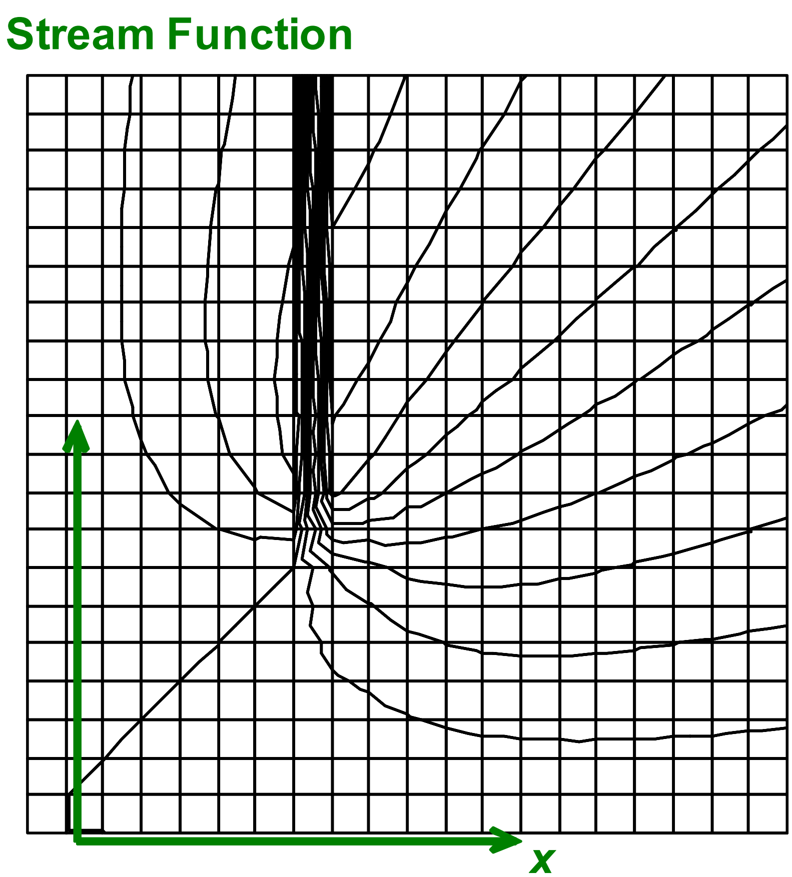
Along the horizontal wall ($y = 0$)

$$u = \frac{q\cdot(x-h)}{2\cdot\pi\left[(x-h)^2+h^2\right]} + \frac{q\cdot(x-h)}{2\cdot\pi\left[(x-h)^2+h^2\right]} + \frac{q\cdot(x+h)}{2\cdot\pi\left[(x+h)^2+h^2\right]} + \frac{q\cdot(x+h)}{2\cdot\pi\left[(x+h)^2+h^2\right]}$$

or $\quad u(x) = \frac{q}{\pi}\cdot\left[\frac{x-h}{(x-h)^2+h^2} + \frac{x+h}{(x+h)^2+h^2}\right]$

The results in *Excel* are:





Problem 6.102

[Difficulty: 3]

6.102 The flow field for a plane source at a distance h above an infinite wall aligned along the x axis is given by

$$\vec{V} = \frac{q}{2\pi[x^2 + (y-h)^2]}[x\hat{i} + (y-h)\hat{j}] + \frac{q}{2\pi[x^2 + (y+h)^2]}[x\hat{i} + (y+h)\hat{j}]$$

where q is the strength of the source. The flow is irrotational and incompressible. Derive the stream function and velocity potential. By choosing suitable values for q and h, plot the streamlines and lines of constant velocity potential. (*Hint*: Use the *Excel* workbook of Example 6.10.)

Given: Velocity field of irrotational and incompressible flow

Find: Stream function and velocity potential; plot

Solution:

The velocity field is
$$u = \frac{q\cdot x}{2\cdot\pi\left[x^2 + (y-h)^2\right]} + \frac{q\cdot x}{2\cdot\pi\left[x^2 + (y+h)^2\right]} \qquad v = \frac{q\cdot(y-h)}{2\cdot\pi\left[x^2 + (y-h)^2\right]} + \frac{q\cdot(y+h)}{2\cdot\pi\left[x^2 + (y+h)^2\right]}$$

The basic equations are
$$u = \frac{\partial}{\partial y}\psi \qquad v = -\frac{\partial}{\partial x}\psi \qquad u = -\frac{\partial}{\partial x}\phi \qquad v = -\frac{\partial}{\partial y}\phi$$

Hence for the stream function
$$\psi = \int u(x,y)\,dy = \frac{q}{2\cdot\pi}\cdot\left(\operatorname{atan}\left(\frac{y-h}{x}\right) + \operatorname{atan}\left(\frac{y+h}{x}\right)\right) + f(x)$$

$$\psi = -\int v(x,y)\,dx = \frac{q}{2\cdot\pi}\cdot\left(\operatorname{atan}\left(\frac{y-h}{x}\right) + \operatorname{atan}\left(\frac{y+h}{x}\right)\right) + g(y)$$

The simplest expression for ψ is
$$\psi(x,y) = \frac{q}{2\cdot\pi}\cdot\left(\operatorname{atan}\left(\frac{y-h}{x}\right) + \operatorname{atan}\left(\frac{y+h}{x}\right)\right)$$

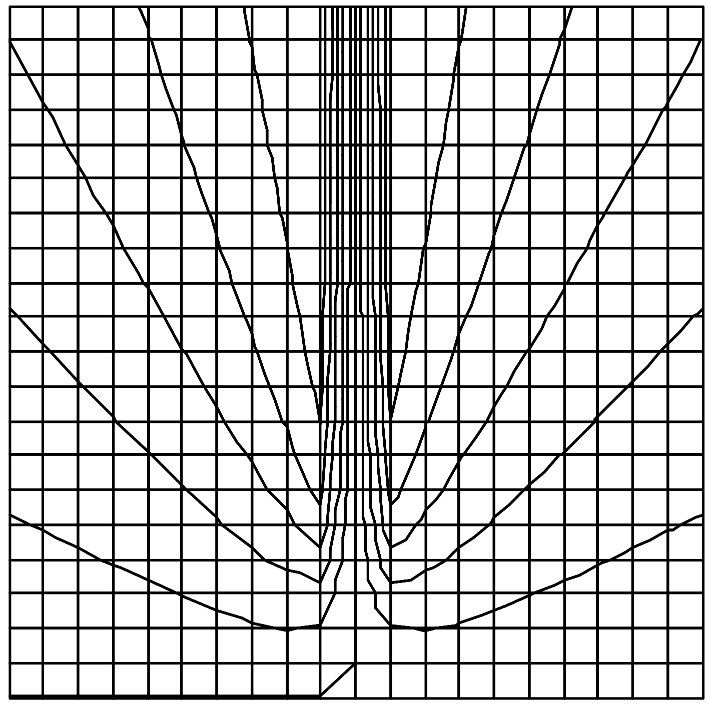
For the stream function
$$\phi = -\int u(x,y)\,dx = -\frac{q}{4\cdot\pi}\cdot\ln\left[\left[x^2 + (y-h)^2\right]\cdot\left[x^2 + (y+h)^2\right]\right] + f(y)$$

$$\phi = -\int v(x,y)\,dy = -\frac{q}{4\cdot\pi}\cdot\ln\left[\left[x^2 + (y-h)^2\right]\cdot\left[x^2 + (y+h)^2\right]\right] + g(x)$$

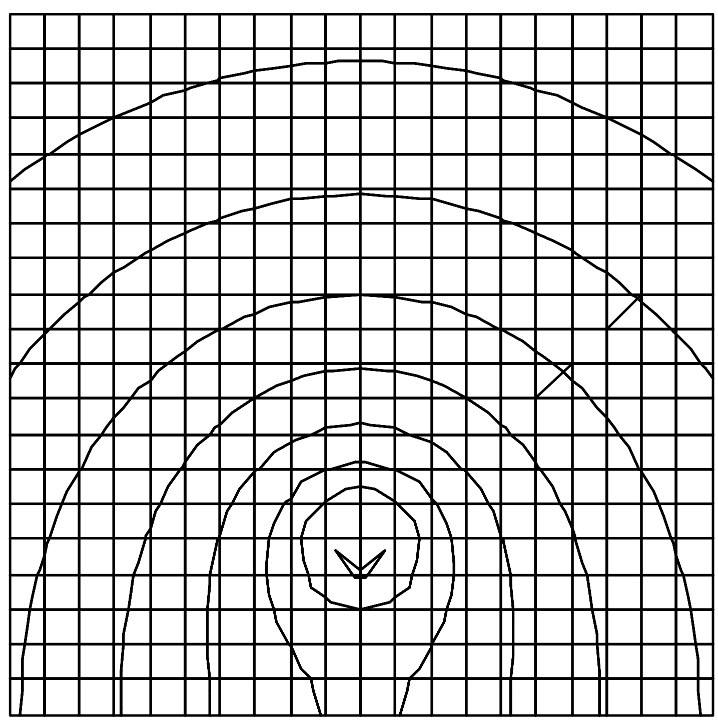
The simplest expression for φ is
$$\phi(x,y) = -\frac{q}{4\cdot\pi}\cdot\ln\left[\left[x^2 + (y-h)^2\right]\cdot\left[x^2 + (y+h)^2\right]\right]$$

In *Excel*:

Stream Function



Velocity Potential



Problem 6.103

[Difficulty: 3]

6.103 Using Table 6.2, find the stream function and velocity potential for a plane vortex, of strength K, near a 90° corner. The vortex is equidistant h from each of the two infinite planes that make up the corner. Find the velocity distribution along one of the planes, assuming $p = p_0$ at infinity. By choosing suitable values for K and h, plot the streamlines and lines of constant velocity potential. (*Hint*: Use the *Excel* workbook of Example Problem 6.10.)

Given: Data from Table 6.2

Find: Stream function and velocity potential for a vortex in a corner; plot; velocity along one plane

Solution:

From Table 6.2, for a vortex at the origin $\quad \phi(r,\theta) = \frac{K}{2\cdot\pi}\cdot\theta \qquad \psi(r,\theta) = -\frac{K}{2\cdot\pi}\cdot\ln(r)$

Expressed in Cartesian coordinates $\quad \phi(x,y) = \frac{q}{2\cdot\pi}\cdot\mathrm{atan}\left(\frac{y}{x}\right) \qquad \psi(x,y) = -\frac{q}{4\cdot\pi}\cdot\ln\left(x^2+y^2\right)$

To build flow in a corner, we need image vortices at three locations so that there is symmetry about both axes. We need vortices at (h,h), (h,- h), (- h,h), and (- h,- h). Note that some of them must have strengths of - K!

Hence the composite velocity potential and stream function are

$$\phi(x,y) = \frac{K}{2\cdot\pi}\cdot\left(\mathrm{atan}\left(\frac{y-h}{x-h}\right) - \mathrm{atan}\left(\frac{y+h}{x-h}\right) + \mathrm{atan}\left(\frac{y+h}{x+h}\right) - \mathrm{atan}\left(\frac{y-h}{x+h}\right)\right)$$

$$\psi(x,y) = -\frac{K}{4\cdot\pi}\cdot\ln\left[\frac{(x-h)^2+(y-h)^2}{(x-h)^2+(y+h)^2}\cdot\frac{(x+h)^2+(y+h)^2}{(x+h)^2+(y-h)^2}\right]$$

By a similar reasoning the horizontal velocity is given by

$$u = -\frac{K\cdot(y-h)}{2\cdot\pi\left[(x-h)^2+(y-h)^2\right]} + \frac{K\cdot(y+h)}{2\cdot\pi\left[(x-h)^2+(y+h)^2\right]} - \frac{K\cdot(y+h)}{2\cdot\pi\left[(x+h)^2+(y+h)^2\right]} + \frac{K\cdot(y-h)}{2\cdot\pi\left[(x+h)^2+(y-h)^2\right]}$$

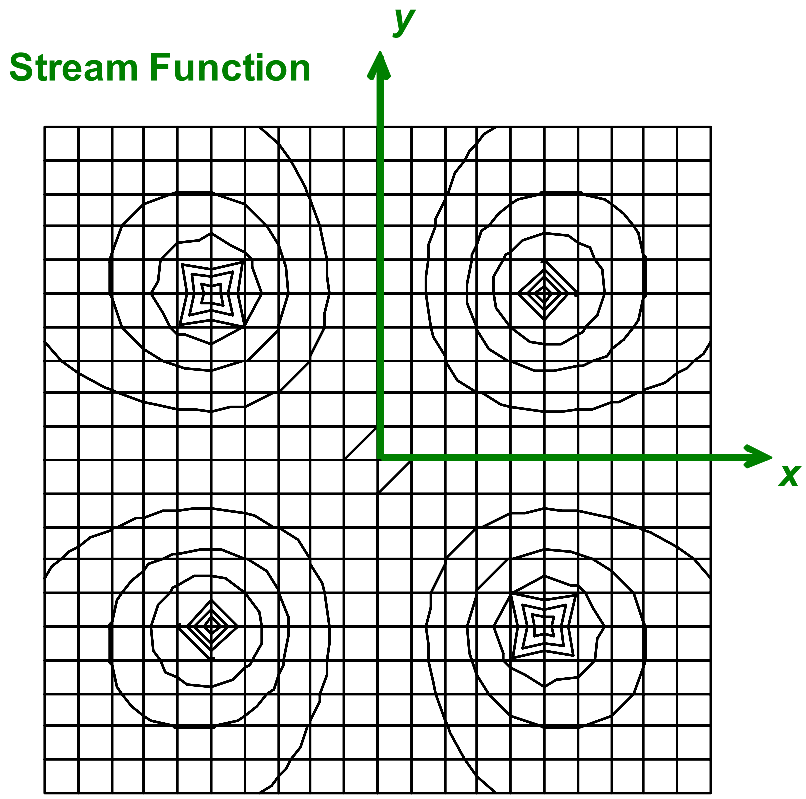
Along the horizontal wall (y = 0)

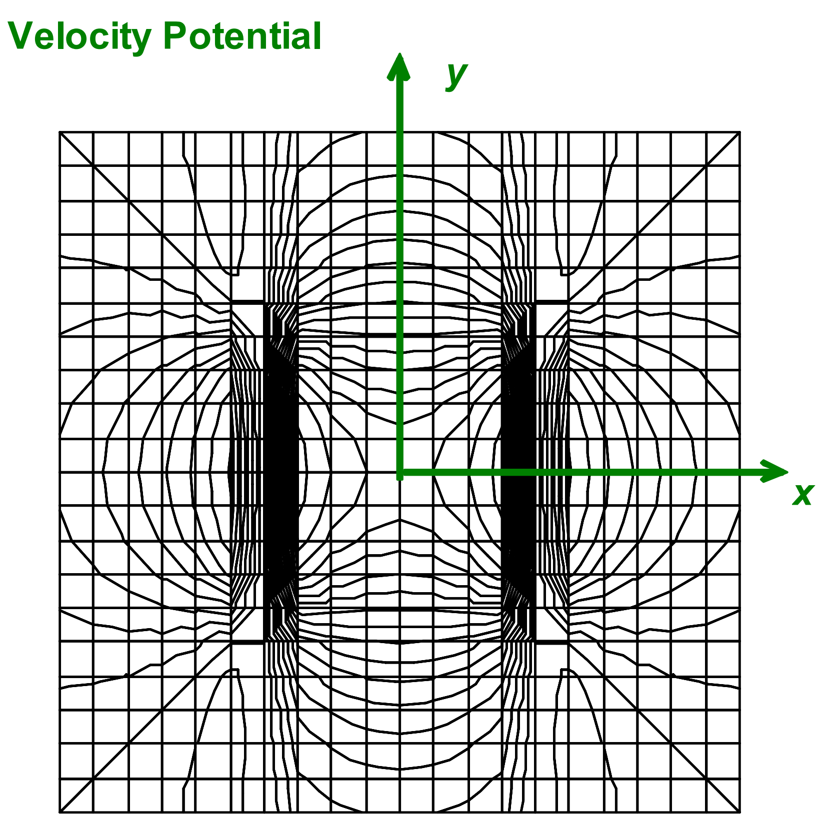
$$u = \frac{K\cdot h}{2\cdot\pi\left[(x-h)^2+h^2\right]} + \frac{K\cdot h}{2\cdot\pi\left[(x-h)^2+h^2\right]} - \frac{K\cdot h}{2\cdot\pi\left[(x+h)^2+h^2\right]} - \frac{K\cdot h}{2\cdot\pi\left[(x+h)^2+h^2\right]}$$

or

$$u(x) = \frac{K\cdot h}{\pi}\cdot\left[\frac{1}{(x-h)^2+h^2} - \frac{1}{(x+h)^2+h^2}\right]$$

In *Excel*:





Problem 6.104

[Difficulty: 2]

6.104 The stream function of a flow field is $\psi = Ax^2y - By^3$, where $A = 1\ \text{m}^{-1}\cdot\text{s}^{-1}$, $B = \frac{1}{3}\ \text{m}^{-1}\cdot\text{s}^{-1}$, and the coordinates are measured in meters. Find an expression for the velocity potential.

Given: Stream function

Find: Velocity potential

Solution:

Basic equations: Incompressibility because ψ exists $\quad u = \frac{\partial}{\partial y}\psi \quad v = -\frac{\partial}{\partial x}\psi \quad u = -\frac{\partial}{\partial x}\varphi \quad v = -\frac{\partial}{\partial y}\varphi$

Irrotationality $\quad \frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u = 0$

We have $\quad \psi(x,y) = A\cdot x^2\cdot y - B\cdot y^3$

Then $\quad u(x,y) = \frac{\partial}{\partial y}\psi(x,y) \quad u(x,y) = A\cdot x^2 - 3\cdot B\cdot y^2$

$v(x,y) = -\frac{\partial}{\partial x}\psi(x,y) \quad v(x,y) = -2\cdot A\cdot x\cdot y$

Then $\quad \frac{\partial}{\partial x}v(x,y) - \frac{\partial}{\partial y}u(x,y) = 6\cdot B\cdot y - 2\cdot A\cdot y \quad$ but $\quad 6\cdot B - 2\cdot A = 0\frac{1}{m\cdot s} \quad$ hence flow is IRROTATIONAL

Hence $\quad u = -\frac{\partial}{\partial x}\varphi \quad$ so $\quad \varphi(x,y) = -\int u(x,y)\,dx + f(y) \rightarrow \varphi(x,y) = f(y) - \frac{A\cdot x^3}{3} + 3\cdot B\cdot x\cdot y^2$

$v = -\frac{\partial}{\partial y}\varphi \quad$ so $\quad \varphi(x,y) = -\int v(x,y)\,dy + g(x) \rightarrow \varphi(x,y) = A\cdot x\cdot y^2 + g(x)$

Comparing, the simplest velocity potential is then $\quad \varphi(x,y) = A\cdot x\cdot y^2 - \frac{A\cdot x^3}{3}$

Problem 6.105

[Difficulty: 2]

6.105 A flow field is represented by the stream function $\psi = x^5 - 10x^3y^2 + 5xy^4$. Find the corresponding velocity field. Show that this flow field is irrotational and obtain the potential function.

Given: Stream function

Find: Velocity field; Show flow is irrotational; Velocity potential

Solution:

Basic equations: Incompressibility because ψ exists $\quad u = \frac{\partial}{\partial y}\psi \quad v = -\frac{\partial}{\partial x}\psi \quad u = -\frac{\partial}{\partial x}\varphi \quad v = -\frac{\partial}{\partial y}\varphi$

Irrotationality $\quad \frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u = 0$

$$\psi(x,y) = x^5 - 10\cdot x^3\cdot y^2 + 5\cdot x\cdot y^4$$

$$u(x,y) = \frac{\partial}{\partial y}\psi(x,y) \qquad u(x,y) = 20\cdot x\cdot y^3 - 20\cdot x^3\cdot y$$

$$v(x,y) = -\frac{\partial}{\partial x}\psi(x,y) \qquad v(x,y) = 30\cdot x^2\cdot y^2 - 5\cdot x^4 - 5\cdot y^4$$

$$\frac{\partial}{\partial x}v(x,y) - \frac{\partial}{\partial y}u(x,y) = 0$$

Hence flow is IRROTATIONAL

Hence $\quad u = -\frac{\partial}{\partial x}\varphi \quad$ so $\quad \varphi(x,y) = -\int u(x,y)\,dx + f(y) = 5\cdot x^4\cdot y - 10\cdot x^2\cdot y^3 + f(y)$

$v = -\frac{\partial}{\partial y}\varphi \quad$ so $\quad \varphi(x,y) = -\int v(x,y)\,dy + g(x) = 5\cdot x^4\cdot y - 10\cdot x^2\cdot y^3 + y^5 + g(x)$

Comparing, the simplest velocity potential is then $\quad \varphi(x,y) = 5\cdot x^4\cdot y - 10\cdot x^2\cdot y^3 + y^5$

Problem 6.106

[Difficulty: 2]

6.106 The stream function of a flow field is $\psi = Ax^3 - Bxy^2$, where $A = 1\ \text{m}^{-1}\cdot\text{s}^{-1}$ and $B = 3\ \text{m}^{-1}\cdot\text{s}^{-1}$, and coordinates are measured in meters. Find an expression for the velocity potential.

Given: Stream function

Find: Velocity potential

Solution:

Basic equations: Incompressibility because ψ exists $\quad u = \frac{\partial}{\partial y}\psi \quad v = -\frac{\partial}{\partial x}\psi \quad u = -\frac{\partial}{\partial x}\varphi \quad v = -\frac{\partial}{\partial y}\varphi$

Irrotationality $\quad \frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u = 0$

We have $\quad \psi(x,y) = A\cdot x^3 - B\cdot x\cdot y^2$

Then $\quad u(x,y) = \frac{\partial}{\partial y}\psi(x,y) \qquad u(x,y) = -2\cdot B\cdot x\cdot y$

$v(x,y) = -\frac{\partial}{\partial x}\psi(x,y) \qquad v(x,y) = B\cdot y^2 - 3\cdot A\cdot x^2$

Then $\quad \frac{\partial}{\partial x}v(x,y) - \frac{\partial}{\partial y}u(x,y) = 2\cdot B\cdot x - 6\cdot A\cdot x \quad$ but $\quad 2\cdot B - 6\cdot A = 0\,\frac{1}{m\cdot s} \quad$ hence flow is IRROTATIONAL

Hence $\quad u = -\frac{\partial}{\partial x}\varphi \quad$ so $\quad \varphi(x,y) = -\int u(x,y)\,dx + f(y) \rightarrow \varphi(x,y) = B\cdot y\cdot x^2 + f(y)$

$v = -\frac{\partial}{\partial y}\varphi \quad$ so $\quad \varphi(x,y) = -\int v(x,y)\,dy + g(x) \rightarrow \varphi(x,y) = g(x) - \frac{B\cdot y^3}{3} + 3\cdot A\cdot x^2\cdot y$

Comparing, the simplest velocity potential is then $\quad \varphi(x,y) = 3\cdot A\cdot x^2\cdot y - \frac{B\cdot y^3}{3}$

Problem 6.107

[Difficulty: 2]

6.107 The stream function of a flow field is $\psi = Ax^3 + B(xy^2 + x^2 - y^2)$, where ψ, x, y, A, and B are all dimensionless. Find the relation between A and B for this to be an irrotational flow. Find the velocity potential.

Given: Stream function

Find: Find A vs B if flow is irrotational; Velocity potential

Solution:

Basic equations: Incompressibility because ψ exists $\quad u = \frac{\partial}{\partial y}\psi \quad v = -\frac{\partial}{\partial x}\psi \quad u = -\frac{\partial}{\partial x}\varphi \quad v = -\frac{\partial}{\partial y}\varphi$

Irrotationality $\quad \frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u = 0$

We have $\quad \psi(x,y) = A\cdot x^3 + B\cdot\left(x\cdot y^2 + x^2 - y^2\right)$

$$u(x,y) = \frac{\partial}{\partial y}\psi(x,y) \qquad u(x,y) = -B\cdot(2\cdot y - 2\cdot x\cdot y)$$

$$v(x,y) = -\frac{\partial}{\partial x}\psi(x,y) \qquad v(x,y) = -3\cdot A\cdot x^2 - B\cdot\left(y^2 + 2\cdot x\right)$$

$$\frac{\partial}{\partial x}v(x,y) - \frac{\partial}{\partial y}u(x,y) = -2\cdot x\cdot(3\cdot A + B)$$

Hence flow is IRROTATIONAL if $\quad B = -3\cdot A$

Hence $\quad u = -\frac{\partial}{\partial x}\varphi \quad$ so $\quad \varphi(x,y) = -\int u(x,y)\,dx + f(y) = 2\cdot B\cdot y\cdot x - B\cdot y\cdot x^2 + f(y)$

$v = -\frac{\partial}{\partial y}\varphi \quad$ so $\quad \varphi(x,y) = -\int v(x,y)\,dy + g(x) = 3\cdot A\cdot x^2\cdot y + 2\cdot B\cdot x\cdot y + \frac{B\cdot y^3}{3} + g(x)$

Comparing, the simplest velocity potential is then $\quad \varphi(x,y) = 2\cdot B\cdot y\cdot x - B\cdot y\cdot x^2 + B\cdot\frac{y^3}{3}$

Problem 6.108

[Difficulty: 2]

6.108 A flow field is represented by the stream function $\psi = x^5 - 15x^4y^2 + 15x^2y^4 - y^6$. Find the corresponding velocity field. Show that this flow field is irrotational and obtain the potential function.

Given: Stream function

Find: Velocity field; Show flow is irrotational; Velocity potential

Solution:

Basic equations: Incompressibility because ψ exists $\quad u = \frac{\partial}{\partial y}\psi \quad v = -\frac{\partial}{\partial x}\psi \quad u = -\frac{\partial}{\partial x}\varphi \quad v = -\frac{\partial}{\partial y}\varphi$

Irrotationality $\quad \frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u = 0$

We have $\quad \psi(x,y) = x^6 - 15\cdot x^4\cdot y^2 + 15\cdot x^2\cdot y^4 - y^6$

$$u(x,y) = \frac{\partial}{\partial y}\psi(x,y) \qquad u(x,y) = 60\cdot x^2\cdot y^3 - 30\cdot x^4\cdot y - 6\cdot y^5$$

$$v(x,y) = -\frac{\partial}{\partial x}\psi(x,y) \qquad v(x,y) = 60\cdot x^3\cdot y^2 - 6\cdot x^5 - 30\cdot x\cdot y^4$$

$$\frac{\partial}{\partial x}v(x,y) - \frac{\partial}{\partial y}u(x,y) = 0$$

Hence flow is IRROTATIONAL

Hence $\quad u = -\frac{\partial}{\partial x}\varphi \quad$ so $\quad \varphi(x,y) = -\int u(x,y)\,dx + f(y) = 6\cdot x^5\cdot y - 20\cdot x^3\cdot y^3 + 6\cdot x\cdot y^5 + f(y)$

$v = -\frac{\partial}{\partial y}\varphi \quad$ so $\quad \varphi(x,y) = -\int v(x,y)\,dy + g(x) = 6\cdot x^5\cdot y - 20\cdot x^3\cdot y^3 + 6\cdot x\cdot y^5 + g(x)$

Comparing, the simplest velocity potential is then $\quad \varphi(x,y) = 6\cdot x^5\cdot y - 20\cdot x^3\cdot y^3 + 6\cdot x\cdot y^5$

Problem 6.109

[Difficulty: 2]

6.109 Consider the flow field represented by the potential function $\phi = Ax^2 + Bxy - Ay^2$. Verify that this is an incompressible flow and determine the corresponding stream function.

Given: Velocity potential

Find: Show flow is incompressible; Stream function

Solution:

Basic equations: Irrotationality because φ exists $\quad u = \frac{\partial}{\partial y}\psi \quad v = -\frac{\partial}{\partial x}\psi \quad u = -\frac{\partial}{\partial x}\varphi \quad v = -\frac{\partial}{\partial y}\varphi$

Incompressibility $\quad \frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$

We have $\quad \varphi(x,y) = A\cdot x^2 + B\cdot x\cdot y - A\cdot y^2$

$u(x,y) = -\frac{\partial}{\partial x}\varphi(x,y) \qquad u(x,y) = -2\cdot A\cdot x - B\cdot y$

$v(x,y) = -\frac{\partial}{\partial y}\varphi(x,y) \qquad v(x,y) = 2\cdot A\cdot y - B\cdot x$

Hence $\quad \frac{\partial}{\partial x}u(x,y) + \frac{\partial}{\partial y}v(x,y) = 0 \qquad$ Hence flow is INCOMPRESSIBLE

Hence $\quad u = \frac{\partial}{\partial y}\psi \quad$ so $\quad \psi(x,y) = \int u(x,y)\,dy + f(x) = -2\cdot A\cdot x\cdot y - \frac{1}{2}\cdot B\cdot y^2 + f(x)$

$v = -\frac{\partial}{\partial x}\psi \quad$ so $\quad \psi(x,y) = -\int v(x,y)\,dx + g(y) = -2\cdot A\cdot x\cdot y + \frac{1}{2}\cdot B\cdot x^2 + g(y)$

Comparing, the simplest stream function is then $\quad \psi(x,y) = -2\cdot A\cdot x\cdot y + \frac{1}{2}\cdot B\cdot x^2 - \frac{1}{2}\cdot B\cdot y^2$

Problem 6.110

[Difficulty: 2]

6.110 Consider the flow field presented by the potential function $\phi = x^5 - 10x^3y^2 + 5xy^4 - x^2 + y^2$. Verify that this is an incompressible flow, and obtain the corresponding stream function.

Given: Velocity potential

Find: Show flow is incompressible; Stream function

Solution:

Basic equations: Irrotationality because φ exists $\quad u = \frac{\partial}{\partial y}\psi \quad v = -\frac{\partial}{\partial x}\psi \quad u = -\frac{\partial}{\partial x}\varphi \quad v = -\frac{\partial}{\partial y}\varphi$

Incompressibility $\quad \frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$

We have $\quad \varphi(x,y) = x^5 - 10\cdot x^3\cdot y^2 + 5\cdot x\cdot y^4 - x^2 + y^2$

$$u(x,y) = -\frac{\partial}{\partial x}\varphi(x,y) \qquad u(x,y) = 30\cdot x^2\cdot y^2 - 5\cdot x^4 + 2\cdot x - 5\cdot y^4$$

$$v(x,y) = -\frac{\partial}{\partial y}\varphi(x,y) \qquad v(x,y) = 20\cdot x^3\cdot y - 20\cdot x\cdot y^3 - 2\cdot y$$

Hence $\quad \frac{\partial}{\partial x}u(x,y) + \frac{\partial}{\partial y}v(x,y) = 0 \qquad$ Hence flow is INCOMPRESSIBLE

Hence $\quad u = \frac{\partial}{\partial y}\psi \quad$ so $\quad \psi(x,y) = \int u(x,y)\,dy + f(x) = 10\cdot x^2\cdot y^3 - 5\cdot x^4\cdot y + 2\cdot x\cdot y - y^5 + f(x)$

$v = -\frac{\partial}{\partial x}\psi \quad$ so $\quad \psi(x,y) = -\int v(x,y)\,dx + g(y) = 10\cdot x^2\cdot y^3 - 5\cdot x^4\cdot y + 2\cdot x\cdot y + g(y)$

Comparing, the simplest stream function is then $\quad \psi(x,y) = 10\cdot x^2\cdot y^3 - 5\cdot x^4\cdot y + 2\cdot x\cdot y - y^5$

Problem 6.111

[Difficulty: 2]

6.111 Consider the flow field presented by the potential function $\phi = x^6 - 15x^4y^2 + 15x^2y^4 - y^6$. Verify that this is an incompressible flow and obtain the corresponding stream function.

Given: Velocity potential

Find: Show flow is incompressible; Stream function

Solution:

Basic equations: Irrotationality because φ exists $\quad u = \frac{\partial}{\partial y}\psi \quad v = -\frac{\partial}{\partial x}\psi \quad u = -\frac{\partial}{\partial x}\varphi \quad v = -\frac{\partial}{\partial y}\varphi$

Incompressibility $\quad \frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$

$$\varphi(x,y) = x^6 - 15\cdot x^4\cdot y^2 + 15\cdot x^2\cdot y^4 - y^6$$

$$u(x,y) = -\frac{\partial}{\partial x}\varphi(x,y) \qquad u(x,y) = 60\cdot x^3\cdot y^2 - 6\cdot x^5 - 30\cdot x\cdot y^4$$

$$v(x,y) = -\frac{\partial}{\partial y}\varphi(x,y) \qquad v(x,y) = 30\cdot x^4\cdot y - 60\cdot x^2\cdot y^3 + 6\cdot y^5$$

Hence $\quad \frac{\partial}{\partial x}u(x,y) + \frac{\partial}{\partial y}v(x,y) = 0 \quad$ Hence flow is INCOMPRESSIBLE

Hence $\quad u = \frac{\partial}{\partial y}\psi \quad$ so $\quad \psi(x,y) = \int u(x,y)\,dy + f(x) = 20\cdot x^3\cdot y^3 - 6\cdot x^5\cdot y - 6\cdot x\cdot y^5 + f(x)$

$v = -\frac{\partial}{\partial x}\psi \quad$ so $\quad \psi(x,y) = -\int v(x,y)\,dx + g(y) = 20\cdot x^3\cdot y^3 - 6\cdot x^5\cdot y - 6\cdot x\cdot y^5 + g(y)$

Comparing, the simplest stream function is then $\quad \psi(x,y) = 20\cdot x^3\cdot y^3 - 6\cdot x^5\cdot y - 6\cdot x\cdot y^5$

Problem 6.112

[Difficulty: 4]

6.112 Show by expanding and collecting real and imaginary terms that $f = z^6$ (where z is the complex number $z = x + iy$) leads to a valid velocity potential (the real part of f) and a corresponding stream function (the negative of the imaginary part of f) of an irrotational and incompressible flow. Then show that the real and imaginary parts of df/dz yield $-u$ and v, respectively.

Given: Complex function

Find: Show it leads to velocity potential and stream function of irrotational incompressible flow; Show that df/dz leads to u and v

Solution:

Basic equations: Irrotationality because φ exists $\quad u = \frac{\partial}{\partial y}\psi \quad v = -\frac{\partial}{\partial x}\psi \quad u = -\frac{\partial}{\partial x}\varphi \quad v = -\frac{\partial}{\partial y}\varphi$

Incompressibility $\quad \frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0 \quad$ Irrotationality $\quad \frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u = 0$

$$f(z) = z^6 = (x + i\cdot y)^6 \qquad \text{Expanding} \qquad f(z) = x^6 - 15\cdot x^4\cdot y^2 + 15\cdot x^2\cdot y^4 - y^6 + i\cdot\left(6\cdot x\cdot y^5 + 6\cdot x^5\cdot y - 20\cdot x^3\cdot y^3\right)$$

We are thus to check the following

$$\varphi(x,y) = x^6 - 15\cdot x^4\cdot y^2 + 15\cdot x^2\cdot y^4 - y^6 \qquad \psi(x,y) = -\left(6\cdot x\cdot y^5 + 6\cdot x^5\cdot y - 20\cdot x^3\cdot y^3\right)$$

$$u(x,y) = -\frac{\partial}{\partial x}\varphi(x,y) \quad \text{so} \quad u(x,y) = 60\cdot x^3\cdot y^2 - 6\cdot x^5 - 30\cdot x\cdot y^4$$

$$v(x,y) = -\frac{\partial}{\partial y}\varphi(x,y) \quad \text{so} \quad v(x,y) = 30\cdot x^4\cdot y - 60\cdot x^2\cdot y^3 + 6\cdot y^5$$

An alternative derivation of u and v is

$$u(x,y) = \frac{\partial}{\partial y}\psi(x,y) \qquad u(x,y) = 60\cdot x^3\cdot y^2 - 6\cdot x^5 - 30\cdot x\cdot y^4$$

$$v(x,y) = -\frac{\partial}{\partial x}\psi(x,y) \qquad v(x,y) = 30\cdot x^4\cdot y - 60\cdot x^2\cdot y^3 + 6\cdot y^5$$

Hence $\quad \frac{\partial}{\partial x}v(x,y) - \frac{\partial}{\partial y}u(x,y) = 0 \quad$ Hence flow is IRROTATIONAL

Hence $\quad \frac{\partial}{\partial x}u(x,y) + \frac{\partial}{\partial y}v(x,y) = 0 \quad$ Hence flow is INCOMPRESSIBLE

Next we find $\quad \frac{df}{dz} = \frac{d\left(z^6\right)}{dz} = 6\cdot z^5 = 6\cdot(x + i\cdot y)^5 = \left(6\cdot x^5 - 60\cdot x^3\cdot y^2 + 30\cdot x\cdot y^4\right) + i\cdot\left(30\cdot x^4\cdot y + 6\cdot y^5 - 60\cdot x^2\cdot y^3\right)$

Hence we see $\quad \frac{df}{dz} = -u + i\cdot v \quad$ Hence the results are verified; $\quad u = -\mathrm{Re}\left(\frac{df}{dz}\right) \quad$ and $\quad v = \mathrm{Im}\left(\frac{df}{dz}\right)$

These interesting results are explained in Problem 6.113!

Problem 6.113

[Difficulty: 4]

6.113 Show that *any* differentiable function $f(z)$ of the complex number $z = x + iy$ leads to a valid potential (the real part of f) and a corresponding stream function (the negative of the imaginary part of f) of an incompressible, irrotational flow. To do so, prove using the chain rule that $f(z)$ automatically satisfies the Laplace equation. Then show that $df/dz = -u + iv$.

Given: Complex function

Find: Show it leads to velocity potential and stream function of irrotational incompressible flow; Show that df/dz leads to u and v

Solution:

Basic equations: $u = \frac{\partial}{\partial y}\psi \qquad v = -\frac{\partial}{\partial x}\psi \qquad u = -\frac{\partial}{\partial x}\varphi \qquad v = -\frac{\partial}{\partial y}\varphi$

First consider $\frac{\partial}{\partial x}f = \frac{\partial}{\partial x}z\cdot\frac{d}{dz}f = 1\cdot\frac{d}{dz}f = \frac{d}{dz}f \qquad (1)$ and also $\frac{\partial}{\partial y}f = \frac{\partial}{\partial y}z\cdot\frac{d}{dz}f = i\cdot\frac{d}{dz}f = i\cdot\frac{d}{dz}f \qquad (2)$

Hence $\frac{\partial^2}{\partial x^2}f = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}f\right) = \frac{d}{dz}\left(\frac{d}{dz}f\right) = \frac{d^2}{dz^2}f$ and $\frac{\partial^2}{\partial y^2}f = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}f\right) = i\cdot\frac{d}{dz}\left(i\cdot\frac{d}{dz}f\right) = -\frac{d^2}{dz^2}f$

Combining $\frac{\partial^2}{\partial x^2}f + \frac{\partial^2}{\partial y^2}f = \frac{d^2}{dz^2}f - \frac{d^2}{dz^2}f = 0$ Any differentiable function f(z) automatically satisfies the Laplace Equation; so do its real and imaginary parts!

We demonstrate derivation of velocities u and v

From Eq 1 $\frac{d}{dz}f = \frac{\partial}{\partial x}f = \frac{\partial}{\partial x}(\varphi - i\cdot\psi) = \frac{\partial}{\partial x}\varphi - i\cdot\frac{\partial}{\partial x}\psi = -u + i\cdot v$

From Eq 2 $\frac{d}{dz}f = \frac{1}{i}\cdot\frac{\partial}{\partial y}f = \frac{1}{i}\cdot\frac{\partial}{\partial y}(\varphi - i\cdot\psi) = -i\cdot\frac{\partial}{\partial y}\varphi - \frac{\partial}{\partial y}\psi = i\cdot v - u$

Hence we have demonstrated that $\frac{df}{dz} = -u + i\cdot v$

6.114 Consider the flow field represented by the velocity potential $\phi = Ax + Bx^2 - By^2$, where $A = 1 \text{ m}\cdot\text{s}^{-1}$, $B = 1 \text{ m}^{-1}\cdot\text{s}^{-1}$, and the coordinates are measured in meters. Obtain expressions for the velocity field and the stream function. Calculate the pressure difference between the origin and point $(x, y) = (1, 2)$.

Solution

The velocity field is determined from the velocity potential

$$\left.\begin{aligned} u &= -\partial\phi/\partial x = -A - 2Bx \\ v &= -\partial\phi/\partial y = 2By \end{aligned}\right\} \quad \vec{V} = -(A+2Bx)\hat{i} + 2By\hat{j} \qquad \vec{V}$$

From the definition of the stream function, $u = \frac{\partial\psi}{\partial y}$ & $v = -\frac{\partial\psi}{\partial x}$

Then

$$\psi = \int u\,dy + f(x) = \int -(A+2Bx)\,dy + f(x)$$

$$\psi = -Ay - 2Bxy + f(x)$$

Also,

$$\psi = \int -v\,dx + g(y) = \int -2By\,dx + g(y)$$

$$\psi = -2Bxy + g(y)$$

Comparing the two expressions for ψ we conclude

$$f(x) = 0 \quad , \quad g(y) = -Ay$$

$$\therefore \quad \psi = -(Ay + 2Bxy) \qquad \psi$$

Since $\nabla^2\phi = 2B - 2B = 0$, the flow is irrotational and the Bernoulli equation can be applied between any two points in the flow field

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + \cancel{gz_2} \qquad \left\{ \begin{aligned} &\text{Assume: } \rho = \text{constant} \\ &z_1 = z_2 \end{aligned} \right\}$$

$$\vec{V}(0,0) = -A\hat{i} = -\hat{i} \text{ m/s} \qquad V_{0,0} = 1 \text{ m/s}$$

$$\vec{V}(1,2) = -(A+2B)\hat{i} + 4B\hat{j} = -3\hat{i} + 4\hat{j} \text{ m/s} \qquad \therefore V_{1,2} = 5 \text{ m/s}$$

$$\therefore \; p_1 - p_2 = \rho\left(\frac{V_2^2}{2} - \frac{V_1^2}{2}\right) = \frac{\rho}{2}(V_2^2 - V_1^2)$$

Assume fluid is water

$$p_1 - p_2 = \frac{1}{2} \times 999 \frac{\text{kg}}{\text{m}^3}\,(25 - 1)\frac{\text{m}^2}{\text{s}^2} \times \frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}} = 12 \text{ kN/m}^2$$

Problem 6.115

6.115 A flow field is represented by the potential function $\phi = Ay^3 - Bx^2y$, where $A = 1/3\,\mathrm{m^{-1}\cdot s^{-1}}$, $B = 1\,\mathrm{m^{-1}\cdot s^{-1}}$, and the coordinates are measured in meters. Obtain an expression for the magnitude of the velocity vector. Find the stream function for the flow. Plot the streamlines and potential lines, and visually verify that they are orthogonal. (*Hint*: Use the *Excel* workbook of Example 6.10.)

Solution:

The velocity field is determined from the velocity potential

$$\left.\begin{aligned} u &= -\partial\phi/\partial x = 2Bxy = 2xy \\ v &= -\partial\phi/\partial y = -3Ay^2 + Bx^2 = x^2 - y^2 \end{aligned}\right\} \vec{V} = 2xy\hat{i} + (x^2 - y^2)\hat{j}$$

$$V = [u^2 + v^2]^{1/2} = [4x^2y^2 + (x^2-y^2)^2]^{1/2} = [4x^2y^2 + x^4 - 2x^2y^2 + y^4]^{1/2}$$

$$V = [x^4 + 2x^2y^2 + y^4]^{1/2} = [(x^2+y^2)^2]^{1/2} = x^2 + y^2 \longleftarrow V$$

The stream function is defined such that $u = \frac{\partial\psi}{\partial y}$ and $v = -\frac{\partial\psi}{\partial x}$

Then, $\psi = \int u\,dy + f(x) = \int 2Bxy\,dy + f(x) = Bxy^2 + f(x)$ ----- (1)

Also, $\psi = \int -v\,dx + g(y) = \int (3Ay^2 - Bx^2)\,dx + g(y) = 3Axy^2 - \frac{B}{3}x^3 + g(y)$ --- (2)

Comparing the two expressions for ψ, we

- note that $Bxy^2 = 3Axy^2$ $(B = 1,\ A = \frac{1}{3})$, and
- conclude that $f(x) = -\frac{B}{3}x^3$ and $g(y) = 0$

$\therefore\ \psi = Bxy^2 - \frac{B}{3}x^3$ or $\psi = 3Axy^2 - \frac{B}{3}x^3 \longleftarrow \psi$

with $A = \frac{1}{3}$, $B = 1$

$$\psi = xy^2 - \frac{x^3}{3} = x\left(y^2 - \frac{x^2}{3}\right)$$

For $\psi = 0$, $x = 0$ or $y = 0.577x$

For $\psi = -4$, $y^2 = \frac{x^2}{3} - \frac{4}{x}$

For $\psi = 4$, $y^2 = \frac{x^2}{3} + \frac{4}{x}$

See the next page for plots

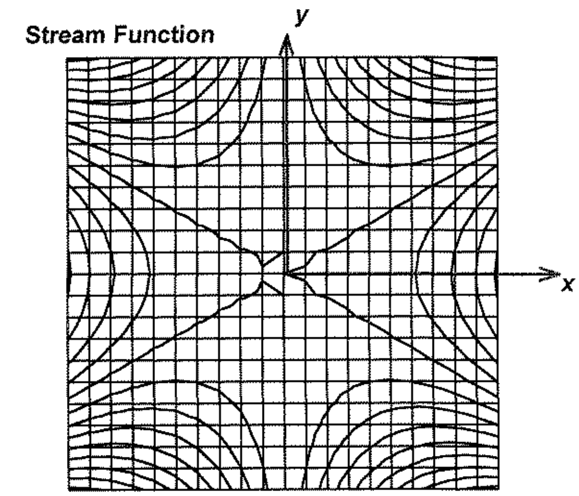
Problem 6.115

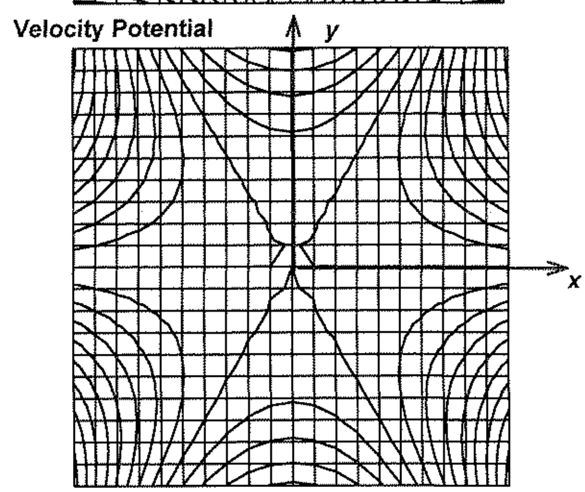
[Difficulty: 2] Part 2/2

Using *Excel*, the stream function and velocity potential can be plotted.
The data below was obtained using the workbook for Example Problem 6.10.
Note the orthogonality of ψ and ϕ!

#NAME?	**Stream Function**
#NAME?	**Velocity Potential**

Note that the plot is
from $x = -5$ to 5 and $y = -5$ to 5





Problem 6.116

[Difficulty: 3]

6.116 An incompressible flow field is characterized by the stream function $\psi = 3Ax^2y - Ay^3$, where $A = 1\ \text{m}^{-1}\cdot\text{s}^{-1}$. Show that this flow field is irrotational. Derive the velocity potential for the flow. Plot the streamlines and potential lines, and visually verify that they are orthogonal. (*Hint*: Use the *Excel* workbook of Example 6.10.)

Solution:

For a 2-D incompressible, irrotational flow $\nabla^2\psi = 0$ (6.30)

For the flow field.

$$\nabla^2\psi = \frac{\partial^2}{\partial x^2}(3Ax^2y - Ay^3) + \frac{\partial^2}{\partial y^2}(3Ax^2y - Ay^3) = 6Ay - 6Ay = 0 \quad \text{irrotational}$$

The velocity field is given by $\vec{V} = u\hat{i} + v\hat{j}$

$$u = \partial\psi/\partial y = 3Ax^2 - 3Ay^2 = 3A(x^2 - y^2)$$
$$v = -\partial\psi/\partial x = -6Axy$$
$$\vec{V} = 3A(x^2-y^2)\hat{i} - 6Axy\hat{j}$$

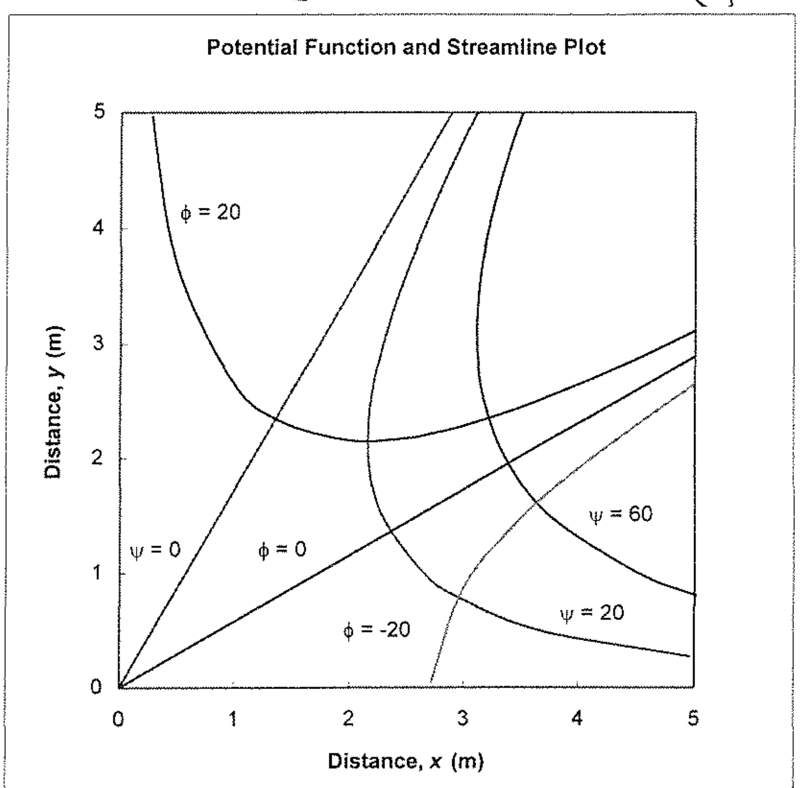
The velocity potential is defined such that $u = -\frac{\partial\phi}{\partial x}$ & $v = -\frac{\partial\phi}{\partial y}$

Then,
$$\phi = -\int u\,dx + f(y) = -\int 3A(x^2-y^2)\,dx + f(y) = -Ax^3 + 3Axy^2 + f(y) \quad (1)$$

Also,
$$\phi = -\int v\,dy + g(x) = \int 6Axy\,dy + g(x) = 3Axy^2 + g(x) \quad (2)$$

Equating expressions for ϕ (Eqs 1 and 2) we see that

$g(x) = -Ax^3$ and $f(y) = 0$ $\quad\therefore\ \phi = 3Axy^2 - Ax^3 \quad \phi$



Problem 6.117

[Difficulty: 3]

6.117 A certain irrotational flow field in the xy plane has the stream function $\psi = Bxy$, where $B = 0.25\ \text{s}^{-1}$, and the coordinates are measured in meters. Determine the rate of flow between points $(x, y) = (2, 2)$ and $(3, 3)$. Find the velocity potential for this flow. Plot the streamlines and potential lines, and visually verify that they are orthogonal. (*Hint*: Use the *Excel* workbook of Example 6.10.)

Solution:

The volume flow rate (per unit depth) between points ① and ② is given by

$$Q_{12} = \psi_2 - \psi_1 = B[x_2 y_2 - x_1 y_1] = 0.25\,\text{s}^{-1}[3\text{m}\times 3\text{m} - 2\text{m}\times 2\text{m}]$$

$$Q_{12} = 1.25\ \text{m}^3/\text{s}/\text{m}$$ Q_{12}

The velocity field is determined from the stream function

$u = \partial\psi/\partial y = Bx \qquad v = -\partial\psi/\partial x = -By \qquad \therefore \vec{V} = Bx\hat{i} - By\hat{j}$

For irrotational flow $\vec{V} = -\nabla\phi$ and $u = -\partial\phi/\partial x$, $v = -\partial\phi/\partial y$

and

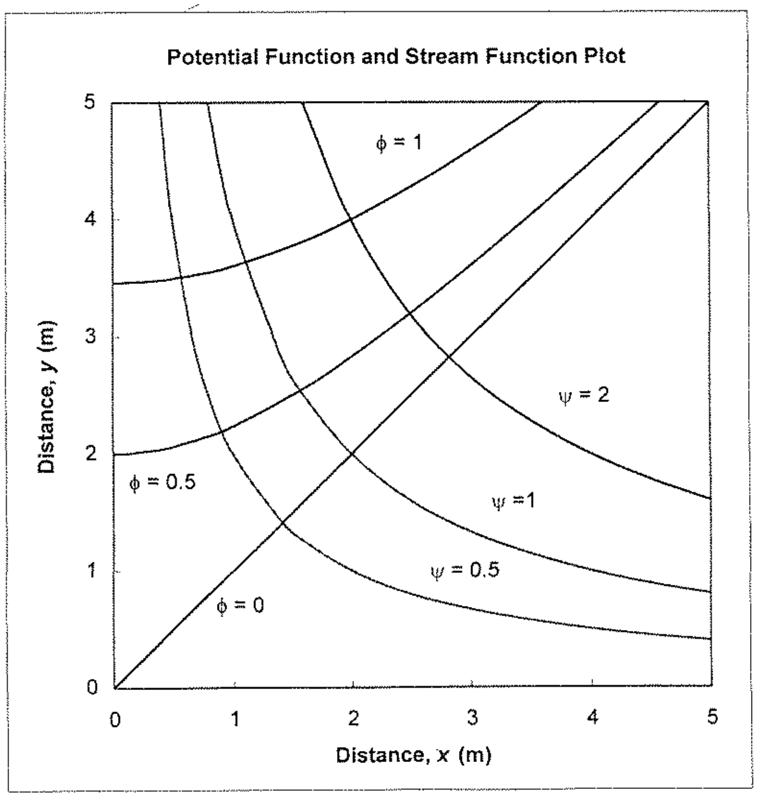
$$\phi = -\int u\,dx + f(y) = -\int Bx\,dx + f(y) = -\frac{B}{2}x^2 + f(y) \quad \ldots(1)$$

Also

$$\phi = -\int v\,dy + g(x) = \int By\,dy + g(x) = \frac{B}{2}y^2 + g(x) \quad \ldots(2)$$

Equating expressions for ϕ (Eqs 1 and 2) we conclude that

$$f(y) = \frac{B}{2}y^2, \quad g(x) = -\frac{B}{2}x^2 \quad \text{and} \quad \phi = \frac{B}{2}(y^2 - x^2)$$ ϕ



6.118 The velocity distribution in a two-dimensional, steady, inviscid flow field in the xy plane is $\vec{V} = (Ax + B)\hat{i} + (C - Ay)\hat{j}$, where $A = 3\ s^{-1}$, $B = 6$ m/s, $C = 4$ m/s, and the coordinates are measured in meters. The body force distribution is $\vec{B} = -g\hat{k}$ and the density is 825 kg/m^3. Does this represent a possible incompressible flow field? Plot a few streamlines in the upper half plane. Find the stagnation point(s) of the flow field. Is the flow irrotational? If so, obtain the potential function. Evaluate the pressure difference between the origin and point $(x, y, z) = (2, 2, 2)$.

Solution:

For incompressible flow $\nabla \cdot \vec{V} = 0$. For this flow

$$\nabla \cdot \vec{V} = \frac{\partial}{\partial x}(Ax+B) + \frac{\partial}{\partial y}(C-Ay) = A - A = 0$$

$\therefore$ velocity field represents possible incompressible flow.

At the stagnation point $u = v = 0$. $(\vec{V} = 0)$

$$u = 0 = (Ax+B) \quad \therefore x = -B/A = \frac{-6\ m \cdot s^{-1}}{3\ s^{-1}} = -2\ m$$

$$v = 0 = (C - Ay) \quad \therefore y = C/A = \frac{4\ m \cdot s^{-1}}{3\ s^{-1}} = 4/3\ m$$

Stagnation point is at $(x,y) = (-2, 4/3)$ m.

The fluid rotation (for a 2-D flow) is given by $\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$

For this flow $\omega_z = \frac{1}{2}\left[\frac{\partial (C-Ay)}{\partial x} - \frac{\partial (Ax+B)}{\partial y}\right] = 0$

$\therefore$ flow is irrotational

Then, $\vec{V} = -\nabla\phi$ and $u = -\partial\phi/\partial x$ and $v = -\partial\phi/\partial y$.

and $\phi = \int -u\,dx + f(y) = -\int (Ax+B)\,dx + f(y) = -A\frac{x^2}{2} - Bx + f(y)$ ...(1)

Also $\phi = -\int v\,dy + g(x) = -\int (C-Ay)\,dy + g(x) = A\frac{y^2}{2} - Cy + g(x)$ ....(2)

Equating the two expressions for ϕ (Eqs. 1 and 2) we note that

$$g(x) = -\left(A\frac{x^2}{2} + Bx\right) \quad \text{and} \quad f(y) = A\frac{y^2}{2} - Cy.$$

$$\therefore \phi = \frac{A}{2}(y^2 - x^2) - Bx - Cy \qquad \phi$$

Since the flow is irrotational we can apply the Bernoulli equation between any two points in the flow field.

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2$$

At point 1 $(0,0,0)$, $\vec{V} = B\hat{i} + C\hat{j} = 6\hat{i} + 4\hat{j}$ m/s, $V_1^2 = 52\ m^2/s^2$

At point 2 $(2,2,2)$ $\vec{V}_2 = [3s^{-1} \times 2m + 6m/s]\hat{i} + [4m/s - 3s^{-1} \times 2m]\hat{j}$

$\vec{V}_2 = 12\hat{i} - 2\hat{j}\ m/s$, $V_2^2 = 148\ m^2/s^2$

$$p_1 - p_2 = \frac{\rho}{2}(V_2^2 - V_1^2) + \rho g(z_2 - z_1) = \rho\left[\frac{(V_2^2 - V_1^2)}{2} + g(z_2 - z_1)\right]$$

$$= 825\ \frac{kg}{m^3} \times \left[\frac{1}{2} \times (148 - 52)\frac{m^2}{s^2} + 9.81\frac{m}{s^2} \times (2m)\right] \times \frac{N \cdot s^2}{kg \cdot m}$$

$p_1 - p_2 = 55.8\ kPa$ ΔP

The stream function is defined such that $u = \frac{\partial\psi}{\partial y}$, $v = -\frac{\partial\psi}{\partial x}$

Then, $\psi = \int u\,dy + f(x) = \int (Ax + B)\,dy + f(x) = Axy + By + f(x)$ ----(1)

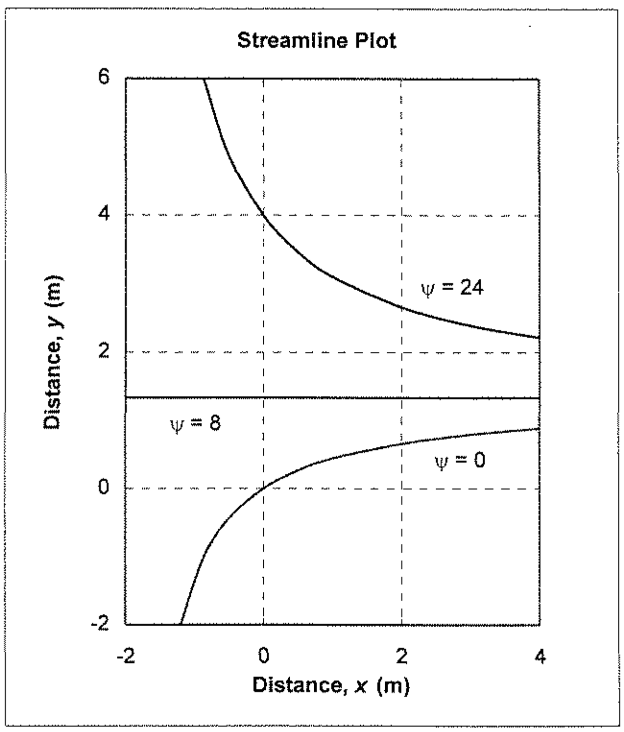
Also, $\psi = -\int v\,dx + g(y) = \int(-C + Ay)\,dx + g(y) = -Cx + Axy + g(y)$ ----(2)

Equating the two expressions for ψ (Eqs 1 and 2) we note that

$f(x) = -Cx$, $g(y) = By$ and $\therefore\ \psi = Axy + By - Cx$ ψ

The stagnation streamline goes through the stagnation point $(-2, \frac{4}{3})$

$\psi_{stag} = 3s^{-1} \times (-2m) \times \frac{4}{3}m + 6m \cdot s^{-1} \times \frac{4}{3}m - 4m \cdot s^{-1} \times (-2m) = 8\ m^2/s$ ψ_{stag}



Problem 6.119

[Difficulty: 3]

6.119 Consider flow around a circular cylinder with freestream velocity from right to left and a counterclockwise free vortex. Show that the lift force on the cylinder can be expressed as $F_L = -\rho U \Gamma$, as illustrated in Example 6.12.

Open-Ended Problem Statement: Consider flow around a circular cylinder with freestream velocity from right to left and a counterclockwise free vortex. Show that the lift force on the cylinder can be expressed as $F_L = -\rho U \Gamma$, as illustrated in Example 6.12.

Discussion: The only change in this flow from the flow of Example 6.12 is that the directions of the freestream velocity and the vortex are changed. This changes the sign of the freestream velocity from U to $-U$ and the sign of the vortex strength from K to $-K$. Consequently the signs of both terms in the equation for lift are changed. Therefore the direction of the lift force remains unchanged.

The analysis of Example 6.12 shows that only the term involving the vortex strength contributes to the lift force. Therefore the expression for lift obtained with the changed freestream velocity and vortex strength is identical to that derived in Example 6.12. Thus the general solution of Example 6.12 holds for any orientation of the freestream and vortex velocities. For the present case, $F_L = -\rho U \Gamma$, as shown for the general case in Example 6.12.

Problem 6.120

6.120 Consider the flow past a circular cylinder, of radius a, used in Example 6.11. Show that $V_r = 0$ along the lines $(r, \theta) = (r, \pm\pi/2)$. Plot V_θ/U versus radius for $r \geq a$, along the line $(r, \theta) = (r, \pi/2)$. Find the distance beyond which the influence of the cylinder is less than 1 percent of U.

Solution:

From Example Problem 6.11

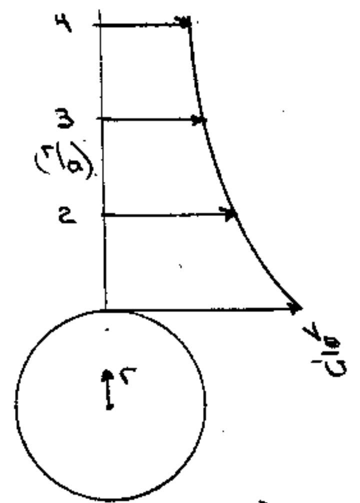
$$\vec{V} = \left(-\frac{\Lambda\cos\theta}{r^2} + U\cos\theta\right)\hat{i}_r + \left(-\frac{\Lambda\sin\theta}{r^2} - U\sin\theta\right)\hat{i}_\theta \quad \ldots\ldots (1)$$

then $V_r = \left(-\frac{\Lambda}{r^2} + U\right)\cos\theta$. For $\theta = \pm\frac{\pi}{2}$, $\cos\theta = 0$ and $V_r = 0$

$V_\theta = -\left(\frac{\Lambda}{r^2} + U\right)\sin\theta$, but $\frac{\Lambda}{U} = a^2$

$\therefore V_\theta = -\left(\frac{a^2}{r^2} + 1\right)U\sin\theta$ For $\theta = \pi/2$.

$$\frac{V_\theta}{U} = -\left(1 + \frac{a^2}{r^2}\right)$$



$$\vec{V} = U\cos\theta\left(1 - \frac{a^2}{r^2}\right)\hat{i} - U\sin\theta\left(1 + \frac{a^2}{r^2}\right)\hat{j}$$

For $\theta = \pi/2$

$\frac{V}{U} = 1 + \frac{a^2}{r^2}$ If $\frac{V}{U} = 1.01$ then $\frac{a^2}{r^2} = 0.01$ or $\frac{a}{r} = 0.1$

$\therefore \frac{V}{U} < 1\%$ for $r > 10a$

Problem 6.121

[Difficulty: 3]

6.121 A crude model of a tornado is formed by combining a sink, of strength $q = 2800\ \text{m}^2/\text{s}$, and a free vortex, of strength $K = 5600\ \text{m}^2/\text{s}$. Obtain the stream function and velocity potential for this flow field. Estimate the radius beyond which the flow may be treated as incompressible. Find the gage pressure at that radius.

Solution:

$$\psi = \psi_{si} + \psi_{vo} = -\frac{q}{2\pi}\theta - \frac{K}{2\pi}\ln r \qquad \psi$$

$$\phi = \phi_{si} + \phi_{vo} = \frac{q}{2\pi}\ln r - \frac{K}{2\pi}\theta \qquad \phi$$

$$\vec{V} = V_r\hat{e}_r + V_\theta\hat{e}_\theta \qquad V_{r\,si} = -\frac{q}{2\pi r},\ V_{r\,vo} = 0;\ V_{\theta\,si} = 0,\ V_{\theta\,vo} = \frac{K}{2\pi r}$$

$$\therefore \vec{V} = -\frac{q}{2\pi r}\hat{e}_r + \frac{K}{2\pi r}\hat{e}_\theta$$

Then

$$V = (V_r^2 + V_\theta^2)^{1/2} = \left[\left(-\frac{q}{2\pi}\right)^2 + \left(\frac{K}{2\pi}\right)^2\right]^{1/2}\frac{1}{r}$$

For incompressible flow $M \leq 0.3$. For standard air this corresponds to $V < 102\ \text{m/sec}$

Then, for incompressible flow

$$V = 102\ \text{m/sec} < \left[q^2 + K^2\right]^{1/2}\frac{1}{2\pi r}$$

or

$$r > \left[q^2 + K^2\right]^{1/2}\frac{1}{2\pi}\times\frac{s}{102\,m} = \left[(2800)^2 + (5600)^2\right]^{1/2}\frac{m^2}{s^2}\times\frac{1}{2\pi}\times\frac{s}{102\,m}$$

$$r > 9.77\ m \qquad r$$

To determine the gage pressure at this radius, apply the Bernoulli equation for irrotational flow

$$\frac{p_\infty}{\rho} + \cancel{\frac{V_\infty^2}{2}}^{\approx 0} + \cancel{g z_\infty} = \frac{p}{\rho} + \frac{V^2}{2} + \cancel{g z} \qquad \text{assume } \Delta z = 0$$

Then

$$p_{gage} = p - p_\infty = -\frac{1}{2}\rho V^2 = -\frac{1}{2}\times 1.225\frac{kg}{m^3}\times(102)^2\frac{m^2}{s^2}\times\frac{N\cdot s^2}{kg\cdot m}$$

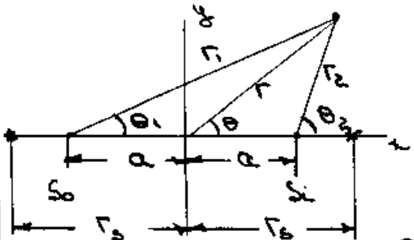
$$p_{gage} = -6.37\ kPa \quad (\text{for standard air}) \qquad p_{gage}$$

Problem 6.122

[Difficulty: 3]

6.122 A source and a sink with strengths of equal magnitude, $q = 3\pi$ m^2/s, are placed on the x axis at $x = -a$ and $x = a$, respectively. A uniform flow, with speed $U = 20$ m/s, in the positive x direction, is added to obtain the flow past a Rankine body. Obtain the stream function, velocity potential, and velocity field for the combined flow. Find the value of $\psi =$ constant on the stagnation streamline. Locate the stagnation points if $a = 0.3$ m.

Solution:



$$\psi = \psi_{so} + \psi_{si} + \psi_{uf} = \frac{q}{2\pi}\theta_1 - \frac{q}{2\pi}\theta_2 + Uy$$

$$\psi = \frac{q}{2\pi}(\theta_1 - \theta_2) + Ur\sin\theta \qquad \psi$$

$$\phi = \phi_{so} + \phi_{si} + \phi_{uf} = -\frac{q}{2\pi}\ln r_1 + \frac{q}{2\pi}\ln r_2 - Ux$$

$$\phi = \frac{q}{2\pi}\ln\frac{r_2}{r_1} - Ur\cos\theta \qquad \phi$$

$$u = u_{so} + u_{si} + u_{uf} = \frac{q}{2\pi r_1}\cos\theta_1 - \frac{q}{2\pi r_2}\cos\theta_2 + U$$

$$v = v_{so} + v_{si} + v_{uf} = \frac{q}{2\pi r_1}\sin\theta_1 - \frac{q}{2\pi r_2}\sin\theta_2$$

$$\vec{V} = u\hat{i} + v\hat{j} = \left\{\frac{q}{2\pi}\left(\frac{\cos\theta_1}{r_1} - \frac{\cos\theta_2}{r_2}\right) + U\right\}\hat{i} + \frac{q}{2\pi}\left(\frac{\sin\theta_1}{r_1} - \frac{\sin\theta_2}{r_2}\right)\hat{j} \qquad \vec{V}$$

At stagnation point $\vec{V} = 0$ $\quad y = 0 \quad \theta_1 = \theta_2 = 0$

$$r_2 = r_s - a, \quad r_1 = r_s + a$$

$$\therefore u = 0 = \frac{q}{2\pi}\left(\frac{1}{r_s + a} - \frac{1}{r_s - a}\right) + U = \frac{q}{2\pi}\left[\frac{(r_s - a) - (r_s + a)}{(r_s^2 - a^2)}\right] + U$$

$$0 = -\frac{qa}{\pi(r_s^2 - a^2)} + U \quad \text{or} \quad (r_s^2 - a^2) = \frac{qa}{\pi U}$$

$$r_s = \left(a^2 + \frac{qa}{\pi U}\right)^{1/2} = a\left(1 + \frac{q}{\pi U a}\right)^{1/2}$$

For $a = 0.3$ m

$$r = 0.3\,\text{m}\left[1 + \frac{3\pi}{\pi}\frac{\text{m}^2}{\text{s}} \cdot \frac{\text{s}}{20\,\text{m}} \cdot \frac{1}{0.3\,\text{m}}\right]^{1/2} = 0.367\,\text{m}$$

Stagnation points located at $\theta = 0, \pi$ $\quad r = 0.367$ m

Since $\psi = \frac{q}{2\pi}(\theta_1 - \theta_2) + Uy$ and $\theta_1 = \theta_2$ & $y = 0$ at stagnation

$$\psi_{stag} = 0$$

Problem 6.123

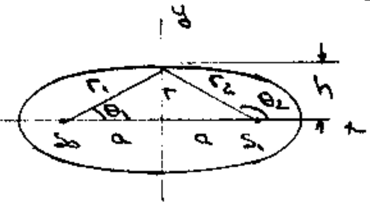
[Difficulty: 3]

6.123 Consider again the flow past a Rankine body of Problem 6.122. The half-width, h, of the body in the y direction is given by the transcendental equation

$$\frac{h}{a} = \cot\left(\frac{\pi U h}{q}\right)$$

Evaluate the half-width, h. Find the local velocity and the pressure at points $(x, y) = (0, \pm h)$. Assume the fluid density is that of standard air.

Solution:



$$\psi = \psi_{so} + \psi_{si} + \psi_{uf} = \frac{q}{2\pi}(\theta_1 - \theta_2) + Ur\sin\theta$$

At stagnation point $\theta_1 = \theta_2$ and $\theta = 0, \pi$.

$\therefore \psi_{stag} = 0$ and equation of stag. streamline is

$$0 = \frac{q}{2\pi}(\theta_1 - \theta_2) + Ur\sin\theta$$

or $$r = \frac{q}{2\pi}\frac{(\theta_2 - \theta_1)}{U\sin\theta}$$

At half width, $\theta = \frac{\pi}{2}$, $\theta_2 = \pi - \theta_1$, and $r = h = \frac{q}{2\pi}\frac{[(\pi - \theta_1) - \theta_1]}{U}$

$\therefore hU = \frac{q}{2\pi}[\pi - 2\theta_1] = \frac{q}{2} - \frac{q\theta_1}{\pi}$ or $\theta_1 = \frac{\pi}{2} - \frac{Uh\pi}{q}$

Since $h = a\tan\theta_1$

$$\frac{h}{a} = \tan\left(\frac{\pi}{2} - \frac{Uh\pi}{q}\right) = \cot\left(\frac{Uh\pi}{q}\right)$$

Substituting values, $\frac{h}{0.3} = \cot\left(\frac{20}{3}h\right)$. Trial and error solution gives

$h = 0.1615\ m$ ——— h

The velocity field is given by $\vec{V} = \hat{i}u + \hat{j}v$

$$\vec{V} = \left\{\frac{q}{2\pi}\left(\frac{\cos\theta_1}{r_1} - \frac{\cos\theta_2}{r_2}\right) + U\right\}\hat{i} + \frac{q}{2\pi}\left(\frac{\sin\theta_1}{r_1} - \frac{\sin\theta_2}{r_2}\right)\hat{j}$$

At $(0, h)$, $r_1 = r_2$, $\theta_2 = \pi - \theta_1$ $\therefore \sin\theta_2 = \sin\theta_1$, $\cos\theta_2 = -\cos\theta_1$

and $$\vec{V} = \left(\frac{q\cos\theta_1}{r_1} + U\right)\hat{i}$$

$\theta_1 = \tan^{-1}\frac{h}{a} = \tan^{-1}\frac{0.1615}{0.3} = 28.3°$ $\qquad r_1 = [a^2 + h^2]^{1/2} = [0.3^2 + 0.1615^2]^{1/2} = 0.341\ m$

$$\vec{V} = \left(\frac{q\cos\theta_1}{r_1} + U\right)\hat{i} = \left(3\pi\ \frac{m^2}{sec} \times \frac{\cos 28.3°}{0.341\ m} + 20\ \frac{m}{s}\right)\hat{i} = 44.3\hat{i}\ m/s$$ ——— V

To find the gage pressure apply the Bernoulli equation between the point at conditions at ∞

$$\frac{p_\infty}{\rho} + \frac{U^2}{2} = \frac{p}{\rho} + \frac{V^2}{2}$$

$$p_{gage} = p - p_\infty = \frac{1}{2}\rho(U^2 - V^2) = \frac{1}{2} \times 1.225\frac{kg}{m^3}\left[(20)^2 - (44.3)^2\right]\frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m}$$

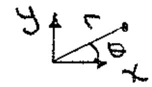
$p_{gag} = -957\ N/m^2$ ——— p_{gag}

Problem 6.124

[Difficulty: 3] Part 1/2

6.124 A flow field is formed by combining a uniform flow in the positive x direction, with $U = 10$ m/s, and a counterclockwise vortex, with strength $K = 16\pi$ m^2/s, located at the origin. Obtain the stream function, velocity potential, and velocity field for the combined flow. Locate the stagnation point(s) for the flow. Plot the streamlines and potential lines. (*Hint*: Use the *Excel* workbook of Example 6.10.)

Solution:



$$\psi = \psi_{uf} + \psi_v = Uy - \frac{K}{2\pi}\ln r = Ur\sin\theta - \frac{K}{2\pi}\ln r \qquad \psi$$

$$\phi = \phi_{uf} + \phi_v = -Ux - \frac{K}{2\pi}\theta = -Ur\cos\theta - \frac{K}{2\pi}\theta \qquad \phi$$

$$V_r = -\frac{\partial\phi}{\partial r} = U\cos\theta \;, \quad V_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = -U\sin\theta + \frac{K}{2\pi r}$$

$$\vec{V} = U\cos\theta\,\hat{e}_r + \left(\frac{K}{2\pi r} - U\sin\theta\right)\hat{e}_\theta \qquad \vec{V}$$

At stagnation point, $\vec{V} = 0$

$V_r = 0$ at $\theta = \pm\frac{\pi}{2}$; $V_\theta = 0$ on $r = \frac{K}{2\pi U\sin\theta}$

$\therefore\ \vec{V} = 0$ at $(r,\theta) = \frac{K}{2\pi U}, \pi/2$ Stagnation

See the next page for plots

Problem 6.124

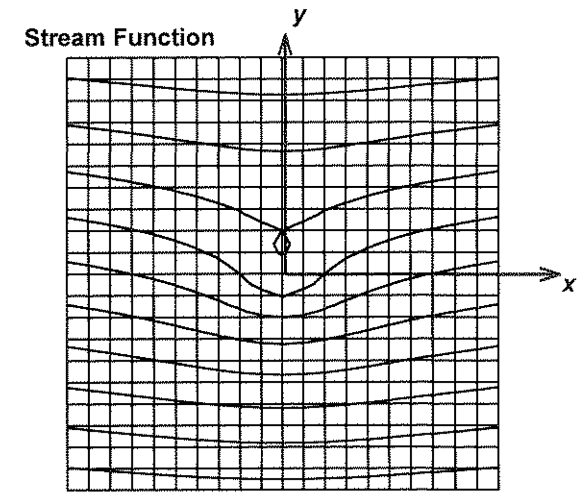
[Difficulty: 3] Part 2/2

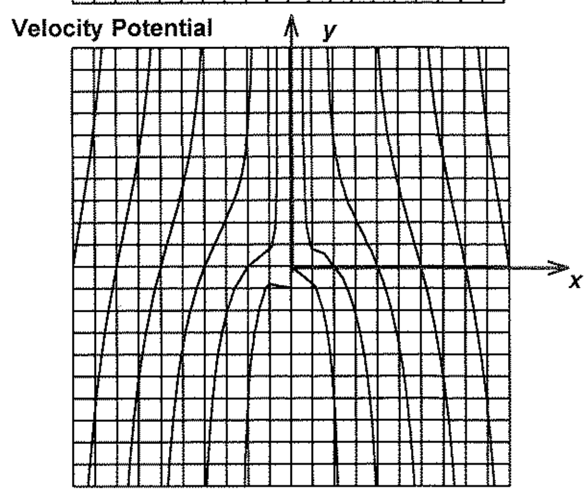
Using *Excel*, the stream function and velocity potential can be plotted.
The data below was obtained using the workbook for Example Problem 6.10.
Note the orthogonality of ψ and ϕ!

#NAME? **Stream Function**

#NAME? **Velocity Potential**

Note that the plot is
from x = -5 to 5 and y = -5 to 5



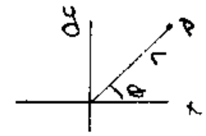


Problem 6.125

[Difficulty: 4]

6.125 Consider the flow field formed by combining a uniform flow in the positive x direction with a sink located at the origin. Let $U = 50$ m/s and $q = 90$ m²/s. Use a suitably chosen control volume to evaluate the net force per unit depth needed to hold in place (in standard air) the surface shape formed by the stagnation streamline.

Solution:



$$\psi = \psi_{uf} + \psi_{si} = Uy - \frac{q}{2\pi}\theta = Ur\sin\theta - \frac{q}{2\pi}\theta \quad \text{----(1)}$$

$u = u_{uf} + u_{si}$; $u_{uf} = U$, $u_{si} = -V_r\cos\theta = -\frac{q}{2\pi r}\frac{x}{r}$ $\quad\therefore u = U - \frac{q}{2\pi}\frac{x}{r^2}$

$v = v_{uf} + v_{si}$; $v_{uf} = 0$, $v_{si} = -V_r\sin\theta = -\frac{q}{2\pi r}\frac{y}{r}$ $\quad\therefore v = -\frac{q}{2\pi}\frac{y}{r^2}$

$$\therefore \vec{V} = u\hat{i} + v\hat{j} = \left(U - \frac{q}{2\pi}\frac{x}{r^2}\right)\hat{i} - \frac{q}{2\pi}\frac{y}{r^2}\hat{j}$$

At the stagnation point , $\vec{V} = 0$

$\therefore -\frac{q}{2\pi}\frac{y}{r^2} = 0$, i.e. $y = 0$. Also $U - \frac{q}{2\pi}\frac{x}{r^2} = 0$ $\quad\therefore U = \frac{q}{2\pi}\frac{x}{(x^2+y^2)}$

and $x_{stag} = \frac{q}{2\pi U} = \frac{90\,m^2}{s} \times \frac{1}{2\pi} \times \frac{s}{50\,m} = 0.286\,m$

At stagnation point, $y = 0$ and $\theta = 0$. From eq(1), then $\psi_{stag} = 0$

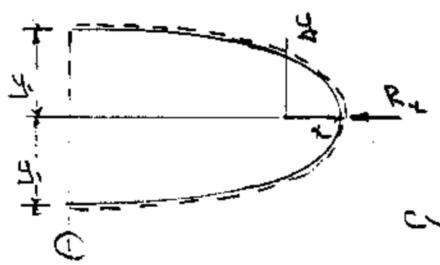
The equation of the stagnation streamline is then,

$\psi = 0 = Ur\sin\theta - \frac{q}{2\pi}\theta$ or $r_{stag} = \frac{q\theta}{2\pi U\sin\theta}$

Since $y = r\sin\theta$, then along the stagnation streamline $y = \frac{q\theta}{2\pi U}$.

For upstream , $\theta \to \pi$ and $y = y_1 \to \frac{q}{2U}$.

The surface shape formed by the stagnation streamline is then as follows:



There is no flow across this streamline. The flow in through the left face must be equal to the flow (q) which leaves through the sink at the origin.

Applying the x momentum equation to the CV shown. R_x is force required to hold shape in place

$$-R_x = \int u\,\rho\vec{V}\cdot d\vec{A} = -U\dot{m}_1 = -U\rho q b$$

$$\therefore \frac{R_x}{b} = \rho q U$$

For standard air $\rho = 1.225$ kg/m³ and

$$\frac{R_x}{b} = 1.225\,\frac{kg}{m^3} \times \frac{90\,m^2}{s} \times \frac{50\,m}{s} \times \frac{N\cdot s^2}{kg\cdot m} = 5.51\,kN/m$$

$$\vec{R}_x/b = -5.51\,\hat{i}\ kN/m \qquad R_x/b$$

6.126 Consider the flow field formed by combining a uniform flow in the positive x direction and a source located at the origin. Obtain expressions for the stream function, velocity potential, and velocity field for the combined flow. If $U = 25$ m/s, determine the source strength if the stagnation point is located at $x = -1$ m. Plot the streamlines and potential lines. (*Hint*: Use the *Excel* workbook of Example 6.10.)

Solution:

$$\psi = \psi_{uf} + \psi_{so} = Uy + \frac{q}{2\pi}\theta = Ur\sin\theta + \frac{q}{2\pi}\theta \qquad \psi$$

$$\phi = \phi_{uf} + \phi_{so} = -Ux - \frac{q}{2\pi}\ln r = -Ur\cos\theta - \frac{q}{2\pi}\ln r \qquad \phi$$

$$u = u_{uf} + u_{so}\,;\; u_{uf} = U\,;\; u_{so} = V_r\cos\theta = \frac{q}{2\pi r}\frac{x}{r} \qquad \therefore u = U + \frac{q}{2\pi}\frac{x}{r^2}$$

$$v = v_{uf} + v_{so}\,;\; v_{uf} = 0\,;\; v_{so} = V_r\sin\theta = \frac{q}{2\pi r}\frac{y}{r} \qquad \therefore v = \frac{q}{2\pi}\frac{y}{r^2}$$

$$\vec{V} = u\hat{i} + v\hat{j} = \left\{U + \frac{q}{2\pi}\frac{x}{(x^2+y^2)}\right\}\hat{i} + \frac{q}{2\pi}\frac{y}{(x^2+y^2)}\hat{j} \qquad \vec{V}$$

At the stagnation point $\vec{V} = 0$ $\quad x = -1.0\text{ m} \quad y = 0\ (v = 0)$.

For $u = 0 = U + \frac{q}{2\pi}\frac{x}{(x^2+y^2)}$ $\quad \therefore q = -2\pi U x_{stag}$

$$q = -2\pi \times 25\frac{\text{m}}{\text{s}} \times (-1.0\text{ m}) = 50\pi\ \text{m}^2/\text{s} \qquad q$$

At the stagnation point, $\theta = \pi$ $\quad \therefore \psi_{stag} = \frac{q}{2\pi}\theta = \frac{q}{2}$

The equation of the stagnation streamline is then

$$q/2 = Ur\sin\theta + \frac{q}{2\pi}\theta \quad \text{and} \quad r = \frac{q(\pi-\theta)}{2\pi U\sin\theta}$$

At $\theta = \pi/2$, $\quad r = \frac{q}{4U} = 50\pi\frac{\text{m}^2}{\text{s}} \times \frac{1}{4} \times \frac{\text{s}}{25\text{ m}} = \pi/2$

Far downstream $\theta \to 0$ and the y coordinate of the body

$$y = r\sin\theta = \frac{q(\pi-\theta)}{2\pi U} \text{ approaches } \frac{q}{2U} = \frac{50\pi}{2 \times 25} = \pm\pi\text{ m} \qquad y_{\theta\to 0}$$

See the next page for plots

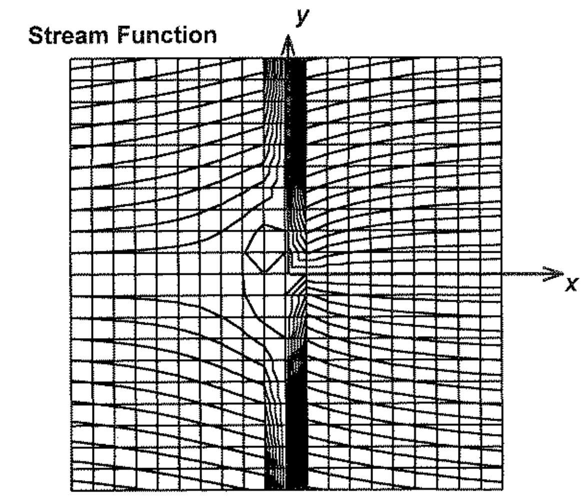
Problem 6.126

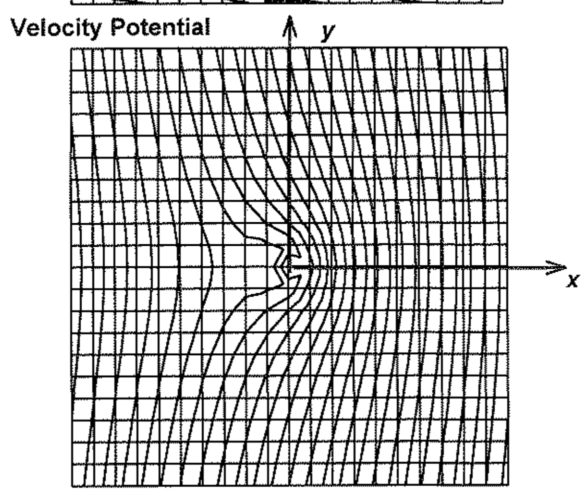
[Difficulty: 3] Part 2/2

Using *Excel*, the stream function and velocity potential can be plotted.
The data below was obtained using the workbook for Example Problem 6.10.
Note the orthogonality of ψ and ϕ!

#NAME?	**Stream Function**
#NAME?	**Velocity Potential**

Note that the plot is
from x = -5 to 5 and y = -5 to 5





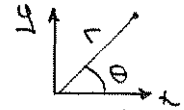
Problem 6.127

[Difficulty: 4] Part 1/2

6.127 Consider the flow field formed by combining a uniform flow in the positive x direction and a source located at the origin. Let $U = 30$ m/s and $q = 150$ m^2/s. Plot the ratio of the local velocity to the freestream velocity as a function of θ along the stagnation streamline. Locate the points on the stagnation streamline where the velocity reaches its maximum value. Find the gage pressure there if the fluid density is 1.2 kg/m^3.

Solution:

Superposition of a uniform flow and source gives flow around a half body.



$$\psi = \psi_{uf} + \psi_{so} = Uy + \frac{q}{2\pi}\theta = Ur\sin\theta + \frac{q}{2\pi}\theta \quad \text{---- (1)}$$

$$u = u_{uf} + u_{so}\ ;\ u_{uf} = U\ ;\ u_{so} = v_r\cos\theta = \frac{q}{2\pi r}\frac{x}{r} \qquad \therefore u = U + \frac{q}{2\pi}\frac{x}{r^2}$$

$$v = v_{uf} + v_{so}\ ;\ v_{uf} = 0\ ;\ v_{so} = v_r\sin\theta = \frac{q}{2\pi r}\frac{y}{r} \qquad \therefore v = \frac{q}{2\pi}\frac{y}{r^2}$$

$$\therefore \vec{V} = u\hat{i} + v\hat{j} = \left(U + \frac{q}{2\pi}\frac{x}{r^2}\right)\hat{i} + \frac{q}{2\pi}\frac{y}{r^2}\hat{j} \quad \text{---- (2)}$$

Then,

$$V^2 = u^2 + v^2 = \left(U + \frac{q}{2\pi r}\cos\theta\right)^2 + \left(\frac{q}{2\pi r}\sin\theta\right)^2$$

$$= U^2 + \left(\frac{q}{2\pi r}\right)^2\cos^2\theta + \frac{Uq}{\pi r}\cos\theta + \left(\frac{q}{2\pi r}\right)^2\sin^2\theta$$

$$V^2 = U^2 + \left(\frac{q}{2\pi r}\right)^2 + \frac{Uq}{\pi r}\cos\theta \quad \text{---- (3)}$$

To determine the equation of the stagnation streamline, we locate the stagnation point ($\vec{V} = 0$). From Eq. 2 $y = 0$ and

$$U + \frac{q}{2\pi}\frac{x}{r^2} = 0 = U + \frac{q}{2\pi}\frac{x}{(x^2+y^2)} = U + \frac{q}{2\pi x} \quad \text{and} \quad x_{stag} = -\frac{q}{2\pi U}$$

$$x_{stag} = -\frac{q}{2\pi U} = -\frac{1}{2\pi} \times 150\frac{m^2}{s} \times \frac{s}{30\,m} = -0.796\,m$$

At the stagnation point $y = 0$ and $\theta = \pi$. From Eq. 1 $\psi_{stag} = \frac{q}{2}$

the equation of the stagnation streamline is then:

$$\psi_{stag} = \frac{q}{2} = Ur\sin\theta + \frac{q}{2\pi}\theta$$

Solving for r, we obtain

$$r = \frac{1}{U\sin\theta}\left(\frac{q}{2} - \frac{q}{2\pi}\theta\right) = \frac{q(\pi - \theta)}{2\pi U\sin\theta} \quad \text{---- (4)}$$

Substituting this value of r into the expression for V^2 [Eq. 3] we obtain

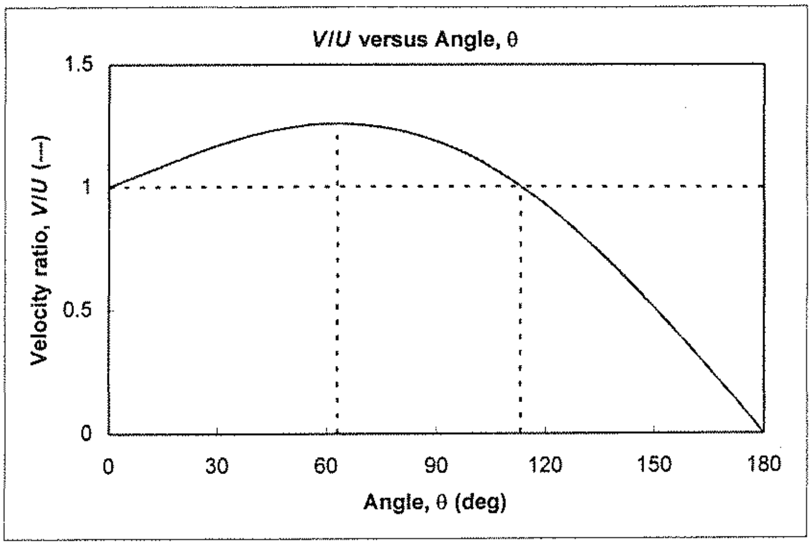
$$V^2 = U^2 + \left[\frac{q}{2\pi} \times \frac{2\pi U\sin\theta}{q(\pi-\theta)}\right]^2 + \frac{Uq\cos\theta}{\pi} \times \frac{2\pi U\sin\theta}{q(\pi-\theta)}$$

$$V^2 = U^2 + \frac{U^2\sin^2\theta}{(\pi-\theta)^2} + \frac{2U^2\sin\theta\cos\theta}{(\pi-\theta)} = U^2\left[1 + \frac{\sin^2\theta}{(\pi-\theta)^2} + \frac{2\sin\theta\cos\theta}{(\pi-\theta)}\right]$$

Along the stagnation streamline

$$\frac{V}{U} = \left[1 + \frac{\sin^2\theta}{(\pi-\theta)^2} + \frac{2\sin\theta\cos\theta}{(\pi-\theta)}\right]^{1/2} \quad \text{---- (5)}$$

V/U is plotted as a function of θ



From the plot we see that V/U is a maximum at $\theta = 63°$ (also at $\theta = 297°$ from symmetry with respect to the x axis.

At $\theta = 63°$, Eq 5 gives $V/U_{max} = 1.26$

, Eq. 4 gives $r = \frac{150\,m^2}{s} \times \frac{(\pi - 0.35\pi)}{2\pi \sin 63°} \times \frac{s}{30\,m} = 1.82\,m$.

Thus $V = V_{max}$ at $r = 1.82\,m$ and $\theta = 63°, 297°$ $\quad (r,\theta)_{V_{max}}$

To determine the gage pressure at this point, write the Bernoulli equation between a point upstream and the point of maximum velocity.

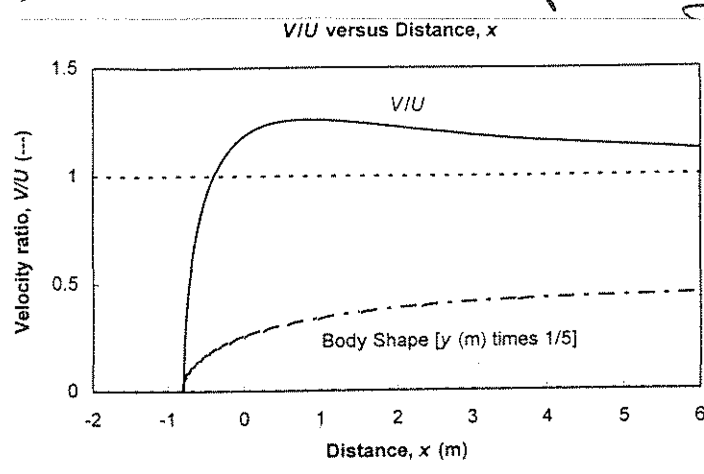
$$\frac{p_\infty}{\rho} + \frac{U^2}{2} = \frac{p}{\rho} + \frac{V_{max}^2}{2}$$

$$\therefore p - p_\infty = \frac{\rho}{2}\left[U^2 - V_{max}^2\right] = \frac{1}{2}\rho U^2\left[1 - \left(\frac{V_{max}}{U}\right)^2\right]$$

$$= \frac{1}{2} \times 1.2\,\frac{kg}{m^3} \times (30)^2\,\frac{m^2}{s^2}\left[1 - \left(1.26\right)^2\right] \times \frac{N \cdot s^2}{kg \cdot m}$$

$$p - p_\infty = 317\ N/m^2 \qquad p_{gage}$$

Note: From the plot we see that $V/U = 1.0$, and hence $p = p_\infty$, at $\theta = 113°$. The corresponding r is 1.01 m.



Problem 7.1

[Difficulty: 2]

7.1 The propagation speed of small-amplitude surface waves in a region of uniform depth is given by

$$c^2 = \left(\frac{\sigma}{\rho}\frac{2\pi}{\lambda} + \frac{g\lambda}{2\pi}\right)\tanh\frac{2\pi h}{\lambda}$$

where h is depth of the undisturbed liquid and λ is wavelength. Using L as a characteristic length and V_0 as a characteristic velocity, obtain the dimensionless groups that characterize the equation.

Given: Equation describing the propagation speed of surface waves in a region of uniform depth

Find: Nondimensionalization for the equation using length scale L and velocity scale Vo. Obtain the dimensionless groups that characterize the flow.

Solution: To nondimensionalize the equation all lengths are divided by the reference length and all velocities are divided by the reference velocity. Denoting the nondimensional quantities by an asterisk:

$$\lambda^* = \frac{\lambda}{L} \qquad h^* = \frac{h}{L} \qquad c^* = \frac{c}{V_0}$$

Substituting into the governing equation: $\left(c^* V_0\right)^2 = \left(\frac{\sigma}{\rho}\frac{2\pi}{\lambda^* L} + \frac{g\lambda^* L}{2\pi}\right)\tanh\frac{2\pi h^* L}{\lambda^* L}$ Simplifying this expression:

$$c^{*2} = \left(\frac{\sigma}{\rho L V_0^2}\frac{2\pi}{\lambda^*} + \frac{gL}{V_0^2}\frac{\lambda^*}{2\pi}\right)\tanh\frac{2\pi h^*}{\lambda^*}$$

The dimensionless group is $\frac{g \cdot L}{V_0^{\,2}}$ which is the reciprocal of the square of the Froude number, and $\frac{\sigma}{\rho \cdot L \cdot V_0^{\,2}}$

which is the inverse of the Weber number.

Problem 7.2

[Difficulty: 2]

7.2 The equation describing small-amplitude vibration of a beam is

$$\rho A \frac{\partial^2 y}{\partial t^2} + EI \frac{\partial^4 y}{\partial x^4} = 0$$

where y is the beam deflection at location x and time t, ρ and E are the density and modulus of elasticity of the beam material, respectively, and A and I are the beam cross-section area and second moment of area, respectively. Use the beam length L, and frequency of vibration ω, to nondimensionalize this equation. Obtain the dimensionless groups that characterize the equation.

Given: Equation for beam

Find: Dimensionless groups

Solution:

Denoting nondimensional quantities by an asterisk

$$A^* = \frac{A}{L^2} \qquad y^* = \frac{y}{L} \qquad t^* = t\omega \qquad I^* = \frac{I}{L^4} \qquad x^* = \frac{x}{L}$$

Hence $\qquad A = L^2 A^* \qquad y = L\,y^* \qquad t = \dfrac{t^*}{\omega} \qquad I = L^4 I^* \qquad x = L\,x^*$

Substituting into the governing equation $\qquad \rho L^2 L \omega^2 A^* \dfrac{\partial^2 y^*}{\partial t^{*2}} + E L^4 \dfrac{1}{L^4} L I^* \dfrac{\partial^4 y^*}{\partial x^{*4}} = 0$

The final dimensionless equation is $\qquad A^* \dfrac{\partial^2 y^*}{\partial t^{*2}} + \left(\dfrac{E}{\rho L^2 \omega^2}\right) I^* \dfrac{\partial^4 y^*}{\partial x^{*4}} = 0$

The dimensionless group is $\qquad \left(\dfrac{E}{\rho L^2 \omega^2}\right)$

Problem 7.3

[Difficulty: 2]

7.3 The slope of the free surface of a steady wave in one-dimensional flow in a shallow liquid layer is described by the equation

$$\frac{\partial h}{\partial x} = -\frac{u}{g}\frac{\partial u}{\partial x}$$

Use a length scale, L, and a velocity scale, V_0, to nondimensionalize this equation. Obtain the dimensionless groups that characterize this flow.

Given: Equation describing the slope of a steady wave in a shallow liquid layer

Find: Nondimensionalization for the equation using length scale L and velocity scale V_o. Obtain the dimensionless groups that characterize the flow.

Solution: To nondimensionalize the equation all lengths are divided by the reference length and all velocities are divided by the reference velocity. Denoting the nondimensional quantities by an asterisk:

$$h^* = \frac{h}{L} \qquad x^* = \frac{x}{L} \qquad u^* = \frac{u}{V_0}$$

Substituting into the governing equation: $$\frac{\partial(h^* L)}{\partial(x^* L)} = -\frac{u^* V_0}{g}\frac{\partial(u^* V_0)}{\partial(x^* L)} \qquad \frac{\partial h^*}{\partial x^*} = -\frac{V_0^2}{gL}u^*\frac{\partial u^*}{\partial x^*}$$

The dimensionless group is $\dfrac{V_0^{\,2}}{g \cdot L}$ which is the square of the Froude number.

Problem 7.4

[Difficulty: 2]

7.4 One-dimensional unsteady flow in a thin liquid layer is described by the equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -g\frac{\partial h}{\partial x}$$

Use a length scale, L, and a velocity scale, V_0, to nondimensionalize this equation. Obtain the dimensionless groups that characterize this flow.

Given: Equation describing one-dimensional unsteady flow in a thin liquid layer

Find: Nondimensionalization for the equation using length scale L and velocity scale Vo. Obtain the dimensionless groups that characterize the flow.

Solution: To nondimensionalize the equation all lengths are divided by the reference length and all velocities are divided by the reference velocity. Denoting the nondimensional quantities by an asterisk:

$$x^* = \frac{x}{L} \qquad h^* = \frac{h}{L} \qquad u^* = \frac{u}{V_0} \qquad t^* = \frac{t}{L/V_0}$$

Substituting into the governing equation: $\dfrac{\partial(u^* V_0)}{\partial(t^* L/V_0)} + u^* V_0 \dfrac{\partial(u^* V_0)}{\partial(x^* L)} = -g\dfrac{\partial(h^* L)}{\partial(x^* L)}$ Simplifying this expression:

$$\frac{V_0^2}{L}\frac{\partial u^*}{\partial t^*} + \frac{V_0^2}{L}u^*\frac{\partial u^*}{\partial x^*} = -g\frac{\partial h^*}{\partial x^*}$$

Thus:

$$\frac{\partial u^*}{\partial t^*} + u^*\frac{\partial u^*}{\partial x^*} = -\frac{gL}{V_0^2}\frac{\partial h^*}{\partial x^*}$$

The dimensionless group is $\dfrac{g \cdot L}{V_0^{\,2}}$ which is the reciprocal of the square of the Froude number.

Problem 7.5

[Difficulty: 2]

7.5 A two-dimensional steady flow in a viscous liquid is described by the equation:

$$u\frac{\partial u}{\partial x} = -g\frac{\partial h}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

Use a length scale, L, and a velocity scale, V_0, to non-dimensionalize this equation. Obtain the dimensionless groups that characterize this flow.

Given: Equation describing two-dimensional steady flow in a liquid

Find: Nondimensionalization for the equation using length scale L and velocity scale V_0. Obtain the dimensionless groups that characterize the flow.

Solution: To nondimensionalize the equation all lengths are divided by the reference length and all velocities are divided by the reference velocity. Denoting the nondimensional quantities by an asterisk:

$$x^* = \frac{x}{L} \qquad h^* = \frac{h}{L} \qquad u^* = \frac{u}{V_0} \qquad y^* = \frac{y}{L}$$

Substituting into the governing equation:

$$u^* V_0 \frac{\partial\left(u^* V_0\right)}{\partial\left(x^* L\right)} = -g\frac{\partial\left(h^* L\right)}{\partial\left(x^* L\right)} + \frac{\mu}{\rho}\left(\frac{\partial^2\left(u^* V_0\right)}{\partial\left(x^* L\right)\partial\left(x^* L\right)} + \frac{\partial^2\left(u^* V_0\right)}{\partial\left(y^* L\right)\partial\left(y^* L\right)}\right)$$

Simplifying this expression:

$$\frac{V_0^2}{L}u^*\frac{\partial u^*}{\partial x^*} = -g\frac{\partial h^*}{\partial x^*} + \frac{\mu V_0}{\rho L^2}\left(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}}\right)$$

Thus: $$u^*\frac{\partial u^*}{\partial x^*} = -\frac{gL}{V_0^2}\frac{\partial h^*}{\partial x^*} + \frac{\mu}{V_0 \rho L}\left(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}}\right)$$

The dimensionless groups are $\dfrac{g\cdot L}{V_0^{\,2}}$ which is the reciprocal of the square of the Froude number, and $\dfrac{\mu}{V_0\cdot\rho\cdot L}$ which is the reciprocal of the Reynolds number.

Problem 7.6

[Difficulty: 2]

7.6 In atmospheric studies the motion of the earth's atmosphere can sometimes be modeled with the equation

$$\frac{D\vec{V}}{Dt} + 2\vec{\Omega} \times \vec{V} = -\frac{1}{\rho}\nabla p$$

where $\vec{V}$ is the large-scale velocity of the atmosphere across the Earth's surface, ∇p is the climatic pressure gradient, and $\vec{\Omega}$ is the Earth's angular velocity. What is the meaning of the term $\vec{\Omega} \times \vec{V}$? Use the pressure difference, Δp, and typical length scale, L (which could, for example, be the magnitude of, and distance between, an atmospheric high and low, respectively), to nondimensionalize this equation. Obtain the dimensionless groups that characterize this flow.

Given: Equations for modeling atmospheric motion

Find: Non-dimensionalized equation; Dimensionless groups

Solution:

Recall that the total acceleration is

$$\frac{D\vec{V}}{Dt} = \frac{\partial \vec{V}}{\partial t} + \vec{V} \cdot \nabla \vec{V}$$

Nondimensionalizing the velocity vector, pressure, angular velocity, spatial measure, and time, (using a typical velocity magnitude V and angular velocity magnitude Ω):

$$\vec{V}^* = \frac{\vec{V}}{V} \qquad p^* = \frac{p}{\Delta p} \qquad \vec{\Omega}^* = \frac{\vec{\Omega}}{\Omega} \qquad x^* = \frac{x}{L} \qquad t^* = t\frac{V}{L}$$

Hence

$$\vec{V} = V\,\vec{V}^* \qquad p = \Delta p\, p^* \qquad \vec{\Omega} = \Omega\,\vec{\Omega}^* \qquad x = L\,x^* \qquad t = \frac{L}{V}t^*$$

Substituting into the governing equation

$$V\frac{V}{L}\frac{\partial \vec{V}^*}{\partial t^*} + V\frac{V}{L}\vec{V}^* \cdot \nabla^* \vec{V}^* + 2\Omega V\,\vec{\Omega}^* \times \vec{V}^* = -\frac{1}{\rho}\frac{\Delta p}{L}\nabla p^*$$

The final dimensionless equation is

$$\frac{\partial \vec{V}^*}{\partial t^*} + \vec{V}^* \cdot \nabla^* \vec{V}^* + 2\left(\frac{\Omega L}{V}\right)\vec{\Omega}^* \times \vec{V} = -\frac{\Delta p}{\rho V^2}\nabla p^*$$

The dimensionless groups are

$$\frac{\Delta p}{\rho V^2} \qquad \frac{\Omega L}{V}$$

The second term on the left of the governing equation is the Coriolis force due to a rotating coordinate system. This is a very significant term in atmospheric studies, leading to such phenomena as geostrophic flow.

Problem 7.7

[Difficulty: 4]

7.7 By using order of magnitude analysis, the continuity and Navier–Stokes equations can be simplified to the Prandtl boundary-layer equations. For steady, incompressible, and two-dimensional flow, neglecting gravity, the result is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}$$

Use L and V_0 as characteristic length and velocity, respectively. Nondimensionalize these equations and identify the similarity parameters that result.

Given: The Prandtl boundary-layer equations for steady, incompressible, two-dimensional flow neglecting gravity

Find: Nondimensionalization for the equation using length scale L and velocity scale V_0. Obtain the dimensionless groups that characterize the flow.

Solution: To nondimensionalize the equation all lengths are divided by the reference length and all velocities are divided by the reference velocity. Denoting the nondimensional quantities by an asterisk:

$$x^* = \frac{x}{L} \qquad y^* = \frac{y}{L} \qquad u^* = \frac{u}{V_0} \qquad v^* = \frac{v}{V_0}$$

Substituting into the continuity equation: $\frac{\partial(u^* V_0)}{\partial(x^* L)} + \frac{\partial(v^* V_0)}{\partial(y^* L)} = 0$ Simplifying this expression: $\frac{V_0}{L}\frac{\partial u^*}{\partial x^*} + \frac{V_0}{L}\frac{\partial v^*}{\partial y^*} = 0$

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0$$

We expand out the second derivative in the momentum equation by writing it as the derivative of the derivative. Upon substitution:

$$u^* V_0 \frac{\partial(u^* V_0)}{\partial(x^* L)} + v^* V_0 \frac{\partial(u^* V_0)}{\partial(y^* L)} = -\frac{1}{\rho}\frac{\partial p}{\partial(x^* L)} + \nu\frac{\partial}{\partial(y^* L)}\frac{\partial(u^* V_0)}{\partial(y^* L)}$$

Simplifying this expression yields:

$$u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = -\frac{1}{\rho V_0^2}\frac{\partial p}{\partial x^*} + \frac{\nu}{V_0 L}\frac{\partial^2 u^*}{\partial y^{*2}}$$

Now every term in this equation has been non-dimensionalized except the pressure gradient. We define a dimensionless pressure as:

$$p^* = \frac{p}{\rho V_0^2}$$

Substituting this into the momentum equation: $u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = -\frac{1}{\rho V_0^2}\frac{\partial(p^* \rho V_0^2)}{\partial x^*} + \frac{\nu}{V_0 L}\frac{\partial^2 u^*}{\partial y^{*2}}$

Simplifying this expression yields:

$$u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{\nu}{V_0 L}\frac{\partial^2 u^*}{\partial y^{*2}}$$

The dimensionless group is $\frac{\nu}{V_0 \cdot L}$ which is the reciprocal of the Reynolds number.

Problem 7.8

[Difficulty: 2]

7.8 An unsteady, two-dimensional, compressible, inviscid flow can be described by the equation

$$\frac{\partial^2\psi}{\partial t^2} + \frac{\partial}{\partial t}(u^2+v^2) + (u^2-c^2)\frac{\partial^2\psi}{\partial x^2} + (v^2-c^2)\frac{\partial^2\psi}{\partial y^2} + 2uv\frac{\partial^2\psi}{\partial x\partial y} = 0$$

where ψ is the stream function, u and v are the x and y components of velocity, respectively, c is the local speed of sound, and t is the time. Using L as a characteristic length and c_0 (the speed of sound at the stagnation point) to nondimensionalize this equation, obtain the dimensionless groups that characterize the equation.

Given: Equation for unsteady, 2D compressible, inviscid flow

Find: Dimensionless groups

Solution:

Denoting nondimensional quantities by an asterisk

$$x^* = \frac{x}{L} \qquad y^* = \frac{y}{L} \qquad u^* = \frac{u}{c_0} \qquad v^* = \frac{v}{c_0} \qquad c^* = \frac{c}{c_0} \qquad t^* = \frac{t\,c_0}{L} \qquad \psi^* = \frac{\psi}{L\,c_0}$$

Note that the stream function indicates volume flow rate/unit depth!

Hence

$$x = L\,x^* \qquad y = L\,y^* \qquad u = c_0\,u^* \qquad v = c_0\,v^* \qquad c = c_0\,c^* \qquad t = \frac{L\,t^*}{c_0} \qquad \psi = L\,c_0\,\psi^*$$

Substituting into the governing equation

$$\left(\frac{c_0^3}{L}\right)\frac{\partial^2\psi^*}{\partial t^{*2}} + \left(\frac{c_0^3}{L}\right)\frac{\partial\left(u^{*2}+v^{*2}\right)}{\partial t} + \left(\frac{c_0^3}{L}\right)\left(u^{*2}-c^{*2}\right)\frac{\partial^2\psi^*}{\partial x^{*2}} + \left(\frac{c_0^3}{L}\right)\left(v^{*2}-c^{*2}\right)\frac{\partial^2\psi^*}{\partial y^{*2}} + \left(\frac{c_0^3}{L}\right)2u^*v^*\frac{\partial^2\psi^*}{\partial x^*\partial y^*} = 0$$

The final dimensionless equation is

$$\frac{\partial^2\psi^*}{\partial t^{*2}} + \frac{\partial\left(u^{*2}+v^{*2}\right)}{\partial t} + \left(u^{*2}-c^{*2}\right)\frac{\partial^2\psi^*}{\partial x^{*2}} + \left(v^{*2}-c^{*2}\right)\frac{\partial^2\psi^*}{\partial y^{*2}} + 2u^*v^*\frac{\partial^2\psi^*}{\partial x^*\partial y^*} = 0$$

No dimensionless group is needed for this equation!

Problem 7.9

[Difficulty: 2]

7.9 The equation describing motion of fluid in a pipe due to an applied pressure gradient, when the flow starts from rest, is

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right)$$

Use the average velocity $\overline{V}$, pressure drop Δp, pipe length L, and diameter D to nondimensionalize this equation. Obtain the dimensionless groups that characterize this flow.

Given: Equations Describing pipe flow

Find: Non-dimensionalized equation; Dimensionless groups

Solution:

Nondimensionalizing the velocity, pressure, spatial measures, and time:

$$u^* = \frac{u}{\overline{V}} \qquad p^* = \frac{p}{\Delta p} \qquad x^* = \frac{x}{L} \qquad r^* = \frac{r}{L} \qquad t^* = t\frac{\overline{V}}{L}$$

Hence

$$u = \overline{V}\,u^* \qquad p = \Delta p\,p^* \qquad x = L\,x^* \qquad r = D\,r^* \qquad t = \frac{L}{\overline{V}}t^*$$

Substituting into the governing equation

$$\frac{\partial u}{\partial t} = \overline{V}\frac{\overline{V}}{L}\frac{\partial u^*}{\partial t^*} = -\frac{1}{\rho}\Delta p\frac{1}{L}\frac{\partial p^*}{\partial x^*} + \nu\overline{V}\frac{1}{D^2}\left(\frac{\partial^2 u^*}{\partial r^{*2}} + \frac{1}{r^*}\frac{\partial u^*}{\partial r^*}\right)$$

The final dimensionless equation is

$$\frac{\partial u^*}{\partial t^*} = -\frac{\Delta p}{\rho\overline{V}^2}\frac{\partial p^*}{\partial x^*} + \left(\frac{\nu}{D\overline{V}}\right)\left(\frac{L}{D}\right)\left(\frac{\partial^2 u^*}{\partial r^{*2}} + \frac{1}{r^*}\frac{\partial u^*}{\partial r^*}\right)$$

The dimensionless groups are

$$\frac{\Delta p}{\rho\overline{V}^2} \qquad \frac{\nu}{D\overline{V}} \qquad \frac{L}{D}$$

Problem 7.10

[Difficulty: 2]

7.10 Experiments show that the pressure drop for flow through an orifice plate of diameter d mounted in a length of pipe of diameter D may be expressed as $\Delta p = p_1 - p_2 = f(\rho, \mu, \overline{V}, d, D)$. You are asked to organize some experimental data. Obtain the resulting dimensionless parameters.

Given: Functional relationship between pressure drop through orifice plate and physical parameters

Find: Appropriate dimensionless parameters

Solution: We will use the Buckingham pi-theorem.

1 $\quad \Delta p \quad \rho \quad \mu \quad V \quad D \quad d$ $\qquad$ n = 6 parameters

2 Select primary dimensions M, L, t:

3 $\quad \Delta p \quad \rho \quad \mu \quad V \quad D \quad d$

$$\frac{M}{Lt^2} \quad \frac{M}{L^3} \quad \frac{M}{Lt} \quad \frac{L}{t} \quad L \quad L$$

r = 3 dimensions

4 $\quad \rho \quad V \quad D$ $\qquad$ m = r = 3 repeating parameters

5 We have n - m = 3 dimensionless groups. Setting up dimensional equations:

$$\Pi_1 = \Delta p \cdot \rho^a \cdot V^b \cdot D^c \quad \text{Thus:} \quad \left(\frac{M}{L \cdot t^2}\right) \cdot \left(\frac{M}{L^3}\right)^a \cdot \left(\frac{L}{t}\right)^b \cdot L^c = M^0 \cdot L^0 \cdot t^0$$

Summing exponents:

M: $1 + a = 0$

L: $-1 - 3 \cdot a + b + c = 0$

t: $-2 - b = 0$

The solution to this system is:

$a = -1 \quad b = -2 \quad c = 0$

$$\Pi_1 = \frac{\Delta p}{\rho \cdot V^2}$$

Check using F, L, t primary dimensions: $\dfrac{F}{L^2} \cdot \dfrac{L^4}{F \cdot t^2} \cdot \dfrac{t^2}{L^2} = 1$ Checks out.

$$\Pi_2 = \mu \cdot \rho^a \cdot V^b \cdot D^c \quad \text{Thus:} \quad \left(\frac{M}{L \cdot t}\right) \cdot \left(\frac{M}{L^3}\right)^a \cdot \left(\frac{L}{t}\right)^b \cdot L^c = M^0 \cdot L^0 \cdot t^0$$

Summing exponents:

M: $1 + a = 0$

L: $-1 - 3 \cdot a + b + c = 0$

t: $-1 - b = 0$

The solution to this system is:

$a = -1 \quad b = -1 \quad c = -1$

$$\Pi_2 = \frac{\mu}{\rho \cdot V \cdot D}$$

(This is the Reynolds number, so it checks out)

$\Pi_3 = d \cdot \rho^a \cdot V^b \cdot D^c$ Thus: $L \cdot \left(\frac{M}{L^3}\right)^a \cdot \left(\frac{L}{t}\right)^b \cdot L^c = M^0 \cdot L^0 \cdot t^0$

Summing exponents:

M: $a = 0$

L: $1 + c = 0$

t: $b = 0$

The solution to this system is:

$a = 0 \quad b = 0 \quad c = -1$

$\Pi_3 = \frac{d}{D}$

(This checks out)

Problem 7.11

[Difficulty: 2]

7.11 At relatively high speeds the drag on an object is independent of fluid viscosity. Thus the aerodynamic drag force, F, on an automobile, is a function only of speed, V, air density ρ, and vehicle size, characterized by its frontal area A. Use dimensional analysis to determine how the drag force F depends on the speed V.

Given:
That drag depends on speed, air density and frontal area

Find:
How drag force depend on speed

Solution:

Apply the Buckingham Π procedure

① $F \quad V \quad \rho \quad A$ $\qquad n = 4$ parameters

② Select primary dimensions M, L, t

③
$$\begin{array}{cccc} F & V & \rho & A \\ \dfrac{ML}{t^2} & \dfrac{L}{t} & \dfrac{M}{L^3} & L^2 \end{array}$$
$r = 3$ primary dimensions

④ $V \quad \rho \quad A$ $\qquad m = r = 3$ repeat parameters

⑤ Then $n - m = 1$ dimensionless groups will result. Setting up a dimensional equation,

$$\Pi_1 = V^a \rho^b A^c F$$

$$= \left(\frac{L}{t}\right)^a \left(\frac{M}{L^3}\right)^b \left(L^2\right)^c \frac{ML}{t^2} = M^0 L^0 t^0$$

Summing exponents,

$$\begin{array}{rc|l} M: & b + 1 = 0 & b = -1 \\ L: & a - 3b + 2c + 1 = 0 & c = -1 \\ t: & -a - 2 = 0 & a = -2 \end{array}$$

Hence

$$\Pi_1 = \frac{F}{\rho V^2 A}$$

⑥ Check using F, L, t as primary dimensions

$$\Pi_1 = \frac{F}{\dfrac{Ft^2}{L^4}\dfrac{L^2}{t^2}L^2} = [1]$$

The relation between drag force F and speed V must then be

$$F \propto \rho V^2 A \propto V^2$$

The drag is proportional to the *square* of the speed.

Problem 7.12

[Difficulty: 2]

7.12 At very low speeds, the drag on an object is independent of fluid density. Thus the drag force, F, on a small sphere is a function only of speed, V, fluid viscosity, μ, and sphere diameter, D. Use dimensional analysis to determine how the drag force F depends on the speed V.

Given: At low speeds, drag F on a sphere is only dependent upon speed V, viscosity μ, and diameter D

Find: Appropriate dimensionless parameters

Solution: We will use the Buckingham pi-theorem.

1 $\quad F \quad V \quad \mu \quad D \qquad$ n = 4 parameters

2 Select primary dimensions M, L, t:

3 $\quad F \quad V \quad \mu \quad D \qquad$ n = 4 parameters

$$\frac{ML}{t^2} \quad \frac{L}{t} \quad \frac{M}{Lt} \quad L \qquad \text{r = 3 dimensions}$$

4 $\quad V \quad \mu \quad D \qquad$ m = r = 3 repeating parameters

5 We have n - m = 1 dimensionless group. Setting up a dimensional equation:

$$\Pi_1 = F \cdot V^a \cdot \mu^b \cdot D^c \qquad \text{Thus:} \quad \left(\frac{M \cdot L}{t^2}\right) \cdot \left(\frac{L}{t}\right)^a \cdot \left(\frac{M}{L \cdot t}\right)^b \cdot L^c = M^0 \cdot L^0 \cdot t^0$$

Summing exponents:

M: $1 + b = 0$

L: $1 + a - b + c = 0$

t: $-2 - a - b = 0$

The solution to this system is:

$a = -1 \quad b = -1 \quad c = -1$

$$\Pi_1 = \frac{F}{\mu \cdot V \cdot D}$$

Check using F, L, t primary dimensions: $F \cdot \frac{t}{L} \cdot \frac{L^2}{F \cdot t} \cdot \frac{1}{L} = 1$ Checks out.

Since the procedure produces only one dimensionless group, it must be a constant. Therefore: $\frac{F}{\mu \cdot V \cdot D} = \text{constant}$

Problem 7.13

[Difficulty: 2]

7.13 The drag force on the International Space Station depends on the mean free path of the molecules λ (a length), the density ρ, a characteristic length L, and the mean speed of the air molecules c. Find a nondimensional form of this functional relationship.

Given: Functional relationship between the drag on a satellite and other physical parameters

Find: Expression for F_D in terms of the other variables

Solution: We will use the Buckingham pi-theorem.

1 $F_D \quad \lambda \quad \rho \quad L \quad c$ — $n = 5$ parameters

2 Select primary dimensions M, L, t:

3 $F_D \quad \lambda \quad \rho \quad L \quad c$

$$\frac{M \cdot L}{t^2} \quad L \quad \frac{M}{L^3} \quad L \quad \frac{L}{t}$$

$r = 3$ dimensions

4 $\rho \quad L \quad c$ — $m = r = 3$ repeating parameters

5 We have n - m = 2 dimensionless groups. Setting up dimensional equations:

$$\Pi_1 = D \cdot \rho^a \cdot L^b \cdot c^d \qquad \text{Thus:} \quad \frac{M \cdot L}{t^2} \cdot \left(\frac{M}{L^3}\right)^a \cdot L^b \cdot \left(\frac{L}{t}\right)^d = M^0 \cdot L^0 \cdot t^0$$

Summing exponents:

M: $1 + a = 0$

L: $1 - 3 \cdot a + b + d = 0$

t: $-2 - d = 0$

The solution to this system is:

$a = -1 \qquad b = -2 \qquad d = -2$

$$\Pi_1 = \frac{F_D}{\rho \cdot L^2 \cdot c^2}$$

Check using F, L, t dimensions: $F \cdot \frac{L^4}{F \cdot t^2} \cdot \frac{1}{L^2} \cdot \frac{t^2}{L^2} = 1$

$$\Pi_2 = \lambda \cdot \rho^a \cdot L^b \cdot c^d \qquad \text{Thus:} \quad L \cdot \left(\frac{M}{L^3}\right)^a \cdot L^b \cdot \left(\frac{L}{t}\right)^d = M^0 \cdot L^0 \cdot t^0$$

Summing exponents:

M: $a = 0$

L: $1 - 3 \cdot a + b + d = 0$

t: $d = 0$

The solution to this system is:

$a = 0 \qquad b = -1 \qquad d = 0$

$$\Pi_2 = \frac{\lambda}{L}$$

(Π_2 is sometimes referred to as the Knudsen number.)

Check using F, L, t dimensions: $L \cdot \frac{1}{L} = 1$

The functional relationship is: $\Pi_1 = f(\Pi_2)$

$$\frac{F_D}{\rho \cdot L^2 \cdot c^2} = f\left(\frac{\lambda}{L}\right) \qquad F_D = \rho \cdot L^2 \cdot c^2 \cdot f\left(\frac{\lambda}{L}\right)$$

Problem 7.14

[Difficulty: 2]

7.14 We saw in Chapter 3 that the buoyant force, F_B, on a body submerged in a fluid is directly proportional to the specific weight of the fluid, γ. Demonstrate this using dimensional analysis, by starting with the buoyant force as a function of the volume of the body and the specific weight of the fluid.

Given: Functional relationship between buoyant force of a fluid and physical parameters

Find: Buoyant force is proportional to the specific weight as demonstrated in Chapter 3.

Solution: We will use the Buckingham pi-theorem.

1 $\quad F_B \quad V \quad \gamma$ $\qquad$ n = 3 parameters

2 Select primary dimensions F, L, t:

3 $\quad F_B \quad V \quad \gamma$

$\quad F \quad L^3 \quad \dfrac{F}{L^3}$ $\qquad$ r = 2 dimensions

4 $\quad V \quad \gamma$ $\qquad$ m = r = 2 repeating parameters

5 We have n - m = 1 dimensionless group. Setting up dimensional equations:

$$\Pi_1 = F_B \cdot V^a \cdot \gamma^b \qquad \text{Thus:} \qquad F \cdot \left(L^3\right)^a \cdot \left(\frac{F}{L^3}\right)^b = F^0 \cdot L^0$$

Summing exponents:

F: $\quad 1 + b = 0$ $\qquad$ The solution to this system is: $\qquad \Pi_1 = \dfrac{F_B}{V \cdot \gamma}$

L: $\quad 3 \cdot a - 3 \cdot b = 0$ $\qquad a = -1 \quad b = -1$

Check using M, L, t dimensions: $\dfrac{M \cdot L}{t^2} \cdot \dfrac{1}{L^3} \cdot \dfrac{t^2 \cdot L^2}{M} = 1$

The functional relationship is: $\Pi_1 = C \quad \dfrac{F_B}{V \cdot \gamma} = C$ $\qquad$ Solving for the buoyant force: $F_B = C \cdot V \cdot \gamma$ $\qquad$ Buoyant force is proportional to γ (Q.E.D.)

Problem 7.15

[Difficulty: 2]

7.15 When an object travels at supersonic speeds, the aerodynamic drag force F acting on the object is a function of the velocity V, air density ρ, object size (characterized by some reference area A), and the speed of sound c (note that all of the variables except c were considered when traveling at subsonic speeds as in Problem 7.11). Develop a functional relationship between a set of dimensionless variables to describe this problem.

Given: Functional relationship between drag on an object in a supersonic flow and physical parameters

Find: Functional relationship for this problem using dimensionless parameters

Solution: We will use the Buckingham pi-theorem.

1 $\quad F_D \quad V \quad \rho \quad A \quad c$ $\qquad$ $n = 5$ parameters

2 $\quad$ Select primary dimensions M, L, t:

3 $\quad F_D \quad V \quad \rho \quad A \quad c$

$$\frac{M\cdot L}{t^2} \quad \frac{L}{t} \quad \frac{M}{L^3} \quad L^2 \quad \frac{L}{t}$$

$r = 3$ dimensions

4 $\quad V \quad \rho \quad A$ $\qquad$ $m = r = 3$ repeating parameters

5 $\quad$ We have $n - m = 2$ dimensionless groups. Setting up dimensional equations:

$$\Pi_1 = F_D\cdot V^a\cdot\rho^b\cdot A^c \qquad \text{Thus:} \qquad \frac{M\cdot L}{t^2}\cdot\left(\frac{L}{t}\right)^a\cdot\left(\frac{M}{L^3}\right)^b\cdot\left(L^2\right)^c = M^0\cdot L^0\cdot t^0$$

Summing exponents:

M: $1 + b = 0$

L: $1 + a - 3\cdot b + 2\cdot c = 0$

t: $-2 - a = 0$

The solution to this system is: $a = -2 \quad b = -1 \quad c = -1$

$$\Pi_1 = \frac{F_D}{V^2\cdot\rho\cdot A}$$

Check using F, L, t dimensions: $F\cdot\frac{t^2}{L^2}\cdot\frac{L^4}{F\cdot t^2}\cdot\frac{1}{L^2} = 1$

$$\Pi_2 = c\cdot V^a\cdot\rho^b\cdot A^c \qquad \text{Thus:} \qquad \frac{L}{t}\cdot\left(\frac{L}{t}\right)^a\cdot\left(\frac{M}{L^3}\right)^b\cdot\left(L^2\right)^c = M^0\cdot L^0\cdot t^0$$

Summing exponents:

M: $b = 0$

L: $1 + a - 3\cdot b + 2\cdot c = 0$

t: $-1 - a = 0$

The solution to this system is: $a = -1 \quad b = 0 \quad c = 0$

$$\Pi_2 = \frac{c}{V}$$

(The reciprocal of Π_2 is also referred to as the Mach number.)

Check using F, L, t dimensions: $\frac{L}{t}\cdot\frac{t}{L} = 1$

The functional relationship is: $\Pi_1 = g(\Pi_2)$

$$\frac{F_D}{V^2\cdot\rho\cdot A} = f\left(\frac{c}{V}\right)$$

Problem 7.16

[Difficulty: 2]

7.16 The speed, V, of a free-surface wave in shallow liquid is a function of depth, D, density, ρ, gravity, g, and surface tension, σ. Use dimensional analysis to find the functional dependence of V on the other variables. Express V in the simplest form possible.

Given: That speed of shallow waves depends on depth, density, gravity and surface tension

Find: Dimensionless groups; Simplest form of V

Solution:

Apply the Buckingham Π procedure

① $V \quad D \quad \rho \quad g \quad \sigma$ $\qquad n = 5$ parameters

② Select primary dimensions M, L, t

③ $$\begin{Bmatrix} V & D & \rho & g & \sigma \\ \dfrac{L}{t} & L & \dfrac{M}{L^3} & \dfrac{L}{t^2} & \dfrac{M}{t^2} \end{Bmatrix}$$ $r = 3$ primary dimensions

④ $g \quad \rho \quad D$ $\qquad m = r = 3$ repeat parameters

⑤ Then $n - m = 2$ dimensionless groups will result. Setting up a dimensional equation,

$$\Pi_1 = g^a \rho^b D^c V = \left(\frac{L}{t^2}\right)^a \left(\frac{M}{L^3}\right)^b (L)^c \frac{L}{t} = M^0 L^0 t^0$$

Summing exponents,

$$\begin{array}{ll|l} M: & b = 0 & b = 0 \\ L: & a - 3b + c + 1 = 0 & c = -\dfrac{1}{2} \\ t: & -2a - 1 = 0 & a = -\dfrac{1}{2} \end{array}$$

Hence $\quad \Pi_1 = \dfrac{V}{\sqrt{gD}}$

$$\Pi_2 = g^a \rho^b D^c \sigma = \left(\frac{L}{t^2}\right)^a \left(\frac{M}{L^3}\right)^b (L)^c \frac{M}{t^2} = M^0 L^0 t^0$$

Summing exponents,

$$\begin{array}{ll|l} M: & b + 1 = 0 & b = -1 \\ L: & a - 3b + c = 0 & c = -2 \\ t: & -2a - 2 = 0 & a = -1 \end{array}$$

Hence $\quad \Pi_2 = \dfrac{\sigma}{g\rho D^2}$

⑥ Check using F, L, t as primary dimensions $\qquad \Pi_1 = \dfrac{\dfrac{L}{t}}{\left(\dfrac{L}{t^2}L\right)^{\frac{1}{2}}} = [1] \qquad \Pi_2 = \dfrac{\dfrac{F}{L}}{\dfrac{L}{t^2}\dfrac{Ft^2}{L^4}L^2} = [1]$

The relation between drag force speed V is $\qquad \Pi_1 = f(\Pi_2) \qquad \dfrac{V}{\sqrt{gD}} = f\left(\dfrac{\sigma}{g\rho D^2}\right) \qquad V = \sqrt{gD}\, f\left(\dfrac{\sigma}{g\rho D^2}\right)$

Problem 7.17

[Difficulty: 2]

7.17 The wall shear stress, τ_w, in a boundary layer depends on distance from the leading edge of the body, x, the density, ρ, and viscosity, μ, of the fluid, and the freestream speed of the flow, U. Obtain the dimensionless groups and express the functional relationship among them.

Given: Functional relationship between wall shear stress in a boundary layer and physical parameters

Find: Appropriate dimensionless parameters

Solution: We will use the Buckingham pi-theorem.

1 $\tau_w \quad x \quad \rho \quad \mu \quad U$ n = 5 parameters

2 Select primary dimensions M, L, t:

3 $\tau_w \quad x \quad \rho \quad \mu \quad U$

$\frac{M}{Lt^2} \quad L \quad \frac{M}{L^3} \quad \frac{M}{Lt} \quad \frac{L}{t}$ r = 3 dimensions

4 $\rho \quad x \quad U$ m = r = 3 repeating parameters

5 We have n - m = 2 dimensionless groups. Setting up dimensional equations:

$\Pi_1 = \tau_W \cdot \rho^a \cdot x^b \cdot U^c$ Thus: $\left(\frac{M}{L \cdot t^2}\right) \cdot \left(\frac{M}{L^3}\right)^a \cdot L^b \cdot \left(\frac{L}{t}\right)^c = M^0 \cdot L^0 \cdot t^0$

Summing exponents:

M: $1 + a = 0$

L: $-1 - 3 \cdot a + b + c = 0$

t: $-2 - c = 0$

The solution to this system is:

$a = -1 \quad b = 0 \quad c = -2$

$$\Pi_1 = \frac{\tau_W}{\rho \cdot U^2}$$

Check using F, L, t dimensions: $\left(\frac{F}{L^2}\right) \cdot \left(\frac{L^4}{F \cdot t^2}\right) \cdot \left(\frac{t^2}{L}\right) = 1$

$\Pi_2 = \mu \cdot \rho^a \cdot x^b \cdot U^c$ Thus: $\left(\frac{M}{L \cdot t}\right) \cdot \left(\frac{M}{L^3}\right)^a \cdot L^b \cdot \left(\frac{L}{t}\right)^c = M^0 \cdot L^0 \cdot t^0$

Summing exponents:

M: $1 + a = 0$

L: $-1 - 3 \cdot a + b + c = 0$

t: $-1 - c = 0$

The solution to this system is:

$a = -1 \quad b = -1 \quad c = -1$

$$\Pi_2 = \frac{\mu}{\rho \cdot x \cdot U}$$

Check using F, L, t dimensions: $\left(\frac{F \cdot t}{L^2}\right) \cdot \left(\frac{L^4}{F \cdot t^2}\right) \cdot \left(\frac{1}{L}\right) \cdot \left(\frac{t^2}{L}\right) = 1$

The functional relationship is:

$$\Pi_1 = f(\Pi_2)$$

Problem 7.18

[Difficulty: 2]

7.18 The boundary-layer thickness, δ, on a smooth flat plate in an incompressible flow without pressure gradients depends on the freestream speed, U, the fluid density, ρ, the fluid viscosity, μ, and the distance from the leading edge of the plate, x. Express these variables in dimensionless form.

Given: Functional relationship between boundary layer thickness and physical parameters

Find: Appropriate dimensionless parameters

Solution: We will use the Buckingham pi-theorem.

1 $\quad \delta \quad x \quad \rho \quad \mu \quad U$ $\qquad$ n = 5 parameters

2 Select primary dimensions M, L, t:

3 $\quad \delta \quad x \quad \rho \quad \mu \quad U$

$\quad L \quad L \quad \dfrac{M}{L^3} \quad \dfrac{M}{Lt} \quad \dfrac{L}{t}$ $\qquad$ r = 3 dimensions

4 $\quad \rho \quad x \quad U$ $\qquad$ m = r = 3 repeating parameters

5 We have n - m = 2 dimensionless groups. Setting up dimensional equations:

$$\Pi_1 = \delta\cdot\rho^a\cdot x^b\cdot U^c \qquad \text{Thus:} \quad L\cdot\left(\frac{M}{L^3}\right)^a\cdot L^b\cdot\left(\frac{L}{t}\right)^c = M^0\cdot L^0\cdot t^0$$

Summing exponents:

M: $0 + a = 0$

L: $1 - 3\cdot a + b + c = 0$

t: $0 - c = 0$

The solution to this system is: $a = 0 \quad b = -1 \quad c = 0$

$$\Pi_1 = \frac{\delta}{x}$$

Check using F, L, t dimensions: $(L)\cdot\left(\frac{1}{L}\right) = 1$

$$\Pi_2 = \mu\cdot\rho^a\cdot x^b\cdot U^c \qquad \text{Thus:} \quad \left(\frac{M}{L\cdot t}\right)\cdot\left(\frac{M}{L^3}\right)^a\cdot L^b\cdot\left(\frac{L}{t}\right)^c = M^0\cdot L^0\cdot t^0$$

Summing exponents:

M: $1 + a = 0$

L: $-1 - 3\cdot a + b + c = 0$

t: $-1 - c = 0$

The solution to this system is: $a = -1 \quad b = -1 \quad c = -1$

$$\Pi_2 = \frac{\mu}{\rho\cdot x\cdot U}$$

Check using F, L, t dimensions: $\left(\frac{F\cdot t}{L^2}\right)\cdot\left(\frac{L^4}{F\cdot t^2}\right)\cdot\left(\frac{1}{L}\right)\cdot\left(\frac{t^2}{L}\right) = 1$

The functional relationship is:

$$\Pi_1 = f(\Pi_2)$$

Problem 7.19

[Difficulty: 2]

7.19 If an object is light enough it can be supported on the surface of a fluid by surface tension. Tests are to be done to investigate this phenomenon. The weight, W, supportable in this way depends on the object's perimeter, p, and the fluid's density, ρ, surface tension σ, and gravity, g. Determine the dimensionless parameters that characterize this problem.

Given: That light objects can be supported by surface tension

Find: Dimensionless groups

Solution:

Apply the Buckingham Π procedure

① $W \quad p \quad \rho \quad g \quad \sigma$ $\qquad n = 5$ parameters

② Select primary dimensions M, L, t

③ $$\begin{Bmatrix} W & p & \rho & g & \sigma \\ \dfrac{ML}{t^2} & L & \dfrac{M}{L^3} & \dfrac{L}{t^2} & \dfrac{M}{t^2} \end{Bmatrix}$$ $r = 3$ primary dimensions

④ $g \quad \rho \quad p$ $\qquad m = r = 3$ repeat parameters

⑤ Then $n - m = 2$ dimensionless groups will result. Setting up a dimensional equation,

$$\Pi_1 = g^a \rho^b p^c W = \left(\frac{L}{t^2}\right)^a \left(\frac{M}{L^3}\right)^b (L)^c \frac{ML}{t^2} = M^0 L^0 t^0$$

Summing exponents,

$$\begin{array}{rl|l} M: & b + 1 = 0 & b = -1 \\ L: & a - 3b + c + 1 = 0 & c = -3 \\ t: & -2a - 2 = 0 & a = -1 \end{array}$$

Hence $\qquad \Pi_1 = \dfrac{W}{g\rho p^3}$

$$\Pi_2 = g^a \rho^b p^c \sigma = \left(\frac{L}{t^2}\right)^a \left(\frac{M}{L^3}\right)^b (L)^c \frac{M}{t^2} = M^0 L^0 t^0$$

Summing exponents,

$$\begin{array}{rl|l} M: & b + 1 = 0 & b = -1 \\ L: & a - 3b + c = 0 & c = -2 \\ t: & -2a - 2 = 0 & a = -1 \end{array}$$

Hence $\qquad \Pi_2 = \dfrac{\sigma}{g\rho p^2}$

⑥ Check using F, L, t as primary dimensions $\qquad \Pi_1 = \dfrac{F}{\dfrac{L}{t^2}\dfrac{Ft^2}{L^4}L^3} = [1] \qquad \Pi_2 = \dfrac{\dfrac{F}{L}}{\dfrac{L}{t^2}\dfrac{Ft^2}{L^4}L^2} = [1]$

Note: Any combination of Π_1 and Π_2 is a Π group, e.g., $\dfrac{\Pi_1}{\Pi_2} = \dfrac{Wp}{\sigma}$, so Π_1 and Π_2 are not unique!

Problem 7.20

[Difficulty: 2]

7.20 The speed, V, of a free-surface gravity wave in deep water is a function of wavelength, λ, depth, D, density, ρ, and acceleration of gravity, g. Use dimensional analysis to find the functional dependence of V on the other variables. Express V in the simplest form possible.

Given: Functional relationship between the speed of a free-surface gravity wave in deep water and physical parameters

Find: The dependence of the speed on the other variables

Solution: We will use the Buckingham pi-theorem.

1 $V \quad \lambda \quad D \quad \rho \quad g$ — $n = 5$ parameters

2 Select primary dimensions M, L, t:

3 $V \quad \lambda \quad D \quad \rho \quad g$

$\frac{L}{t} \quad L \quad L \quad \frac{M}{L^3} \quad \frac{L}{t^2}$ — $r = 3$ dimensions

4 $D \quad \rho \quad g$ — $m = r = 3$ repeating parameters

5 We have n - m = 2 dimensionless groups. Setting up dimensional equations:

$$\Pi_1 = V \cdot D^a \cdot \rho^b \cdot g^c \qquad \text{Thus:} \qquad \frac{L}{t} \cdot L^a \cdot \left(\frac{M}{L^3}\right)^b \cdot \left(\frac{L}{t^2}\right)^c = M^0 \cdot L^0 \cdot t^0$$

Summing exponents:

M: $b = 0$

L: $1 + a - 3 \cdot b + c = 0$

t: $-1 - 2 \cdot c = 0$

The solution to this system is: $a = -\frac{1}{2} \quad b = 0 \quad c = -\frac{1}{2}$

$$\Pi_1 = \frac{V}{\sqrt{g \cdot D}}$$

Check using F, L, t dimensions: $\left(\frac{L}{t}\right) \cdot \left(\frac{t}{L}\right) = 1$

$$\Pi_2 = \lambda \cdot D^a \cdot \rho^b \cdot g^c \qquad \text{Thus:} \qquad L \cdot L^a \cdot \left(\frac{M}{L^3}\right)^b \cdot \left(\frac{L}{t^2}\right)^c = M^0 \cdot L^0 \cdot t^0$$

Summing exponents:

M: $b = 0$

L: $1 + a - 3 \cdot b + c = 0$

t: $-2 \cdot c = 0$

The solution to this system is: $a = -1 \quad b = 0 \quad c = 0$

$$\Pi_2 = \frac{\lambda}{D}$$

Check using F, L, t dimensions: $L \cdot \frac{1}{L} = 1$

The functional relationship is: $\Pi_1 = f(\Pi_2) \qquad \frac{V}{\sqrt{g \cdot D}} = f\left(\frac{\lambda}{D}\right)$ Therefore the velocity is: $V = \sqrt{g \cdot D} \cdot f\left(\frac{\lambda}{D}\right)$

Problem 7.21

[Difficulty: 2]

7.21 The mean velocity, $\overline{u}$, for turbulent flow in a pipe or a boundary layer may be correlated using the wall shear stress, τ_w, distance from the wall, y, and the fluid properties, ρ and μ. Use dimensional analysis to find one dimensionless parameter containing $\overline{u}$ and one containing y that are suitable for organizing experimental data. Show that the result may be written

$$\frac{\overline{u}}{u_*} = f\left(\frac{yu_*}{\nu}\right)$$

where $u_* = (\tau_w/\rho)^{1/2}$ is the *friction velocity*.

Given: Functional relationship between mean velocity for turbulent flow in a pipe or boundary layer and physical parameters

Find: (a) Appropriate dimensionless parameters containing mean velocity and one containing the distance from the wall that are suitable for organizing experimental data.
(b) Show that the result may be written as:

$$\frac{\overline{u}}{u_*} = f\left(\frac{yu_*}{\nu}\right) \quad \text{where} \quad u_* = \sqrt{\frac{\tau_w}{\rho}} \quad \text{is the friction velocity}$$

Solution: We will use the Buckingham pi-theorem.

1 $\quad \overline{u} \quad \tau_w \quad y \quad \rho \quad \mu$ $\qquad$ n = 5 parameters

2 Select primary dimensions M, L, t:

3 $\quad \overline{u} \quad \tau_w \quad y \quad \rho \quad \mu$

$\quad \frac{L}{t} \quad \frac{M}{Lt^2} \quad L \quad \frac{M}{L^3} \quad \frac{M}{Lt}$ $\qquad$ r = 3 dimensions

4 $\quad \rho \quad y \quad \tau_w$ $\qquad$ m = r = 3 repeating parameters

5 We have n - m = 2 dimensionless groups. Setting up dimensional equations:

$$\Pi_1 = \overline{u}\rho^a y^b \tau_w^c \qquad \text{Thus:} \quad \frac{L}{t}\cdot\left(\frac{M}{L^3}\right)^a\cdot L^b\cdot\left(\frac{M}{L\cdot t^2}\right)^c = M^0\cdot L^0\cdot t^0$$

Summing exponents:

M: $a + c = 0$

L: $1 - 3\cdot a + b - c = 0$

t: $-1 - 2\cdot c = 0$

The solution to this system is:

$$a = \frac{1}{2} \quad b = 0 \quad c = -\frac{1}{2}$$

$$\Pi_1 = \overline{u}\sqrt{\frac{\rho}{\tau_w}} = \frac{\overline{u}}{u_*}$$

Check using F, L, t dimensions: $\left(\frac{L}{t}\right)\cdot\left(\frac{t}{L}\right) = 1$

$$\Pi_2 = \mu\rho^a y^b \tau_w^c \qquad \text{Thus:} \quad \frac{M}{L\cdot t}\cdot\left(\frac{M}{L^3}\right)^a\cdot L^b\cdot\left(\frac{M}{L\cdot t^2}\right)^c = M^0\cdot L^0\cdot t^0$$

Summing exponents:

M: $1 + a + c = 0$

L: $-1 - 3 \cdot a + b - c = 0$

t: $-1 - 2 \cdot c = 0$

The solution to this system is:

$a = -\dfrac{1}{2}$ $\quad b = -1$ $\quad c = -\dfrac{1}{2}$

$$\Pi_2 = \frac{\mu}{y\sqrt{\rho\tau_w}} = \frac{\mu}{\rho y}\sqrt{\frac{\rho}{\tau_w}} = \frac{\mu}{\rho y u_*} = \frac{\nu}{y u_*}$$

Π_2 is the reciprocal of the Reynolds number, so we know that it checks out.

The functional relationship is: $\Pi_1 = g(\Pi_2)$ $\quad \dfrac{\overline{u}}{u_*} = g\left(\dfrac{\nu}{y u_*}\right)$ which may be rewritten as: $\dfrac{\overline{u}}{u_*} = f\left(\dfrac{y u_*}{\nu}\right)$

Problem 7.22

[Difficulty: 2]

7.22 The energy released during an explosion, E, is a function of the time after detonation t, the blast radius R at time t, and the ambient air pressure p, and density ρ. Determine, by dimensional analysis, the general form of the expression for E in terms of the other variables.

(The solution to this problem was first devised by G.I. Taylor in the paper "The formation of a blast wave by a very intense explosion. I. Theoretical discussion," Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences, Vol. 201, No. 1065, pages 159 - 174 (22 March 1950).)

Given: Functional relationship between the energy released by an explosion and other physical parameters

Find: Expression for E in terms of the other variables

Solution: We will use the Buckingham pi-theorem.

1 $\quad E \quad t \quad R \quad p \quad \rho$ $\qquad$ n = 5 parameters

2 $\quad$ Select primary dimensions M, L, t:

3 $\quad E \quad t \quad R \quad p \quad \rho$

$$\frac{M\cdot L^2}{t^2} \quad t \quad L \quad \frac{M}{L\cdot t^2} \quad \frac{M}{L^3}$$ r = 3 dimensions

4 $\quad \rho \quad t \quad R$ $\qquad$ m = r = 3 repeating parameters

5 $\quad$ We have n - m = 2 dimensionless groups. Setting up dimensional equations:

$$\Pi_1 = E\cdot\rho^a\cdot t^b\cdot R^c \qquad \text{Thus:} \qquad \frac{M\cdot L^2}{t^2}\cdot\left(\frac{M}{L^3}\right)^a\cdot t^b\cdot L^c = M^0\cdot L^0\cdot t^0$$

Summing exponents:

M: $\quad 1 + a = 0$

L: $\quad 2 - 3\cdot a + c = 0$

t: $\quad -2 + b = 0$

The solution to this system is:

$a = -1 \qquad b = 2 \qquad c = -5$

$$\Pi_1 = \frac{E\cdot t^2}{\rho\cdot R^5}$$

Check using F, L, t dimensions: $\quad F\cdot L\cdot\frac{L^4}{F\cdot t^2}\cdot t^2\cdot\frac{1}{L^5} = 1$

$$\Pi_2 = p\cdot\rho^a\cdot t^b\cdot R^c \qquad \text{Thus:} \qquad \frac{M}{L\cdot t^2}\cdot\left(\frac{M}{L^3}\right)^a\cdot t^b\cdot L^c = M^0\cdot L^0\cdot t^0$$

Summing exponents:

M: $\quad 1 + a = 0$

L: $\quad -1 - 3\cdot a + c = 0$

t: $\quad -2 + b = 0$

The solution to this system is:

$a = -1 \qquad b = 2 \qquad c = -2$

$$\Pi_2 = \frac{p\cdot t^2}{\rho\cdot R^2}$$

Check using F, L, t dimensions: $\quad \frac{F}{L^2}\cdot\frac{L^4}{F\cdot t^2}\cdot t^2\cdot\frac{1}{L^2} = 1$

The functional relationship is: $\quad \Pi_1 = f(\Pi_2) \qquad \frac{E\cdot t^2}{\rho\cdot R^5} = f\left(\frac{p\cdot t^2}{\rho\cdot R^2}\right) \qquad E = \frac{\rho\cdot R^5}{t^2}\cdot f\left(\frac{p\cdot t^2}{\rho\cdot R^2}\right)$

Problem 7.23

[Difficulty: 2]

7.23 Capillary waves are formed on a liquid free surface as a result of surface tension. They have short wavelengths. The speed of a capillary wave depends on surface tension, σ, wavelength, λ, and liquid density, ρ. Use dimensional analysis to express wave speed as a function of these variables.

Given: Functional relationship between the speed of a capillary wave and other physical parameters

Find: An expression for V based on the other variables

Solution: We will use the Buckingham pi-theorem.

1 $\quad V \quad \sigma \quad \lambda \quad \rho$ $\qquad$ n = 4 parameters

2 $\quad$ Select primary dimensions M, L, t:

3 $\quad V \quad \sigma \quad \lambda \quad \rho$

$$\frac{L}{t} \qquad \frac{M}{t^2} \qquad L \qquad \frac{M}{L^3}$$

r = 3 dimensions

4 $\quad \sigma \quad \lambda \quad \rho$ $\qquad$ m = r = 3 repeating parameters

5 $\quad$ We have n - m = 1 dimensionless group. Setting up dimensional equations:

$$\Pi_1 = V\cdot\sigma^a\cdot\lambda^b\cdot\rho^c \qquad \text{Thus:} \qquad \frac{L}{t}\cdot\left(\frac{M}{t^2}\right)^a\cdot L^b\cdot\left(\frac{M}{L^3}\right)^c = L^0\cdot t^0$$

Summing exponents:

M: $\quad a + c = 0$

L: $\quad 1 + b - 3\cdot c = 0$

t: $\quad -1 - 2\cdot a = 0$

The solution to this system is:

$$a = -\frac{1}{2} \qquad b = \frac{1}{2} \qquad c = \frac{1}{2}$$

$$\Pi_1 = V\cdot\sqrt{\frac{\lambda\cdot\rho}{\sigma}}$$

Check using F, L, t dimensions: $\quad \left(\frac{L}{t}\right)\cdot\sqrt{L\cdot\frac{F\cdot t^2}{L^4}\cdot\frac{L}{F}} = 1$

The functional relationship is: $\quad \Pi_1 = C \qquad V\cdot\sqrt{\frac{\lambda\cdot\rho}{\sigma}} = C$

Therefore the velocity is: $\quad V = C\cdot\sqrt{\frac{\sigma}{\lambda\cdot\rho}}$

Problem 7.24

[Difficulty: 2]

7.24 Measurements of the liquid height upstream from an obstruction placed in an open-channel flow can be used to determine volume flow rate. (Such obstructions, designed and calibrated to measure rate of open-channel flow, are called *weirs*.) Assume the volume flow rate, Q, over a weir is a function of upstream height, h, gravity, g, and channel width, b. Use dimensional analysis to find the functional dependence of Q on the other variables.

Given: Functional relationship between the flow rate over a weir and physical parameters

Find: An expression for Q based on the other variables

Solution: We will use the Buckingham pi-theorem.

1 $\quad Q \quad h \quad g \quad b \qquad\qquad n = 5$ parameters

2 Select primary dimensions L, t:

3 $\quad Q \quad h \quad g \quad b$

$$\frac{L^3}{t} \qquad L \qquad \frac{L}{t^2} \qquad L \qquad\qquad r = 2 \text{ dimensions}$$

4 $\quad h \quad g \qquad\qquad m = r = 2$ repeating parameters

5 We have n - m = 2 dimensionless groups. Setting up dimensional equations:

$$\Pi_1 = Q\cdot h^a\cdot g^b \qquad \text{Thus:} \qquad \frac{L^3}{t}\cdot L^a\cdot\left(\frac{L}{t^2}\right)^b = L^0\cdot t^0$$

Summing exponents:

L: $\quad 3 + a + b = 0$

t: $\quad -1 - 2\cdot b = 0$

The solution to this system is:

$$a = -\frac{5}{2} \qquad b = -\frac{1}{2}$$

$$\Pi_1 = \frac{Q}{h^2\cdot\sqrt{g\cdot h}}$$

Check: $\left(\frac{L^3}{t}\right)\cdot\left(\frac{1}{L}\right)^2\cdot\left(\frac{t}{L}\right) = 1$

$$\Pi_2 = b\cdot h^a\cdot g^b \qquad \text{Thus:} \qquad L\cdot L^a\cdot\left(\frac{L}{t^2}\right)^b = L^0\cdot t^0$$

Summing exponents:

L: $\quad 1 + a + b = 0$

t: $\quad -2\cdot b = 0$

The solution to this system is:

$$a = -1 \qquad b = 0$$

$$\Pi_2 = \frac{b}{h}$$

Check: $L\cdot\frac{1}{L} = 1$

The functional relationship is: $\Pi_1 = f(\Pi_2) \qquad \frac{Q}{h^2\cdot\sqrt{g\cdot D}} = f\left(\frac{b}{h}\right)$ Therefore the flow rate is: $Q = h^2\cdot\sqrt{g\cdot h}\cdot f\left(\frac{b}{h}\right)$

Problem 7.25

[Difficulty: 2]

7.25 The torque, T, of a handheld automobile buffer is a function of rotational speed, ω, applied normal force, F, automobile surface roughness, e, buffing paste viscosity, μ, and surface tension, σ. Determine the dimensionless parameters that characterize this problem.

Given: That automobile buffer depends on several parameters

Find: Dimensionless groups

Solution:

Apply the Buckingham Π procedure

① $T \quad \omega \quad F \quad e \quad \mu \quad \sigma$ $\qquad n = 6$ parameters

② Select primary dimensions M, L, t

③ $$\left\{ \begin{matrix} T & \omega & F & e & \mu & \sigma \\ \dfrac{ML^2}{t^2} & \dfrac{1}{t} & \dfrac{ML}{t^2} & L & \dfrac{M}{Lt} & \dfrac{M}{t^2} \end{matrix} \right\}$$ $r = 3$ primary dimensions

④ $F \quad e \quad \omega$ $\qquad m = r = 3$ repeat parameters

⑤ Then $n - m = 3$ dimensionless groups will result. Setting up a dimensional equation,

$$\Pi_1 = F^a e^b \omega^c T = \left(\frac{ML}{t^2}\right)^a (L)^b \left(\frac{1}{t}\right)^c \frac{ML^2}{t^2} = M^0 L^0 t^0$$

Summing exponents,

$$\begin{array}{rr|l} M: & a+1=0 & a=-1 \\ L: & a+b+2=0 & b=-1 \\ t: & -2a-c-2=0 & c=0 \end{array} \qquad \text{Hence} \qquad \Pi_1 = \frac{T}{Fe}$$

$$\Pi_2 = F^a e^b \omega^c \mu = \left(\frac{ML}{t^2}\right)^a (L)^b \left(\frac{1}{t}\right)^c \frac{M}{Lt} = M^0 L^0 t^0$$

Summing exponents,

$$\begin{array}{rr|l} M: & a+1=0 & a=-1 \\ L: & a+b-1=0 & b=2 \\ t: & -2a-c-1=0 & c=1 \end{array} \qquad \text{Hence} \qquad \Pi_2 = \frac{\mu e^2 \omega}{F}$$

$$\Pi_3 = F^a e^b \omega^c \sigma = \left(\frac{ML}{t^2}\right)^a (L)^b \left(\frac{1}{t}\right)^c \frac{M}{t^2} = M^0 L^0 t^0$$

Summing exponents,

$$\begin{array}{rr|l} M: & a+1=0 & a=-1 \\ L: & a+b=0 & b=1 \\ t: & -2a-c-2=0 & c=0 \end{array} \qquad \text{Hence} \qquad \Pi_3 = \frac{\sigma e}{F}$$

⑥ Check using F, L, t as primary dimensions

$$\Pi_1 = \frac{FL}{FL} = [1] \qquad \Pi_2 = \frac{\frac{Ft}{L^2} L^2 \frac{1}{t}}{F} = [1] \qquad \Pi_3 = \frac{\frac{F}{L} L}{F} = [1]$$

Note: Any combination of Π_1, Π_2 and Π_3 is a Π group, e.g., $\dfrac{\Pi_1}{\Pi_2} = \dfrac{T}{\mu \omega e^3}$, so Π_1, Π_2 and Π_3 are not unique!

Problem 7.26

[Difficulty: 2]

7.26 The power, $\mathscr{P}$, used by a vacuum cleaner is to be correlated with the amount of suction provided (indicated by the pressure drop, Δp, below the ambient room pressure). It also depends on impeller diameter, D, and width, d, motor speed, ω, air density, ρ, and cleaner inlet and exit widths, d_i and d_o, respectively. Determine the dimensionless parameters that characterize this problem.

Given: That the power of a vacuum depends on various parameters

Find: Dimensionless groups

Solution:

Apply the Buckingham Π procedure

① $\mathscr{P}$ Δp D d ω ρ d_i d_o $n = 8$ parameters

② Select primary dimensions M, L, t

③
$$\left\{\begin{array}{cccccccc} \mathscr{P} & \Delta p & D & d & \omega & \rho & d_i & d_o \\ \frac{ML^2}{t^3} & \frac{M}{Lt^2} & L & L & \frac{1}{t} & \frac{M}{L^3} & L & L \end{array}\right\}$$
$r = 3$ primary dimensions

④ ρ D ω $m = r = 3$ repeat parameters

⑤ Then $n - m = 5$ dimensionless groups will result. Setting up a dimensional equation,

$$\Pi_1 = \rho^a D^b \omega^c \mathscr{P} = \left(\frac{M}{L^3}\right)^a (L)^b \left(\frac{1}{t}\right)^c \frac{ML^2}{t^3} = M^0 L^0 t^0$$

Summing exponents,
$$\begin{array}{ll|l} M: & a + 1 = 0 & a = -1 \\ L: & -3a + b + 2 = 0 & b = -5 \\ t: & -c - 3 = 0 & c = -3 \end{array}$$
Hence $\Pi_1 = \dfrac{\mathscr{P}}{\rho D^5 \omega^3}$

$$\Pi_2 = \rho^a D^b \omega^c \Delta p = \left(\frac{M}{L^3}\right)^a (L)^b \left(\frac{1}{t}\right)^c \frac{M}{Lt^2} = M^0 L^0 t^0$$

Summing exponents,
$$\begin{array}{ll|l} M: & a + 1 = 0 & a = -1 \\ L: & -3a + b - 1 = 0 & b = -2 \\ t: & -c - 2 = 0 & c = -2 \end{array}$$
Hence $\Pi_2 = \dfrac{\Delta p}{\rho D^2 \omega^2}$

The other Π groups can be found by inspection: $\Pi_3 = \dfrac{d}{D}$ $\Pi_4 = \dfrac{d_i}{D}$ $\Pi_5 = \dfrac{d_o}{D}$

⑥ Check using F, L, t as primary dimensions

$$\Pi_1 = \frac{\frac{FL}{t}}{\frac{Ft^2}{L^4} L^5 \frac{1}{t^3}} = [1] \qquad \Pi_2 = \frac{\frac{F}{L^2}}{\frac{Ft^2}{L^4} L^2 \frac{1}{t^2}} = [1] \qquad \Pi_3 = \Pi_4 = \Pi_5 = \frac{L}{L} = [1]$$

Note: Any combination of Π_1, Π_2 and Π_3 is a Π group, e.g., $\dfrac{\Pi_1}{\Pi_2} = \dfrac{\mathscr{P}}{\Delta p D^3 \omega}$, so the Π's are not unique!

Problem 7.27

[Difficulty: 2]

7.27 The load-carrying capacity, W, of a journal bearing is known to depend on its diameter, D, length, l, and clearance, c, in addition to its angular speed, ω, and lubricant viscosity, μ. Determine the dimensionless parameters that characterize this problem.

Given: Functional relationship between the load bearing capacity of a journal bearing and other physical parameters

Find: Dimensionless parameters that characterize the problem.

Solution: We will use the Buckingham pi-theorem.

1 $W \quad D \quad l \quad c \quad \omega \quad \mu$ $\qquad$ $n = 6$ parameters

2 Select primary dimensions F, L, t:

3

W	D	l	c	ω	μ	
F	L	L	L	$\frac{1}{t}$	$\frac{F\cdot t}{L^2}$	$r = 3$ dimensions

4 $D \quad \omega \quad \mu$ $\qquad$ $m = r = 3$ repeating parameters

5 We have n - m = 3 dimensionless groups. Setting up dimensional equations:

$$\Pi_1 = W\cdot D^a\cdot\omega^b\cdot\mu^c \qquad \text{Thus:} \qquad F\cdot L^a\cdot\left(\frac{1}{t}\right)^b\cdot\left(\frac{F\cdot t}{L^2}\right)^c = F^0\cdot L^0\cdot t^0$$

Summing exponents:

F: $1 + c = 0$

L: $a - 2\cdot c = 0$

t: $-b + c = 0$

The solution to this system is:

$a = -2 \qquad b = -1 \qquad c = -1$

$$\Pi_1 = \frac{W}{D^2\cdot\omega\cdot\mu}$$

Check using M, L, t dimensions: $\frac{M\cdot L}{t^2}\cdot\frac{1}{L^2}\cdot t\cdot\frac{L\cdot t}{M} = 1$ $\qquad$ By inspection, we can see that: $\Pi_2 = \frac{l}{D} \qquad \Pi_3 = \frac{c}{D}$

The functional relationship is: $\Pi_1 = f(\Pi_2,\Pi_3)$

$$\frac{W}{D^2\cdot\omega\cdot\mu} = f\left(\frac{l}{D},\frac{c}{D}\right)$$

Problem 7.28 (In Excel)

[Difficulty: 2]

7.28 The time, t, for oil to drain out of a viscosity calibration container depends on the fluid viscosity, μ, and density, ρ, the orifice diameter, d, and gravity, g. Use dimensional analysis to find the functional dependence of t on the other variables. Express t in the simplest possible form.

Given: That drain time depends on fluid viscosity and density, orifice diameter, and gravity

Find: Functional dependence of t on other variables

Solution:

We will use the workbook of Example 7.1, modified for the current problem

The number of parameters is: $n = 5$

The number of primary dimensions is: $r = 3$

The number of repeat parameters is: $m = r = 3$

The number of Π groups is: $n - m = 2$

Enter the dimensions (**M**, **L**, **t**) of
the repeating parameters, and of up to
four other parameters (for up to four Π groups).
The spreadsheet will compute the exponents a, b, and c for each.

REPEATING PARAMETERS: Choose ρ, g, d

	M	L	t
ρ	1	-3	
g		1	-2
d		1	

Π GROUPS:

	M	L	t
t	0	0	1

Π_1: $a = 0$, $b = 0.5$, $c = -0.5$

	M	L	t
μ	1	-1	-1

Π_2: $a = -1$, $b = -0.5$, $c = -1.5$

The following Π groups from Example 7.1 are not used:

M	L	t
0	0	0

Π_3: $a = 0$, $b = 0$, $c = 0$

M	L	t
0	0	0

Π_4: $a = 0$, $b = 0$, $c = 0$

Hence $\Pi_1 = t\sqrt{\frac{g}{d}}$ and $\Pi_2 = \frac{\mu}{\rho g^{\frac{1}{2}} d^{\frac{3}{2}}} \rightarrow \frac{\mu^2}{\rho^2 g d^3}$ with $\Pi_1 = f(\Pi_2)$

The final result is $t = \sqrt{\frac{d}{g}}\, f\left(\frac{\mu^2}{\rho^2 g d^3}\right)$

Problem 7.29

[Difficulty: 2]

7.29 The power per unit cross-sectional area, E, transmitted by a sound wave is a function of wave speed, V, medium density, ρ, wave amplitude, r, and wave frequency, n. Determine, by dimensional analysis, the general form of the expression for E in terms of the other variables.

Given: Functional relationship between the power transmited by a sound wave and other physical parameters

Find: Expression for E in terms of the other variables

Solution: We will use the Buckingham pi-theorem.

1 $E \quad V \quad \rho \quad r \quad n$ n = 5 parameters

2 Select primary dimensions M, L, t:

3 $E \quad V \quad \rho \quad r \quad n$

$$\frac{M}{t^3} \quad \frac{L}{t} \quad \frac{M}{L^3} \quad L \quad \frac{1}{t}$$

r = 3 dimensions

4 $\rho \quad V \quad r$ m = r = 3 repeating parameters

5 We have n - m = 2 dimensionless groups. Setting up dimensional equations:

$$\Pi_1 = E\cdot\rho^a\cdot V^b\cdot r^c \qquad \text{Thus:} \qquad \frac{M}{t^3}\cdot\left(\frac{M}{L^3}\right)^a\cdot\left(\frac{L}{t}\right)^b\cdot L^c = M^0\cdot L^0\cdot t^0$$

Summing exponents:

M: $1 + a = 0$

L: $-3\cdot a + b + c = 0$

t: $-3 - b = 0$

The solution to this system is:

$a = -1 \qquad b = -3 \qquad c = 0$

$$\Pi_1 = \frac{E}{\rho\cdot V^3}$$

Check using F, L, t dimensions: $\frac{F}{L\cdot t}\cdot\frac{L^4}{F\cdot t^2}\cdot\frac{t^3}{L^3} = 1$

$$\Pi_2 = n\cdot\rho^a\cdot V^b\cdot r^c \qquad \text{Thus:} \qquad \frac{1}{t}\cdot\left(\frac{M}{L^3}\right)^a\cdot\left(\frac{L}{t}\right)^b\cdot L^c = M^0\cdot L^0\cdot t^0$$

Summing exponents:

M: $a = 0$

L: $-3\cdot a + b + c = 0$

t: $-1 - b = 0$

The solution to this system is:

$a = 0 \qquad b = -1 \qquad c = 1$

$$\Pi_2 = \frac{n\cdot r}{V}$$

Check using F, L, t dimensions: $\frac{1}{t}\cdot L\cdot\frac{t}{L} = 1$

The functional relationship is: $\Pi_1 = f(\Pi_2)$

$$\frac{E}{\rho\cdot V^3} = f\left(\frac{n\cdot r}{V}\right) \qquad E = \rho\cdot V^3\cdot f\left(\frac{n\cdot r}{V}\right)$$

Problem 7.30

[Difficulty: 2]

7.30 You are asked to find a set of dimensionless parameters to organize data from a laboratory experiment, in which a tank is drained through an orifice from initial liquid level h_0. The time, τ, to drain the tank depends on tank diameter, D, orifice diameter, d, acceleration of gravity, g, liquid density, ρ, and liquid viscosity, μ. How many dimensionless parameters will result? How many repeating variables must be selected to determine the dimensionless parameters? Obtain the Π parameter that contains the viscosity.

Given: Functional relationship between the time needed to drain a tank through an orifice plate and other physical parameters

Find:
(a) the number of dimensionless parameters
(b) the number of repeating variables
(c) the Π term which contains the viscosity

Solution: We will use the Buckingham pi-theorem.

1 $\tau \quad h_0 \quad D \quad d \quad g \quad \rho \quad \mu$ $\qquad$ n = 7 parameters

2 Select primary dimensions M, L, t:

3 $\tau \quad h_0 \quad D \quad d \quad g \quad \rho \quad \mu$

$$T \quad L \quad L \quad L \quad \frac{L}{t^2} \quad \frac{M}{L^3} \quad \frac{M}{L\cdot t}$$

r = 3 dimensions

4 $\rho \quad d \quad g$ $\qquad$ m = r = 3 repeating parameters $\qquad$ We have n - m = 4 dimensionless groups.

5 Setting up dimensional equation including the viscosity:

$$\Pi_1 = \mu\cdot\rho^a\cdot d^b\cdot g^c \qquad \text{Thus:} \qquad \frac{M}{L\cdot t}\cdot\left(\frac{M}{L^3}\right)^a\cdot L^b\cdot\left(\frac{L}{t^2}\right)^c = M^0\cdot L^0\cdot t^0$$

Summing exponents:

M: $1 + a = 0$

L: $-1 - 3\cdot a + b + c = 0$

t: $-1 - 2\cdot c = 0$

The solution to this system is:

$$a = -1 \qquad b = -\frac{3}{2} \qquad c = -\frac{1}{2}$$

$$\Pi_1 = \frac{\mu}{\rho\cdot d^{\frac{3}{2}}\cdot g^{\frac{1}{2}}}$$

Check using F, L, t dimensions:

$$\frac{F\cdot t}{L^2}\cdot\frac{L^4}{F\cdot t^2}\cdot\frac{1}{L^{\frac{3}{2}}}\cdot\frac{t}{L^{\frac{1}{2}}} = 1$$

Problem 7.31

[Difficulty: 3]

7.31 A continuous belt moving vertically through a bath of viscous liquid drags a layer of liquid, of thickness h, along with it. The volume flow rate of liquid, Q, is assumed to depend on μ, ρ, g, h, and V, where V is the belt speed. Apply dimensional analysis to predict the form of dependence of Q on the other variables.

Given: Functional relationship between the flow rate of viscous liquid dragged out of a bath and other physical parameters

Find: Expression for Q in terms of the other variables

Solution: We will use the Buckingham pi-theorem.

1 $\quad Q \quad \mu \quad \rho \quad g \quad h \quad V$ $\qquad$ n = 6 parameters

2 $\quad$ Select primary dimensions M, L, t:

3 $\quad Q \quad \mu \quad \rho \quad g \quad h \quad V$

$$\frac{L^3}{t} \quad \frac{M}{L\cdot t} \quad \frac{M}{L^3} \quad \frac{L}{t^2} \quad L \quad \frac{L}{t}$$

r = 3 dimensions

4 $\quad \rho \quad V \quad h$ $\qquad$ m = r = 3 repeating parameters

5 $\quad$ We have n - m = 3 dimensionless groups. Setting up dimensional equations:

$$\Pi_1 = Q\cdot\rho^a\cdot V^b\cdot h^c \qquad \text{Thus:} \qquad \frac{L^3}{t}\cdot\left(\frac{M}{L^3}\right)^a\cdot\left(\frac{L}{t}\right)^b\cdot L^c = M^0\cdot L^0\cdot t^0$$

Summing exponents:

M: $a = 0$

L: $3 - 3\cdot a + b + c = 0$

t: $-1 - b = 0$

The solution to this system is:

$a = 0 \qquad b = -1 \qquad c = -2$

$$\Pi_1 = \frac{Q}{V\cdot h^2}$$

Check using F, L, t dimensions: $\quad \dfrac{L^3}{t}\cdot\dfrac{t}{L}\cdot\dfrac{1}{L^2} = 1$

$$\Pi_2 = \mu\cdot\rho^a\cdot V^b\cdot h^c \qquad \text{Thus:} \qquad \frac{M}{L\cdot t}\cdot\left(\frac{M}{L^3}\right)^a\cdot\left(\frac{L}{t}\right)^b\cdot L^c = M^0\cdot L^0\cdot t^0$$

Summing exponents:

M: $1 + a = 0$

L: $-1 - 3\cdot a + b + c = 0$

t: $-1 - b = 0$

The solution to this system is:

$a = -1 \qquad b = -1 \qquad c = -1$

$$\Pi_2 = \frac{\mu}{\rho\cdot V\cdot h}$$

Check using F, L, t dimensions: $\quad \dfrac{F\cdot t}{L^2}\cdot\dfrac{L^4}{F\cdot t^2}\cdot\dfrac{t}{L}\cdot\dfrac{1}{L} = 1$

$\Pi_3 = g \cdot \rho^a \cdot V^b \cdot h^c$ Thus: $\frac{L}{t^2} \cdot \left(\frac{M}{L^3}\right)^a \cdot \left(\frac{L}{t}\right)^b \cdot L^c = M^0 \cdot L^0 \cdot t^0$

Summing exponents:

M: $a = 0$

L: $1 - 3 \cdot a + b + c = 0$

t: $-2 - b = 0$

The solution to this system is:

$a = 0 \qquad b = -2 \qquad c = 1$

$$\Pi_3 = \frac{g \cdot h}{V^2}$$

Check using F, L, t dimensions: $\frac{L}{t^2} \cdot L \cdot \frac{t^2}{L^2} = 1$

The functional relationship is: $\Pi_1 = f(\Pi_2, \Pi_3)$ $\qquad \frac{Q}{V \cdot h^2} = f\left(\frac{\rho \cdot V \cdot h}{\mu}, \frac{V^2}{g \cdot h}\right)$ $\qquad Q = V \cdot h^2 \cdot f\left(\frac{\rho \cdot V \cdot h}{\mu}, \frac{V^2}{g \cdot h}\right)$

Problem 7.32

[Difficulty: 2]

7.32 The power, $\mathcal{P}$, required to drive a fan is believed to depend on fluid density, ρ, volume flow rate, Q, impeller diameter, D, and angular velocity, ω. Use dimensional analysis to determine the dependence of $\mathcal{P}$ on the other variables.

Given: Functional relationship between the power required to drive a fan and other physical parameters

Find: Expression for P in terms of the other variables

Solution: We will use the Buckingham pi-theorem.

1 $P \quad \rho \quad Q \quad D \quad \omega$ — n = 5 parameters

2 Select primary dimensions M, L, t:

3 $P \quad \rho \quad Q \quad D \quad \omega$

$$\frac{M\cdot L^2}{t^3} \quad \frac{M}{L^3} \quad \frac{L^3}{t} \quad L \quad \frac{1}{t}$$

r = 3 dimensions

4 $\rho \quad D \quad \omega$ — m = r = 3 repeating parameters

5 We have n - m = 2 dimensionless groups. Setting up dimensional equations:

$$\Pi_1 = P\cdot\rho^a\cdot D^b\cdot\omega^c \qquad \text{Thus:} \quad \frac{M\cdot L^2}{t^3}\cdot\left(\frac{M}{L^3}\right)^a\cdot L^b\cdot\left(\frac{1}{t}\right)^c = M^0\cdot L^0\cdot t^0$$

Summing exponents:

M: $1 + a = 0$

L: $2 - 3\cdot a + b = 0$

t: $-3 - c = 0$

The solution to this system is:

$a = -1 \qquad b = -5 \qquad c = -3$

$$\Pi_1 = \frac{P}{\rho\cdot D^5\cdot\omega^3}$$

Check using F, L, t dimensions: $\frac{F\cdot L}{t}\cdot\frac{L^4}{F\cdot t^2}\cdot\frac{1}{L^5}\cdot t^3 = 1$

$$\Pi_2 = Q\cdot\rho^a\cdot D^b\cdot\omega^c \qquad \text{Thus:} \quad \frac{L^3}{t}\cdot\left(\frac{M}{L^3}\right)^a\cdot L^b\cdot\left(\frac{1}{t}\right)^c = M^0\cdot L^0\cdot t^0$$

Summing exponents:

M: $a = 0$

L: $3 - 3\cdot a + b = 0$

t: $-1 - c = 0$

The solution to this system is:

$a = 0 \qquad b = -3 \qquad c = -1$

$$\Pi_2 = \frac{Q}{D^3\cdot\omega}$$

Check using F, L, t dimensions: $\frac{1}{t}\cdot L\cdot\frac{t}{L} = 1$

The functional relationship is: $\Pi_1 = f(\Pi_2)$

$$\frac{P}{\rho\cdot D^5\cdot\omega^3} = f\left(\frac{Q}{D^3\cdot\omega}\right) \qquad P = \rho\cdot D^5\cdot\omega^3\cdot f\left(\frac{Q}{D^3\cdot\omega}\right)$$

Problem 7.33

[Difficulty: 2]

7.33 In a fluid mechanics laboratory experiment a tank of water, with diameter D, is drained from initial level h_0. The smoothly rounded drain hole has diameter d. Assume the mass flow rate from the tank is a function of h, D, d, g, ρ, and μ, where g is the acceleration of gravity and ρ and μ are fluid properties. Measured data are to be correlated in dimensionless form. Determine the number of dimensionless parameters that will result. Specify the number of repeating parameters that must be selected to determine the dimensionless parameters. Obtain the Π parameter that contains the viscosity.

Given: Functional relationship between the mass flow rate exiting a tank through a rounded drain hole and other physical parameters

Find:
(a) Number of dimensionless parameters that will result
(b) Number of repeating parameters
(c) The Π term that contains the viscosity

Solution: We will use the Buckingham pi-theorem.

1 $\quad m \quad h_0 \quad D \quad d \quad g \quad \rho \quad \mu$ $\qquad$ $n = 7$ parameters

2 $\quad$ Select primary dimensions M, L, t:

3

m	h_0	D	d	g	ρ	μ
$\frac{M}{t}$	L	L	L	$\frac{L}{t^2}$	$\frac{M}{L^3}$	$\frac{M}{L\cdot t}$

$r = 3$ dimensions
We have $n - r = 4$ dimensionless groups.

4 $\quad \rho \quad d \quad g$ $\qquad$ $m = r = 3$ repeating parameters

5 $\quad$ Setting up dimensional equation involving the viscosity:

$$\Pi_1 = \mu\cdot\rho^a\cdot d^b\cdot g^c \qquad \text{Thus:} \qquad \frac{M}{L\cdot t}\cdot\left(\frac{M}{L^3}\right)^a\cdot L^b\cdot\left(\frac{L}{t^2}\right)^c = M^0\cdot L^0\cdot t^0$$

Summing exponents:

M: $1 + a = 0$

L: $-1 - 3\cdot a + b + c = 0$

t: $-1 - 2\cdot c = 0$

The solution to this system is:

$$a = -1 \qquad b = -\frac{3}{2} \qquad c = -\frac{1}{2}$$

$$\Pi_1 = \frac{\mu}{\rho\cdot\sqrt{d^3\cdot g}}$$

Check using F, L, t dimensions:

$$\frac{F\cdot t}{L^2}\cdot\frac{L^4}{F\cdot t^2}\cdot\frac{1}{L^{\frac{3}{2}}}\cdot\frac{t}{L^{\frac{1}{2}}} = 1$$

Problem 7.34

[Difficulty: 3]

7.34 Cylindrical water tanks are frequently found on the tops of tall buildings. When a tank is filled with water, the bottom of the tank typically deflects under the weight of the water inside. The deflection δ is a function of the tank diameter D, the height of water h, the thickness of the tank bottom d, the specific weight of the water γ, and the modulus of elasticity of the tank material E. Determine the functional relationship among these parameters using dimensionless groups.

Given: Functional relationship between the deflection of the bottom of a cylindrical tank and other physical parameters

Find: Functional relationship between these parameters using dimensionless groups.

Solution: We will use the Buckingham pi-theorem.

1 $\delta \quad D \quad h \quad d \quad \gamma \quad E$ — $n = 6$ parameters

2 Select primary dimensions F, L, t:

3

δ	D	h	d	γ	E
L	L	L	L	$\frac{F}{L^3}$	$\frac{F}{L^2}$

$r = 2$ dimensions

4 $D \quad \gamma$ — $m = r = 2$ repeating parameters — We have $n - m = 4$ dimensionless groups.

5 Setting up dimensional equations:

$$\Pi_1 = \delta \cdot D^a \cdot \gamma^b \qquad \text{Thus:} \qquad L \cdot L^a \cdot \left(\frac{F}{L^3}\right)^b = F^0 \cdot L^0$$

Summing exponents:

F: $b = 0$

L: $1 + a - 3 \cdot b = 0$

The solution to this system is:

$a = -1 \qquad b = 0$

$$\Pi_1 = \frac{\delta}{D}$$

Check using M, L, t dimensions: $L \cdot \frac{1}{L} = 1$

Now since h and d have the same dimensions as δ, it would follow that the the next two pi terms would be: $\Pi_2 = \frac{h}{D} \qquad \Pi_3 = \frac{d}{D}$

$$\Pi_4 = E \cdot D^a \cdot \gamma^b \qquad \text{Thus:} \qquad \frac{F}{L^2} \cdot L^a \cdot \left(\frac{F}{L^3}\right)^b = F^0 \cdot L^0$$

Summing exponents:

F: $1 + b = 0$

L: $-2 + a - 3 \cdot b = 0$

The solution to this system is:

$a = -1 \qquad b = -1$

$$\Pi_4 = \frac{E}{D \cdot \gamma}$$

Check using M, L, t dimensions: $\frac{M}{L \cdot t^2} \cdot \frac{1}{L} \cdot \frac{L^2 \cdot t^2}{M} = 1$

The functional relationship is: $\Pi_1 = f(\Pi_2, \Pi_3, \Pi_4)$

$$\frac{\delta}{D} = f\left(\frac{h}{D}, \frac{d}{D}, \frac{E}{D \cdot \gamma}\right)$$

(For further reading, one should consult an appropriate text, such as *Advanced Strength of Materials* by Cook and Young)

Problem 7.35

[Difficulty: 3]

7.35 Small droplets of liquid are formed when a liquid jet breaks up in spray and fuel injection processes. The resulting droplet diameter, d, is thought to depend on liquid density, viscosity, and surface tension, as well as jet speed, V, and diameter, D. How many dimensionless ratios are required to characterize this process? Determine these ratios.

Given: Functional relationship between the diameter of droplets formed during jet breakup and other physical parameters

Find: (a) The number of dimensionless parameters needed to characterize the process
(b) The ratios (Π-terms)

Solution: We will use the Buckingham pi-theorem.

1 $\quad d \quad \rho \quad \mu \quad \sigma \quad V \quad D$ $\qquad$ $n = 6$ parameters

2 $\quad$ Select primary dimensions M, L, t:

3 $\quad d \quad \rho \quad \mu \quad \sigma \quad V \quad D$

$$L \qquad \frac{M}{L^3} \qquad \frac{M}{L\cdot t} \qquad \frac{M}{t^2} \qquad \frac{L}{t} \qquad L$$

$r = 3$ dimensions

4 $\quad \rho \quad V \quad D$ $\qquad$ $m = r = 3$ repeating parameters $\qquad$ We have $n - m = 3$ dimensionless groups.

5 $\quad$ Setting up dimensional equations:

$$\Pi_1 = d\cdot\rho^a\cdot V^b\cdot D^c \qquad \text{Thus:} \qquad L\cdot\left(\frac{M}{L^3}\right)^a\cdot\left(\frac{L}{t}\right)^b\cdot L^c = M^0\cdot L^0\cdot t^0$$

Summing exponents:

M: $a = 0$

L: $1 - 3\cdot a + b + c = 0$

t: $-b = 0$

The solution to this system is:

$a = 0 \qquad b = 0 \qquad c = -1$

$$\Pi_1 = \frac{d}{D}$$

Check using F, L, t dimensions: $L\cdot\frac{1}{L} = 1$

$$\Pi_2 = \mu\cdot\rho^a\cdot V^b\cdot D^c \qquad \text{Thus:} \qquad \frac{M}{L\cdot t}\cdot\left(\frac{M}{L^3}\right)^a\cdot\left(\frac{L}{t}\right)^b\cdot L^c = M^0\cdot L^0\cdot t^0$$

Summing exponents:

M: $1 + a = 0$

L: $-1 - 3\cdot a + b + c = 0$

t: $-1 - b = 0$

The solution to this system is:

$a = -1 \qquad b = -1 \qquad c = -1$

$$\Pi_2 = \frac{\mu}{\rho\cdot V\cdot D}$$

Check using F, L, t dimensions: $\frac{F\cdot t}{L^2}\cdot\frac{L^4}{F\cdot t^2}\cdot\frac{t}{L}\cdot\frac{1}{L} = 1$

$$\Pi_3 = \sigma\cdot\rho^a\cdot V^b\cdot D^c \qquad \text{Thus:} \qquad \frac{M}{t^2}\cdot\left(\frac{M}{L^3}\right)^a\cdot\left(\frac{L}{t}\right)^b\cdot L^c = M^0\cdot L^0\cdot t^0$$

Summing exponents:

M: $1 + a = 0$

L: $-3\cdot a + b + c = 0$

t: $-2 - b = 0$

The solution to this system is:

$a = -1 \qquad b = -2 \qquad c = -1$

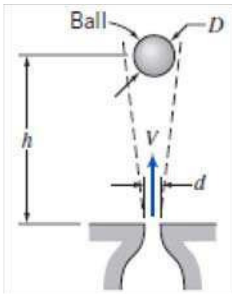
$$\Pi_3 = \frac{\sigma}{\rho\cdot V^2\cdot D}$$

Check using F, L, t dimensions: $\frac{F}{L}\cdot\frac{L^4}{F\cdot t^2}\cdot\frac{t^2}{L^2}\cdot\frac{1}{L} = 1$

Problem 7.36

[Difficulty: 3]

7.36 The sketch shows an air jet discharging vertically. Experiments show that a ball placed in the jet is suspended in a stable position. The equilibrium height of the ball in the jet is found to depend on D, d, V, ρ, μ, and W, where W is the weight of the ball. Dimensional analysis is suggested to correlate experimental data. Find the Π parameters that characterize this phenomenon.



Given: Functional relationship between the height of a ball suported by a vertical air jet and other physical parameters

Find: The Π terms that characterize this phenomenon

Solution: We will use the Buckingham pi-theorem.

1 $h \quad D \quad d \quad V \quad \rho \quad \mu \quad W$ $\qquad n = 7$ parameters

2 Select primary dimensions M, L, t:

3 $h \quad D \quad d \quad V \quad \rho \quad \mu \quad W$

$L \quad L \quad L \quad \frac{L}{t} \quad \frac{M}{L^3} \quad \frac{M}{L \cdot t} \quad \frac{M \cdot L}{t^2}$ $\qquad r = 3$ dimensions

4 $\rho \quad V \quad d$ $\qquad m = r = 3$ repeating parameters

5 We have n - m = 4 dimensionless groups. Setting up dimensional equations:

$$\Pi_1 = h \cdot \rho^a \cdot V^b \cdot d^c \qquad \text{Thus:} \qquad L \cdot \left(\frac{M}{L^3}\right)^a \cdot \left(\frac{L}{t}\right)^b \cdot L^c = M^0 \cdot L^0 \cdot t^0$$

Summing exponents:

M: $a = 0$

L: $1 - 3 \cdot a + b + c = 0$

t: $-b = 0$

The solution to this system is:

$a = 0 \qquad b = 0 \qquad c = -1$

$$\Pi_1 = \frac{h}{d}$$

Check using F, L, t dimensions: $L \cdot \frac{1}{L} = 1$

$$\Pi_2 = D \cdot \rho^a \cdot V^b \cdot d^c \qquad \text{Thus:} \qquad L \cdot \left(\frac{M}{L^3}\right)^a \cdot \left(\frac{L}{t}\right)^b \cdot L^c = M^0 \cdot L^0 \cdot t^0$$

Summing exponents:

M: $a = 0$

L: $1 - 3 \cdot a + b + c = 0$

t: $-b = 0$

The solution to this system is:

$a = 0 \qquad b = 0 \qquad c = -1$

$$\Pi_2 = \frac{D}{d}$$

Check using F, L, t dimensions: $L \cdot \frac{1}{L} = 1$

$$\Pi_3 = \mu\cdot\rho^a\cdot V^b\cdot d^c$$

Thus: $$\frac{M}{L\cdot t}\cdot\left(\frac{M}{L^3}\right)^a\cdot\left(\frac{L}{t}\right)^b\cdot L^c = M^0\cdot L^0\cdot t^0$$

Summing exponents:

M: $1 + a = 0$

L: $-1 - 3\cdot a + b + c = 0$

t: $-1 - b = 0$

The solution to this system is:

$a = -1 \qquad b = -1 \qquad c = -1$

$$\Pi_3 = \frac{\mu}{\rho\cdot V\cdot d}$$

Check using F, L, t dimensions: $$\frac{F\cdot t}{L^2}\cdot\frac{L^4}{F\cdot t^2}\cdot\frac{t}{L}\cdot\frac{1}{L} = 1$$

$$\Pi_4 = W\cdot\rho^a\cdot V^b\cdot d^c$$

Thus: $$\frac{M\cdot L}{t^2}\cdot\left(\frac{M}{L^3}\right)^a\cdot\left(\frac{L}{t}\right)^b\cdot L^c = M^0\cdot L^0\cdot t^0$$

Summing exponents:

M: $1 + a = 0$

L: $1 - 3\cdot a + b + c = 0$

t: $-2 - b = 0$

The solution to this system is:

$a = -1 \qquad b = -2 \qquad c = -2$

$$\Pi_4 = \frac{W}{\rho\cdot V^2\cdot d^2}$$

Check using F, L, t dimensions: $$F\cdot\frac{L^4}{F\cdot t^2}\cdot\frac{t^2}{L^2}\cdot\frac{1}{L^2} = 1$$

Problem 7.37 (In Excel)

[Difficulty: 3]

7.37 The diameter, d, of the dots made by an ink jet printer depends on the ink viscosity, μ, density, ρ, and surface tension, σ, the nozzle diameter, D, the distance, L, of the nozzle from the paper surface, and the ink jet velocity, V. Use dimensional analysis to find the Π parameters that characterize the ink jet's behavior.

Given: That dot size depends on ink viscosity, density, and surface tension, and geometry

Find: Π groups

Solution:

We will use the workbook of Example 7.1, modified for the current problem

The number of parameters is: $n = 7$

The number of primary dimensions is: $r = 3$

The number of repeat parameters is: $m = r = 3$

The number of Π groups is: $n - m = 4$

Enter the dimensions (**M**, **L**, **t**) of
the repeating parameters, and of up to
four other parameters (for up to four Π groups).
The spreadsheet will compute the exponents a, b, and c for each.

REPEATING PARAMETERS: **Choose ρ, V, D**

	M	L	t
ρ	1	-3	
V		1	-1
D		1	

Π GROUPS:

	M	L	t
d	0	1	0

Π_1: a = **0**, b = **0**, c = **-1**

	M	L	t
μ	1	-1	-1

Π_2: a = **-1**, b = **-1**, c = **-1**

	M	L	t
σ	1	0	-2

Π_3: a = **-1**, b = **-2**, c = **-1**

	M	L	t
L	0	1	0

Π_4: a = **0**, b = **0**, c = **-1**

Hence $\quad \Pi_1 = \dfrac{d}{D} \qquad \Pi_2 = \dfrac{\mu}{\rho V D} \rightarrow \dfrac{\rho V D}{\mu} \qquad \Pi_3 = \dfrac{\sigma}{\rho V^2 D} \qquad \Pi_4 = \dfrac{L}{D}$

Note that groups Π_1 and Π_4 can be obtained by inspection

Problem 7.38 (In Excel)

[Difficulty: 3]

7.38 The diameter, d, of bubbles produced by a bubble-making toy depends on the soapy water viscosity, μ, density, ρ, and surface tension, σ, the ring diameter, D, and the pressure differential, Δp, generating the bubbles. Use dimensional analysis to find the Π parameters that characterize this phenomenon.

Given: Bubble size depends on viscosity, density, surface tension, geometry and pressure

Find: Π groups

Solution:

We will use the workbook of Example 7.1, modified for the current problem

The number of parameters is:	$n = 6$
The number of primary dimensions is:	$r = 3$
The number of repeat parameters is:	$m = r = 3$
The number of Π groups is:	$n - m = 3$

Enter the dimensions (**M**, **L**, **t**) of
the repeating parameters, and of up to
four other parameters (for up to four Π groups).
The spreadsheet will compute the exponents a, b, and c for each.

REPEATING PARAMETERS: **Choose ρ, Δp, D**

	M	L	t
ρ	1	-3	
Δp	1	-1	-2
D		1	

Π GROUPS:

	M	L	t
d	0	1	0

Π_1: $a = 0$, $b = 0$, $c = -1$

	M	L	t
μ	1	-1	-1

Π_2: $a = -0.5$, $b = -0.5$, $c = -1$

	M	L	t
σ	1	0	-2

Π_3: $a = 0$, $b = -1$, $c = -1$

	M	L	t
	0	0	0

Π_4: $a = 0$, $b = 0$, $c = 0$

Hence $\quad \Pi_1 = \dfrac{d}{D} \qquad \Pi_2 = \dfrac{\mu}{\rho^{\frac{1}{2}} \Delta p^{\frac{1}{2}} D} \rightarrow \dfrac{\mu^2}{\rho \Delta p D^2} \qquad \Pi_3 = \dfrac{\sigma}{D \Delta p}$

Note that the Π_1 group can be obtained by inspection

Problem 7.39 (In Excel)

[Difficulty: 3]

7.39 The terminal speed V of shipping boxes sliding down an incline on a layer of air (injected through numerous pinholes in the incline surface) depends on the box mass, m, and base area, A, gravity, g, the incline angle, θ, the air viscosity, μ, and the air layer thickness, δ. Use dimensional analysis to find the Π parameters that characterize this phenomenon.

Given: Speed depends on mass, area, gravity, slope, and air viscosity and thickness

Find: Π groups

Solution:

We will use the workbook of Example 7.1, modified for the current problem

The number of parameters is:	$n = 7$
The number of primary dimensions is:	$r = 3$
The number of repeat parameters is:	$m = r = 3$
The number of Π groups is:	$n - m = 4$

Enter the dimensions (**M**, **L**, **t**) of
the repeating parameters, and of up to
four other parameters (for up to four Π groups).
The spreadsheet will compute the exponents a, b, and c for each.

REPEATING PARAMETERS: **Choose g, δ, m**

	M	L	t
g		1	-2
δ		1	
m	1		

Π GROUPS:

	M	L	t
V	0	1	-1

Π_1: $a = -0.5$, $b = -0.5$, $c = 0$

	M	L	t
μ	1	-1	-1

Π_2: $a = -0.5$, $b = 1.5$, $c = -1$

	M	L	t
θ	0	0	0

Π_3: $a = 0$, $b = 0$, $c = 0$

	M	L	t
A	0	2	0

Π_4: $a = 0$, $b = -2$, $c = 0$

Hence $\Pi_1 = \dfrac{V}{g^{\frac{1}{2}}\delta^{\frac{1}{2}}} \rightarrow \dfrac{V^2}{g\delta}$ $\quad \Pi_2 = \dfrac{\mu\delta^{\frac{3}{2}}}{g^{\frac{1}{2}}m} \rightarrow \dfrac{\mu^2\delta^3}{m^2 g}$ $\quad \Pi_3 = \theta$ $\quad \Pi_4 = \dfrac{A}{\delta^2}$

Note that the Π_1, Π_3 and Π_4 groups can be obtained by inspection

Problem 7.40

[Difficulty: 2]

7.40 The length of the wake w behind an airfoil is a function of the flow speed V, chord length L, thickness t, and fluid density ρ and viscosity μ. Find the dimensionless parameters that characterize this phenomenon.

Given: Functional relationship between the length of a wake behind an airfoil and other physical parameters

Find: The Π terms that characterize this phenomenon

Solution: We will use the Buckingham pi-theorem.

1 $\quad w \quad V \quad L \quad t \quad \rho \quad \mu$ $\qquad$ $n = 6$ parameters

2 $\quad$ Select primary dimensions M, L, t:

3

w	V	L	t	ρ	μ
L	$\frac{L}{t}$	L	L	$\frac{M}{L^3}$	$\frac{M}{L\cdot t}$

$r = 3$ dimensions

4 $\quad \rho \quad V \quad L$ $\qquad$ $m = r = 3$ repeating parameters

5 $\quad$ We have n - m = 3 dimensionless groups. Setting up dimensional equations:

$$\Pi_1 = w\cdot\rho^a\cdot V^b\cdot L^c \qquad \text{Thus:} \qquad L\cdot\left(\frac{M}{L^3}\right)^a\cdot\left(\frac{L}{t}\right)^b\cdot L^c = M^0\cdot L^0\cdot t^0$$

Summing exponents:

M: $a = 0$

L: $1 - 3\cdot a + b + c = 0$

t: $-b = 0$

The solution to this system is:

$a = 0 \qquad b = 0 \qquad c = -1$

$$\Pi_1 = \frac{w}{L}$$

Check using F, L, t dimensions: $L\cdot\frac{1}{L} = 1$

$$\Pi_2 = t\cdot\rho^a\cdot V^b\cdot L^c \qquad \text{Thus:} \qquad L\cdot\left(\frac{M}{L^3}\right)^a\cdot\left(\frac{L}{t}\right)^b\cdot L^c = M^0\cdot L^0\cdot t^0$$

Summing exponents:

M: $a = 0$

L: $1 - 3\cdot a + b + c = 0$

t: $-b = 0$

The solution to this system is:

$a = 0 \qquad b = 0 \qquad c = -1$

$$\Pi_2 = \frac{t}{L}$$

Check using F, L, t dimensions: $L\cdot\frac{1}{L} = 1$

$$\Pi_3 = \mu\cdot\rho^a\cdot V^b\cdot L^c \qquad \text{Thus:} \qquad \frac{M}{L\cdot t}\cdot\left(\frac{M}{L^3}\right)^a\cdot\left(\frac{L}{t}\right)^b\cdot L^c = M^0\cdot L^0\cdot t^0$$

Summing exponents:

M: $1 + a = 0$

L: $-1 - 3\cdot a + b + c = 0$

t: $-1 - b = 0$

The solution to this system is:

$a = -1 \qquad b = -1 \qquad c = -1$

$$\Pi_3 = \frac{\mu}{\rho\cdot V\cdot L}$$

Check using F, L, t dimensions: $\frac{F\cdot t}{L^2}\cdot\frac{L^4}{F\cdot t^2}\cdot\frac{t}{L}\cdot\frac{1}{L} = 1$

Problem 7.41

[Difficulty: 2]

7.41 A washing machine agitator is to be designed. The power, $\mathscr{P}$, required for the agitator is to be correlated with the amount of water used (indicated by the depth, H, of the water). It also depends on the agitator diameter, D, height, h, maximum angular velocity, ω_{max}, and frequency of oscillations, f, and water density, ρ, and viscosity, μ. Determine the dimensionless parameters that characterize this problem.

Given:

That the power of a washing machine agitator depends on various parameters

Find:

Dimensionless groups

Solution:

Apply the Buckingham Π procedure

① $\mathscr{P}$ H D h ω_{max} f ρ μ $n = 8$ parameters

② Select primary dimensions M, L, t

③
$$\left\{\begin{array}{cccccccc} \mathscr{P} & H & D & h & \omega_{max} & f & \rho & \mu \\ \\ \dfrac{ML^2}{t^3} & L & L & L & \dfrac{1}{t} & \dfrac{1}{t} & \dfrac{M}{L^3} & \dfrac{M}{Lt} \end{array}\right\} \quad r = 3 \text{ primary dimensions}$$

④ ρ D ω_{max} $m = r = 3$ repeat parameters

⑤ Then $n - m = 5$ dimensionless groups will result. Setting up a dimensional equation,

$$\Pi_1 = \rho^a D^b \omega_{max}^c \mathscr{P} = \left(\frac{M}{L^3}\right)^a (L)^b \left(\frac{1}{t}\right)^c \frac{ML^2}{t^3} = M^0 L^0 t^0$$

Summing exponents,
$$\begin{array}{ll|l} M: & a+1=0 & a=-1 \\ L: & -3a+b+2=0 & b=-5 \\ t: & -c-3=0 & c=-3 \end{array}$$
Hence $\Pi_1 = \dfrac{\mathscr{P}}{\rho D^5 \omega_{max}^3}$

$$\Pi_2 = \rho^a D^b \omega_{max}^c \mu = \left(\frac{M}{L^3}\right)^a (L)^b \left(\frac{1}{t}\right)^c \frac{M}{Lt} = M^0 L^0 t^0$$

Summing exponents,
$$\begin{array}{ll|l} M: & a+1=0 & a=-1 \\ L: & -3a+b-1=0 & b=-2 \\ t: & -c-1=0 & c=-1 \end{array}$$
Hence $\Pi_2 = \dfrac{\mu}{\rho D^2 \omega_{max}}$

The other Π groups can be found by inspection: $\Pi_3 = \dfrac{H}{D}$ $\Pi_4 = \dfrac{h}{D}$ $\Pi_5 = \dfrac{f}{\omega_{max}}$

⑥ Check using F, L, t as primary dimensions

$$\Pi_1 = \frac{\dfrac{FL}{t}}{\dfrac{Ft^2}{L^4} L^5 \dfrac{1}{t^3}} = [1] \qquad \Pi_2 = \frac{\dfrac{Ft}{L^2}}{\dfrac{Ft^2}{L^4} L^2 \dfrac{1}{t}} = [1] \qquad \Pi_3 = \Pi_4 = \Pi_5 = [1]$$

Note: Any combination of Π's is a Π group, e.g., $\dfrac{\Pi_1}{\Pi_2} = \dfrac{\mathscr{P}}{D^3 \omega_{max}^2 \mu}$, so the Π's are not unique!

Problem 7.42

[Difficulty: 3]

7.42 Choked-flow nozzles are often used to meter the flow of gases through piping systems. The mass flow rate of gas is thought to depend on nozzle area A, pressure p, and temperature T upstream of the meter, and the gas constant R. Determine how many independent Π parameters can be formed for this problem. State the functional relationship for the mass flow rate in terms of the dimensionless parameters.

Given: Functional relationship between the mass flow rate of gas through a choked-flow nozzle and other physical parameters

Find:
(a) How many independent Π terms that characterize this phenomenon
(b) Find the Π terms
(c) State the functional relationship for the mass flow rate in terms of the Π terms

Solution: We will use the Buckingham pi-theorem.

1 $m \quad A \quad p \quad T \quad R$ (Mathcad can't render dots!) $n = 5$ parameters

2 Select primary dimensions M, L, t:

3 $m \quad A \quad p \quad T \quad R$

$\frac{M}{t} \quad L^2 \quad \frac{M}{L\cdot t^2} \quad T \quad \frac{L^2}{t^2\cdot T}$ $r = 4$ dimensions

4 $p \quad A \quad T \quad R$ $m = r = 4$ repeating parameters We have $n - m = 1$ dimensionless group.

5 Setting up dimensional equations:

$$\Pi_1 = m\cdot p^a\cdot A^b\cdot T^c\cdot R^d \qquad \text{Thus:} \qquad \frac{M}{t}\cdot\left(\frac{M}{L\cdot t^2}\right)^a\cdot\left(L^2\right)^b\cdot T^c\cdot\left(\frac{L^2}{t^2\cdot T}\right)^d = M^0\cdot L^0\cdot t^0\cdot T^0$$

Summing exponents:

M: $1 + a = 0$

L: $-a + 2\cdot b + 2\cdot d = 0$

t: $-1 - 2\cdot a - 2\cdot d = 0$

T: $c - d = 0$

The solution to this system is:

$a = -1 \quad b = -1 \quad c = \frac{1}{2} \quad d = \frac{1}{2}$

$$\Pi_1 = \frac{m}{p\cdot A}\cdot\sqrt{R\cdot T}$$

Check using F, L, t dimensions: $\frac{F\cdot t}{L}\cdot\frac{L^2}{F}\cdot\frac{1}{L^2}\cdot\frac{L}{t\cdot T^{\frac{1}{2}}}\cdot T^{\frac{1}{2}} = 1$

The functional relationship is: $\Pi_1 = C \qquad \frac{m}{p\cdot A}\cdot\sqrt{R\cdot T} = C$ So the mass flow rate is: $m = C\cdot\frac{p\cdot A}{\sqrt{R\cdot T}}$

Problem 7.43 (In Excel)

[Difficulty: 3]

7.43 The time, t, for a flywheel, with moment of inertia, I, to reach angular velocity, ω, from rest, depends on the applied torque, T, and the following flywheel bearing properties: the oil viscosity, μ, gap, δ, diameter, D, and length, L. Use dimensional analysis to find the Π parameters that characterize this phenomenon.

Given: Time to speed up depends on inertia, speed, torque, oil viscosity and geometry

Find: Π groups

Solution:

We will use the workbook of Example 7.1, modified for the current problem

The number of parameters is:	$n = 8$
The number of primary dimensions is:	$r = 3$
The number of repeat parameters is:	$m = r = 3$
The number of Π groups is:	$n - m = 5$

Enter the dimensions (**M**, **L**, **t**) of
the repeating parameters, and of up to
four other parameters (for up to four Π groups).
The spreadsheet will compute the exponents a, b, and c for each.

REPEATING PARAMETERS: **Choose ω, D, T**

	M	L	t
ω			-1
D		1	
T	1	2	-2

Π GROUPS:

Two Π groups can be obtained by inspection: δ/D and L/D. The others are obtained below

	M	L	t
t	0	0	1

Π_1: a = **1**, b = **0**, c = **0**

	M	L	t
μ	1	-1	-1

Π_2: a = **1**, b = **3**, c = **-1**

	M	L	t
I	1	2	0

Π_3: a = **2**, b = **0**, c = **-1**

	M	L	t
	0	0	0

Π_4: a = **0**, b = **0**, c = **0**

Hence the Π groups are

$$t\omega \qquad \frac{\delta}{D} \qquad \frac{L}{D} \qquad \frac{\mu\omega D^3}{T} \qquad \frac{I\omega^2}{T}$$

Note that the Π_1 group can also be easily obtained by inspection

Problem 7.44

[Difficulty: 3]

7.44 A large tank of liquid under pressure is drained through a smoothly contoured nozzle of area *A*. The mass flow rate is thought to depend on nozzle area, *A*, liquid density, *ρ*, difference in height between the liquid surface and nozzle, *h*, tank gage pressure, *Δp*, and gravitational acceleration, *g*. Determine how many independent Π parameters can be formed for this problem. Find the dimensionless parameters. State the functional relationship for the mass flow rate in terms of the dimensionless parameters.

Given: Functional relationship between the mass flow rate of liquid from a pressurized tank through a contoured nozzle and other physical parameters

Find:
(a) How many independent Π terms that characterize this phenomenon
(b) Find the Π terms
(c) State the functional relationship for the mass flow rate in terms of the Π terms

Solution: We will use the Buckingham pi-theorem.

1 $\quad m \quad A \quad \rho \quad h \quad \Delta p \quad g$ $\qquad$ n = 6 parameters

2 Select primary dimensions M, L, t:

3 $\quad m \quad A \quad \rho \quad h \quad \Delta p \quad g$

$$\frac{M}{t} \quad L^2 \quad \frac{M}{L^3} \quad L \quad \frac{M}{L\cdot t^2} \quad \frac{L}{t^2}$$

r = 3 dimensions

4 $\quad \rho \quad A \quad g$ $\qquad$ m = r = 3 repeating parameters $\qquad$ We have n - m = 3 dimensionless groups.

5 Setting up dimensional equations:

$\Pi_1 = m\cdot\rho^a\cdot A^b\cdot g^c$ $\qquad$ Thus: $\frac{M}{t}\cdot\left(\frac{M}{L^3}\right)^a\cdot\left(L^2\right)^b\cdot\left(\frac{L}{t^2}\right)^c = M^0\cdot L^0\cdot t^0$

Summing exponents:

M: $1 + a = 0$

L: $-3\cdot a + 2\cdot b + c = 0$

t: $-1 - 2\cdot c = 0$

The solution to this system is:

$a = -1 \qquad b = -\frac{5}{4} \qquad c = -\frac{1}{2}$

$$\Pi_1 = \frac{m}{\rho\cdot A^{\frac{5}{4}}\cdot g^{\frac{1}{2}}}$$

Check using F, L, t dimensions: $\frac{F\cdot t}{L}\cdot\frac{L^4}{F\cdot t^2}\cdot\frac{1}{L^{\frac{5}{2}}}\cdot\frac{t}{L^{\frac{1}{2}}} = 1$

$\Pi_2 = h\cdot\rho^a\cdot A^b\cdot g^c$ $\qquad$ Thus: $L\cdot\left(\frac{M}{L^3}\right)^a\cdot\left(L^2\right)^b\cdot\left(\frac{L}{t^2}\right)^c = M^0\cdot L^0\cdot t^0$

Summing exponents:

M: $a = 0$

L: $1 - 3\cdot a + 2\cdot b + c = 0$

t: $-2\cdot c = 0$

The solution to this system is:

$a = 0 \qquad b = -\frac{1}{2} \qquad c = 0$

$$\Pi_2 = \frac{h}{\sqrt{A}}$$

Check using F, L, t dimensions: $L\cdot\frac{1}{L} = 1$

$$\Pi_3 = \Delta p \cdot \rho^a \cdot A^b \cdot g^c$$ Thus: $$\frac{M}{L \cdot t^2} \cdot \left(\frac{M}{L^3}\right)^a \cdot \left(L^2\right)^b \cdot \left(\frac{L}{t^2}\right)^c = M^0 \cdot L^0 \cdot t^0$$

Summing exponents:

M: $1 + a = 0$

L: $-1 - 3 \cdot a + 2 \cdot b + c = 0$

t: $-2 - 2 \cdot c = 0$

The solution to this system is:

$a = -1 \qquad b = -\frac{1}{2} \qquad c = -1$

$$\Pi_3 = \frac{\Delta p}{\rho \cdot g \cdot \sqrt{A}}$$

Check using F, L, t dimensions: $$\frac{F}{L^2} \cdot \frac{L^4}{F \cdot t^2} \cdot \frac{t^2}{L} \cdot \frac{1}{L} = 1$$

The functional relationship is: $$\Pi_1 = f\left(\Pi_2, \Pi_3\right) \qquad \frac{m}{\rho \cdot A^{\frac{5}{4}} \cdot g^{\frac{1}{2}}} = f\left(\frac{h}{\sqrt{A}}, \frac{\Delta p}{\rho \cdot g \cdot \sqrt{A}}\right)$$

So the mass flow rate is:

$$m = \rho \cdot A^{\frac{5}{4}} \cdot g^{\frac{1}{2}} \cdot f\left(\frac{h}{\sqrt{A}}, \frac{\Delta p}{\rho \cdot g \cdot \sqrt{A}}\right)$$

Problem 7.45

[Difficulty: 3]

7.45 Spin plays an important role in the flight trajectory of golf, Ping-Pong, and tennis balls. Therefore, it is important to know the rate at which spin decreases for a ball in flight. The aerodynamic torque, T, acting on a ball in flight, is thought to depend on flight speed, V, air density, ρ, air viscosity, μ, ball diameter, D, spin rate (angular speed), ω, and diameter of the dimples on the ball, d. Determine the dimensionless parameters that result.

Given: Functional relationship between the aerodynamic torque on a spinning ball and other physical parameters

Find: The Π terms that characterize this phenomenon

Solution: We will use the Buckingham pi-theorem.

1 $\quad T \quad V \quad \rho \quad \mu \quad D \quad \omega \quad d \qquad$ n = 7 parameters

2 Select primary dimensions M, L, t:

3 $\quad T \quad V \quad \rho \quad \mu \quad D \quad \omega \quad d$

$$\frac{M\cdot L^2}{t^2} \quad \frac{L}{t} \quad \frac{M}{L^3} \quad \frac{M}{L\cdot t} \quad L \quad \frac{1}{t} \quad L$$

r = 3 dimensions

4 $\quad \rho \quad V \quad D \qquad$ m = r = 3 repeating parameters

5 We have n - m = 4 dimensionless groups. Setting up dimensional equations:

$$\Pi_1 = T\cdot\rho^a\cdot V^b\cdot D^c \qquad \text{Thus:} \qquad \frac{M\cdot L^2}{t^2}\cdot\left(\frac{M}{L^3}\right)^a\cdot\left(\frac{L}{t}\right)^b\cdot L^c = M^0\cdot L^0\cdot t^0$$

Summing exponents:

M: $1 + a = 0$

L: $2 - 3\cdot a + b + c = 0$

t: $-2 - b = 0$

The solution to this system is:

$a = -1 \qquad b = -2 \qquad c = -3$

$$\Pi_1 = \frac{T}{\rho\cdot V^2\cdot D^3}$$

Check using F, L, t dimensions: $F\cdot L\cdot\frac{L^4}{F\cdot t^2}\cdot\frac{t^2}{L^2}\cdot\frac{1}{L^3} = 1$

$$\Pi_2 = \mu\cdot\rho^a\cdot V^b\cdot D^c \qquad \text{Thus:} \qquad \frac{M}{L\cdot t}\cdot\left(\frac{M}{L^3}\right)^a\cdot\left(\frac{L}{t}\right)^b\cdot L^c = M^0\cdot L^0\cdot t^0$$

Summing exponents:

M: $1 + a = 0$

L: $-1 - 3\cdot a + b + c = 0$

t: $-1 - b = 0$

The solution to this system is:

$a = -1 \qquad b = -1 \qquad c = -1$

$$\Pi_2 = \frac{\mu}{\rho\cdot V\cdot D}$$

Check using F, L, t dimensions: $\frac{F\cdot t}{L^2}\cdot\frac{L^4}{F\cdot t^2}\cdot\frac{t}{L}\cdot\frac{1}{L} = 1$

$\Pi_3 = \omega\cdot\rho^a\cdot V^b\cdot D^c$ Thus: $\frac{1}{t}\cdot\left(\frac{M}{L^3}\right)^a\cdot\left(\frac{L}{t}\right)^b\cdot L^c = M^0\cdot L^0\cdot t^0$

Summing exponents:

M: $a = 0$

L: $-3\cdot a + b + c = 0$

t: $-1 - b = 0$

The solution to this system is:

$a = 0 \qquad b = -1 \qquad c = 1$

$$\Pi_3 = \frac{\omega\cdot D}{V}$$

Check using F, L, t dimensions: $\frac{1}{t}\cdot L\cdot\frac{t}{L} = 1$

$\Pi_4 = d\cdot\rho^a\cdot V^b\cdot D^c$ Thus: $L\cdot\left(\frac{M}{L^3}\right)^a\cdot\left(\frac{L}{t}\right)^b\cdot L^c = M^0\cdot L^0\cdot t^0$

Summing exponents:

M: $a = 0$

L: $1 - 3\cdot a + b + c = 0$

t: $-b = 0$

The solution to this system is:

$a = 0 \qquad b = 0 \qquad c = -1$

$$\Pi_4 = \frac{d}{D}$$

Check using F, L, t dimensions: $\frac{1}{L}\cdot L = 1$

The functional relationship is: $\Pi_1 = f(\Pi_2, \Pi_3, \Pi_4)$

$$\frac{T}{\rho\cdot V^2\cdot D^3} = f\left(\frac{\mu}{\rho\cdot V\cdot D}, \frac{\omega\cdot D}{V}, \frac{d}{D}\right)$$

Problem 7.46

[Difficulty: 2]

7.46 The ventilation in the clubhouse on a cruise ship is insufficient to clear cigarette smoke (the ship is not yet completely smoke-free). Tests are to be done to see if a larger extractor fan will work. The concentration of smoke, c (particles per cubic meter) depends on the number of smokers, N, the pressure drop produced by the fan, Δp, the fan diameter, D, motor speed, ω, the particle and air densities, ρ_p, and ρ, respectively, gravity, g, and air viscosity, μ. Determine the dimensionless parameters that characterize this problem.

Given: Ventilation system of cruise ship clubhouse

Find: Dimensionless groups

Solution:

Apply the Buckingham Π procedure

① $c \quad N \quad \Delta p \quad D \quad \omega \quad \rho_p \quad \rho \quad g \quad \mu$ $\qquad n = 9$ parameters

② Select primary dimensions M, L, t

③ $$\left\{\begin{matrix} c & N & \Delta p & D & \omega & \rho_p & \rho & g & \mu \\ \dfrac{1}{L^3} & 1 & \dfrac{M}{Lt^2} & L & \dfrac{1}{t} & \dfrac{M}{L^3} & \dfrac{M}{L^3} & \dfrac{L}{t^2} & \dfrac{M}{Lt} \end{matrix}\right\}$$ $r = 3$ primary dimensions

④ $\rho \quad D \quad \omega$ $\qquad m = r = 3$ repeat parameters

⑤ Then $n - m = 6$ dimensionless groups will result. Setting up a dimensional equation,

$$\Pi_1 = \rho^a D^b \omega^c \Delta p = \left(\frac{M}{L^3}\right)^a (L)^b \left(\frac{1}{t}\right)^c \frac{M}{Lt^2} = M^0 L^0 t^0$$

Summing exponents,

$$\begin{array}{ll|l} M: & a+1=0 & a=-1 \\ L: & -3a+b-1=0 & b=-2 \\ t: & -c-2=0 & c=-2 \end{array}$$

Hence $\Pi_1 = \dfrac{\Delta p}{\rho D^2 \omega^2}$

$$\Pi_2 = \rho^a D^b \omega^c \mu = \left(\frac{M}{L^3}\right)^a (L)^b \left(\frac{1}{t}\right)^c \frac{M}{Lt} = M^0 L^0 t^0$$

Summing exponents,

$$\begin{array}{ll|l} M: & a+1=0 & a=-1 \\ L: & -3a+b-1=0 & b=-2 \\ t: & -c-1=0 & c=-1 \end{array}$$

Hence $\Pi_2 = \dfrac{\mu}{\rho D^2 \omega}$

The other Π groups can be found by inspection: $\Pi_3 = cD^3 \qquad \Pi_4 = N \qquad \Pi_5 = \dfrac{\rho_p}{\rho} \qquad \Pi_6 = \dfrac{g}{D\omega^2}$

⑥ Check using F, L, t as primary dimensions

$$\Pi_1 = \frac{\dfrac{F}{L^2}}{\dfrac{Ft^2}{L^4} L^2 \dfrac{1}{t^2}} = [1] \qquad \Pi_2 = \frac{\dfrac{Ft}{L^2}}{\dfrac{Ft^2}{L^4} L^2 \dfrac{1}{t}} = [1] \qquad \Pi_3 = \Pi_4 = \Pi_5 = \Pi_6 = [1]$$

Note: Any combination of Π's is a Π group, e.g., $\dfrac{\Pi_1}{\Pi_2} = \dfrac{\Delta p}{\omega \mu}$, so the Π's are not unique!

Problem 7.47

[Difficulty: 3]

7.47 The mass burning rate of flammable gas $\dot{m}$ is a function of the thickness of the flame δ, the gas density ρ, the thermal diffusivity α, and the mass diffusivity D. Using dimensional analysis, determine the functional form of this dependence in terms of dimensionless parameters. Note that α and D have the dimensions L^2/t.

Given: Functional relationship between the mass burning rate of a combustible mixture and other physical parameters

Find: The dependence of mass burning rate

Solution: We will use the Buckingham pi-theorem.

1 $\quad m \quad \delta \quad \rho \quad \alpha \quad D$ $\qquad$ (Mathcad can't render dots!)

2 Select primary dimensions M, L, t: $\qquad$ n = 5 parameters

3 $\quad m \quad \delta \quad \rho \quad \alpha \quad D$

$$\frac{M}{t} \quad L \quad \frac{M}{L^3} \quad \frac{L^2}{t} \quad \frac{L^2}{t}$$

r = 3 dimensions

4 $\quad \delta \quad \rho \quad \alpha$ $\qquad$ m = r = 3 repeating parameters

5 We have n - m = 2 dimensionless groups. Setting up dimensional equations:

$$\Pi_1 = m \cdot \delta^a \cdot \rho^b \cdot \alpha^c \qquad \text{Thus:} \qquad \frac{M}{t} \cdot L^a \cdot \left(\frac{M}{L^3}\right)^b \cdot \left(\frac{L^2}{t}\right)^c = M^0 \cdot L^0 \cdot t^0$$

Summing exponents: $\qquad$ The solution to this system is:

M: $1 + b = 0$ $\qquad$ $a = -1 \quad b = -1 \quad c = -1$ $\qquad$ $\Pi_1 = \dfrac{m}{\delta \cdot \rho \cdot \alpha}$

L: $a - 3 \cdot b + 2 \cdot c = 0$

t: $-1 - c = 0$

$$\Pi_2 = D \cdot \delta^a \cdot \rho^b \cdot \alpha^c \qquad \text{Thus:} \qquad \frac{L^2}{t} \cdot L^a \cdot \left(\frac{M}{L^3}\right)^b \cdot \left(\frac{L^2}{t}\right)^c = M^0 \cdot L^0 \cdot t^0$$

Summing exponents: $\qquad$ The solution to this system is:

M: $b = 0$ $\qquad$ $a = 0 \quad b = 0 \quad c = -1$ $\qquad$ $\Pi_2 = \dfrac{D}{\alpha}$

L: $2 + a - 3 \cdot b + 2 \cdot c = 0$

t: $-1 - c = 0$

6 Check using F, L, t dimensions: $\quad \dfrac{F \cdot t}{L} \cdot \dfrac{1}{L} \cdot \dfrac{L^4}{F \cdot t^2} \cdot \dfrac{t}{L^2} = 1 \qquad \dfrac{L^2}{t} \cdot \dfrac{t}{L^2} = 1$

The functional relationship is: $\quad \Pi_1 = f(\Pi_2)$ $\qquad$ $\dfrac{m}{\delta \cdot \rho \cdot \alpha} = f\left(\dfrac{D}{\alpha}\right)$

Problem 7.48

[Difficulty: 3]

7.48 The power loss, $\mathcal{P}$, in a journal bearing depends on length, l, diameter, D, and clearance, c, of the bearing, in addition to its angular speed, ω. The lubricant viscosity and mean pressure are also important. Obtain the dimensionless parameters that characterize this problem. Determine the functional form of the dependence of $\mathcal{P}$ on these parameters.

Given: Functional relationship between the power loss in a journal bearing and other physical parameters

Find: The Π terms that characterize this phenomenon and the function form of the dependence of P on these parameters

Solution: We will use the Buckingham pi-theorem.

1 $\quad P \quad l \quad D \quad c \quad \omega \quad \mu \quad p$ $\qquad$ n = 7 parameters

2 $\quad$ Select primary dimensions F, L, t:

3 $\quad P \quad l \quad D \quad c \quad \omega \quad \mu \quad p$

$$\frac{F\cdot L}{t} \quad L \quad L \quad L \quad \frac{1}{t} \quad \frac{F\cdot t}{L^2} \quad \frac{F}{L^2}$$

r = 3 dimensions

4 $\quad D \quad \omega \quad p$ $\qquad$ m = r = 3 repeating parameters

5 $\quad$ We have n - m = 4 dimensionless groups. Setting up dimensional equations:

$$\Pi_1 = P\cdot D^a\cdot\omega^b\cdot p^c \qquad \text{Thus:} \qquad \frac{F\cdot L}{t}\cdot L^a\cdot\left(\frac{1}{t}\right)^b\cdot\left(\frac{F}{L^2}\right)^c = F^0\cdot L^0\cdot t^0$$

Summing exponents: $\qquad$ The solution to this system is:

F: $\quad 1 + c = 0$ $\qquad a = -3 \quad b = -1 \quad c = -1$ $\qquad \Pi_1 = \dfrac{P}{D^3\cdot\omega\cdot p}$

L: $\quad 1 + a - 2\cdot c = 0$

t: $\quad -1 - b = 0$

$$\Pi_2 = l\cdot D^a\cdot\omega^b\cdot p^c \qquad \text{Thus:} \qquad L\cdot L^a\cdot\left(\frac{1}{t}\right)^b\cdot\left(\frac{F}{L^2}\right)^c = F^0\cdot L^0\cdot t^0$$

Summing exponents: $\qquad$ The solution to this system is:

F: $\quad c = 0$ $\qquad a = -1 \quad b = 0 \quad c = 0$ $\qquad \Pi_2 = \dfrac{l}{D}$

L: $\quad 1 + a - 2\cdot c = 0$

t: $\quad -b = 0$

$$\Pi_3 = c\cdot D^a\cdot\omega^b\cdot p^c \qquad \text{Thus:} \qquad L\cdot L^a\cdot\left(\frac{1}{t}\right)^b\cdot\left(\frac{F}{L^2}\right)^c = F^0\cdot L^0\cdot t^0$$

Summing exponents: $\qquad$ The solution to this system is:

F: $\quad c = 0$ $\qquad a = -1 \quad b = 0 \quad c = 0$ $\qquad \Pi_3 = \dfrac{c}{D}$

L: $\quad 1 + a - 2\cdot c = 0$

t: $\quad -b = 0$

$\Pi_4 = \mu \cdot D^a \cdot \omega^b \cdot p^c$ Thus: $\frac{F \cdot t}{L^2} \cdot L^a \cdot \left(\frac{1}{t}\right)^b \cdot \left(\frac{F}{L^2}\right)^c = F^0 \cdot L^0 \cdot t^0$

Summing exponents:

F: $1 + c = 0$

L: $-2 + a - 2 \cdot c = 0$

t: $1 - b = 0$

The solution to this system is:

$a = 0 \qquad b = 1 \qquad c = -1$

$$\Pi_4 = \frac{\mu \cdot \omega}{p}$$

6 Check using M, L, t dimensions: $\frac{M \cdot L^2}{t^3} \cdot \frac{1}{L^3} \cdot t \cdot \frac{L \cdot t^2}{M} = 1 \quad L \cdot \frac{1}{L} = 1 \quad L \cdot \frac{1}{L} = 1 \quad \frac{M}{L \cdot t} \cdot \frac{1}{t} \cdot \frac{L \cdot t^2}{M} = 1$

The functional relationship is: $\Pi_1 = f(\Pi_2, \Pi_3, \Pi_4)$ $\quad \frac{P}{\omega \cdot p \cdot D^3} = f\left(\frac{l}{D}, \frac{c}{D}, \frac{\mu \cdot \omega}{p}\right)$ $\quad P = \omega \cdot p \cdot D^3 \cdot f\left(\frac{l}{D}, \frac{c}{D}, \frac{\mu \cdot \omega}{p}\right)$

Problem 7.49

[Difficulty: 4]

7.49 In a fan-assisted convection oven, the heat transfer rate to a roast, $\dot{Q}$ (energy per unit time), is thought to depend on the specific heat of air, c_p, temperature difference, Θ, a length scale, L, air density, ρ, air viscosity, μ, and air speed, V. How many basic dimensions are included in these variables? Determine the number of Π parameters needed to characterize the oven. Evaluate the Π parameters.

Given: Functional relationship between the heat transfer rate in a convection oven and other physical parameters

Find: The number of Π terms that characterize this phenomenon and the Π terms

Solution: We will use the Buckingham pi-theorem.

1 $\quad Q \quad c_p \quad \Theta \quad L \quad \rho \quad \mu \quad V$ $\qquad$ $n = 7$ parameters

2 Select primary dimensions F, L, t, T (temperature):

3 $\quad Q \quad c_p \quad \Theta \quad L \quad \rho \quad \mu \quad V$

$$\frac{F\cdot L}{t} \quad \frac{L^2}{t^2\cdot T} \quad T \quad L \quad \frac{F\cdot t^2}{L^4} \quad \frac{F\cdot t}{L^2} \quad \frac{L}{t}$$

$r = 4$ dimensions

4 $\quad \rho \quad V \quad L \quad \Theta$ $\qquad$ $m = r = 4$ repeating parameters $\qquad$ We have $n - m = 3$ dimensionless groups.

5 Setting up dimensional equations:

$$\Pi_1 = Q\cdot\rho^a\cdot V^b\cdot L^c\cdot\Theta^d \qquad \text{Thus:} \qquad \frac{F\cdot L}{t}\cdot\left(\frac{F\cdot t^2}{L^4}\right)^a\cdot\left(\frac{L}{t}\right)^b\cdot L^c\cdot T^d = F^0\cdot L^0\cdot t^0\cdot T^0$$

Summing exponents:

F: $1 + a = 0$

L: $1 - 4\cdot a + b + c = 0$

t: $-1 + 2\cdot a - b = 0$

T: $d = 0$

The solution to this system is:

$a = -1 \quad b = -3 \quad c = -2 \quad d = 0$

$$\Pi_1 = \frac{Q}{\rho\cdot V^3\cdot L^2}$$

$$\Pi_2 = c_p\cdot\rho^a\cdot V^b\cdot L^c\cdot\Theta^d \qquad \text{Thus:} \qquad \frac{L^2}{t^2\cdot T}\cdot\left(\frac{F\cdot t^2}{L^4}\right)^a\cdot\left(\frac{L}{t}\right)^b\cdot L^c\cdot T^d = F^0\cdot L^0\cdot t^0\cdot T^0$$

Summing exponents:

F: $a = 0$

L: $2 - 4\cdot a + b + c = 0$

t: $-2 + 2\cdot a - b = 0$

T: $-1 + d = 0$

The solution to this system is:

$a = 0 \quad b = -2 \quad c = 0 \quad d = 1$

$$\Pi_2 = \frac{c_p\cdot\Theta}{V^2}$$

$$\Pi_3 = \mu\cdot\rho^a\cdot V^b\cdot L^c\cdot\Theta^d \qquad \text{Thus:} \qquad \frac{F\cdot t}{L^2}\cdot\left(\frac{F\cdot t^2}{L^4}\right)^a\cdot\left(\frac{L}{t}\right)^b\cdot L^c\cdot T^d = F^0\cdot L^0\cdot t^0\cdot T^0$$

Summing exponents:

F: $1 + a = 0$

L: $-2 - 4\cdot a + b + c = 0$

t: $1 + 2\cdot a - b = 0$

T: $d = 0$

The solution to this system is:

$a = -1 \quad b = -1 \quad c = -1 \quad d = 0$

$$\Pi_3 = \frac{\mu}{\rho\cdot V\cdot L}$$

6 Check using M, L, t, T dimensions: $\frac{M\cdot L^2}{t^3}\cdot\frac{L^3}{M}\cdot\frac{t^2}{L^2}\cdot\frac{1}{L^3} = 1$ $\quad \frac{L^2}{t^2\cdot T}\cdot\frac{t^2}{L^2}\cdot T = 1$ $\quad \frac{M}{L\cdot t}\cdot\frac{L^3}{M}\cdot\frac{t}{L}\cdot\frac{1}{L} = 1$

The functional relationship is: $\Pi_1 = f\left(\Pi_2,\Pi_3\right)$ $\quad \frac{Q}{\rho\cdot V^3\cdot L^2} = f\left(\frac{c_p\cdot\Theta}{V^2},\frac{\mu}{\rho\cdot V\cdot L}\right)$ $\quad Q = \rho\cdot V^3\cdot L^2\cdot f\left(\frac{c_p\cdot\Theta}{V^2},\frac{\mu}{\rho\cdot V\cdot L}\right)$

Problem 7.50

[Difficulty: 3]

7.50 The thrust of a marine propeller is to be measured during "open-water" tests at a variety of angular speeds and forward speeds ("speeds of advance"). The thrust, F_T, is thought to depend on water density, ρ, propeller diameter, D, speed of advance, V, acceleration of gravity, g, angular speed, ω, pressure in the liquid, p, and liquid viscosity, μ. Develop a set of dimensionless parameters to characterize the performance of the propeller. (One of the resulting parameters, gD/V^2, is known as the *Froude speed of advance*.)

Given: Functional relationship between the thrust of a marine propeller and other physical parameters

Find: The Π terms that characterize this phenomenon

Solution: We will use the Buckingham pi-theorem.

1 $F_T \quad \rho \quad D \quad V \quad g \quad \omega \quad p \quad \mu$ n = 8 parameters

2 Select primary dimensions F, L, t:

3 $F_T \quad \rho \quad D \quad V \quad g \quad \omega \quad p \quad \mu$

$$\frac{M\cdot L}{t^2} \quad \frac{M}{L^3} \quad L \quad \frac{L}{t} \quad \frac{L}{t^2} \quad \frac{1}{t} \quad \frac{M}{L\cdot t^2} \quad \frac{M}{L\cdot t}$$

r = 3 dimensions

4 $\rho \quad V \quad D$ m = r = 3 repeating parameters

5 We have n - m = 5 dimensionless groups. Setting up dimensional equations:

$\Pi_1 = F_T\cdot\rho^a\cdot V^b\cdot D^c$ Thus: $\frac{M\cdot L}{t^2}\cdot\left(\frac{M}{L^3}\right)^a\cdot\left(\frac{L}{t}\right)^b\cdot L^c = M^0\cdot L^0\cdot t^0$

Summing exponents:

M: $1 + a = 0$

L: $1 - 3\cdot a + b + c = 0$

t: $-2 - b = 0$

The solution to this system is:

$a = -1 \quad b = -2 \quad c = -2$

$$\Pi_1 = \frac{F_T}{\rho\cdot V^2\cdot D^2}$$

$\Pi_2 = g\cdot\rho^a\cdot V^b\cdot D^c$ Thus: $\frac{L}{t^2}\cdot\left(\frac{M}{L^3}\right)^a\cdot\left(\frac{L}{t}\right)^b\cdot L^c = M^0\cdot L^0\cdot t^0$

Summing exponents:

M: $a = 0$

L: $1 - 3\cdot a + b + c = 0$

t: $-2 - b = 0$

The solution to this system is:

$a = 0 \quad b = -2 \quad c = 1$

$$\Pi_2 = \frac{g\cdot D}{V^2}$$

$\Pi_3 = \omega\cdot\rho^a\cdot V^b\cdot D^c$ Thus: $\frac{1}{t}\cdot\left(\frac{M}{L^3}\right)^a\cdot\left(\frac{L}{t}\right)^b\cdot L^c = M^0\cdot L^0\cdot t^0$

Summing exponents:

M: $a = 0$

L: $-3\cdot a + b + c = 0$

t: $-1 - b = 0$

The solution to this system is:

$a = 0 \quad b = -1 \quad c = 1$

$$\Pi_3 = \frac{\omega\cdot D}{V}$$

$$\Pi_4 = p\cdot\rho^a\cdot V^b\cdot D^c \qquad \text{Thus:} \qquad \frac{M}{L\cdot t^2}\cdot\left(\frac{M}{L^3}\right)^a\cdot\left(\frac{L}{t}\right)^b\cdot L^c = M^0\cdot L^0\cdot t^0$$

Summing exponents:

M: $1 + a = 0$

L: $-1 - 3\cdot a + b + c = 0$

t: $-2 - b = 0$

The solution to this system is:

$a = -1 \quad b = -2 \quad c = 0$

$$\Pi_4 = \frac{p}{\rho\cdot V^2}$$

$$\Pi_5 = \mu\cdot\rho^a\cdot V^b\cdot D^c \qquad \text{Thus:} \qquad \frac{M}{L\cdot t}\cdot\left(\frac{M}{L^3}\right)^a\cdot\left(\frac{L}{t}\right)^b\cdot L^c = M^0\cdot L^0\cdot t^0$$

Summing exponents:

M: $1 + a = 0$

L: $-1 - 3\cdot a + b + c = 0$

t: $-1 - b = 0$

The solution to this system is:

$a = -1 \quad b = -1 \quad c = -1$

$$\Pi_5 = \frac{\mu}{\rho\cdot V\cdot D}$$

6 Check using F, L, t dimensions: $F\cdot\frac{L^4}{F\cdot t^2}\cdot\frac{t^2}{L^2}\cdot\frac{1}{L^2} = 1 \quad \frac{L}{t^2}\cdot L\cdot\frac{t^2}{L^2} = 1 \quad \frac{1}{t}\cdot L\cdot\frac{t}{L} = 1 \quad \frac{F}{L^2}\cdot\frac{L^4}{F\cdot t^2}\cdot\frac{t^2}{L^2} = 1 \quad \frac{F\cdot t}{L^2}\cdot\frac{L^4}{F\cdot t^2}\cdot\frac{t}{L}\cdot\frac{1}{L} = 1$

Problem 7.51

[Difficulty: 3]

7.51 The rate dT/dt at which the temperature T at the center of a rice kernel falls during a food technology process is critical—too high a value leads to cracking of the kernel, and too low a value makes the process slow and costly. The rate depends on the rice specific heat, c, thermal conductivity, k, and size, L, as well as the cooling air specific heat, c_p, density, ρ, viscosity, μ, and speed, V. How many basic dimensions are included in these variables? Determine the Π parameters for this problem.

Given:

That the cooling rate depends on rice properties and air properties

Find:

The Π groups

Solution:

Apply the Buckingham Π procedure

① $dT/dt \quad c \quad k \quad L \quad c_p \quad \rho \quad \mu \quad V$ $\qquad n = 8$ parameters

② Select primary dimensions M, L, t and T (temperature)

③ $$\begin{array}{cccccccc} dT/dt & c & k & L & c_p & \rho & \mu & V \\ \dfrac{T}{t} & \dfrac{L^2}{t^2 T} & \dfrac{ML}{t^2 T} & L & \dfrac{L^2}{t^2 T} & \dfrac{M}{L^3} & \dfrac{M}{Lt} & \dfrac{L}{t} \end{array}$$ $r = 4$ primary dimensions

④ $V \quad \rho \quad L \quad c_p$ $\qquad m = r = 4$ repeat parameters

Then $n - m = 4$ dimensionless groups will result. By inspection, one Π group is c/c_p. Setting up a dimensional equation,

$$\Pi_1 = V^a \rho^b L^c c_p^d \frac{dT}{dt} = \left(\frac{L}{t}\right)^a \left(\frac{M}{L^3}\right)^b (L)^c \left(\frac{L^2}{t^2 T}\right)^d \frac{T}{t} = T^0 M^0 L^0 t^0$$

Summing exponents,

$$\begin{array}{rc|c} T: & -d + 1 = 0 & d = 1 \\ M: & b = 0 & b = 0 \\ L: & a - 3b + c + 2d = 0 & a + c = -2 \rightarrow c = 1 \\ t: & -a - 2d - 1 = 0 & a = -3 \end{array}$$

Hence $\Pi_1 = \dfrac{dT}{dt}\dfrac{Lc_p}{V^3}$

By a similar process, we find $\Pi_2 = \dfrac{k}{\rho L^2 c_p}$ and $\Pi_3 = \dfrac{\mu}{\rho L V}$

Hence

$$\frac{dT}{dt}\frac{Lc_p}{V^3} = f\left(\frac{c}{c_p}, \frac{k}{\rho L^2 c_p}, \frac{\mu}{\rho L V}\right)$$

Problem 7.52

[Difficulty: 3]

7.52 The power, $\mathcal{P}$, required to drive a propeller is known to depend on the following variables: freestream speed, V, propeller diameter, D, angular speed, ω, fluid viscosity, μ, fluid density, ρ, and speed of sound in the fluid, c. How many dimensionless groups are required to characterize this situation? Obtain these dimensionless groups.

Given: Functional relationship between the power to drive a marine propeller and other physical parameters

Find: (a) The number of Π terms that characterize this phenomenon
(b) The Π terms

Solution: We will use the Buckingham pi-theorem.

1 $\quad P \quad \rho \quad D \quad V \quad c \quad \omega \quad \mu$ $\qquad$ n = 7 parameters

2 $\quad$ Select primary dimensions F, L, t:

3 $\quad P \quad \rho \quad D \quad V \quad c \quad \omega \quad \mu$

$$\frac{M\cdot L^2}{t^3} \quad \frac{M}{L^3} \quad L \quad \frac{L}{t} \quad \frac{L}{t} \quad \frac{1}{t} \quad \frac{M}{L\cdot t}$$

r = 3 dimensions

4 $\quad \rho \quad V \quad D$ $\qquad$ m = r = 3 repeating parameters

5 $\quad$ We have n - m = 4 dimensionless groups. Setting up dimensional equations:

$$\Pi_1 = P\cdot\rho^a\cdot V^b\cdot D^c \qquad \text{Thus:} \qquad \frac{M\cdot L^2}{t^3}\cdot\left(\frac{M}{L^3}\right)^a\cdot\left(\frac{L}{t}\right)^b\cdot L^c = M^0\cdot L^0\cdot t^0$$

Summing exponents:

M: $1 + a = 0$

L: $2 - 3\cdot a + b + c = 0$

t: $-3 - b = 0$

The solution to this system is:

$a = -1 \quad b = -3 \quad c = -2$

$$\Pi_1 = \frac{P}{\rho\cdot V^3\cdot D^2}$$

$$\Pi_2 = c\cdot\rho^a\cdot V^b\cdot D^c \qquad \text{Thus:} \qquad \frac{L}{t}\cdot\left(\frac{M}{L^3}\right)^a\cdot\left(\frac{L}{t}\right)^b\cdot L^c = M^0\cdot L^0\cdot t^0$$

Summing exponents:

M: $a = 0$

L: $1 - 3\cdot a + b + c = 0$

t: $-1 - b = 0$

The solution to this system is:

$a = 0 \quad b = -1 \quad c = 0$

$$\Pi_2 = \frac{c}{V}$$

$$\Pi_3 = \omega\cdot\rho^a\cdot V^b\cdot D^c \qquad \text{Thus:} \qquad \frac{1}{t}\cdot\left(\frac{M}{L^3}\right)^a\cdot\left(\frac{L}{t}\right)^b\cdot L^c = M^0\cdot L^0\cdot t^0$$

Summing exponents:

M: $a = 0$

L: $-3\cdot a + b + c = 0$

t: $-1 - b = 0$

The solution to this system is:

$a = 0 \quad b = -1 \quad c = 1$

$$\Pi_3 = \frac{\omega\cdot D}{V}$$

$$\Pi_4 = \mu\cdot\rho^a\cdot V^b\cdot D^c \qquad \text{Thus:} \qquad \frac{M}{L\cdot t}\cdot\left(\frac{M}{L^3}\right)^a\cdot\left(\frac{L}{t}\right)^b\cdot L^c = M^0\cdot L^0\cdot t^0$$

Summing exponents:

M: $1 + a = 0$

L: $-1 - 3\cdot a + b + c = 0$

t: $-1 - b = 0$

The solution to this system is:

$a = -1 \qquad b = -1 \qquad c = -1$

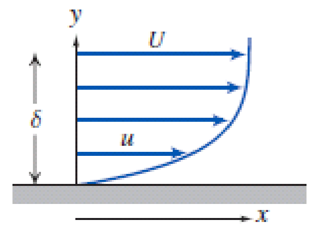
$$\Pi_4 = \frac{\mu}{\rho\cdot V\cdot D}$$

6 Check using F, L, t dimensions: $\frac{F\cdot L}{t}\cdot\frac{L^4}{F\cdot t^2}\cdot\frac{t^3}{L^3}\cdot\frac{1}{L^2} = 1 \qquad \frac{L}{t}\cdot\frac{t}{L} = 1 \qquad \frac{1}{t}\cdot L\cdot\frac{t}{L} = 1 \qquad \frac{F\cdot t}{L^2}\cdot\frac{L^4}{F\cdot t^2}\cdot\frac{t}{L}\cdot\frac{1}{L} = 1$

Problem 7.53

[Difficulty: 2]

7.53 The fluid velocity u at any point in a boundary layer depends on the distance y of the point above the surface, the free-stream velocity U and free-stream velocity gradient dU/dx, the fluid kinematic viscosity ν, and the boundary layer thickness δ. How many dimensionless groups are required to describe this problem? Find: (a) two Π groups by inspection, (b) one Π that is a standard fluid mechanics group, and (c) any remaining Π groups using the Buckingham Pi theorem.



Given: Boundary layer profile

Find: Two Π groups by inspection; One Π that is a standard fluid mechanics group; Dimensionless groups

Solution:

Two obvious Π groups are u/U and y/δ. A dimensionless group common in fluid mechanics is $U\delta/\nu$ (Reynolds number)

Apply the Buckingham Π procedure

① $u \quad y \quad U \quad dU/dx \quad \nu \quad \delta$ $\qquad n = 6$ parameters

② Select primary dimensions M, L, t

③ $$\begin{Bmatrix} u & y & U & dU/dx & \nu & \delta \\ \frac{L}{t} & L & \frac{L}{t} & \frac{1}{t} & \frac{L^2}{t} & L \end{Bmatrix}$$ $m = r = 3$ primary dimensions

④ $U \quad \delta$ $\qquad m = r = 2$ repeat parameters

⑤ Then $n - m = 4$ dimensionless groups will result. We can easily do these by inspection

$$\Pi_1 = \frac{u}{U} \qquad \Pi_2 = \frac{y}{\delta} \qquad \Pi_3 = \frac{(dU/dy)\delta}{U} \qquad \Pi_4 = \frac{\nu}{\delta U}$$

⑥ Check using F, L, t as primary dimensions, is not really needed here

Note: Any combination of Π's can be used; they are not unique!

Problem 7.54

[Difficulty: 4]

7.54 When a valve is closed suddenly in a pipe with flowing water, a water hammer pressure wave is set up. The very high pressures generated by such waves can damage the pipe. The maximum pressure, p_{max}, generated by water hammer is a function of liquid density, ρ, initial flow speed, U_0, and liquid bulk modulus, E_v. How many dimensionless groups are needed to characterize water hammer? Determine the functional relationship among the variables in terms of the necessary Π groups.

Given: Functional relationship between the maximum pressure experienced in a water hammer wave and other physical parameters

Find: (a) The number of Π terms that characterize this phenomenon
(b) The functional relationship between the Π terms

Solution: We will use the Buckingham pi-theorem.

1 $\quad p_{max} \quad \rho \quad U_0 \quad E_V \qquad$ n = 4 parameters

2 Select primary dimensions M, L, t:

3 $\quad p_{max} \quad \rho \quad U_0 \quad E_V$

$$\frac{M}{L\cdot t^2} \quad \frac{M}{L^3} \quad \frac{L}{t} \quad \frac{M}{L\cdot t^2}$$

r = 3 dimensions

4 $\quad \rho \quad U_0 \qquad$ m = 2 repeating parameters because p_{max} and E_v have the same dimensions.

We have n - m = 2 dimensionless groups.

5 Setting up dimensional equations:

$$\Pi_1 = p_{max}\cdot\rho^a\cdot U_0^{\,b} \qquad \text{Thus:} \qquad \frac{M}{L\cdot t^2}\cdot\left(\frac{M}{L^3}\right)^a\cdot\left(\frac{L}{t}\right)^b = M^0\cdot L^0\cdot t^0$$

Summing exponents:

M: $1 + a = 0$

L: $-1 - 3\cdot a + b = 0$

t: $-2 - b = 0$

The solution to this system is:

$a = -1 \qquad b = -2$

$$\Pi_1 = \frac{p_{max}}{\rho\cdot U_0^{\,2}}$$

$$\Pi_2 = E_V\cdot\rho^a\cdot U_0^{\,b} \qquad \text{Thus:} \qquad \frac{M}{L\cdot t^2}\cdot\left(\frac{M}{L^3}\right)^a\cdot\left(\frac{L}{t}\right)^b = M^0\cdot L^0\cdot t^0$$

Summing exponents:

M: $1 + a = 0$

L: $-1 - 3\cdot a + b = 0$

t: $-2 - b = 0$

The solution to this system is:

$a = -1 \qquad b = -2$

$$\Pi_2 = \frac{E_v}{\rho\cdot U_0^{\,2}}$$

6 Check using F, L, t dimensions: $\frac{F}{L^2}\cdot\frac{L^4}{F\cdot t^2}\cdot\frac{t^2}{L^2} = 1 \qquad \frac{F}{L^2}\cdot\frac{L^4}{F\cdot t^2}\cdot\frac{t^2}{L^2} = 1$

The functional relationship is: $\Pi_1 = f(\Pi_2)$ Thus:

$$\frac{p_{max}}{\rho\cdot U_0^{\,2}} = f\left(\frac{E_v}{\rho\cdot U_0^{\,2}}\right)$$

Problem 7.55

[Difficulty: 3]

7.55 The designers of a large tethered pollution-sampling balloon wish to know what the drag will be on the balloon for the maximum anticipated wind speed of 5 m/s (the air is assumed to be at 20°C). A $\frac{1}{20}$-scale model is built for testing in water at 20°C. What water speed is required to model the prototype? At this speed the model drag is measured to be 2 kN. What will be the corresponding drag on the prototype?

Given: Model scale for on balloon

Find: Required water model water speed; drag on protype based on model drag

Solution:

From Appendix A (inc. Fig. A.2) $\rho_{air} = 1.24 \cdot \frac{kg}{m^3}$ $\mu_{air} = 1.8 \times 10^{-5} \cdot \frac{N \cdot s}{m^2}$ $\rho_w = 999 \cdot \frac{kg}{m^3}$ $\mu_w = 10^{-3} \cdot \frac{N \cdot s}{m^2}$

The given data is $V_{air} = 5 \cdot \frac{m}{s}$ $L_{ratio} = 20$ $F_w = 2 \cdot kN$

For dynamic similarity we assume $\frac{\rho_w \cdot V_w \cdot L_w}{\mu_w} = \frac{\rho_{air} \cdot V_{air} \cdot L_{air}}{\mu_{air}}$

Then $V_w = V_{air} \cdot \frac{\mu_w}{\mu_{air}} \cdot \frac{\rho_{air}}{\rho_w} \cdot \frac{L_{air}}{L_w} = V_{air} \cdot \frac{\mu_w}{\mu_{air}} \cdot \frac{\rho_{air}}{\rho_w} \cdot L_{ratio} = 5 \cdot \frac{m}{s} \times \left(\frac{10^{-3}}{1.8 \times 10^{-5}} \right) \times \left(\frac{1.24}{999} \right) \times 20$ $V_w = 6.90 \frac{m}{s}$

For the same Reynolds numbers, the drag coefficients will be the same so we have $\frac{F_{air}}{\frac{1}{2} \cdot \rho_{air} \cdot A_{air} \cdot V_{air}^2} = \frac{F_w}{\frac{1}{2} \cdot \rho_w \cdot A_w \cdot V_w^2}$

where $\frac{A_{air}}{A_w} = \left(\frac{L_{air}}{L_w} \right)^2 = L_{ratio}^2$

Hence the prototype drag is $F_{air} = F_w \cdot \frac{\rho_{air}}{\rho_w} \cdot L_{ratio}^2 \cdot \left(\frac{V_{air}}{V_w} \right)^2 = 2000 \cdot N \times \left(\frac{1.24}{999} \right) \times 20^2 \times \left(\frac{5}{6.9} \right)^2$ $F_{air} = 522\,N$

Problem 7.56

[Difficulty: 3]

7.56 An airship is to operate at 20 m/s in air at standard conditions. A model is constructed to $\frac{1}{20}$ scale and tested in a wind tunnel at the same air temperature to determine drag. What criterion should be considered to obtain dynamic similarity? If the model is tested at 75 m/s, what pressure should be used in the wind tunnel? If the model drag force is 250 N, what will be the drag of the prototype?

Given: Airship is to operate at 20 m/s in air at standard conditions. A 1/20 scale model is to be tested in a wind tunnel at the same temperature to determine drag.

Find:
(a) Criterion needed to obtain dynamic similarity
(b) Air pressure required if air speed in wind tunnel is 75 m/s
(c) Prototype drag if the drag on the model is 250 N

Solution: Dimensional analysis predicts: $\frac{F}{\rho \cdot V^2 \cdot L^2} = f\left(\frac{\rho \cdot V \cdot L}{\mu}\right)$ Therefore, for dynamic similarity, it would follow that:

$$\frac{\rho_m \cdot V_m \cdot L_m}{\mu_m} = \frac{\rho_p \cdot V_p \cdot L_p}{\mu_p}$$

Since the tests are performed at the same temperature, the viscosities are the same. Solving for the ratio of densities:

$\frac{\rho_m}{\rho_p} = \frac{V_p}{V_m} \cdot \frac{L_p}{L_m} \cdot \frac{\mu_m}{\mu_p} = \frac{20}{75} \times 20 \times 1 = 5.333$ Now from the ideal gas equation of state: $\rho = \frac{p}{R \cdot T}$ Thus:

$$p_m = p_p \cdot \frac{\rho_m}{\rho_p} \cdot \frac{T_p}{T_m} \qquad p_m = 101 \cdot \text{kPa} \times 5.333 \times 1 \qquad p_m = 5.39 \times 10^5 \, \text{Pa}$$

From the force ratios: $\frac{F_p}{\rho_p \cdot V_p^2 \cdot L_p^2} = \frac{F_m}{\rho_m \cdot V_m^2 \cdot L_m^2}$ Thus: $F_p = F_m \cdot \frac{\rho_p}{\rho_m} \cdot \left(\frac{V_p}{V_m}\right)^2 \cdot \left(\frac{L_p}{L_m}\right)^2$

Substituting known values: $F_p = 250 \cdot \text{N} \times \frac{1}{5.333} \times \left(\frac{20}{75}\right)^2 \times (20)^2$ $\qquad F_p = 1.333 \cdot \text{kN}$

Problem 7.57

[Difficulty: 2]

7.57 To match the Reynolds number in an air flow and a water flow using the same size model, which flow will require the higher flow speed? How much higher must it be?

Given: A model is to be subjected to the same Reynolds number in air flow and water flow

Find:
(a) Which flow will require the higher flow speed
(b) How much higher the flow speed needs to be

Solution: For dynamic similarity: $\frac{\rho_w \cdot V_w \cdot L_w}{\mu_w} = \frac{\rho_a \cdot V_a \cdot L_a}{\mu_a}$ We know that $L_w = L_a$ Thus: $\frac{V_a}{V_w} = \frac{\rho_w}{\rho_a} \cdot \frac{\mu_a}{\mu_w} = \frac{\nu_a}{\nu_w}$

From Tables A.8 and A.10 at 20 deg C: $\nu_w = 1.00 \times 10^{-6} \cdot \frac{m^2}{s}$ and $\nu_a = 1.51 \times 10^{-5} \cdot \frac{m^2}{s}$ Therefore:

$$\frac{V_a}{V_w} = \frac{1.51 \times 10^{-5}}{1.00 \times 10^{-6}} = 15.1$$

Air speed must be higher than water speed.

To match Reynolds number: $V_a = 15.1 \cdot V_w$

Problem 7.58

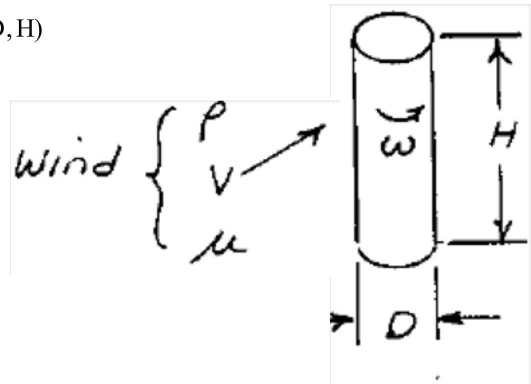
[Difficulty: 5]

7.58 An ocean-going vessel is to be powered by a rotating circular cylinder. Model tests are planned to estimate the power required to rotate the prototype cylinder. A dimensional analysis is needed to scale the power requirements from model test results to the prototype. List the parameters that should be included in the dimensional analysis. Perform a dimensional analysis to identify the important dimensionless groups.

Given: Vessel to be powered by a rotating circular cylinder. Model tests are planned to determine the required power for the prototype.

Find: (a) List of parameters that should be included in the analysis
(b) Perform dimensional analysis to identify the important dimensionless groups

Solution: From an inspection of the physical problem: $P = f(\rho, \mu, V, \omega, D, H)$



We will now use the Buckingham pi-theorem to find the dimensionless groups.

1 $\quad P \quad \rho \quad \mu \quad V \quad \omega \quad D \quad H \qquad$ n = 7 parameters

2 Select primary dimensions M, L, t:

3

P	ρ	μ	V	ω	D	H	
$\frac{M \cdot L^2}{t^3}$	$\frac{M}{L^3}$	$\frac{M}{L \cdot t}$	$\frac{L}{t}$	$\frac{1}{t}$	L	L	r = 3 dimensions

4 $\quad \rho \quad \omega \quad D \qquad$ m = r = 3 repeating parameters

5 We have n - m = 4 dimensionless groups. Setting up dimensional equations:

$$\Pi_1 = P \cdot \rho^a \cdot \omega^b \cdot D^c \qquad \text{Thus:} \qquad \frac{M \cdot L^2}{t^3} \cdot \left(\frac{M}{L^3}\right)^a \cdot \left(\frac{1}{t}\right)^b \cdot L^c = M^0 \cdot L^0 \cdot t^0$$

Summing exponents: The solution to this system is:

M: $1 + a = 0$ $\qquad a = -1 \quad b = -3 \quad c = -5$ $\qquad \Pi_1 = \frac{P}{\rho \cdot \omega^3 \cdot D^5}$

L: $2 - 3 \cdot a + c = 0$

t: $-3 - b = 0$

$$\Pi_2 = \mu \cdot \rho^a \cdot \omega^b \cdot D^c \qquad \text{Thus:} \qquad \frac{M}{L \cdot t} \cdot \left(\frac{M}{L^3}\right)^a \cdot \left(\frac{1}{t}\right)^b \cdot L^c = M^0 \cdot L^0 \cdot t^0$$

Summing exponents: The solution to this system is:

M: $1 + a = 0$ $\qquad a = -1 \quad b = -1 \quad c = -2$ $\qquad \Pi_2 = \frac{\mu}{\rho \cdot \omega \cdot D^2}$

L: $-1 - 3 \cdot a + c = 0$

t: $-1 - b = 0$

$$\Pi_3 = V \cdot \rho^a \cdot \omega^b \cdot D^c \qquad \text{Thus:} \qquad \frac{L}{t} \cdot \left(\frac{M}{L^3}\right)^a \cdot \left(\frac{1}{t}\right)^b \cdot L^c = M^0 \cdot L^0 \cdot t^0$$

Summing exponents: The solution to this system is:

M: $a = 0$ $\qquad a = 0 \quad b = -1 \quad c = -1$ $\qquad \Pi_3 = \frac{V}{\omega \cdot D}$

L: $1 - 3 \cdot a + c = 0$

t: $-1 - b = 0$

$\Pi_4 = H \cdot \rho^a \cdot \omega^b \cdot D^c$ Thus: $L \cdot \left(\frac{M}{L^3}\right)^a \cdot \left(\frac{1}{t}\right)^b \cdot L^c = M^0 \cdot L^0 \cdot t^0$

Summing exponents:

M: $a = 0$

L: $1 - 3 \cdot a + c = 0$

t: $-b = 0$

The solution to this system is:

$a = 0 \qquad b = 0 \qquad c = -1$

$$\Pi_4 = \frac{H}{D}$$

6 Check using F, L, t dimensions: $\frac{F \cdot L}{t} \cdot \frac{L^4}{F \cdot t^2} \cdot t^3 \cdot \frac{1}{L^5} = 1 \quad \frac{F \cdot t}{L^2} \cdot \frac{L^4}{F \cdot t^2} \cdot t \cdot \frac{1}{L^2} = 1 \quad \frac{L}{t} \cdot t \cdot \frac{1}{L} = 1 \quad L \cdot \frac{1}{L} = 1$

The functional relationship is: $\Pi_1 = f(\Pi_2, \Pi_3, \Pi_4)$

$$\frac{P}{\rho \cdot \omega^3 \cdot D^5} = f\left(\frac{\mu}{\rho \cdot \omega \cdot D^2}, \frac{V}{\omega \cdot D}, \frac{H}{D}\right)$$

Problem 7.59

[Difficulty: 3]

7.59 Measurements of drag force are made on a model automobile in a towing tank filled with fresh water. The model length scale is $\frac{1}{5}$ that of the prototype. State the conditions required to ensure dynamic similarity between the model and prototype. Determine the fraction of the prototype speed in air at which the model test should be made in water to ensure dynamically similar conditions. Measurements made at various speeds show that the dimensionless force ratio becomes constant at model test speeds above $V_m = 4$ m/s. The drag force measured during a test at this speed is $F_{D_m} = 182$ N. Calculate the drag force expected on the prototype vehicle operating at 90 km/hr in air.

Given: Measurements of drag are made on a model car in a fresh water tank. The model is 1/5-scale.

Find: (a) Conditions requred to ensure dynamic similarity between the model and the prototype.
(b) Required fraction of speed in air at which the model needs to be tested in water to ensure dynamically similar conditions.
(c) Drag force on the prototype model traveling at 90 kph in air if the model drag is 182 N traveling at 4 m/s in water.

Solution: The flows must be geometrically and kinematically similar, and have equal Reynolds numbers to be dynamically similar:

Geometric similarity requires a true model in all respects.
Kinematic similarity requires the same flow pattern, i.e., no free-surface or cavitation effects.
The problem may be stated as $F = f(\rho,V,L,\mu)$

Dimensional analysis gives this relation: $\dfrac{F}{\rho\cdot V^2\cdot L^2} = g(Re)$ where $Re = \dfrac{\rho\cdot V\cdot L}{\mu} = \dfrac{V\cdot L}{\nu}$

Matching Reynolds numbers between the model and prototype flows: $\dfrac{V_m\cdot L_m}{\nu_m} = \dfrac{V_p\cdot L_p}{\nu_p}$ Thus: $\dfrac{V_m}{V_p} = \dfrac{\nu_m}{\nu_p}\cdot\dfrac{L_p}{L_m}$

From Tables A.8 and A.10 at 20 deg C: $\nu_w = 1.00\times10^{-6}\cdot\dfrac{m^2}{s}$ and $\nu_a = 1.51\times10^{-5}\cdot\dfrac{m^2}{s}$ Therefore:

$$\frac{V_m}{V_p} = \frac{1.00\times10^{-6}}{1.51\times10^{-5}}\times\frac{5}{1} = 0.331 \qquad \frac{V_m}{V_p} = 0.331$$

If the conditions are dynamically similar: $\dfrac{F_m}{\rho_m\cdot V_m^2\cdot L_m^2} = \dfrac{F_p}{\rho_p\cdot V_p^2\cdot L_p^2}$ Thus: $F_p = F_m\cdot\dfrac{\rho_p}{\rho_m}\cdot\left(\dfrac{V_p}{V_m}\right)^2\cdot\left(\dfrac{L_p}{L_m}\right)^2$

Substituting in known values: $$F_p = 182\cdot N\times\frac{1.20}{999}\times\left(90\cdot\frac{km}{hr}\times\frac{1000\cdot m}{km}\times\frac{hr}{3600\cdot s}\times\frac{s}{4\cdot m}\right)^2\times\left(\frac{5}{1}\right)^2 \qquad F_p = 213\,N$$

Problem 7.60

[Difficulty: 2]

7.60 On a cruise ship, passengers complain about the noise emanating from the ship's propellers (probably due to turbulent flow effects between the propeller and the ship). You have been hired to find out the source of this noise. You will study the flow pattern around the propellers and have decided to use a 1:9-scale water tank. If the ship's propellers rotate at 100 rpm, estimate the model propeller rotation speed if (a) the Froude number or (b) the Reynolds number is the governing dimensionless group. Which is most likely to lead to the best modeling?

Given: Flow around ship's propeller

Find: Model propeller speed using Froude number and Reynolds number

Solution:

Basic equations: $Fr = \frac{V}{\sqrt{g \cdot L}}$ $\qquad Re = \frac{V \cdot L}{\nu}$

Assumptions: (a) The model and the actual propeller are geometrically similar
(b) The flows about the propellers are kinematically and dynamically similar

Using the Froude number $\quad Fr_m = \frac{V_m}{\sqrt{g \cdot L_m}} = Fr_p = \frac{V_p}{\sqrt{g \cdot L_p}} \quad$ or $\quad \frac{V_m}{V_p} = \sqrt{\frac{L_m}{L_p}} \qquad (1)$

But the angular velocity is given by $\quad V = L \cdot \omega \quad$ so $\quad \frac{V_m}{V_p} = \frac{L_m}{L_p} \cdot \frac{\omega_m}{\omega_p} \qquad (2)$

Comparing Eqs. 1 and 2 $\quad \frac{L_m}{L_p} \cdot \frac{\omega_m}{\omega_p} = \sqrt{\frac{L_m}{L_p}} \qquad \frac{\omega_m}{\omega_p} = \sqrt{\frac{L_p}{L_m}}$

The model rotation speed is then $\quad \omega_m = \omega_p \cdot \sqrt{\frac{L_p}{L_m}} \qquad \omega_m = 100 \cdot rpm \times \sqrt{\frac{9}{1}} \qquad \omega_m = 300 \cdot rpm$

Using the Reynolds number $\quad Re_m = \frac{V_m \cdot L_m}{\nu_m} = Re_p = \frac{V_p \cdot L_p}{\nu_p} \quad$ or $\quad \frac{V_m}{V_p} = \frac{L_p}{L_m} \cdot \frac{\nu_m}{\nu_p} = \frac{L_p}{L_m} \qquad (3)$

(We have assumed the viscosities of the sea water and model water are comparable)

Comparing Eqs. 2 and 3 $\quad \frac{L_m}{L_p} \cdot \frac{\omega_m}{\omega_p} = \frac{L_p}{L_m} \qquad \frac{\omega_m}{\omega_p} = \left(\frac{L_p}{L_m}\right)^2$

The model rotation speed is then $\quad \omega_m = \omega_p \cdot \left(\frac{L_p}{L_m}\right)^2 \qquad \omega_m = 100 \cdot rpm \times \left(\frac{9}{1}\right)^2 \qquad \omega_m = 8100 \cdot rpm$

Of the two models, the Froude number appears most realistic; at 8100 rpm serious cavitation will occur, which would invalidate the similarity assumptions. Both flows will likely have high Reynolds numbers so that the flow becomes independent of Reynolds number; the Froude number is likely to be a good indicator of static pressure to dynamic pressure for this (although cavitation number would be better).

Problem 7.61

[Difficulty: 3]

7.61 A $\frac{1}{5}$-scale model of a torpedo is tested in a wind tunnel to determine the drag force. The prototype operates in water, has 533 mm diameter, and is 6.7 m long. The desired operating speed of the prototype is 28 m/s. To avoid compressibility effects in the wind tunnel, the maximum speed is limited to 110 m/s. However, the pressure in the wind tunnel can be varied while holding the temperature constant at 20°C. At what minimum pressure should the wind tunnel be operated to achieve a dynamically similar test? At dynamically similar test conditions, the drag force on the model is measured as 618 N. Evaluate the drag force expected on the full-scale torpedo.

Given: A torpedo with D = 533 mm and L = 6.7 m is to travel at 28 m/s in water. A 1/5 scale model of the torpedo is to be tested in a wind tunnel. The maximum speed in the tunnel is fixed at 110 m/s, but the pressure can be varied at a constant temperature of 20 deg C.

Find: (a) Minimum pressure required in the wind tunnel for dynamically similar testing.
(b) The expected drag on the prototype if the model drag is 618 N.

Solution: The problem may be stated as: $F = f(\rho, V, D, \mu)$ From the Buckingham pi theorem, we expect 2 Π terms:

$$\frac{F}{\rho \cdot V^2 \cdot D^2} = g(Re) \quad \text{where} \quad Re = \frac{\rho \cdot V \cdot D}{\mu}$$

Matching Reynolds numbers between the model and prototype flows: $\frac{\rho_m \cdot V_m \cdot D_m}{\mu_m} = \frac{\rho_p \cdot V_p \cdot D_p}{\mu_p}$ Thus: $\rho_m = \rho_p \cdot \frac{V_p}{V_m} \cdot \frac{D_p}{D_m} \cdot \frac{\mu_m}{\mu_p}$

At 20 deg C: $\mu_p = 1.00 \times 10^{-3} \cdot \frac{N \cdot s}{m^2}$ and $\mu_m = 1.81 \times 10^{-5} \cdot \frac{N \cdot s}{m^2}$ So substituting in values yields:

$$\rho_m = 998 \cdot \frac{kg}{m^3} \times \frac{28}{110} \times \frac{5}{1} \times \frac{1.81 \times 10^{-5}}{1.00 \times 10^{-3}} \qquad \rho_m = 23.0 \cdot \frac{kg}{m^3}$$

From the ideal gas equation of state: $p_m = \rho_m \cdot R \cdot T_m$

Substituting in values: $p_m = 23.0 \cdot \frac{kg}{m^3} \times 287 \cdot \frac{N \cdot m}{kg \cdot K} \times 293 \cdot K \times \frac{Pa \cdot m^2}{N}$ $\qquad p_m = 1.934 \cdot MPa$

If the conditions are dynamically similar: $\frac{F_m}{\rho_m \cdot V_m^2 \cdot D_m^2} = \frac{F_p}{\rho_p \cdot V_p^2 \cdot D_p^2}$ Thus: $F_p = F_m \cdot \frac{\rho_p}{\rho_m} \cdot \left(\frac{V_p}{V_m}\right)^2 \cdot \left(\frac{D_p}{D_m}\right)^2$

Substituting in known values: $F_p = 618 \cdot N \times \frac{998}{23.0} \times \left(\frac{28}{110}\right)^2 \times \left(\frac{5}{1}\right)^2$ $\qquad F_p = 43.4 \cdot kN$

Problem 7.62

[Difficulty: 3]

7.62 The drag of an airfoil at zero angle of attack is a function of density, viscosity, and velocity, in addition to a length parameter. A 1:5-scale model of an airfoil was tested in a wind tunnel at a speed of 130 ft/s, temperature of 59°F, and 5 atmospheres absolute pressure. The prototype airfoil has a chord length of 6 ft and is to be flown in air at standard conditions. Determine the Reynolds number at which the wind tunnel model was tested and the corresponding prototype speed at the same Reynolds number.

Given: A 1/10 scale airfoil was tested in a wind tunnel at known test conditions. Prototype airfoil has a chord length of 6 ft and is to be flown at standard conditions.

Find: (a) Reynolds number at which the model was tested
(b) Corresponding prototype speed

Solution:

Assumptions: (a) The viscosity of air does not vary appreciably between 1 and 5 atmospheres
(b) Geometric, kinematic, and dynamic similarity applies

The problem may be stated as: $F = f(\rho, V, L, \mu)$ From the Buckingham pi theorem, we expect 2 Π terms:

$$\frac{F}{\rho \cdot V^2 \cdot L^2} = g(Re) \quad \text{where} \quad Re = \frac{\rho \cdot V \cdot L}{\mu} \quad \text{The model chord length is} \quad L_m = \frac{6 \cdot ft}{5} = 1.20 \cdot ft$$

We can calculate the model flow density from the ideal gas equation of state: $\rho_m = \frac{p_m}{R \cdot T_m}$ Substituting known values:

$$\rho_m = \left(5 \cdot atm \times \frac{2116 \cdot lbf}{atm \cdot ft^2}\right) \times \frac{lbm \cdot R}{53.33 \cdot ft \cdot lbf} \times \frac{1}{519 \cdot R} \times \frac{slug}{32.2 \cdot lbm} \qquad \rho_m = 0.0119 \cdot \frac{slug}{ft^3}$$

At 59 deg F: $\mu_m = 3.74 \times 10^{-7} \cdot \frac{lbf \cdot s}{ft^2}$ Therefore:

$$Re_m = 0.0119 \cdot \frac{slug}{ft^3} \times 130 \cdot \frac{ft}{s} \times 1.2 \cdot ft \times \frac{ft^2}{3.74 \times 10^{-7} \cdot lbf \cdot s} \times \frac{lbf \cdot s^2}{slug \cdot ft}$$

$$Re_m = 5.0 \times 10^6$$

Matching Reynolds numbers between the model and prototype flows: $\frac{\rho_m \cdot V_m \cdot L_m}{\mu_m} = \frac{\rho_p \cdot V_p \cdot L_p}{\mu_p}$ Thus: $V_p = V_m \cdot \frac{\rho_m}{\rho_p} \cdot \frac{L_m}{L_p} \cdot \frac{\mu_p}{\mu_m}$

From the ideal gas equation of state: $\frac{\rho_m}{\rho_p} = \frac{p_m}{p_p} \cdot \frac{T_p}{T_m}$ Therefore: $V_p = V_m \cdot \frac{p_m}{p_p} \cdot \frac{T_p}{T_m} \cdot \frac{L_m}{L_p} \cdot \frac{\mu_p}{\mu_m}$ So substituting in values yields:

$$V_p = 130 \cdot \frac{ft}{s} \times \frac{5}{1} \times \frac{519}{519} \times \frac{1}{5} \times \frac{3.74 \times 10^{-7}}{3.74 \times 10^{-7}} \qquad V_p = 130.0 \cdot \frac{ft}{s}$$

Problem 7.63

[Difficulty: 2]

7.63 Consider a smooth sphere, of diameter D, immersed in a fluid moving with speed V. The drag force on a 10-ft-diameter weather balloon in air moving at 5 ft/s is to be calculated from test data. The test is to be performed in water using a 2-in.-diameter model. Under dynamically similar conditions, the model drag force is measured as 0.85 lbf. Evaluate the model test speed and the drag force expected on the full-scale balloon.

Given: Model of weather balloon

Find: Model test speed; drag force expected on full-scale balloon

Solution:

From Buckingham Π

$$\frac{F}{\rho \cdot V^2 \cdot D^2} = f\left(\frac{\nu}{V \cdot D}, \frac{V}{c}\right) = F(Re, M)$$

For similarity $\quad Re_p = Re_m \quad$ and $\quad M_p = M_m \quad$ (Mach number criterion satisified because M<<1)

Hence

$$Re_p = \frac{V_p \cdot D_p}{\nu_p} = Re_m = \frac{V_m \cdot D_m}{\nu_m}$$

$$V_m = V_p \cdot \frac{\nu_m}{\nu_p} \cdot \frac{D_p}{D_m}$$

From Table A.7 at 68°F $\quad \nu_m = 1.08 \times 10^{-5} \cdot \frac{ft^2}{s}$ $\qquad$ From Table A.9 at 68°F $\quad \nu_p = 1.62 \times 10^{-4} \cdot \frac{ft^2}{s}$

$$V_m = 5 \cdot \frac{ft}{s} \times \left(\frac{1.08 \times 10^{-5} \cdot \frac{ft^2}{s}}{1.62 \times 10^{-4} \cdot \frac{ft^2}{s}}\right) \times \left(\frac{10 \cdot ft}{\frac{1}{6} \cdot ft}\right) \qquad V_m = 20.0 \cdot \frac{ft}{s}$$

Then

$$\frac{F_m}{\rho_m \cdot V_m{}^2 \cdot D_m{}^2} = \frac{F_p}{\rho_p \cdot V_p{}^2 \cdot D_p{}^2} \qquad F_p = F_m \cdot \frac{\rho_p}{\rho_m} \cdot \frac{V_p{}^2}{V_m{}^2} \cdot \frac{D_p{}^2}{D_m{}^2}$$

$$F_p = 0.85 \cdot lbf \times \left(\frac{0.00234 \cdot \frac{slug}{ft^3}}{1.94 \cdot \frac{slug}{ft^3}}\right) \times \left(\frac{5 \cdot \frac{ft}{s}}{20 \cdot \frac{ft}{s}}\right)^2 \times \left(\frac{10 \cdot ft}{\frac{1}{6} \cdot ft}\right)^2 \qquad F_p = 0.231 \cdot lbf$$

Problem 7.64

[Difficulty: 2]

7.64 An airplane wing, with chord length of 1.5 m and span of 9 m, is designed to move through standard air at a speed of 7.5 m/s. A $\frac{1}{10}$-scale model of this wing is to be tested in a water tunnel. What speed is necessary in the water tunnel to achieve dynamic similarity? What will be the ratio of forces measured in the model flow to those on the prototype wing?

Given: Model of wing

Find: Model test speed for dynamic similarity; ratio of model to prototype forces

Solution:

We would expect $F = F(l, s, V, \rho, \mu)$ where F is the force (lift or drag), l is the chord and s the span

From Buckingham Π
$$\frac{F}{\rho \cdot V^2 \cdot l \cdot s} = f\left(\frac{\rho \cdot V \cdot l}{\mu}, \frac{l}{s}\right)$$

For dynamic similarity
$$\frac{\rho_m \cdot V_m \cdot l_m}{\mu_m} = \frac{\rho_p \cdot V_p \cdot l_p}{\mu_p}$$

Hence
$$V_m = V_p \cdot \frac{\rho_p}{\rho_m} \cdot \frac{l_p}{l_m} \cdot \frac{\mu_m}{\mu_p}$$

From Table A.8 at 20ºC $\mu_m = 1.01 \times 10^{-3} \cdot \frac{N \cdot s}{m^2}$ From Table A.10 at 20ºC $\mu_p = 1.81 \times 10^{-5} \cdot \frac{N \cdot s}{m^2}$

$$V_m = 7.5 \cdot \frac{m}{s} \times \left(\frac{1.21 \cdot \frac{kg}{m^3}}{998 \cdot \frac{kg}{m^3}}\right) \times \left(\frac{10}{1}\right) \times \left(\frac{1.01 \times 10^{-3} \cdot \frac{N \cdot s}{m^2}}{1.81 \times 10^{-5} \cdot \frac{N \cdot s}{m^2}}\right) \qquad V_m = 5.07 \frac{m}{s}$$

Then
$$\frac{F_m}{\rho_m \cdot V_m^{\ 2} \cdot l_m \cdot s_m} = \frac{F_p}{\rho_p \cdot V_p^{\ 2} \cdot l_p \cdot s_p} \qquad \frac{F_m}{F_p} = \frac{\rho_m}{\rho_p} \cdot \frac{V_m^{\ 2}}{V_p^{\ 2}} \cdot \frac{l_m \cdot s_m}{l_p \cdot s_p} = \frac{998}{1.21} \times \left(\frac{5.07}{7.5}\right)^2 \times \frac{1}{10} \times \frac{1}{10} = 3.77$$

Problem 7.65

[Difficulty: 3]

7.65 The fluid dynamic characteristics of a golf ball are to be tested using a model in a wind tunnel. Dependent parameters are the drag force, F_D, and lift force, F_L, on the ball. The independent parameters should include angular speed, ω, and dimple depth, d. Determine suitable dimensionless parameters and express the functional dependence among them. A golf pro can hit a ball at $V = 75$ m/s and $\omega = 8100$ rpm. To model these conditions in a wind tunnel with a maximum speed of 25 m/s, what diameter model should be used? How fast must the model rotate? (The diameter of a U.S. golf ball is 4.27 cm.)

Given: The fluid dynamic charachteristics of a gold ball are the be tested using a model in a wind tunnel. The dependent variables are the drag and lift forces. Independent variables include the angular speed and dimple depth. A pro golfer can hit a ball at a speed of 75 m/s and 8100 rpm. Wind tunnel maximum speed is 25 m/s.

Find: (a) Suitable dimensionless parameters and express the functional dependence between them.
(b) Required diameter of model
(c) Required rotational speed of model

Solution:

Assumption: Wind tunnel is at standard conditions

The problem may be stated as: $F_D = F_D(D, V, \omega, d, \rho, \mu)$ $\quad F_L = F_L(D, V, \omega, d, \rho, \mu)$ n = 7 and m = r = 3, so

from the Buckingham pi theorem, we expect two sets of four Π terms. The application of the Buckingham pi theorem will not be shown here, but the functional dependences would be:

$$\frac{F_D}{\rho \cdot V^2 \cdot D^2} = f\left(\frac{\rho \cdot V \cdot D}{\mu}, \frac{\omega \cdot D}{V}, \frac{d}{D}\right) \qquad \frac{F_L}{\rho \cdot V^2 \cdot D^2} = g\left(\frac{\rho \cdot V \cdot D}{\mu}, \frac{\omega \cdot D}{V}, \frac{d}{D}\right)$$

To determine the required model diameter, we match Reynolds numbers between the model and prototype flows:

$$\frac{\rho_m \cdot V_m \cdot D_m}{\mu_m} = \frac{\rho_p \cdot V_p \cdot D_p}{\mu_p}$$ Thus: $$D_m = D_p \cdot \frac{\rho_p}{\rho_m} \cdot \frac{V_p}{V_m} \cdot \frac{\mu_m}{\mu_p}$$ Substituting known values: $$D_m = 4.27 \cdot cm \times 1 \times \frac{75}{25} \times 1$$

$$D_m = 12.81 \cdot cm$$

To determine the required angular speed of the model, we match the dimensionless rotational speed between the flows:

$$\frac{\omega_m \cdot D_m}{V_m} = \frac{\omega_p \cdot D_p}{V_p}$$ Thus: $$\omega_m = \omega_p \cdot \frac{D_p}{D_m} \cdot \frac{V_m}{V_p}$$ Substituting known values: $$\omega_m = 8100 \cdot rpm \times \frac{4.27}{12.81} \times \frac{25}{75}$$ $$\omega_m = 900 \cdot rpm$$

Problem 7.66

[Difficulty: 3]

7.66 A water pump with impeller diameter 24 in. is to be designed to move 15 ft^3/s when running at 750 rpm. Testing is performed on a 1:4 scale model running at 2400 rpm using air (68°F) as the fluid. For similar conditions (neglecting Reynolds number effects), what will be the model flow rate? If the model draws 0.1 hp, what will be the power requirement of the prototype?

Given: Model of water pump

Find: Model flow rate for dynamic similarity (ignoring Re); Power of prototype

Solution:

From Buckingham Π $\quad \dfrac{Q}{\omega \cdot D^3} \quad$ and $\quad \dfrac{P}{\rho \cdot \omega^3 \cdot D^5}$ $\quad$ where Q is flow rate, ω is angular speed, d is diameter, and ρ is density (these Π groups will be discussed in Chapter 10)

For dynamic similarity
$$\frac{Q_m}{\omega_m \cdot D_m^{\ 3}} = \frac{Q_p}{\omega_p \cdot D_p^{\ 3}}$$

Hence
$$Q_m = Q_p \cdot \frac{\omega_m}{\omega_p} \cdot \left(\frac{D_m}{D_p}\right)^3$$

$$Q_m = 15 \cdot \frac{ft^3}{s} \times \left(\frac{2400}{750}\right) \times \left(\frac{1}{4}\right)^3 \qquad Q_m = 0.750 \cdot \frac{ft^3}{s}$$

From Table A.8 at 68°F $\quad \rho_p = 1.94 \cdot \dfrac{slug}{ft^3}$ $\quad$ From Table A.9 at 68°F $\quad \rho_m = 0.00234 \cdot \dfrac{slug}{ft^3}$

Then
$$\frac{P_m}{\rho_m \cdot \omega_m^{\ 3} \cdot D_m^{\ 5}} = \frac{P_p}{\rho_p \cdot \omega_p^{\ 3} \cdot D_p^{\ 5}}$$

$$P_p = P_m \cdot \frac{\rho_p}{\rho_m} \cdot \left(\frac{\omega_p}{\omega_m}\right)^3 \cdot \left(\frac{D_p}{D_m}\right)^5$$

$$P_p = 0.1 \cdot hp \times \frac{1.94}{0.00234} \times \left(\frac{750}{2400}\right)^3 \times \left(\frac{4}{1}\right)^5 \qquad P_p = 2.59 \times 10^3 \cdot hp$$

Note that if we had used water instead of air as the working fluid for the model pump, it would have drawn 83 hp. Water would have been an acceptable working fluid for the model, and there would have been less discrepancy in the Reynolds number.

Problem 7.67

[Difficulty: 2]

7.67 A model test is performed to determine the flight characteristics of a Frisbee. Dependent parameters are drag force, F_D, and lift force, F_L. The independent parameters should include angular speed, ω, and roughness height, h. Determine suitable dimensionless parameters, and express the functional dependence among them. The test (using air) on a 1:7-scale model Frisbee is to be geometrically, kinematically, and dynamically similar to the prototype. The wind tunnel test conditions are $V_m = 140$ ft/s and $\omega_m = 5000$ rpm. What are the corresponding values of V_p and ω_p?

Given: Model of Frisbee

Find: Dimensionless parameters; Prototype speed and angular speed

Solution:

Assumption: Geometric, kinematic, and dynamic similarity between model and prototype.

The functional dependence is $F = F(D, V, \omega, h, \rho, \mu)$ where F represents lift or drag

From Buckingham Π $\quad \dfrac{F}{\rho \cdot V^2 \cdot D^2} = f\left(\dfrac{\rho \cdot V \cdot D}{\mu}, \dfrac{\omega \cdot D}{V}, \dfrac{h}{D}\right)$

For dynamic similarity $\quad \dfrac{\rho_m \cdot V_m \cdot D_m}{\mu_m} = \dfrac{\rho_p \cdot V_p \cdot D_p}{\mu_p} \qquad V_p = V_m \cdot \dfrac{\rho_m}{\rho_p} \cdot \dfrac{D_m}{D_p} \cdot \dfrac{\mu_p}{\mu_m} \qquad V_p = 140 \cdot \dfrac{ft}{s} \times (1) \times \left(\dfrac{1}{7}\right) \times (1)$

$$V_p = 20 \cdot \frac{ft}{s}$$

Also $\quad \dfrac{\omega_m \cdot D_m}{V_m} = \dfrac{\omega_p \cdot D_p}{V_p} \qquad \omega_p = \omega_m \cdot \dfrac{D_m}{D_p} \cdot \dfrac{V_p}{V_m} \qquad \omega_p = 5000 \cdot rpm \times \left(\dfrac{1}{7}\right) \times \left(\dfrac{20}{140}\right)$

$$\omega_p = 102 \cdot rpm$$

Problem 7.68

[Difficulty: 3]

7.68 A model hydrofoil is to be tested at 1:20 scale. The test speed is chosen to duplicate the Froude number corresponding to the 60-knot prototype speed. To model cavitation correctly, the cavitation number also must be duplicated. At what ambient pressure must the test be run? Water in the model test basin can be heated to 130°F, compared to 45°F for the prototype.

Given: A 1:20 model of a hydrofoil is to be tested in water at 130 deg F. The prototype operates at a speed of 60 knots in water at 45 deg F. To model the cavitation, the cavitation number must be duplicated.

Find: Ambient pressure at which the test must be run

Solution: To duplicate the Froude number between the model and the prototype requires: $\frac{V_m}{\sqrt{g \cdot L_m}} = \frac{V_p}{\sqrt{g \cdot L_p}}$ Thus:

$$V_m = V_p \cdot \sqrt{\frac{L_m}{L_p}} \qquad V_m = 60 \cdot knot \cdot \sqrt{\frac{1}{20}} \qquad V_m = 13.42 \cdot knot$$

To match the cavitation number between the model and the prototype: $\frac{p_m - p_{vm}}{\frac{1}{2} \cdot \rho_m \cdot V_m^2} = \frac{p_p - p_{vp}}{\frac{1}{2} \cdot \rho_p \cdot V_p^2}$ Therefore:

$$p_m = p_{vm} + (p_p - p_{vp}) \cdot \frac{\rho_m}{\rho_p} \cdot \left(\frac{V_m}{V_p}\right)^2$$

Assuming that the densities are equal: $p_m = p_{vm} + (p_p - p_{vp}) \cdot \left(\frac{V_m}{V_p}\right)^2$

From table A.7: at 130 deg F $p_{vm} = 2.23 \cdot psi$ at 45 deg F $p_{vp} = 0.15 \cdot psi$ Thus the model pressure is:

$$p_m = 2.23 \cdot psi + (14.7 \cdot psi - 0.15 \cdot psi) \cdot \left(\frac{13.42}{60}\right)^2 \qquad p_m = 2.96 \cdot psi$$

Problem 7.69

[Difficulty: 2]

7.69 SAE 10W oil at 77°F flowing in a 1-in.-diameter horizontal pipe, at an average speed of 3 ft/s, produces a pressure drop of 7 psi (gage) over a 500-ft length. Water at 60°F flows through the same pipe under dynamically similar conditions. Using the results of Example 7.2, calculate the average speed of the water flow and the corresponding pressure drop.

Given: Oil flow in pipe and dynamically similar water flow

Find: Average water speed and pressure drop

Solution:

From Example 7.2 $$\frac{\Delta p}{\rho \cdot V^2} = f\left(\frac{\mu}{\rho \cdot V \cdot D}, \frac{l}{D}, \frac{e}{D}\right)$$

For dynamic similarity $$\frac{\mu_{H2O}}{\rho_{H2O} \cdot V_{H2O} \cdot D_{H2O}} = \frac{\mu_{Oil}}{\rho_{Oil} \cdot V_{Oil} \cdot D_{Oil}}$$ so $$V_{H2O} = \frac{\mu_{H2O}}{\rho_{H2O}} \cdot \frac{\rho_{Oil}}{\mu_{Oil}} \cdot V_{oil} = \frac{\nu_{H2O}}{\nu_{Oil}} \cdot V_{Oil}$$

From Fig. A.3 at 77°F $$\nu_{Oil} = 10.8 \times 8 \times 10^{-5} \cdot \frac{ft^2}{s} = 8.64 \times 10^{-4} \cdot \frac{ft^2}{s}$$

From Table A.8 at 60°F $$\nu_{H2O} = 1.21 \times 10^{-5} \cdot \frac{ft^2}{s}$$

Hence $$V_{H2O} = \frac{1.21 \times 10^{-5} \cdot \frac{ft^2}{s}}{8.64 \times 10^{-4} \cdot \frac{ft^2}{s}} \times 3 \cdot \frac{ft}{s}$$ $$V_{H2O} = 0.0420 \cdot \frac{ft}{s}$$

Then $$\frac{\Delta p_{Oil}}{\rho_{Oil} \cdot V_{Oil}^2} = \frac{\Delta p_{H2O}}{\rho_{H2O} \cdot V_{H2O}^2}$$ $$\Delta p_{H2O} = \frac{\rho_{H2O} \cdot V_{H2O}^2}{\rho_{Oil} \cdot V_{Oil}^2} \cdot \Delta p_{Oil}$$

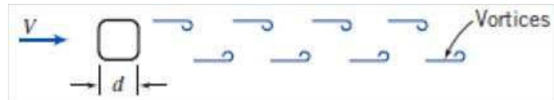
From Table A.2 $$SG_{Oil} = 0.92$$

$$\Delta p_{H2O} = \frac{1}{0.92} \times \left(\frac{0.0420}{3}\right)^2 \times 7 \cdot psi$$ $$\Delta p_{H2O} = 1.49 \times 10^{-3} \cdot psi$$

Problem 7.70

[Difficulty: 3]

7.70 In some speed ranges, vortices are shed from the rear of bluff cylinders placed across a flow. The vortices alternately leave the top and bottom of the cylinder, as shown, causing an alternating force normal to the freestream velocity. The vortex shedding frequency, f, is thought to depend on ρ, d, V, and μ. Use dimensional analysis to develop a functional relationship for f. Vortex shedding occurs in standard air on two cylinders with a diameter ratio of 2. Determine the velocity ratio for dynamic similarity, and the ratio of vortex shedding frequencies.



Given: The frequency of vortex shedding from the rear of a bluff cylinder is a function of ρ, d, V, and μ. Vortex shedding occurs in standard air on two cylinders with a diameter ratio of 2.

Find:
(a) Functional relationship for f using dimensional analysis
(b) Velocity ratio for vortex shedding
(c) Frequency ratio for vortex shedding

Solution: We will use the Buckingham pi-theorem.

1 $\quad f \quad \rho \quad d \quad V \quad \mu$ $\qquad$ n = 5 parameters

2 Select primary dimensions F, L, t:

3 $\quad f \quad \rho \quad d \quad V \quad \mu$

$$\frac{1}{t} \quad \frac{M}{L^3} \quad L \quad \frac{L}{t} \quad \frac{M}{L\cdot t}$$

r = 3 dimensions

4 $\quad \rho \quad V \quad d$ $\qquad$ m = r = 3 repeating parameters

5 We have n - m = 2 dimensionless groups. Setting up dimensional equations:

$$\Pi_1 = f\cdot\rho^a\cdot V^b\cdot d^c \qquad \text{Thus:} \qquad \frac{1}{t}\cdot\left(\frac{M}{L^3}\right)^a\cdot\left(\frac{L}{t}\right)^b\cdot L^c = M^0\cdot L^0\cdot t^0$$

Summing exponents: $\qquad$ The solution to this system is: $\qquad \Pi_1 = \frac{f\cdot d}{V}$

M: $a = 0$ $\qquad a = 0 \quad b = -1 \quad c = 1$

L: $-3\cdot a + b + c = 0$

t: $-1 - b = 0$

$$\Pi_2 = \mu\cdot\rho^a\cdot V^b\cdot d^c \qquad \text{Thus:} \qquad \frac{M}{L\cdot t}\cdot\left(\frac{M}{L^3}\right)^a\cdot\left(\frac{L}{t}\right)^b\cdot L^c = M^0\cdot L^0\cdot t^0$$

Summing exponents: $\qquad$ The solution to this system is: $\qquad \Pi_2 = \frac{\mu}{\rho\cdot V\cdot d}$

M: $1 + a = 0$ $\qquad a = -1 \quad b = -1 \quad c = -1$

L: $-1 - 3\cdot a + b + c = 0$

t: $-1 - b = 0$

6 Check using F, L, t dimensions: $\quad \frac{1}{t}\cdot\frac{t}{L}\cdot L = 1 \qquad \frac{F\cdot t}{L^2}\cdot\frac{L^4}{F\cdot t^2}\cdot\frac{t}{L}\cdot\frac{1}{L} = 1$

The functional relationship is: $\Pi_1 = f(\Pi_2)$ $$\frac{f \cdot d}{V} = f\left(\frac{\rho \cdot V \cdot d}{\mu}\right)$$

To achieve dynamic similarity between geometrically similar flows, we must duplicate all but one of the dimensionless groups:

$$\frac{\rho_1 \cdot V_1 \cdot d_1}{\mu_1} = \frac{\rho_2 \cdot V_2 \cdot d_2}{\mu_2} \qquad \frac{V_1}{V_2} = \frac{\rho_2}{\rho_1} \cdot \frac{d_2}{d_1} \cdot \frac{\mu_1}{\mu_2} = 1 \times \frac{1}{2} \times 1 \qquad \frac{V_1}{V_2} = \frac{1}{2}$$

Now if $\dfrac{\rho_1 \cdot V_1 \cdot d_1}{\mu_1} = \dfrac{\rho_2 \cdot V_2 \cdot d_2}{\mu_2}$ it follows that: $\dfrac{f_1 \cdot d_1}{V_1} = \dfrac{f_2 \cdot d_2}{V_2}$ and $\dfrac{f_1}{f_2} = \dfrac{d_2}{d_1} \cdot \dfrac{V_1}{V_2} = \dfrac{1}{2} \times \dfrac{1}{2}$ $\qquad \dfrac{f_1}{f_2} = \dfrac{1}{4}$

Problem 7.71

[Difficulty: 3]

7.71 A $\frac{1}{8}$-scale model of a tractor-trailer rig is tested in a pressurized wind tunnel. The rig width, height, and length are $W = 0.305$ m, $H = 0.476$ m, and $L = 2.48$ m, respectively. At wind speed $V = 75.0$ m/s, the model drag force is $F_D = 128$ N. (Air density in the tunnel is $\rho = 3.23$ kg/m^3.) Calculate the aerodynamic drag coefficient for the model. Compare the Reynolds numbers for the model test and for the prototype vehicle at 55 mph. Calculate the aerodynamic drag force on the prototype vehicle at a road speed of 55 mph into a headwind of 10 mph.

Given: 1/8-scale model of a tractor-trailer rig was tested in a pressurized wind tunnel.

Find:
(a) Aerodynamic drag coefficient for the model
(b) Compare the Reynolds numbers for the model and the prototype vehicle at 55 mph
(c) Calculate aerodynamic drag on the prototype at a speed of 55 mph into a headwind of 10 mph

Solution: We will use definitions of the drag coefficient and Reynolds number.

Governing Equations:

$$C_D = \frac{F_D}{\frac{1}{2}\cdot\rho\cdot V^2\cdot A} \qquad \text{(Drag Coefficient)}$$

$$Re = \frac{\rho\cdot V\cdot L}{\mu} \qquad \text{(Reynolds Number)}$$

Assume that the frontal area for the model is: $A_m = W_m\cdot H_m \quad A_m = 0.305\cdot m \times 0.476\cdot m \qquad A_m = 0.1452\cdot m^2$

The drag coefficient would then be: $C_{Dm} = 2 \times 128\cdot N \times \frac{m^3}{3.23\cdot kg} \times \left(\frac{s}{75.0\cdot m}\right)^2 \times \frac{1}{0.1452\cdot m^2} \times \frac{kg\cdot m}{N\cdot s^2} \qquad C_{Dm} = 0.0970$

From the definition of Re: $\frac{Re_m}{Re_p} = \frac{\rho_m}{\rho_p}\cdot\frac{V_m}{V_p}\cdot\frac{L_m}{L_p}\cdot\frac{\mu_p}{\mu_m}$ Assuming standard conditions and equal viscosities:

$$\frac{Re_m}{Re_p} = \frac{3.23}{1.23} \times \left(75\cdot\frac{m}{s} \times \frac{hr}{55\cdot mi} \times \frac{mi}{5280\cdot ft} \times \frac{ft}{0.3048\cdot m} \times \frac{3600\cdot s}{hr}\right) \times \frac{1}{8} \times 1 = 1 \qquad Re_m = Re_p$$

Since the Reynolds numbers match, assuming geometric and kinetic similarity we can say that the drag coefficients are equal:

$F_{Dp} = \frac{1}{2}\cdot C_D\cdot\rho_p\cdot V_p^{\,2}\cdot A_p$ Susbstituting known values yields:

$$F_{Dp} = \frac{1}{2} \times 0.0970 \times 1.23\cdot\frac{kg}{m^3} \times \left[(55 + 10)\frac{mi}{hr} \times \frac{5280\cdot ft}{mi} \times \frac{0.3048\cdot m}{ft} \times \frac{hr}{3600\cdot s}\right]^2 \times 0.1452\cdot m^2 \times 8^2 \times \frac{N\cdot s^2}{kg\cdot m} \qquad F_{Dp} = 468\,N$$

Problem 7.72

[Difficulty: 2]

7.72 On a cruise ship, passengers complain about the amount of smoke that becomes entrained behind the cylindrical smoke stack. You have been hired to study the flow pattern around the stack, and have decided to use a 1:15 scale model of the 15-ft smoke stack. What range of wind tunnel speeds could you use if the ship speed for which the problem occurs is 12 to 24 knots?

Given: Flow around cruise ship smoke stack

Find: Range of wind tunnel speeds

Solution:

For dynamic similarity $\dfrac{V_m \cdot D_m}{\nu_m} = \dfrac{V_p \cdot D_p}{\nu_p}$ or $V_m = \dfrac{D_p}{D_m} \cdot V_p = \dfrac{15}{1} \cdot V_p = 15 \cdot V_p$

Since $1 \cdot \text{knot} = 1 \cdot \dfrac{\text{nmi}}{\text{hr}}$ and $1 \cdot \text{nmi} = 6076.1 \cdot \text{ft}$

Hence for $V_p = 12 \cdot \dfrac{\text{nmi}}{\text{hr}} \times \dfrac{6076.1 \cdot \text{ft}}{\text{nmi}} \times \dfrac{\text{hr}}{3600 \cdot \text{s}}$ $\quad V_p = 20.254 \cdot \dfrac{\text{ft}}{\text{s}}$ $\quad V_m = 15 \times 20.254 \cdot \dfrac{\text{ft}}{\text{s}}$ $\quad V_m = 304 \cdot \dfrac{\text{ft}}{\text{s}}$

$V_p = 24 \cdot \dfrac{\text{nmi}}{\text{hr}} \times \dfrac{6076.1 \cdot \text{ft}}{\text{nmi}} \times \dfrac{\text{hr}}{3600 \cdot \text{s}}$ $\quad V_p = 40.507 \cdot \dfrac{\text{ft}}{\text{s}}$ $\quad V_m = 15 \times 40.507 \cdot \dfrac{\text{ft}}{\text{s}}$ $\quad V_m = 608 \cdot \dfrac{\text{ft}}{\text{s}}$

Note that these speeds are very high - compressibility effects may become important, since the Mach number is no longer much less than 1!

Problem 7.73

[Difficulty: 2]

7.73 The aerodynamic behavior of a flying insect is to be investigated in a wind tunnel using a 1:8-scale model. If the insect flaps its wings 60 times per second when flying at 1.5 m/s, determine the wind tunnel air speed and wing oscillation required for dynamic similarity. Do you expect that this would be a successful or practical model for generating an easily measurable wing lift? If not, can you suggest a different fluid (e.g., water, or air at a different pressure or temperature) that would produce a better modeling?

Given: Model of flying insect

Find: Wind tunnel speed and wing frequency; select a better model fluid

Solution: For dynamic similarity the following dimensionless groups must be the same in the insect and model (these are Reynolds number and Strouhal number, and can be obtained from a Buckingham Π analysis)

$$\frac{V_{insect}\cdot L_{insect}}{\nu_{air}} = \frac{V_m\cdot L_m}{\nu_m} \qquad \frac{\omega_{insect}\cdot L_{insect}}{V_{insect}} = \frac{\omega_m\cdot L_m}{V_m}$$

From Table A.9 (68°F) $\rho_{air} = 1.21\cdot\frac{kg}{m^3}$ $\qquad \nu_{air} = 1.50\times 10^{-5}\cdot\frac{m^2}{s}$

The given data is $\omega_{insect} = 60\cdot Hz$ $\qquad V_{insect} = 1.5\cdot\frac{m}{s}$ $\qquad \frac{L_{insect}}{L_m} = \frac{1}{8}$

Hence in the wind tunnel $V_m = V_{insect}\cdot\frac{L_{insect}}{L_m}\cdot\frac{\nu_m}{\nu_{air}} = V_{insect}\cdot\frac{L_{insect}}{L_m}$ $\qquad V_m = 1.5\cdot\frac{m}{s}\times\frac{1}{8}$ $\qquad V_m = 0.1875\frac{m}{s}$

Also $\omega_m = \omega_{insect}\cdot\frac{V_m}{V_{insect}}\cdot\frac{L_{insect}}{L_m}$ $\qquad \omega_m = 60\cdot Hz\times\frac{0.1875}{1.5}\times\frac{1}{8}$ $\qquad \omega_m = 0.9375\cdot Hz$

It is unlikely measurable wing lift can be measured at such a low wing frequency (unless the measured lift was averaged, using an integrator circuit, or perhaps a load cell and data acquisition system). Maybe try hot air (100°C) for the model

For hot air try $\nu_{hot} = 2.29\times 10^{-5}\cdot\frac{m^2}{s}$ instead of $\nu_{air} = 1.50\times 10^{-5}\cdot\frac{m^2}{s}$

Hence $\frac{V_{insect}\cdot L_{insect}}{\nu_{air}} = \frac{V_m\cdot L_m}{\nu_{hot}}$ $\qquad V_m = V_{insect}\cdot\frac{L_{insect}}{L_m}\cdot\frac{\nu_{hot}}{\nu_{air}}$ $\qquad V_m = 1.5\cdot\frac{m}{s}\times\frac{1}{8}\times\frac{2.29\times 10^{-5}}{1.50\times 10^{-5}}$ $\qquad V_m = 0.286\frac{m}{s}$

Also $\omega_m = \omega_{insect}\cdot\frac{V_m}{V_{insect}}\cdot\frac{L_{insect}}{L_m}$ $\qquad \omega_m = 60\cdot Hz\times\frac{0.286}{1.5}\times\frac{1}{8}$ $\qquad \omega_m = 1.43\cdot Hz$

Hot air does not improve things much. Try modeling in water $\nu_w = 1.01\times 10^{-6}\cdot\frac{m^2}{s}$

Hence $\frac{V_{insect}\cdot L_{insect}}{\nu_{air}} = \frac{V_m\cdot L_m}{\nu_w}$ $\qquad V_m = V_{insect}\cdot\frac{L_{insect}}{L_m}\cdot\frac{\nu_w}{\nu_{air}}$ $\qquad V_m = 1.5\cdot\frac{m}{s}\times\frac{1}{8}\times\frac{1.01\times 10^{-6}}{1.50\times 10^{-5}}$ $\qquad V_m = 0.01262\frac{m}{s}$

Also $\omega_m = \omega_{insect}\cdot\frac{V_m}{V_{insect}}\cdot\frac{L_{insect}}{L_m} = \omega_{insect}\cdot\frac{V_m}{V_{insect}}\cdot L_{ratio}$ $\qquad \omega_m = 60\cdot Hz\times\frac{0.01262}{1.5}\times\frac{1}{8}$ $\qquad \omega_m = 0.0631\cdot Hz$

This is even worse! It seems the best bet is hot (very hot) air for the wind tunnel. Alternatively, choose a much smaller wind tunnel model, e.g., a 2.5 X model would lead to V_m = 0.6 m/s and ω_m = 9.6 Hz

Problem 7.74

[Difficulty: 3]

7.74 A model test of a tractor-trailer rig is performed in a wind tunnel. The drag force, F_D, is found to depend on frontal area A, wind speed V, air density ρ, and air viscosity μ. The model scale is 1:4; frontal area of the model is 7 ft^2. Obtain a set of dimensionless parameters suitable to characterize the model test results. State the conditions required to obtain dynamic similarity between model and prototype flows. When tested at wind speed V = 300 ft/s in standard air, the measured drag force on the model was F_D = 550 lbf. Assuming dynamic similarity, estimate the aerodynamic drag force on the full-scale vehicle at V = 75 ft/s. Calculate the power needed to overcome this drag force if there is no wind.

Given: A model test of a 1:4 scale tractor-trailer rig is performed in standard air. The drag force is a function of A, V, ρ, and μ.

Find:
(a) Dimensionless parameters to characterize the model test results
(b) Conditions for dynamic similarity
(c) Drag force on the prototype vehicle based on test results
(d) Power needed to overcome the drag force

Solution: We will use the Buckingham pi-theorem.

1 $\quad F_D \quad A \quad V \quad \rho \quad \mu$ $\qquad$ n = 5 parameters

2 Select primary dimensions F, L, t:

3 $\quad F_D \quad A \quad V \quad \rho \quad \mu$

$$\frac{M\cdot L}{t^2} \quad L^2 \quad \frac{L}{t} \quad \frac{M}{L^3} \quad \frac{M}{L\cdot t}$$

r = 3 dimensions

4 $\quad \rho \quad V \quad A$ $\qquad$ m = r = 3 repeating parameters

5 We have n - m = 2 dimensionless groups. Setting up dimensional equations:

$$\Pi_1 = F\cdot\rho^a\cdot V^b\cdot A^c \qquad \text{Thus:} \qquad \frac{M\cdot L}{t^2}\cdot\left(\frac{M}{L^3}\right)^a\cdot\left(\frac{L}{t}\right)^b\cdot\left(L^2\right)^c = M^0\cdot L^0\cdot t^0$$

Summing exponents: $\qquad$ The solution to this system is:

M: $1 + a = 0$ $\qquad$ $a = -1 \quad b = -2 \quad c = -1$ $\qquad$ $\Pi_1 = \dfrac{F_D}{\rho\cdot V^2\cdot A}$

L: $1 - 3\cdot a + b + 2c = 0$

t: $-2 - b = 0$

$$\Pi_2 = \mu\cdot\rho^a\cdot V^b\cdot A^c \qquad \text{Thus:} \qquad \frac{M}{L\cdot t}\cdot\left(\frac{M}{L^3}\right)^a\cdot\left(\frac{L}{t}\right)^b\cdot\left(L^2\right)^c = M^0\cdot L^0\cdot t^0$$

Summing exponents: $\qquad$ The solution to this system is:

M: $1 + a = 0$ $\qquad$ $a = -1 \quad b = -1 \quad c = -\dfrac{1}{2}$ $\qquad$ $\Pi_2 = \dfrac{\mu}{\rho\cdot V\cdot\sqrt{A}}$

L: $-1 - 3\cdot a + b + 2\cdot c = 0$

t: $-1 - b = 0$

6 Check using F, L, t dimensions: $\quad F\cdot\dfrac{L^4}{F\cdot t^2}\cdot\dfrac{t^2}{L^2}\cdot\dfrac{1}{L^2} = 1 \qquad \dfrac{F\cdot t}{L^2}\cdot\dfrac{L^4}{F\cdot t^2}\cdot\dfrac{t}{L}\cdot\dfrac{1}{L} = 1$

For dynamic similarity: We must have geometric and kinematic similarity, and The Reynolds numbers must match.

Once dynamic similarity is insured, the drag coefficients must be equal: $\dfrac{F_{Dm}}{\frac{1}{2}\cdot\rho_m\cdot V_m^{\,2}\cdot A_m} = \dfrac{F_{Dp}}{\frac{1}{2}\cdot\rho_p\cdot V_p^{\,2}\cdot A_p}$

So for the prototype: $F_{Dp} = F_{Dm}\cdot\dfrac{\rho_p}{\rho_m}\cdot\left(\dfrac{V_p}{V_m}\right)^2\cdot\dfrac{A_p}{A_m}$ $\quad F_{Dp} = 550\cdot\text{lbf}\times\dfrac{0.00237}{0.00237}\times\left(\dfrac{75}{300}\right)^2\times 4^2$ $\quad F_{Dp} = 550\cdot\text{lbf}$

The power requirement would be: $P = F_{Dp}\cdot V_p$ $\quad P = 550\cdot\text{lbf}\times 75\cdot\dfrac{\text{ft}}{\text{s}}\times\dfrac{\text{hp}\cdot\text{s}}{550\cdot\text{ft}\cdot\text{lbf}}$ $\quad P = 75.0\cdot\text{hp}$

$P = 55.9\cdot\text{kW}$

Problem 7.75

[Difficulty: 2]

7.75 Tests are performed on a 1:10-scale boat model. What must be the kinematic viscosity of the model fluid if friction and wave drag phenomena are to be correctly modeled? The full-size boat will be used in a freshwater lake where the average water temperature is 50°F.

Given: Model of boat

Find: Model kinematic viscosity for dynamic similarity

Solution:

For dynamic similarity $\frac{V_m \cdot L_m}{\nu_m} = \frac{V_p \cdot L_p}{\nu_p}$ (1) $\frac{V_m}{\sqrt{g \cdot L_m}} = \frac{V_p}{\sqrt{g \cdot L_p}}$ (2) (from Buckingham Π; the first is the Reynolds number, the second the Froude number)

Hence from Eq 2 $\frac{V_m}{V_p} = \sqrt{\frac{g \cdot L_m}{g \cdot L_p}} = \sqrt{\frac{L_m}{L_p}}$

Using this in Eq 1 $\nu_m = \nu_p \cdot \frac{V_m}{V_p} \cdot \frac{L_m}{L_p} = \nu_p \cdot \sqrt{\frac{L_m}{L_p}} \cdot \frac{L_m}{L_p} = \nu_p \cdot \left(\frac{L_m}{L_p}\right)^{\frac{3}{2}}$

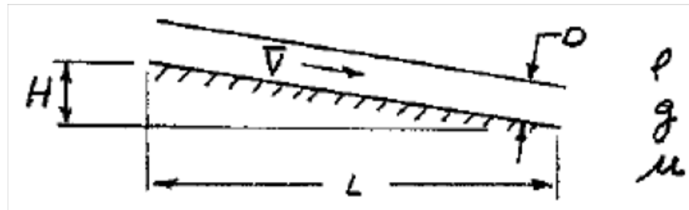
From Table A.8 at 50°F $\nu_p = 1.41 \times 10^{-5} \cdot \frac{ft^2}{s}$ $\nu_m = 1.41 \times 10^{-5} \cdot \frac{ft^2}{s} \times \left(\frac{1}{10}\right)^{\frac{3}{2}}$ $\nu_m = 4.46 \times 10^{-7} \cdot \frac{ft^2}{s}$

Note that there aren't any fluids in Figure A.3 with viscosities that low!

Problem 7.76

[Difficulty: 4]

7.76 Your favorite professor likes mountain climbing, so there is always a possibility that the professor may fall into a crevasse in some glacier. If that happened today, and the professor was trapped in a slowly moving glacier, you are curious to know whether the professor would reappear at the downstream drop-off of the glacier during this academic year. Assuming ice is a Newtonian fluid with the density of glycerine but a million times as viscous, you decide to build a glycerin model and use dimensional analysis and similarity to estimate when the professor would reappear. Assume the real glacier is 15 m deep and is on a slope that falls 1.5 m in a horizontal distance of 1850 m. Develop the dimensionless parameters and conditions expected to govern dynamic similarity in this problem. If the model professor reappears in the laboratory after 9.6 hours, when should you return to the end of the real glacier to provide help to your favorite professor?



Given: Model the motion of a glacier using glycerine. Assume ice as Newtonian fluid with density of glycerine but one million times as viscous. In laboratory test the professor reappears in 9.6 hours.

Find: (a) Dimensionless parameters to characterize the model test results
(b) Time needed for professor to reappear

Solution: We will use the Buckingham pi-theorem.

1 $\quad V \quad \rho \quad g \quad \mu \quad D \quad H \quad L$ $\qquad$ n = 7 parameters

2 $\quad$ Select primary dimensions F, L, t:

3 $\quad V \quad \rho \quad g \quad \mu \quad D \quad H \quad L$

$$\frac{L}{t} \quad \frac{M}{L^3} \quad \frac{L}{t^2} \quad \frac{M}{L\cdot t} \quad L \quad L \quad L$$

r = 3 dimensions

4 $\quad \rho \quad g \quad D$ $\qquad$ m = r = 3 repeating parameters

5 $\quad$ We have n - m = 4 dimensionless groups. Setting up dimensional equations:

$$\Pi_1 = V\cdot\rho^a\cdot g^b\cdot D^c \qquad \text{Thus:} \quad \frac{L}{t}\cdot\left(\frac{M}{L^3}\right)^a\cdot\left(\frac{L}{t^2}\right)^b\cdot L^c = M^0\cdot L^0\cdot t^0$$

Summing exponents:

M: $\quad a = 0$

L: $\quad 1 - 3\cdot a + b + c = 0$

t: $\quad -1 - 2\cdot b = 0$

The solution to this system is:

$$a = 0 \qquad b = -\frac{1}{2} \qquad c = -\frac{1}{2}$$

$$\Pi_1 = \frac{V}{\sqrt{g\cdot D}}$$

$$\Pi_2 = \mu\cdot\rho^a\cdot g^b\cdot D^c \qquad \text{Thus:} \quad \frac{M}{L\cdot t}\cdot\left(\frac{M}{L^3}\right)^a\cdot\left(\frac{L}{t^2}\right)^b\cdot L^c = M^0\cdot L^0\cdot t^0$$

Summing exponents:

M: $\quad 1 + a = 0$

L: $\quad -1 - 3\cdot a + b + c = 0$

t: $\quad -1 - 2\cdot b = 0$

The solution to this system is:

$$a = -1 \qquad b = -\frac{1}{2} \qquad c = -\frac{3}{2}$$

$$\Pi_2 = \frac{\mu}{\rho\sqrt{g\cdot D^3}}$$

(This is a gravity-driven version of Reynolds #)

$\Pi_3 = H \cdot \rho^a \cdot g^b \cdot D^c$ Thus: $L \cdot \left(\frac{M}{L^3}\right)^a \cdot \left(\frac{L}{t^2}\right)^b \cdot L^c = M^0 \cdot L^0 \cdot t^0$

Summing exponents:

M: $a = 0$

L: $-1 - 3 \cdot a + b + c = 0$

t: $-2 \cdot b = 0$

The solution to this system is:

$a = 0 \qquad b = 0 \qquad c = -1$

$$\Pi_3 = \frac{H}{D}$$

By inspection we can see that $\Pi_4 = \frac{L}{D}$

6 Check using F, L, t dimensions: $\frac{L}{t} \cdot \frac{t}{L^{\frac{1}{2}}} \cdot \frac{1}{L^{\frac{1}{2}}} = 1 \qquad \frac{F \cdot t}{L^2} \cdot \frac{L^4}{F \cdot t^2} \cdot \frac{t}{L^{\frac{1}{2}}} \cdot \frac{1}{L^{\frac{3}{2}}} = 1 \qquad L \cdot \frac{1}{L} = 1$

The functional relationship would be: $\Pi_1 = f(\Pi_2, \Pi_3, \Pi_4)$ Matching the last two terms insures geometric similarity.

For dynamic similarity: $\frac{\mu_m}{\rho_m \cdot \sqrt{g_m \cdot D_m^3}} = \frac{\mu_p}{\rho_p \cdot \sqrt{g_p \cdot D_p^3}}$ From Tables A.1 and A.2: $SG_{ice} = 0.92 \quad SG_{glycerine} = 1.26$

Therefore: $\frac{D_m}{D_p} = \left(\frac{\mu_m}{\mu_p} \cdot \frac{\rho_p}{\rho_m}\right)^{\frac{2}{3}} = \left(\frac{1}{10^6} \times \frac{0.92}{1.26}\right)^{\frac{2}{3}} = 8.11 \times 10^{-5}$ Since we have geometric similarity, the last two terms must match for model and prototype:

So $\frac{L_m}{L_p} = 8.11 \times 10^{-5}$ $L_m = 1850 \cdot m \times 8.11 \times 10^{-5}$ Matching the first Π term: $\frac{V_m}{V_p} = \sqrt{\frac{D_m}{D_p}} = 0.00900$

$L_m = 0.1500\,m$

The time needed to reappear would be: $\tau = \frac{L}{V}$ Thus: $\tau_m = \frac{L_m}{V_m} \quad V_m = \frac{L_m}{\tau_m}$ Solving for the actual time:

$$\tau_p = \frac{L_p}{V_p} = \frac{L_m}{V_m} \cdot \frac{L_p}{L_m} \cdot \frac{V_m}{V_p} = \tau_m \cdot \frac{L_p}{L_m} \cdot \frac{V_m}{V_p} \qquad \tau_p = 9.6 \cdot hr \times \frac{1}{8.11 \cdot 10^{-5}} \times 0.00900 \times \frac{day}{24 \cdot hr}$$

$$\tau_p = 44.4 \cdot day$$

Your professor will be back before the end of the semester!

Problem 7.77

[Difficulty: 3]

7.77 An automobile is to travel through standard air at 60 mph. To determine the pressure distribution, a $\frac{1}{5}$-scale model is to be tested in water. What factors must be considered to ensure kinematic similarity in the tests? Determine the water speed that should be used. What is the corresponding ratio of drag force between prototype and model flows? The lowest pressure coefficient is $C_p = -1.4$ at the location of the minimum static pressure on the surface. Estimate the minimum tunnel pressure required to avoid cavitation, if the onset of cavitation occurs at a cavitation number of 0.5.

Given: Model of automobile

Find: Factors for kinematic similarity; Model speed; ratio of protype and model drags; minimum pressure for no cavitation

Solution:

For dynamic similarity
$$\frac{\rho_m \cdot V_m \cdot L_m}{\mu_m} = \frac{\rho_p \cdot V_p \cdot L_p}{\mu_p} \qquad V_m = V_p \cdot \frac{\rho_p}{\rho_m} \cdot \frac{L_p}{L_m} \cdot \frac{\mu_m}{\mu_p}$$

For air (Table A.9) and water (Table A.7) at 68°F

$$\rho_p = 0.00234 \cdot \frac{slug}{ft^3} \qquad \mu_p = 3.79 \times 10^{-7} \cdot \frac{lbf \cdot s}{ft^2}$$

$$\rho_m = 1.94 \cdot \frac{slug}{ft^3} \qquad \mu_m = 2.10 \times 10^{-5} \cdot \frac{lbf \cdot s}{ft^2}$$

$$V_m = 60 \cdot mph \times \frac{88 \cdot \frac{ft}{s}}{60 \cdot mph} \times \left(\frac{0.00234}{1.94}\right) \times \left(\frac{5}{1}\right) \times \left(\frac{2.10 \times 10^{-5}}{3.79 \times 10^{-7}}\right) \qquad V_m = 29.4 \cdot \frac{ft}{s}$$

Then
$$\frac{F_m}{\rho_m \cdot V_m^2 \cdot L_m^2} = \frac{F_p}{\rho_p \cdot V_p^2 \cdot L_p^2}$$

Hence
$$\frac{F_p}{F_m} = \frac{\rho_p \cdot V_p^2 \cdot L_p^2}{\rho_m \cdot V_m^2 \cdot L_m^2} = \left(\frac{0.00234}{1.94}\right) \times \left(\frac{88}{29.4}\right)^2 \times \left(\frac{5}{1}\right)^2 \qquad \frac{F_p}{F_m} = 0.270$$

For Ca = 0.5
$$\frac{p_{min} - p_v}{\frac{1}{2} \cdot \rho \cdot V^2} = 0.5 \qquad \text{so we get} \qquad p_{min} = p_v + \frac{1}{4} \cdot \rho \cdot V^2 \qquad \text{for the water tank}$$

From steam tables, for water at 68°F $\quad p_v = 0.339 \cdot psi \quad$ so

$$p_{min} = 0.339 \cdot psi + \frac{1}{4} \times 1.94 \cdot \frac{slug}{ft^3} \times \left(29.4 \cdot \frac{ft}{s}\right)^2 \times \frac{lbf \cdot s^2}{slug \cdot ft} \times \left(\frac{1 \cdot ft}{12 \cdot in}\right)^2 \qquad p_{min} = 3.25 \cdot psi$$

This is the minimum allowable pressure in the water tank; we can use it to find the required tank pressure

$$C_p = -1.4 = \frac{p_{min} - p_{tank}}{\frac{1}{2} \cdot \rho \cdot V^2} \qquad p_{tank} = p_{min} + \frac{1.4}{2} \cdot \rho \cdot V^2 = p_{min} + 0.7 \cdot \rho \cdot V^2$$

$$p_{tank} = 3.25 \cdot psi + 0.7 \times 1.94 \cdot \frac{slug}{ft^3} \times \left(29.4 \cdot \frac{ft}{s}\right)^2 \times \frac{lbf \cdot s^2}{slug \cdot ft} \times \left(\frac{1 \cdot ft}{12 \cdot in}\right)^2 \qquad p_{tank} = 11.4 \cdot psi$$

Problem 7.78

[Difficulty: 3]

7.78 A 1:50-scale model of a submarine is to be tested in a towing tank under two conditions: motion at the free surface and motion far below the surface. The tests are performed in freshwater. On the surface, the submarine cruises at 24 knots. At what speed should the model be towed to ensure dynamic similarity? Far below the surface, the sub cruises at 0.35 knot. At what speed should the model be towed to ensure dynamic similarity? What must the drag of the model be multiplied by under each condition to give the drag of the full-scale submarine?

Given: A scale model of a submarine is to be tested in fresh water under two conditions:
1 - on the surface
2 - far below the surface

Find:
(a) Speed for the model test on the surface
(b) Speed for the model test submerged
(c) Ratio of full-scale drag to model drag

Solution: On the surface, we need to match Froude numbers: $\frac{V_m}{\sqrt{g \cdot L_m}} = \frac{V_p}{\sqrt{g \cdot L_p}}$ or: $V_m = V_p \cdot \sqrt{\frac{L_m}{L_p}}$

Thus for 1:50 scale: $V_m = 24 \cdot \text{knot} \times \sqrt{\frac{1}{50}}$ $\qquad V_m = 3.39 \cdot \text{knot}$ or $V_m = 1.75 \cdot \frac{m}{s}$

When submerged, we need to match Reynolds numbers: $\frac{\rho_m \cdot V_m \cdot L_m}{\mu_m} = \frac{\rho_p \cdot V_p \cdot L_p}{\mu_p}$ or: $V_m = V_p \cdot \frac{\rho_p}{\rho_m} \cdot \frac{L_p}{L_m} \cdot \frac{\mu_m}{\mu_p}$

From Table A.2, $SG_{seawater} = 1.025$ and $\mu_{seawater} = 1.08 \times 10^{-3} \cdot \frac{N \cdot s}{m^2}$ at 20°C. Thus for 1:50 scale:

$V_m = 0.35 \cdot \text{knot} \times \frac{1.025}{0.998} \times \frac{50}{1} \times \frac{1.08 \times 10^{-3}}{1.00 \times 10^{-3}}$ $\qquad V_m = 19.41 \cdot \text{knot}$ or $V_m = 9.99 \cdot \frac{m}{s}$

Under dynamically similar conditions, the drag coefficients will match: $\frac{F_{Dm}}{\frac{1}{2} \cdot \rho_m \cdot V_m^2 \cdot A_m} = \frac{F_{Dp}}{\frac{1}{2} \cdot \rho_p \cdot V_p^2 \cdot A_p}$

Solving for the ratio of forces: $\frac{F_{Dp}}{F_{Dm}} = \frac{\rho_p}{\rho_m} \cdot \left(\frac{V_p}{V_m}\right)^2 \cdot \frac{A_p}{A_m} = \frac{\rho_p}{\rho_m} \cdot \left(\frac{V_p}{V_m} \cdot \frac{L_p}{L_m}\right)^2$ Substituting in known values:

For surface travel: $\frac{F_{Dp}}{F_{Dm}} = \frac{1.025}{0.998} \times \left(\frac{24}{3.39} \times \frac{50}{1}\right)^2 = 1.29 \times 10^5$ $\qquad \frac{F_{Dp}}{F_{Dm}} = 1.29 \times 10^5$ (on surface)

For submerged travel: $\frac{F_{Dp}}{F_{Dm}} = \frac{1.025}{0.998} \times \left(\frac{0.35}{19.41} \times \frac{50}{1}\right)^2 = 0.835$ $\qquad \frac{F_{Dp}}{F_{Dm}} = 0.835$ (submerged)

Problem 7.79

[Difficulty: 2]

7.79 A wind tunnel is being used to study the aerodynamics of a full-scale model rocket that is 12 in. long. Scaling for drag calculations are based on the Reynolds number. The rocket has an expected maximum velocity of 120 mph. What is the Reynolds number at this speed? Assume ambient air is at 68°F. The wind tunnel is capable of speeds up to 100 mph; so an attempt is made to improve this top speed by varying the air temperature. Calculate the equivalent speed for the wind tunnel using air at 40°F and 150°F. Would replacing air with carbon dioxide provide higher equivalent speeds?

Given: Model size, model speed, and air temperatures.

Find: Equivalent speed of the full scale vehicle corresponding to the different air temperatures.

Solution:

Governing Equation:

$$Re_L = \frac{VL}{\nu} \qquad \text{(Reynolds Number)}$$

where V is the air velocity, L is the length of the rocket or model, and , ν is the kinematic viscosity of air. Subscript m corresponds to the model and r is the rocket.

Assumption: Modeling follows the Reynolds equivalency.

The given or available data is: $L_m = L_R = 12\text{in}$ $\quad V_T = 100\text{mph}$ $\quad V_R = 120\text{mph}$

$$\nu_{40°\text{F}} = 1.47\times10^{-4}\,\frac{\text{ft}^2}{\text{s}} \text{ (Table A.9)} \qquad \nu_{68°\text{F}} = 1.62\times10^{-4}\,\frac{\text{ft}^2}{\text{s}} \text{ (Table A.9)}$$

$$\nu_{150°\text{F}} = 2.09\times10^{-4}\,\frac{\text{ft}^2}{\text{s}} \text{ (Table A.9)}$$

$$\nu_{CO_2} = 8.3\times10^{-6}\,\frac{\text{m}^2}{\text{s}} \text{ (Figure A.3 or other source)}$$

Determine the Reynolds Number for expected maximum speed at ambient temperature:

$$Re_L = \frac{V_R L_R}{\nu_R} = \frac{120\,\frac{\text{mile}}{\text{hr}}\times\frac{5280\text{ft}}{\text{mile}}\times\frac{\text{hr}}{3600\text{s}}\times12\text{in}\times\frac{\text{ft}}{12\text{in}}}{1.62\times10^{-4}\times\frac{\text{ft}^2}{\text{s}}} \qquad Re_L = 1.09\times10^6$$

Re-arrange the Reynolds Number Equation for speed equivalents: $Re_L = \dfrac{VL}{\nu} \Rightarrow V_R = V_T\times\dfrac{L_M}{L_R}\times\dfrac{\nu_{68°F}}{\nu_T}$

In this problem, the only term that changes is ν_T

Solve for speed at the low temperature: $V_R = V_T\times\dfrac{L_M}{L_R}\times\dfrac{\nu_{68°\text{F}}}{\nu_{40°\text{F}}} = 100\text{mph}\times\dfrac{12\text{in}}{12\text{in}}\times\dfrac{1.62\times10^{-4}\,\frac{\text{ft}^2}{\text{s}}}{1.47\times10^{-4}\,\frac{\text{ft}^2}{\text{s}}} = 110\text{mph}$

Solve for speed at the high temperature: $V_R = V_T \times \dfrac{L_M}{L_R} \times \dfrac{\nu_{68°\mathrm{F}}}{\nu_{150°\mathrm{F}}} = 100\mathrm{mph} \times \dfrac{12\mathrm{in}}{12\mathrm{in}} \times \dfrac{1.62\times10^{-4}\,\dfrac{\mathrm{ft}^2}{\mathrm{s}}}{2.09\times10^{-4}\,\dfrac{\mathrm{ft}^2}{\mathrm{s}}} = 77.5\mathrm{mph}$

Solve for CO_2: $V_R = V_T \times \dfrac{L_M}{L_R} \times \dfrac{\nu_{68°\mathrm{F}}}{\nu_{CO_2}} = 100\mathrm{mph} \times \dfrac{12\mathrm{in}}{12\mathrm{in}} \times \dfrac{1.62\times10^{-4}\,\dfrac{\mathrm{ft}^2}{\mathrm{s}}}{8.3\times10^{-6}\,\dfrac{\mathrm{m}^2}{\mathrm{s}} \times \left(\dfrac{\mathrm{ft}}{0.305\mathrm{m}}\right)^2} = 181\mathrm{mph}$

$V_{R@40°\mathrm{F}} = 110\mathrm{mph}$ $\quad V_{R@150°\mathrm{F}} = 77.5\mathrm{mph}$ $\quad V_{R@CO_2} = 181\mathrm{mph}$

Chilling the air to 40°F increases the model speed, but not enough to achieve the target. Heating the air works against the desired outcome.

This shows that the equivalent speed can be increased by decreasing the kinematic viscosity. An inspection of figure A.3 shows that cooling air decreases the kinematic viscosity. It also shows that CO_2 has a lower kinematic viscosity than air resulting in much higher model speeds.

Problem 7.80

[Difficulty: 3]

7.80 Consider water flow around a circular cylinder, of diameter D and length l. In addition to geometry, the drag force is known to depend on liquid speed, V, density, ρ, and viscosity, μ. Express drag force, F_D, in dimensionless form as a function of all relevant variables. The static pressure distribution on a circular cylinder, measured in the laboratory, can be expressed in terms of the dimensionless pressure coefficient; the lowest pressure coefficient is $C_p = -2.4$ at the location of the minimum static pressure on the cylinder surface. Estimate the maximum speed at which a cylinder could be towed in water at atmospheric pressure, without causing cavitation, if the onset of cavitation occurs at a cavitation number of 0.5.

Given: The drag force on a circular cylinder immersed in a water flow can be expressed as a function of D, l, V, ρ, and μ. Static pressure distribution can be expressed in terms of the pressure coefficient. At the minimum static pressure, the pressure coefficient is equal to -2.4. Cavitation onset occurs at a cavitation number of 0.5.

Find: (a) Drag force in dimensionless form as a function of all relevant variables
(b) Maximum speed at which a cylinder could be towed in water at atmospheric pressure without cavitation

Solution: The functional relationship for drag force is: $F_D = F_D(D, l, V, \rho, \mu)$ From the Buckingham Π-theorem, we have 6 variables and 3 repeating parameters. Therefore, we will have 3 dimensionless groups. The functional form of these groups is:

$$\frac{F_D}{\rho \cdot V^2 \cdot D^2} = g\left(\frac{l}{D}, \frac{\rho \cdot V \cdot D}{\mu}\right)$$

The pressure coefficient is: $C_P = \dfrac{p - p_{inf}}{\frac{1}{2} \cdot \rho \cdot V^2}$ and the cavitation number is: $Ca = \dfrac{p - p_v}{\frac{1}{2} \cdot \rho \cdot V^2}$

At the minimum pressure point $p_{min} = p_{inf} + \frac{1}{2} \cdot \rho \cdot V_{max}^2 \cdot C_{Pmin}$ where $C_{Pmin} = -2.4$

At the onset of cavitation $p_{min} = p_v + \frac{1}{2} \cdot \rho \cdot V_{max}^2 \cdot Ca$ where $Ca = 0.5$

Equating these two expressions: $p_{inf} + \frac{1}{2} \cdot \rho \cdot V_{max}^2 \cdot C_{Pmin} = p_v + \frac{1}{2} \cdot \rho \cdot V_{max}^2 \cdot Ca$ and if we solve for Vmax:

$V_{max} = \sqrt{\dfrac{2(p_{inf} - p_v)}{\rho \cdot (Ca - C_{Pmin})}}$ At room temperature (68 deg F): $p_v = 0.339 \cdot psi$ $\rho = 1.94 \cdot \dfrac{slug}{ft^3}$

Substituting values we get:

$$V_{max} = \sqrt{2 \times (14.7 - 0.339) \cdot \frac{lbf}{in^2} \times \frac{ft^3}{1.94 \cdot slug} \times \frac{1}{[0.5 - (-2.4)]} \times \frac{slug \cdot ft}{lbf \cdot s^2} \times \frac{144 \cdot in^2}{ft^2}} \qquad V_{max} = 27.1 \cdot \frac{ft}{s}$$

Problem 7.81

[Difficulty: 4]

7.81 A circular container, partially filled with water, is rotated about its axis at constant angular speed, ω. At any time, τ, from the start of rotation, the speed, V_θ, at distance r from the axis of rotation, was found to be a function of τ, ω, and the properties of the liquid. Write the dimensionless parameters that characterize this problem. If, in another experiment, honey is rotated in the same cylinder at the same angular speed, determine from your dimensionless parameters whether honey will attain steady motion as quickly as water. Explain why the Reynolds number would not be an important dimensionless parameter in scaling the steady-state motion of liquid in the container.

Given: A circular container partiall filled with water is rotate about its axis at constant angular velocity ω. Velocity in the θ direction is a function of r, τ, ω, ρ, and μ.

Find:
(a) Dimensionless parameters that characterize this problem
(b) If honey would attain steady motion as quickly as water if rotated at the same angular speed
(c) Why Reynolds number is not an important parameter in scaling the steady-state motion of liquid in the container.

Solution: The functional relationship for drag force is: $V_\theta = V_\theta(\omega, r, \tau, \rho, \mu)$ From the Buckingham Π-theorem, we have 6 variables and 3 repeating parameters. Therefore, we will have 3 dimensionless groups. The functional form of these groups is:

$$\frac{V_\theta}{\omega \cdot r} = g\left(\frac{\mu}{\rho \cdot \omega \cdot r^2}, \omega \cdot \tau\right)$$

From the above result $\Pi_2 = \dfrac{\mu}{\rho \cdot \omega \cdot r^2}$ containing the properties μ and ρ, and $\Pi_3 = \omega \cdot \tau$ containing the time τ

$\Pi_2 \cdot \Pi_3 = \dfrac{\mu}{\rho \cdot \omega \cdot r^2} \cdot \omega \cdot \tau = \dfrac{\mu \cdot \tau}{\rho \cdot r^2} = \dfrac{\nu \cdot \tau}{r^2}$ Now for steady flow: $\dfrac{\nu_h \cdot \tau_h}{r^2} = \dfrac{\nu_w \cdot \tau_w}{r^2}$ and at the same radius:

$\nu_h \cdot \tau_h = \nu_w \cdot \tau_w \qquad \tau_h = \tau_w \cdot \dfrac{\nu_w}{\nu_h}$ Now since honey is more viscous than water, it follows that: $\tau_h < \tau_w$

At steady state, solid body rotation exists. There are no viscous forces, and therefore, the Reynolds number would not be important.

Problem 7.82

[Difficulty: 3]

7.82 A $\frac{1}{10}$-scale model of a tractor-trailer rig is tested in a wind tunnel. The model frontal area is $A_m = 0.1\ m^2$. When tested at $V_m = 75$ m/s in standard air, the measured drag force is $F_D = 350$ N. Evaluate the drag coefficient for the model conditions given. Assuming that the drag coefficient is the same for model and prototype, calculate the drag force on a prototype rig at a highway speed of 90 km/hr. Determine the air speed at which a model should be tested to ensure dynamically similar results if the prototype speed is 90 km/hr. Is this air speed practical? Why or why not?

Given: Model of tractor-trailer truck

Find: Drag coefficient; Drag on prototype; Model speed for dynamic similarity

Solution:

For kinematic similarity we need to ensure the geometries of model and prototype are similar, as is the incoming flow field

The drag coefficient is
$$C_D = \frac{F_m}{\frac{1}{2}\cdot\rho_m\cdot V_m^{\ 2}\cdot A_m}$$

For air (Table A.10) at 20°C
$$\rho_m = 1.21\cdot\frac{kg}{m^3} \qquad \mu_p = 1.81\times 10^{-5}\cdot\frac{N\cdot s}{m^2}$$

$$C_D = 2\times 350\cdot N\times\frac{m^3}{1.21\cdot kg}\times\left(\frac{s}{75\cdot m}\right)^2\times\frac{1}{0.1\cdot m^2}\times\frac{N\cdot s^2}{kg\cdot m} \qquad C_D = 1.028$$

This is the drag coefficient for model and prototype

For the rig
$$F_p = \frac{1}{2}\cdot\rho_p\cdot V_p^{\ 2}\cdot A_p\cdot C_D \quad \text{with} \quad \frac{A_p}{A_m} = \left(\frac{L_p}{L_m}\right)^2 = 100 \qquad A_p = 10\cdot m^2$$

$$F_p = \frac{1}{2}\times 1.21\cdot\frac{kg}{m^3}\times\left(90\cdot\frac{km}{hr}\times\frac{1000\cdot m}{1\cdot km}\times\frac{1\cdot hr}{3600\cdot s}\right)^2\times 10\cdot m^2\times 1.028\times\frac{N\cdot s^2}{kg\cdot m} \qquad F_p = 3.89\cdot kN$$

For dynamic similarity
$$\frac{\rho_m\cdot V_m\cdot L_m}{\mu_m} = \frac{\rho_p\cdot V_p\cdot L_p}{\mu_p} \qquad V_m = V_p\cdot\frac{\rho_p}{\rho_m}\cdot\frac{L_p}{L_m}\cdot\frac{\mu_m}{\mu_p} = V_p\cdot\frac{L_p}{L_m}$$

$$V_m = 90\cdot\frac{km}{hr}\times\frac{1000\cdot m}{1\cdot km}\times\frac{1\cdot hr}{3600\cdot s}\times\frac{10}{1} \qquad V_m = 250\,\frac{m}{s}$$

For air at standard conditions, the speed of sound is $c = \sqrt{k\cdot R\cdot T}$

$$c = \sqrt{1.40\times 286.9\cdot\frac{N\cdot m}{kg\cdot K}\times(20+273)\cdot K\times\frac{kg\cdot m}{s^2\cdot N}} \qquad c = 343\,\frac{m}{s}$$

Hence we have
$$M = \frac{V_m}{c} = \frac{250}{343} = 0.729$$
which indicates compressibility is significant - this model speed is impractical (and unnecessary)

Problem 7.83

[Difficulty: 3]

7.83 It is recommended in [8] that the frontal area of a model be less than 5 percent of the wind tunnel test section area and $Re = Vw/\nu > 2 \times 10^6$, where w is the model width. Further, the model height must be less than 30 percent of the test section height, and the maximum projected width of the model at maximum yaw (20°) must be less than 30 percent of the test section width. The maximum air speed should be less than 300 ft/s to avoid compressibility effects. A model of a tractor-trailer rig is to be tested in a wind tunnel that has a test section 1.5 ft high and 2 ft wide. The height, width, and length of the full-scale rig are 13 ft 6 in., 8 ft, and 65 ft, respectively. Evaluate the scale ratio of the largest model that meets the recommended criteria. Assess whether an adequate Reynolds number can be achieved in this test facility.

Given: Recommended procedures for wind tunnel tests of trucks and buses suggest:
-Model frontal area less than 5% of test section area
-Reynolds number based on model width greater than 2,000,000
-Model height less than 30% of test section height
-Model projected width at maximum yaw (20 deg) less than 30% of test section width
-Air speed less than 300 ft/s to avoid compressibility effects
Model of a tractor-trailer to be tested in a tunnel 1.5 ft high x 2 ft wide. Full scale rig is 13'6" high, 8' wide, and 65' long.

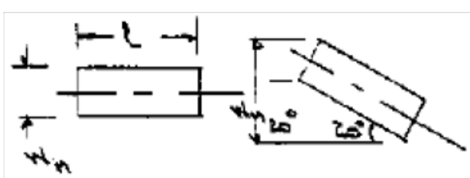
Find: (a) Max scale for tractor-trailer model in this tunnel
(b) If adequate Reynolds number can be achieved in this facility.

Solution: Let s be the scale ratio. Then: $h_m = s \cdot h_p \quad w_m = s \cdot w_p \quad l_m = s \cdot l_p$

Area criterion: $A_m = 0.05 \times 1.5 \cdot ft \times 2.0 \cdot ft \quad A_m = 0.15 \cdot ft^2$ Therefore: $s = \sqrt{\dfrac{0.15}{13.5 \times 8}}$ $\quad s = 0.0373$

Height criterion: $h_m = 0.30 \times 1.5 \cdot ft \quad h_m = 0.45 \cdot ft$ Therefore: $s = \dfrac{0.45}{13.5}$ $\quad s = 0.0333$

Width criterion: we need to account for the yaw in the model. We make a relationship for the maximum width as a function of the model dimensions and the yaw angle and relate that to the full-scale dimensions.



$$w_{m20deg} = w_m \cdot \cos(20 \cdot deg) + l_m \cdot \sin(20 \cdot deg) = s \cdot \left(w_p \cdot \cos(20 \cdot deg) + l_p \cdot \sin(20 \cdot deg)\right)$$

$$w_{m20deg} = 0.30 \times 2.0 \cdot ft \qquad w_{m20deg} = 0.60 \cdot ft$$

Therefore: $s = \dfrac{0.60}{8 \times \cos(20 \cdot deg) + 65 \times \sin(20 \cdot deg)}$ $\quad s = 0.0202$

To determine the acceptable scale for the model, we take the smallest of these scale factors: $s = 0.0202$

$\dfrac{1}{s} = 49.58$ We choose a round number to make the model scale easier to calculate: $\text{Model} = \dfrac{1}{50} \text{Prototype}$

For the current model conditions: $Re = \dfrac{V_m \cdot w_m}{\nu_m}$ For standard air: $\nu_m = 1.57 \times 10^{-4} \cdot \dfrac{ft^2}{s}$ Substituting known values:

$Re = 300 \cdot \dfrac{ft}{s} \times \left(\dfrac{1}{50} \times 8 \cdot ft\right) \times \dfrac{s}{1.57 \times 10^{-4} \cdot ft^2}$ $\quad Re = 3.06 \times 10^5$ This is less than the minimum stipulated in the problem, thus:

An adequate Reynolds number can not be achieved.

Problem 7.84

[Difficulty: 4]

7.84 The power, $\mathcal{P}$, required to drive a fan is assumed to depend on fluid density ρ, volume flow rate Q, impeller diameter D, and angular speed ω. If a fan with $D_1 = 8$ in. delivers $Q_1 = 15$ ft^3/s of air at $\omega_1 = 2500$ rpm, what size diameter fan could be expected to deliver $Q_2 = 88$ ft^3/s of air at $\omega_2 = 1800$ rpm, provided they were geometrically and dynamically similar?

Given: Power to drive a fan is a function of ρ, Q, D, and ω.

Condition 1: $D_1 = 8\cdot in$ $\quad \omega_1 = 2500\cdot rpm$ $\quad Q_1 = 15\cdot\frac{ft^3}{s}$ $\qquad$ Condition 2: $Q_2 = 88\cdot\frac{ft^3}{s}$ $\quad \omega_2 = 1800\cdot rpm$

Find: Fan diameter for condition 2 to insure dynamic similarity

Solution: We will use the Buckingham pi-theorem.

1 $\quad P \quad \rho \quad Q \quad D \quad \omega$ $\qquad$ n = 5 parameters

2 $\quad$ Select primary dimensions M, L, t:

3 $\quad P \quad \rho \quad Q \quad D \quad \omega$

$$\frac{M\cdot L^2}{t^3} \quad \frac{M}{L^3} \quad \frac{L^3}{t} \quad L \quad \frac{1}{t}$$

r = 3 dimensions

4 $\quad \rho \quad D \quad \omega$ $\qquad$ m = r = 3 repeating parameters

5 $\quad$ We have n - m = 2 dimensionless groups. Setting up dimensional equations:

$$\Pi_1 = P\cdot\rho^a\cdot D^b\cdot\omega^c \qquad \text{Thus:} \qquad \frac{M\cdot L^2}{t^3}\cdot\left(\frac{M}{L^3}\right)^a\cdot L^b\cdot\left(\frac{1}{t}\right)^c = M^0\cdot L^0\cdot t^0$$

Summing exponents: $\qquad$ The solution to this system is:

M: $1 + a = 0$ $\qquad a = -1 \quad b = -5 \quad c = -3$ $\qquad \Pi_1 = \frac{P}{\rho\cdot\omega^3\cdot D^5}$

L: $2 - 3\cdot a + b = 0$

t: $-3 - c = 0$

$$\Pi_2 = Q\cdot\rho^a\cdot D^b\cdot\omega^c \qquad \text{Thus:} \qquad \frac{L^3}{t}\cdot\left(\frac{M}{L^3}\right)^a\cdot L^b\cdot\left(\frac{1}{t}\right)^c = M^0\cdot L^0\cdot t^0$$

Summing exponents: $\qquad$ The solution to this system is:

M: $a = 0$ $\qquad a = 0 \quad b = -3 \quad c = -1$ $\qquad \Pi_2 = \frac{Q}{\omega\cdot D^3}$

L: $3 - 3\cdot a + b = 0$

t: $-1 - c = 0$

6 $\quad$ Check using F, L, t dimensions: $\frac{F\cdot L}{t}\cdot\frac{L^4}{F\cdot t^2}\cdot\frac{1}{L^5}\cdot t^3 = 1 \quad \frac{L^3}{t}\cdot t\cdot\frac{1}{L^3} = 1$ $\qquad$ Thus the relationship is: $\frac{P}{\rho\cdot\omega^3\cdot D^5} = f\left(\frac{Q}{\omega\cdot D^3}\right)$

For dynamic similarity we must have geometric and kinematic similarity, and: $\frac{Q_1}{\omega_1\cdot D_1^3} = \frac{Q_2}{\omega_2\cdot D_2^3}$ $\quad$ Solving for D_2

$$D_2 = D_1\cdot\left(\frac{Q_2}{Q_1}\cdot\frac{\omega_1}{\omega_2}\right)^{\frac{1}{3}} \qquad D_2 = 8\cdot in\times\left(\frac{88}{15}\times\frac{2500}{1800}\right)^{\frac{1}{3}}$$

$$D_2 = 16.10\cdot in$$

Problem 7.85 (In Excel)

[Difficulty: 3]

7.85 Over a certain range of air speeds, V, the lift, F_L, produced by a model of a complete aircraft in a wind tunnel depends on the air speed, air density, ρ, and a characteristic length (the wing base chord length, $c = 150$ mm). The following experimental data is obtained for air at standard atmospheric conditions:

V (m/s)	10	15	20	25	30	35	40	45	50
F_L (N)	2.2	4.8	8.7	13.3	19.6	26.5	34.5	43.8	54

Plot the lift versus speed curve. By using *Excel* to perform a trendline analysis on this curve, generate and plot data for the lift produced by the prototype, which has a wing base chord length of 5 m, over a speed range of 75 m/s to 250 m/s.

Given: Data on model of aircraft

Find: Plot of lift vs speed of model; also of prototype

Solution:

V_m (m/s)	10	15	20	25	30	35	40	45	50
F_m (N)	2.2	4.8	8.7	13.3	19.6	26.5	34.5	43.8	54.0

This data can be fit to

$$F_m = \frac{1}{2}\cdot\rho\cdot A_m\cdot C_D\cdot V_m^2 \qquad \text{or} \qquad F_m = k_m\cdot V_m^2$$

From the trendline, we see that

$k_m = 0.0219$ N/(m/s)^2

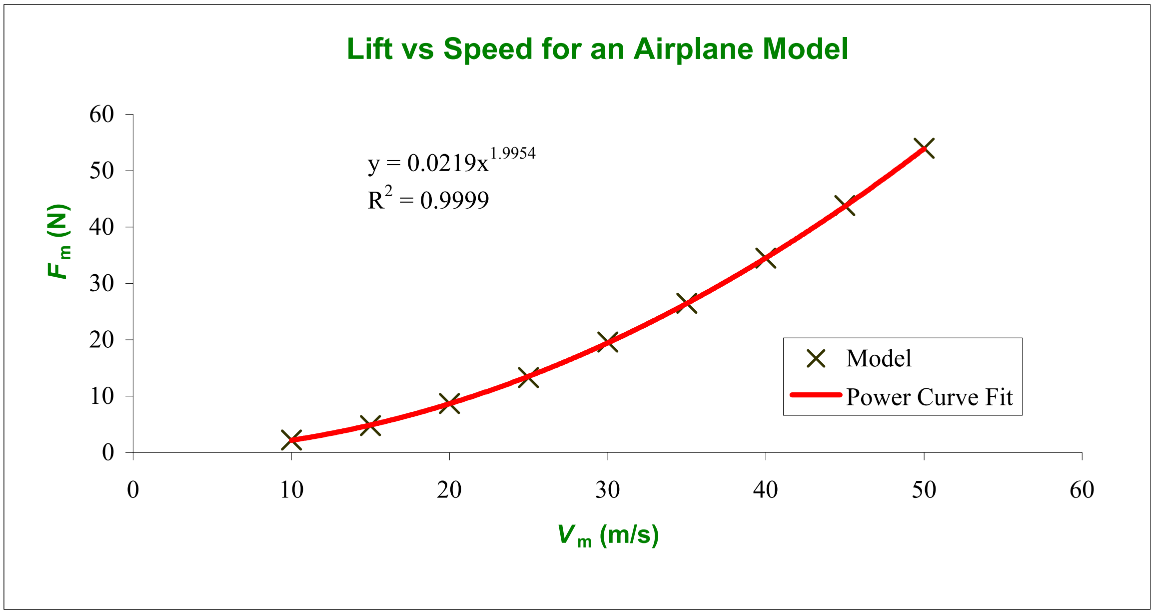
(And note that the power is 1.9954 or 2.00 to three signifcant figures, confirming the relation is quadratic)

Also, $k_p = 1110\ k_m$

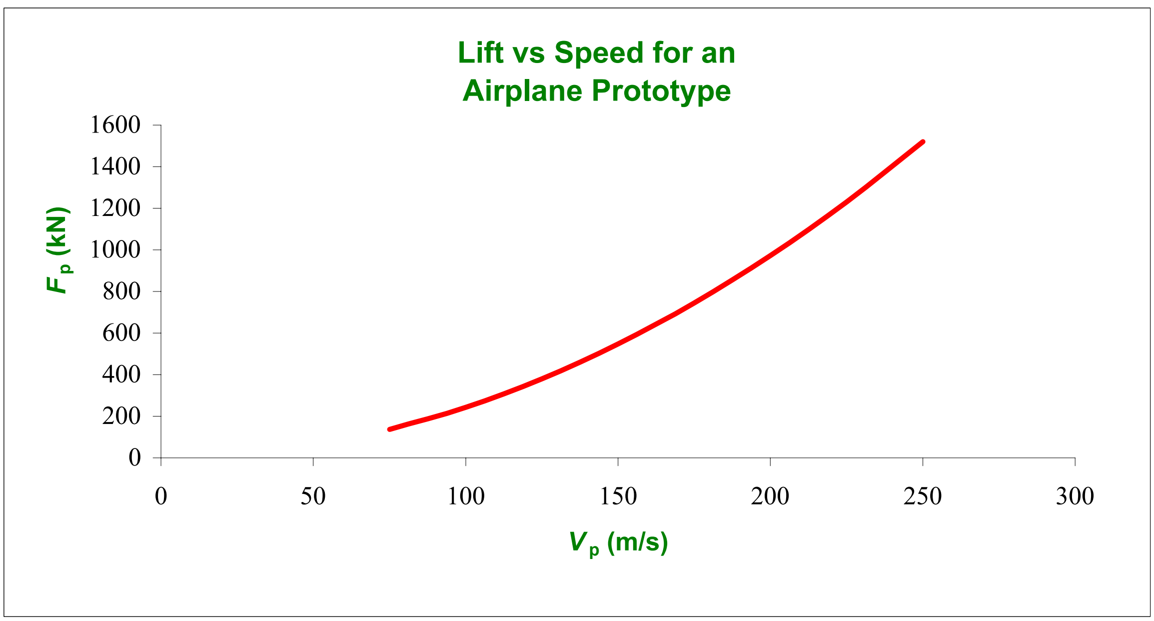
Hence,

$k_p =$ 24.3 N/(m/s)^2 $\qquad F_p = k_p V_m^2$

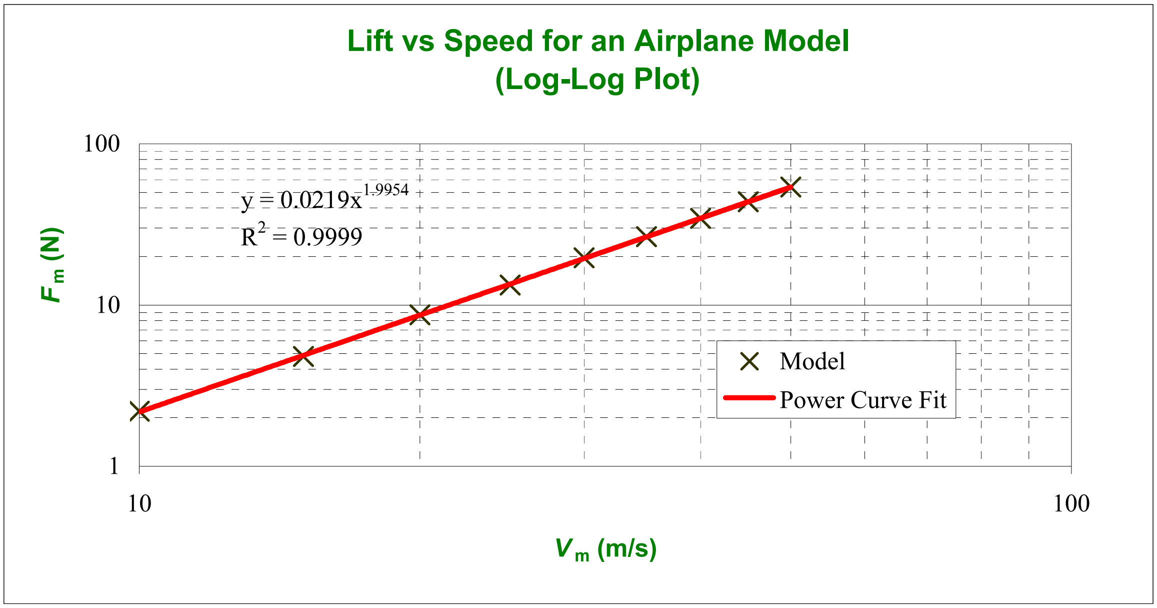
V_p (m/s)	75	100	125	150	175	200	225	250
F_p (kN) (Trendline)	137	243	380	547	744	972	1231	1519



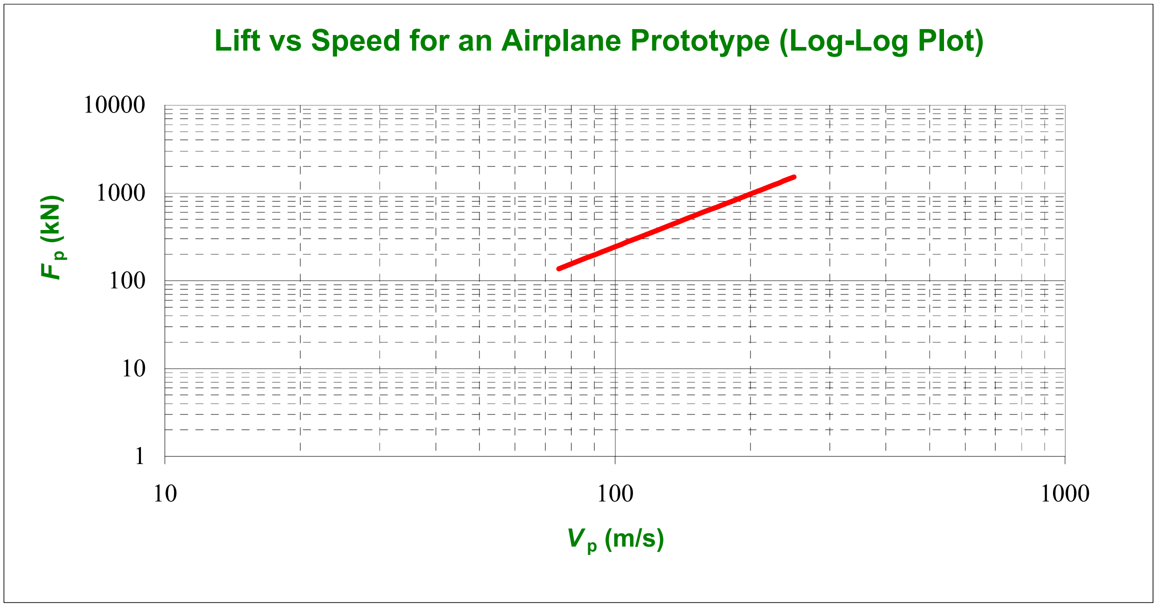
Lift vs Speed for an Airplane Model
y = 0.0219x^1.9954
R^2 = 0.9999
Model
Power Curve Fit
F_m (N)
V_m (m/s)
0
10
20
30
40
50
60



Lift vs Speed for an
Airplane Prototype
F_p (kN)
V_p (m/s)
0
200
400
600
800
1000
1200
1400
1600
50
100
150
200
250
300



Lift vs Speed for an Airplane Model
(Log-Log Plot)
y = 0.0219x^1.9954
R^2 = 0.9999
Model
Power Curve Fit
100
10
1
10
100
F_m (N)
V_m (m/s)



Lift vs Speed for an Airplane Prototype (Log-Log Plot)
10000
1000
100
10
1
10
100
1000
F_p (kN)
V_p (m/s)

Problem 7.86

[Difficulty: 2]

7.86 The pressure rise, Δp, of a liquid flowing steadily through a centrifugal pump depends on pump diameter D, angular speed of the rotor ω, volume flow rate Q, and density ρ. The table gives data for the prototype and for a geometrically similar model pump. For conditions corresponding to dynamic similarity between the model and prototype pumps, calculate the missing values in the table.

Variable	Prototype	Model
Δp	52.5 kPa	
Q		0.0928 m³/min
ρ	800 kg/m³	999 kg/m³
ω	183 rad/s	367 rad/s
D	150 mm	50 mm

Given: Information relating to geometrically similar model test for a centrifugal pump.

Find: The missing values in the table

Solution: We will use the Buckingham pi-theorem.

1 $\quad \Delta p \quad Q \quad \rho \quad \omega \quad D$ $\qquad$ n = 5 parameters

2 Select primary dimensions M, L, t:

3 $\quad \Delta p \quad Q \quad \rho \quad \omega \quad D$

$$\frac{M}{L \cdot t^2} \quad \frac{L^3}{t} \quad \frac{M}{L^3} \quad \frac{1}{t} \quad L$$

r = 3 dimensions

4 $\quad \rho \quad \omega \quad D$ $\qquad$ m = r = 3 repeating parameters

5 We have n - m = 2 dimensionless groups. Setting up dimensional equations:

$$\Pi_1 = \Delta p \cdot \rho^a \cdot \omega^b \cdot D^c \qquad \text{Thus:} \quad \frac{M}{L \cdot t^2} \cdot \left(\frac{M}{L^3}\right)^a \cdot \left(\frac{1}{t}\right)^b \cdot L^c = M^0 \cdot L^0 \cdot t^0$$

Summing exponents: $\qquad$ The solution to this system is:

M: $1 + a = 0$ $\qquad a = -1 \quad b = -2 \quad c = -2$ $\qquad$ $\Pi_1 = \dfrac{\Delta p}{\rho \cdot \omega^2 \cdot D^2}$

L: $-1 - 3 \cdot a + c = 0$

t: $-2 - b = 0$

$$\Pi_2 = Q \cdot \rho^a \cdot \omega^b \cdot D^c \qquad \text{Thus:} \quad \frac{L^3}{t} \cdot \left(\frac{M}{L^3}\right)^a \cdot \left(\frac{1}{t}\right)^b \cdot L^c = M^0 \cdot L^0 \cdot t^0$$

Summing exponents: $\qquad$ The solution to this system is:

M: $a = 0$ $\qquad a = 0 \quad b = -1 \quad c = -3$ $\qquad$ $\Pi_2 = \dfrac{Q}{\omega \cdot D^3}$

L: $3 - 3 \cdot a + c = 0$

t: $-1 - b = 0$

6 Check using F, L, t dimensions: $\dfrac{F}{L^2} \cdot \dfrac{L^4}{F \cdot t^2} \cdot t^2 \cdot \dfrac{1}{L^2} = 1 \qquad \dfrac{L^3}{t} \cdot t \cdot \dfrac{1}{L^3} = 1$ $\qquad$ Thus the relationship is: $\dfrac{\Delta p}{\rho \cdot \omega^2 \cdot D^2} = f\left(\dfrac{Q}{\omega \cdot D^3}\right)$

The flows are geometrically similar, and we assume kinematic similarity. Thus, for dynamic similarity:

$$\text{If} \quad \frac{Q_m}{\omega_m \cdot D_m^{\ 3}} = \frac{Q_p}{\omega_p \cdot D_p^{\ 3}} \quad \text{then} \quad \frac{\Delta p_m}{\rho_m \cdot \omega_m^{\ 2} \cdot D_m^{\ 2}} = \frac{\Delta p_p}{\rho_p \cdot \omega_p^{\ 2} \cdot D_p^{\ 2}}$$

From the first relation: $Q_p = Q_m \cdot \frac{\omega_p}{\omega_m} \cdot \left(\frac{D_p}{D_m}\right)^3$ $\quad Q_p = 0.0928 \cdot \frac{m^3}{min} \times \frac{183}{367} \times \left(\frac{150}{50}\right)^3$ $\quad Q_p = 1.249 \cdot \frac{m^3}{min}$

From the second relation: $\Delta p_m = \Delta p_p \cdot \frac{\rho_m}{\rho_p} \cdot \left(\frac{\omega_m}{\omega_p} \cdot \frac{D_m}{D_p}\right)^2$ $\quad \Delta p_m = 52.5 \cdot kPa \times \frac{999}{800} \times \left(\frac{367}{183} \times \frac{50}{150}\right)^2$ $\quad \Delta p_m = 29.3 \cdot kPa$

Problem 7.87

[Difficulty: 3]

7.87 Tests are performed on a 3-ft-long ship model in a water tank. Results obtained (after doing some data analysis) are as follows:

V (ft/s)	10	20	30	40	50	60	70
D_{Wave} (lbf)	0	0.028	0.112	0.337	0.674	0.899	1.237
$D_{Friction}$ (lbf)	0.022	0.079	0.169	0.281	0.45	0.618	0.731

The assumption is that wave drag is done using the Froude number and friction drag by the Reynolds number. The full-size ship will be 150 ft long when built. Estimate the total drag when it is cruising at 15 knots and at 20 knots in a freshwater lake.

For drag we can use $C_D = \dfrac{D}{\frac{1}{2}\rho V^2 A}$ As a suitable scaling area for A we use L^2 $C_D = \dfrac{D}{\frac{1}{2}\rho V^2 L^2}$

Model: $L =$ 3 ft

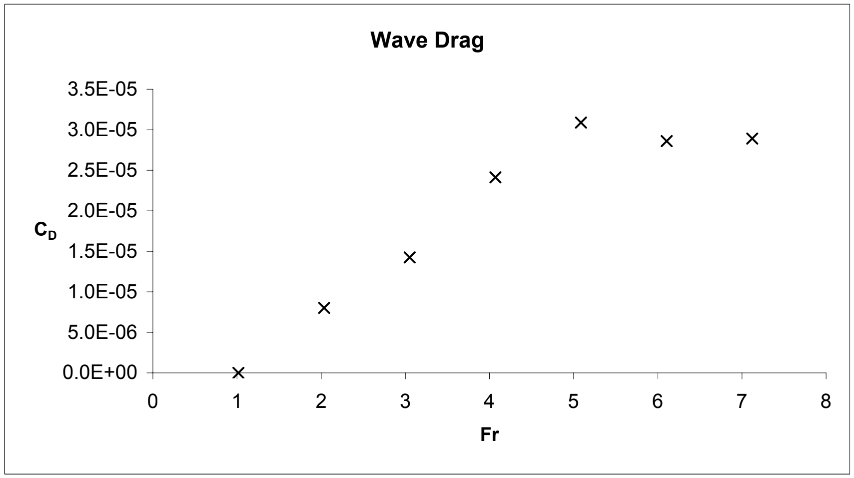
For water $\rho =$ 1.94 slug/ft^3

$\mu =$ 2.10E-05 lbf s/ft^2

The data is:

V (ft/s)	10	20	30	40	50	60	70
D_{Wave} (lbf)	0	0.028	0.112	0.337	0.674	0.899	1.237
$D_{Friction}$ (lbf)	0.022	0.079	0.169	0.281	0.45	0.618	0.731

Fr	1.017	2.035	3.052	4.070	5.087	6.105	7.122
Re	2.77E+06	5.54E+06	8.31E+06	1.11E+07	1.39E+07	1.66E+07	1.94E+07
$C_{D(Wave)}$	0.00E+00	8.02E-06	1.43E-05	2.41E-05	3.09E-05	2.86E-05	2.89E-05
$C_{D(Friction)}$	2.52E-05	2.26E-05	2.15E-05	2.01E-05	2.06E-05	1.97E-05	1.71E-05

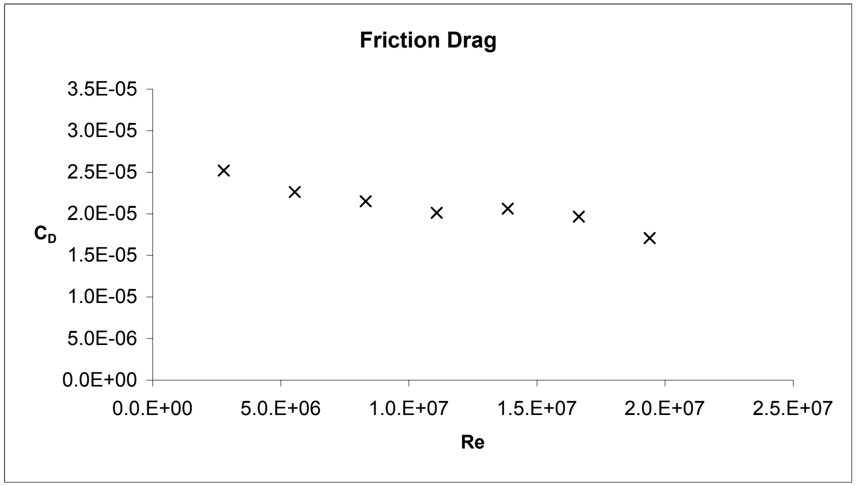


The friction drag coefficient becomes a constant, as expected, at high Re.
The wave drag coefficient appears to be linear with Fr, over most values

Ship: $L =$ 150 ft

V (knot)	15	20
V (ft/s)	25.32	33.76
Fr	0.364	0.486
Re	3.51E+08	4.68E+08

$$D = \frac{1}{2}\rho V^2 L^2 C_D$$



Hence for the ship we have very high Re, and low Fr.
From the graph we see the friction C_D levels out at about 1.9 x 10^{-5}
From the graph we see the wave C_D is negligibly small

$C_{D(Wave)}$	0	0
$C_{D(Friction)}$	1.90E-05	1.90E-05

D_{Wave} (lbf)	0	0
$D_{Friction}$ (lbf)	266	473

D_{Total} (lbf)	266	473

Problem 7.88 (In Excel)

[Difficulty: 4]

7.88 A centrifugal water pump running at speed $\omega = 800$ rpm has the following data for flow rate, Q, and pressure head, Δp.

Q (ft^3/min)	0	50	75	100	120	140	150	165
Δp (psf)	7.54	7.29	6.85	6.12	4.80	3.03	2.38	1.23

The pressure head is a function of the flow rate, speed, impeller diameter D, and water density ρ. Plot the pressure head versus flow rate curve. Find the two Π parameters for this problem, and, from the above data, plot one against the other. By using *Excel* to perform a trendline analysis on this latter curve, generate and plot data for pressure head versus flow rate for impeller speeds of 600 rpm and 1200 rpm.

Given: Data on centrifugal water pump

Find: Π groups; plot pressure head vs flow rate for range of speeds

Solution:

We will use the workbook of Example 7.1, modified for the current problem

The number of parameters is:	$n = 5$
The number of primary dimensions is:	$r = 3$
The number of repeat parameters is:	$m = r = 3$
The number of Π groups is:	$n - m = 2$

Enter the dimensions (**M**, **L**, **t**) of
the repeating parameters, and of up to
four other parameters (for up to four Π groups).
The spreadsheet will compute the exponents a, b, and c for each.

REPEATING PARAMETERS: **Choose ρ, g, d**

	M	**L**	**t**
ρ	1	-3	
ω			-1
D		1	

Π GROUPS:

	M	**L**	**t**
Δp	1	-1	-2

Π_1: $a =$ **-1**, $b =$ **-2**, $c =$ **-2**

	M	**L**	**t**
Q	0	3	-1

Π_2: $a =$ **0**, $b =$ **-1**, $c =$ **-3**

The following Π groups from Example 7.1 are not used:

M	**L**	**t**
0	0	0

Π_3: $a =$ **0**, $b =$ **0**, $c =$ **0**

M	**L**	**t**
0	0	0

Π_4: $a =$ **0**, $b =$ **0**, $c =$ **0**

Hence $\quad \Pi_1 = \dfrac{\Delta p}{\rho \omega^2 D^2}$ and $\Pi_2 = \dfrac{Q}{\omega D^3}$ with $\Pi_1 = f(\Pi_2)$.

Based on the plotted data, it looks like the relation between Π_1 and Π_2 may be parabolic

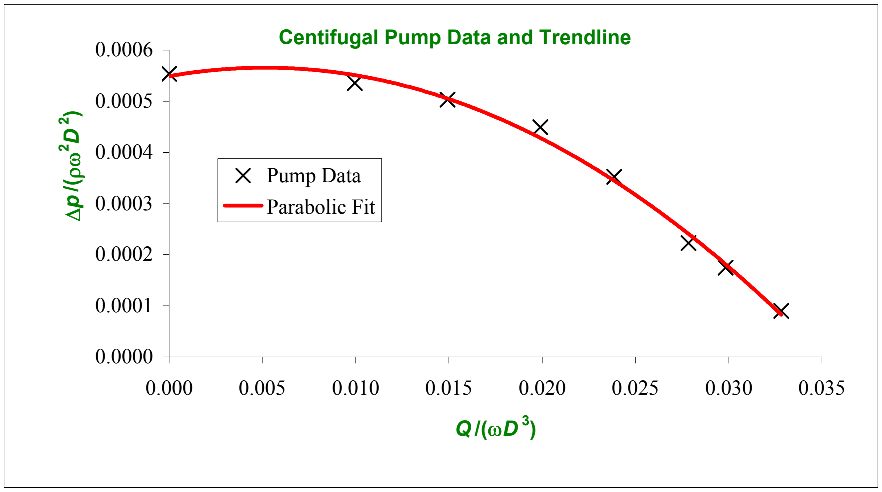
Hence
$$\frac{\Delta p}{\rho \omega^2 D^2} = a + b\left(\frac{Q}{\omega D^3}\right) + c\left(\frac{Q}{\omega D^3}\right)^2$$

The data is

Q (ft^3/min)	0	50	75	100	120	140	150	165
Δp (psf)	7.54	7.29	6.85	6.12	4.80	3.03	2.38	1.23

$\rho =$ 1.94 slug/ft^3
$\omega =$ 800 rpm
$D =$ 1 ft (*D* is not given; use *D* = 1 ft as a scale)

$Q/(\omega D^3)$	0.00000	0.00995	0.01492	0.01989	0.02387	0.02785	0.02984	0.03283
$\Delta p/(\rho\omega^2 D^2)$	0.000554	0.000535	0.000503	0.000449	0.000353	0.000223	0.000175	0.000090



The curve fit result is: $\Delta p/(\rho\omega^2 D^2) = -0.6302\,(Q/(\omega D^3))^2 + 0.006476\,(Q/(\omega D^3)) + 0.0005490$

From the *Trendline* analysis

a = 0.000549
b = 0.006476
c = -0.6302

and $$\Delta p = \rho\omega^2 D^2\left[a + b\left(\frac{Q}{\omega D^3}\right) + c\left(\frac{Q}{\omega D^3}\right)^2\right]$$

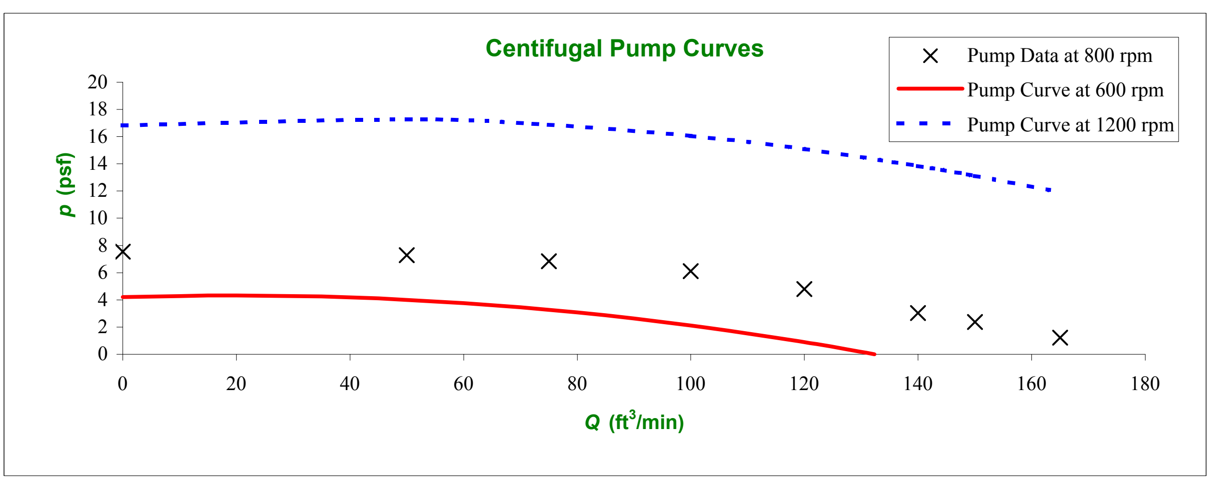
Finally, data at 500 and 1000 rpm can be calculated and plotted

$\omega =$ 600 rpm

Q (ft^3/min)	0	20	40	60	80	100	120	132
Δp (kPa)	4.20	4.33	4.19	3.77	3.08	2.12	0.89	0.00

$\omega =$ 1200 rpm

Q (ft^3/min)	0	50	75	100	120	140	150	165
Δp (kPa)	16.82	17.29	16.88	16.05	15.09	13.85	13.12	11.91



Problem 7.89

[Difficulty: 3]

7.89 An axial-flow pump is required to deliver 0.75 m^3/s of water at a head of 15 J/kg. The diameter of the rotor is 0.25 m, and it is to be driven at 500 rpm. The prototype is to be modeled on a small test apparatus having a 2.25 kW, 1000 rpm power supply. For similar performance between the prototype and the model, calculate the head, volume flow rate, and diameter of the model.

Given: Model of water pump

Find: Model head, flow rate and diameter

Solution:

From Buckingham Π $\quad \dfrac{h}{\omega^2 \cdot D^2} = f\left(\dfrac{Q}{\omega \cdot D^3}, \dfrac{\rho \cdot \omega \cdot D^2}{\mu}\right)$ and $\dfrac{P}{\omega^3 \cdot D^5} = f\left(\dfrac{Q}{\omega \cdot D^3}, \dfrac{\rho \cdot \omega \cdot D^2}{\mu}\right)$

Neglecting viscous effects $\quad \dfrac{Q_m}{\omega_m \cdot D_m^{\,3}} = \dfrac{Q_p}{\omega_p \cdot D_p^{\,3}}$ then $\dfrac{h_m}{\omega_m^{\,2} \cdot D_m^{\,2}} = \dfrac{h_p}{\omega_p^{\,2} \cdot D_p^{\,2}}$ and $\dfrac{P_m}{\omega_m^{\,3} \cdot D_m^{\,5}} = \dfrac{P_p}{\omega_p^{\,3} \cdot D_p^{\,5}}$

Hence if $\quad \dfrac{Q_m}{Q_p} = \dfrac{\omega_m}{\omega_p} \cdot \left(\dfrac{D_m}{D_p}\right)^3 = \dfrac{1000}{500} \cdot \left(\dfrac{D_m}{D_p}\right)^3 = 2 \cdot \left(\dfrac{D_m}{D_p}\right)^3 \qquad (1)$

then $\quad \dfrac{h_m}{h_p} = \dfrac{\omega_m^{\,2}}{\omega_p^{\,2}} \cdot \dfrac{D_m^{\,2}}{D_p^{\,2}} = \left(\dfrac{1000}{500}\right)^2 \cdot \dfrac{D_m^{\,2}}{D_p^{\,2}} = 4 \cdot \dfrac{D_m^{\,2}}{D_p^{\,2}} \qquad (2)$

and $\quad \dfrac{P_m}{P_p} = \dfrac{\omega_m^{\,3}}{\omega_p^{\,3}} \cdot \dfrac{D_m^{\,5}}{D_p^{\,5}} = \left(\dfrac{1000}{500}\right)^3 \cdot \dfrac{D_m^{\,5}}{D_p^{\,5}} = 8 \cdot \dfrac{D_m^{\,5}}{D_p^{\,5}} \qquad (3)$

We can find P_p from $\quad P_p = \rho \cdot Q \cdot h = 1000 \cdot \dfrac{kg}{m^3} \times 0.75 \cdot \dfrac{m^3}{s} \times 15 \cdot \dfrac{J}{kg} = 11.25 \cdot kW$

From Eq 3 $\quad \dfrac{P_m}{P_p} = 8 \cdot \dfrac{D_m^{\,5}}{D_p^{\,5}}$ so $D_m = D_p \cdot \left(\dfrac{1}{8} \cdot \dfrac{P_m}{P_p}\right)^{\frac{1}{5}}$ $\quad D_m = 0.25 \cdot m \times \left(\dfrac{1}{8} \times \dfrac{2.25}{11.25}\right)^{\frac{1}{5}}$ $\quad D_m = 0.120\,m$

From Eq 1 $\quad \dfrac{Q_m}{Q_p} = 2 \cdot \left(\dfrac{D_m}{D_p}\right)^3$ so $Q_m = Q_p \cdot 2 \cdot \left(\dfrac{D_m}{D_p}\right)^3$ $\quad Q_m = 0.75 \cdot \dfrac{m^3}{s} \times 2 \times \left(\dfrac{0.12}{0.25}\right)^3$ $\quad Q_m = 0.166 \dfrac{m^3}{s}$

From Eq 2 $\quad \dfrac{h_m}{h_p} = 4 \cdot \left(\dfrac{D_m}{D_p}\right)^2$ so $h_m = h_p \cdot 4 \cdot \left(\dfrac{D_m}{D_p}\right)^2$ $\quad h_m = 15 \cdot \dfrac{J}{kg} \times 4 \times \left(\dfrac{0.12}{0.25}\right)^2$ $\quad h_m = 13.8 \cdot \dfrac{J}{kg}$

Problem 7.90

[Difficulty: 3]

7.90 A model propeller 1 m in diameter is tested in a wind tunnel. Air approaches the propeller at 50 m/s when it rotates at 1800 rpm. The thrust and torque measured under these conditions are 100 N and 10 N·m, respectively. A prototype 8 times as large as the model is to be built. At a dynamically similar operating point, the approach air speed is to be 130 m/s. Calculate the speed, thrust, and torque of the prototype propeller under these conditions, neglecting the effect of viscosity but including density.

Given: Data on model propeller

Find: Speed, thrust and torque on prototype

Solution: We will use the Buckingham Pi-theorem to find the functional relationships between these variables. Neglecting the effects of viscosity:

1 $F \quad T \quad \rho \quad V \quad D \quad \omega$ $\qquad$ n = 6 parameters

2 Select primary dimensions M, L, t:

3 $F \quad T \quad \rho \quad V \quad D \quad \omega$

$$\frac{M \cdot L}{t^2} \quad \frac{M \cdot L^2}{t^2} \quad \frac{M}{L^3} \quad \frac{L}{t} \quad L \quad \frac{1}{t}$$ r = 3 dimensions

4 $\rho \quad D \quad \omega$ $\qquad$ m = r = 3 repeating parameters

5 We have n - m = 3 dimensionless groups. Setting up dimensional equations:

$\Pi_1 = F \cdot \rho^a \cdot D^b \cdot \omega^c$ $\qquad$ Thus: $\frac{M \cdot L}{t^2} \cdot \left(\frac{M}{L^3}\right)^a \cdot L^b \cdot \left(\frac{1}{t}\right)^c = M^0 \cdot L^0 \cdot t^0$

Summing exponents: $\qquad$ The solution to this system is:

M: $1 + a = 0$ $\qquad$ $a = -1 \quad b = -4 \quad c = -2$ $\qquad$ $\Pi_1 = \frac{F}{\rho \cdot D^4 \cdot \omega^2}$

L: $1 - 3 \cdot a + b = 0$

t: $-2 - c = 0$

$\Pi_2 = T \cdot \rho^a \cdot D^b \cdot \omega^c$ $\qquad$ Thus: $\frac{M \cdot L^2}{t^2} \cdot \left(\frac{M}{L^3}\right)^a \cdot L^b \cdot \left(\frac{1}{t}\right)^c = M^0 \cdot L^0 \cdot t^0$

Summing exponents: $\qquad$ The solution to this system is:

M: $1 + a = 0$ $\qquad$ $a = -1 \quad b = -5 \quad c = -2$ $\qquad$ $\Pi_2 = \frac{T}{\rho \cdot D^5 \cdot \omega^2}$

L: $2 - 3 \cdot a + b = 0$

t: $-2 - c = 0$

$\Pi_3 = V \cdot \rho^a \cdot D^b \cdot \omega^c$ $\qquad$ Thus: $\frac{L}{t} \cdot \left(\frac{M}{L^3}\right)^a \cdot L^b \cdot \left(\frac{1}{t}\right)^c = M^0 \cdot L^0 \cdot t^0$

Summing exponents:

M: $0 + a = 0$

L: $1 - 3 \cdot a + b = 0$

t: $-1 - c = 0$

The solution to this system is:

$a = 0 \qquad b = -1 \qquad c = -1$

$$\Pi_3 = \frac{V}{D \cdot \omega}$$

6 Check using F, L, t dimensions: $F \cdot \frac{L^4}{F \cdot t^2} \cdot \frac{1}{L^4} \cdot t^2 = 1 \qquad F \cdot L \cdot \frac{L^4}{F \cdot t^2} \cdot \frac{1}{L^5} \cdot t^2 = 1 \qquad \frac{L}{t} \cdot \frac{1}{L} \cdot t = 1$

For dynamically similar conditions:

$$\frac{V_m}{D_m \cdot \omega_m} = \frac{V_p}{D_p \cdot \omega_p} \qquad \text{Thus:} \qquad \omega_p = \omega_m \cdot \frac{V_p}{V_m} \cdot \frac{D_m}{D_p} \qquad \omega_p = 1800 \cdot rpm \times \frac{130}{50} \times \frac{1}{8} \qquad \omega_p = 585 \cdot rpm$$

$$\frac{F_m}{\rho_m \cdot D_m^{\;4} \cdot \omega_m^{\;2}} = \frac{F_p}{\rho_p \cdot D_p^{\;4} \cdot \omega_p^{\;2}} \qquad \text{Thus:} \qquad F_p = F_m \cdot \frac{\rho_p}{\rho_m} \cdot \left(\frac{D_p}{D_m}\right)^4 \cdot \left(\frac{\omega_p}{\omega_m}\right)^2 \qquad F_p = 100 \cdot N \times \frac{1}{1} \times \left(\frac{8}{1}\right)^4 \times \left(\frac{585}{1800}\right)^2$$

$$F_p = 43.3 \cdot kN$$

$$\frac{T_m}{\rho_m \cdot D_m^{\;5} \cdot \omega_m^{\;2}} = \frac{T_p}{\rho_p \cdot D_p^{\;5} \cdot \omega_p^{\;2}} \qquad \text{Thus:} \qquad T_p = T_m \cdot \frac{\rho_p}{\rho_m} \cdot \left(\frac{D_p}{D_m}\right)^5 \cdot \left(\frac{\omega_p}{\omega_m}\right)^2 \qquad T_p = 10 \cdot N \cdot m \times \frac{1}{1} \times \left(\frac{8}{1}\right)^5 \times \left(\frac{585}{1800}\right)^2$$

$$T_p = 34.6 \cdot kN \cdot m$$

Problem 7.91

[Difficulty: 3]

7.91 Consider again Problem 7.51. Experience shows that for ship-size propellers, viscous effects on scaling are small. Also, when cavitation is not present, the nondimensional parameter containing pressure can be ignored. Assume that torque, T, and power, $\mathcal{P}$, depend on the same parameters as thrust. For conditions under which effects of μ and p can be neglected, derive scaling "laws" for propellers, similar to the pump "laws" of Section 7.6, that relate thrust, torque, and power to the angular speed and diameter of the propeller.

Given: For a marine propeller (Problem 7.40) the thrust force is: $F_T = F_T(\rho, D, V, g, \omega, p, \mu)$

For ship size propellers viscous and pressure effects can be neglected. Assume that power and torque depend on the same parameters as thrust.

Find: Scaling laws for propellers that relate thrust, power and torque to other variables

Solution: We will use the Buckingham pi-theorem. Based on the simplifications given above:

1 $\quad F_T \quad P \quad T \quad \rho \quad D \quad V \quad g \quad \omega \qquad n = 8$ parameters

2 Select primary dimensions F, L, t:

3

F_T	P	T	ρ	D	V	g	ω
F	$\frac{F\cdot L}{t}$	$F\cdot L$	$\frac{F\cdot t^2}{L^4}$	L	$\frac{L}{t}$	$\frac{L}{t^2}$	$\frac{1}{t}$

$r = 3$ dimensions

4 $\quad \rho \quad \omega \quad D \qquad m = r = 3$ repeating parameters

5 We have n - m = 5 dimensionless groups (3 dependent, 2 independent). Setting up dimensional equations:

$$\Pi_1 = F_T\cdot\rho^a\cdot\omega^b\cdot D^c \qquad \text{Thus:} \qquad F\cdot\left(\frac{F\cdot t^2}{L^4}\right)^a\cdot\left(\frac{1}{t}\right)^b\cdot L^c = F^0\cdot L^0\cdot t^0$$

Summing exponents:

F: $\quad 1 + a = 0$

L: $\quad -4\cdot a + c = 0$

t: $\quad 2\cdot a - b = 0$

The solution to this system is:

$a = -1 \qquad b = -2 \qquad c = -2$

$$\Pi_1 = \frac{F_T}{\rho\cdot\omega^2\cdot D^4}$$

$$\Pi_2 = P\cdot\rho^a\cdot\omega^b\cdot D^c \qquad \text{Thus:} \qquad \frac{F\cdot L}{t}\cdot\left(\frac{F\cdot t^2}{L^4}\right)^a\cdot\left(\frac{1}{t}\right)^b\cdot L^c = F^0\cdot L^0\cdot t^0$$

Summing exponents:

F: $\quad 1 + a = 0$

L: $\quad 1 - 4\cdot a + c = 0$

t: $\quad -1 + 2\cdot a - b = 0$

The solution to this system is:

$a = -1 \qquad b = -3 \qquad c = -5$

$$\Pi_2 = \frac{P}{\rho\cdot\omega^3\cdot D^5}$$

$\Pi_3 = T \cdot \rho^a \cdot \omega^b \cdot D^c$ Thus: $F \cdot L \cdot \left(\frac{F \cdot t^2}{L^4}\right)^a \cdot \left(\frac{1}{t}\right)^b \cdot L^c = F^0 \cdot L^0 \cdot t^0$

Summing exponents:

F: $1 + a = 0$

L: $1 - 4 \cdot a + c = 0$

t: $2 \cdot a - b = 0$

The solution to this system is:

$a = -1 \quad b = -2 \quad c = -5$

$$\Pi_3 = \frac{T}{\rho \cdot \omega^2 \cdot D^5}$$

$\Pi_4 = V \cdot \rho^a \cdot \omega^b \cdot D^c$ Thus: $\frac{L}{t} \cdot \left(\frac{F \cdot t^2}{L^4}\right)^a \cdot \left(\frac{1}{t}\right)^b \cdot L^c = F^0 \cdot L^0 \cdot t^0$

Summing exponents:

F: $a = 0$

L: $1 - 4 \cdot a + c = 0$

t: $-1 + 2 \cdot a - b = 0$

The solution to this system is:

$a = 0 \quad b = -1 \quad c = -1$

$$\Pi_4 = \frac{V}{\omega \cdot D}$$

$\Pi_5 = g \cdot \rho^a \cdot \omega^b \cdot D^c$ Thus: $\frac{L}{t} \cdot \left(\frac{F \cdot t^2}{L^4}\right)^a \cdot \left(\frac{1}{t}\right)^b \cdot L^c = F^0 \cdot L^0 \cdot t^0$

Summing exponents:

F: $a = 0$

L: $1 - 4 \cdot a + c = 0$

t: $-1 + 2 \cdot a - b = 0$

The solution to this system is:

$a = 0 \quad b = -1 \quad c = -1$

$$\Pi_5 = \frac{g}{\omega^2 \cdot D}$$

6 Check using M, L, t dimensions: $\frac{M \cdot L}{t^2} \cdot \frac{L^3}{M} \cdot t^2 \cdot \frac{1}{L^4} = 1 \quad \frac{M \cdot L^2}{t^3} \cdot \frac{L^3}{M} \cdot t^3 \cdot \frac{1}{L^5} = 1 \quad \frac{M \cdot L^2}{t^2} \cdot \frac{L^3}{M} \cdot t^2 \cdot \frac{1}{L^5} = 1$

$\frac{L}{t} \cdot t \cdot \frac{1}{L} = 1 \quad \frac{L}{t^2} \cdot t^2 \cdot \frac{1}{L^2} = 1$

Based on the dependent and independent variables, the "scaling laws" are:

$$\frac{F_T}{\rho \cdot \omega^2 \cdot D^4} = f_1\left(\frac{V}{\omega \cdot D}, \frac{g}{\omega^2 \cdot D}\right)$$

$$\frac{P}{\rho \cdot \omega^3 \cdot D^5} = f_2\left(\frac{V}{\omega \cdot D}, \frac{g}{\omega^2 \cdot D}\right)$$

$$\frac{T}{\rho \cdot \omega^2 \cdot D^5} = f_3\left(\frac{V}{\omega \cdot D}, \frac{g}{\omega^2 \cdot D}\right)$$

Problem 7.92

[Difficulty: 2]

7.92 Water drops are produced by a mechanism that it is believed follows the pattern $d_p = D(\text{We})^{-3/5}$. In this formula, d_p is the drop size, D is proportional to a length scale, and We is the Weber number. In scaling up, if the large-scale characteristic length scale was increased by a factor of 20 and the large-scale velocity decreased by a factor of 5, how would the small- and large-scale drops differ from each other for the same material, for example, water?

Given: Water drop mechanism

Find: Difference between small and large scale drops

Solution:

Given relation
$$d = D \cdot (\text{We})^{-\frac{3}{5}} = D \cdot \left(\frac{\rho \cdot V^2 \cdot D}{\sigma}\right)^{-\frac{3}{5}}$$

For dynamic similarity
$$\frac{d_m}{d_p} = \frac{D_m \cdot \left(\frac{\rho \cdot V_m{}^2 \cdot D_m}{\sigma}\right)^{-\frac{3}{5}}}{D_p \cdot \left(\frac{\rho \cdot V_p{}^2 \cdot D_p}{\sigma}\right)^{-\frac{3}{5}}} = \left(\frac{D_m}{D_p}\right)^{\frac{2}{5}} \cdot \left(\frac{V_m}{V_p}\right)^{-\frac{6}{5}}$$

where d_p stands for $d_{prototype}$ not the original d_p!

Hence
$$\frac{d_m}{d_p} = \left(\frac{1}{20}\right)^{\frac{2}{5}} \times \left(\frac{5}{1}\right)^{-\frac{6}{5}}$$

$$\frac{d_m}{d_p} = 0.044$$

The small scale droplets are 4.4% of the size of the large scale

Problem 7.93

[Difficulty: 2]

7.93 Closed-circuit wind tunnels can produce higher speeds than open-circuit tunnels with the same power input because energy is recovered in the diffuser downstream from the test section. The *kinetic energy ratio* is a figure of merit defined as the ratio of the kinetic energy flux in the test section to the drive power. Estimate the kinetic energy ratio for the 40 ft × 80 ft wind tunnel at NASA-Ames described on page 318.

Given: Kinetic energy ratio for a wind tunnel is the ratio of the kinetic energy flux in the test section to the drive power

Find: Kinetic energy ratio for the 40 ft x 80 ft tunnel at NASA-Ames

Solution: From the text: $P = 125000 \cdot hp \quad V_{max} = 300 \cdot \frac{nmi}{hr} \times \frac{6080 \cdot ft}{nmi} \times \frac{hr}{3600 \cdot s} \quad V_{max} = 507 \cdot \frac{ft}{s}$

Therefore, the kinetic energy ratio is: $KE_{ratio} = \frac{m \cdot \frac{V^2}{2}}{P} = \frac{(\rho \cdot V \cdot A) \cdot V^2}{2 \cdot P} = \frac{\rho \cdot A \cdot V^3}{2 \cdot P}$ Assuming standard conditions and substituting values:

$$KE_{ratio} = \frac{1}{2} \times 0.00238 \cdot \frac{slug}{ft^3} \times (40 \cdot ft \times 80 \cdot ft) \times \left(507 \cdot \frac{ft}{s}\right)^3 \times \frac{1}{125000 \cdot hp} \times \frac{hp \cdot s}{550 \cdot ft \cdot lbf} \times \frac{lbf \cdot s^2}{slug \cdot ft}$$

$$KE_{ratio} = 7.22$$

Problem 7.94

[Difficulty: 3]

7.94 A 1:16 model of a 60-ft-long truck is tested in a wind tunnel at a speed of 250 ft/s, where the axial static pressure gradient is $-0.07\ \text{lbf/ft}^2$ per foot. The frontal area of the prototype is $110\ \text{ft}^2$. Estimate the horizontal buoyancy correction for this situation. Express the correction as a fraction of the measured C_D, of $C_D = 0.85$.

Given: A scale model of a truck is tested in a wind tunnel. The axial pressure gradient and frontal area of the prototype are known. Drag coefficient is 0.85.

Find:
(a) Horizontal buoyancy correction
(b) Express this correction as a fraction of the measured drag force.

Solution: The horizontal buoyancy force is the difference in the pressure force between the front and back of the model due to the pressure gradient in the tunnel:

$$F_B = (p_f - p_b)\cdot A = \frac{\Delta p}{\Delta L}\cdot L_m\cdot A_m \quad \text{where:} \quad L_m = \frac{L_p}{16} \quad A_m = \frac{A_p}{16^2}$$

Thus: $$F_B = -0.07\cdot\frac{\text{lbf}}{\text{ft}^2\cdot\text{ft}} \times \frac{60\cdot\text{ft}}{16} \times \frac{110\cdot\text{ft}^2}{16^2} \qquad F_B = -0.113\cdot\text{lbf}$$

The horizontal buoyancy correction should be added to the measured drag force on the model. The measured drag force on the model is given by:

$$F_{Dm} = \frac{1}{2}\cdot\rho\cdot V^2\cdot A_m\cdot C_D = \frac{1}{2}\cdot\rho\cdot V^2\cdot\frac{A_p}{16^2}\cdot C_D$$ When we substitute in known values we get:

$$F_{Dm} = \frac{1}{2} \times 0.00238\cdot\frac{\text{slug}}{\text{ft}^3} \times \left(250\cdot\frac{\text{ft}}{\text{s}}\right)^2 \times \frac{110\cdot\text{ft}^2}{16^2} \times 0.85 \times \frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}} \qquad F_{Dm} = 27.16\cdot\text{lbf}$$

Therefore the ratio of the forces is: $$\text{DragRatio} = \frac{-0.113}{27.16} \qquad \text{DragRatio} = -0.42\cdot\%$$

Problem 7.95

[Difficulty: 4]

7.95 Frequently one observes a flag on a pole flapping in the wind. Explain why this occurs.

Given: Flapping flag on a flagpole

Find: Explanation of the flapping

Solution:

Discussion: The natural wind contains significant fluctuations in air speed and direction. These fluctuations tend to disturb the flag from an initially plane position.

When the flag is bent or curved from the plane position, the flow nearby must follow its contour. Flow over a convex surface tends to be faster, and have lower pressure, than flow over a concave curved surface. The resulting pressure forces tend to exaggerate the curvature of the flag. The result is a seemingly random "flapping" motion of the flag.

The rope or chain used to raise the flag may also flap in the wind. It is much more likely to exhibit a periodic motion than the flag itself. The rope is quite close to the flag pole, where it is influenced by any vortices shed from the pole. If the Reynolds number is such that periodic vortices are shed from the pole, they will tend to make the rope move with the same frequency. This accounts for the periodic thump of a rope or clank of a chain against the pole.

The vortex shedding phenomenon is characterized by the Strouhal number, $St = fD/V_\infty$, where f is the vortex shedding frequency, D is the pole diameter, and D is the wind speed. The Strouhal number is constant at approximately 0.2 over a broad range of Reynolds numbers.

Problem 7.96

[Difficulty: 3]

7.96 A 1:16 model of a bus is tested in a wind tunnel in standard air. The model is 152 mm wide, 200 mm high, and 762 mm long. The measured drag force at 26.5 m/s wind speed is 6.09 N. The longitudinal pressure gradient in the wind tunnel test section is $-11.8\ \text{N/m}^2\text{/m}$. Estimate the correction that should be made to the measured drag force to correct for horizontal buoyancy caused by the pressure gradient in the test section. Calculate the drag coefficient for the model. Evaluate the aerodynamic drag force on the prototype at 100 km/hr on a calm day.

Given: A 1:16 scale model of a bus (152 mm x 200 mm x 762 mm) is tested in a wind tunnel at 26.5 m/s. Drag force is 6.09 N. The axial pressure gradient is $-11.8\ \text{N/m}^2\text{/m}$.

Find:
(a) Horizontal buoyancy correction
(b) Drag coefficient for the model
(c) Aerodynamic drag on the prototype at 100 kph on a calm day.

Solution: The horizontal buoyancy force is the difference in the pressure force between the front and back of the model due to the pressure gradient in the tunnel:

$$F_B = (p_f - p_b)\cdot A = \frac{dp}{dx}\cdot L_m\cdot A_m \quad \text{where:} \quad A_m = 152\cdot\text{mm}\times 200\cdot\text{mm} \quad A_m = 30400\cdot\text{mm}^2$$

Thus:
$$F_B = -11.8\cdot\frac{\text{N}}{\text{m}^2\cdot\text{m}}\times 762\cdot\text{mm}\times 30400\cdot\text{mm}^2\times\left(\frac{\text{m}}{1000\cdot\text{mm}}\right)^3 \qquad F_B = -0.273\ \text{N}$$

So the corrected drag force is: $F_{Dc} = 6.09\cdot\text{N} - 0.273\cdot\text{N}$ $\quad F_{Dc} = 5.817\ \text{N}$

The corrected model drag coefficient would then be: $C_{Dm} = \dfrac{F_{Dc}}{\frac{1}{2}\cdot\rho\cdot V^2\cdot A_m}$ Substituting in values:

$$C_{Dm} = 2\times 5.82\cdot\text{N}\times\frac{\text{m}^3}{1.23\cdot\text{kg}}\times\left(\frac{\text{s}}{26.5\cdot\text{m}}\right)^2\times\frac{1}{30400\cdot\text{mm}^2}\times\left(\frac{1000\cdot\text{mm}}{\text{m}}\right)^2\times\frac{\text{kg}\cdot\text{m}}{\text{N}\cdot\text{s}^2} \qquad C_{Dm} = 0.443$$

If we assume that the test was conducted at high enough Reynolds number, then the drag coefficient should be the same at both scales, i.e.: $C_{Dp} = C_{Dm}$

$$F_{Dp} = \frac{1}{2}\cdot\rho\cdot V^2\cdot A_p\cdot C_{Dp} \quad \text{where} \quad A_p = 30400\cdot\text{mm}^2\cdot 16^2\cdot\left(\frac{\text{m}}{1000\cdot\text{mm}}\right)^2 \qquad A_p = 7.782\cdot\text{m}^2$$

$$F_{Dp} = \frac{1}{2}\times 1.23\cdot\frac{\text{kg}}{\text{m}^3}\times\left(100\cdot\frac{\text{km}}{\text{hr}}\times\frac{1000\cdot\text{m}}{\text{km}}\cdot\frac{\text{hr}}{3600\times\text{s}}\right)^2\times 7.782\cdot\text{m}^2\times 0.443\times\frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}} \qquad F_{Dp} = 1.636\cdot\text{kN}$$

(The rolling resistance must also be included to obtain the total tractive effort needed to propel the vehicle.)

Problem 7.97

[Difficulty: 5]

7.97 Explore the variation in wave propagation speed given by the equation of Problem 7.1 for a free-surface flow of water. Find the operating depth to minimize the speed of capillary waves (waves with small wavelength, also called ripples). First assume wavelength is much smaller than water depth. Then explore the effect of depth. What depth do you recommend for a water table used to visualize compressible-flow wave phenomena? What is the effect of reducing surface tension by adding a surfactant?

Discussion: The equation given in Problem 7.2 contains three terms. The first term contains surface tension and gives a speed inversely proportional to wavelength. These terms will be important when small wavelengths are considered.

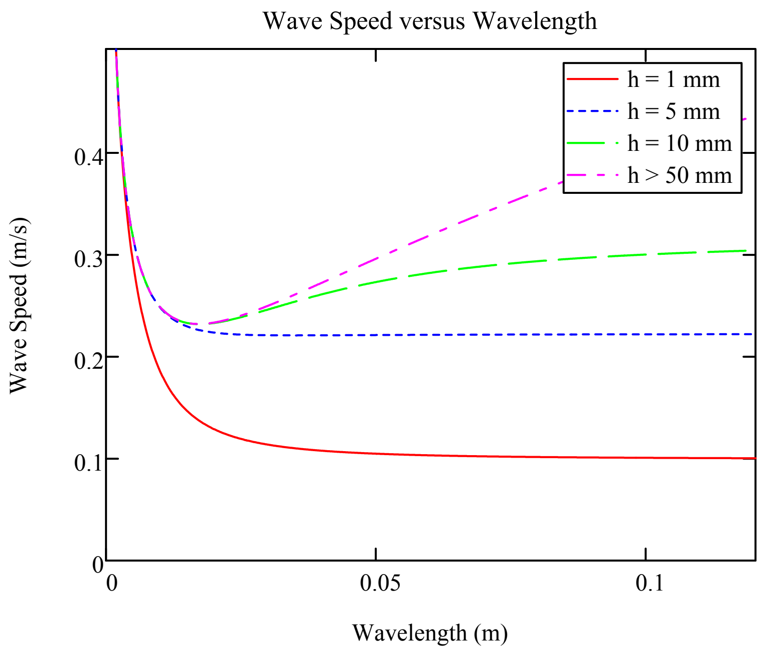
The second term contains gravity and gives a speed proportional to wavelength. This term will be important when long wavelengths are considered.

The argument of the hyperbolic tangent is proportional to water depth and inversely proportional to wavelength. For small wavelengths this term should approach unity since the hyperbolic tangent of a large number approaches one.

The governing equation is: $$c^2 = \left(\frac{\sigma}{\rho}\cdot\frac{2\cdot\pi}{\lambda} + \frac{g\cdot\lambda}{2\cdot\pi}\right)\cdot\tanh\left(\frac{2\cdot\pi\cdot h}{\lambda}\right)$$

The relevant physical parameters are: $g = 9.81\cdot\frac{m}{s^2}$ $\quad\rho = 999\cdot\frac{kg}{m^3}$ $\quad\sigma = 0.0728\cdot\frac{N}{m}$

A plot of the wave speed versus wavelength at different depths is shown here:



Problem 8.1

[Difficulty: 1]

8.1 Air at 100°C enters a 125-mm-diameter duct. Find the volume flow rate at which the flow becomes turbulent. At this flow rate, estimate the entrance length required to establish fully developed flow.

Given: Air entering duct

Find: Flow rate for turbulence; Entrance length

Solution:

The basic equations are $\quad Re = \frac{V \cdot D}{\nu} \quad Re_{crit} = 2300 \quad Q = \frac{\pi}{4} \cdot D^2 \cdot V$

The given data is $\quad D = 125 \cdot mm \quad$ From Table A.10 $\quad \nu = 2.29 \times 10^{-5} \cdot \frac{m^2}{s}$

$L_{laminar} = 0.06 \cdot Re_{crit} \cdot D \quad$ or, for turbulent, $L_{turb} = 25D$ to $40D$

Hence $\quad Re_{crit} = \frac{\frac{Q}{\frac{\pi}{4} \cdot D^2} \cdot D}{\nu}$ or $\quad Q = \frac{Re_{crit} \cdot \pi \cdot \nu \cdot D}{4} \quad Q = 5.171 \times 10^{-3} \frac{m^3}{s}$

For laminar flow $\quad L_{laminar} = 0.06 \cdot Re_{crit} \cdot D \quad L_{laminar} = 17.3\,m$

For turbulent flow $\quad L_{min} = 25 \cdot D \quad L_{min} = 3.13\,m \quad L_{max} = 40 \cdot D \quad L_{max} = 5.00\,m$

Problem 8.2

[Difficulty: 2]

8.2 Consider incompressible flow in a circular channel. Derive general expressions for Reynolds number in terms of (a) volume flow rate and tube diameter and (b) mass flow rate and tube diameter. The Reynolds number is 1800 in a section where the tube diameter is 10 mm. Find the Reynolds number for the same flow rate in a section where the tube diameter is 6 mm.

Solution:

Assume steady, incompressible flow

Definitions: $Re = \frac{\rho D \bar{V}}{\mu}$, $Q = A\bar{V}$, $\dot{m} = \rho A \bar{V}$ and $A = \frac{\pi D^2}{4}$

Then,

$$Re = \frac{\rho D \bar{V}}{\mu} = \frac{\rho D}{\mu}\frac{Q}{A} = \frac{\rho D}{\mu}\frac{Q4}{\pi D^2} = \frac{4Q}{\pi D}\frac{\rho}{\mu} = \frac{4Q}{\pi D \nu}$$ Re

Also

$$Re = \frac{\rho D \bar{V}}{\mu} = \frac{D}{\mu}\frac{\rho \bar{V} A}{A} = \frac{D}{\mu}\frac{\dot{m}4}{\pi D^2} = \frac{4\dot{m}}{\pi D \mu}$$ Re

From Eq (i) a

$$Q = \frac{\pi D \nu Re}{4}$$

Then for same flow rate in sections with different channel diameter,

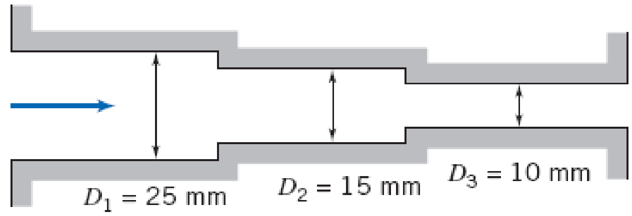
$$D_1 Re_1 = D_2 Re_2$$

$$Re_2 = \frac{D_1}{D_2} Re_1 = \frac{10\,mm}{6\,mm} \times 1800 = 3000$$ Re_2

Problem 8.3

[Difficulty: 3]

8.3 Air at 40°C flows in a pipe system in which diameter is decreased in two stages from 25 mm to 15 mm to 10 mm. Each section is 2 m long. As the flow rate is increased, which section will become turbulent first? Determine the flow rates at which one, two, and then all three sections first become turbulent. At each of these flow rates, determine which sections, if any, attain fully developed flow.



Given: Air entering pipe system

Find: Flow rate for turbulence in each section; Which become fully developed

Solution:

From Table A.10 $\nu = 1.69 \times 10^{-5} \cdot \frac{m^2}{s}$

The given data is $L = 2 \cdot m$ $\quad D_1 = 25 \cdot mm$ $\quad D_2 = 15 \cdot mm$ $\quad D_3 = 10 \cdot mm$

The critical Reynolds number is $Re_{crit} = 2300$

Writing the Reynolds number as a function of flow rate

$$Re = \frac{V \cdot D}{\nu} = \frac{Q}{\frac{\pi}{4} \cdot D^2} \cdot \frac{D}{\nu} \qquad \text{or} \qquad Q = \frac{Re \cdot \pi \cdot \nu \cdot D}{4}$$

Then the flow rates for turbulence to begin in each section of pipe are

$$Q_1 = \frac{Re_{crit} \cdot \pi \cdot \nu \cdot D_1}{4} \qquad Q_1 = 7.63 \times 10^{-4} \frac{m^3}{s}$$

$$Q_2 = \frac{Re_{crit} \cdot \pi \cdot \nu \cdot D_2}{4} \qquad Q_2 = 4.58 \times 10^{-4} \frac{m^3}{s}$$

$$Q_3 = \frac{Re_{crit} \cdot \pi \cdot \nu \cdot D_3}{4} \qquad Q_3 = 3.05 \times 10^{-4} \frac{m^3}{s}$$

Hence, **smallest** pipe becomes turbulent first, then second, then the largest.

For the smallest pipe transitioning to turbulence (Q_3)

For pipe 3 $\quad Re_3 = 2300 \quad L_{laminar} = 0.06 \cdot Re_3 \cdot D_3 \quad L_{laminar} = 1.38\,m \quad L_{laminar} < L$: Fully developed

or, for turbulent, $\quad L_{min} = 25 \cdot D_3 \quad L_{min} = 0.25\,m \quad L_{max} = 40 \cdot D_3 \quad L_{max} = 0.4\,m \quad L_{max/min} < L$: Fully developed

For pipes 1 and 2 $\quad L_{laminar} = 0.06 \cdot \left(\frac{4 \cdot Q_3}{\pi \cdot \nu \cdot D_1}\right) \cdot D_1 \quad L_{laminar} = 1.38\,m \quad L_{laminar} < L$: Fully developed

$$L_{laminar} = 0.06 \cdot \left(\frac{4 \cdot Q_3}{\pi \cdot \nu \cdot D_2} \right) \cdot D_2 \qquad L_{laminar} = 1.38\,m \qquad L_{laminar} < L\text{: Fully developed}$$

For the middle pipe transitioning to turbulence (*Q_2*)

For pipe 2 $\quad Re_2 = 2300 \quad L_{laminar} = 0.06 \cdot Re_2 \cdot D_2 \quad L_{laminar} = 2.07\,m \quad L_{laminar} > L$: Not fully developed

or, for turbulent, $\quad L_{min} = 25 \cdot D_2 \quad L_{min} = 1.23 \cdot ft \quad L_{max} = 40 \cdot D_2 \quad L_{max} = 0.6\,m$

$L_{max/min} < L$: Fully developed

For pipes 1 and 3 $\quad L_1 = 0.06 \cdot \left(\frac{4 \cdot Q_2}{\pi \cdot \nu \cdot D_1} \right) \cdot D_1 \quad L_1 = 2.07 \cdot m \quad L_{laminar} > L$: Not fully developed

$L_{3min} = 25 \cdot D_3 \quad L_{3min} = 0.25 \cdot m \quad L_{3max} = 40 \cdot D_3 \quad L_{3max} = 0.4\,m$

$L_{max/min} < L$: Fully developed

For the large pipe transitioning to turbulence (*Q_1*)

For pipe 1 $\quad Re_1 = 2300 \quad L_{laminar} = 0.06 \cdot Re_1 \cdot D_1 \quad L_{laminar} = 3.45\,m \quad L_{laminar} > L$: Not fully developed

or, for turbulent, $\quad L_{min} = 25 \cdot D_1 \quad L_{min} = 2.05 \cdot ft \quad L_{max} = 40 \cdot D_1 \quad L_{max} = 1.00\,m$

$L_{max/min} < L$: Fully developed

For pipes 2 and 3 $\quad L_{2min} = 25 \cdot D_2 \quad L_{2min} = 1.23 \cdot ft \quad L_{2max} = 40 \cdot D_2 \quad L_{2max} = 0.6\,m$

$L_{max/min} < L$: Fully developed

$L_{3min} = 25 \cdot D_3 \quad L_{3min} = 0.82 \cdot ft \quad L_{3max} = 40 \cdot D_3 \quad L_{3max} = 0.4\,m$

$L_{max/min} < L$: Fully developed

Problem 8.4

[Difficulty: 2]

8.4 For flow in circular tubes, transition to turbulence usually occurs around $Re \approx 2300$. Investigate the circumstances under which the flows of (a) standard air and (b) water at 15°C become turbulent. On log-log graphs, plot: the average velocity, the volume flow rate, and the mass flow rate, at which turbulence first occurs, as functions of tube diameter.

Given: That transition to turbulence occurs at about *Re* = 2300

Find: Plots of average velocity and volume and mass flow rates for turbulence for air and water

Solution:

The basic equations are $\quad Re = \frac{V \cdot D}{\nu} \qquad Re_{crit} = 2300$

From Tables A.8 and A.10 $\quad \rho_{air} = 1.23 \cdot \frac{kg}{m^3} \qquad \nu_{air} = 1.45 \times 10^{-5} \cdot \frac{m^2}{s} \qquad \rho_w = 999 \cdot \frac{kg}{m^3} \qquad \nu_w = 1.14 \times 10^{-6} \cdot \frac{m^2}{s}$

For the average velocity $\quad V = \frac{Re_{crit} \cdot \nu}{D}$

Hence for air $\quad V_{air} = \frac{2300 \times 1.45 \times 10^{-5} \cdot \frac{m^2}{s}}{D} \qquad V_{air} = \frac{0.0334 \cdot \frac{m^2}{s}}{D}$

For water $\quad V_w = \frac{2300 \times 1.14 \times 10^{-6} \cdot \frac{m^2}{s}}{D} \qquad V_w = \frac{0.00262 \cdot \frac{m^2}{s}}{D}$

For the volume flow rates $\quad Q = A \cdot V = \frac{\pi}{4} \cdot D^2 \cdot V = \frac{\pi}{4} \cdot D^2 \cdot \frac{Re_{crit} \cdot \nu}{D} = \frac{\pi \cdot Re_{crit} \cdot \nu}{4} \cdot D$

Hence for air $\quad Q_{air} = \frac{\pi}{4} \times 2300 \times 1.45 \cdot 10^{-5} \cdot \frac{m^2}{s} \cdot D \qquad Q_{air} = 0.0262 \cdot \frac{m^2}{s} \times D$

For water $\quad Q_w = \frac{\pi}{4} \times 2300 \times 1.14 \cdot 10^{-6} \cdot \frac{m^2}{s} \cdot D \qquad Q_w = 0.00206 \cdot \frac{m^2}{s} \times D$

Finally, the mass flow rates are obtained from volume flow rates

$$m_{air} = \rho_{air} \cdot Q_{air} \qquad m_{air} = 0.0322 \cdot \frac{kg}{m \cdot s} \times D$$

$$m_w = \rho_w \cdot Q_w \qquad m_w = 2.06 \cdot \frac{kg}{m \cdot s} \times D$$

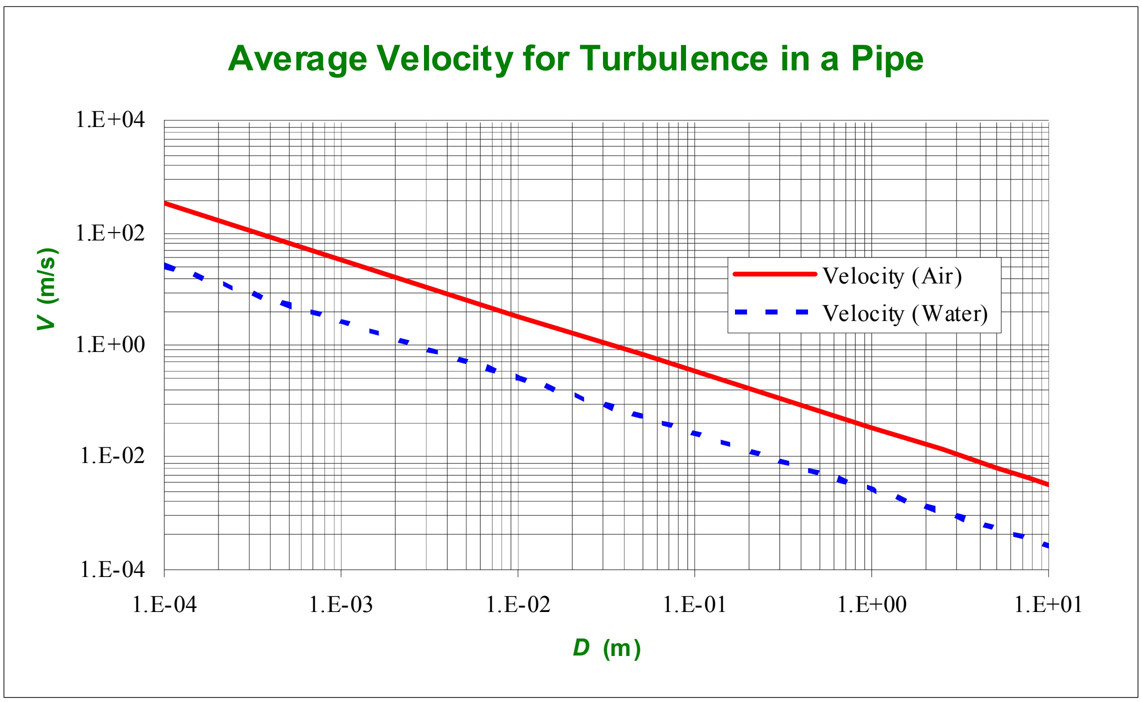
These results can be plotted in *Excel* as shown below in the next two pages

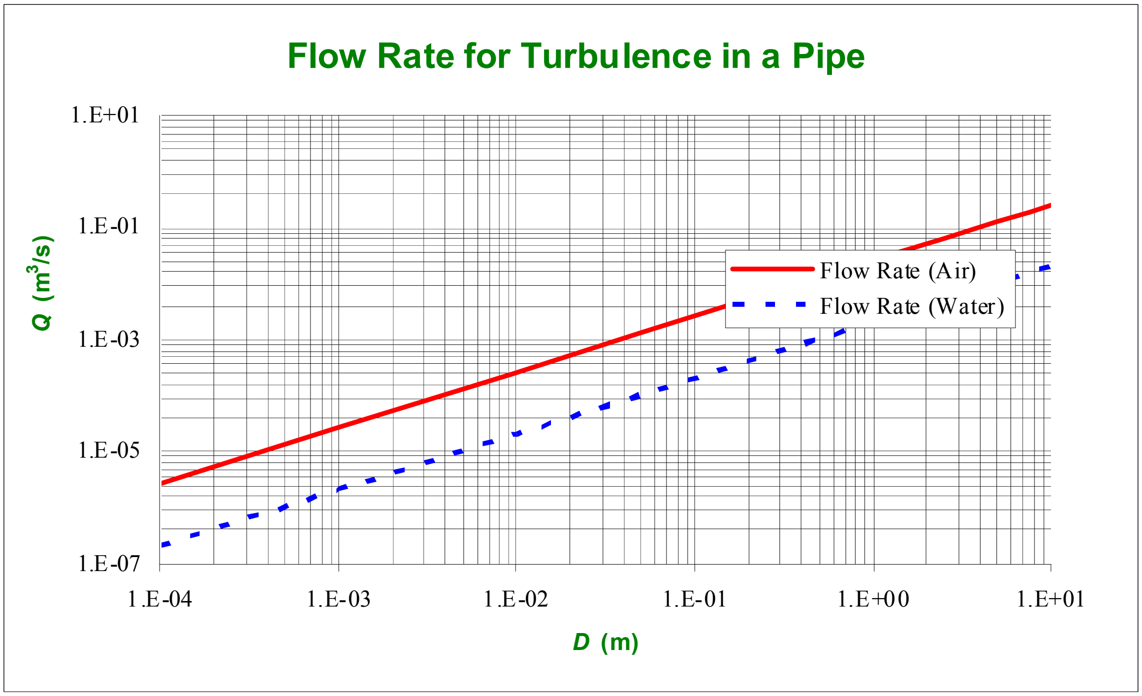
From Tables A.8 and A.10 the data required is

$\Diamond_{air} =$ 1.23 kg/m^3 $\qquad \Diamond_{air} =$ 1.45E-05 m^2/s

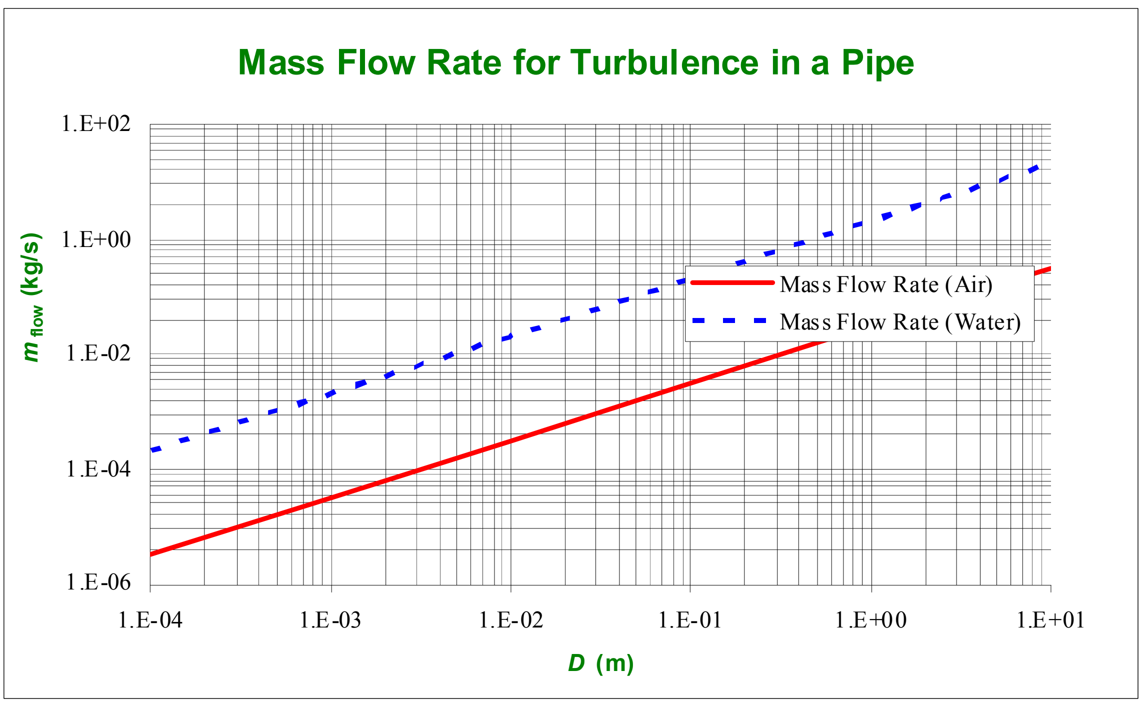
$\Diamond_{w} =$ 999 kg/m^3 $\qquad \Diamond_{w} =$ 1.14E-06 m^2/s

D **(m)**	0.0001	0.001	0.01	0.05	1.0	2.5	5.0	7.5	10.0
V_{air} **(m/s)**	333.500	33.350	3.335	0.667	3.34E-02	1.33E-02	6.67E-03	4.45E-03	3.34E-03
V_{w} **(m/s)**	26.2	2.62	0.262	5.24E-02	2.62E-03	1.05E-03	5.24E-04	3.50E-04	2.62E-04
Q_{air} **(m^3/s)**	2.62E-06	2.62E-05	2.62E-04	1.31E-03	2.62E-02	6.55E-02	1.31E-01	1.96E-01	2.62E-01
Q_{w} **(m^3/s)**	2.06E-07	2.06E-06	2.06E-05	1.03E-04	2.06E-03	5.15E-03	1.03E-02	1.54E-02	2.06E-02
m_{air} **(kg/s)**	3.22E-06	3.22E-05	3.22E-04	1.61E-03	3.22E-02	8.05E-02	1.61E-01	2.42E-01	3.22E-01
m_{w} **(kg/s)**	2.06E-04	2.06E-03	2.06E-02	1.03E-01	2.06E+00	5.14E+00	1.03E+01	1.54E+01	2.06E+01





Flow Rate for Turbulence in a Pipe
1.E+01
1.E-01
1.E-03
1.E-05
1.E-07
Q (m3/s)
1.E-04
1.E-03
1.E-02
1.E-01
1.E+00
1.E+01
D (m)
Flow Rate (Air)
Flow Rate (Water)



Mass Flow Rate for Turbulence in a Pipe
1.E+02
1.E+00
1.E-02
1.E-04
1.E-06
m flow (kg/s)
1.E-04
1.E-03
1.E-02
1.E-01
1.E+00
1.E+01
D (m)
Mass Flow Rate (Air)
Mass Flow Rate (Water)

Problem 8.5

8.5 For the laminar flow in the section of pipe shown in Fig. 8.1, sketch the expected wall shear stress, pressure, and centerline velocity as functions of distance along the pipe. Explain significant features of the plots, comparing them with fully developed flow. Can the Bernoulli equation be applied anywhere in the flow field? If so, where? Explain briefly.

Discussion: The centerline velocity, static pressure, and wall shear stress variations are sketched on the next page. Each variation sketch is aligned vertically with the corresponding sections of the developing pipe flow in Fig. 8.1.

Boundary layers grow on the tube wall, reducing the velocity near the wall. The velocity reduction becomes more pronounced farther downstream. Consequently the centerline velocity must increase in the streamwise direction to carry the same mass flow rate across each section of the tube. (When laminar flow becomes fully developed, the centerline velocity becomes twice the average velocity at any cross-section.)

Frictional effects are concentrated within the boundary layers. The boundary layers do not join at the tube centerline for some distance along the tube. Therefore in the center region outside the boundary layers flow may still be considered to behave as though it were inviscid.

Flow outside the boundary layers is steady, frictionless, incompressible, and along a streamline. These are the restrictions required to apply the Bernoulli equation. Therefore the Bernoulli equation may be applied as a reasonable model for the actual flow outside the boundary layers. The Bernoulli equation predicts that pressure decreases as flow speed increases.

After the boundary layers merge at the centerline of the channel the entire flow is affected by friction. Therefore it is no longer possible to apply the Bernoulli equation.

When flow becomes fully developed the rate of change of pressure with distance becomes constant. In the entrance region the pressure falls more rapidly; the increased pressure gradient is caused by increased shear stress at the wall (larger than for fully developed flow) and by the developing velocity profile, which causes momentum flux to increase.

In fully developed flow the pressure curve becomes linear; the pressure drops the same amount for each length along the tube. The pressure distribution curve at the end of the entrance length becomes asymptotic to the linear variation for fully developed flow.

The wall shear stress initially is large, because the boundary layers are thin. The shear stress decreases as the boundary layers become thicker. At the end of the entrance length the shear stress asymptotically approaches the constant value for fully developed flow.

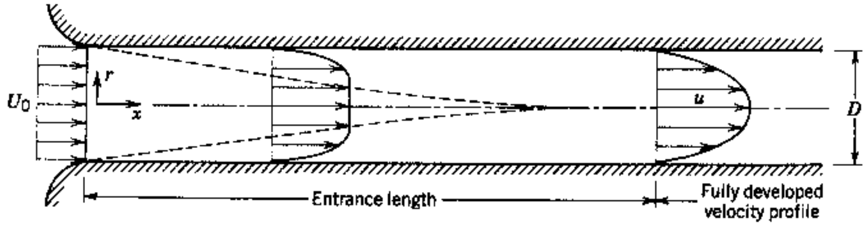
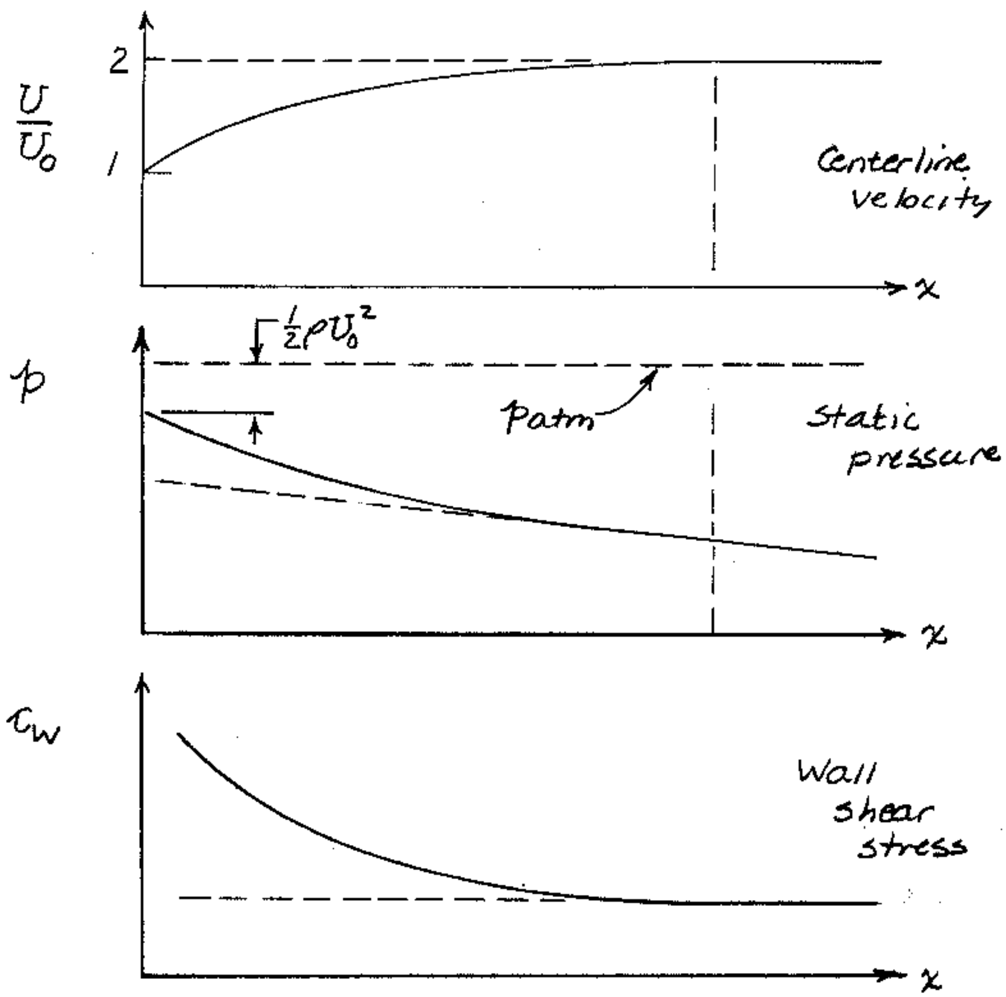


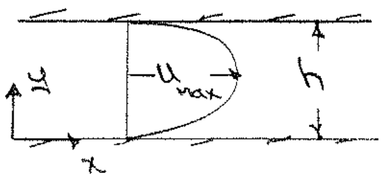
Fig. 8.1 Flow in the entrance region of a pipe.



Problem 8.6

[Difficulty: 2]

8.6 An incompressible fluid flows between two infinite stationary parallel plates. The velocity profile is given by $u = u_{max}(Ay^2 + By + C)$, where A, B, and C are constants and y is measured upward from the lower plate. The total gap width is h units. Use appropriate boundary conditions to express the magnitude and units of the constants in terms of h. Develop an expression for volume flow rate per unit depth and evaluate the ratio $\overline{V}/u_{max}$.



Solution:

(a) Available boundary conditions: (1) $y=0,\ u=0$
(2) $y=h,\ u=0$
(3) $y=h/2,\ u=u_{max}$

From B.C (1) $u(0) = 0 = u_{max}\, C \qquad \therefore\ C = 0$ — C

From B.C (2) $u(h) = 0 = u_{max}(Ah^2 + Bh)$ ---(i)

From B.C (3) $u(h/2) = u_{max} = u_{max}\left(A\frac{h^2}{4} + B\frac{h}{2}\right)$ ---(ii)

From Eq (i), $B = -Ah$. Substituting into Eq (ii) gives

$$\cancel{u_{max}} = \cancel{u_{max}}\left(A\frac{h^2}{4} - A\frac{h^2}{2}\right) \qquad \therefore\ A = -\frac{4}{h^2}$$ — A

and $B = -Ah = \frac{4}{h}$ — B

Then

$$u = u_{max}(Ay^2 + By + c) = u_{max}\left(-4\frac{y^2}{h^2} + 4\frac{y}{h}\right) = 4u_{max}\left[\frac{y}{h} - \left(\frac{y}{h}\right)^2\right]$$

(b) $$Q = \int_0^h u\,b\,dy = \int_0^h 4u_{max}\left[\frac{y}{h} - \frac{y^2}{h^2}\right]b\,dy = 4u_{max}b\left[\frac{y^2}{2h} - \frac{y^3}{3h^2}\right]_0^h$$

$$Q = 4b\,u_{max}\left[\frac{h}{2} - \frac{h}{3}\right] = \frac{2}{3}u_{max}bh$$

$$Q/b = \frac{2}{3}u_{max}h$$ — Q/b

(c) Since $Q = \overline{V}A = \overline{V}bh$

$$\frac{Q}{b} = \overline{V}h = \frac{2}{3}u_{max}h$$

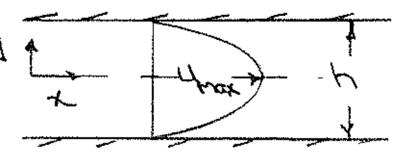
and

$$\frac{\overline{V}}{u_{max}} = \frac{2}{3}$$ — $\frac{\overline{V}}{u_{max}}$

Problem 8.7

[Difficulty: 2]

8.7 The velocity profile for fully developed flow between stationary parallel plates is given by $u = a(h^2/4 - y^2)$, where a is a constant, h is the total gap width between plates, and y is the distance measured from the center of the gap. Determine the ratio $\overline{V}/u_{max}$.



<u>Solution:</u> First find u_{max}, by setting $\frac{du}{dy} = 0$

$$\frac{du}{dy} = -2ay \quad ; \quad \frac{du}{dy} = 0 \text{ at } y = 0$$

$$u_{max} = u(0) = a\frac{h^2}{4}$$

From the definition of $\overline{V}$,

$$\overline{V} = \frac{Q}{A} = \frac{1}{A}\int u\,dA = \frac{1}{h}\int_{-h/2}^{h/2} u\,dy$$

$$= \frac{1}{h}\int_{-h/2}^{h/2} a\left(\frac{h^2}{4} - y^2\right) = \frac{a}{h}\left[\frac{h^2 y}{4} - \frac{y^3}{3}\right]_{-h/2}^{h/2}$$

$$\overline{V} = \frac{a}{h}\left[\left(\frac{h^3}{8} - \frac{h^3}{24}\right) - \left(-\frac{h^3}{8} + \frac{h^3}{24}\right)\right] = \frac{a}{h}\left[\frac{h^3}{4} - \frac{h^3}{12}\right]$$

$$\overline{V} = \frac{1}{6}ah^2$$

and

$$\frac{\overline{V}}{u_{max}} = \frac{ah^2}{6}\,\frac{4}{ah^2} = \frac{2}{3} \qquad \frac{\overline{V}}{u_{max}}$$

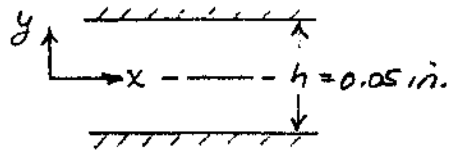
Problem 8.8

[Difficulty: 3]

8.8 A fluid flows steadily between two parallel plates. The flow is fully developed and laminar. The distance between the plates is h.

(a) Derive an equation for the shear stress as a function of y. Sketch this function.

(b) For $\mu = 2.4 \times 10^{-5}$ lbf · s/ft², $\partial p/\partial x = -4.0$ lbf/ft²/ft, and $h = 0.05$ in., calculate the maximum shear stress, in lbf/ft².



Solution: From Eq. 8.7, with $a = h$, $u = -\frac{h^2}{8\mu}\frac{\partial p}{\partial x}\left[1-\left(\frac{2y}{h}\right)^2\right]$

By symmetry, the origin for y must be located at the channel centerline. Apply Newton's law of viscosity.

$$\tau_{yx} = \mu \frac{du}{dy}$$

Assumption: Newtonian fluid

Then

$$\tau_{yx} = \mu \frac{d}{dy}\left\{-\frac{h^2}{8\mu}\frac{\partial p}{\partial x}\left[1-\left(\frac{2y}{h}\right)^2\right]\right\} = y\frac{\partial p}{\partial x}$$

τ_{yx}

For $u>0$, $\partial p/\partial x<0$. Thus $\tau_{yx}<0$ for $y>0$ and $\tau_{yx}>0$ for $y<0$.

On the upper plate (a minus y surface), $\tau_{yx}<0$, so shear stress acts to the right.

On the lower plate (a plus y surface), $\tau_{yx}>0$, so shear stress acts to the right.

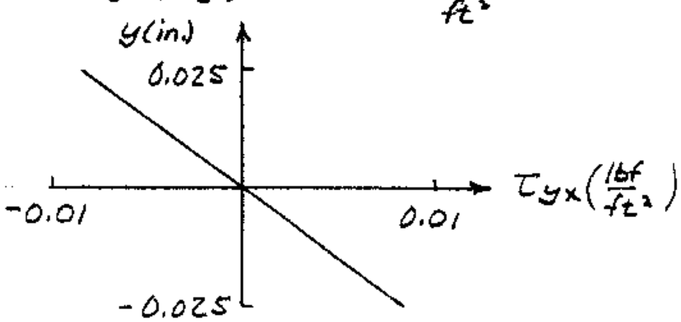
The maximum stress occurs when $y = \pm h/2$. Thus

$$\tau_{max} = \tau_{yx}\left(\frac{h}{2}\right) = \frac{h}{2}\frac{\partial p}{\partial x} = \frac{1}{2}\times 0.05\,\text{in.}\times\frac{\text{ft}}{12\,\text{in.}}\times\left(-4.0\,\frac{\text{lbf}}{\text{ft}^3}\right) = -0.00835\,\frac{\text{lbf}}{\text{ft}^2}$$

τ_{max}

or $\tau_{max} = \tau_{yx}\left(-\frac{h}{2}\right) = 0.00835\,\frac{\text{lbf}}{\text{ft}^2}$

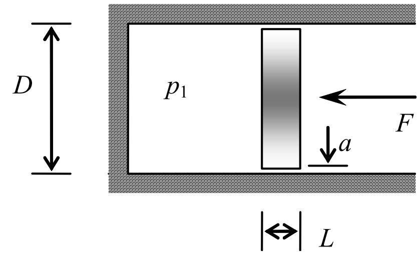
Plot:



Problem 8.9

[Difficulty: 3]

8.9 Oil is confined in a 4-in.-diameter cylinder by a piston having a radial clearance of 0.001 in. and a length of 2 in. A steady force of 4500 lbf is applied to the piston. Assume the properties of SAE 30 oil at 120°F. Estimate the rate at which oil leaks past the piston.



Given: Piston cylinder assembly

Find: Rate of oil leak

Solution:

Basic equation $\quad \dfrac{Q}{l} = \dfrac{a^3 \cdot \Delta p}{12 \cdot \mu \cdot L} \qquad Q = \dfrac{\pi \cdot D \cdot a^3 \cdot \Delta p}{12 \cdot \mu \cdot L}$ (from Eq. 8.6c; we assume laminar flow and verify this is correct after solving)

For the system $\quad \Delta p = p_1 - p_{atm} = \dfrac{F}{A} = \dfrac{4 \cdot F}{\pi \cdot D^2}$

$$\Delta p = \frac{4}{\pi} \times 4500 \cdot lbf \times \left(\frac{1}{4 \cdot in} \times \frac{12 \cdot in}{1 \cdot ft} \right)^2 \qquad \Delta p = 358 \cdot psi$$

At 120°F (about 50°C), from Fig. A.2 $\quad \mu = 0.06 \times 0.0209 \cdot \dfrac{lbf \cdot s}{ft^2} \qquad \mu = 1.25 \times 10^{-3} \cdot \dfrac{lbf \cdot s}{ft^2}$

$$Q = \frac{\pi}{12} \times 4 \cdot in \times \left(0.001 \cdot in \times \frac{1 \cdot ft}{12 \cdot in} \right)^3 \times 358 \cdot \frac{lbf}{in^2} \times \frac{144 \cdot in^2}{1 \cdot ft^2} \times \frac{ft^2}{1.25 \times 10^{-3} \ lbf \cdot s} \times \frac{1}{2} \quad Q = 1.25 \times 10^{-5} \cdot \frac{ft^3}{s} \quad Q = 0.0216 \cdot \frac{in^3}{s}$$

Check Re: $\quad V = \dfrac{Q}{A} = \dfrac{Q}{a \cdot \pi \cdot D} \qquad V = \dfrac{1}{\pi} \times 1.25 \times 10^{-5} \dfrac{ft^3}{s} \times \dfrac{1}{.001 \cdot in} \times \dfrac{1}{4 \cdot in} \times \left(\dfrac{12 \cdot in}{1 \cdot ft} \right)^2 \qquad V = 0.143 \cdot \dfrac{ft}{s}$

$$Re = \frac{V \cdot a}{\nu} \qquad \nu = 6 \times 10^{-5} \times 10.8 \frac{ft^2}{s} \qquad \nu = 6.48 \times 10^{-4} \cdot \frac{ft^2}{s}$$ (at 120°F, from Fig. A.3)

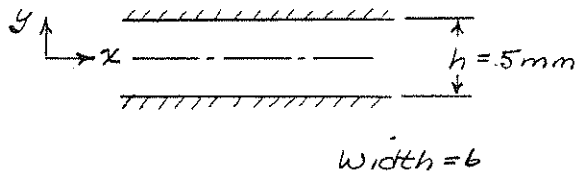
$$Re = 0.143 \cdot \frac{ft}{s} \times 0.001 \cdot in \times \frac{1 \cdot ft}{12 \cdot in} \times \frac{s}{6.48 \times 10^{-4} \ ft^2} \quad Re = 0.0184$$ so flow is very much laminar

The speed of the piston is approximately

$$V_p = \frac{Q}{\left(\frac{\pi \cdot D^2}{4} \right)} \qquad V_p = \frac{4}{\pi} \times 1.25 \times 10^{-5} \frac{ft^3}{s} \times \left(\frac{1}{4 \cdot in} \times \frac{12 \cdot in}{1 \cdot ft} \right)^2 \qquad V_p = 1.432 \times 10^{-4} \cdot \frac{ft}{s}$$

The piston motion is negligible so our assumption of flow between parallel plates is reasonable

8.10 A viscous oil flows steadily between stationary parallel plates. The flow is laminar and fully developed. The total gap width between the plates is $h = 5$ mm. The oil viscosity is $0.5\ \mathrm{N\cdot s/m^2}$ and the pressure gradient is $-1000\ \mathrm{N/m^2/m}$. Find the magnitude and direction of the shear stress on the upper plate and the volume flow rate through the channel, per meter of width.



Solution: From Eq. 8.7 with $a = h$,

$$u = -\frac{h^2}{8\mu}\frac{\partial p}{\partial x}\left[1-\left(\frac{2y}{h}\right)^2\right]$$

Then

$$\tau_{yx} = \mu\frac{du}{dy} = -\frac{h^2}{8}\frac{\partial p}{\partial x}\left(-\frac{8y}{h^2}\right) = y\frac{\partial p}{\partial x}$$

At upper surface, $y = h/2$, and

$$\tau_{yx} = \frac{0.005\ \mathrm{m}}{2} \times -1000\ \frac{\mathrm{N}}{\mathrm{m^3}} = -2.5\ \mathrm{N/m^2}$$

The upper plate is a negative y surface. Thus since $\tau_{yx} < 0$, stress acts to right, in $+x$ direction. τ_{yx}

The volume flow rate is

$$Q = \int_A u\,dA = \int_{-h/2}^{h/2} u\,b\,dy = 2\int_0^{h/2} u\,b\,dy = 2\left(\frac{h}{2}\right)b\int_0^1 u\,d\left(\frac{2y}{h}\right)$$

or

$$\frac{Q}{b} = h\int_0^1 u\,d\eta \quad \text{where } \eta = \frac{2y}{h} \text{ and } u = -\frac{h^2}{8\mu}\frac{\partial p}{\partial x}(1-\eta^2)$$

Thus

$$\frac{Q}{b} = h\int_0^1 -\frac{h^2}{8\mu}\frac{\partial p}{\partial x}(1-\eta^2)\,d\eta = -\frac{h^3}{8\mu}\frac{\partial p}{\partial x}\left(\eta - \frac{1}{3}\eta^3\right)\Big|_0^1 = -\frac{h^3}{12\mu}\frac{\partial p}{\partial x}$$

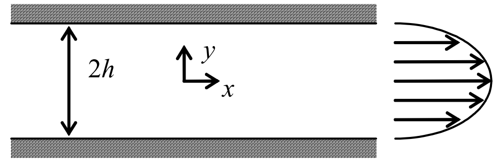
$$\frac{Q}{b} = -\frac{1}{12} \times (0.005)^3\,\mathrm{m^3} \times \frac{\mathrm{m^2}}{0.5\ \mathrm{N\cdot s}} \times -1000\ \frac{\mathrm{N}}{\mathrm{m^3}} = 20.8 \times 10^{-6}\ \mathrm{m^2/s}$$ Q/b

Note $u > 0$, so flow is from left to right.

Problem 8.11

[Difficulty: 2]

8.11 Viscous oil flows steadily between parallel plates. The flow is fully developed and laminar. The pressure gradient is 1.25 kPa/m and the channel half-width is $h = 1.5$ mm. Calculate the magnitude and direction of the wall shear stress at the upper plate surface. Find the volume flow rate through the channel ($\mu = 0.50\ \mathrm{N \cdot s/m^2}$).



Given: Laminar flow between flat plates

Find: Shear stress on upper plate; Volume flow rate per width

Solution:

Basic equation $\quad \tau_{yx} = \mu \cdot \frac{du}{dy} \qquad u(y) = -\frac{h^2}{2 \cdot \mu} \cdot \frac{dp}{dx} \cdot \left[1 - \left(\frac{y}{h}\right)^2\right]$ (from Eq. 8.7)

Then $\quad \tau_{yx} = \frac{-h^2}{2} \cdot \frac{dp}{dx} \cdot \left(-\frac{2 \cdot y}{h^2}\right) = -y \cdot \frac{dp}{dx}$

At the upper surface $\quad y = h \qquad \tau_{yx} = -1.5 \cdot mm \times \frac{1 \cdot m}{1000 \cdot mm} \times 1.25 \times 10^3 \cdot \frac{N}{m^2 \cdot m} \qquad \tau_{yx} = -1.88 Pa$

The volume flow rate is $\quad Q = \int u\, dA = \int_{-h}^{h} u \cdot b\, dy = -\frac{h^2 \cdot b}{2 \cdot \mu} \cdot \frac{dp}{dx} \cdot \int_{-h}^{h} \left[1 - \left(\frac{y}{h}\right)^2\right] dy \qquad Q = -\frac{2 \cdot h^3 \cdot b}{3 \cdot \mu} \cdot \frac{dp}{dx}$

$$\frac{Q}{b} = -\frac{2}{3} \times \left(1.5 \cdot mm \times \frac{1 \cdot m}{1000 \cdot mm}\right)^3 \times 1.25 \times 10^3 \cdot \frac{N}{m^2 \cdot m} \times \frac{m^2}{0.5 \cdot N \cdot s} \qquad \frac{Q}{b} = -5.63 \times 10^{-6} \frac{m^2}{s}$$

Problem 8.12

[Difficulty: 3]

8.12 A large mass is supported by a piston of diameter $D = 4$ in. and length $L = 4$ in. The piston sits in a cylinder closed at the bottom, and the gap $a = 0.001$ in. between the cylinder wall and piston is filled with SAE 10 oil at 68°F. The piston slowly sinks due to the mass, and oil is forced out at a rate of 0.1 gpm. What is the mass (slugs)?

Given: Piston-cylinder assembly

Find: Mass supported by piston

Solution:

Basic equation $\dfrac{Q}{l} = \dfrac{a^3 \cdot \Delta p}{12 \cdot \mu \cdot L}$ This is the equation for pressure-driven flow between parallel plates; for a small gap a, the flow between the piston and cylinder can be modeled this way, with l = πD

Available data $L = 4 \cdot in$ $D = 4 \cdot in$ $a = 0.001 \cdot in$ $Q = 0.1 \cdot gpm$ $68\,°F = 20\,°C$

From Fig. A.2, SAE10 oil at 20ºF is $\mu = 0.1 \cdot \dfrac{N \cdot s}{m^2}$ or $\mu = 2.089 \times 10^{-3} \cdot \dfrac{lbf \cdot s}{ft^2}$

Hence, solving for Δp $\Delta p = \dfrac{12 \cdot \mu \cdot L \cdot Q}{\pi \cdot D \cdot a^3}$ $\Delta p = 2.133 \times 10^4 \cdot psi$

A force balance for the piston involves the net pressure force $F = \Delta p \cdot A = \Delta p \cdot \dfrac{\pi}{4} \cdot D^2$ and the weight $W = M \cdot g$

Hence $M = \dfrac{\pi \cdot D^2}{4} \cdot \dfrac{\Delta p}{g}$ $M = 8331 \cdot slug$ $M = 2.68 \times 10^5 \cdot lb$

Note the following $V_{ave} = \dfrac{Q}{a \cdot \pi \cdot D}$ $V_{ave} = 2.55 \cdot \dfrac{ft}{s}$ $\nu = 10^{-4} \cdot \dfrac{m^2}{s}$ $\nu = 1.076 \times 10^{-3} \cdot \dfrac{ft^2}{s}$

Hence an estimate of the Reynolds number in the gap is $Re = \dfrac{a \cdot V_{ave}}{\nu}$ $Re = 0.198$

This is a highly viscous flow; it can be shown that the force on the piston due to this motion is much less than that due to Δp!

Note also that the piston speed is $V_{piston} = \dfrac{4 \cdot Q}{\pi \cdot D^2}$ $V_{piston} = 0.00255 \cdot \dfrac{ft}{s}$

$\dfrac{V_{piston}}{V_{ave}} = 0.1 \cdot \%$ so the approximation of stationary walls is valid

Problem 8.13

[Difficulty: 3]

8.13 A high pressure in a system is created by a small piston-cylinder assembly. The piston diameter is 6 mm and it extends 50 mm into the cylinder. The radial clearance between the piston and cylinder is 0.002 mm. Neglect elastic deformations of the piston and cylinder caused by pressure. Assume the fluid properties are those of SAE 10W oil at 35°C. When the pressure in the cylinder is 600 MPa, estimate the leakage rate.

Solution: Computing equation: $\frac{Q}{l} = \frac{a^3 \Delta p}{12 \mu L}$ (8.6c)

Assumptions: (1) Laminar flow

(2) Fully developed flow ($L \gg a$)

For SAE 10W oil at 35°C, $\mu = 3.8 \times 10^{-2}$ N·s/m² (Fig. A.2)

For this configuration, $l = \pi D$, since $a \ll D$. Then

$$Q = \frac{a^3 \Delta p\, l}{12 \mu L} = \frac{\pi a^3 \Delta p D}{12 \mu L}$$

$$Q = \frac{\pi}{12} \times (2 \times 10^{-6}\,\text{m})^3 \times 6 \times 10^8 \frac{\text{N}}{\text{m}^2} \times 0.006\,\text{m} \times \frac{1}{3.8 \times 10^{-2}} \frac{\text{m}^2}{\text{N}\cdot\text{s}} \times \frac{1}{0.05\,\text{m}}$$

$$Q = 3.97 \times 10^{-9}\,\text{m}^3/\text{s} = 3.97 \times 10^{-6}\,\text{L/s}$$ Q

Check Re to assure laminar flow

$$\bar{V} = \frac{Q}{A} = \frac{Q}{\pi D a} = \frac{1}{\pi} \times 3.97 \times 10^{-9} \frac{\text{m}^3}{\text{s}} \times \frac{1}{0.006\,\text{m}} \times \frac{1}{2 \times 10^{-6}\,\text{m}} = 0.105\,\text{m/s}$$

SG = 0.88 (Table A.2) ; $\rho = SG\, \rho_{H_2O}$

$$Re = \frac{\rho \bar{V} a}{\mu} = \frac{SG\, \rho_{H_2O} \bar{V} a}{\mu}$$

$$= 0.88 \times 999 \frac{\text{kg}}{\text{m}^3} \times 0.105 \frac{\text{m}}{\text{s}} \times 2 \times 10^{-6}\,\text{m} \times \frac{1}{3.8 \times 10^{-2}} \frac{\text{m}^2}{\text{N}\cdot\text{s}}$$

$Re = 0.005 \ll 2300$ so flow is definitely laminar!

Problem 8.14

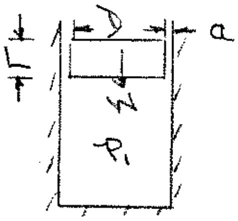
[Difficulty: 3]

8.14 A hydraulic jack supports a load of 9000 kg. The following data are given:

Diameter of piston	100 mm
Radial clearance between piston and cylinder	0.05 mm
Length of piston	120 mm

Estimate the rate of leakage of hydraulic fluid past the piston, assuming the fluid is SAE 30 oil at 30°C.

Solution:



Model the flow as steady, fully developed laminar flow between stationary parallel plates, i.e., neglect motion of the piston.

Then, the leakage flow rate can be evaluated from Eq. 8.6c (in the text)

$$\frac{Q}{l} = \frac{a^3 \Delta p}{12\mu L} \qquad \text{where } l = \pi D$$

From Fig. A.2 at $T = 30°C$, $\mu = 3.0\times10^{-1}\ N\cdot s/m^2$

$\Delta p = p_1 - p_{atm}$ and $p_1 = \frac{W}{A} = \frac{Mg}{A} = \frac{4Mg}{\pi D^2}$

$$p_1 = \frac{4}{\pi} \times 9000\,kg \times 9.81\,\frac{m}{s^2} \times \frac{1}{(0.1m)^2} \times \frac{N\cdot s^2}{kg\cdot m} = 11.2\ MPa$$

$$Q = \frac{\pi D a^3 \Delta p}{12\mu L} = \frac{\pi}{12} \times (0.1m) \times (5\times10^{-5}m)^3 \times 11.2\times10^6\,\frac{N}{m^2} \times \frac{1}{0.3}\,\frac{m^2}{N\cdot s} \times \frac{1}{0.12m}$$

$$Q = 1.01\times10^{-6}\ m^3/s = 1.01\times10^{-3}\ L/s \qquad Q$$

Check $Re = \frac{\rho a \bar{V}}{\mu} = \frac{a\bar{V}}{\nu}$ where $\nu = 2.8\times10^{-4}\ m^2/s$ (Fig. A.3)

$$\bar{V} = \frac{Q}{A} = \frac{Q}{al} = \frac{Q}{a\pi D} = \frac{1}{\pi} \times 1.01\times10^{-6}\,\frac{m^3}{s} \times \frac{1}{5\times10^{-5}m} \times \frac{1}{0.1m} = 0.0643\ m/s$$

$$Re = \frac{a\bar{V}}{\nu} = 5\times10^{-5}m \times 0.0643\,\frac{m}{s} \times \frac{1}{2.8\times10^{-4}}\,\frac{s}{m^2} = 0.011$$

$\therefore$ flow is definitely laminar

Piston moving down at speed v displaces liquid at rate Q where

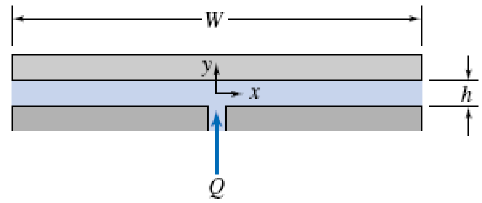
$$Q = \frac{\pi D^2}{4} v$$

Then

$$v = \frac{4Q}{\pi D^2} = \frac{4}{\pi} \times 1.01\times10^{-6}\,\frac{m^3}{s} \times \frac{1}{(0.1m)^2} = 1.29\times10^{-4}\ m/s$$

Since $\frac{v}{\bar{V}} = \frac{1.29\times10^{-4}\ m/s}{0.0643\ m/s} = 2.0\times10^{-3}$, motion of piston can be neglected.

8.15 A hydrostatic bearing is to support a load of 1000 lbf/ft of length perpendicular to the diagram. The bearing is supplied with SAE 10W-30 oil at 212°F and 35 psig through the central slit. Since the oil is viscous and the gap is small, the flow may be considered fully developed. Calculate (a) the required width of the bearing pad, (b) the resulting pressure gradient, *dp/dx*, and (c) the gap height, if the flow rate is $Q = 2.5$ gal/hr/ft.



Given: Hydrostatic bearing

Find: Required pad width; Pressure gradient; Gap height

Solution:

Basic equation $\frac{Q}{l} = -\frac{h^3}{12\cdot\mu}\cdot\left(\frac{dp}{dx}\right)$

Available data $F = 1000\cdot lbf$ $\quad l = 1\cdot ft$ (F is the load on width l) $\quad p_i = 35\cdot psi$ $\quad Q = 2.5\cdot\frac{gal}{hr}$ per ft

$212\ °F = 100\ °C$ At 100°C from Fig. A.2, for SAE 10-30 $\mu = 0.01\cdot\frac{N\cdot s}{m^2}$ $\quad \mu = 2.089\times 10^{-4}\cdot\frac{lbf\cdot s}{ft^2}$

For a laminar flow (we will verify this assumption later), the pressure gradient is constant

$$p(x) = p_i\cdot\left(1-\frac{2\cdot x}{W}\right)$$ where p_i = 35 psi is the inlet pressure (gage), and x = 0 to W/2

Hence the total force in the y direction due to pressure is $F = l\cdot\int p\,dx$ where b is the pad width into the paper

$$F = 2\cdot l\cdot\int_0^{\frac{W}{2}} p_i\cdot\left(1-\frac{2\cdot x}{W}\right)dx \qquad F = \frac{1}{2}\cdot p_i\cdot l\cdot W$$

This must be equal to the applied load F. Hence $W = \frac{2}{p_i}\cdot\frac{F}{l}$ $\quad W = 0.397\cdot ft$

The pressure gradient is then $\frac{dp}{dx} = -\frac{\Delta p}{\frac{W}{2}} = -\frac{2\cdot\Delta p}{W} = -2\times\frac{35\cdot lbf}{in^2}\times\frac{1}{0.397\cdot ft} = -176\cdot\frac{psia}{ft}$

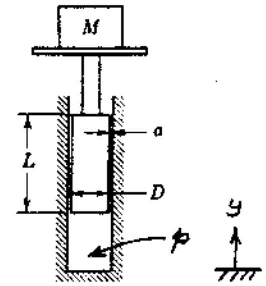
From the basic equation $\frac{Q}{l} = -\frac{h^3}{12\cdot\mu}\cdot\left(\frac{dp}{dx}\right)$ we can solve for $h = \left(-\frac{12\cdot\mu\cdot\frac{Q}{l}}{\frac{dp}{dx}}\right)^{\frac{1}{3}}$ $\quad h = 2.51\times 10^{-3}\cdot in$

Check Re: $Re = \frac{V\cdot D}{\nu} = \frac{D}{\nu}\cdot\frac{Q}{A} = \frac{h}{\nu}\cdot\frac{Q}{l\cdot h}$

From Fig. A.3 $\nu = 1.2\cdot 10^{-5}\cdot\frac{m^2}{s}$ $\quad \nu = 1.29\times 10^{-4}\cdot\frac{ft^2}{s}$ $\quad Re = \frac{Q}{\nu\cdot l}$ $\quad Re = 0.72$

so flow is very laminar

8.16 The basic component of a pressure gage tester consists of a piston-cylinder apparatus as shown. The piston, 6 mm in diameter, is loaded to develop a pressure of known magnitude. (The piston length is 25 mm.) Calculate the mass, M, required to produce 1.5 MPa (gage) in the cylinder. Determine the leakage flow rate as a function of radial clearance, a, for this load if the liquid is SAE 30 oil at 20°C. Specify the maximum allowable radial clearance so the vertical movement of the piston due to leakage will be less than 1 mm/min.



Solution: The mass may be found from a force balance on the piston.

$$\Sigma F_y = \frac{\pi D^2}{4}(p - p_{atm}) - Mg = 0 \quad \text{so} \quad M = \frac{\pi D^2}{4g} p_{gage}$$

$$M = \frac{\pi}{4} \times (0.006)^2 m^2 \times 1.5\times10^6 \frac{N}{m^2} \times \frac{s^2}{9.81\, m} \times \frac{kg\cdot m}{N\cdot s^2} = 4.32 \text{ kg}$$ M

The leakage flow rate may be evaluated for flow between flat plates. From Eq. 8.6c, neglecting motion of the piston,

$$\frac{Q}{l} = \frac{a^3 \Delta p}{12 \mu L} \quad \text{or, since } l = \pi D, \quad Q = \frac{\pi}{12} \frac{a^3 \Delta p D}{\mu L} \sim a^3$$ Q

The piston, moving downward at speed, v, displaces liquid at rate

$$Q = \frac{\pi D^2}{4} v = \frac{\pi}{4} (0.006)^2 m^2 \times 0.001 \frac{m}{min} \times \frac{min}{60\, s} = 4.71\times10^{-10}\, m^3/s$$

Then, with $\mu = 0.42\, N\cdot sec/m^2$ (at 20°C, Fig. A.2),

$$a = \left[\frac{12 \mu Q L}{\pi D \Delta p}\right]^{1/3} = \left[\frac{12}{\pi} \times 0.42 \frac{N\cdot s}{m^2} \times 4.71\times10^{-10} \frac{m^3}{s} \times 0.025\, m \times \frac{1}{0.006\, m} \times \frac{m^2}{1.5\times10^6 N}\right]^{1/3}$$

$$a = 1.28\times10^{-5}\, m \quad (12.8\, \mu m)$$ a

Check assumptions: $\bar{V} = \frac{Q}{A} = \frac{Q}{\pi D a} = \frac{1}{\pi} \times 4.71\times10^{-10} \frac{m^3}{s} \times \frac{1}{0.006\, m} \times \frac{1}{1.28\times10^{-5} m} = 1.95 \frac{mm}{s}$

Thus $\frac{v}{\bar{V}} = 1 \frac{mm}{min} \times \frac{sec}{1.95\, mm} \times \frac{min}{60\, s} = 0.00855 < 0.01$

Therefore piston motion is negligible.

Also $Re = \frac{\bar{V} a}{\nu}$; $\nu = \frac{\mu}{\rho} = \frac{\mu}{SG\, \rho_{H_2O}}$. From Table A.2 (Appendix A), SG = 0.92

$$\nu = 0.42 \frac{N\cdot s}{m^2} \times \frac{m^3}{(0.92)1000\, kg} \times \frac{kg\cdot m}{N\cdot s^2} = 4.57\times10^{-4}\, m^2/s$$

$$Re = 1.95\times10^{-3} \frac{m}{s} \times 1.28\times10^{-5} m \times \frac{s}{4.57\times10^{-4}\, m^2} = 5.46\times10^{-5} \lll 1$$

Therefore flow is surely laminar!

Problem 8.17

[Difficulty: 2]

8.17 In Section 8.2 we derived the velocity profile between parallel plates (Eq. 8.5) by using a differential control volume. Instead, following the procedure we used in Example 5.9, derive Eq. 8.5 by starting with the Navier–Stokes equations (Eqs. 5.27). Be sure to state all assumptions.

Given: Navier-Stokes Equations

Find: Derivation of Eq. 8.5

Solution:

The Navier-Stokes equations are

$$\cancelto{4}{\frac{\partial u}{\partial x}} + \frac{\partial v}{\partial y} + \cancelto{3}{\frac{\partial w}{\partial z}} = 0 \qquad (5.1c)$$

$$\rho\left(\cancelto{1}{\frac{\partial u}{\partial t}} + u\cancelto{4}{\frac{\partial u}{\partial x}} + v\cancelto{5}{\frac{\partial u}{\partial y}} + w\cancelto{3}{\frac{\partial u}{\partial z}}\right) = \cancelto{6}{\rho g_x} - \frac{\partial p}{\partial x} + \mu\left(\cancelto{4}{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2} + \cancelto{3}{\frac{\partial^2 u}{\partial z^2}}\right) \qquad (5.27a)$$

$$\rho\left(\cancelto{1}{\frac{\partial v}{\partial t}} + u\cancelto{4}{\frac{\partial v}{\partial x}} + v\cancelto{5}{\frac{\partial v}{\partial y}} + w\cancelto{3}{\frac{\partial v}{\partial z}}\right) = \rho g_y - \frac{\partial p}{\partial y} + \mu\left(\cancelto{4}{\frac{\partial^2 v}{\partial x^2}} + \cancelto{5}{\frac{\partial^2 v}{\partial y^2}} + \cancelto{3}{\frac{\partial^2 v}{\partial z^2}}\right) \qquad (5.27b)$$

$$\rho\left(\cancelto{1}{\frac{\partial w}{\partial t}} + u\cancelto{3}{\frac{\partial w}{\partial x}} + v\cancelto{3}{\frac{\partial w}{\partial y}} + w\cancelto{3}{\frac{\partial w}{\partial z}}\right) = \cancelto{3}{\rho g_z} - \cancelto{3}{\frac{\partial p}{\partial z}} + \mu\left(\cancelto{3}{\frac{\partial^2 w}{\partial x^2}} + \cancelto{3}{\frac{\partial^2 w}{\partial y^2}} + \cancelto{3}{\frac{\partial^2 w}{\partial z^2}}\right) \qquad (5.27c)$$

The following assumptions have been applied:

(1) Steady flow (given).
(2) Incompressible flow; ρ = constant.
(3) No flow or variation of properties in the z direction; $w = 0$ and $\partial/\partial z = 0$.
(4) Fully developed flow, so no properties except pressure p vary in the x direction; $\partial/\partial x = 0$.
(5) See analysis below.
(6) No body force in the x direction; $g_x = 0$

Assumption (1) eliminates time variations in any fluid property. Assumption (2) eliminates space variations in density. Assumption (3) states that there is no z component of velocity and no property variations in the z direction. All terms in the z component of the Navier–Stokes equation cancel. After assumption (4) is applied, the continuity equation reduces to $\partial v/\partial y = 0$. Assumptions (3) and (4) also indicate that $\partial v/\partial z = 0$ and $\partial v/\partial x = 0$. Therefore v must be constant. Since v is zero at the solid surface, then v must be zero everywhere. The fact that $v = 0$ reduces the Navier–Stokes equations further, as indicated by (5). Hence for the y direction

$$\frac{\partial p}{\partial y} = \rho g$$

which indicates a hydrostatic variation of pressure. In the x direction, after assumption (6) we obtain

$$\mu \frac{\partial^2 u}{\partial y^2} - \frac{\partial p}{\partial x} = 0$$

Integrating twice

$$u = \frac{1}{2\mu}\frac{\partial p}{\partial x}y^2 + \frac{c_1}{\mu}y + c_2$$

To evaluate the constants, c_1 and c_2, we must apply the boundary conditions. At $y = 0$, $u = 0$. Consequently, $c_2 = 0$. At $y = a$, $u = 0$. Hence

$$0 = \frac{1}{2\mu}\frac{\partial p}{\partial x}a^2 + \frac{c_1}{\mu}a$$

which gives

$$c_1 = -\frac{1}{2\mu}\frac{\partial p}{\partial x}a$$

and finally

$$u = \frac{a^2}{2\mu}\frac{\partial p}{\partial x}\left[\left(\frac{y}{a}\right)^2 - \left(\frac{y}{a}\right)\right]$$

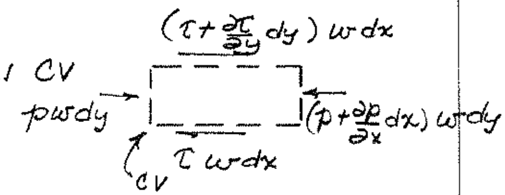
Problem 8.18

[Difficulty: 4]

8.18 Consider the simple power-law model for a non-Newtonian fluid given by Eq. 2.16. Extend the analysis of Section 8.2 to show that the velocity profile for fully developed laminar flow of a power-law fluid between stationary parallel plates separated by distance $2h$ may be written

$$u = \left(\frac{h}{k}\frac{\Delta p}{L}\right)^{1/n} \frac{nh}{n+1}\left[1-\left(\frac{y}{h}\right)^{(n+1)/n}\right]$$

where y is the coordinate measured from the channel centerline. Plot the profiles u/U_{max} versus y/h for $n = 0.7, 1.0,$ and 1.3.



Solution: Apply momentum equation to differential CV

Basic equation:

$$\underset{=0(1)}{F_{S_x}} + \cancel{F_{B_x}} = \underset{=0(2)}{\cancel{\frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall}} + \underset{=0(3)}{\cancel{\int_{CS} u\rho \vec{V}\cdot d\vec{A}}}$$

Assumptions: (1) Horizontal flow
(2) Steady flow
(3) Fully developed flow

Then

$$p w dy + \left(\tau + \frac{\partial \tau}{\partial y} dy\right) w dx - \left(p + \frac{\partial p}{\partial x} dx\right) w dy - \tau w dx = 0 \quad \text{or} \quad \frac{\partial \tau}{\partial y} = \frac{\partial p}{\partial x}$$

Since $\tau = \tau(y)$ and $p = p(x)$, then $\frac{d\tau}{dy} = \frac{\partial p}{\partial x} = \text{constant}$ and $\tau = y\frac{\partial p}{\partial x}$ or

$$\tau_{yx} = k\left(\frac{du}{dy}\right)^n = y\frac{\partial p}{\partial x} = -y\frac{\Delta p}{L}$$

Thus

$$\frac{du}{dy} = -\left(\frac{1}{k}\frac{\Delta p}{L}\right)^{1/n} y^{1/n}$$

Integrating

$$u = -\left(\frac{1}{k}\frac{\Delta p}{L}\right)^{1/n} \frac{1}{1/n+1}\, y^{1/n+1} + c = -\left(\frac{1}{k}\frac{\Delta p}{L}\right)^{\frac{1}{n}} \frac{n}{n+1}\, y^{\frac{n+1}{n}} + c$$

But $u = 0$ at $y = h$, so

$$c = \left(\frac{1}{k}\frac{\Delta p}{L}\right)^{\frac{1}{n}} \frac{n}{n+1}\, h^{\frac{n+1}{n}}$$

and

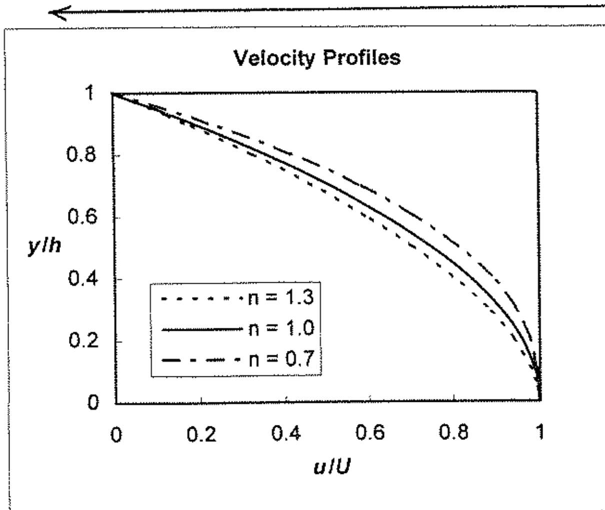
$$u = \left(\frac{1}{k}\frac{\Delta p}{L}\right)^{\frac{1}{n}} \frac{n}{n+1}\, h^{\frac{n+1}{n}}\left[1-\left(\frac{y}{h}\right)^{\frac{n+1}{n}}\right]$$

or

$$u = \left(\frac{h}{k}\frac{\Delta p}{L}\right)^{\frac{1}{n}} \frac{nh}{n+1}\left[1-\left(\frac{y}{h}\right)^{\frac{n+1}{n}}\right]$$

u

	$n = 0.7$	$n = 1.0$	$n = 1.3$
y/h	u/U	u/U	u/U
0	1	1	1
0.03	1.000	0.999	0.998
0.06	0.999	0.996	0.993
0.1	0.996	0.990	0.983
0.2	0.980	0.960	0.942
0.3	0.946	0.910	0.881
0.4	0.892	0.840	0.802
0.5	0.814	0.750	0.707
0.6	0.711	0.640	0.595
0.7	0.580	0.510	0.468
0.8	0.418	0.360	0.326
0.9	0.226	0.190	0.170
1	0	0	0

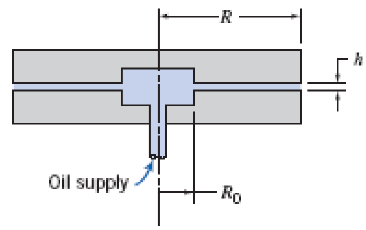


Problem 8.19

[Difficulty: 5]

8.19 Viscous liquid, at volume flow rate Q, is pumped through the central opening into the narrow gap between the parallel disks shown. The flow rate is low, so the flow is laminar, and the pressure gradient due to convective acceleration in the gap is negligible compared with the gradient caused by viscous forces (this is termed *creeping flow*). Obtain a general expression for the variation of average velocity in the gap between the disks. For creeping flow, the velocity profile at any cross section in the gap is the same as for fully developed flow between stationary parallel plates. Evaluate the pressure gradient, dp/dr, as a function of radius. Obtain an expression for $p(r)$. Show that the net force required to hold the upper plate in the position shown is

$$F = \frac{3\mu Q R^2}{h^3}\left[1 - \left(\frac{R_0}{R}\right)^2\right]$$



Solution: From the definition of mean velocity, $Q = \bar{V} 2\pi r h$ so $\bar{V} = \dfrac{Q}{2\pi r h}$ $\qquad \bar{V}(r)$

The pressure change with radius can be evaluated by analogy to Eq. 8.6b

$$\frac{Q}{l} = -\frac{1}{12\mu}\left(\frac{\partial p}{\partial x}\right)h^3 \quad \text{with } l = 2\pi r \quad \text{so } \frac{Q}{2\pi r} = -\frac{1}{12\mu}\left(\frac{\partial p}{\partial r}\right)h^3$$

Thus

$$\frac{dp}{dr} = -\frac{6\mu Q}{\pi h^3 r} \qquad \frac{dp}{dr}$$

Integrating to find $p(r)$,

$$\int_p^{p_{atm}} dp = p_{atm} - p = \int_r^R -\frac{6\mu Q}{\pi h^3 r}\,dr = -\frac{6\mu Q}{\pi h^3}\ln r\Big]_r^R = \frac{6\mu Q}{\pi h^3}\ln(r/R)$$

Thus $\quad p(r) = p_{atm} - \dfrac{6\mu Q}{\pi h^3}\ln(r/R) \quad (R_0 < r < R)\ ;\ p = p_0 \quad r < R_0$ $\qquad p(r)$

The force on the upper plate is $\quad dF_z = (p(r) - p_{atm})2\pi r\,dr$

Integrating and using gage pressures (note $p_{0g} = -\dfrac{6\mu Q}{\pi h^3}\ln\left(\dfrac{R_0}{R}\right)$)

$$F_z = p_0 \pi R_0^2 + \int_{R_0}^R p(r)\,2\pi r\,dr = p_0 \pi R_0^2 + 2\pi R^2 \int_{R_0/R}^1 p(r)\left(\frac{r}{R}\right)d\left(\frac{r}{R}\right)$$

$$= p_0 \pi R_0^2 + 2\pi R^2 \int_{R_0/R}^1 -\frac{6\mu Q}{\pi h^3}\ln\left(\frac{r}{R}\right)\left(\frac{r}{R}\right)d\left(\frac{r}{R}\right) = p_0 \pi R_0^2 - \frac{12\mu Q R^2}{h^3}\left(\frac{r}{R}\right)^2\left[\frac{1}{2}\ln\left(\frac{r}{R}\right) - \frac{1}{4}\right]_{R_0/R}^1$$

$$= p_0 \pi R_0^2 - \frac{12\mu Q R^2}{h^3}\left\{(1)\left[\frac{1}{2}(0) - \frac{1}{4}\right] - \left(\frac{R_0}{R}\right)^2\left[\frac{1}{2}\ln\left(\frac{R_0}{R}\right) - \frac{1}{4}\right]\right\}$$

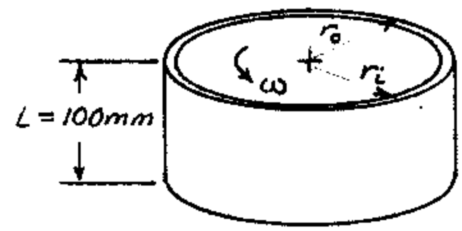
$$= -\frac{6\mu Q R^2}{h^3}\left(\frac{R_0}{R}\right)^2 \ln\left(\frac{R_0}{R}\right) - \frac{6\mu Q R^2}{h^3}\left[-\frac{1}{2} - \left(\frac{R_0}{R}\right)^2\ln\left(\frac{R_0}{R}\right) + \frac{1}{2}\left(\frac{R_0}{R}\right)^2\right]$$

$$F_z = \frac{3\mu Q R^2}{h^3}\left[1 - \left(\frac{R_0}{R}\right)^2\right] \qquad F_z$$

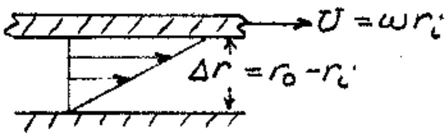
Problem 8.20

[Difficulty: 2]

8.20 A sealed journal bearing is formed from concentric cylinders. The inner and outer radii are 25 and 26 mm, the journal length is 100 mm, and it turns at 2800 rpm. The gap is filled with oil in laminar motion. The velocity profile is linear across the gap. The torque needed to turn the journal is 0.2 N · m. Calculate the viscosity of the oil. Will the torque increase or decrease with time? Why?



Solution: "Unfold" bearing since gap is small, and consider as flow between parallel plates. Apply Newton's law of viscosity.



Basic equation: $\tau_{yx} = \mu \frac{du}{dy}$

Assumption: Linear velocity profile

Then $\tau_{yx} = \mu \frac{U}{\Delta r} = \frac{\mu \omega r_i}{\Delta r}$

and

$$T = r_i \left(2\pi r_i L \tau_{yx}\right) = 2\pi r_i^2 L \tau_{yx} = \frac{2\pi \mu \omega r_i^3 L}{\Delta r}$$

Solving, $\mu = \frac{\Delta r\, T}{2\pi \omega r_i^3 L}$

$$\mu = \frac{1}{2\pi} \times 0.001\,m \times 0.2\,N\cdot m \times \frac{min}{2800\,rev} \times \frac{1}{(0.025)^3 m^3} \times \frac{1}{0.1\,m} \times \frac{rev}{2\pi\,rad} \times 60\frac{s}{min}$$

$$\mu = 0.0695\,N\cdot s / m^2$$

μ

Bearing is sealed, so oil temperature will increase as energy is dissipated by friction. For liquids, μ decreases as T increases. Thus torque will decrease, since it is proportional to μ.

Problem 8.21

[Difficulty: 3]

8.21 Using the profile of Problem 8.18, show that the flow rate for fully developed laminar flow of a power-law fluid between stationary parallel plates may be written as

$$Q = \left(\frac{h}{k}\frac{\Delta p}{L}\right)^{1/n}\frac{2nwh^2}{2n+1}$$

Here w is the plate width. In such an experimental setup the following data on applied pressure difference Δp and flow rate Q were obtained:

Δp (kPa)	10	20	30	40	50	60	70	80	90	100
Q (L/min)	0.451	0.759	1.01	1.15	1.41	1.57	1.66	1.85	2.05	2.25

Determine if the fluid is pseudoplastic or dilatant, and obtain an experimental value for n.

Given: Laminar velocity profile of power-law fluid flow between parallel plates

Find: Expression for flow rate; from data determine the type of fluid

Solution:

The velocity profile is
$$u = \left(\frac{h}{k}\cdot\frac{\Delta p}{L}\right)^{\frac{1}{n}}\cdot\frac{n\cdot h}{n+1}\cdot\left[1-\left(\frac{y}{h}\right)^{\frac{n+1}{n}}\right]$$

The flow rate is then
$$Q = w\cdot\int_{-h}^{h} u\,dy$$
or, because the flow is symmetric
$$Q = 2\cdot w\cdot\int_{0}^{h} u\,dy$$

The integral is computed as
$$\int 1-\left(\frac{y}{h}\right)^{\frac{n+1}{n}}\,dy = y\cdot\left[1-\frac{n}{2\cdot n+1}\cdot\left(\frac{y}{h}\right)^{\frac{2\cdot n+1}{n}}\right]$$

Using this with the limits
$$Q = 2\cdot w\cdot\left(\frac{h}{k}\cdot\frac{\Delta p}{L}\right)^{\frac{1}{n}}\cdot\frac{n\cdot h}{n+1}\cdot h\cdot\left[1-\frac{n}{2\cdot n+1}\cdot(1)^{\frac{2\cdot n+1}{n}}\right]$$
$$Q = \left(\frac{h}{k}\cdot\frac{\Delta p}{L}\right)^{\frac{1}{n}}\cdot\frac{2\cdot n\cdot w\cdot h^2}{2\cdot n+1}$$

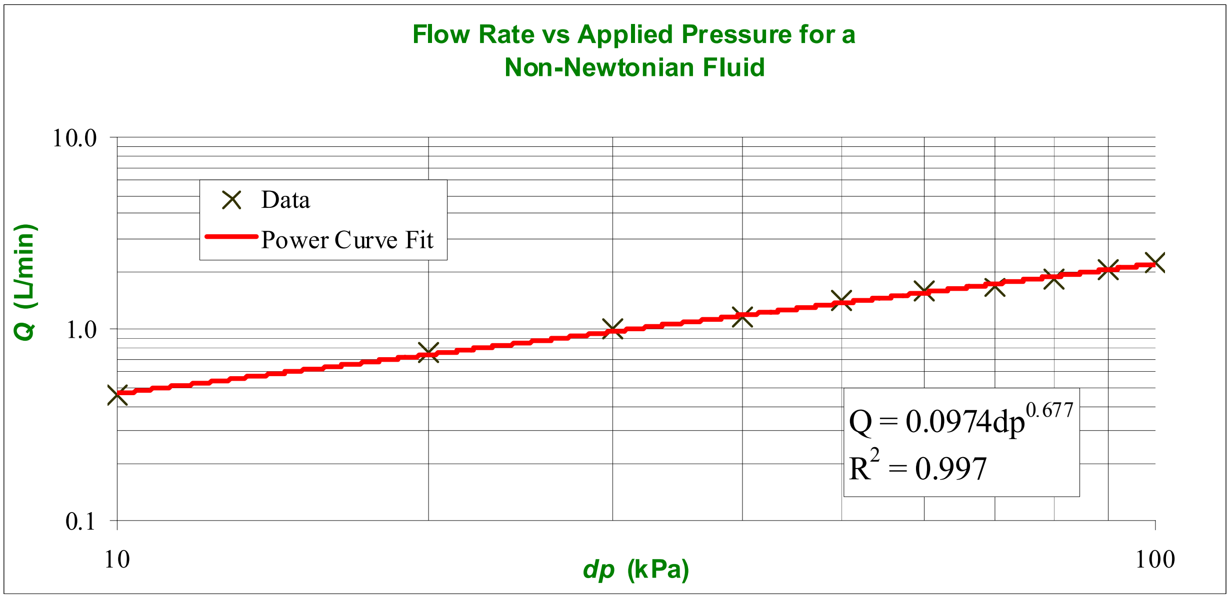
An *Excel* spreadsheet can be used for computation of *n*.

The data is

dp **(kPa)**	10	20	30	40	50	60	70	80	90	100
Q **(L/min)**	0.451	0.759	1.01	1.15	1.41	1.57	1.66	1.85	2.05	2.25

This must be fitted to
$$Q = \left(\frac{h}{k}\cdot\frac{\Delta p}{L}\right)^{\frac{1}{n}}\cdot\frac{2\cdot n\cdot w\cdot h^2}{2\cdot n+1}$$
or
$$Q = k\cdot\Delta p^{\frac{1}{n}}$$

We can fit a power curve to the data



Hence $1/n$ = 0.677 n = **1.48**

It's a dilatant fluid

Problem 8.22

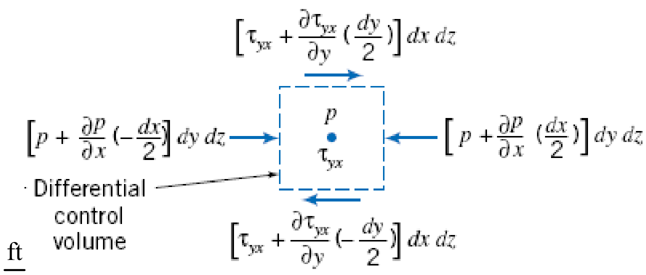
[Difficulty: 2]

8.22 Consider fully developed laminar flow between infinite parallel plates separated by gap width $d = 0.2$ in. The upper plate moves to the right with speed $U_2 = 5$ ft/s; the lower plate moves to the left with speed $U_1 = 2$ ft/s. The pressure gradient in the direction of flow is zero. Develop an expression for the velocity distribution in the gap. Find the volume flow rate per unit depth (gpm/ft) passing a given cross section.

Given: Laminar flow between moving plates

Find: Expression for velocity; Volume flow rate per depth

Solution:



Given data $\quad d = 0.2 \cdot \text{in} \qquad U_1 = 5 \cdot \frac{\text{ft}}{\text{s}} \qquad U_2 = 2 \cdot \frac{\text{ft}}{\text{s}}$

Using the analysis of Section 8.2, the sum of forces in the x direction is

$$\left[\tau + \frac{\partial}{\partial y}\tau \cdot \frac{dy}{2} - \left(\tau - \frac{\partial}{\partial y}\tau \cdot \frac{dy}{2}\right)\right] \cdot b \cdot dx + \left(p - \frac{\partial}{\partial x}p \cdot \frac{dx}{2} - p + \frac{\partial}{\partial x}p \cdot \frac{dx}{2}\right) \cdot b \cdot dy = 0$$

Simplifying $\quad \frac{d\tau}{dy} = \frac{dp}{dx} = 0 \qquad \text{or} \qquad \mu \cdot \frac{d^2u}{dy^2} = 0$

Integrating twice $\quad u = c_1 \cdot y + c_2$

Boundary conditions: $\quad u(0) = -U_1 \qquad c_2 = -U_1 \qquad c_2 = -5\frac{\text{ft}}{\text{s}} \qquad u(y = d) = U_2 \qquad c_1 = \frac{U_1 + U_2}{d} \qquad c_1 = 420\,\text{s}^{-1}$

Hence $\quad u(y) = (U_1 + U_2) \cdot \frac{y}{d} - U_1 \qquad u(y) = 420 \cdot y - 5 \qquad$ u in ft/s, y in ft

The volume flow rate is $\quad Q = \int u\, dA = b \cdot \int u\, dy \qquad Q = b \cdot \int_0^d \left[(U_1 + U_2) \cdot \frac{y}{d} - U_1\right] dx = b \cdot \left(\frac{U_1 + U_2}{d} \cdot \frac{d^2}{2} - U_1 \cdot d\right)$

Hence $\quad Q = b \cdot d \cdot \frac{(U_2 - U_1)}{2} \qquad \frac{Q}{b} = d \cdot \frac{(U_2 - U_1)}{2}$

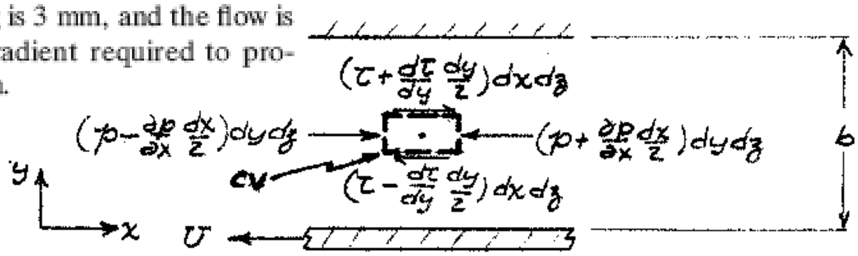
$$\frac{Q}{b} = 0.2 \cdot \text{in} \times \frac{1 \cdot \text{ft}}{12 \cdot \text{in}} \times \frac{1}{2} \times (2 - 5) \times \frac{\text{ft}}{\text{s}} \qquad \frac{Q}{b} = -.025 \cdot \frac{\frac{\text{ft}^3}{\text{s}}}{\text{ft}}$$

$$\frac{Q}{b} = -.025 \cdot \frac{\frac{\text{ft}^3}{\text{s}}}{\text{ft}} \times \frac{7.48 \cdot \text{gal}}{1 \cdot \text{ft}^3} \times \frac{60 \cdot \text{s}}{1 \cdot \text{min}} \qquad \frac{Q}{b} = -11.2 \cdot \frac{\text{gpm}}{\text{ft}}$$

Problem 8.23

[Difficulty: 2]

8.23 Water at 60°C flows between two large flat plates. The lower plate moves to the left at a speed of 0.3 m/s; the upper plate is stationary. The plate spacing is 3 mm, and the flow is laminar. Determine the pressure gradient required to produce zero net flow at a cross section.



Solution: Apply momentum equation using CV and coordinates shown.

Basic equations:

$$F_{S_x} + \underbrace{F_{B_x}}_{=0(1)} = \underbrace{\frac{\partial}{\partial t}\int_{CV} u\rho\, dV}_{=0(2)} + \underbrace{\int_{CS} u\rho \vec{V}\cdot d\vec{A}}_{=0(3)}, \quad \tau = \tau_{yx} = \mu\frac{du}{dy}$$

Assumptions: (1) $F_{Bx} = 0$
(2) Steady flow
(3) Fully-developed flow
(4) Newtonian fluid

Then $F_{Sx} = 0$. Substituting the force terms (see page 315 for details) gives

$$\frac{\partial p}{\partial x} = \frac{d\tau_{yx}}{dy} = \frac{d}{dy}\left(\mu\frac{du}{dy}\right) = \mu\frac{d^2u}{dy^2} \quad \text{or} \quad \frac{d^2u}{dy^2} = \frac{1}{\mu}\frac{\partial p}{\partial x}$$

Integrating twice,

$$u = \frac{1}{2\mu}\frac{\partial p}{\partial x}y^2 + c_1 y + c_2$$

To evaluate the constants c_1 and c_2, we must use the boundary conditions. At $y=0$, $u=-U$, so $c_2 = -U$. At $y=b$, $u=0$, so

$$0 = \frac{1}{2\mu}\frac{\partial p}{\partial x}b^2 + c_1 b - U \quad \text{or} \quad c_1 = \frac{U}{b} - \frac{1}{2\mu}\frac{\partial p}{\partial x}b$$

Thus

$$u = \frac{1}{2\mu}\frac{\partial p}{\partial x}(y^2 - by) + U\left(\frac{y}{b} - 1\right)$$

To find the flowrate, we integrate

$$\frac{Q}{w} = \int_0^b u\,dy = \int_0^b \left[\frac{1}{2\mu}\frac{\partial p}{\partial x}(y^2 - by) + U\left(\frac{y}{b} - 1\right)\right]dy = -\frac{1}{12\mu}\frac{\partial p}{\partial x}b^3 - \frac{Ub}{2}$$

For $Q = 0$, with $\mu = 4.63\times10^{-4}\ \frac{N\cdot s}{m^2}$ from Table A.8,

$$\frac{\partial p}{\partial x} = -\frac{6U\mu}{b^2} = -6\times 0.3\frac{m}{s}\times 4.63\times10^{-4}\frac{N\cdot s}{m^2}\times\frac{1}{(0.003)^2 m^2} = -92.6\ N/m^2\cdot m$$

$\frac{\partial p}{\partial x}$

Thus pressure must <u>decrease</u> in x direction for zero net flowrate.

Problem 8.24

[Difficulty: 3]

8.24 Two immiscible fluids are contained between infinite parallel plates. The plates are separated by distance $2h$, and the two fluid layers are of equal thickness $h = 5$ mm. The dynamic viscosity of the upper fluid is four times that of the lower fluid, which is $\mu_{\text{lower}} = 0.1\ \text{N}\cdot\text{s/m}^2$. If the plates are stationary and the applied pressure gradient is -50 kPa/m, find the velocity at the interface. What is the maximum velocity of the flow? Plot the velocity distribution.

Given: Properties of two fluids flowing between parallel plates; applied pressure gradient

Find: Velocity at the interface; maximum velocity; plot velocity distribution

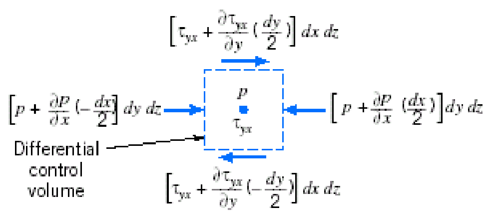
Solution:

Given data $\quad \dfrac{dp}{dx} = k \qquad k = -50\cdot\dfrac{\text{kPa}}{\text{m}} \qquad h = 5\cdot\text{mm} \qquad \mu_1 = 0.1\cdot\dfrac{\text{N}\cdot\text{s}}{\text{m}^2} \qquad \mu_2 = 4\cdot\mu_1 \qquad \mu_2 = 0.4\cdot\dfrac{\text{N}\cdot\text{s}}{\text{m}^2}$

(Lower fluid is fluid 1; upper is fluid 2)

Following the analysis of Section 8.2, analyse the forces on a differential CV of either fluid

The net force is zero for steady flow, so



$$\left[\tau + \frac{d\tau}{dy}\cdot\frac{dy}{2} - \left(\tau - \frac{d\tau}{dy}\cdot\frac{dy}{2}\right)\right]\cdot dx\cdot dz + \left[p - \frac{dp}{dx}\cdot\frac{dx}{2} - \left(p + \frac{dp}{dx}\cdot\frac{dx}{2}\right)\right]\cdot dy\cdot dz = 0$$

Simplifying $\qquad \dfrac{d\tau}{dy} = \dfrac{dp}{dx} = k \qquad$ so for each fluid $\qquad \mu\cdot\dfrac{d^2}{dy^2}u = k$

Applying this to fluid 1 (lower fluid) and fluid 2 (upper fluid), integrating twice yields

$$u_1 = \frac{k}{2\cdot\mu_1}\cdot y^2 + c_1\cdot y + c_2 \qquad\qquad u_2 = \frac{k}{2\cdot\mu_2}\cdot y^2 + c_3\cdot y + c_4$$

For convenience the origin of coordinates is placed at the centerline

We need four BCs. Three are obvious $\qquad y = -h \quad u_1 = 0 \quad (1) \qquad y = 0 \quad u_1 = u_2 \quad (2) \qquad y = h \quad u_2 = 0 \quad (3)$

The fourth BC comes from the fact that the stress at the interface generated by each fluid is the same

$$y = 0 \qquad\qquad \mu_1\cdot\frac{du_1}{dy} = \mu_2\cdot\frac{du_2}{dy} \qquad (4)$$

Using these four BCs $\qquad 0 = \dfrac{k}{2\cdot\mu_1}\cdot h^2 - c_1\cdot h + c_2 \qquad c_2 = c_4 \qquad 0 = \dfrac{k}{2\cdot\mu_2}\cdot h^2 + c_3\cdot h + c_4 \qquad \mu_1\cdot c_1 = \mu_2\cdot c_3$

Hence, after some algebra

$$c_1 = \frac{k\cdot h}{2\cdot\mu_1}\cdot\frac{(\mu_2 - \mu_1)}{(\mu_2 + \mu_1)} \qquad c_4 = -\frac{k\cdot h^2}{\mu_2 + \mu_1} \qquad c_2 = c_4 \qquad c_3 = \frac{k\cdot h}{2\cdot\mu_2}\cdot\frac{(\mu_2 - \mu_1)}{(\mu_2 + \mu_1)}$$

$$c_1 = -750\,\frac{1}{s} \qquad c_2 = 2.5\,\frac{m}{s} \qquad c_3 = -187.5\,\frac{1}{s} \qquad c_4 = 2.5\,\frac{m}{s}$$

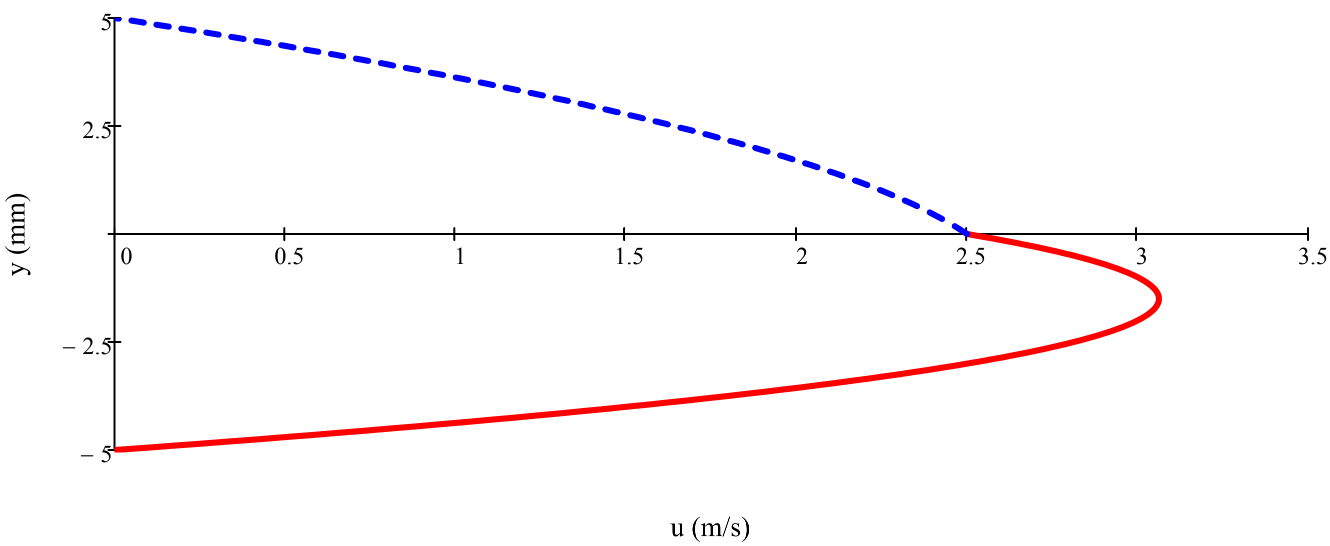
The velocity distributions are then

$$u_1(y) = \frac{k}{2\cdot\mu_1}\cdot\left[y^2 + y\cdot h\cdot\frac{(\mu_2-\mu_1)}{(\mu_2+\mu_1)}\right] - \frac{k\cdot h^2}{\mu_2+\mu_1} \qquad u_2(y) = \frac{k}{2\cdot\mu_2}\cdot\left[y^2 + y\cdot h\cdot\frac{(\mu_2-\mu_1)}{(\mu_2+\mu_1)}\right] - \frac{k\cdot h^2}{\mu_2+\mu_1}$$

Evaluating either velocity at $y = 0$, gives the velocity at the interface

$$u_{interface} = -\frac{k\cdot h^2}{\mu_2+\mu_1} \qquad u_{interface} = 2.5\,\frac{m}{s}$$

The plots of these velocity distributions can be done in *Excel*. Typical curves are shown below



Clearly, u_1 has the maximum velocity, when $\frac{du_1}{dy} = 0$ or $2\cdot y_{max} + h\cdot\frac{(\mu_2-\mu_1)}{(\mu_2+\mu_1)} = 0$

$$y_{max} = -\frac{h}{2}\cdot\frac{(\mu_2-\mu_1)}{(\mu_2+\mu_1)} \qquad y_{max} = -1.5\,mm \qquad u_{max} = u_1(y_{max}) \qquad u_{max} = 3.06\,\frac{m}{s}$$

(We could also have used *Excel*'s Solver for this.)

Problem 8.25

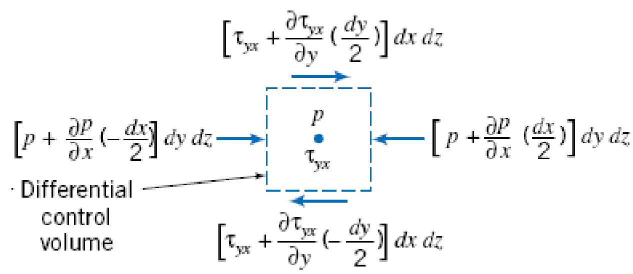
[Difficulty: 3]

8.25 Two immiscible fluids are contained between infinite parallel plates. The plates are separated by distance $2h$, and the two fluid layers are of equal thickness h; the dynamic viscosity of the upper fluid is three times that of the lower fluid. If the lower plate is stationary and the upper plate moves at constant speed $U = 20$ ft/s, what is the velocity at the interface? Assume laminar flows, and that the pressure gradient in the direction of flow is zero.

Given: Laminar flow of two fluids between plates

Find: Velocity at the interface

Solution:



Using the analysis of Section 8.2, the sum of forces in the x direction is

$$\left[\tau + \frac{\partial}{\partial y}\tau\cdot\frac{dy}{2} - \left(\tau - \frac{\partial}{\partial y}\tau\cdot\frac{dy}{2}\right)\right]\cdot b\cdot dx + \left(p - \frac{\partial}{\partial x}p\cdot\frac{dx}{2} - p + \frac{\partial}{\partial x}p\cdot\frac{dx}{2}\right)\cdot b\cdot dy = 0$$

Simplifying $\quad \frac{d\tau}{dy} = \frac{dp}{dx} = 0 \quad$ or $\quad \mu\cdot\frac{d^2u}{dy^2} = 0$

Applying this to fluid 1 (lower fluid) and fluid 2 (upper fluid), integrating twice yields $\quad u_1 = c_1\cdot y + c_2 \qquad u_2 = c_3\cdot y + c_4$

We need four BCs. Three are obvious $\quad y = 0 \quad u_1 = 0 \qquad y = h \quad u_1 = u_2 \qquad y = 2\cdot h \quad u_2 = U$

The fourth BC comes from the fact that the stress at the interface generated by each fluid is the same

$$y = h \qquad \mu_1\cdot\frac{du_1}{dy} = \mu_2\cdot\frac{du_2}{dy}$$

Using these four BCs $\quad 0 = c_2 \qquad c_1\cdot h + c_2 = c_3\cdot h + c_4 \qquad U = c_3\cdot 2\cdot h + c_4 \qquad \mu_1\cdot c_1 = \mu_2\cdot c_3$

Hence $\quad c_2 = 0$

From the 2nd and 3rd equations $\quad c_1\cdot h - U = -c_3\cdot h \quad$ and $\quad \mu_1\cdot c_1 = \mu_2\cdot c_3$

Hence $\quad c_1\cdot h - U = -c_3\cdot h = -\frac{\mu_1}{\mu_2}\cdot h\cdot c_1 \qquad c_1 = \frac{U}{h\cdot\left(1 + \frac{\mu_1}{\mu_2}\right)}$

Hence for fluid 1 (we do not need to complete the analysis for fluid 2) $\quad u_1 = \frac{U}{h\cdot\left(1 + \frac{\mu_1}{\mu_2}\right)}\cdot y$

Evaluating this at $y = h$, where $u_1 = u_{interface}$ $\quad u_{interface} = \frac{20\cdot\frac{ft}{s}}{\left(1 + \frac{1}{3}\right)} \qquad u_{interface} = 15\cdot\frac{ft}{s}$

Problem 8.26

[Difficulty: 2]

8.26 The record-read head for a computer disk-drive memory storage system rides above the spinning disk on a very thin film of air (the film thickness is 0.25 μm). The head location is 25 mm from the disk centerline; the disk spins at 8500 rpm. The record-read head is 5 mm square. For standard air in the gap between the head and disk, determine (a) the Reynolds number of the flow, (b) the viscous shear stress, and (c) the power required to overcome viscous shear.

Given: Computer disk drive

Find: Flow Reynolds number; Shear stress; Power required

Solution:

For a distance R from the center of a disk spinning at speed ω

$$V = R\cdot\omega \qquad V = 25\cdot mm \times \frac{1\cdot m}{1000\cdot mm} \times 8500\cdot rpm \times \frac{2\cdot\pi\cdot rad}{rev} \times \frac{1\cdot min}{60\cdot s} \qquad V = 22.3\cdot\frac{m}{s}$$

The gap Reynolds number is $Re = \frac{\rho\cdot V\cdot a}{\mu} = \frac{V\cdot a}{\nu}$ $\qquad \nu = 1.45\times 10^{-5}\cdot\frac{m^2}{s}$ from Table A.10 at 15ºC

$$Re = 22.3\cdot\frac{m}{s} \times 0.25\times 10^{-6}\cdot m \times \frac{s}{1.45\times 10^{-5}\cdot m^2} \qquad Re = 0.384$$

The flow is definitely laminar

The shear stress is then $\tau = \mu\cdot\frac{du}{dy} = \mu\cdot\frac{V}{a}$ $\qquad \mu = 1.79\times 10^{-5}\cdot\frac{N\cdot s}{m^2}$ from Table A.10 at 15ºC

$$\tau = 1.79\times 10^{-5}\cdot\frac{N\cdot s}{m^2} \times 22.3\cdot\frac{m}{s} \times \frac{1}{0.25\times 10^{-6}\cdot m} \qquad \tau = 1.60\cdot kPa$$

The power required is $P = T\cdot\omega$ where torque T is given by

$$T = \tau\cdot A\cdot R \quad \text{with} \quad A = (5\cdot mm)^2 \qquad A = 2.5\times 10^{-5}\,m^2$$

$$P = \tau\cdot A\cdot R\cdot\omega \qquad P = 1600\cdot\frac{N}{m^2} \times 2.5\times 10^{-5}\cdot m^2 \times 25\cdot mm \times \frac{1\cdot m}{1000\cdot mm} \times 8500\cdot rpm \times \frac{2\cdot\pi\cdot rad}{rev} \times \frac{1\cdot min}{60\cdot s} \qquad P = 0.890\ W$$

Problem 8.27

[Difficulty: 2]

8.27 The dimensionless velocity profile for fully developed laminar flow between infinite parallel plates with the upper plate moving at constant speed U is shown in Fig. 8.6. Find the pressure gradient $\partial p/\partial x$ at which (a) the upper plate and (b) the lower plate experience zero shear stress, in terms of U, a, and μ. Plot the dimensionless velocity profiles for these cases.

Given: Velocity profile between parallel plates

Find: Pressure gradients for zero stress at upper/lower plates; plot

Solution:

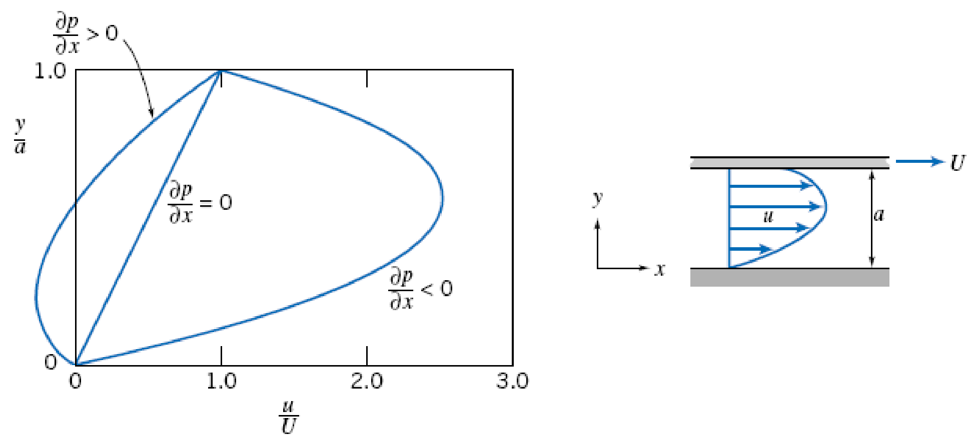


Fig. 8.6 Dimensionless velocity profile for fully developed laminar flow between infinite parallel plates: upper plate moving with constant speed, U.

From Eq. 8.8, the velocity distribution is

$$u = \frac{U \cdot y}{a} + \frac{a^2}{2 \cdot \mu} \cdot \left(\frac{\partial}{\partial x} p\right) \cdot \left[\left(\frac{y}{a}\right)^2 - \frac{y}{a}\right]$$

The shear stress is

$$\tau_{yx} = \mu \cdot \frac{du}{dy} = \mu \cdot \frac{U}{a} + \frac{a^2}{2} \cdot \left(\frac{\partial}{\partial x} p\right) \cdot \left(2 \cdot \frac{y}{a^2} - \frac{1}{a}\right)$$

(a) For $\tau_{yx} = 0$ at $y = a$

$$0 = \mu \cdot \frac{U}{a} + \frac{a}{2} \cdot \frac{\partial}{\partial x} p \qquad \frac{\partial}{\partial x} p = -\frac{2 \cdot U \cdot \mu}{a^2}$$

The velocity distribution is then

$$u = \frac{U \cdot y}{a} - \frac{a^2}{2 \cdot \mu} \cdot \frac{2 \cdot U \cdot \mu}{a^2} \cdot \left[\left(\frac{y}{a}\right)^2 - \frac{y}{a}\right] \qquad \frac{u}{U} = 2 \cdot \frac{y}{a} - \left(\frac{y}{a}\right)^2$$

(b) For $\tau_{yx} = 0$ at $y = 0$

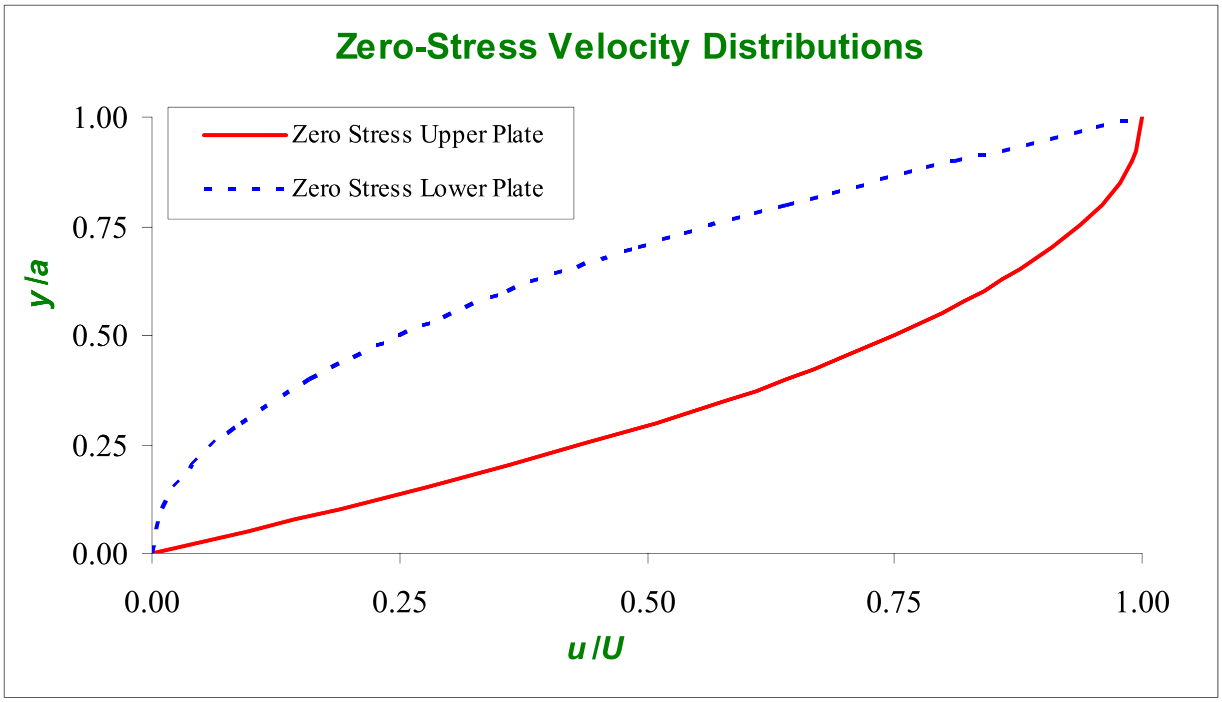
$$0 = \mu \cdot \frac{U}{a} - \frac{a}{2} \cdot \frac{\partial}{\partial x} p \qquad \frac{\partial}{\partial x} p = \frac{2 \cdot U \cdot \mu}{a^2}$$

The velocity distribution is then

$$u = \frac{U \cdot y}{a} + \frac{a^2}{2 \cdot \mu} \cdot \frac{2 \cdot U \cdot \mu}{a^2} \cdot \left[\left(\frac{y}{a}\right)^2 - \frac{y}{a}\right] \qquad \frac{u}{U} = \left(\frac{y}{a}\right)^2$$

The velocity distributions can be plotted in *Excel*.

y/a	(a) u/U	(b) u/U
0.0	0.000	0.000
0.1	0.190	0.010
0.2	0.360	0.040
0.3	0.510	0.090
0.4	0.640	0.160
0.5	0.750	0.250
0.6	0.840	0.360
0.7	0.910	0.490
0.8	0.960	0.640
0.9	0.990	0.810
1.0	1.00	1.000

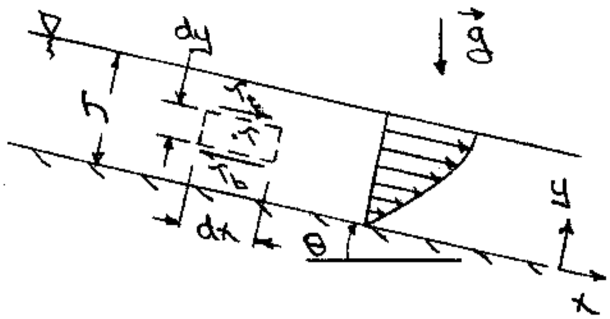


Problem 8.28

[Difficulty: 2]

8.28 Consider steady, fully developed laminar flow of a viscous liquid down an inclined surface. The liquid layer is of constant thickness, h. Use a suitably chosen differential control volume to obtain the velocity profile. Develop an expression for the volume flow rate.

Solution: Flow is fully developed, so $u = u(y)$ and $\tau = \tau(y)$. Expand τ in a Taylor series about the center of the differential CV



$$\tau_t = \tau + \frac{d\tau}{dy}\frac{dy}{2}$$

$$\tau_b = \tau + \frac{d\tau}{dy}\left(-\frac{dy}{2}\right)$$

The boundary conditions on the velocity profile are:

@ $y = 0$, $u = 0$ (no slip).

@ $y = h$, $\frac{du}{dy} = 0$ (no shear stress).

Apply the x component of the momentum equation to the differential CV shown

$$F_{S_x} + F_{B_x} = \cancel{\frac{\partial}{\partial t}\int_{CV} u\rho\, dV}^{\,0(1)} + \cancel{\int_{CS} u\rho\vec{V}\cdot d\vec{A}}^{\,0(2)}$$

Assumptions: (1) steady flow

(2) fully developed flow, so u and τ are functions of y only

(3) no variation of pressure in the x direction

Then

$$F_{S_x} + F_{B_x} = 0 = \left(\tau + \frac{d\tau}{dy}\frac{dy}{2}\right)dx\,dz - \left(\tau - \frac{d\tau}{dy}\frac{dy}{2}\right)dx\,dz + \rho g\sin\theta\, dx\,dy\,dz$$

or

$$\frac{d\tau}{dy} = -\rho g \sin\theta$$

Integrating, $\tau = -\rho g\sin\theta\, y + c_1$

But $\tau = 0$ @ $y = h$, $\therefore$ $c_1 = \rho g \sin\theta\, h$, and

$$\frac{du}{dy} = \frac{\rho g\sin\theta}{\mu}(h - y)$$

Integrating again, $u = \frac{\rho g\sin\theta}{\mu}\left(hy - \frac{y^2}{2}\right) + c_2$

At $y = 0$, $u = 0$, so $c_2 = 0$ and hence

$$u = \frac{\rho g\sin\theta}{\mu}\left(hy - \frac{y^2}{2}\right) \quad u$$

$$Q/w = \int_0^h u\,dy = \frac{\rho g\sin\theta}{\mu}\int_0^h\left(hy - \frac{y^2}{2}\right)dy = \frac{\rho g\sin\theta}{\mu}\left[\frac{hy^2}{2} - \frac{y^3}{6}\right]_0^h$$

$$Q/w = \rho g\sin\theta\, h^3/3\mu \quad Q/w$$

Problem 8.29

[Difficulty: 2]

8.29 Consider steady, incompressible, and fully developed laminar flow of a viscous liquid down an incline with no pressure gradient. The velocity profile was derived in Example 5.9. Plot the velocity profile. Calculate the kinematic viscosity of the liquid if the film thickness on a $30°$ slope is 0.8 mm and the maximum velocity is 15.7 mm/s.

Solution:

$$u = \frac{\rho g \sin\theta}{\mu}\left(hy - \frac{y^2}{2}\right) = \frac{g\sin\theta}{\nu}\left(hy - \frac{y^2}{2}\right)$$

$u = u_{max}$ at $y = h$

$$\therefore\ u_{max} = \frac{g\sin\theta}{\nu}\left(h^2 - \frac{h^2}{2}\right) = \frac{g\sin\theta h^2}{2\nu}$$

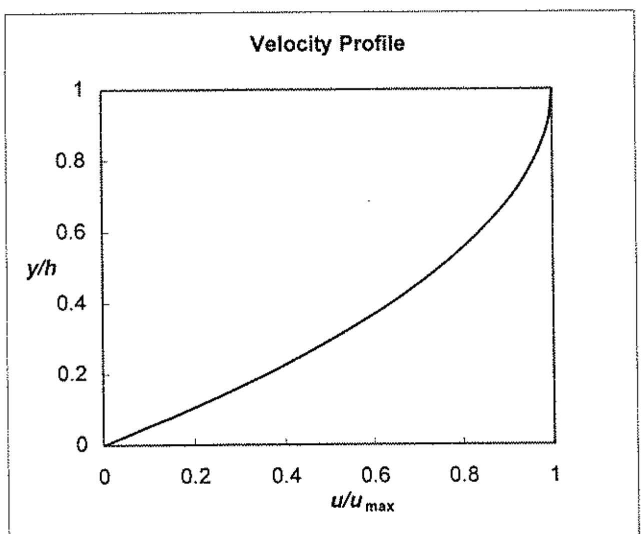
and

$$\nu = \frac{g\sin\theta h^2}{2u_{max}} = \frac{\sin 30°}{2} \times 9.81\frac{m}{s^2} \times (0.8\times10^{-3}\,m)^2 \times \frac{s}{15.7\times10^{-3}\,m}$$

$$\nu = 1.00\times10^{-4}\ m^2/s \qquad \nu$$

Plot $$\frac{u}{u_{max}} = \frac{g\sin\theta}{\nu}\left(hy - \frac{y^2}{2}\right) \times \frac{2\nu}{g\sin\theta h^2} = 2\frac{y}{h} - \left(\frac{y}{h}\right)^2$$

u/u_{max}	y/h
0	0
0.0396	0.02
0.098	0.05
0.190	0.1
0.360	0.2
0.510	0.3
0.640	0.4
0.750	0.5
0.840	0.6
0.910	0.7
0.960	0.8
0.990	0.9
1.00	1.0



Problem 8.30

[Difficulty: 3]

8.30 Two immiscible fluids of equal density are flowing down a surface inclined at a 60° angle. The two fluid layers are of equal thickness $h = 10$ mm; the kinematic viscosity of the upper fluid is 1/5th that of the lower fluid, which is $\nu_{\text{lower}} = 0.01\ \text{m}^2/\text{s}$. Find the velocity at the interface and the velocity at the free surface. Plot the velocity distribution.

Given: Data on flow of liquids down an incline

Find: Velocity at interface; velocity at free surface; plot

Solution:

Given data $\quad h = 10\cdot\text{mm} \quad \theta = 60\cdot\text{deg} \quad \nu_1 = 0.01\cdot\frac{\text{m}^2}{\text{s}} \quad \nu_2 = \frac{\nu_1}{5} \quad \nu_2 = 2\times10^{-3}\,\frac{\text{m}^2}{\text{s}}$

(The lower fluid is designated fluid 1, the upper fluid 2)

From Example 5.9 (or Exanple 8.3 with *g* replaced with *g*sinθ), a free body analysis leads to (for either fluid)

$$\frac{d^2}{dy^2}u = -\frac{\rho\cdot g\cdot\sin(\theta)}{\mu}$$

Applying this to fluid 1 (lower fluid) and fluid 2 (upper fluid), integrating twice yields

$$u_1 = -\frac{\rho\cdot g\cdot\sin(\theta)}{2\cdot\mu_1}\cdot y^2 + c_1\cdot y + c_2 \qquad u_2 = -\frac{\rho\cdot g\cdot\sin(\theta)}{2\cdot\mu_2}\cdot y^2 + c_3\cdot y + c_4$$

We need four BCs. Two are obvious

$$y = 0 \qquad u_1 = 0 \qquad (1)$$

$$y = h \qquad u_1 = u_2 \qquad (2)$$

The third BC comes from the fact that there is no shear stress at the free surface

$$y = 2\cdot h \qquad \mu_2\cdot\frac{du_2}{dy} = 0 \qquad (3)$$

The fourth BC comes from the fact that the stress at the interface generated by each fluid is the same

$$y = h \qquad \mu_1\cdot\frac{du_1}{dy} = \mu_2\cdot\frac{du_2}{dy} \qquad (4)$$

Using these four BCs $\quad c_2 = 0$

$$-\frac{\rho\cdot g\cdot\sin(\theta)}{2\cdot\mu_1}\cdot h^2 + c_1\cdot h + c_2 = -\frac{\rho\cdot g\cdot\sin(\theta)}{2\cdot\mu_2}\cdot h^2 + c_3\cdot h + c_4$$

$$-\rho\cdot g\cdot\sin(\theta)\cdot 2\cdot h + \mu_2\cdot c_3 = 0 \qquad -\rho\cdot g\cdot\sin(\theta)\cdot h + \mu_1\cdot c_1 = -\rho\cdot g\cdot\sin(\theta)\cdot h + \mu_2\cdot c_3$$

Hence, after some algebra $\quad c_1 = \frac{2\cdot\rho\cdot g\cdot\sin(\theta)\cdot h}{\mu_1} \quad c_2 = 0 \quad c_3 = \frac{2\cdot\rho\cdot g\cdot\sin(\theta)\cdot h}{\mu_2} \quad c_4 = 3\cdot\rho\cdot g\cdot\sin(\theta)\cdot h^2\cdot\frac{(\mu_2 - \mu_1)}{2\cdot\mu_1\cdot\mu_2}$

The velocity distributions are then $\quad u_1 = \frac{\rho \cdot g \cdot \sin(\theta)}{2 \cdot \mu_1} \cdot \left(4 \cdot y \cdot h - y^2\right) \qquad u_2 = \frac{\rho \cdot g \cdot \sin(\theta)}{2 \cdot \mu_2} \cdot \left[3 \cdot h^2 \cdot \frac{(\mu_2 - \mu_1)}{\mu_1} + 4 \cdot y \cdot h - y^2\right]$

Rewriting in terms of ν_1 and ν_2 (ρ is constant and equal for both fluids)

$$u_1 = \frac{g \cdot \sin(\theta)}{2 \cdot \nu_1} \cdot \left(4 \cdot y \cdot h - y^2\right) \qquad u_2 = \frac{g \cdot \sin(\theta)}{2 \cdot \nu_2} \cdot \left[3 \cdot h^2 \cdot \frac{(\nu_2 - \nu_1)}{\nu_1} + 4 \cdot y \cdot h - y^2\right]$$

(Note that these result in the same expression if $\nu_1 = \nu_2$, i.e., if we have one fluid)

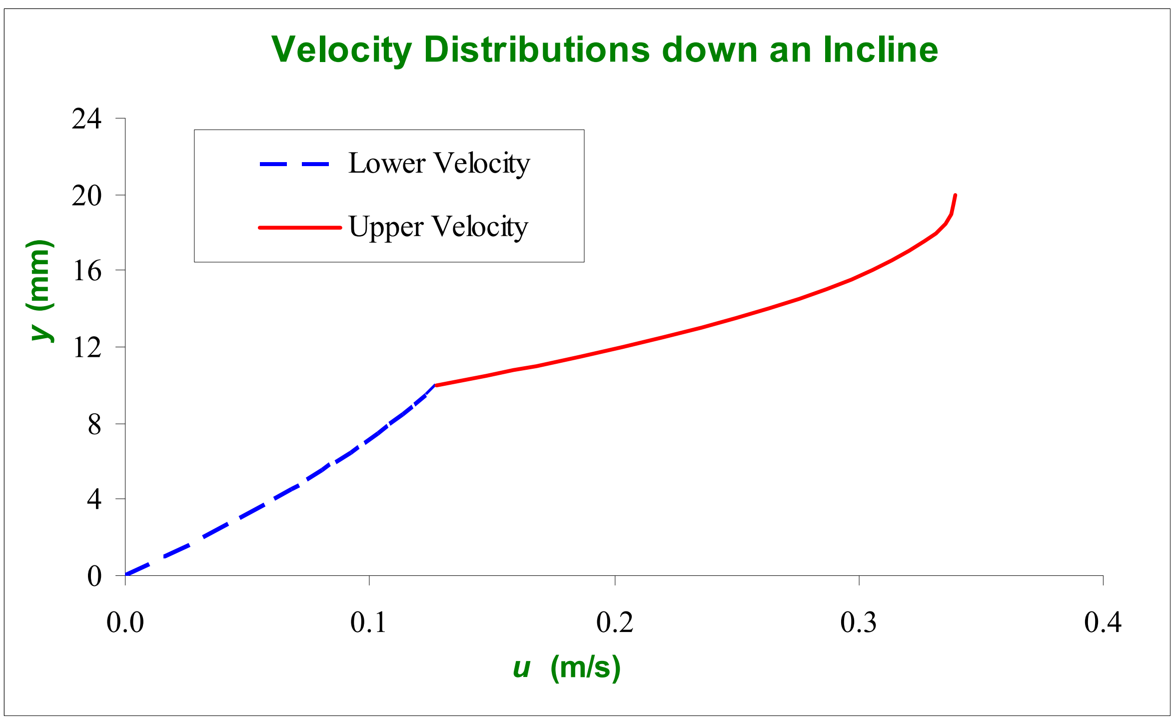
Evaluating either velocity at $y = h$, gives the velocity at the interface $\quad u_{interface} = \frac{3 \cdot g \cdot h^2 \cdot \sin(\theta)}{2 \cdot \nu_1} \qquad u_{interface} = 0.127 \frac{m}{s}$

Evaluating u_2 at $y = 2h$ gives the velocity at the free surface $\quad u_{freesurface} = g \cdot h^2 \cdot \sin(\theta) \cdot \frac{(3 \cdot \nu_2 + \nu_1)}{2 \cdot \nu_1 \cdot \nu_2} \qquad u_{freesurface} = 0.340 \frac{m}{s}$

Note that a Reynolds number based on the free surface velocity is $\quad \frac{u_{freesurface} \cdot h}{\nu_2} = 1.70 \quad$ indicating laminar flow

The velocity distributions can be plotted in *Excel*.

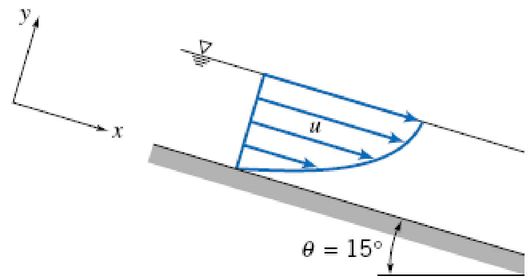
y (mm)	u_1 (m/s)	u_2 (m/s)
0.000	0.000	
1.000	0.0166	
2.000	0.0323	
3.000	0.0472	
4.000	0.061	
5.000	0.074	
6.000	0.087	
7.000	0.098	
8.000	0.109	
9.000	0.119	
10.000	**0.127**	**0.127**
11.000		0.168
12.000		0.204
13.000		0.236
14.000		0.263
15.000		0.287
16.000		0.306
17.000		0.321
18.000		0.331
19.000		0.338
20.000		**0.340**



Problem 8.31

[Difficulty: 2]

8.31 The velocity distribution for flow of a thin viscous film down an inclined plane surface was developed in Example 5.9. Consider a film 7 mm thick, of liquid with SG = 1.2 and dynamic viscosity of 1.60 N·s/m^2. Derive an expression for the shear stress distribution within the film. Calculate the maximum shear stress within the film and indicate its direction. Evaluate the volume flow rate in the film, in mm^3/s per millimeter of surface width. Calculate the film Reynolds number based on average velocity.



Given: Velocity distribution on incline

Find: Expression for shear stress; Maximum shear; volume flow rate/mm width; Reynolds number

Solution:

From Example 5.9 $$u(y) = \frac{\rho\cdot g\cdot \sin(\theta)}{\mu}\cdot\left(h\cdot y - \frac{y^2}{2}\right)$$

For the shear stress $$\tau = \mu\cdot\frac{du}{dy} = \rho\cdot g\cdot\sin(\theta)\cdot(h-y)$$

τ is a maximum at y = 0 $$\tau_{max} = \rho\cdot g\cdot\sin(\theta)\cdot h = SG\cdot\rho_{H2O}\cdot g\cdot\sin(\theta)\cdot h$$

$$\tau_{max} = 1.2\times 1000\frac{kg}{m^3}\times 9.81\cdot\frac{m}{s^2}\times\sin(15\cdot deg)\times 0.007\cdot m\times\frac{N\cdot s^2}{kg\cdot m} \qquad \tau_{max} = 21.3\ Pa$$

This stress is in the x direction on the wall

The flow rate is $$Q = \int u\,dA = w\cdot\int_0^h u(y)\,dy = w\cdot\int_0^h \frac{\rho\cdot g\cdot\sin(\theta)}{\mu}\cdot\left(h\cdot y - \frac{y^2}{2}\right)dy \qquad Q = \frac{\rho\cdot g\cdot\sin(\theta)\cdot w\cdot h^3}{3\cdot\mu}$$

$$\frac{Q}{w} = \frac{1}{3}\times 1.2\times 1000\frac{kg}{m^3}\times 9.81\cdot\frac{m}{s^2}\times\sin(15\cdot deg)\times(0.007\cdot m)^3\times\frac{m^2}{1.60\cdot N\cdot s}\cdot\frac{N\cdot s^2}{kg\cdot m} = 2.18\times 10^{-4}\frac{\frac{m^3}{s}}{m} \qquad \frac{Q}{w} = 217\frac{\frac{mm^3}{s}}{mm}$$

The average velocity is $$V = \frac{Q}{A} = \frac{Q}{w\cdot h} \qquad V = 217\cdot\frac{\frac{mm^3}{s}}{mm}\times\frac{1}{7\cdot mm} \qquad V = 31.0\cdot\frac{mm}{s}$$

The gap Reynolds number is $$Re = \frac{\rho\cdot V\cdot h}{\mu}$$

$$Re = 1.2\times 1000\frac{kg}{m^3}\times 31\cdot\frac{mm}{s}\times 7\cdot mm\times\frac{m^2}{1.60\cdot N\cdot s}\times\left(\frac{1\cdot m}{1000\cdot mm}\right)^2 \qquad Re = 0.163$$

The flow is definitely laminar

Problem 8.32

[Difficulty: 3]

8.32 Consider fully developed flow between parallel plates with the upper plate moving at $U = 5$ ft/s; the spacing between the plates is $a = 0.1$ in. Determine the flow rate per unit depth for the case of zero pressure gradient. If the fluid is air, evaluate the shear stress on the lower plate and plot the shear stress distribution across the channel for the zero pressure gradient case. Will the flow rate increase or decrease if the pressure gradient is adverse? Determine the pressure gradient that will give zero shear stress at $y = 0.25a$. Plot the shear stress distribution across the channel for the latter case.

Given: Flow between parallel plates

Find: Shear stress on lower plate; Plot shear stress; Flow rate for pressure gradient; Pressure gradient for zero shear; Plot

Solution:

From Section 8-2 $\quad u(y) = \frac{U \cdot y}{a} + \frac{a^2}{2 \cdot \mu} \cdot \frac{dp}{dx} \cdot \left[\left(\frac{y}{a}\right)^2 - \frac{y}{a}\right]$

For dp/dx = 0 $\quad u = U \cdot \frac{y}{a} \qquad \frac{Q}{l} = \int_0^a u(y)\,dy = w \cdot \int_0^a U \cdot \frac{y}{a}\,dy = \frac{U \cdot a}{2} \qquad Q = \frac{1}{2} \times 5 \cdot \frac{ft}{s} \times \frac{0.1}{12} \cdot ft \qquad Q = 0.0208 \cdot \frac{\frac{ft^3}{s}}{ft}$

For the shear stress $\quad \tau = \mu \cdot \frac{du}{dy} = \frac{\mu \cdot U}{a}$ when dp/dx = 0 $\qquad \mu = 3.79 \times 10^{-7} \cdot \frac{lbf \cdot s}{ft^2}$ (Table A.9)

The shear stress is constant - no need to plot!

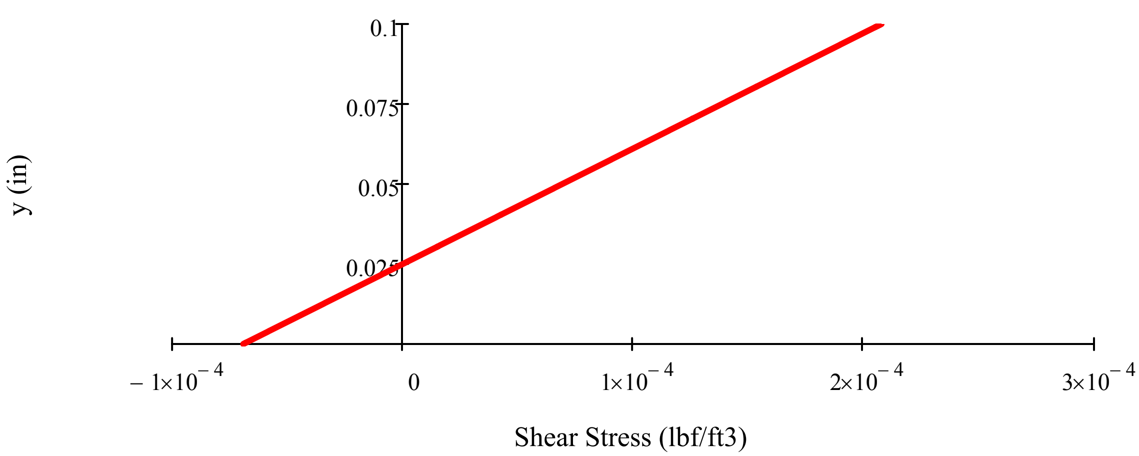
$$\tau = 3.79 \times 10^{-7} \cdot \frac{lbf \cdot s}{ft^2} \times 5 \cdot \frac{ft}{s} \times \frac{12}{0.1 \cdot ft} \times \left(\frac{1 \cdot ft}{12 \cdot in}\right)^2 \qquad \tau = 1.58 \times 10^{-6} \cdot psi$$

Q will decrease if dp/dx > 0; it will increase if dp/dx < 0.

For non- zero dp/dx: $\quad \tau = \mu \cdot \frac{du}{dy} = \frac{\mu \cdot U}{a} + a \cdot \frac{dp}{dx} \cdot \left(\frac{y}{a} - \frac{1}{2}\right)$

At y = 0.25a, we get $\quad \tau(y = 0.25 \cdot a) = \mu \cdot \frac{U}{a} + a \cdot \frac{dp}{dx} \cdot \left(\frac{1}{4} - \frac{1}{2}\right) = \mu \cdot \frac{U}{a} - \frac{a}{4} \cdot \frac{dp}{dx}$

Hence this stress is zero when $\quad \frac{dp}{dx} = \frac{4 \cdot \mu \cdot U}{a^2} = 4 \times 3.79 \times 10^{-7} \cdot \frac{lbf \cdot s}{ft^2} \times 5 \cdot \frac{ft}{s} \times \left(\frac{12}{0.1 \cdot ft}\right)^2 = 0.109 \cdot \frac{\frac{lbf}{ft^2}}{ft} = 7.58 \times 10^{-4} \frac{psi}{ft}$



Problem 8.33

[Difficulty: 3]

8.33 Glycerin at 59°F flows between parallel plates with gap width $b = 0.1$ in. The upper plate moves with speed $U = 2$ ft/s in the positive x direction. The pressure gradient is $\partial p/\partial x = -50$ psi/ft. Locate the point of maximum velocity and determine its magnitude (let $y = 0$ at the bottom plate). Determine the volume of flow (gal/ft) that passes a given cross section ($x =$ constant) in 10 s. Plot the velocity and shear stress distributions.

Given: Flow between parallel plates

Find: Location and magnitude of maximum velocity; Volume flow in 10 s; Plot velocity and shear stress

Solution:

From Section 8.2
$$u(y) = \frac{U\cdot y}{b} + \frac{b^2}{2\cdot\mu}\cdot\frac{dp}{dx}\cdot\left[\left(\frac{y}{b}\right)^2 - \frac{y}{b}\right]$$

For u_{max} set du/dx = 0
$$\frac{du}{dy} = 0 = \frac{U}{b} + \frac{b^2}{2\cdot\mu}\cdot\frac{dp}{dx}\cdot\left(\frac{2\cdot y}{b^2} - \frac{1}{a}\right) = \frac{U}{b} + \frac{1}{2\cdot\mu}\cdot\frac{dp}{dx}\cdot(2\cdot y - b)$$

Hence
$$u = u_{max} \quad \text{at} \quad y = \frac{b}{2} - \frac{\mu\cdot U}{b\cdot\frac{dp}{dx}}$$

From Fig. A.2 at
$$59\,°F = 15\cdot°C \qquad \mu = 4\cdot\frac{N\cdot s}{m^2} \qquad \mu = 0.0835\cdot\frac{lbf\cdot s}{ft^2}$$

$$y = \frac{0.1\cdot in}{2} + 0.0835\cdot\frac{lbf\cdot s}{ft^2} \times 2\cdot\frac{ft}{s} \times \frac{1}{0.1\cdot in} \times \frac{in^2\cdot ft}{50\cdot lbf} \qquad y = 0.0834\cdot in$$

Hence
$$u_{max} = \frac{U\cdot y}{b} + \frac{b^2}{2\cdot\mu}\cdot\frac{dp}{dx}\cdot\left[\left(\frac{y}{b}\right)^2 - \frac{y}{b}\right] \qquad \text{with} \qquad y = 0.0834\cdot in$$

$$u_{max} = 2\cdot\frac{ft}{s} \times \left(\frac{.0834}{0.1}\right) + \frac{1}{2} \times \left(\frac{0.1}{12}\cdot ft\right)^2 \times \frac{ft^2}{.0835\cdot lbf\cdot s} \times -\frac{50\cdot psi}{ft} \times \left(\frac{12\cdot in}{1\cdot ft}\right)^2 \times \left[\left(\frac{.0834}{0.1}\right)^2 - \left(\frac{.0834}{0.1}\right)\right]$$

$$u_{max} = 2.083\cdot\frac{ft}{s}$$

$$\frac{Q}{w} = \int_0^b u(y)\,dy = w\cdot\int_0^b \left[\frac{U\cdot y}{b} + \frac{b^2}{2\cdot\mu}\cdot\frac{dp}{dx}\cdot\left[\left(\frac{y}{b}\right)^2 - \frac{y}{b}\right]\right] dy = \frac{U\cdot b}{2} - \frac{b^3}{12\cdot\mu}\cdot\frac{dp}{dx}$$

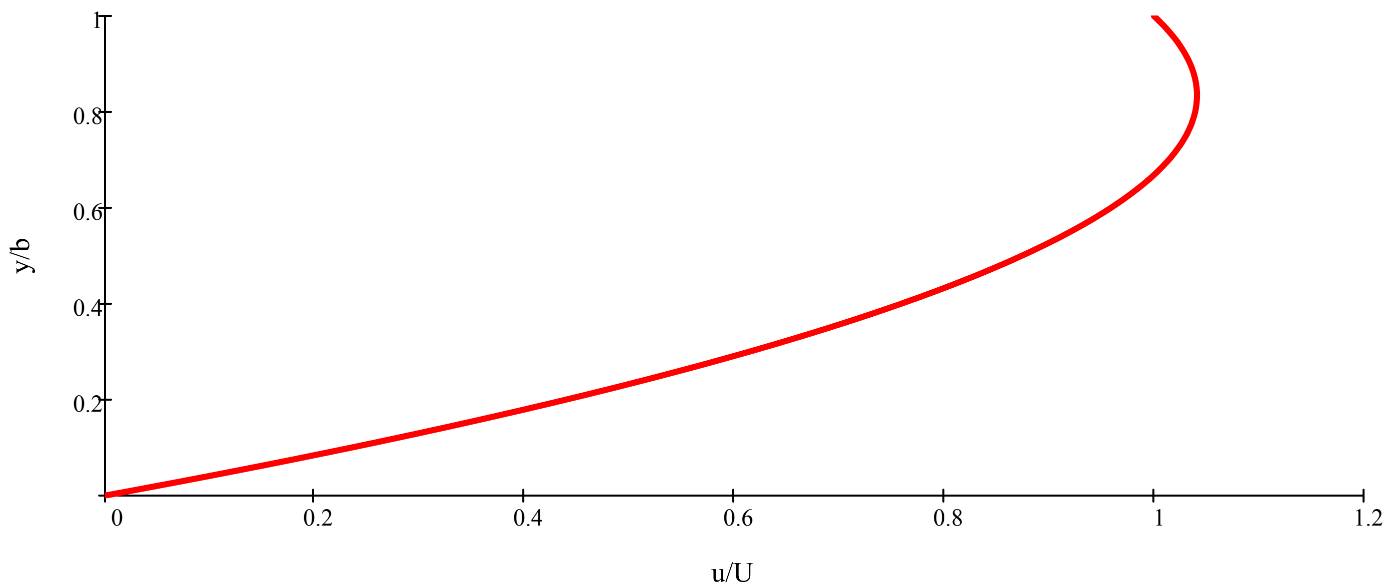
$$\frac{Q}{w} = \frac{1}{2} \times 2\cdot\frac{ft}{s} \times \frac{0.1}{12}\cdot ft - \frac{1}{12} \times \left(\frac{0.1}{12}\cdot ft\right)^3 \times \frac{ft^2}{.0835\cdot lbf\cdot s} \times \left(-\frac{50\cdot psi}{ft}\right) \times \left(\frac{12\cdot in}{1\cdot ft}\right)^2$$

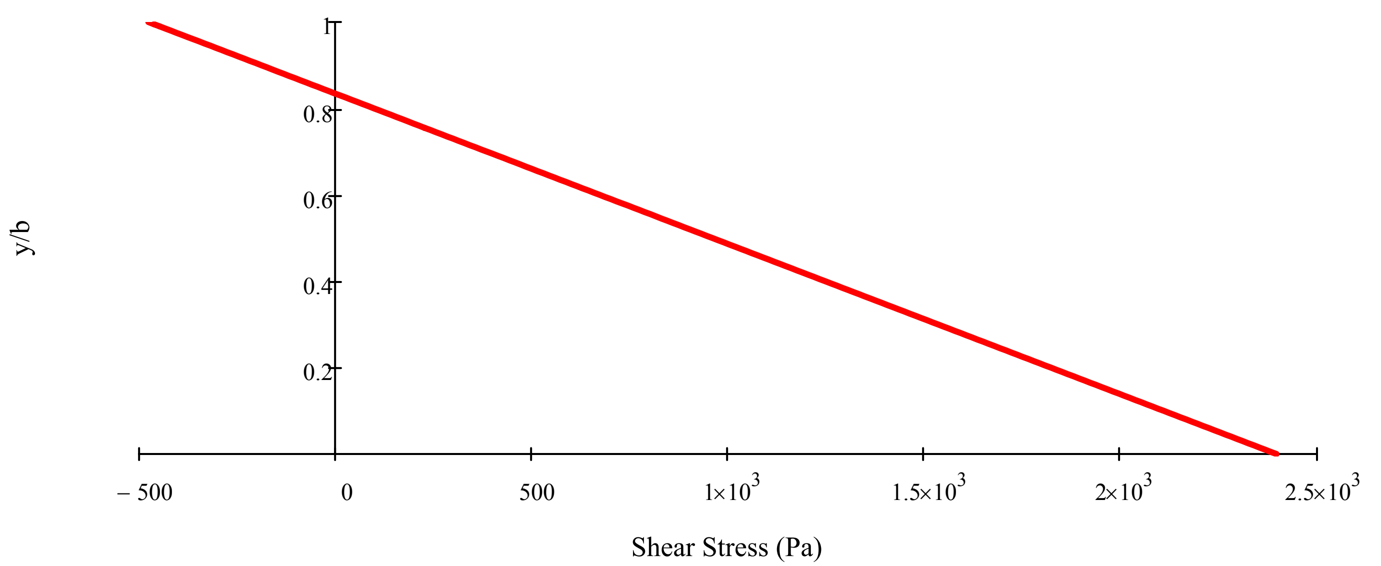
$$\frac{Q}{w} = 0.0125\,\frac{\frac{ft^3}{s}}{ft} \qquad \frac{Q}{w} = 5.61\cdot\frac{gpm}{ft}$$

$$Flow = \frac{Q}{w}\cdot\Delta t = 5.61\cdot\frac{gpm}{ft}\times\frac{1\cdot min}{60\cdot s}\times 10\cdot s \qquad Flow = 0.935\cdot\frac{gal}{ft}$$

The velocity profile is $\frac{u}{U} = \frac{y}{b} + \frac{b^2}{2\cdot\mu\cdot U}\cdot\frac{dp}{dx}\cdot\left[\left(\frac{y}{b}\right)^2 - \frac{y}{b}\right]$ For the shear stress $\tau = \mu\cdot\frac{du}{dy} = \mu\cdot\frac{U}{b} + \frac{b}{2}\cdot\frac{dp}{dx}\cdot\left[2\cdot\left(\frac{y}{b}\right) - 1\right]$

The graphs below can be plotted in *Excel*





Problem 8.34

[Difficulty: 3]

8.34 The velocity profile for fully developed flow of castor oil at 20°C between parallel plates with the upper plate moving is given by Eq. 8.8. Assume $U = 1.5$ m/s and $a = 5$ mm. Find the pressure gradient for which there is no net flow in the x direction. Plot the expected velocity distribution and the expected shear stress distribution across the channel for this flow. For the case where $u = \frac{1}{2}U$ at $y/a = 0.5$, plot the expected velocity distribution and shear stress distribution across the channel. Comment on features of the plots.

Given: Flow between parallel plates

Find: Pressure gradient for no flow; plot velocity and stress distributions; also plot for u = U at y = a/2

Solution:

Basic equations

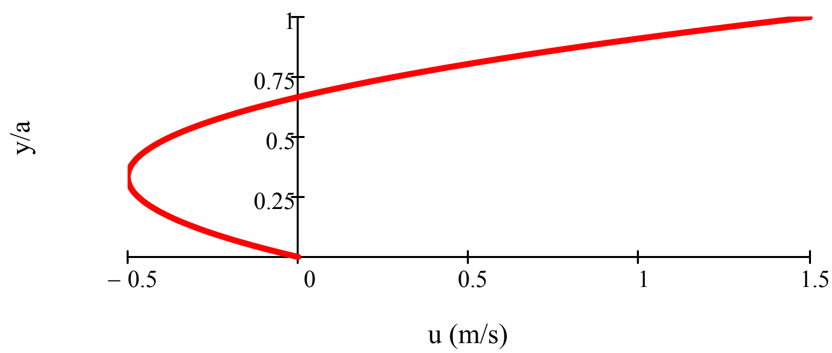
$$u(y) = \frac{U\cdot y}{a} + \frac{a^2}{2\cdot\mu}\cdot\frac{dp}{dx}\cdot\left[\left(\frac{y}{a}\right)^2 - \frac{y}{a}\right] \quad (1) \qquad \frac{Q}{l} = \frac{U\cdot a}{2} - \frac{a^3}{12\cdot\mu}\cdot\frac{dp}{dx} \quad (2) \qquad \tau = \mu\cdot\frac{U}{a} + a\cdot\frac{dp}{dx}\cdot\left(\frac{y}{a} - \frac{1}{2}\right) \quad (3)$$

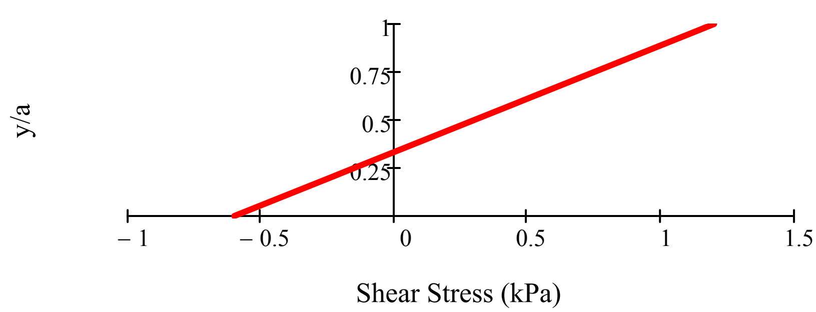
Available data $\quad U = 1.5\cdot\frac{m}{s} \qquad a = 5\cdot mm \qquad$ From Fig. A.2 for castor oil at 20°C $\qquad \mu = 1\cdot\frac{N\cdot s}{m^2}$

From Eq 2 for Q = 0

$$\frac{dp}{dx} = \frac{6\cdot\mu\cdot U}{a^2} = 6 \times 1\cdot\frac{N\cdot s}{m^2} \times 1.5\cdot\frac{m}{s} \times \frac{1}{(0.005\cdot m)^2} \qquad \frac{dp}{dx} = 360\cdot\frac{kPa}{m}$$

The graphs below, using Eqs. 1 and 3, can be plotted in *Excel*



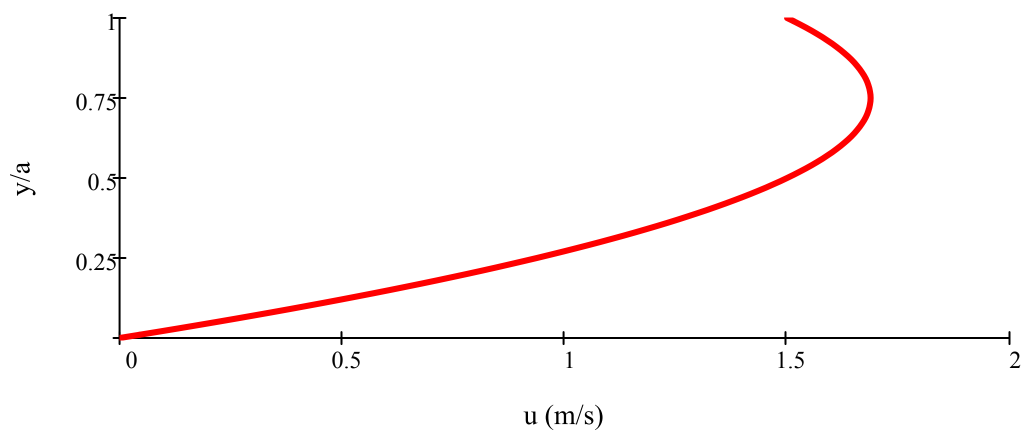


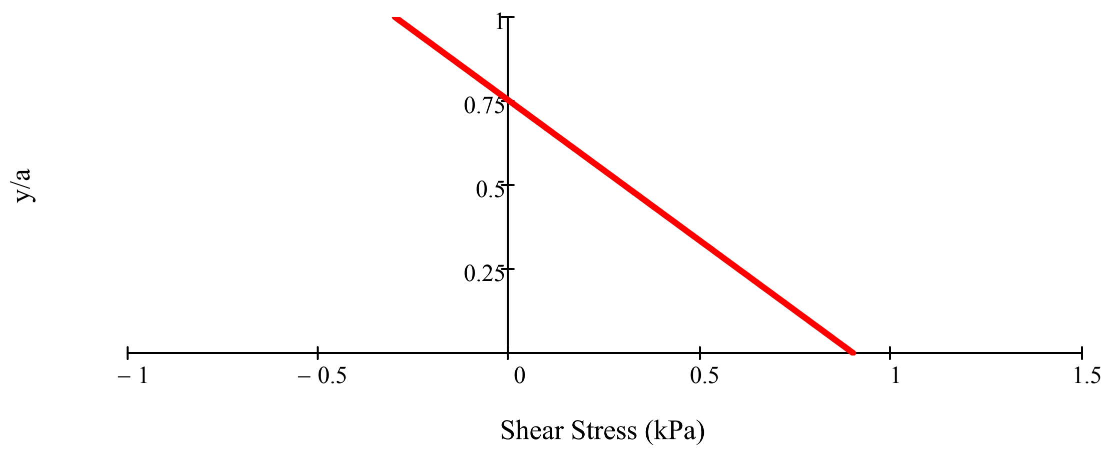
The pressure gradient is adverse, to counteract the flow generated by the upper plate motion

For u = U at y = a/2 we need to adjust the pressure gradient. From Eq. 1 $u(y) = \frac{U \cdot y}{a} + \frac{a^2}{2 \cdot \mu} \cdot \frac{dp}{dx} \cdot \left[\left(\frac{y}{a} \right)^2 - \frac{y}{a} \right]$

Hence $U = \frac{U \cdot \frac{a}{2}}{a} + \frac{a^2}{2 \cdot \mu} \cdot \frac{dp}{dx} \cdot \left[\left(\frac{\frac{a}{2}}{a} \right)^2 - \frac{\frac{a}{2}}{a} \right]$ or $\frac{dp}{dx} = -\frac{4 \cdot U \cdot \mu}{a^2} = -4 \times 1 \cdot \frac{N \cdot s}{m^2} \times 1.5 \cdot \frac{m}{s} \times \frac{1}{(0.005 \cdot m)^2}$

$$\frac{dp}{dx} = -240 \cdot \frac{kPa}{m}$$





The pressure gradient is positive to provide the "bulge" needed to satisfy the velocity requirement

Problem 8.35

[Difficulty: 3]

8.35 The velocity profile for fully developed flow of carbon tetrachloride at 68°F between parallel plates (gap $a = 0.05$ in.), with the upper plate moving, is given by Eq. 8.8. Assuming a volume flow rate per unit depth is 1.5 gpm/ft for zero pressure gradient, find U. Evaluate the shear stress on the lower plate. Would the volume flow rate increase or decrease with a mild adverse pressure gradient? Calculate the pressure gradient that will give zero shear stress at $y/a = 0.25$. Plot the velocity distribution and the shear stress distribution for this case.

Given: Flow between parallel plates

Find: Shear stress on lower plate; pressure gradient for zero shear stress at y/a = 0.25; plot velocity and shear stress

Solution:

Basic equations

$$u(y) = \frac{U\cdot y}{a} + \frac{a^2}{2\cdot\mu}\cdot\frac{dp}{dx}\cdot\left[\left(\frac{y}{a}\right)^2 - \frac{y}{a}\right] \quad (1) \qquad \frac{Q}{l} = \frac{U\cdot a}{2} - \frac{a^3}{12\cdot\mu}\cdot\frac{dp}{dx} \quad (2) \qquad \tau = \mu\cdot\frac{U}{a} + a\cdot\frac{dp}{dx}\cdot\left(\frac{y}{a} - \frac{1}{2}\right) \quad (3)$$

Available data $\quad q = 1.5\cdot\frac{gpm}{ft} \quad a = 0.05\cdot in \quad 68°F = 20°C$

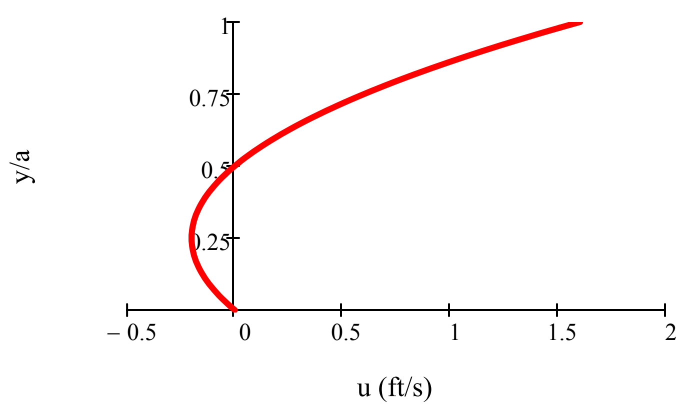
From Fig. A.2, Carbon tetrachloride at 20°C $\quad \mu = 0.001\cdot\frac{N\cdot s}{m^2} \quad \mu = 2.089\times 10^{-5}\cdot\frac{lbf\cdot s}{ft^2}$

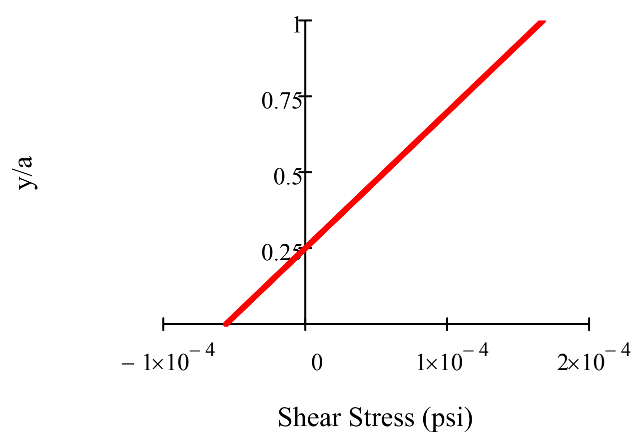
From Eq. 2, for zero pressure gradient $\quad U = \frac{2\cdot Q}{a\cdot l}$ or $U = \frac{2\cdot q}{a} \quad U = 1.60\cdot\frac{ft}{s}$

From Eq. 3, when y = 0, with $U = 1.60\frac{ft}{s} \quad \tau_{yx} = \frac{\mu\cdot U}{a} \quad \tau_{yx} = 5.58\times 10^{-5}\cdot psi$

A mild adverse pressure gradient would reduce the flow rate.

For zero shear stress at y/a = 0.25, from Eq. 3 $\quad 0 = \mu\cdot\frac{U}{a} + a\cdot\frac{dp}{dx}\cdot\left(\frac{1}{4} - \frac{1}{2}\right)$ or $\frac{dp}{dx} = \frac{4\cdot\mu\cdot U}{a^2} \quad \frac{dp}{dx} = 0.0536\cdot\frac{psi}{ft}$





Note that the location of zero shear is also where u is maximum!

Problem 8.36

[Difficulty: 3]

8.36 Free-surface waves begin to form on a laminar liquid film flowing down an inclined surface whenever the Reynolds number, based on mass flow per unit width of film, is larger than about 33. Estimate the maximum thickness of a laminar film of water that remains free from waves while flowing down a vertical surface.

Solution: The mass flow rate is $\dot{m} = \rho \bar{V} A = \rho \bar{V} w \delta$, so $\dot{m}/w = \rho \bar{V} \delta$.

Thus

$$Re = \frac{\rho \bar{V} \delta}{\mu} = \frac{\bar{V}\delta}{\nu} = 33 \text{ (maximum)}$$

Using the result for average velocity from Example 8.3

$$\bar{V} = \frac{\rho g \delta^2}{3\mu}$$

Thus

$$\frac{\rho \bar{V} \delta}{\mu} = \frac{\rho^2 g \delta^3}{3\mu^2} = 33$$

Solving for δ,

$$\delta = \left[\frac{99\mu^2}{\rho^2 g}\right]^{1/3}$$

At $T = 20°C$, $\mu = 1.00 \times 10^{-3}$ kg/m·s and $\rho = 998$ kg/m³ (Table A.8). Substituting,

$$\delta = \left[99 \times (1.00 \times 10^{-3})^2 \frac{kg^2}{m^2 \cdot s^2} \times \frac{m^6}{(998)^2\, kg^2} \times \frac{s^2}{9.81\, m}\right]^{1/3}$$

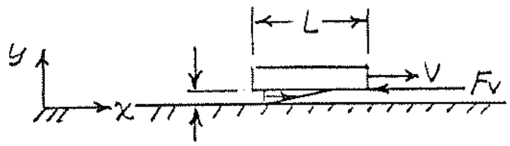
$\delta = 2.16 \times 10^{-4}$ m or 0.216 mm $\qquad \delta_{max}$

Problem 8.37

[Difficulty: 3]

8.37 Microchips are supported on a thin air film on a smooth horizontal surface during one stage of the manufacturing process. The chips are 11.7 mm long and 9.35 mm wide and have a mass of 0.325 g. The air film is 0.125 mm thick. The initial speed of a chip is $V_0 = 1.75$ mm/s; the chip slows as the result of viscous shear in the air film. Analyze the chip motion during deceleration to develop a differential equation for chip speed V versus time t. Calculate the time required for a chip to lose 5 percent of its initial speed. Plot the variation of chip speed versus time during deceleration. Explain why it looks as you have plotted it.

Solution: Apply Newton's law of viscosity



Basic equations: $\tau_{yx} = \mu \dfrac{du}{dy}$

$F_V = \tau A \qquad \Sigma F = m a_x$

Assume: (1) Newtonian fluid (3) Air at STP

(2) Linear velocity profile in narrow gap

Then

$$\tau_{yx} = \mu \frac{du}{dy} = \mu \frac{V}{h}; \quad F_V = \tau A = \mu \frac{V}{h} w L = \frac{\mu V w L}{h}$$

The free-body diagram for the chip is

$$\Sigma F_x = -F_V = -\frac{\mu V w L}{h} = m \frac{dV}{dt} \quad ; \quad \frac{dV}{V} = -\frac{\mu w L}{m h} dt$$

D.E.

Integrating, $\displaystyle\int_{V_0}^{V} \frac{dV}{V} = \ln \frac{V}{V_0} = -\frac{\mu w L}{m h} t$

Thus

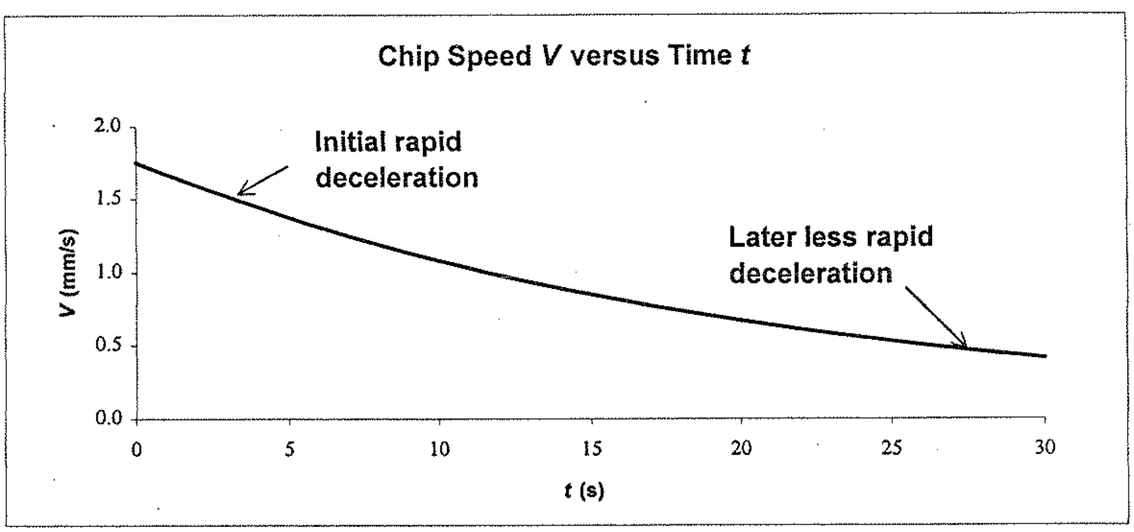
$$t = -\frac{m h}{\mu w L} \ln \frac{V}{V_0}$$

$$t = -0.325\,g \times 0.125\,mm \times \frac{m \cdot s}{1.79 \times 10^{-5}\,kg} \times \frac{1}{9.35\,mm} \times \frac{1}{11.7\,mm} \times \ln 0.95 \times \frac{kg}{1000\,g} \times \frac{1000\,mm}{m}$$

$t = 1.06\,s$

t

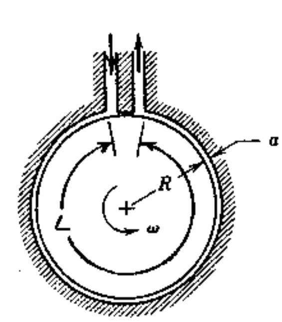
From *Excel*, the plot of speed vs. time is:



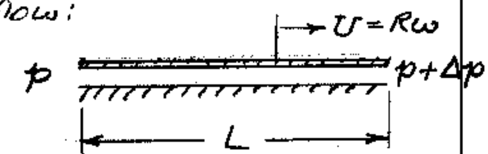
Problem 8.38

[Difficulty: 5]

8.38 A viscous-shear pump is made from a stationary housing with a close-fitting rotating drum inside. The clearance is small compared with the diameter of the drum, so flow in the annular space may be treated as flow between parallel plates. Fluid is dragged around the annulus by viscous forces. Evaluate the performance characteristics of the shear pump (pressure differential, input power, and efficiency) as functions of volume flow rate. Assume that the depth normal to the diagram is b.



Solution: Since $a \ll R$, unwrap to form flow between parallel plates. Apply Eqs. 8.9 to fully developed flow:



Volume flow rate is $\dfrac{Q}{b} = \dfrac{Ua}{2} - \dfrac{1}{12\mu}\left(\dfrac{\partial p}{\partial x}\right)a^3$

Substituting $U = R\omega$ and $\dfrac{\partial p}{\partial x} = \dfrac{\Delta p}{L}$, then

$$\Delta p = \frac{12\mu L}{a^3}\left(\frac{\omega R a}{2} - \frac{Q}{b}\right) = \frac{6\mu L R\omega}{a^2}\left(1 - \frac{2Q}{abR\omega}\right)$$ Δp

Torque is $T = \tau R(bL) = RLb\tau$. Power is $P = T\omega$. From Eq. 8.9a, at $y = a$,

$$P = RLb\omega\left[\frac{\mu R\omega}{a} + \frac{\Delta p}{L}\frac{a}{2}\right] = RLb\omega\left[\frac{\mu R\omega}{a} + \frac{6\mu L R\omega}{a^2}\left(1 - \frac{2Q}{abR\omega}\right)\frac{a}{2L}\right]$$

$$P = RLb\omega\left[\frac{\mu R\omega}{a}\left(4 - \frac{6Q}{abR\omega}\right)\right] = \frac{\mu L b (R\omega)^2}{a}\left(4 - \frac{6Q}{abR\omega}\right)$$ P

Output power is $Q\Delta p$, so efficiency is

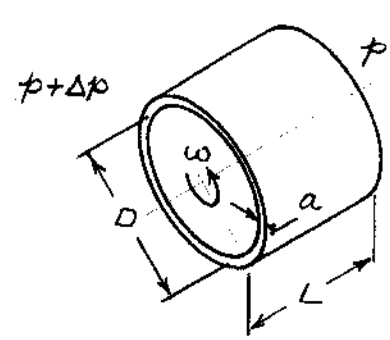
$$\eta = \frac{Q\Delta p}{P} = \frac{6\mu Q L R\omega}{a^2}\left(1 - \frac{2Q}{abR\omega}\right)\frac{a}{\mu L b (R\omega)^2}\frac{1}{\left(4 - \frac{6Q}{abR\omega}\right)}$$

$$\eta = \frac{6Q}{abR\omega}\,\frac{\left(1 - \frac{2Q}{abR\omega}\right)}{\left(4 - \frac{6Q}{abR\omega}\right)}$$ η

Problem 8.39

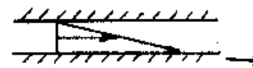
[Difficulty: 5]

8.39 The clamping force to hold a part in a metal-turning operation is provided by high-pressure oil supplied by a pump. Oil leaks axially through an annular gap with diameter D, length L, and radial clearance a. The inner member of the annulus rotates at angular speed ω. Power is required both to pump the oil and to overcome viscous dissipation in the annular gap. Develop expressions in terms of the specified geometry for the pump power, $\mathcal{P}_p$, and the viscous dissipation power, $\mathcal{P}_v$. Show that the total power requirement is minimized when the radial clearance, a, is chosen such that $\mathcal{P}_v = 3\mathcal{P}_p$.



Solution: Apply Eqs. 8.6 and 8.9 for flow between parallel plates.

Assumptions: (1) $a \ll D$, so unfold to flat plates
(2) No pressure gradient circumferentially



$V = \omega R = \frac{\omega D}{2}$

The viscous power is the product of viscous torque times ω:

$$P_v = T\omega = \tau(2\pi R L)R\omega = \mu\frac{V}{a}\left(2\pi\frac{D}{2}L\right)\frac{D}{2}\omega = \mu\frac{\omega D}{2a}\pi D L\frac{D}{2}\omega = \frac{\pi\mu\omega^2 D^3 L}{4a}$$ P_v

The pump power is the product of flow rate times pressure drop.

$$P_p = Q\Delta p$$

From Eq. 8.6c, $Q = \frac{l a^3 \Delta p}{12\mu L} = \frac{\pi D a^3 \Delta p}{12\mu L}$, so $P_p = \frac{\pi D a^3 \Delta p^2}{12\mu L}$ P_p

The total power required is $P_T = P_v + P_p = \frac{\pi\mu\omega^2 D^3 L}{4a} + \frac{\pi D a^3 \Delta p^2}{12\mu L}$

It may be minimized by setting $\frac{dP_T}{da} = 0$. Thus

$$\frac{dP_T}{da} = -\frac{\pi\mu\omega^2 D^3 L}{4a^2} + \frac{\pi D a^2 \Delta p^2}{4\mu L} = 0 \qquad (1)$$

This can be written

$$\frac{dP_T}{da} = -\frac{1}{a}P_v + \frac{3}{a}P_p = 0$$

which is satisfied when $3P_p - P_v = 0$ or $P_v = 3P_p$ Optimum

Equation 1 also can be solved for a at optimum conditions:

$$a^4 = \frac{\mu^2\omega^2 D^2 L^2}{\Delta p^2} \quad \text{or} \quad a^2 = \frac{\mu\omega D L}{\Delta p} \quad \text{or} \quad \frac{a}{D} = \sqrt{\frac{\mu\omega L}{D\Delta p}} \text{ (optimum)}$$

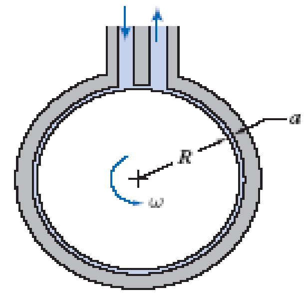
Problem 8.40

[Difficulty: 2]

8.40 The efficiency of the viscous-shear pump of Fig. P8.39 is given by

$$\eta = 6q\,\frac{(1-2q)}{(4-6q)}$$

where $q = Q/abR\omega$ is a dimensionless flow rate (Q is the flow rate at pressure differential Δp, and b is the depth normal to the diagram). Plot the efficiency versus dimensionless flow rate, and find the flow rate for maximum efficiency. Explain why the efficiency peaks, and why it is zero at certain values of q.

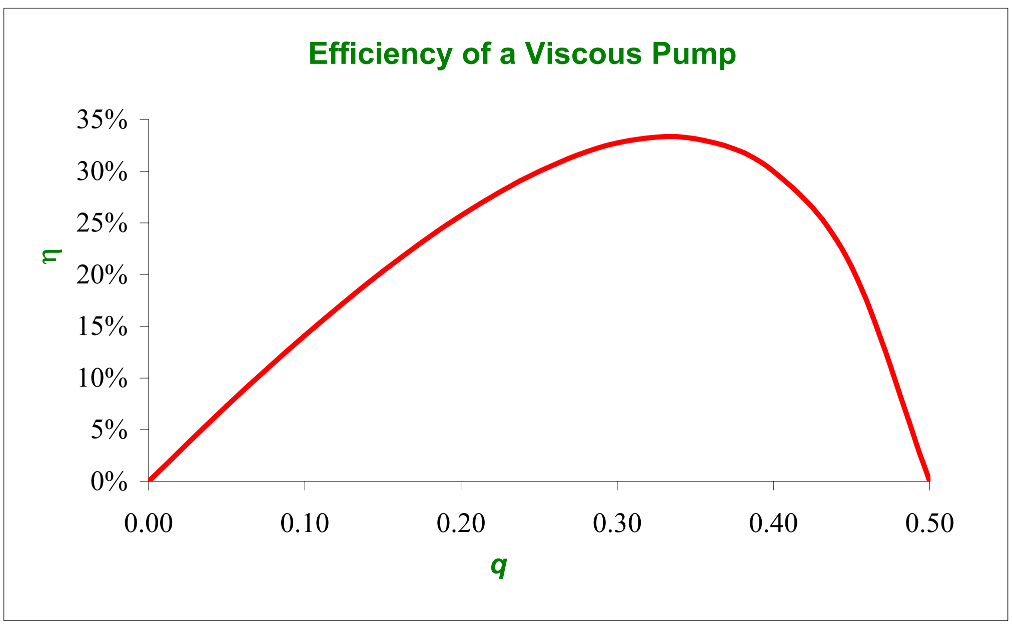


Given: Expression for efficiency

Find: Plot; find flow rate for maximum efficiency; explain curve

Solution:

q	η
0.00	0.0%
0.05	7.30%
0.10	14.1%
0.15	20.3%
0.20	25.7%
0.25	30.0%
0.30	32.7%
0.35	33.2%
0.40	30.0%
0.45	20.8%
0.50	0.0%



For the maximum efficiency point we can use *Solver* (or alternatively differentiate)

q	η
0.333	33.3%

The efficiency is zero at zero flow rate because there is no output at all
The efficiency is zero at maximum flow rate $\Delta p = 0$ so there is no output
The efficiency must therefore peak somewhere between these extremes

Problem 8.41

[Difficulty: 5]

8.41 Automotive design is tending toward all-wheel drive to improve vehicle performance and safety when traction is poor. An all-wheel drive vehicle must have an interaxle differential to allow operation on dry roads. Numerous vehicles are being built using multiplate viscous drives for interaxle differentials. Perform the analysis and design needed to define the torque transmitted by the differential for a given speed difference, in terms of the design parameters. Identify suitable dimensions for a viscous differential to transmit a torque of 150 N · m at a speed loss of 125 rpm, using lubricant with the properties of SAE 30 oil. Discuss how to find the minimum material cost for the viscous differential, if the plate cost per square meter is constant.

Solution: From Problem 2.66, $dT = r\,dF = r\tau\,dA$

But $\tau = \mu \frac{du}{dy} = \mu \frac{U}{h} = \mu \frac{r\Delta\omega}{h}$; $dA = 2\pi r\,dr$

Thus $dT = r\mu \frac{r\Delta\omega}{h} 2\pi r\,dr = \frac{2\pi\mu\Delta\omega}{h} r^3\,dr$; $T = \frac{\pi\mu\Delta\omega}{2h}\left[R_o^4 - R_i^4\right]$

or $T = \frac{\pi\mu\Delta\omega}{2h} R^4(1-\alpha^4)$ where $\alpha = R_i/R$ — T

This value is per gap. Each rotor has 2 gaps to a housing. For n gaps

$$T_n = \frac{n\pi\mu\Delta\omega}{2h} R^4(1-\alpha^4) \qquad (1)$$ T_n

From Eq. 1, assuming $\mu = 0.18$ kg/m·s (Fig. A.2) and $\alpha = \frac{1}{2}$, so $1-\alpha^4 = 1-\frac{1}{16} \approx 1$, then

$$\frac{nR^4}{h} = \frac{2T_n}{\pi\mu\Delta\omega} = \frac{2}{\pi} \times 150\ \text{N}\cdot\text{m} \times \frac{\text{m}^2}{0.18\ \text{N}\cdot\text{s}} \times \frac{\text{min}}{125\ \text{rev}} \times \frac{\text{rev}}{2\pi\ \text{rad}} \times \frac{60\ \text{s}}{\text{min}} = 40.5\ \text{m}^3 = C$$

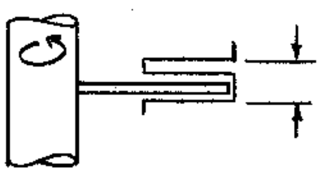
or

$$R^4 = C\frac{h}{n}$$

For $n = 100$ and $h = 0.2$ mm, $R^4 = 40.5\ \text{m}^3 \times 0.0002\ \text{m} \times \frac{1}{100} = 8.11\times10^{-5}\ \text{m}^4$

$$R = \left[8.11\times10^{-5}\right]^{1/4}\text{m} = 0.0949\ \text{m} \quad (\text{or } D \approx 190\ \text{mm})$$ R

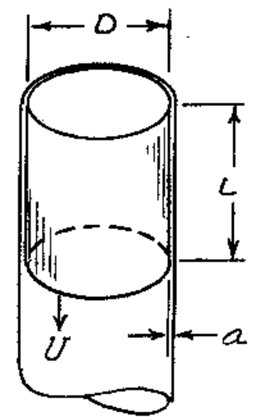
The stack length might be



≈ 2.5 mm for $n = 2$, or 125 mm for $n = 100$

Problem 8.42

8.42 An inventor proposes to make a "viscous timer" by placing a weighted cylinder inside a slightly larger cylinder containing viscous liquid, creating a narrow annular gap close to the wall. Analyze the flow field created when the apparatus is inverted and the mass begins to fall under gravity. Would this system make a satisfactory timer? If so, for what range of time intervals? What would be the effect of a temperature change on measured time?



Solution: Apply conservation of mass to a CV enclosing the cylinder and the moving mass:

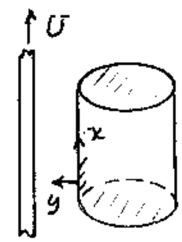
Then: $$Q = U\frac{\pi D^2}{4} = \bar{V}\,\pi D a = \bar{V} \ell a \qquad (1)$$

Assume: (1) Gap is narrow, $a \ll D$
(2) Unroll gap so flat, $\ell = \pi D$
(3) Steady flow
(4) Fully developed laminar flow

Under these assumptions, the flow field in the gap is that for flow between parallel plates with one plate moving.

Flow Field

Place coordinates on the moving mass:

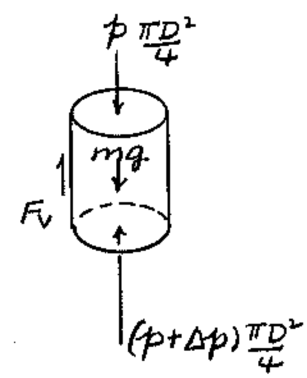


Then the volume flow rate (Eq. 8.9b) is

$$\frac{Q}{\ell} = \frac{Q}{\pi D} = \frac{Ua}{2} - \frac{1}{12\mu}\left(\frac{\partial p}{\partial x}\right)a^3$$

But $\frac{\partial p}{\partial x} = -\frac{\Delta p_v}{L}$, where Δp_v is the pressure drop driving viscous flow, so

$$\frac{Q}{\ell} = \frac{Ua}{2} - \frac{1}{12\mu}\left(-\frac{\Delta p_v}{L}\right)a^3 = \frac{Ua}{2} + \frac{\Delta p_v a^3}{12\mu L} \qquad (2)$$



The pressure change across the moving mass is

$$\Delta p = \rho_\ell g L + \Delta p_v \qquad (3)$$

Summing forces on the moving mass gives

$$\Sigma F_x = \Delta p\frac{\pi D^2}{4} - mg + F_v = m\cancel{\frac{dU}{dt}}^{=0(3)}$$

But $mg = \rho_m \frac{\pi D^2}{4} L$ and $F_v = \tau_s \pi D L$

From Eq. 8.9a, $\tau_s = \mu\frac{U}{a} - \frac{a}{2}\left(\frac{\partial p}{\partial x}\right) = \mu\frac{U}{a} + \frac{a}{2}\frac{\Delta p_v}{L}$

Substituting, $$\Delta p\frac{\pi D^2}{4} - \rho_m\frac{\pi D^2}{4}Lg + \left[\mu\frac{U}{a} + \frac{a}{2}\frac{\Delta p_v}{L}\right]\pi D L = 0$$

or $$\Delta p = \rho_m g L - \left[\mu\frac{U}{a} + \frac{a}{2}\frac{\Delta p_v}{L}\right]\frac{4L}{D} \qquad (4)$$

Combining Eqs. 1 and 2 gives $\frac{UD}{4} = \frac{Ua}{2} + \frac{\Delta p_v a^3}{12\mu L}$

Thus

$$\Delta p_v = \frac{12\mu L}{a^3}\left[\frac{UD}{4} - \overset{\ll D}{\cancel{\frac{Ua}{2}}}\right] = \frac{3\mu U L D}{a^3} \qquad (5)$$ Δp_v

Combining Eqs. 3 and 4 gives $\Delta p = \rho_\ell g L + \Delta p_v = \rho_m g L - \left[\mu \frac{U}{a} + \frac{a}{2}\frac{\Delta p_v}{L}\right]\frac{4L}{D}$

Using Eq. 5,

$$\rho_\ell g L + \frac{3\mu U L D}{a^3} = \rho_m g L - \mu \frac{U}{a}\frac{4L}{D} - \frac{a}{2}\,\frac{3\mu U L D}{L a^3}\,\frac{4L}{D}$$

Simplifying and re-arranging,

$$(\rho_m - \rho_\ell) g L = \frac{3\mu U L D}{a^3} + \frac{4\mu U L}{aD} + \frac{6\mu U L}{a^2} \approx \frac{3\mu U L D}{a^3}$$

Finally, using $\rho = SG\,\rho_{H_2O}$,

$$U = \frac{(SG_m - SG_\ell)\,\rho_{H_2O}\, g\, a^3}{3\mu D}$$ U

The time interval for the mass to move distance H is

$$\Delta t = \frac{H}{U} = \frac{3\mu D}{(SG_m - SG_\ell)\,\rho_{H_2O}\, g\, a^3} \qquad (6)$$ Δt

Equation 6 shows that the time interval for the mass to fall any distance H is proportional to liquid viscosity μ and inversely proportional to gap width a cubed. A temperature change would affect the diameter of the measuring tube and the diameter of the falling mass. A temperature change also would affect the viscosity of the liquid in the tube.

Speed of the falling mass is proportional to the cube of gap width. If the coefficient of thermal expansion of the falling mass were greater than that of the glass measuring tube (which seems likely), then the width of the annular gap would decrease with increasing temperature. This would tend to slow the falling mass. The total amount of thermal expansion would depend on the diameter of the mass and tube. The effect on gap width would be greater, the larger the tube diameter compared to the initial gap width.

It might be possible to "tailor" the thermal expansion coefficient of the cylinder, by using a suitable material, to closely match that of the falling mass. Then there would be no differential thermal expansion between the mass and tube, and changes in temperature would not affect the gap width.

Speed of the falling mass is inversely proportional to liquid viscosity. Liquid viscosity decreases sharply as temperature increases (the viscosity of SAE 30 oil drops more than 10 percent as its temperature increases from 20°C to 25°C, see Fig. A.2). This would tend to increase the speed of the falling mass.

The entire device could be maintained at constant temperature.

Problem 8.43

[Difficulty: 3]

8.43 A journal bearing consists of a shaft of diameter $D = 35$ mm and length $L = 50$ mm (moment of inertia $I = 0.125$ kg·m^2) installed symmetrically in a stationary housing such that the annular gap is $\delta = 1$ mm. The fluid in the gap has viscosity $\mu = 0.1$ N·s/m^2. If the shaft is given an initial angular velocity of $\omega = 500$ rpm, determine the time for the shaft to slow to 100 rpm. On another day, an unknown fluid is tested in the same way, but takes 10 minutes to slow from 500 to 100 rpm. What is its viscosity?

Given: Data on a journal bearing

Find: Time for the bearing to slow to 100 rpm; visocity of new fluid

Solution:

The given data is $D = 35 \cdot mm$ $\quad L = 50 \cdot mm$ $\quad \delta = 1 \cdot mm$ $\quad I = 0.125 \cdot kg \cdot m^2$

$\omega_i = 500 \cdot rpm$ $\quad \omega_f = 100 \cdot rpm$ $\quad \mu = 0.1 \cdot \frac{N \cdot s}{m^2}$

The equation of motion for the slowing bearing is $\quad I \cdot \alpha = \text{Torque} = -\tau \cdot A \cdot \frac{D}{2}$

where α is the angular acceleration and τ is the viscous stress, and $A = \pi \cdot D \cdot L$ is the surface area of the bearing

As in Example 8.2 the stress is given by $\quad \tau = \mu \cdot \frac{U}{\delta} = \frac{\mu \cdot D \cdot \omega}{2 \cdot \delta}$

where U and ω are the instantaneous linear and angular velocities.

Hence $\quad I \cdot \alpha = I \cdot \frac{d\omega}{dt} = -\frac{\mu \cdot D \cdot \omega}{2 \cdot \delta} \cdot \pi \cdot D \cdot L \cdot \frac{D}{2} = -\frac{\mu \cdot \pi \cdot D^3 \cdot L}{4 \cdot \delta} \cdot \omega$

Separating variables $\quad \frac{d\omega}{\omega} = -\frac{\mu \cdot \pi \cdot D^3 \cdot L}{4 \cdot \delta \cdot I} \cdot dt$

Integrating and using IC $\omega = \omega_0$ $\quad \omega(t) = \omega_i \cdot e^{-\frac{\mu \cdot \pi \cdot D^3 \cdot L}{4 \cdot \delta \cdot I} \cdot t}$

The time to slow down to $\omega_f = 10$ rpm is obtained from solving $\quad \omega_f = \omega_i \cdot e^{-\frac{\mu \cdot \pi \cdot D^3 \cdot L}{4 \cdot \delta \cdot I} \cdot t}$

so $\quad t = -\frac{4 \cdot \delta \cdot I}{\mu \cdot \pi \cdot D^3 \cdot L} \cdot \ln\left(\frac{\omega_f}{\omega_i}\right)$ $\quad$ Hence $\quad t = 1.19 \times 10^3\, s$ $\quad t = 19.9 \cdot min$

For the new fluid, the time to slow down is $\quad t = 10 \cdot min$

Rearranging the equation $\quad \mu = -\frac{4 \cdot \delta \cdot I}{\pi \cdot D^3 \cdot L \cdot t} \cdot \ln\left(\frac{\omega_f}{\omega_i}\right)$ $\quad \mu = 0.199 \frac{kg}{m \cdot s}$

It is more viscous as it slows down the rotation in a shorter time

Problem 8.44

[Difficulty: 2]

8.44 In Example 8.3 we derived the velocity profile for laminar flow on a vertical wall by using a differential control volume. Instead, following the procedure we used in Example 5.9, derive the velocity profile by starting with the Navier–Stokes equations (Eqs. 5.27). Be sure to state all assumptions.

Given: Navier-Stokes Equations

Find: Derivation of Example 8.3 result

Solution:

The Navier-Stokes equations are (using the coordinates of Example 8.3, so that x is vertical, y is horizontal)

$$\cancelto{4}{\frac{\partial u}{\partial x}} + \frac{\partial v}{\partial y} + \cancelto{3}{\frac{\partial w}{\partial z}} = 0 \tag{5.1c}$$

$$\rho\left(\cancelto{1}{\frac{\partial u}{\partial t}} + u\cancelto{4}{\frac{\partial u}{\partial x}} + v\cancelto{5}{\frac{\partial u}{\partial y}} + w\cancelto{3}{\frac{\partial u}{\partial z}}\right) = \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\cancelto{4}{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2} + \cancelto{3}{\frac{\partial^2 u}{\partial z^2}}\right) \tag{5.27a}$$

$$\rho\left(\cancelto{1}{\frac{\partial v}{\partial t}} + u\cancelto{4}{\frac{\partial v}{\partial x}} + v\cancelto{5}{\frac{\partial v}{\partial y}} + w\cancelto{3}{\frac{\partial v}{\partial z}}\right) = \cancelto{6}{\rho g_y} - \frac{\partial p}{\partial y} + \mu\left(\cancelto{4}{\frac{\partial^2 v}{\partial x^2}} + \cancelto{5}{\frac{\partial^2 v}{\partial y^2}} + \cancelto{3}{\frac{\partial^2 v}{\partial z^2}}\right) \tag{5.27b}$$

$$\rho\left(\cancelto{1}{\frac{\partial w}{\partial t}} + u\cancelto{3}{\frac{\partial w}{\partial x}} + v\cancelto{3}{\frac{\partial w}{\partial y}} + w\cancelto{3}{\frac{\partial w}{\partial z}}\right) = \cancelto{3}{\rho g_z} - \cancelto{3}{\frac{\partial p}{\partial z}} + \mu\left(\cancelto{3}{\frac{\partial^2 w}{\partial x^2}} + \cancelto{3}{\frac{\partial^2 w}{\partial y^2}} + \cancelto{3}{\frac{\partial^2 w}{\partial z^2}}\right) \tag{5.27c}$$

The following assumptions have been applied:

(1) Steady flow (given).
(2) Incompressible flow; ρ = constant.
(3) No flow or variation of properties in the z direction; $w = 0$ and $\partial/\partial z = 0$.
(4) Fully developed flow, so no properties except possibly pressure p vary in the x direction; $\partial/\partial x = 0$.
(5) See analysis below.
(6) No body force in the y direction; $g_y = 0$

Assumption (1) eliminates time variations in any fluid property. Assumption (2) eliminates space variations in density. Assumption (3) states that there is no z component of velocity and no property variations in the z direction. All terms in the z component of the Navier–Stokes equation cancel. After assumption (4) is applied, the continuity equation reduces to $\partial v/\partial y = 0$. Assumptions (3) and (4) also indicate that $\partial v/\partial z = 0$ and $\partial v/\partial x = 0$. Therefore v must be constant. Since v is zero at the solid surface, then v must be zero everywhere. The fact that $v = 0$ reduces the Navier–Stokes equations further, as indicated by (5). Hence for the y direction

$$\frac{\partial p}{\partial y} = 0$$

which indicates the pressure is a constant across the layer. However, at the free surface $p = p_{atm}$ = constant. Hence we conclude that p = constant throughout the fluid, and so

$$\frac{\partial p}{\partial x} = 0$$

In the x direction, we obtain

$$\mu \frac{\partial^2 u}{\partial y^2} + \rho g = 0$$

Integrating twice

$$u = -\frac{1}{2\mu}\rho g y^2 + \frac{c_1}{\mu} y + c_2$$

To evaluate the constants, c_1 and c_2, we must apply the boundary conditions. At $y = 0$, $u = 0$. Consequently, $c_2 = 0$. At $y = a$, $du/dy = 0$ (we assume air friction is negligible). Hence

$$\tau(y = \delta) = \mu \left.\frac{du}{dy}\right|_{y=\delta} = -\frac{1}{\mu}\rho g \delta + \frac{c_1}{\mu} = 0$$

which gives

$$c_1 = \rho g \delta$$

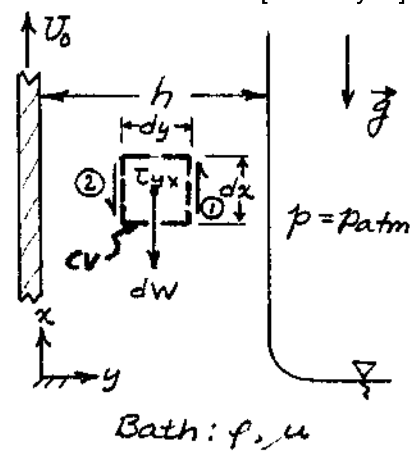
and finally

$$u = -\frac{1}{2\mu}\rho g y^2 + \frac{\rho g}{\mu} y = \frac{\rho g}{\mu}\delta^2\left[\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^2\right]$$

Problem 8.45

[Difficulty: 3]

8.45 A continuous belt, passing upward through a chemical bath at speed U_0, picks up a liquid film of thickness h, density ρ, and viscosity μ. Gravity tends to make the liquid drain down, but the movement of the belt keeps the liquid from running off completely. Assume that the flow is fully developed and laminar with zero pressure gradient, and that the atmosphere produces no shear stress at the outer surface of the film. State clearly the boundary conditions to be satisfied by the velocity at $y = 0$ and $y = h$. Obtain an expression for the velocity profile.



Solution: Choose CV $dx\,dy\,dz$ as shown.
Apply x component of momentum equation.

Basic equations:

$$F_{S_x} + F_{B_x} = \underbrace{\frac{\partial}{\partial t}\int_{CV} u\rho\, dV}_{=0(2)} + \underbrace{\int_{CS} u\rho\vec{V}\cdot d\vec{A}}_{=0(3)} \quad ; \quad \tau_{yx} = \mu\frac{du}{dy} = \tau$$

Assumptions: (1) F_{S_x} due to shear forces only
(2) Steady flow
(3) Fully-developed flow

Then

$$F_{S_x} + F_{B_x} = F_{①} - F_{②} + F_{B_x} = \left(\tau + \frac{d\tau}{dy}\frac{dy}{2}\right)dx\,dz - \left(\tau - \frac{d\tau}{dy}\frac{dy}{2}\right)dx\,dz - \rho g\, dx\,dy\,dz = 0$$

or $\dfrac{d\tau}{dy} = \rho g$. Integrating

$$\tau = \rho g y + c_1 = \mu\frac{du}{dy} \quad \text{or} \quad \frac{du}{dy} = \frac{\rho g y}{\mu} + \frac{c_1}{\mu}. \text{ Integrating again,}$$

$$u = \frac{\rho g y^2}{2\mu} + \frac{c_1}{\mu}y + c_2$$

To evaluate the constants c_1 and c_2, apply the boundary conditions:

At $y = 0$, $u = U_0$, so $c_2 = U_0$

At $y = h$, $\tau = 0$, so $\dfrac{du}{dy} = 0$, and $c_1 = -\rho g h$ ← BC

Substituting,

$$u = \frac{\rho g y^2}{2\mu} - \frac{\rho g h y}{\mu} + U_0 = \frac{\rho g}{\mu}\left(\frac{y^2}{2} - hy\right) + U_0$$ ← u

Note that at $y = h$,

$$u = \frac{\rho g}{\mu}\left(-\frac{h^2}{2}\right) + U_0 \neq 0$$

Thus the solution is determined only when U_0 and h are known.

Problem 8.46

[Difficulty: 3]

8.46 A wet paint film of uniform thickness, δ, is painted on a vertical wall. The wet paint can be approximated as a Bingham fluid with a yield stress, τ_y, and density, ρ. Derive an expression for the maximum value of δ that can be sustained without having the paint flow down the wall. Calculate the maximum thickness for lithographic ink whose yield stress $\tau_y = 40$ Pa and density is approximately 1000 kg/m^3.

Given: Paint flow (Bingham fluid)

Find: Maximum thickness of paint film before flow occur

Solution:

Basic equations: Bingham fluid: $\tau_{yx} = \tau_y + \mu_p \cdot \frac{du}{dy}$

Use the analysis of Example 8.3, where we obtain a force balance between gravity and shear stresses: $\frac{d\tau_{yx}}{dy} = -\rho \cdot g$

The given data is $\tau_y = 40 \cdot \text{Pa}$ $\quad \rho = 1000 \cdot \frac{\text{kg}}{\text{m}^3}$

From the force balance equation, itegrating $\tau_{yx} = -\rho \cdot g \cdot y + c$ and we have boundary condition $\tau_{yx}(y = \delta) = 0$

Hence $\tau_{yx} = -\rho \cdot g \cdot (\delta - y)$ and this is a maximum at the wall $|\tau_{max}| = \rho \cdot g \cdot \delta$

Motion occurs when $|\tau_{max}| \geq |\tau_y|$ or $\rho \cdot g \cdot \delta \geq \tau_y$

Hence the maximum thickness is $\delta = \frac{\tau_y}{\rho \cdot g}$ $\quad \delta = 4.08 \times 10^{-3}\,\text{m}$ $\quad \delta = 4.08\,\text{mm}$

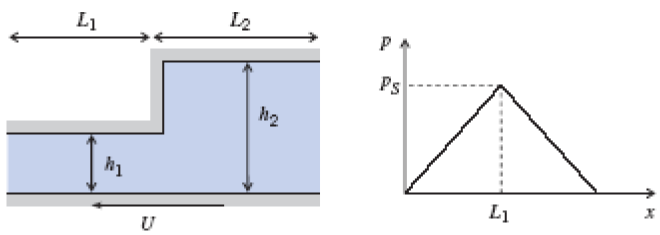
Problem 8.47 [Difficulty: 4]

8.47 When dealing with the lubrication of bearings, the governing equation describing pressure is the Reynolds equation, generally written in 1D as

$$\frac{d}{dx}\left(\frac{h^3}{\mu}\frac{dp}{dx}\right) + 6U\frac{dh}{dx} = 0$$

where h is the step height and U is the velocity of the lower surface. Step bearings have a relatively simple design and are used with low-viscosity fluids such as water, gasoline, and solvents. The minimum film thickness in these applications is quite small. The step height must be small enough for good load capacity, yet large enough for the bearing to accommodate some wear without losing its load capacity by becoming smooth and flat. Beginning with the 1D equation for fluid motion in the x direction, show that the pressure distribution in the step bearing is as shown, where

$$p_s = \frac{6\mu(h_2 - h_1)}{\dfrac{h_1^3}{L_1} + \dfrac{h_2^3}{L_2}}$$



Given: Equation for fluid motion in the x-direction.

Find: Expression for peak pressure

Solution: Begin with the steady-state Navier-Stokes equation – x-direction

Governing equation:
The Navier-Stokes equations are

$$\overset{4}{\cancel{\frac{\partial u}{\partial x}}} + \frac{\partial v}{\partial y} + \overset{3}{\cancel{\frac{\partial w}{\partial z}}} = 0 \qquad (5.1c)$$

$$\rho\left(\overset{1}{\cancel{\frac{\partial u}{\partial t}}} + u\overset{4}{\cancel{\frac{\partial u}{\partial x}}} + v\overset{5}{\cancel{\frac{\partial u}{\partial y}}} + w\overset{3}{\cancel{\frac{\partial u}{\partial z}}}\right) = \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\overset{4}{\cancel{\frac{\partial^2 u}{\partial x^2}}} + \frac{\partial^2 u}{\partial y^2} + \overset{3}{\cancel{\frac{\partial^2 u}{\partial z^2}}}\right) \qquad (5.27a)$$

$$\rho\left(\overset{1}{\cancel{\frac{\partial v}{\partial t}}} + u\overset{4}{\cancel{\frac{\partial v}{\partial x}}} + v\overset{5}{\cancel{\frac{\partial v}{\partial y}}} + w\overset{3}{\cancel{\frac{\partial v}{\partial z}}}\right) = \overset{6}{\cancel{\rho g_y}} - \frac{\partial p}{\partial y} + \mu\left(\overset{4}{\cancel{\frac{\partial^2 v}{\partial x^2}}} + \overset{5}{\cancel{\frac{\partial^2 v}{\partial y^2}}} + \overset{3}{\cancel{\frac{\partial^2 v}{\partial z^2}}}\right) \qquad (5.27b)$$

$$\rho\left(\overset{1}{\cancel{\frac{\partial w}{\partial t}}} + u\overset{3}{\cancel{\frac{\partial w}{\partial x}}} + v\overset{3}{\cancel{\frac{\partial w}{\partial y}}} + w\overset{3}{\cancel{\frac{\partial w}{\partial z}}}\right) = \overset{3}{\cancel{\rho g_z}} - \overset{3}{\cancel{\frac{\partial p}{\partial z}}} + \mu\left(\overset{3}{\cancel{\frac{\partial^2 w}{\partial x^2}}} + \overset{3}{\cancel{\frac{\partial^2 w}{\partial y^2}}} + \overset{3}{\cancel{\frac{\partial^2 w}{\partial z^2}}}\right) \qquad (5.27c)$$

The following assumptions have been applied:

(1) Steady flow (given).
(2) Incompressible flow; ρ = constant.
(3) No flow or variation of properties in the z direction; w= 0 and $\partial/\partial z = 0$.
(4) Fully developed flow, so no properties except possibly pressure p vary in the x direction; $\partial/\partial x = 0$.
(5) See analysis below.
(6) No body force in the y direction; $g_y = 0$

Assumption (1) eliminates time variations in any fluid property. Assumption (2) eliminates space variations in density. Assumption (3) states that there is no z component of velocity and no property variations in the z direction. All terms in the z component of the Navier–Stokes equation cancel. After assumption (4) is applied, the continuity equation reduces to $\partial v/\partial y = 0$. Assumptions (3) and (4) also indicate that $\partial v/\partial z = 0$ and $\partial v/\partial x = 0$. Therefore v must be constant (except of course in a more realistic model $v \neq 0$ near the transition. Since v is zero at the solid surface, then v must be zero everywhere. The fact that v = 0 reduces the Navier–Stokes equations further, as indicated by (5). Hence for the y direction

$$\frac{\partial p}{\partial y} = 0$$

which indicates the pressure is a constant across the flow. Hence we conclude that p is a function at most of x.

In the x direction, we obtain

$$0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} \tag{1}$$

Integrating this twice for the first region

$$u_1 = \frac{1}{2\mu} \left.\frac{dp}{dx}\right|_1 y^2 + \frac{c_1}{\mu} y + c_2$$

where $\left.\frac{dp}{dx}\right|_1$ denotes the pressure gradient in region 1. Note that we change to regular derivative as p is a function of x only. Note that Eq 1 implies that we have a function of x only $\left(\frac{\partial p}{\partial x}\right)$ and a function of y only $\left(\frac{\partial^2 u}{\partial y^2}\right)$ that must add up to be a constant (0); hence EACH is a constant! This means that

$$\left.\frac{dp}{dx}\right|_1 = const = \frac{p_s}{L_1}$$

using the notation of the figure.

To evaluate the constants, c_1 and c_2, we must apply the boundary conditions. We do this separately for each region.

In the first region, at $y = 0$, $u = U$. Consequently, $c_2 = U$. At $y = h_1$, $u = 0$. Hence

$$0 = \frac{1}{2\mu} \left.\frac{dp}{dx}\right|_1 h_1^2 + \frac{c_1}{\mu} h_1 + U$$

so

$$c_1 = -\frac{1}{2} \left.\frac{dp}{dx}\right|_1 h_1 - \frac{\mu U}{h_1}$$

Hence, combining results

$$u_1 = \frac{1}{2\mu} \left.\frac{dp}{dx}\right|_1 \left(y^2 - h_1 y\right) + U\left(\frac{y}{h_1} - 1\right)$$

Exactly the same reasoning applies to the second region, so

$$u_2 = \frac{1}{2\mu} \left.\frac{dp}{dx}\right|_2 \left(y^2 - h_2 y\right) + U\left(\frac{y}{h_2} - 1\right)$$

where

$$\left.\frac{dp}{dx}\right|_2 = const = -\frac{p_s}{L_2}$$

What connects these flow is the flow rate Q.

$$q = \int_0^{h_1} u_1 dy = \int_0^{h_2} u_2 dy = -\frac{1}{12\mu}\left.\frac{dp}{dx}\right|_1 h_1^3 - \frac{Uh_1}{2} = -\frac{1}{12\mu}\left.\frac{dp}{dx}\right|_2 h_2^3 - \frac{Uh_2}{2}$$

Hence

$$\frac{1}{12\mu}\frac{p_s}{L_1}h_1^3 + \frac{Uh_1}{2} = -\frac{1}{12\mu}\frac{p_s}{L_2}h_2^3 + \frac{Uh_2}{2}$$

Solving for p_s,

$$\frac{p_s}{12\mu}\left(\frac{h_1^3}{L_1} + \frac{h_2^3}{L_2}\right) = \frac{Uh_2}{2} - \frac{Uh_1}{2}$$

or

$$p_s = \frac{6\mu U(h_2 - h_1)}{\left(\frac{h_1^3}{L_1} + \frac{h_2^3}{L_2}\right)}$$

Problem 8.48

[Difficulty: 2]

8.48 Consider first water and then SAE 10W lubricating oil flowing at 40°C in a 6-mm-diameter tube. Determine the maximum flow rate (and the corresponding pressure gradient, $\partial p/\partial x$) for each fluid at which laminar flow would be expected.

Solution: Laminar flow is expected for $Re \le 2300$. Expressing this in terms of flowrate,

$$Re = \frac{\rho \bar{V} D}{\mu} = \frac{\bar{V} D}{\nu} = \frac{Q D}{A \nu} = \frac{4}{\pi D^2}\frac{QD}{\nu} = \frac{4Q}{\pi \nu D} \quad \text{or} \quad Q = \frac{\pi \nu D Re}{4}$$

Thus

$$Q_{max} = \frac{\pi \nu D Re_{max}}{4} = \frac{\pi}{4} \times 2300 \times 0.006\,m \times \nu \frac{m^2}{s} = 10.8\,\nu\left(\frac{m^3}{s}\right)$$

Also, $Q = -\frac{\pi R^4}{8\mu}\frac{\partial p}{\partial x}$ for laminar flow, according to Eq. 8.13b. Then

$$\frac{\partial p}{\partial x} = -\frac{8\mu Q}{\pi R^4} = -\frac{128\,\mu Q}{\pi D^4}$$

so

$$\frac{\partial p}{\partial x} = -\frac{128}{\pi} \times \mu \frac{N\cdot s}{m^2} \times Q\frac{m^3}{s} \times \frac{1}{(0.006)^4\,m^4} = -3.14 \times 10^{10}\,\mu Q \left(\frac{N}{m^3}\right)$$

Using data from Appendix A, at 40°C,

Fluid	$\nu\left(\frac{m^2}{s}\right)$	$Q\left(\frac{m^3}{s}\right)$	$\mu\left(\frac{N\cdot s}{m^2}\right)$	$\mu Q\,(N\cdot m)$	$\frac{\partial p}{\partial x}\left(\frac{N}{m^3}\right)$
Water	6.57×10^{-7}	7.10×10^{-6}	6.51×10^{-4}	4.62×10^{-9}	-145
SAE 10W oil	3.8×10^{-5}	4.10×10^{-4}	3.4×10^{-2}	1.39×10^{-5}	-4.36×10^{5}

$\frac{\partial p}{\partial x}$

Q_{ma}

$\left\{\text{Note } Q \sim \nu = \frac{\mu}{\rho} \text{ and } \frac{\partial p}{\partial x} \sim \mu Q \sim \frac{\mu^2}{\rho}.\right\}$

Problem 8.49

[Difficulty: 2]

8.49 For fully developed laminar flow in a pipe, determine the radial distance from the pipe axis at which the velocity equals the average velocity.

Solution: First determine $\bar{V}$.

$$\bar{V} = \frac{Q}{A} = \frac{1}{\pi R^2}\int_A u\,dA = \frac{1}{\pi R^2}\int_0^R \left\{-\frac{R^2}{4\mu}\frac{\partial p}{\partial x}\left[1-\left(\frac{r}{R}\right)^2\right]\right\} 2\pi r\,dr$$

$$= -\frac{R^2}{2\mu}\frac{\partial p}{\partial x}\int_0^1 \left[1-\left(\frac{r}{R}\right)^2\right]\left(\frac{r}{R}\right) d\left(\frac{r}{R}\right) = -\frac{R^2}{2\mu}\frac{\partial p}{\partial x}\left[\frac{1}{2}\left(\frac{r}{R}\right)^2 - \frac{1}{4}\left(\frac{r}{R}\right)^4\right]_0^1$$

$$\bar{V} = -\frac{R^2}{8\mu}\frac{\partial p}{\partial x}$$

Then $u = \bar{V}$ when

$$u = -\frac{R^2}{4\mu}\frac{\partial p}{\partial x}\left[1-\left(\frac{r}{R}\right)^2\right] = \bar{V} = -\frac{R^2}{8\mu}\frac{\partial p}{\partial x}$$

or

$$1-\left(\frac{r}{R}\right)^2 = \frac{1}{2}$$

or

$$\left(\frac{r}{R}\right)^2 = \frac{1}{2}$$

$$r = \frac{R}{\sqrt{2}} = 0.707R \quad r$$

Problem 8.50

[Difficulty: 3]

8.50 Using Eq. A.3 in Appendix A for the viscosity of water, find the viscosity at −20°C and 120°C. Plot the viscosity over this range. Find the maximum laminar flow rate (L/hr) in a 7.5-mm-diameter tube at these temperatures. Plot the maximum laminar flow rate over this temperature range.

Given: Data on water temperature and tube

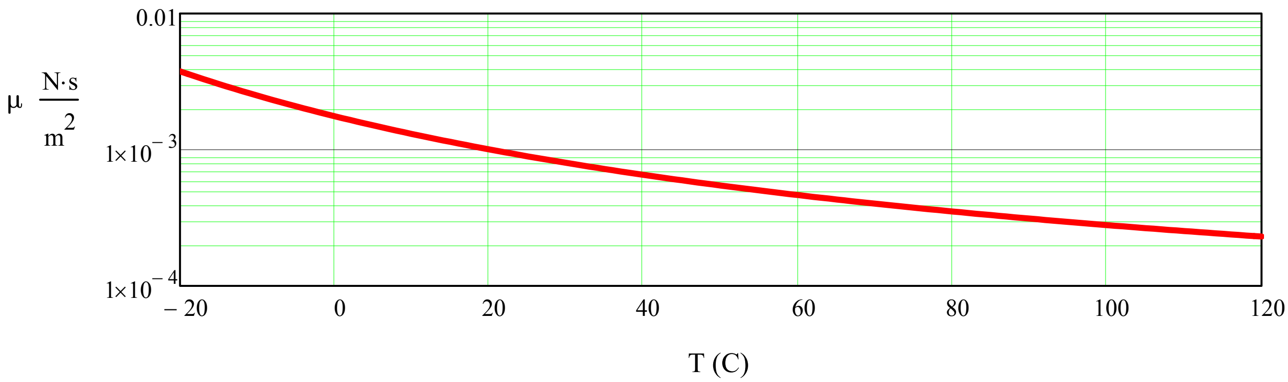
Find: Maximum laminar flow; plot

Solution: From Appendix A $\quad A = 2.414 \cdot 10^{-5} \cdot \frac{N \cdot s}{m^2} \quad B = 247.8 \cdot K \quad C = 140 \cdot K \quad$ in $\quad \mu(T) = A \cdot 10^{\frac{B}{T-C}}$

$D = 7.5 \cdot mm \qquad \rho = 1000 \frac{kg}{m^3} \qquad Re_{crit} = 2300$

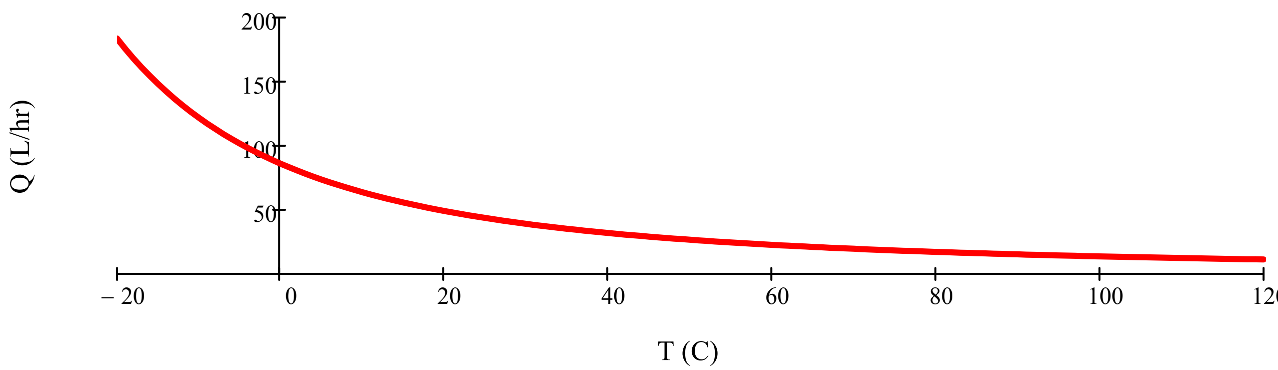
$T_1 = -20\,°C \quad T_1 = 253\,K \quad \mu(T_1) = 3.74 \times 10^{-3} \frac{N \cdot s}{m^2} \qquad T_2 = 120\,°C \quad T_2 = 393\,K \quad \mu(T_2) = 2.3 \times 10^{-4} \frac{N \cdot s}{m^2}$

The plot of viscosity is



For the flow rate $\quad Re_{crit} = \frac{\rho \cdot V_{crit} \cdot D}{\mu} = \frac{\rho \cdot Q_{max} \cdot D}{\mu \cdot \frac{\pi}{4} \cdot D^2} = \frac{4 \cdot \rho \cdot Q_{max}}{\mu \cdot \pi \cdot D} \qquad Q_{max}(T) = \frac{\pi \cdot \mu(T) \cdot D \cdot Re_{crit}}{4 \cdot \rho}$

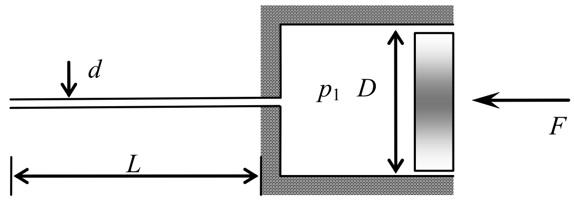
$Q_{max}(T_1) = 5.07 \times 10^{-5} \frac{m^{3.00}}{s} \qquad Q_{max}(T_1) = 182 \frac{L}{hr} \qquad Q_{max}(T_2) = 3.12 \times 10^{-6} \frac{m^3}{s} \qquad Q_{max}(T_2) = 11.2 \frac{L}{hr}$



Problem 8.51

[Difficulty: 2]

8.51 A hypodermic needle, with inside diameter $d = 0.005$ in. and length $L = 1$ in., is used to inject saline solution with viscosity five times that of water. The plunger diameter is $D = 0.375$ in.; the maximum force that can be exerted by a thumb on the plunger is $F = 7.5$ lbf. Estimate the volume flow rate of saline that can be produced.



Given: Hyperdermic needle

Find: Volume flow rate of saline

Solution:

Basic equation $Q = \frac{\pi \cdot \Delta p \cdot d^4}{128 \cdot \mu \cdot L}$ (Eq. 8.13c; we assume laminar flow and verify this is correct after solving)

For the system $\Delta p = p_1 - p_{atm} = \frac{F}{A} = \frac{4 \cdot F}{\pi \cdot D^2}$

$$\Delta p = \frac{4}{\pi} \times 7.5 \cdot lbf \times \left(\frac{1}{0.375 in} \times \frac{12 \cdot in}{1 \cdot ft} \right)^2 \qquad \Delta p = 67.9 \cdot psi$$

At 68°F, from Table A.7 $\mu_{H2O} = 2.1 \times 10^{-5} \cdot \frac{lbf \cdot s}{ft^2}$ $\mu = 5 \cdot \mu_{H2O}$ $\mu = 1.05 \times 10^{-4} \cdot \frac{lbf \cdot s}{ft^2}$

$$Q = \frac{\pi}{128} \times 67.9 \cdot \frac{lbf}{in^2} \times \frac{144 \cdot in^2}{1 \cdot ft^2} \times \left(0.005 \cdot in \times \frac{1 \cdot ft}{12 \cdot in} \right)^4 \times \frac{ft^2}{1.05 \times 10^{-4} \, lbf \cdot s} \times \frac{1}{1 \cdot in} \times \frac{12 \cdot in}{1 \cdot ft}$$

$Q = 8.27 \times 10^{-7} \cdot \frac{ft^3}{s}$ $Q = 1.43 \times 10^{-3} \cdot \frac{in^3}{s}$ $Q = 0.0857 \cdot \frac{in^3}{min}$

Check Re: $V = \frac{Q}{A} = \frac{Q}{\frac{\pi \cdot d^2}{4}}$ $V = \frac{4}{\pi} \times 8.27 \times 10^{-7} \frac{ft^3}{s} \times \left(\frac{1}{.005 \cdot in} \right)^2 \times \left(\frac{12 \cdot in}{1 \cdot ft} \right)^2$ $V = 6.07 \cdot \frac{ft}{s}$

$Re = \frac{\rho \cdot V \cdot d}{\mu}$ $\rho = 1.94 \cdot \frac{slug}{ft^3}$ (assuming saline is close to water)

$$Re = 1.94 \cdot \frac{slug}{ft^3} \times 6.07 \cdot \frac{ft}{s} \times 0.005 \cdot in \times \frac{1 \cdot ft}{12 \cdot in} \times \frac{ft^2}{1.05 \times 10^{-4} \cdot lbf \cdot s} \times \frac{slug \cdot ft}{s^2 \cdot lbf}$$

$Re = 46.7$

Flow is laminar

Problem 8.52

[Difficulty: 3]

8.52 In engineering science, there are often analogies to be made between disparate phenomena. For example, the applied pressure difference, Δp, and corresponding volume flow rate, Q, in a tube can be compared to the applied DC voltage, V, across and current, I, through an electrical resistor, respectively. By analogy, find a formula for the "resistance" of laminar flow of fluid of viscosity, μ, in a tube length of L and diameter D, corresponding to electrical resistance, R. For a tube 250 mm long with inside diameter 7.5 mm, find the maximum flow rate and pressure difference for which this analogy will hold for (a) kerosene and (b) castor oil (both at 40°C). When the flow exceeds this maximum, why does the analogy fail?

Given: Data on a tube

Find: "Resistance" of tube; maximum flow rate and pressure difference for which electrical analogy holds for (a) kerosine and (b) castor oil

Solution:

The given data is $\quad L = 250\cdot mm \quad D = 7.5\cdot mm$

From Fig. A.2 and Table A.2

Kerosene: $\quad \mu = 1.1\times 10^{-3}\cdot\frac{N\cdot s}{m^2} \quad \rho = 0.82\times 990\cdot\frac{kg}{m^3} = 812\cdot\frac{kg}{m^3}$

Castor oil: $\quad \mu = 0.25\cdot\frac{N\cdot s}{m^2} \quad \rho = 2.11\times 990\cdot\frac{kg}{m^3} = 2090\cdot\frac{kg}{m^3}$

For an electrical resistor $\quad V = R\cdot I \quad (1)$

The governing equation for the flow rate for laminar flow in a tube is Eq. 8.13c

$$Q = \frac{\pi\cdot\Delta p\cdot D^4}{128\cdot\mu\cdot L}$$

or

$$\Delta p = \frac{128\cdot\mu\cdot L}{\pi\cdot D^4}\cdot Q \quad (2)$$

By analogy, current I is represented by flow rate Q, and voltage V by pressure drop Δp.
Comparing Eqs. (1) and (2), the "resistance" of the tube is

$$R = \frac{128\cdot\mu\cdot L}{\pi\cdot D^4}$$

The "resistance" of a tube is directly proportional to fluid viscosity and pipe length, and strongly dependent on the inverse of diameter

The analogy is only valid for $\quad Re < 2300 \quad$ or $\quad \frac{\rho\cdot V\cdot D}{\mu} < 2300$

Writing this constraint in terms of flow rate $\quad \frac{\rho\cdot\frac{Q}{\frac{\pi}{4}\cdot D^2}\cdot D}{\mu} < 2300 \quad$ or $\quad Q_{max} = \frac{2300\cdot\mu\cdot\pi\cdot D}{4\cdot\rho}$

The corresponding maximum pressure gradient is then obtained from Eq. (2)

$$\Delta p_{max} = \frac{128 \cdot \mu \cdot L}{\pi \cdot D^4} \cdot Q_{max} = \frac{32 \cdot 2300 \cdot \mu^2 \cdot L}{\rho \cdot D^3}$$

Substituting values

(a) For kerosine $\quad Q_{max} = 1.84 \times 10^{-5} \frac{m^3}{s} \quad Q_{max} = 1.10 \cdot \frac{l}{min} \quad \Delta p_{max} = 65.0 \cdot Pa$

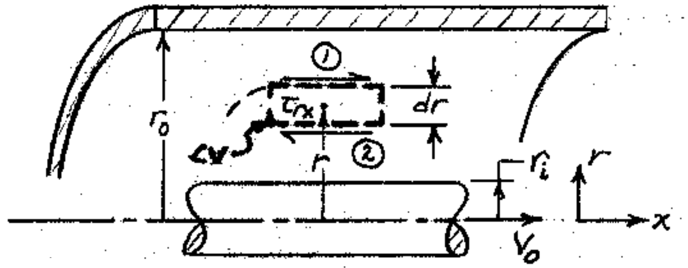
(b) For castor oil $\quad Q_{max} = 1.62 \times 10^{-3} \frac{m^3}{s} \quad Q_{max} = 97.3 \cdot \frac{l}{min} \quad \Delta p_{max} = 1.30 \cdot MPa$

The analogy fails when Re > 2300 because the flow becomes turbulent, and "resistance" to flow is then no longer linear with flow rate

Problem 8.53

[Difficulty: 3]

8.53 Consider fully developed laminar flow in the annulus between two concentric pipes. The outer pipe is stationary, and the inner pipe moves in the x direction with speed V. Assume the axial pressure gradient is zero ($\partial p/\partial x = 0$). Obtain a general expression for the shear stress, τ, as a function of the radius, r, in terms of a constant, C_1. Obtain a general expression for the velocity profile, $u(r)$, in terms of two constants, C_1 and C_2. Obtain expressions for C_1 and C_2.



Solution: Apply x component of momentum equation, using annular CV shown.

Basic Equations: $F_{S_x} + \cancel{F_{B_x}}^{=0(1)} = \cancel{\frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall}^{=0(2)} + \cancel{\int_{CS} u\rho \vec{V}\cdot d\vec{A}}^{=0(3)}\,; \quad \tau_{rx} = \mu \frac{du}{dr} = \tau$

Assumptions: (1) $F_{Bx} = 0$
(2) steady flow
(3) Fully-developed flow

Then

$$F_{S_x} = F_{①} - F_{②} = \left(\tau + \frac{d\tau}{dr}\frac{dr}{2}\right)2\pi\left(r+\frac{dr}{2}\right)dx - \left(\tau - \frac{d\tau}{dr}\frac{dr}{2}\right)2\pi\left(r - \frac{dr}{2}\right)dx = 0$$

Neglecting products of differentials, this reduces to

$$\tau + r\frac{d\tau}{dr} = 0 \qquad \text{or} \qquad \frac{d}{dr}(r\tau) = 0$$

Thus $\quad r\tau = C_1 \qquad$ or $\qquad \tau = \dfrac{C_1}{r}$ $\qquad \tau(r)$

But $\quad \tau = \mu\dfrac{du}{dr}$, so $\qquad \dfrac{du}{dr} = \dfrac{C_1}{\mu r}$

and $\quad u = \dfrac{C_1}{\mu}\ln r + C_2$ $\qquad u(r)$

To evaluate constants C_1 and C_2, use boundary conditions.

At $r = r_i$, $u = V_0$, so $\quad V_0 = \dfrac{C_1}{\mu}\ln r_i + C_2$

At $r = r_0$, $u = 0$, so $\quad 0 = \dfrac{C_1}{\mu}\ln r_0 + C_2 \quad$ and $\quad C_2 = -\dfrac{C_1}{\mu}\ln r_0$

Thus, subtracting, $\quad V_0 = \dfrac{C_1}{\mu}\ln\left(\dfrac{r_i}{r_0}\right) \quad$ or $\quad C_1 = \dfrac{\mu V_0}{\ln(r_i/r_0)} \quad$ so $\quad C_2 = -\dfrac{V_0 \ln r_0}{\ln(r_i/r_0)}$

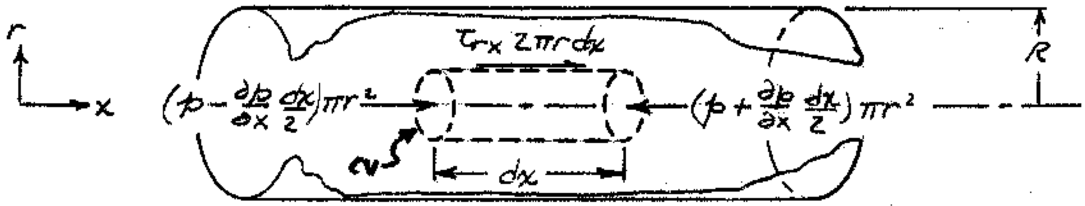
Finally

$$u = \frac{V_0}{\ln(r_i/r_0)}(\ln r - \ln r_i) = V_0\frac{\ln(r/r_0)}{\ln(r_i/r_0)}$$ $\qquad u(r)$

Problem 8.54

[Difficulty: 3]

8.54 Consider fully developed laminar flow in a circular pipe. Use a cylindrical control volume as shown. Indicate the forces acting on the control volume. Using the momentum equation, develop an expression for the velocity distribution.



Solution: The forces on a CV of radius r are shown above.

Apply the x component of momentum to CV shown.

Basic equations:

$$F_{S_x} + \underbrace{F_{B_x}}_{=0(1)} = \underbrace{\frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall}_{=0(2)} + \underbrace{\int_{CS} u\rho \vec{V}\cdot d\vec{A}}_{=0(3)}, \quad \tau_{rx} = \mu\frac{du}{dr}$$

Assumptions: (1) $F_{Bx} = 0$
(2) Steady flow
(3) Fully-developed flow

Then

$$F_{S_x} = \left(p - \frac{\partial p}{\partial x}\frac{dx}{2}\right)\pi r^2 + \tau_{rx}\,2\pi r\,dx - \left(p + \frac{\partial p}{\partial x}\frac{dx}{2}\right)\pi r^2 = 0$$

Cancelling and combining terms,

$$-r\frac{\partial p}{\partial x} + 2\tau_{rx} = 0 \qquad \text{or} \qquad \tau_{rx} = \mu\frac{du}{dr} = \frac{r}{2}\frac{\partial p}{\partial x}$$

Thus $\displaystyle \frac{du}{dr} = \frac{r}{2\mu}\frac{\partial p}{\partial x}$

and

$$u = \frac{r^2}{4\mu}\frac{\partial p}{\partial x} + c_1$$

To evaluate c_1, apply the boundary condition $u = 0$ at $r = R$. Thus

$$c_1 = -\frac{R^2}{4\mu}\frac{\partial p}{\partial x}$$

and

$$u = \frac{1}{4\mu}\frac{\partial p}{\partial x}(r^2 - R^2) = -\frac{R^2}{4\mu}\frac{\partial p}{\partial x}\left[1 - \left(\frac{r}{R}\right)^2\right]$$ u

which is identical to Eq. 8.12.

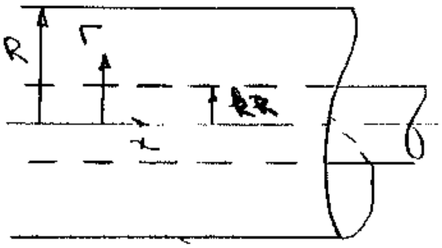
Problem 8.56

[Difficulty: 4] Part 1/2

8.56 For the flow of Problem 8.55 show that the volume flow rate is given by

$$Q = -\frac{\pi R^4}{8\mu}\frac{\partial p}{\partial x}\left[(1-k^4) - \frac{(1-k^2)^2}{\ln(1/k)}\right]$$

Find an expression for the average velocity. Compare the limiting case, $k \to 0$, with the corresponding expression for flow in a circular pipe.



Solution: The volume flow rate is given by

$$Q = \int u\,dA = \int_{kR}^{R} u\, 2\pi r\,dr = 2\pi \int_{kR}^{R} u\, r\,dr$$

$$= 2\pi\left(-\frac{R^2}{4\mu}\frac{\partial p}{\partial x}\right)\int_{kR}^{R}\left[r - \frac{r^3}{R^2} + \frac{(1-k^2)}{\ln(1/k)}\, r\ln\frac{r}{R}\right]dr$$

$$= -\frac{\pi R^4}{2\mu}\frac{\partial p}{\partial x}\int_{k}^{1}\left[\frac{r}{R} - \left(\frac{r}{R}\right)^3 + \frac{(1-k^2)}{\ln(1/k)}\frac{r}{R}\ln\frac{r}{R}\right]d\left(\frac{r}{R}\right)$$

$$= -\frac{\pi R^4}{2\mu}\frac{\partial p}{\partial x}\left[\frac{1}{2}\left(\frac{r}{R}\right)^2 - \frac{1}{4}\left(\frac{r}{R}\right)^4 + \frac{(1-k^2)}{\ln(1/k)}\left\{\left(\frac{r}{R}\right)^2\left[\frac{1}{2}\ln\left(\frac{r}{R}\right) - \frac{1}{4}\right]\right\}\right]_k^1$$

$$= -\frac{\pi R^4}{2\mu}\frac{\partial p}{\partial x}\left[\frac{1}{2} - \frac{k^2}{2} - \frac{1}{4} + \frac{k^4}{4} + \frac{(1-k^2)}{\ln(1/k)}\left\{-\frac{1}{4} - k^2\left[\frac{1}{2}\ln k - \frac{1}{4}\right]\right\}\right]$$

$$= -\frac{\pi R^4}{2\mu}\frac{\partial p}{\partial x}\left[\frac{1}{4} - \frac{k^2}{2} + \frac{k^4}{4} + \frac{(1-k^2)}{\ln(1/k)}\left\{-\frac{1}{4} + \frac{k^2}{4} - k^2\frac{1}{2}\ln k\right\}\right]$$

$$= -\frac{\pi R^4}{2\mu}\frac{\partial p}{\partial x}\left[\frac{1-2k^2+k^4}{4} + \frac{(1-k^2)}{\ln(1/k)}\frac{(k^2-1)}{4} - \frac{(1-k^2)}{\ln(1/k)}k^2\frac{1}{2}\ln k\right]$$

$$= -\frac{\pi R^4}{2\mu}\frac{\partial p}{\partial x}\left[\frac{1-2k^2+k^4}{4} - \frac{(1-k^2)^2}{4\ln(1/k)} + \frac{k^2-k^4}{2}\right]$$

$$= -\frac{\pi R^4}{2\mu}\frac{\partial p}{\partial x}\left[\frac{1-\cancel{2k^2}+k^4+\cancel{2k^2}-2k^4}{4} - \frac{(1-k^2)^2}{4\ln(1/k)}\right]$$

$$Q = -\frac{\pi R^4}{8\mu}\frac{\partial p}{\partial x}\left[(1-k^4) - \frac{(1-k^2)^2}{\ln(1/k)}\right] \longleftarrow Q$$

The average velocity, $\bar{V} = \frac{Q}{A}$

The area is given by

$$A = \int dA = \int 2\pi r\,dr = 2\pi R^2 \int_k^1 \frac{r}{R}\,d\left(\frac{r}{R}\right)$$

$$A = 2\pi R^2 \left[\frac{1}{2}\left(\frac{r}{R}\right)^2\right]_k^1 = 2\pi R^2 \times \frac{1}{2}(1-k^2) = \pi R^2(1-k^2)$$

Thus

$$\bar{V} = \frac{Q}{A} = -\frac{\pi R^4}{8\mu}\frac{\partial p}{\partial x} \times \frac{1}{\pi R^2}\left[\frac{(1-k^4)}{(1-k^2)} - \frac{(1-k^2)}{\ln(1/k)}\right]$$

$$\bar{V} = -\frac{R^2}{8\mu}\frac{\partial p}{\partial x}\left[\frac{(1-k^4)}{(1-k^2)} - \frac{(1-k^2)}{\ln(1/k)}\right] \longleftarrow \bar{V}$$

For $k \to 0$

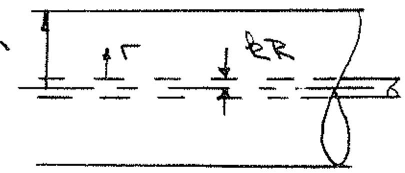
$$Q = -\frac{\pi R^4}{8\mu}\frac{\partial p}{\partial x} \quad \text{and} \quad \bar{V} = -\frac{R^2}{8\mu}\frac{\partial p}{\partial x}$$

These agree with the results for flow in a circular pipe.

Problem 8.57

[Difficulty: 4]

8.57 It has been suggested in the design of an agricultural sprinkler that a structural member be held in place by a wire placed along the centerline of a pipe; it is surmised that a relatively small wire would have little effect on the pressure drop for a given flow rate. Using the result of Problem 8.56, derive an expression giving the percentage change in pressure drop as a function of the ratio of wire diameter to pipe diameter for laminar flow. Plot the percentage change in pressure drop as a function of radius ratio k for $0.001 \leq k \leq 0.10$.



Solution: The results of problem 8.56 give

$$Q = -\frac{\pi R^4}{8\mu}\frac{\partial p}{\partial x}\left[(1-k^4) - \frac{(1-k^2)^2}{\ln(1/k)}\right]$$

Thus

$$\frac{\Delta p}{L} = -\frac{\partial p}{\partial x} = \frac{8\mu Q}{\pi R^4} \times \frac{1}{\left[(1-k^4) - \frac{(1-k^2)^2}{\ln(1/k)}\right]}$$

or $k = 0$, $\quad \dfrac{\Delta p}{L} = \dfrac{8\mu Q}{\pi R^4}$

$$\text{Percent change} = \frac{\Delta p/L - \Delta p/L)_{k=0}}{\Delta p/L)_{k=0}} = \frac{1}{\left[(1-k^4) - \frac{(1-k^2)^2}{\ln(1/k)}\right]} - 1$$

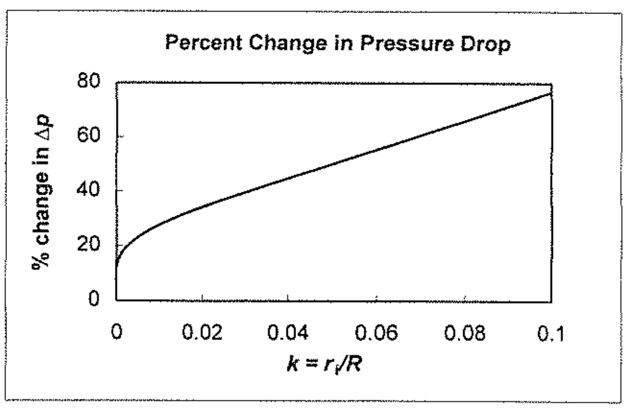
$$\%\text{ change} = \frac{1 - \left[(1-k^4) - \frac{(1-k^2)}{\ln(1/k)}\right]}{\left[(1-k^4) - \frac{(1-k^2)}{\ln(1/k)}\right]}$$

For small k,

$$\%\text{ change} = \frac{1 - \left[1 - \frac{1}{\ln(1/k)}\right]}{\left[1 - \frac{1}{\ln(1/k)}\right]} = \frac{1 - \left[1 + \frac{1}{\ln k}\right]}{\left[1 + \frac{1}{\ln k}\right]} = \frac{-\frac{1}{\ln k}}{\left[1 + \frac{1}{\ln k}\right]}$$

$$\%\text{ change} = -\frac{1}{\ln k\left(1 + \frac{1}{\ln k}\right)} \times 100 \qquad \text{\% change}$$

$k = r_i/R$	% change in Δp
0.0001	12.2
0.0002	13.3
0.0005	15.1
0.001	16.9
0.002	19.2
0.005	23.3
0.01	27.7
0.02	34.3
0.05	50.1
0.1	76.8



The plot shows that even the smallest of wires causes a significant increase in pressure drop for a given flow rate.

Problem 8.58

[Difficulty: 2]

8.58 Consider fully developed pressure-driven flow in a cylindrical tube of radius, R, and length, $L = 10$ mm, with flow generated by an applied pressure gradient, Δp. Tests are performed with room temperature water for various values of R, with a fixed flow rate of $Q = 10$ μL/min. The hydraulic resistance is defined as $R_{hyd} = \Delta p/Q$ (by analogy with the electrical resistance $R_{elec} = \Delta V/I$, where ΔV is the electrical potential drop and I is the electric current). Calculate the required pressure gradient and hydraulic resistance for the range of tube radii listed in the table. Based on the results, is it appropriate to use a pressure gradient to pump fluids in microchannels, or should some other driving mechanism be used?

R (mm)	Δp (Pa)	R_{hyd} (Pa · s/m^3)
1		
10^{-1}		
10^{-2}		
10^{-3}		
10^{-4}		

Given:

Tube dimensions and volumetric flow rate

Find:

Pressure difference and hydraulic resistance

Solution:

The flow rate of a fully developed pressure-driven flow in a pipe is $Q = \dfrac{\pi \Delta p R^4}{8\mu L}$. Rearranging it, one obtains $\Delta p = \dfrac{8\mu L Q}{\pi R^4}$. For a flow rate $Q = 10\mu l/\min$, L=1 cm, $\mu = 1.0\times 10^{-3}\ Pa.s$, and R = 1 mm,

$$\Delta p = \frac{8}{\pi}\times\frac{10\times 10^{-9}}{60}\times\frac{m^3}{s}\times\frac{0.01}{1\times 10^{-12}}\times\frac{m}{m^4}\times 1.0\times 10^{-3}\ Pa.s = 0.00424\ Pa$$

Similarly, the required pressure drop for other values of R can be obtained.

The hydraulic resistance $R_{hyd} = \dfrac{\Delta p}{Q} = \dfrac{8\mu L}{\pi R^4}$. Substituting the values of the viscosity, length and radius of the tube, one obtains the value of the hydraulic resistance.

R **(mm)**	***Δp***	$\boldsymbol{R_{hyd}}$ **(Pa.s/m^3)**
1	0.00424 Pa	2.55×10^{7}
10^{-1}	42.4 Pa	2.55×10^{11}
10^{-2}	424 kPa	2.55×10^{15}
10^{-3}	4.24 GPa	2.55×10^{19}
10^{-4}	4.24×10^{4} GPa	2.55×10^{23}

(3) To achieve a reasonable flow rate in microscale or nanoscale channel, a very high pressure difference is required since Δp is proportional to R^{-4}. Therefore, the widely used pressure-driven flow in large scale systems is not appropriate in microscale or nanoscale channel applications. Other means to manipulate fluids in microscale or nanoscale channel applications are required.

Problem 8.59

[Difficulty: 2]

8.59 The figure schematically depicts a conical diffuser, which is designed to increase pressure and reduce kinetic energy. We assume the angle α is small ($\alpha < 10°$) so that $\tan\alpha \approx \alpha$ and $r_e = r_i + \alpha l$, where r_i is the radius at the diffuser inlet, r_e is the radius at the exit, and l is the length of the diffuser. The flow in a diffuser is complex, but here we assume that each layer of fluid in the diffuser flow is laminar, as in a cylindrical tube with constant cross-sectional area. Based on reasoning similar to that in Section 8.3, the pressure difference Δp between the ends of a cylindrical pipe is

$$\Delta p = \frac{8\mu}{\pi} Q \int_0^x \frac{1}{r^4} dx$$

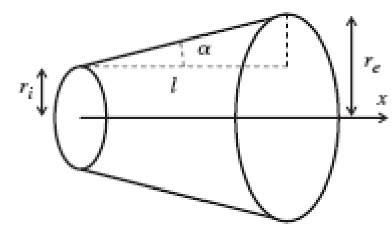
where x is the location in the diffuser, μ is the fluid dynamic viscosity, and Q is the flow rate. The equation above is applicable to flows in a diffuser assuming that the inertial force and exit effects are negligible. Derive the hydraulic resistance, $R_{hyd} = \Delta p/Q$, of the diffuser.

Given: Definition of hydraulic resistance

Find: Hydraulic resistance in a diffuser

Solution:

Basic equation: $R_{hyd} = \dfrac{\Delta p}{Q}$



$$R_{hyd} = \frac{\Delta p}{Q} = \frac{8\mu}{\pi}\int_{z_1}^{z_2} \frac{1}{r^4} dz = \frac{8\mu}{\pi}\int_{z_1}^{z_2} \frac{1}{(r_i + \alpha z)^4} dz$$

$$= \frac{8\mu}{\pi}\int_0^z \frac{1}{(r_i + \alpha z)^4} d(r_i + \alpha z)$$

$$= -\frac{8\mu}{\pi\alpha}\frac{1}{3}(r_i + \alpha z)^{-3}\Big|_0^z = -\frac{8\mu}{3\pi\alpha}[(r_i + \alpha z)^{-3} - r_i^{-3}]$$

$$R_{hyd} = -\frac{8\mu}{3\pi\alpha}\left[\frac{1}{(r_i + \alpha z)^3} - \frac{1}{r_i^3}\right]$$

Problem 8.60

[Difficulty: 4]

8.60 Consider blood flow in an artery. Blood is non-Newtonian; the shear stress versus shear rate is described by the Casson relationship:

$$\begin{cases} \sqrt{\tau} = \sqrt{\tau_c} + \sqrt{\mu \dfrac{du}{dr}}, & \text{for } \tau \geq \tau_c \\ \tau = 0 & \text{for } \tau < \tau_c \end{cases}$$

where τ_c is the critical shear stress, and μ is a constant having the same dimensions as dynamic viscosity. The Casson relationship shows a linear relationship between $\sqrt{\tau}$ and $\sqrt{du/dr}$, with intercept $\sqrt{\tau_c}$ and slope $\sqrt{\mu}$. The Casson relationship approaches Newtonian behavior at high values of deformation rate. Derive the velocity profile of steady fully developed blood flow in an artery of radius R. Determine the flow rate in the blood vessel. Calculate the flow rate due to a pressure gradient $dp/dx = -100$ Pa/m, in an artery of radius $R = 1$ mm, using the following blood data: $\mu = 3.5$ cP, $\tau_c = 0.05$ dynes/cm^2.

Given: Relationship between shear stress and deformation rate; fully developed flow in a cylindrical blood vessel

Find: Velocity profile; flow rate

Solution:

Similar to the Example Problem described in Section 8.3, based on the force balance, one obtains

$$\tau_{rx} = \frac{r}{2}\frac{dp}{dx} \qquad (1)$$

This result is valid for all types of fluids, since it is based on a simple force balance without any assumptions about fluid rheology. Since the axial pressure gradient in a steady fully developed flow is a constant, Equation (1) shows that $\tau = 0 < \tau_c$ at $r = 0$. Therefore, there must be a small region near the center line of the blood vessel for which $\tau < \tau_c$. If we call R_c the radial location at which $\tau = \tau_c$, the flow can then be divided into two regions:

$r > R_c$: The shear stress vs. shear rate is governed by

$$\sqrt{\tau} = \sqrt{\tau_c} + \sqrt{\mu \frac{du}{dr}} \qquad (2)$$

$r < R_c$: $\tau = 0 < \tau_c$.

We first determine the velocity profile in the region $r > R_c$. Substituting (1) into (2), one obtains:

$$\sqrt{\frac{r}{2}\frac{dp}{dx}} = \sqrt{\tau_c} + \sqrt{\mu \frac{du}{dr}} \qquad (3)$$

Using equation (3) and the fact that du/dr at $r = R_c$ is zero, the critical shear stress can be written as

$$\sqrt{\frac{R_c}{2}\frac{dp}{dx}} = \sqrt{\tau_c}\ . \qquad (4)$$

Rearranging eq. (4), R_c is

$$R_c = 2\tau_c / \frac{dp}{dx}. \qquad (5)$$

Inserting (4) into (3), rearranging, and squaring both sides, one obtains

$$\mu\frac{du}{dr}=\frac{1}{2}\frac{dp}{dx}[r-2\sqrt{rR_c}+R_c] \qquad (6)$$

Integrating the above first-order differential equation using the non-slip boundary condition, $u = 0$ at $r = R$:

$$u=-\frac{1}{4\mu}\frac{dp}{dx}\left[(R^2-r^2)-\frac{8}{3}\sqrt{R_c}(R_c^{\ 3/2}-r^{3/2})+2R_c(R-r)\right] \text{ for } R_c\le r\le R \qquad (7)$$

In the region $r < R_c$, since the shear stress is zero, fluid travels as a plug with a plug velocity. Since the plug velocity must match the velocity at $r = R_c$, we set $r = R_c$ in equation (7) to obtain the plug velocity:

$$u=-\frac{1}{4\mu}\frac{dp}{dx}\left[(R^2-R_c^{\ 2})+2R_c(R-R_c)\right] \text{ for } r\le R_c \qquad (8)$$

The flow rate is obtained by integrating *u(r)* across the vessel cross section:

$$Q=\int_0^R u(r)2\pi r dr=\int_0^{R_c} u(r)2\pi r dr+\int_{R_c}^R u(r)2\pi r dr$$
$$=-\frac{\pi R^4}{8\mu}\frac{dp}{dx}\left[1-\frac{16}{7}\sqrt{\frac{R_c}{R}}+\frac{4}{3}\frac{R_c}{R}-\frac{1}{21}\left(\frac{R_c}{R}\right)^4\right] \qquad (9)$$

Given $R = 1\text{mm} = 10^{-3}$ m, $\mu = 3.5$ cP $= 3.5\times10^{-3}$ Pa·s, and $\tau_c = 0.05$ dynes/cm$^2 = 0.05\times10^{-1}$ Pa, and $\frac{dp}{dx}=-100 Pa/m$.

From eq. (5), $R_c=2\tau_c/\frac{dp}{dx}$

$$R_c=\frac{2\times0.05}{100}\frac{10\times10^{-6}N/m^2}{Pa/m}=0.1mm$$

Substituting the values of R, μ, R_c, and $\frac{dp}{dx}$ into eq. (9),

$$Q=-\frac{\pi}{8}\frac{1\times10^{-12}}{3.5\times10^{-3}}\frac{m^4}{Pa.s}\times(-100)Pa/m\times[1-\frac{16}{7}\sqrt{\frac{0.1}{1}\frac{mm}{mm}}+\frac{4}{3}\frac{0.1}{1}\frac{mm}{mm}-\frac{1}{21}(\frac{0.1}{1}\frac{mm}{mm})^4]=3.226\times10^{-9}m^3/s$$

Problem 8.61

[Difficulty: 4]

8.61 Using Eq. 2.16, derive the velocity profile, flow rate, and average velocity of a non-Newtonian fluid in a circular tube. For a flow rate of $Q = 1\ \mu$L/min and $R = 1$ mm, with k having a value of unity in standard SI units, compare the required pressure gradients for $n = 0.5$, 1.0, and 1.5. Which fluid requires the smallest pump for the same pipe length?

Given: Fully developed flow, Navier-Stokes equations; Non-Newtonian fluid

Find: Velocity profile, flow rate and average velocity

Solution:

According to equation (8.10), we can write the governing equation for Non-Newtonian fluid velocity in a circular tube

$$\tau_{rx} = k\left(\frac{du}{dr}\right)^n = \frac{r}{2}\frac{\partial p}{\partial x} + \frac{c_1}{r} \tag{1}$$

However, as for the Newtonian fluid case, we must set $c_1 = 0$ as otherwise we'd have infinite stress at $r = 0$. Hence, equation (1) becomes

$$k\left(\frac{du}{dr}\right)^n = \frac{r}{2}\frac{\partial p}{\partial x} \tag{2}$$

The general solution for equation (2), obtained by integrating, is given by

$$u = \left(\frac{1}{2k}\frac{\partial p}{\partial x}\right)^{\frac{1}{n}} \frac{1}{\left(1+\frac{1}{n}\right)} r^{1+\frac{1}{n}} + c_2 \tag{3}$$

Apply the no slip boundary condition at $r = R$ into equation (3), we get

$$c_2 = -\left(\frac{1}{2k}\frac{\partial p}{\partial x}\right)^{\frac{1}{n}} \frac{1}{\left(1+\frac{1}{n}\right)} R^{1+\frac{1}{n}} \tag{4}$$

The fluid velocity then is given as

$$u(r) = \frac{n}{(n+1)}\left(\frac{1}{2k}\frac{\partial p}{\partial x}\right)^{\frac{1}{n}}\left(r^{\frac{n+1}{n}} - R^{\frac{n+1}{n}}\right) \tag{5}$$

The volume flow rate is

$$Q = \int_A u\,dA == \int_0^R 2\pi r \frac{n}{(n+1)}\left(\frac{1}{2k}\frac{\partial p}{\partial x}\right)^{\frac{1}{n}}\left(r^{\frac{n+1}{n}} - R^{\frac{n+1}{n}}\right)dr \tag{6}$$

Hence

$$Q=\frac{2n\pi}{(n+1)}\left(\frac{1}{2k}\frac{\partial p}{\partial x}\right)^{\frac{1}{n}}\left[\frac{n}{3n+1}r^{\frac{3n+1}{n}}-\frac{r^2}{2}R^{\frac{n+1}{n}}\right]_0^R=\frac{2n\pi}{(n+1)}\left(\frac{1}{2k}\frac{\partial p}{\partial x}\right)^{\frac{1}{n}}R^{\frac{3n+1}{n}}\left(\frac{n}{3n+1}-\frac{1}{2}\right) \tag{7}$$

Simplifying

$$Q=-\frac{n\pi}{(3n+1)}\left(\frac{1}{2k}\frac{\partial p}{\partial x}\right)^{\frac{1}{n}}R^{\frac{3n+1}{n}} \tag{8}$$

When $n = 1$, then $k = \mu$, and $Q=-\frac{\pi a^4}{8\mu}\frac{\partial p}{\partial x}$, just like equation (8.13b) in the textbook.

The average velocity is given by

$$\bar{V}=\frac{Q}{A}=\frac{Q}{\pi R^2}=-\frac{n}{(3n+1)}\left(\frac{1}{2k}\frac{\partial p}{\partial x}\right)^{\frac{1}{n}}R^{\frac{n+1}{n}} \tag{9}$$

Based on Eq. (7), the pressure gradient is

$$\frac{\partial p}{\partial x}=-2k\left[\frac{Q(3n+1)}{n\pi R^{\frac{3n+1}{n}}}\right]^n \tag{10}$$

Substituting $Q = 1\mu\text{L/min} = 1\times 10^{-9}/60\ \text{m}^3/\text{s}$, $R = 1\text{mm} = 10^{-3}$ m, and $n = 0.5$ into eq.(10):

$$\frac{\partial p}{\partial x}=-2k\left[\frac{\frac{10^{-9}}{60}(3(0.5)+1)}{0.5\pi^{\frac{3(0.5)+1}{0.5}}}\right]^{0.5}=-325k\ \text{Pa/m}\quad \text{for } n=0.5 \tag{11}$$

Similarly, the required pressure gradients for $n = 1$ and $n = 1.5$ can be obtained:

$$\frac{\partial p}{\partial x}=-42.4k\ \text{Pa/m, for } n=1 \tag{12}$$

$$\frac{\partial p}{\partial x}\approx-5.42k\ \text{Pa/m, for } n=1.5 \tag{12}$$

Obviously, the magnitude of the required pressure gradient increases as n decreases. Among the three types of fluids (pseudoplastic for $n = 0.5$, Newtonian for $n = 1$, and dilatant for $n = 1.5$), the dilatant fluid requires the lowest pressure pump for the same pipe length.

Problem 8.62

[Difficulty: 2]

8.62 The classic Poiseuille flow (Eq. 8.12), is for no-slip conditions at the walls. If the fluid is a gas, and when the mean free path, l (the average distance a molecule travels before collision with another molecule), is comparable to the length-scale L of the flow, slip will occur at the walls, and the flow rate and velocity will be increased for a given pressure gradient. In Eq. 8.11, c_1 will still be zero, but c_2 must satisfy the slip condition $u = l\,\partial u/\partial r$ at $r = R$. Derive the velocity profile and flow rate of gas flow in a micro- or nanotube which has such a slip velocity on the wall. Calculate the flow rate when $R = 10\,\text{m}$, $\mu = 1.84 \times 10^{-5}\ \text{N}\cdot\text{s/m}^2$, the mean free path $l = 68$ nm, and $-\partial p/\partial x = 1.0 \times 10^6$ Pa/m.

Given: Fully developed flow in a pipe; slip boundary condition on the wall

Find: Velocity profile and flow rate

Solution:

Similar to the example described in Section 8.3, one obtained

$$u = \frac{r^2}{4\mu}\frac{\partial p}{\partial x} + c_2 \qquad (1)$$

The constant c_2 will be determined by the slip velocity boundary condition at $r = R$:

$$u = l\frac{\partial u}{\partial r} \qquad (2)$$

and one obtains

$$c_2 = \frac{R^2}{4\mu}\frac{\partial p}{\partial x}\left(2\frac{l}{R} - 1\right) \qquad (3)$$

Substituting c_2 into Eq.(1), one obtains

$$u = -\frac{1}{4\mu}\frac{\partial p}{\partial x}\left(R^2 - r^2 + 2lR\right) \qquad (4)$$

The volume flow rate is

$$Q = \int_0^R u 2\pi r dr = -\frac{\pi R^4}{8\mu}\frac{\partial p}{\partial x}\left[1 + 4\frac{l}{R}\right] \qquad (5)$$

Substituting $R = 10$ μm, $\mu = 1.84 \times 10^{-5}$ N·s/m^2, mean free path $l = 68$ nm, and $-\dfrac{\partial p}{\partial x} = 1.0\times10^6$ Pa/m into eq. (5),

$$Q = -\frac{\pi}{8}\frac{(10\times10^{-6})^4}{1.84\times10^{-5}}\frac{\text{m}^4}{\text{Pa.s}}\times(-1.0\times10^6)\text{Pa/m}\times[1 + 4\frac{68\times10^{-9}}{10\times10^{-6}}\frac{\text{m}}{\text{m}}] = 2.19\times10^{-10}\,\text{m}^3/\text{s}\,.$$

Problem 8.63

[Difficulty: 3]

8.63 The following solution:

$$u = u_0\left(1 - \frac{y^2}{a^2} - \frac{z^2}{b^2}\right)$$

can be used as a model for the velocity profile of fully developed pressure-driven flow in a channel with an elliptic cross section. The center of the ellipse is $(y, z) = (0, 0)$, and the major axis of length a and the minor axis of length b are parallel to the y axis and z axes, respectively. The axial pressure gradient, $\partial p/\partial x$, is constant. Based on the Navier–Stokes equations, determine the maximum velocity u_0 in terms of a, b, viscosity μ, and $\partial p/\partial x$. Letting (ρ, ϕ) be the radial and azimuthal polar coordinates, respectively, of a unit disk ($0 \le \rho \le 1$ and $0 \le \phi \le 2\pi$), the coordinates (y, z) and the velocity $u(y, z)$ can be expressed as functions of (ρ, ϕ):

$$y(\rho,\phi) = a\rho\cos\phi \qquad z(\rho,\phi) = b\rho\sin\phi \qquad u(\rho,\phi) = u_0(1-\rho^2)$$

The flow rate is $Q = \int u(y,z)dydz = ab\int_0^{2\pi}\int_0^1 \rho u(\rho,\phi)d\rho d\phi$. Derive the flow rate of fully developed pressure-driven flow in an elliptic pipe. Compare the flow rates in a channel with an elliptic cross section with $a = 1.5R$ and $b = R$ and in a pipe of radius R with the same pressure gradient.

Given: Fully developed flow, velocity profile, and expression to calculate the flow rate

Find: Velocity and flow rate

Solution:

For the fully developed flow, the N-S equations can be simplified to

$$\mu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \frac{\partial p}{\partial x} = \text{constant} \qquad (1)$$

Substituting the trial solution in equation (1), one obtains

$$-2\mu u_0\left(\frac{1}{a^2} + \frac{1}{b^2}\right) = \frac{\partial p}{\partial x} \qquad (2)$$

Rearrange it and one obtains

$$u_0 = -\frac{a^2b^2}{2\mu(a^2+b^2)}\frac{\partial p}{\partial x} \qquad (3)$$

The flow rate is

$$Q = \int u(y,z)dydz = ab\int_0^{2\pi}\int_0^1 \rho u(\rho,\phi)d\rho d\phi \qquad (4)$$

Substituting $u(\rho,\phi) = u_0(1-\rho^2)$ into Eq. (4) and integrating twice:

$$Q = ab\int_0^{2\pi}\int_0^1 \rho u_0(1-\rho^2)d\rho d\phi = \frac{1}{2}\pi abu_0 \qquad (5)$$

Substituting u_0 into (5), one obtains

$$Q = \frac{1}{2}\pi abu_0 = -\frac{\pi a^3b^3}{4\mu(a^2+b^2)}\frac{\partial p}{\partial x} \qquad (6)$$

For a pipe radius R, $a = b = R$, from equation (6),

$$Q_{pipe} = \frac{1}{8}\left(-\frac{\pi R^4}{\mu}\frac{\partial p}{\partial x}\right)$$

which is the same as equation (8.13b) in the book.

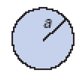
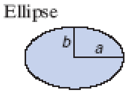
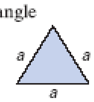
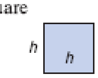
For a channel with an elliptic cross-section with $a = R$ and $b = 1.5R$, from equation (6), one has

$$Q_{pipe} = \frac{29}{104}\left(-\frac{\pi R^4}{\mu}\frac{\partial p}{\partial x}\right).$$

Problem 8.64

[Difficulty: 2]

8.64 For pressure-driven, steady, fully developed laminar flow of an incompressible fluid through a straight channel of length L, we can define the hydraulic resistance as $R_{\text{hyd}} = \Delta p/Q$, where Δp is the pressure drop and Q is the flow rate (analogous to the electrical resistance $R_{\text{elec}} = \Delta V/I$, where ΔV is the electrical potential drop and I is the electric current). The following table summarizes the hydraulic resistance of channels with different cross sectional shapes [30]:

Shape	Formula for R_{hyd}	Computed R_{hyd}
Circle 	$\dfrac{8\mu L}{\pi a^4}$	
Ellipse 	$\dfrac{4\mu L[1+(b/a)^2]}{\pi a b^3}$	
Triangle 	$\dfrac{320\mu L}{\sqrt{3}a^4}$	
Two plates 	$\dfrac{12\mu L}{h^3 w}$	
Rectangle 	$\dfrac{12\mu L}{h^3 w[1-0.63(h/w)]}$	
Square 	$\dfrac{12\mu L}{0.37h^4}$	

Calculate the hydraulic resistance of a straight channel with the listed cross-sectional shapes using the following parameters: $\mu = 1$ mPa·s (water), $L = 10$ mm, $a = 100$ μm, $b = 33$ μm, $h = 100$ μm, and $w = 300$ μm. Based on the calculated hydraulic resistance, which shape is the most energy efficient to pump water?

Given:

The expression of hydraulic resistance of straight channels with different cross sectional shapes

Find:

Hydraulic resistance

Solution:

Based on the expressions of hydraulic resistance listed in the table, one obtains

Using the circle as the example,

$$R_{hyd} = \frac{8}{\pi}\mu L \frac{1}{a^4} = \frac{8}{\pi}\frac{1\times 10^{-3}\times 10\times 10^{-3}}{(1\times 10^{-4})^4}\frac{\text{Pa}\cdot\text{s}\times\text{m}}{\text{m}^4} = 0.254\times 10^{12}\ \text{Pa}\cdot\text{s/m}^3$$

The results are

Shape	R_{hyd} (10^{12} Pa s/m^3)
Circle	0.25
Ellipse	3.93
Triangle	18.48
Two plates	0.40
Rectangle	0.51
Square	3.24

Comparing the values of the hydraulic resistances, a straight channel with a circular cross section is the most energy efficient to pump fluid with a fixed volumetric flow rate; the triangle is the worst.

Problem 8.65

[Difficulty: 3]

8.65 In a food industry plant, two immiscible fluids are pumped through a tube such that fluid 1 ($\mu_1 = 0.5\ \text{N}\cdot\text{s/m}^2$) forms an inner core and fluid 2 ($\mu_2 = 5\ \text{N}\cdot\text{s/m}^2$) forms an outer annulus. The tube has $D = 5$ mm diameter and length $L = 5$ m. Derive and plot the velocity distribution if the applied pressure difference, Δp, is 5 MPa.

Given: Two-fluid flow in tube

Find: Velocity distribution; Plot

Solution:

Given data $\quad D = 5\cdot\text{mm} \quad L = 5\cdot\text{m} \quad \Delta p = -5\cdot\text{MPa} \quad \mu_1 = 0.5\cdot\frac{\text{N}\cdot\text{s}}{\text{m}^2} \quad \mu_2 = 5\cdot\frac{\text{N}\cdot\text{s}}{\text{m}^2}$

From Section 8-3 for flow in a pipe, Eq. 8.11 can be applied to either fluid

$$u = \frac{r^2}{4\cdot\mu}\cdot\left(\frac{\partial}{\partial x}p\right) + \frac{c_1}{\mu}\cdot\ln(r) + c_2$$

Applying this to fluid 1 (inner fluid) and fluid 2 (outer fluid)

$$u_1 = \frac{r^2}{4\cdot\mu_1}\cdot\frac{\Delta p}{L} + \frac{c_1}{\mu_1}\cdot\ln(r) + c_2 \qquad u_2 = \frac{r^2}{4\cdot\mu_2}\cdot\frac{\Delta p}{L} + \frac{c_3}{\mu_2}\cdot\ln(r) + c_4$$

We need four BCs. Two are obvious $\quad r = \frac{D}{2} \quad u_2 = 0 \quad (1) \qquad r = \frac{D}{4} \quad u_1 = u_2 \quad (2)$

The third BC comes from the fact that the axis is a line of symmetry

$$r = 0 \qquad \frac{du_1}{dr} = 0 \qquad (3)$$

The fourth BC comes from the fact that the stress at the interface generated by each fluid is the same

$$r = \frac{D}{4} \qquad \mu_1\cdot\frac{du_1}{dr} = \mu_2\cdot\frac{du_2}{dr} \qquad (4)$$

Using these four BCs

$$\frac{\left(\frac{D}{2}\right)^2}{4\cdot\mu_2}\cdot\frac{\Delta p}{L} + \frac{c_3}{\mu_2}\cdot\ln\left(\frac{D}{2}\right) + c_4 = 0 \qquad \frac{\left(\frac{D}{4}\right)^2}{4\cdot\mu_1}\cdot\frac{\Delta p}{L} + \frac{c_1}{\mu_1}\cdot\ln\left(\frac{D}{4}\right) + c_2 = \frac{\left(\frac{D}{4}\right)^2}{4\cdot\mu_2}\cdot\frac{\Delta p}{L} + \frac{c_3}{\mu_2}\cdot\ln\left(\frac{D}{4}\right) + c_4$$

$$\lim_{r \to 0} \frac{c_1}{\mu_1\cdot r} = 0 \qquad \frac{D}{8}\cdot\frac{\Delta p}{L} + \frac{4\cdot c_1}{D} = \frac{D}{8}\cdot\frac{\Delta p}{L} + \frac{4\cdot c_3}{D}$$

Hence, after some algebra

$$c_1 = 0 \quad \text{(To avoid singularity)} \qquad c_2 = -\frac{D^2\cdot\Delta p}{64\cdot L}\frac{\left(\mu_2 + 3\cdot\mu_1\right)}{\mu_1\cdot\mu_2} \qquad c_3 = 0 \qquad c_4 = -\frac{D^2\cdot\Delta p}{16\cdot L\cdot\mu_2}$$

The velocity distributions are then

$$u_1(r) = \frac{\Delta p}{4 \cdot \mu_1 \cdot L} \cdot \left[r^2 - \left(\frac{D}{2}\right)^2 \cdot \frac{(\mu_2 + 3 \cdot \mu_1)}{4 \cdot \mu_2} \right] \qquad u_2(r) = \frac{\Delta p}{4 \cdot \mu_2 \cdot L} \cdot \left[r^2 - \left(\frac{D}{2}\right)^2 \right]$$

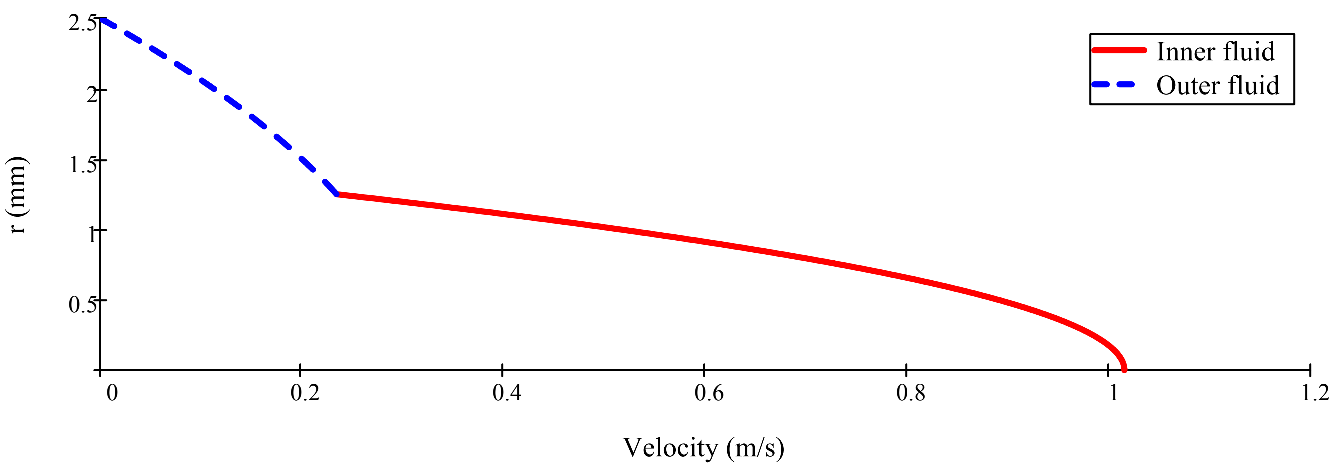
(Note that these result in the same expression if $\mu_1 = \mu_2$, i.e., if we have one fluid)

Evaluating either velocity at $r = D/4$ gives the velocity at the interface

$$u_{interface} = -\frac{3 \cdot D^2 \cdot \Delta p}{64 \cdot \mu_2 \cdot L} \qquad u_{interface} = -\frac{3}{64} \times (0.005 \cdot m)^2 \times \left(-5 \times 10^6 \cdot \frac{N}{m^2}\right) \times \frac{m^2}{5 \cdot N \cdot s} \times \frac{1}{5 \cdot m} \qquad u_{interface} = 0.234 \frac{m}{s}$$

Evaluating u_1 at $r = 0$ gives the maximum velocity

$$u_{max} = -\frac{D^2 \cdot \Delta p \cdot (\mu_2 + 3 \cdot \mu_1)}{64 \cdot \mu_1 \cdot \mu_2 \cdot L} \quad u_{max} = -\frac{1}{64} \times (0.005 \cdot m)^2 \times \left(-5 \times 10^6 \cdot \frac{N}{m^2}\right) \times \frac{5 + 3 \times 0.5}{5 \times .5} \cdot \frac{m^2}{N \cdot s} \times \frac{1}{5 \cdot m} \quad u_{max} = 1.02 \frac{m}{s}$$



The velocity distributions can be plotted in *Excel*

Problem 8.66

[Difficulty: 2]

8.66 A horizontal pipe carries fluid in fully developed turbulent flow. The static pressure difference measured between two sections is 750 psi. The distance between the sections is 15 ft, and the pipe diameter is 3 in. Calculate the shear stress, τ_w, that acts on the walls.

Given: Turbulent pipe flow

Find: Wall shear stress

Solution:

Basic equation
$$F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\, d\mathcal{V} + \int_{CS} u\,\rho\,\vec{V}\cdot d\vec{A} \qquad \text{(Eq. 4.18a)}$$

Assumptions 1) Horizontal pipe 2) Steady flow 3) Fully developed flow

With these assumptions the x momentum equation becomes

$$p_1\cdot\frac{\pi\cdot D^2}{4} + \tau_w\cdot\pi\cdot D\cdot L - p_2\cdot\frac{\pi\cdot D^2}{4} = 0 \qquad \text{or} \qquad \tau_w = \frac{(p_2 - p_1)\cdot D}{4\cdot L} = -\frac{\Delta p\cdot D}{4\cdot L}$$

$$\tau_w = -\frac{1}{4}\times 750\cdot\text{psi}\times\frac{\frac{3}{12}}{15} \qquad \tau_w = -3.13\cdot\text{psi}$$

Since τ_w is negative it acts to the left on the fluid, to the right on the pipe wall

Problem 8.67

[Difficulty: 2]

8.67 One end of a horizontal pipe is attached using glue to a pressurized tank containing liquid, and the other has a cap attached. The inside diameter of the pipe is 3 in., and the tank pressure is 30 psig. Find the force the glue must withstand with the cap on, and the force it must withstand when the cap is off and the liquid is discharging to atmosphere.

Given: Pipe glued to tank

Find: Force glue must hold when cap is on and off

Solution:

Basic equation $$F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\,dV\!\!\!\!-\, + \int_{CS} u\,\rho\,\vec{V}\cdot d\vec{A}$$ (Eq. 4.18a)

First solve when the cap is on. In this static case

$$F_{glue} = \frac{\pi \cdot D^2}{4}\cdot p_1$$ where p_1 is the tank pressure

Second, solve for when flow is occuring:

Assumptions 1) Horizontal pipe 2) Steady flow 3) Fully developed flow

With these assumptions the x momentum equation becomes

$$p_1 \cdot \frac{\pi \cdot D^2}{4} + \tau_w \cdot \pi \cdot D \cdot L - p_2 \cdot \frac{\pi \cdot D^2}{4} = 0$$

Here p_1 is again the tank pressure and p_2 is the pressure at the pipe exit; the pipe exit pressure is p_{atm} = 0 kPa gage. Hence

$$F_{pipe} = F_{glue} = -\tau_w \cdot \pi \cdot D \cdot L = \frac{\pi \cdot D^2}{4}\cdot p_1$$

We conclude that in each case the force on the glue is the same! When the cap is on the glue has to withstand the tank pressure; when the cap is off, the glue has to hold the pipe in place against the friction of the fluid on the pipe, which is equal in magnitude to the pressure drop.

$$F_{glue} = \frac{\pi}{4} \times (3 \cdot in)^2 \times 30 \cdot \frac{lbf}{in^2}$$ $$F_{glue} = 212 \cdot lbf$$

Problem 8.68

[Difficulty: 2]

8.68 Kerosene is pumped through a smooth tube with inside diameter $D = 30$ mm at close to the critical Reynolds number. The flow is unstable and fluctuates between laminar and turbulent states, causing the pressure gradient to intermittently change from approximately -4.5 kPa/m to -11 kPa/m. Which pressure gradient corresponds to laminar, and which to turbulent, flow? For each flow, compute the shear stress at the tube wall, and sketch the shear stress distributions.

Given: Data on pressure drops in flow in a tube

Find: Which pressure drop is laminar flow, which turbulent

Solution:

Given data $\quad \frac{\partial}{\partial x}p_1 = -4.5\cdot\frac{\text{kPa}}{\text{m}} \quad \frac{\partial}{\partial x}p_2 = -11\cdot\frac{\text{kPa}}{\text{m}} \quad D = 30\cdot\text{mm}$

From Section 8-4, a force balance on a section of fluid leads to

$$\tau_w = -\frac{R}{2}\cdot\frac{\partial}{\partial x}p = -\frac{D}{4}\cdot\frac{\partial}{\partial x}p$$

Hence for the two cases

$$\tau_{w1} = -\frac{D}{4}\cdot\frac{\partial}{\partial x}p_1 \qquad \tau_{w1} = 33.8\,\text{Pa}$$

$$\tau_{w2} = -\frac{D}{4}\cdot\frac{\partial}{\partial x}p_2 \qquad \tau_{w2} = 82.5\,\text{Pa}$$

Because both flows are at the same nominal flow rate, the higher pressure drop must correspond to the turbulent flow, because, as indicated in Section 8-4, turbulent flows experience additional stresses. Also indicated in Section 8-4 is that for both flows the shear stress varies from zero at the centerline to the maximums computed above at the walls.

The stress distributions are linear in both cases: Maximum at the walls and zero at the centerline.

Problem 8.69

[Difficulty: 2]

8.69 The pressure drop between two taps separated in the streamwise direction by 30 ft in a horizontal, fully developed channel flow of water is 1 psi. The cross section of the channel is a $1 \text{ in.} \times 9\frac{1}{2}$ in. rectangle. Calculate the average wall shear stress.

Given: Flow through channel

Find: Average wall stress

Solution:

Basic equation $$F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\,d\forall + \int_{CS} u\,\rho\,\vec{V}\cdot d\vec{A}$$ (Eq. 4.18a)

Assumptions 1) Horizontal pipe 2) Steady flow 3) Fully developed flow

With these assumptions the x momentum equation becomes

$$p_1 \cdot W \cdot H + \tau_w \cdot 2 \cdot L \cdot (W + H) - p_2 \cdot W \cdot H = 0 \qquad \text{or} \qquad \tau_w = (p_2 - p_1) \cdot \frac{W \cdot H}{2 \cdot (W + H) \cdot L} \qquad \tau_w = -\Delta p \cdot \frac{\frac{H}{L}}{2 \cdot \left(1 + \frac{H}{W}\right)}$$

$$\tau_w = -\frac{1}{2} \times 1 \cdot \frac{\text{lbf}}{\text{in}^2} \times \frac{144 \cdot \text{in}^2}{\text{ft}^2} \times \frac{1 \cdot \text{in} \times \frac{1 \cdot \text{ft}}{12 \cdot \text{in}}}{30 \cdot \text{ft}} \times \left(\frac{1}{1 + \frac{9.5 \cdot \text{in} \times \frac{1 \cdot \text{ft}}{12 \cdot \text{in}}}{30 \cdot \text{ft}}} \right) \qquad \tau_w = -0.195 \cdot \frac{\text{lbf}}{\text{ft}^2} \qquad \tau_w = -1.35 \times 10^{-3} \cdot \text{psi}$$

Since $\tau_w < 0$, it acts to the left on the fluid, to the right on the channel wall

8.70 A liquid drug, with the viscosity and density of water, is to be administered through a hypodermic needle. The inside diameter of the needle is 0.25 mm and its length is 50 mm. Determine (a) the maximum volume flow rate for which the flow will be laminar, (b) the pressure drop required to deliver the maximum flow rate, and (c) the corresponding wall shear stress.

Solution: Flow will be laminar for $Re < 2300$.

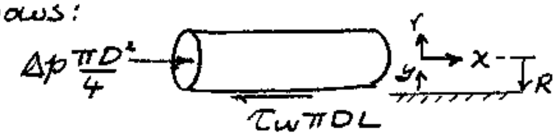
$$Re = \frac{\rho \bar{V} D}{\mu} = \frac{\bar{V} D}{\nu} = \frac{Q}{A}\frac{D}{\nu} = \frac{4Q}{\pi D^2}\frac{D}{\nu} = \frac{4Q}{\pi \nu D} < 2300$$

Thus (at $T = 20°C$)

$$Q < \frac{2300 \pi \nu D}{4} = \frac{2300\pi}{4} \times 1.0\times10^{-6}\frac{m^2}{s} \times 0.00025\,m = 4.52\times10^{-7}\,m^3/s$$ Q

(This flow rate corresponds to 27.1 mL/min.)

A force balance on a fluid element shows:



$$\sum F_x = \Delta p \frac{\pi D^2}{4} - \tau_w \pi D L = 0$$

or

$$\Delta p = \tau_w \frac{4L}{D}$$

For laminar pipe flow, $u = u_{max}\left[1 - \left(\frac{r}{R}\right)^2\right]$, from Eq. 8.14. Thus

$$\tau_w = \mu \left.\frac{\partial u}{\partial y}\right)_{y=0} = -\mu \left.\frac{\partial u}{\partial r}\right)_{r=R} = -\mu u_{max}\left(-\frac{2r}{R^2}\right)_{r=R} = \frac{2\mu u_{max}}{R}$$

But $u_{max} = 2\bar{V}$, so $\tau_w = \frac{2\mu 2\bar{V}}{D/2} = \frac{8\mu\bar{V}}{D} = \frac{8\rho\nu\bar{V}}{D}$

Also

$$\bar{V} = \frac{Q}{A} = \frac{4Q}{\pi D^2} = \frac{4}{\pi} \times 4.52\times10^{-7}\frac{m^3}{s} \times \frac{1}{(0.00025)^2 m^2} = 9.21\,m/s$$

Thus

$$\tau_w = 8 \times 999\frac{kg}{m^3} \times 1.0\times10^{-6}\frac{m^2}{s} \times 9.21\frac{m}{s} \times \frac{1}{0.00025\,m} \times \frac{N\cdot s^2}{kg\cdot m} = 294\,N/m^2\ (294\,Pa)$$ τ_w

and

$$\Delta p = 4 \times 0.05\,m \times \frac{1}{0.00025\,m} \times 294\frac{N}{m^2} = 235\,kPa$$ Δp

Problem 8.71

[Difficulty: 1]

8.71 The "pitch-drop" experiment has been running continuously at the University of Queensland since 1927 (http://www.physics.uq.edu.au/physics_museum/pitchdrop.shtml). In this experiment, a funnel pitch is being used to measure the viscosity of pitch. Flow averages at about one drop—*per decade!* Viscosity is calculated using the volume flow rate equation

$$Q = \frac{V\!\!\!/}{t} = \frac{\pi D^4 \rho g}{128\mu}\left(1 + \frac{h}{L}\right)$$

where D is the diameter of the flow from the funnel, h is the depth to the pitch in the main body of the funnel, L is the length of the funnel stem, and t is the elapsed time. Compare this equation with Eq. 8.13c using hydrostatic force instead of a pressure gradient. After the 6th drop in 1979, they measured that it took 17,708 days for 4.7×10^{-5} m^3 of pitch to fall. Given the measurements $D = 9.4$ mm, $h = 75$ mm, $L = 29$ mm, and $\rho_{\text{pitch}} = 1.1 \times 10^3$ kg/m^3, what is the viscosity of the pitch?

Given:

Data from a funnel viscometer filled with pitch.

Find:

Viscosity of pitch.

Solution:

Basic equation: $Q = \dfrac{V\!\!\!/}{t} = \dfrac{\pi D^4 \rho g}{128\mu}\left(1 + \dfrac{h}{L}\right)$ (Volume flow rate)

where Q is the volumetric flow rate, $V\!\!\!/$ flow volume, t is the time of flow, D is the diameter of the funnel stem, ρ is the density of the pitch, μ is the viscosity of the pitch, h is the depth of the pitch in the funnel body, and L is the length of the funnel stem.

Assumption: Viscous effects above the stem are negligible and the stem has a uniform diameter.

The given or available data is: $V\!\!\!/ = 4.7 \times 10^{-5}\,\text{m}^3$ $\quad t = 17{,}708\,\text{days}$ $\quad D = 9.4\,\text{mm}$

$h = 75\,\text{mm}$ $\quad L = 29\,\text{mm}$ $\quad \rho = 1.1 \times 10^3\,\dfrac{\text{kg}}{\text{m}^3}$

Calculate the flow rate:

$$Q = \frac{V\!\!\!/}{t} = \frac{4.7 \times 10^{-5}\,\text{m}^3}{17708\,\text{day} \times \dfrac{24\,\text{hour}}{\text{day}} \times \dfrac{3600\,s}{\text{hour}}} = 3.702 \times 10^{-14}\,\frac{\text{m}^3}{\text{s}}$$

Solve the governing equation for viscosity:

$$\mu = \frac{\pi D^4 \rho g}{128 Q}\left(1 + \frac{h}{L}\right)$$

$$\mu = \frac{\pi \times (9.4\,\text{mm})^4 \times \left(\dfrac{\text{m}}{1000\,\text{mm}}\right)^4 \times 1.1 \times 10^3\,\dfrac{\text{kg}}{\text{m}^3} \times 9.81\,\dfrac{\text{m}}{\text{s}^2}}{128 \times 3.702 \times 10^{-14}\,\dfrac{\text{m}^3}{\text{s}}}\left(1 + \frac{75\,\text{mm}}{29\,\text{mm}}\right) \times \frac{\text{N} \times \text{s}^2}{\text{kg} \times \text{m}}$$

$$\mu = 2.41 \times 10^8\,\frac{\text{N} \cdot \text{s}}{\text{m}^2}$$

Compare this to the viscosity of water, which is 10^{-3} N·s/m^2!

Relate this equation to 8.13c for flow driven by a pressure gradient:

$$Q = \frac{\pi \Delta p D^4}{128 \mu L} = \frac{\pi D^4}{128 \mu} \times \frac{\Delta p}{L} .$$

For this problem, the pressure (Δp) is replaced by the hydrostatic force of the pitch.
Consider the pressure variation in a static fluid.

$$\frac{dp}{dz} = -\rho g = -\rho g = \frac{\Delta p}{\Delta z} = \frac{\Delta p}{L+h} .$$

Replacing the term in 8.13c

$$Q = \frac{\pi D^4}{128 \mu} \times \frac{\Delta p}{L} = \frac{\pi D^4}{128 \mu} \times \frac{\rho g \times (L+h)}{L} = \frac{\pi D^4}{128 \mu} \times \rho g \times \left(1 + \frac{h}{L}\right)$$

Hence

$$Q = \frac{\forall}{t} = \frac{\pi D^4 \rho g}{128 \mu} \left(1 + \frac{h}{L}\right)$$

which is the same as the given equation.

8.72 Consider the empirical "power-law" profile for turbulent pipe flow, Eq. 8.22. For $n = 7$ determine the value of r/R at which u is equal to the average velocity, $\overline{V}$. Plot the results over the range $6 \le n \le 10$ and compare with the case of fully developed laminar pipe flow, Eq. 8.14.

Solution:

Definition: $\bar{V} = \frac{Q}{A} = \frac{1}{A}\int u\,dA$

For laminar flow, $\bar{V} = \frac{1}{\pi R^2}\int_0^R U\left[1-\left(\frac{r}{R}\right)^2\right]2\pi r\,dr = 2U\int_0^1\left[1-\left(\frac{r}{R}\right)^2\right]\frac{r}{R}\,d\left(\frac{r}{R}\right)$

$$\bar{V} = 2U\left[\frac{1}{2}\left(\frac{r}{R}\right)^2 - \frac{1}{4}\left(\frac{r}{R}\right)^4\right]_0^1 = \frac{U}{2}$$

Thus $u = \bar{V}$ when $1-\left(\frac{r}{R}\right)^2 = \frac{\bar{V}}{U} = \frac{1}{2}$ or $\frac{r}{R} = 0.707$ laminar

For turbulent flow, $\bar{V} = \frac{1}{\pi R^2}\int_0^R U\left(1-\frac{r}{R}\right)^{\frac{1}{n}} 2\pi r\,dr$

$$\bar{V} = 2U\int_0^1 \left(1-\frac{r}{R}\right)^{\frac{1}{n}} \frac{r}{R}\,d\left(\frac{r}{R}\right)$$

To integrate let $m = 1-\frac{r}{R}$. Then $\frac{r}{R} = 1-m$, $d\left(\frac{r}{R}\right) = -dm$

and

$$\bar{V} = 2U\int_1^0 m^{\frac{1}{n}}(1-m)(-dm) = 2U\int_0^1 \left(m^{\frac{1}{n}} - m^{1+\frac{1}{n}}\right)dm$$

$$= 2U\left[\frac{n}{(n+1)} m^{\frac{1}{n}+1} - \frac{n}{2n+1} m^{2+\frac{1}{n}}\right]_0^1 = 2U\left[\frac{n}{n+1} - \frac{n}{2n+1}\right]$$

$$\bar{V} = 2U\left[\frac{n(2n+1) - n(n+1)}{(n+1)(2n+1)}\right] = U\frac{2n^2}{(n+1)(2n+1)} \quad \text{(8.24)}$$

For $n = 7$, $\bar{V} = U\frac{2(7)^2}{8\times 15} = 0.817U$

Thus $u = \bar{V}$ when $\left(1-\frac{r}{R}\right)^{1/7} = 0.817$ or $\frac{r}{R} = 1-(0.817)^7 = 0.758$ turb

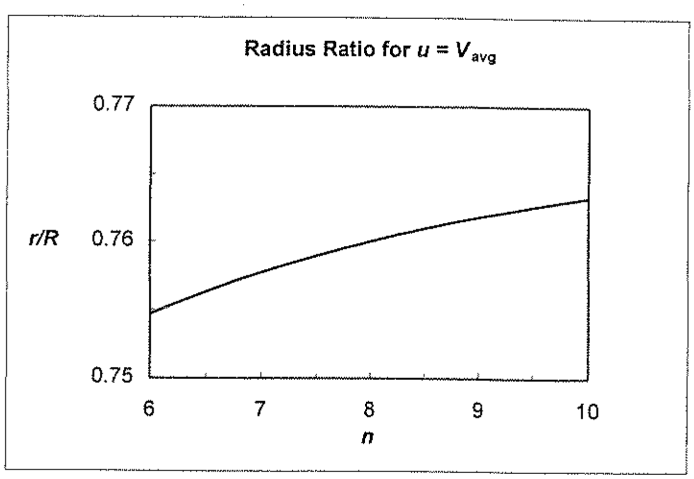
From Eq 8.24, $u = \bar{V}$ when

$$\left(1-\frac{r}{R}\right)^{\frac{1}{n}} = \frac{2n^2}{(n+1)(2n+1)}$$

or

$$\frac{r}{R} = 1-\left[\frac{2n^2}{(n+1)(2n+1)}\right]^n$$

r/R is plotted vs n.



Problem 8.73

[Difficulty: 3]

8.73 Laufer [5] measured the following data for mean velocity in fully developed turbulent pipe flow at $Re_U = 50{,}000$:

$\bar{u}/U$	0.996	0.981	0.963	0.937	0.907	0.866	0.831
y/r	0.898	0.794	0.691	0.588	0.486	0.383	0.280
$\bar{u}/U$	0.792	0.742	0.700	0.650	0.619	0.551	
y/R	0.216	0.154	0.093	0.062	0.041	0.024	

In addition, Laufer measured the following data for mean velocity in fully developed turbulent pipe flow at $Re_U = 500{,}000$:

$\bar{u}/U$	0.997	0.988	0.975	0.959	0.934	0.908
y/R	0.898	0.794	0.691	0.588	0.486	0.383
$\bar{u}/U$	0.874	0.847	0.818	0.771	0.736	0.690
y/R	0.280	0.216	0.154	0.093	0.062	0.037

Using *Excel*'s trendline analysis, fit each set of data to the "power-law" profile for turbulent flow, Eq. 8.22, and obtain a value of n for each set. Do the data tend to confirm the validity of Eq. 8.22? Plot the data and their corresponding trendlines on the same graph.

Given: Data on mean velocity in fully developed turbulent flow

Find: Trendlines for each set; values of n for each set; plot

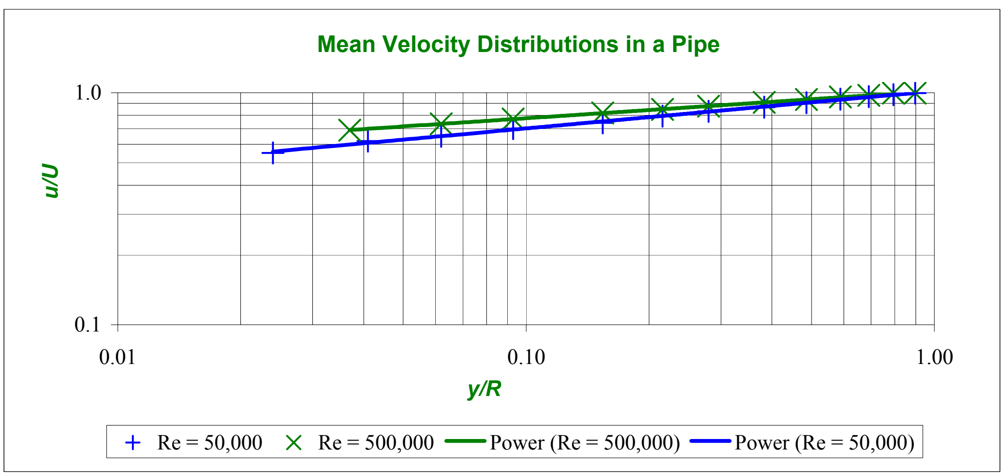
Solution:

y/R	*u/U*
0.898	0.996
0.794	0.981
0.691	0.963
0.588	0.937
0.486	0.907
0.383	0.866
0.280	0.831
0.216	0.792
0.154	0.742
0.093	0.700
0.062	0.650
0.041	0.619
0.024	0.551

y/R	*u/U*
0.898	0.997
0.794	0.998
0.691	0.975
0.588	0.959
0.486	0.934
0.383	0.908
0.280	0.874
0.216	0.847
0.154	0.818
0.093	0.771
0.062	0.736
0.037	0.690

Equation 8.22 is

$$\frac{\bar{u}}{U} = \left(\frac{y}{R}\right)^{1/n} = \left(1 - \frac{y}{R}\right)^{1/n}$$



Applying the *Trendline* analysis to each set of data:

At Re = 50,000

$u/U = 1.017(y/R)^{0.161}$

with $R^2 = 0.998$ (high confidence)

Hence $1/n = 0.161$

$n =$ **6.21**

At Re = 500,000

$u/U = 1.017(y/R)^{0.117}$

with $R^2 = 0.999$ (high confidence)

Hence $1/n = 0.117$

$n =$ **8.55**

Both sets of data tend to confirm the validity of Eq. 8.22

Problem 8.74

[Difficulty: 3]

8.74 Equation 8.23 gives the power-law velocity profile exponent, n, as a function of centerline Reynolds number, Re_U, for fully developed turbulent flow in smooth pipes. Equation 8.24 relates mean velocity, $\overline{V}$, to centerline velocity, U, for various values of n. Prepare a plot of $\overline{V}/U$ as a function of Reynolds number, $Re_{\overline{V}}$.

Solution:

Prepare a Table of values

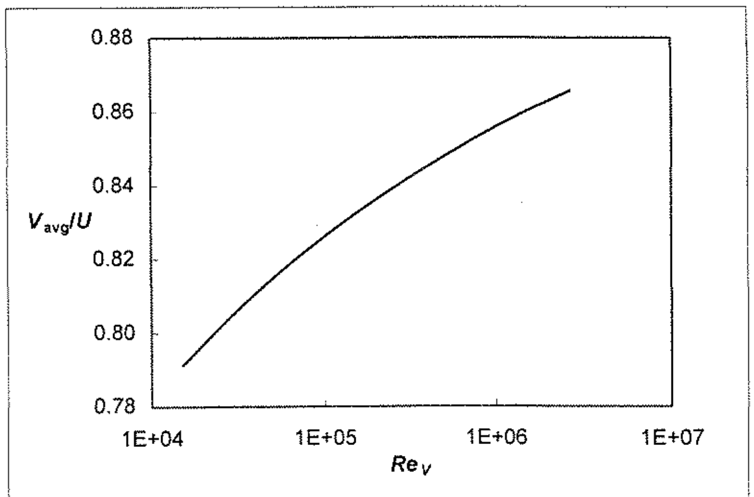
Re_U

n from Eq 8.23

$\bar{V}/U$ from Eq 8.24

$Re_{\bar{V}} = \frac{\bar{V}}{U} \times Re_U$

Re_U	n	Re_V	V_{avg}/U
1.90E+04	6.00	1.50E+04	0.791
3.60E+04	6.50	2.90E+04	0.805
6.85E+04	7.00	5.59E+04	0.817
1.29E+05	7.50	1.07E+05	0.827
2.45E+05	8.00	2.05E+05	0.837
4.65E+05	8.50	3.93E+05	0.845
8.80E+05	9.00	7.50E+05	0.853
1.67E+06	9.50	1.44E+06	0.860
3.16E+06	10.0	2.74E+06	0.866



Problem 8.75

8.75 A momentum coefficient, β, is defined by

$$\int_A u\,\rho u\,dA = \beta \int_A \bar{V}\,\rho u\,dA = \beta \dot{m} \bar{V}$$

Evaluate β for a laminar velocity profile, Eq. 8.14, and for a "power-law" turbulent velocity profile, Eq. 8.22. Plot β as a function of n for turbulent power-law profiles over the range $6 \le n \le 10$ and compare with the case of fully developed laminar pipe flow.

Solution:

$$\beta = \frac{1}{\dot{m}\bar{V}} \int_A u\rho u\,dA = \frac{1}{\rho \bar{V}\pi R^2 \bar{V}} \int_0^R u\,\rho u\,2\pi r\,dr$$

Noting that $\frac{u}{U} = f(r/R)$

$$\beta = \frac{1}{\rho\pi R^2}\left[\frac{U}{\bar{V}}\right]^2 \int_0^R \left(\frac{u}{U}\right)^2 2\rho\pi r\,dr = 2\left[\frac{U}{\bar{V}}\right]^2 \int_0^1 \left(\frac{u}{U}\right)^2 \left(\frac{r}{R}\right) d\left(\frac{r}{R}\right) \quad \text{--- } \beta$$

For laminar flow, $\frac{u}{U} = 1 - \left(\frac{r}{R}\right)^2$, so $\left(\frac{u}{U}\right)^2 = 1 - 2\left(\frac{r}{R}\right)^2 + \left(\frac{r}{R}\right)^4$, and

$$\beta = 2\left[\frac{U}{\bar{V}}\right]^2 \int_0^1 \left[\left(\frac{r}{R}\right) - 2\left(\frac{r}{R}\right)^3 + \left(\frac{r}{R}\right)^5\right] d\left(\frac{r}{R}\right) = 2\left[\frac{U}{\bar{V}}\right]^2 \left[\frac{1}{2} - \frac{1}{2} + \frac{1}{6}\right]$$

$\beta = \frac{1}{3}\left[\frac{U}{\bar{V}}\right]^2$. For this case $U = 2\bar{V}$ so

$$\beta = \frac{1}{3}[2]^2 = \frac{4}{3} \quad \longleftarrow \beta_{laminar}$$

For turbulent flow, $\frac{u}{U} = \left(1 - \frac{r}{R}\right)^{\frac{1}{n}}$, so $\left(\frac{u}{U}\right)^2 = \left(1 - \frac{r}{R}\right)^{\frac{2}{n}}$, and

$$\beta = 2\left[\frac{U}{\bar{V}}\right]^2 \int_0^1 \left(1 - \frac{r}{R}\right)^{\frac{2}{n}} \left(\frac{r}{R}\right) d\left(\frac{r}{R}\right)$$

To integrate, let $m = 1 - \frac{r}{R}$. Then $\frac{r}{R} = 1 - m$, $d\left(\frac{r}{R}\right) = -dm$, so

$$\beta = 2\left[\frac{U}{\bar{V}}\right]^2 \int_1^0 m^{\frac{2}{n}} (1-m)(-dm) = 2\left[\frac{U}{\bar{V}}\right]^2 \int_0^1 \left(m^{\frac{2}{n}} - m^{1+\frac{2}{n}}\right) dm$$

$$\beta = 2\left[\frac{U}{\bar{V}}\right]^2 \left[\frac{n}{(n+2)} m^{\frac{2}{n}+1} - \frac{n}{(2n+2)} m^{2+\frac{2}{n}}\right]_0^1 = 2\left[\frac{U}{\bar{V}}\right]^2 \left[\frac{n}{(n+2)} - \frac{n}{(2n+2)}\right]$$

$$\beta = 2\left[\frac{U}{\bar{V}}\right]^2 \left[\frac{(2n+2)n - (n+2)n}{(n+2)(2n+2)}\right] = 2\left[\frac{U}{\bar{V}}\right]^2 \left[\frac{n^2}{(n+2)(2n+2)}\right] \quad \text{--- (1)}$$

From Eq. 8.24, $\frac{\bar{V}}{U} = \frac{2n^2}{(n+1)(2n+1)}$

For $n = 7$, $\frac{\bar{V}}{U} = 0.817$, so

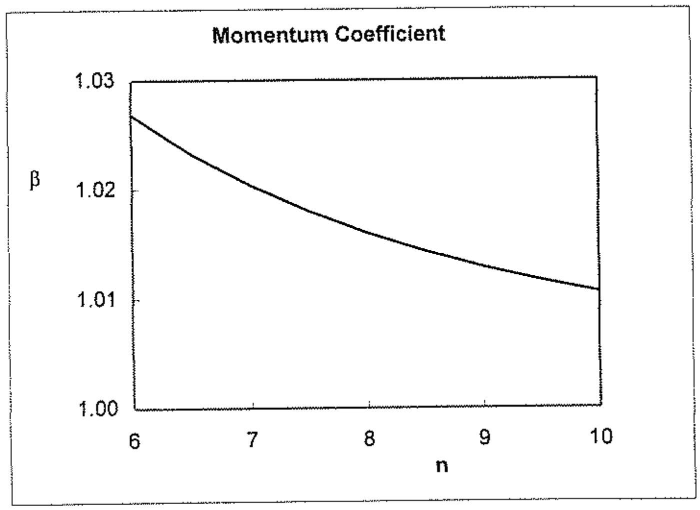
$$\beta = \left[\frac{1}{0.817}\right]^2 \frac{2(7)^2}{(9)(16)} = 1.02 \quad \longleftarrow \beta_{turb}$$

Problem 8.75

To plot β vs n

- $\dfrac{\bar{V}}{U} = \dfrac{2n^2}{(n+1)(2n+1)}$ .... (8.24)
- $\beta = \left[\dfrac{U}{\bar{V}}\right]^2 \dfrac{n^2}{(n+2)(n+1)}$
- $\beta = \dfrac{(n+1)(2n+1)^2}{4n^2(n+2)}$

n	β
6.0	1.027
6.5	1.023
7.0	1.020
7.5	1.018
8.0	1.016
8.5	1.014
9.0	1.013
9.5	1.012
10.0	1.011



Problem 8.76

[Difficulty: 3]

8.76 Consider fully developed laminar flow of water between stationary parallel plates. The maximum flow speed, plate spacing, and width are 20 ft/s, 0.075 in. and 1.25 in., respectively. Find the kinetic energy coefficient, α.

Given: Laminar flow between parallel plates

Find: Kinetic energy coefficient, α

Solution:

Basic Equation: The kinetic energy coefficient, α is given by

$$\alpha = \frac{\int_A \rho V^3 dA}{\dot{m}\bar{V}^2} \tag{8.26b}$$

From Section 8-2, for flow between parallel plates

$$u = u_{\max}\left[1 - \left(\frac{y}{a/2}\right)^2\right] = \frac{3}{2}\bar{V}\left[1 - \left(\frac{y}{a/2}\right)^2\right]$$

since $u_{\max} = \frac{3}{2}\bar{V}$.

Substituting

$$\alpha = \frac{\int_A \rho V^3 dA}{\dot{m}\bar{V}^2} = \frac{\int_A \rho u^3 dA}{\rho \bar{V} A \bar{V}^2} = \frac{1}{A}\int_A \left(\frac{u}{\bar{V}}\right)^3 dA = \frac{1}{wa}\int_{-\frac{a}{2}}^{\frac{a}{2}} \left(\frac{u}{\bar{V}}\right)^3 w\,dy = \frac{2}{a}\int_0^{\frac{a}{2}} \left(\frac{u}{\bar{V}}\right)^3 dy$$

Then

$$\alpha = \frac{2}{a}\frac{a}{2}\int_0^1 \left(\frac{u}{u_{\max}}\right)^3 \left(\frac{u_{\max}}{\bar{V}}\right)^3 d\left(\frac{y}{a/2}\right) = \left(\frac{3}{2}\right)^3 \int_0^1 \left(1-\eta^2\right)^3 d\eta$$

where $\eta = \frac{y}{a/2}$

Evaluating,

$$\left(1-\eta^2\right)^3 = 1 - 3\eta^2 + 3\eta^4 - \eta^6$$

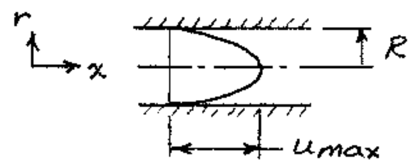
The integral is then

$$\alpha = \left(\frac{3}{2}\right)^3 \int_0^1 \left(1 - 3\eta^2 + 3\eta^4 - \eta^6\right) d\eta = \left(\frac{3}{2}\right)^3 \left[\eta - \eta^3 + \frac{3}{5}\eta^5 - \frac{1}{7}\eta^7\right]_0^1 = \frac{27}{8}\frac{16}{35} = 1.54$$

Problem 8.77

[Difficulty: 3]

8.77 Consider fully developed laminar flow in a circular tube. Evaluate the kinetic energy coefficient for this flow.



Solution: Apply definition of kinetic energy coefficient,

$$\alpha = \frac{\int_A \rho V^3 dA}{\dot{m}\bar{V}^2}, \quad \dot{m} = \rho \bar{V} A \qquad (8.26b)$$

From the analysis of Section 8-3, for flow in a circular tube,

$$u = u_{max}\left[1-\left(\frac{r}{R}\right)^2\right] = 2\bar{V}\left[1-\left(\frac{r}{R}\right)^2\right] \quad \text{since} \quad u_{max} = 2\bar{V}$$

Substituting into Eq. 8.25b,

$$\alpha = \frac{\int_A \rho V^3 dA}{\dot{m}\bar{V}^2} = \frac{\int_A \rho u^3 dA}{\rho \bar{V} A \bar{V}^2} = \frac{1}{A}\int_A \left(\frac{u}{\bar{V}}\right)^3 dA = \frac{1}{\pi R^2}\int_0^R \left(\frac{u}{\bar{V}}\right)^3 2\pi r\,dr = 2\int_0^1 \left(\frac{u}{\bar{V}}\right)^3 \left(\frac{r}{R}\right) d\left(\frac{r}{R}\right)$$

Then

$$\alpha = 2\int_0^1 \left(\frac{u}{u_{max}}\right)^3 \left(\frac{u_{max}}{\bar{V}}\right)^3 \left(\frac{r}{R}\right) d\left(\frac{r}{R}\right) = 2(2)^3 \int_0^1 (1-\eta^2)^3 \eta\, d\eta \quad \text{where } \eta = \frac{r}{R}$$

Evaluating,

$$(1-\eta^2)^3 \eta = \eta - 3\eta^3 + 3\eta^5 - \eta^7$$

The integral is

$$\int_0^1 (1-\eta^2)^3 \eta\, d\eta = \left[\frac{\eta^2}{2} - \frac{3}{4}\eta^4 + \frac{3}{6}\eta^6 - \frac{1}{8}\eta^8\right]_0^1 = \frac{1}{2} - \frac{3}{4} + \frac{1}{2} - \frac{1}{8} = \frac{1}{8}$$

Substituting,

$$\alpha = 16\int_0^1 (1-\eta^2)^3 \eta\, d\eta = 16 \times \frac{1}{8} = 2$$

α

Problem 8.78

[Difficulty: 3]

8.78 Show that the kinetic energy coefficient, α, for the "power law" turbulent velocity profile of Eq. 8.22 is given by Eq. 8.27. Plot α as a function of $Re_{\bar{V}}$, for $Re_{\bar{V}} = 1\times 10^4$ to 1×10^7. When analyzing pipe flow problems it is common practice to assume $\alpha \approx 1$. Plot the error associated with this assumption as a function of $Re_{\bar{V}}$, for $Re_{\bar{V}} = 1\times 10^4$ to 1×10^7.

Given: Definition of kinetic energy correction coefficient α

Find: α for the power-law velocity profile; plot

Solution:

Equation 8.26b is

$$\alpha = \frac{\int \rho\cdot V^3\,dA}{m_{rate}\cdot V_{av}^{\ 2}}$$

where V is the velocity, m_{rate} is the mass flow rate and V_{av} is the average velocity

For the power-law profile (Eq. 8.22)

$$V = U\cdot\left(1 - \frac{r}{R}\right)^{\frac{1}{n}}$$

For the mass flow rate

$$m_{rate} = \rho\cdot\pi\cdot R^{2}\cdot V_{av}$$

Hence the denominator of Eq. 8.26b is

$$m_{rate}\cdot V_{av}^{\ 2} = \rho\cdot\pi\cdot R^2\cdot V_{av}^{\ 3}$$

We next must evaluate the numerator of Eq. 8.26b

$$\int \rho\cdot V^3\,dA = \int \rho\cdot 2\cdot\pi\cdot r\cdot U^3\cdot\left(1 - \frac{r}{R}\right)^{\frac{3}{n}} dr$$

$$\int_0^R \rho\cdot 2\cdot\pi\cdot r\cdot U^3\cdot\left(1 - \frac{r}{R}\right)^{\frac{3}{n}} dr = \frac{2\cdot\pi\cdot\rho\cdot R^2\cdot n^2\cdot U^3}{(3 + n)\cdot(3 + 2\cdot n)}$$

To integrate substitute

$$m = 1 - \frac{r}{R} \qquad dm = -\frac{dr}{R}$$

Then

$$r = R\cdot(1 - m) \qquad dr = -R\cdot dm$$

$$\int_0^R \rho\cdot 2\cdot\pi\cdot r\cdot U^3\cdot\left(1 - \frac{r}{R}\right)^{\frac{3}{n}} dr = -\int_1^0 \rho\cdot 2\cdot\pi\cdot R\cdot(1 - m)\cdot m^{\frac{3}{n}}\cdot R\,dm$$

Hence $$\int \rho \cdot V^3\, dA = \int_0^1 \rho \cdot 2 \cdot \pi \cdot R \cdot \left(m^{\frac{3}{n}} - m^{\frac{3}{n}+1} \right) \cdot R\, dm$$

$$\int \rho \cdot V^3\, dA = \frac{2 \cdot R^2 \cdot n^2 \cdot \rho \cdot \pi \cdot U^3}{(3+n) \cdot (3+2 \cdot n)}$$

Putting all these results together $$\alpha = \frac{\int \rho \cdot V^3\, dA}{m_{rate} \cdot V_{av}^{\ 2}} = \frac{\frac{2 \cdot R^2 \cdot n^2 \cdot \rho \cdot \pi \cdot U^3}{(3+n) \cdot (3+2 \cdot n)}}{\rho \cdot \pi \cdot R^2 \cdot V_{av}^{\ 3}}$$

$$\alpha = \left(\frac{U}{V_{av}} \right)^3 \cdot \frac{2 \cdot n^2}{(3+n) \cdot (3+2 \cdot n)}$$

To plot α versus Re_{Vav} we use the following parametric relations

$$n = -1.7 + 1.8 \cdot \log\left(Re_u\right) \qquad \text{(Eq. 8.23)}$$

$$\frac{V_{av}}{U} = \frac{2 \cdot n^2}{(n+1) \cdot (2 \cdot n + 1)} \qquad \text{(Eq. 8.24)}$$

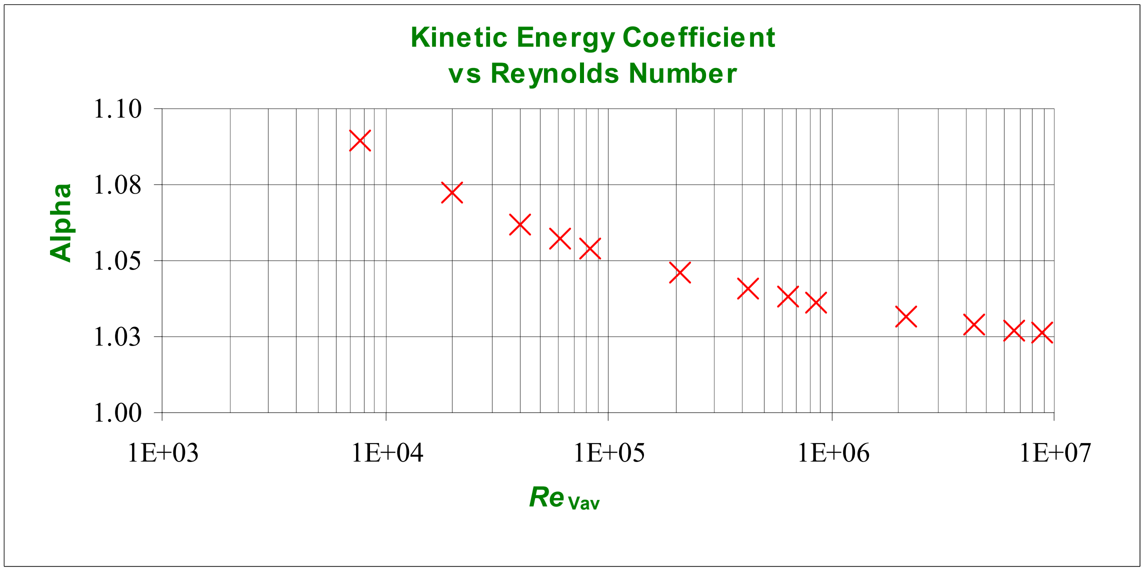
$$Re_{Vav} = \frac{V_{av}}{U} \cdot Re_U$$

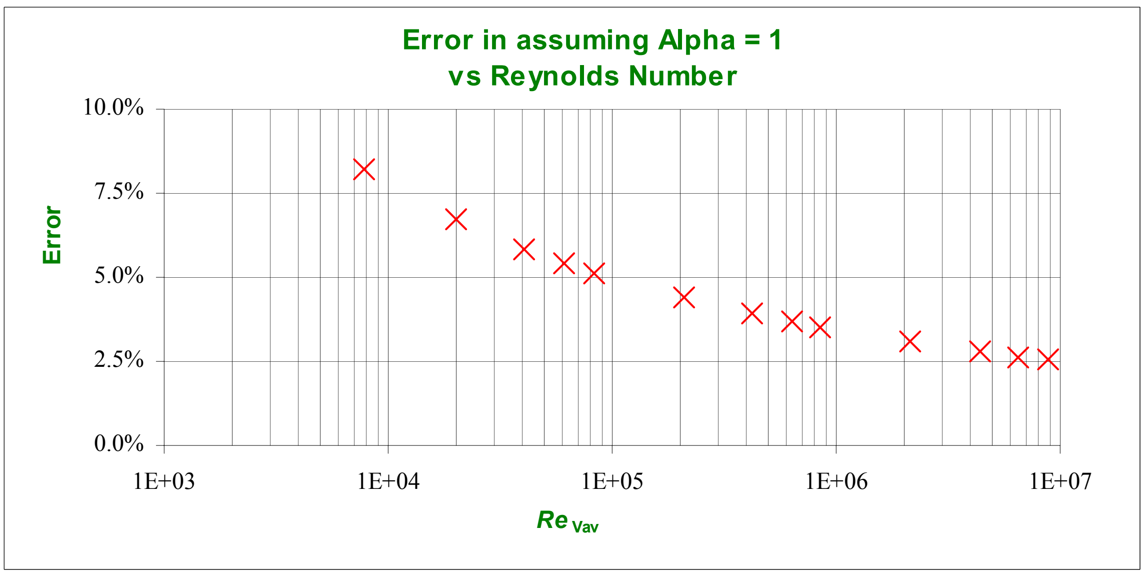
$$\alpha = \left(\frac{U}{V_{av}} \right)^3 \cdot \frac{2 \cdot n^2}{(3+n) \cdot (3+2 \cdot n)} \qquad \text{(Eq. 8.27)}$$

A value of Re_U leads to a value for n; this leads to a value for V_{av}/U; these lead to a value for Re_{Vav} and α

The plots of α, and the error in assuming α = 1, versus Re_{Vav} can be done in *Excel*.

Re_U	n	V_{av}/U	Re_{Vav}	*Alpha*	Error
1.00E+04	5.50	0.776	7.76E+03	1.09	8.2%
2.50E+04	6.22	0.797	1.99E+04	1.07	6.7%
5.00E+04	6.76	0.811	4.06E+04	1.06	5.9%
7.50E+04	7.08	0.818	6.14E+04	1.06	5.4%
1.00E+05	7.30	0.823	8.23E+04	1.05	5.1%
2.50E+05	8.02	0.837	2.09E+05	1.05	4.4%
5.00E+05	8.56	0.846	4.23E+05	1.04	3.9%
7.50E+05	8.88	0.851	6.38E+05	1.04	3.7%
1.00E+06	9.10	0.854	8.54E+05	1.04	3.5%
2.50E+06	9.82	0.864	2.16E+06	1.03	3.1%
5.00E+06	10.4	0.870	4.35E+06	1.03	2.8%
7.50E+06	10.7	0.873	6.55E+06	1.03	2.6%
1.00E+07	10.9	0.876	8.76E+06	1.03	2.5%

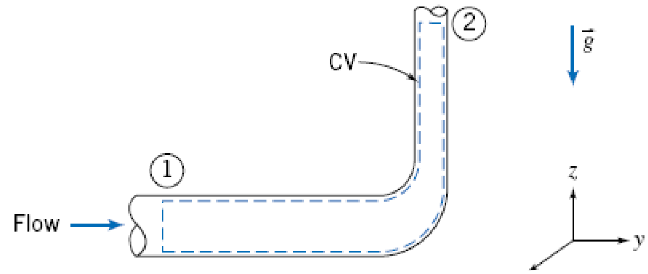

Kinetic Energy Coefficient
vs Reynolds Number
Alpha
1.10
1.08
1.05
1.03
1.00
1E+03
1E+04
1E+05
1E+06
1E+07
Re Vav


Error in assuming Alpha = 1
vs Reynolds Number
Error
10.0%
7.5%
5.0%
2.5%
0.0%
1E+03
1E+04
1E+05
1E+06
1E+07
Re Vav

Problem 8.79

[Difficulty: 2]

8.79 Measurements are made for the flow configuration shown in Fig. 8.12. At the inlet, section (1), the pressure is 70 kPa (gage), the average velocity is 1.75 m/s, and the elevation is 2.25 m. At the outlet, section (2), the pressure, average velocity, and elevation are 45 kPa (gage), 3.5 m/s, and 3 m, respectively. Calculate the head loss in meters. Convert to units of energy per unit mass.



Given: Data on flow through elbow

Find: Head loss

Solution:

Basic equation
$$\left(\frac{p_1}{\rho\cdot g}+\alpha\cdot\frac{V_1^2}{2\cdot g}+z_1\right)-\left(\frac{p_2}{\rho\cdot g}+\alpha\cdot\frac{V_2^2}{2\cdot g}+z_2\right)=\frac{h_{lT}}{g}=H_{lT}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) α at 1 and 2 is approximately 1

Then
$$H_{lT}=\frac{p_1-p_2}{\rho\cdot g}+\frac{V_1^2-V_2^2}{2\cdot g}+z_1-z_2$$

$$H_{lT}=(70-45)\times 10^3\cdot\frac{N}{m^2}\times\frac{m^3}{1000\cdot kg}\times\frac{kg\cdot m}{s^2\cdot N}\times\frac{s^2}{9.81\cdot m}+\frac{1}{2}\times\left(1.75^2-3.5^2\right)\cdot\left(\frac{m}{s}\right)^2\times\frac{s^2}{9.81\cdot m}+(2.25-3)\cdot m \qquad H_{lT}=1.33\,m$$

In terms of energy/mass $\quad h_{lT}=g\cdot H_{lT} \qquad h_{lT}=9.81\cdot\frac{m}{s^2}\times 1.33\cdot m\times\frac{N\cdot s^2}{kg\cdot m} \qquad h_{lT}=13.0\cdot\frac{N\cdot m}{kg}$

Problem 8.80

[Difficulty: 2]

8.80 Water flows in a horizontal constant-area pipe; the pipe diameter is 75 mm and the average flow speed is 5 m/s. At the pipe inlet, the gage pressure is 275 kPa, and the outlet is at atmospheric pressure. Determine the head loss in the pipe. If the pipe is now aligned so that the outlet is 15 m above the inlet, what will the inlet pressure need to be to maintain the same flow rate? If the pipe is now aligned so that the outlet is 15 m below the inlet, what will the inlet pressure need to be to maintain the same flow rate? Finally, how much lower than the inlet must the outlet be so that the same flow rate is maintained if both ends of the pipe are at atmospheric pressure (i.e., gravity feed)?

Given: Data on flow in a pipe

Find: Head loss for horizontal pipe; inlet pressure for different alignments; slope for gravity feed

Solution:

The basic equation between inlet (1) and exit (2) is
$$\left(\frac{p_1}{\rho} + \alpha_1 \cdot \frac{V_1^2}{2} + g \cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \cdot \frac{V_2^2}{2} + g \cdot z_2\right) = h_{lT} \qquad (8.29)$$

Given or available data $\quad D = 75 \cdot mm \quad V = 5 \cdot \frac{m}{s} \quad \rho = 999 \cdot \frac{kg}{m^3} \quad \mu = 0.001 \cdot \frac{N \cdot s}{m^2}$

Horizontal pipe data $\quad p_1 = 275 \cdot kPa \quad p_2 = 0 \cdot kPa \quad$ (Gage pressures) $\quad z_1 = z_2 \quad V_1 = V_2$

Equation 8.29 becomes $\quad h_{lT} = \frac{p_1 - p_2}{\rho} \qquad h_{lT} = 275 \cdot \frac{J}{kg}$

For an inclined pipe with the same flow rate, the head loss will be the same as above; in addition we have the following new data

$z_1 = 0 \cdot m \qquad z_2 = 15 \cdot m$

Equation 8.29 becomes $\quad p_1 = p_2 + \rho \cdot g \cdot (z_2 - z_1) + \rho \cdot h_{lT} \qquad p_1 = 422 \cdot kPa$

For a declining pipe with the same flow rate, the head loss will be the same as above; in addition we have the following new data

$z_1 = 0 \cdot m \qquad z_2 = -15 \cdot m$

Equation 8.29 becomes $\quad p_1 = p_2 + \rho \cdot g \cdot (z_2 - z_1) + \rho \cdot h_{lT} \qquad p_1 = 128 \cdot kPa$

For a gravity feed with the same flow rate, the head loss will be the same as above; in addition we have the following new data

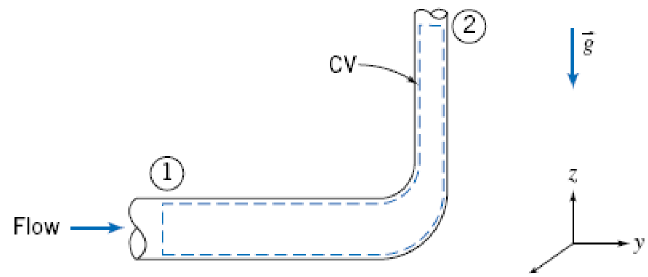
$p_1 = 0 \cdot kPa \qquad$ (Gage)

Equation 8.29 becomes $\quad z_2 = z_1 - \frac{h_{lT}}{g} \qquad z_2 = -28.1\, m$

Problem 8.81

[Difficulty: 2]

8.81 For the flow configuration of Fig. 8.12, it is known that the head loss is 1 m. The pressure drop from inlet to outlet is 50 kPa, the velocity doubles from inlet to outlet, and the elevation increase is 2 m. Compute the inlet water velocity.



Given: Data on flow through elbow

Find: Inlet velocity

Solution:

Basic equation
$$\left(\frac{p_1}{\rho\cdot g}+\alpha\cdot\frac{V_1^{\,2}}{2\cdot g}+z_1\right)-\left(\frac{p_2}{\rho\cdot g}+\alpha\cdot\frac{V_2^{\,2}}{2\cdot g}+z_2\right)=\frac{h_{lT}}{g}=H_{lT}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) α at 1 and 2 is approximately 1

Then
$$V_2^{\,2}-V_1^{\,2}=\left(2\cdot V_1\right)^2-V_1^{\,2}=3\cdot V_1^{\,2}=\frac{2\cdot\left(p_1-p_2\right)}{\rho}+2\cdot g\cdot\left(z_1-z_2\right)-2\cdot g\cdot H_{lT}$$

$$V_1=\sqrt{\frac{2}{3}\cdot\left[\frac{\left(p_1-p_2\right)}{\rho}+g\cdot\left(z_1-z_2\right)-g\cdot H_{lT}\right]}$$

$$V_1=\sqrt{\frac{2}{3}\times\left[50\times10^3\cdot\frac{N}{m^2}\times\frac{m^3}{1000\cdot kg}\times\frac{kg\cdot m}{s^2\cdot N}+\frac{9.81\cdot m}{s^2}\times(-2)\cdot m-9.81\cdot\frac{m}{s^2}\times1\cdot m\right]}\qquad V_1=3.70\,\frac{m}{s}$$

Problem 8.82

[Difficulty: 2]

8.82 For a given volume flow rate and piping system, will the pressure loss be greater for hot water or cold water? Explain.

Given:

A given piping system and volume flow rate with two liquid choices.

Find:

Which liquid has greater pressure loss

Solution:

Governing equation:

$$\left(\frac{P_1}{\rho}+\alpha_1\frac{\overline{V}_1^2}{2}+gz_1\right)-\left(\frac{P_2}{\rho}+\alpha_2\frac{\overline{V}_2^2}{2}+gz_2\right)=h_{l_T}$$

$$h_{l_T}=h_l+h_{l_m}=f\frac{L}{D}\frac{\overline{V}^2}{2}+K\frac{\overline{V}^2}{2}$$

Assumption: 1) Steady flow 2) Incompressible 3) Neglect elevation effects 4)Neglect velocity effects

$$\Delta P=\rho f\frac{L}{D}\frac{\overline{V}^2}{2}+\rho K\frac{\overline{V}^2}{2}$$

From Table A.8 it is seen that hot water has a lower density and lower kinematic viscosity than cold water.

The lower density means that for a constant minor loss coefficient (*K*) and velocity the pressure loss due to minor losses will be less for hot water.

The lower kinematic viscosity means that for a constant diameter and velocity the Reynolds number will increase. From Figure 8.13 it is seen that increasing the Reynolds number will either result in a decreased friction factor (*f*) or no change in the friction factor. This potential decrease in friction factor combined with a lower density for hot water means that the pressure loss due to major losses will be less for hot water as well.

Cold water has a greater pressure drop

Problem 8.83

[Difficulty: 2]

8.83 Consider the pipe flow from the water tower of Example 8.7. After another 5 years the pipe roughness has increased such that the flow is fully turbulent and $f = 0.035$. Find by how much the flow rate is decreased.

Given: Increased friction factor for water tower flow

Find: How much flow is decreased

Solution:

Basic equation from Example 8.7

$$V_2 = \sqrt{\frac{2 \cdot g \cdot (z_1 - z_2)}{f \cdot \left(\frac{L}{D} + 8\right) + 1}}$$

where $L = 680 \cdot ft$ $\quad D = 4 \cdot in$ $\quad z_1 - z_2 = 80 \cdot ft$

With f = 0.0308, we obtain $V_2 = 8.97 \cdot \frac{ft}{s}$ and Q = 351 gpm

We need to recompute with f = 0.035

$$V_2 = \sqrt{2 \times 32.2 \cdot \frac{ft}{s^2} \times 80 \cdot ft \times \frac{1}{0.035 \cdot \left(\frac{680}{\frac{4}{12}} + 8\right) + 1}} \qquad V_2 = 8.42 \cdot \frac{ft}{s}$$

Hence

$$Q = V_2 \cdot A = V_2 \cdot \frac{\pi \cdot D^2}{4}$$

$$Q = 8.42 \cdot \frac{ft}{s} \times \frac{\pi}{4} \times \left(\frac{4}{12} \cdot ft\right)^2 \times \frac{7.48 \cdot gal}{1 \cdot ft^3} \times \frac{60 \cdot s}{1 \cdot min} \qquad Q = 330 \cdot gpm$$

(From Table G.2 1 ft^3 = 7.48 gal)

Hence the flow is decreased by $(330 - 309) \cdot gpm = 21 \cdot gpm$

Problem 8.84

[Difficulty: 2]

8.84 Consider the pipe flow from the water tower of Problem 8.83. To increase delivery, the pipe length is reduced from 600 ft to 450 ft (the flow is still fully turbulent and $f = 0.035$). What is the flow rate?

Given: Increased friction factor for water tower flow, and reduced length

Find: How much flow is decreased

Solution:

Basic equation from Example 8.7

$$V_2 = \sqrt{\frac{2 \cdot g \cdot (z_1 - z_2)}{f \cdot \left(\frac{L}{D} + 8\right) + 1}}$$

where now we have $L = 530 \cdot ft$ $\quad D = 4 \cdot in$ $\quad z_1 - z_2 = 80 \cdot ft$

We need to recompute with f = 0.04

$$V_2 = \sqrt{2 \times 32.2 \cdot \frac{ft}{s^2} \times 80 \cdot ft \times \frac{1}{0.035 \cdot \left(\frac{530}{\frac{4}{12}} + 8\right) + 1}} \qquad V_2 = 9.51 \cdot \frac{ft}{s}$$

Hence

$$Q = V_2 \cdot A = V_2 \cdot \frac{\pi \cdot D^2}{4}$$

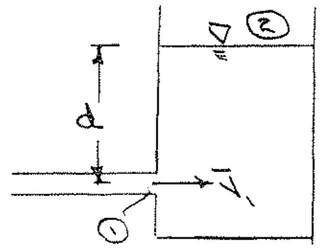
$$Q = 9.51 \cdot \frac{ft}{s} \times \frac{\pi}{4} \times \left(\frac{4}{12} \cdot ft\right)^2 \times \frac{7.48 \cdot gal}{1 \cdot ft^3} \times \frac{60 \cdot s}{1 \cdot min} \qquad Q = 372 \cdot gpm$$

(From Table G.2 1 ft^3 = 7.48 gal)

Problem 8.85

[Difficulty: 2]

8.85 Water flows from a horizontal tube into a large tank. The tube is located 2.5 m below the free surface of water in the tank. The head loss is 2 J/kg. Compute the average flow speed in the tube.



Solution:

Apply definition of head loss, Eq 8.29,

$$\left(\frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2} + g z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \frac{\bar{V}_2^2}{2} + g z_2\right) = h_{l_T}$$

At free surface, $V_2 = 0$, $p_2 = p_{atm}$

At tube discharge $p_1 = \rho g d$, $z_1 = 0$. Assume $\alpha_1 \approx 1$

Then

$$g d + \frac{\bar{V}_1^2}{2} - g d = h_{l_T}$$

$$\bar{V}_1^2 = 2 h_{l_T} = 2 \times 2 \frac{N \cdot m}{kg} \times \frac{kg \cdot m}{N \cdot s^2} = 4 \ m^2/s^2$$

$$\bar{V}_1 = 2 \ m/s$$

$\bar{V}_1$

Problem 8.86

[Difficulty: 2]

8.86 The average flow speed in a constant-diameter section of the Alaskan pipeline is 2.5 m/s. At the inlet, the pressure is 8.25 MPa (gage) and the elevation is 45 m; at the outlet, the pressure is 350 kPa (gage) and the elevation is 115 m. Calculate the head loss in this section of pipeline.

Given: Data on flow through Alaskan pipeline

Find: Head loss

Solution:

Basic equation

$$\left(\frac{p_1}{\rho_{oil} \cdot g} + \alpha \cdot \frac{V_1^2}{2 \cdot g} + z_1\right) - \left(\frac{p_2}{\rho_{oil} \cdot g} + \alpha \cdot \frac{V_2^2}{2 \cdot g} + z_2\right) = \frac{h_{lT}}{g} = H_{lT}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) α at 1 and 2 is approximately 1 4) SG = 0.9 (Table A.2)

Then

$$H_{lT} = \frac{p_1 - p_2}{SG_{oil} \cdot \rho_{H2O} \cdot g} + z_1 - z_2$$

$$H_{lT} = (8250 - 350) \times 10^3 \cdot \frac{N}{m^2} \times \frac{1}{0.9} \times \frac{m^3}{1000 \cdot kg} \times \frac{kg \cdot m}{s^2 \cdot N} \times \frac{s^2}{9.81 \cdot m} + (45 - 115) \cdot m \qquad H_{lT} = 825\,m$$

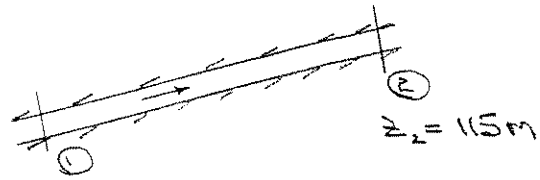
In terms of energy/mass $\quad h_{lT} = g \cdot H_{lT}$

$$h_{lT} = 9.81 \cdot \frac{m}{s^2} \times 825 \cdot m \times \frac{N \cdot s^2}{kg \cdot m} \qquad h_{lT} = 8.09 \cdot \frac{kN \cdot m}{kg}$$

Problem 8.87

[Difficulty: 2]

8.87 At the inlet to a constant-diameter section of the Alaskan pipeline, the pressure is 8.5 MPa and the elevation is 45 m; at the outlet the elevation is 115 m. The head loss in this section of pipeline is 6.9 kJ/kg. Calculate the outlet pressure.



$p_1 = 8.5$ MPa

$z_1 = 45$ m

Solution:

Computing equation: $$\left(\frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2} + g z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \frac{\bar{V}_2^2}{2} + g z_2\right) = h_{l_T} \quad (8.29)$$

Assumptions: (1) incompressible flow, so $\bar{V}_1 = \bar{V}_2$
(2) fully developed so $\alpha_1 = \alpha_2$
(3) SG crude oil = 0.90 (Table A.2)

then

$$p_2 = p_1 + \rho g (z_1 - z_2) - \rho h_{l_T}$$

$$= 8.5 \times 10^6\ N/m^2 + 0.9 \times 999\ \frac{kg}{m^3} \times 9.81\ \frac{m}{s^2} \times (-70\ m) \times \frac{N \cdot s^2}{kg \cdot m} - 0.9 \times 999\ \frac{kg}{m^3} \times 6.9 \times 10^3\ \frac{N \cdot m}{kg}$$

$p_2 = 1.68$ MPa ← p_2

Problem 8.88

[Difficulty: 2]

8.88 Water flows at 10 L/min through a horizontal 15-mm-diameter tube. The pressure drop along a 20-m length of tube is 85 kPa. Calculate the head loss.

Given: Data on flow through a tube

Find: Head loss

Solution:

Basic equation $$\left(\frac{p_1}{\rho\cdot g} + \alpha\cdot\frac{V_1^{\,2}}{2\cdot g} + z_1\right) - \left(\frac{p_2}{\rho\cdot g} + \alpha\cdot\frac{V_2^{\,2}}{2\cdot g} + z_2\right) = \frac{h_{lT}}{g} = H_{lT}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) α at 1 and 2 is approximately 1

Given or available data $Q = 10\cdot\frac{L}{min}$ $D = 15\cdot mm$ $\Delta p = 85\cdot kPa$ $\rho = 999\cdot\frac{kg}{m^3}$

The basic equation reduces to $h_{lT} = \frac{\Delta p}{\rho}$ $h_{lT} = 85.1\,\frac{m^2}{s^2}$ $H_{lT} = \frac{h_{lT}}{g}$ $H_{lT} = 8.68\,m$

Problem 8.89

[Difficulty: 2]

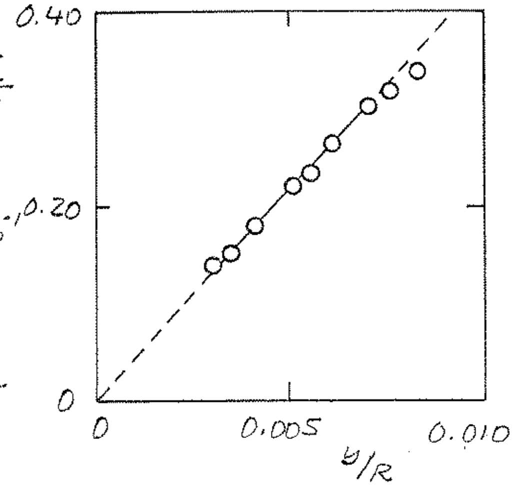
8.89 Laufer [5] measured the following data for mean velocity near the wall in fully developed turbulent pipe flow at $Re_U = 50,000$ ($U = 9.8$ ft/s and $R = 4.86$ in.) in air:

$\bar{u}/U$	0.343	0.318	0.300	0.264	0.228	0.221	0.179	0.152	0.140
y/R	0.0082	0.0075	0.0071	0.0061	0.0055	0.0051	0.0041	0.0034	0.0030

Plot the data and obtain the best-fit slope, $d\bar{u}/dy$. Use this to estimate the wall shear stress from $\tau_w = \mu\, d\bar{u}/dy$. Compare this value to that obtained using the friction factor f computed using (a) the Colebrook formula (Eq. 8.37), and (b) the Blasius correlation (Eq. 8.38).

Solution: "Best-fit" slope is {from analysis in Excel file}

$$\frac{d(\bar{u}/U)}{d(y/R)} \approx \frac{\Delta(\bar{u}/U)}{\Delta(y/R)} = 39.8$$



$$\frac{d\bar{u}}{dy} = \frac{U\, d(\bar{u}/U)}{R\, d(y/R)} = 39.8 \times 9.8\frac{ft}{s} \times \frac{1}{4.86\, in.} \times \frac{12\, in.}{ft} = 963\ s^{-1}$$ $\frac{d\bar{u}}{dy}$

For standard air, $\mu = 3.72\times10^{-7}\ lbf\cdot s/ft^2$, so

$$\tau_w = \mu\frac{d\bar{u}}{dy} = 3.72\times10^{-7}\frac{lbf\cdot s}{ft^2}\times\frac{963}{s} = 3.58\times10^{-4}\ lbf/ft^2$$ τ_w

Friction factor is $f = f(Re, e/D)$. For $Re_U = 50{,}000$, $n = 6.8$ from Eq. 8.23. Then from Eq. 8.24,

$$\frac{\bar{V}}{U} = \frac{2n^2}{(n+1)(2n+1)} = 0.812 \text{ and } Re_{\bar{V}} = 0.812\, Re_U = 0.812\times 50{,}000 = 40{,}600$$

Assuming smooth pipe, $f = 0.0219$ from Eq. 8.37

Balancing forces on a fluid element: $(p+\Delta p)\frac{\pi D^2}{4} \rightarrow$ $\leftarrow p\frac{\pi D^2}{4}$

$\tau_w \pi D L$

Then $(p+\Delta p)\frac{\pi D^2}{4} - \tau_w \pi D L - p\frac{\pi D^2}{4} = 0$

$$\tau_w = \frac{R}{2}\frac{\Delta p}{L} = \frac{D}{4L} f \frac{L}{D}\frac{\rho\bar{V}^2}{2} = \frac{f}{8}\rho\bar{V}^2\ ;\ \bar{V} = 0.812U = 0.812\times 9.8\frac{ft}{sec} = 7.96\ ft/sec$$

Substituting,

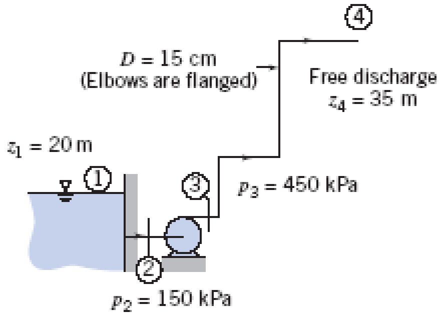
$$\tau_w = \frac{0.0219}{8}\times 0.00238\frac{slug}{ft^3}\times(7.96)^2\frac{ft^2}{s^2}\times\frac{lbf\cdot s^2}{slug\cdot ft} = 4.13\times10^{-4}\ lbf/ft^2$$ τ_w

The result calculated from the friction factor is 15% higher than that evaluated graphically.

Problem 8.90

[Difficulty: 3]

8.90 Water is pumped at the rate of 0.075 m^3/s from a reservoir 20 m above a pump to a free discharge 35 m above the pump. The pressure on the intake side of the pump is 150 kPa and the pressure on the discharge side is 450 kPa. All pipes are commercial steel of 15 cm diameter. Determine (a) the head supplied by the pump and (b) the total head loss between the pump and point of free discharge.



Given: Data on flow from reservoir

Find: Head from pump; head loss

Solution:

Basic equations

$$\left(\frac{p_3}{\rho \cdot g} + \alpha \cdot \frac{V_3^2}{2 \cdot g} + z_3\right) - \left(\frac{p_4}{\rho \cdot g} + \alpha \cdot \frac{V_4^2}{2 \cdot g} + z_4\right) = \frac{h_{lT}}{g} = H_{lT} \qquad \text{for flow from 3 to 4}$$

$$\left(\frac{p_3}{\rho \cdot g} + \alpha \cdot \frac{V_3^2}{2 \cdot g} + z_3\right) - \left(\frac{p_2}{\rho \cdot g} + \alpha \cdot \frac{V_2^2}{2 \cdot g} + z_2\right) = \frac{\Delta h_{pump}}{g} = H_{pump} \qquad \text{for flow from 2 to 3}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) α at 1 and 2 is approximately 1 4) $V_2 = V_3 = V_4$ (constant area pipe)

Then for the pump

$$H_{pump} = \frac{p_3 - p_2}{\rho \cdot g}$$

$$H_{pump} = (450 - 150) \times 10^3 \cdot \frac{N}{m^2} \times \frac{m^3}{1000 \cdot kg} \times \frac{kg \cdot m}{s^2 \cdot N} \times \frac{s^2}{9.81 \cdot m} \qquad H_{pump} = 30.6\,m$$

In terms of energy/mass

$$h_{pump} = g \cdot H_{pump} \qquad h_{pump} = 9.81 \cdot \frac{m}{s^2} \times 30.6 \cdot m \times \frac{N \cdot s^2}{kg \cdot m} \qquad h_{pump} = 300 \cdot \frac{N \cdot m}{kg}$$

For the head loss from 3 to 4

$$H_{lT} = \frac{p_3 - p_4}{\rho \cdot g} + z_3 - z_4$$

$$H_{lT} = (450 - 0) \times 10^3 \cdot \frac{N}{m^2} \times \frac{m^3}{1000 \cdot kg} \times \frac{kg \cdot m}{s^2 \cdot N} \times \frac{s^2}{9.81 \cdot m} + (0 - 35) \cdot m \qquad H_{lT} = 10.9\,m$$

In terms of energy/mass

$$h_{lT} = g \cdot H_{lT} \qquad h_{lT} = 9.81 \cdot \frac{m}{s^2} \times 10.9 \cdot m \times \frac{N \cdot s^2}{kg \cdot m} \qquad h_{lT} = 107 \cdot \frac{N \cdot m}{kg}$$

Problem 8.91

[Difficulty: 2]

8.91 A smooth, 75-mm-diameter pipe carries water (65°C) horizontally. When the mass flow rate is 0.075 kg/s, the pressure drop is measured to be 7.5 Pa per 100 m of pipe. Based on these measurements, what is the friction factor? What is the Reynolds number? Does this Reynolds number generally indicate laminar or turbulent flow? Is the flow actually laminar or turbulent?

Given: Data on flow in a pipe

Find: Friction factor; Reynolds number; if flow is laminar or turbulent

Solution:

Given data $D = 75\cdot mm$ $\qquad \frac{\Delta p}{L} = 0.075\cdot\frac{Pa}{m}$ $\qquad m_{rate} = 0.075\cdot\frac{kg}{s}$

From Appendix A $\rho = 1000\cdot\frac{kg}{m^3}$ $\qquad \mu = 4\cdot 10^{-4}\cdot\frac{N\cdot s}{m^2}$

The governing equations between inlet (1) and exit (2) are

$$\left(\frac{p_1}{\rho} + \alpha_1\cdot\frac{V_1^{\,2}}{2} + g\cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2\cdot\frac{V_2^{\,2}}{2} + g\cdot z_2\right) = h_l \qquad (8.29)$$

$$h_l = f\cdot\frac{L}{D}\cdot\frac{V^2}{2} \qquad (8.34)$$

For a constant area pipe $V_1 = V_2 = V$

Hence Eqs. 8.29 and 8.34 become

$$f = \frac{2\cdot D}{L\cdot V^2}\cdot\frac{(p_1 - p_2)}{\rho} = \frac{2\cdot D}{\rho\cdot V^2}\cdot\frac{\Delta p}{L}$$

For the velocity $V = \frac{m_{rate}}{\rho\cdot\frac{\pi}{4}\cdot D^2}$ $\qquad V = 0.017\,\frac{m}{s}$

Hence $f = \frac{2\cdot D}{\rho\cdot V^2}\cdot\frac{\Delta p}{L}$ $\qquad f = 0.0390$

The Reynolds number is $Re = \frac{\rho\cdot V\cdot D}{\mu}$ $\qquad Re = 3183$

This Reynolds number indicates the flow is turbulent.

(From Eq. 8.37, at this Reynolds number the friction factor for a smooth pipe is $f = 0.043$; the friction factor computed above thus indicates that, within experimental error, the flow corresponds to turbulent flow in a smooth pipe)

Problem 8.92

[Difficulty: 2]

8.92 A small-diameter capillary tube made from drawn aluminum is used in place of an expansion valve in a home refrigerator. The inside diameter is 0.5 mm. Calculate the corresponding relative roughness. Comment on whether this tube may be considered "smooth" with regard to fluid flow.

Solution:

For drawn tubing, from Table 8.1, $e = 0.0015$ mm

Then with $D = 0.5$ mm, $\frac{e}{D} = \frac{0.0015}{0.5} = 0.003$

Looking at the Moody diagram (Fig. 8.13), it is clear that this tube cannot be considered smooth for turbulent flow through the tube.

For laminar flow ($Re < 2300$) the relative roughness has no effect on the flow.

Problem 8.93

[Difficulty: 3]

8.93 The Colebrook equation (Eq. 8.37) for computing the turbulent friction factor is implicit in f. An explicit expression [31] that gives reasonable accuracy is

$$f_0 = 0.25\left[\log\left(\frac{e/D}{3.7} + \frac{5.74}{Re^{0.9}}\right)\right]^{-2}$$

Compare the accuracy of this expression for f with Eq. 8.37 by computing the percentage discrepancy as a function of Re and e/D, for $Re = 10^4$ to 10^8, and $e/D = 0$, 0.0001, 0.001, 0.01, and 0.05. What is the maximum discrepancy for these Re and e/D values? Plot f against Re with e/D as a parameter.

Using the above formula for f_0, and Eq. 8.37 for f_1

e/D =	0	0.0001	0.0002	0.0005	0.001	0.002	0.005	0.01	0.02	0.05
Re	f_0									
1.00E+04	0.0310	0.0311	0.0313	0.0318	0.0327	0.0342	0.0383	0.0440	0.0534	0.0750
2.50E+04	0.0244	0.0247	0.0250	0.0258	0.0270	0.0291	0.0342	0.0407	0.0508	0.0731
5.00E+04	0.0208	0.0212	0.0216	0.0226	0.0242	0.0268	0.0325	0.0395	0.0498	0.0724
7.50E+04	0.0190	0.0195	0.0200	0.0212	0.0230	0.0258	0.0319	0.0390	0.0494	0.0721
1.00E+05	0.0179	0.0185	0.0190	0.0204	0.0223	0.0253	0.0316	0.0388	0.0493	0.0720
2.50E+05	0.0149	0.0158	0.0167	0.0186	0.0209	0.0243	0.0309	0.0383	0.0489	0.0717
5.00E+05	0.0131	0.0145	0.0155	0.0178	0.0204	0.0239	0.0307	0.0381	0.0488	0.0717
7.50E+05	0.0122	0.0139	0.0150	0.0175	0.0201	0.0238	0.0306	0.0380	0.0487	0.0716
1.00E+06	0.0116	0.0135	0.0148	0.0173	0.0200	0.0237	0.0305	0.0380	0.0487	0.0716
5.00E+06	0.0090	0.0124	0.0140	0.0168	0.0197	0.0235	0.0304	0.0379	0.0487	0.0716
1.00E+07	0.0081	0.0122	0.0139	0.0168	0.0197	0.0235	0.0304	0.0379	0.0486	0.0716
5.00E+07	0.0066	0.0120	0.0138	0.0167	0.0196	0.0234	0.0304	0.0379	0.0486	0.0716
1.00E+08	0.0060	0.0120	0.0137	0.0167	0.0196	0.0234	0.0304	0.0379	0.0486	0.0716

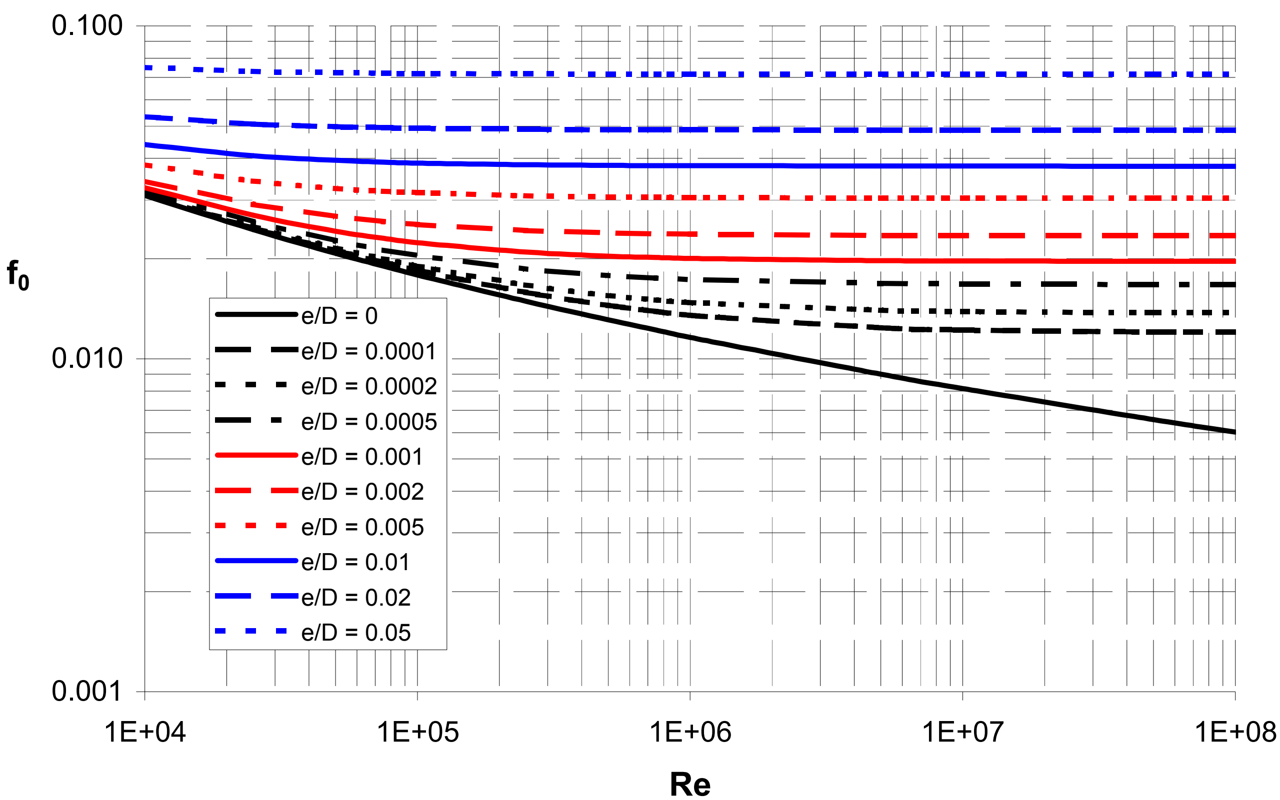
Using the add-in function *Friction factor* from the Web

e/D =	0	0.0001	0.0002	0.0005	0.001	0.002	0.005	0.01	0.02	0.05
Re	f									
1.00E+04	0.0309	0.0310	0.0312	0.0316	0.0324	0.0338	0.0376	0.0431	0.0523	0.0738
2.50E+04	0.0245	0.0248	0.0250	0.0257	0.0268	0.0288	0.0337	0.0402	0.0502	0.0725
5.00E+04	0.0209	0.0212	0.0216	0.0226	0.0240	0.0265	0.0322	0.0391	0.0494	0.0720
7.50E+04	0.0191	0.0196	0.0200	0.0212	0.0228	0.0256	0.0316	0.0387	0.0492	0.0719
1.00E+05	0.0180	0.0185	0.0190	0.0203	0.0222	0.0251	0.0313	0.0385	0.0490	0.0718
2.50E+05	0.0150	0.0158	0.0166	0.0185	0.0208	0.0241	0.0308	0.0381	0.0488	0.0716
5.00E+05	0.0132	0.0144	0.0154	0.0177	0.0202	0.0238	0.0306	0.0380	0.0487	0.0716
7.50E+05	0.0122	0.0138	0.0150	0.0174	0.0200	0.0237	0.0305	0.0380	0.0487	0.0716
1.00E+06	0.0116	0.0134	0.0147	0.0172	0.0199	0.0236	0.0305	0.0380	0.0487	0.0716
5.00E+06	0.0090	0.0123	0.0139	0.0168	0.0197	0.0235	0.0304	0.0379	0.0486	0.0716
1.00E+07	0.0081	0.0122	0.0138	0.0168	0.0197	0.0234	0.0304	0.0379	0.0486	0.0716
5.00E+07	0.0065	0.0120	0.0138	0.0167	0.0196	0.0234	0.0304	0.0379	0.0486	0.0716
1.00E+08	0.0059	0.0120	0.0137	0.0167	0.0196	0.0234	0.0304	0.0379	0.0486	0.0716

The error can now be computed

e/D =	0	0.0001	0.0002	0.0005	0.001	0.002	0.005	0.01	0.02	0.05
Re	Error (%)									
1.00E+04	0.29%	0.36%	0.43%	0.61%	0.88%	1.27%	1.86%	**2.12%**	2.08%	1.68%
2.50E+04	0.39%	0.24%	0.11%	0.21%	0.60%	1.04%	1.42%	1.41%	1.21%	0.87%
5.00E+04	0.63%	0.39%	0.19%	0.25%	0.67%	1.00%	1.11%	0.98%	0.77%	0.52%
7.50E+04	0.69%	0.38%	0.13%	0.35%	0.73%	0.95%	0.93%	0.77%	0.58%	0.38%
1.00E+05	0.71%	0.33%	0.06%	0.43%	0.76%	0.90%	0.81%	0.64%	0.47%	0.30%
2.50E+05	0.65%	0.04%	0.28%	0.64%	0.72%	0.66%	0.48%	0.35%	0.24%	0.14%
5.00E+05	0.52%	0.26%	0.51%	0.64%	0.59%	0.47%	0.31%	0.21%	0.14%	0.08%
7.50E+05	0.41%	0.41%	0.58%	0.59%	0.50%	0.37%	0.23%	0.15%	0.10%	0.06%
1.00E+06	0.33%	0.49%	0.60%	0.54%	0.43%	0.31%	0.19%	0.12%	0.08%	0.05%
5.00E+06	0.22%	0.51%	0.39%	0.24%	0.16%	0.10%	0.06%	0.03%	0.02%	0.01%
1.00E+07	0.49%	0.39%	0.27%	0.15%	0.10%	0.06%	0.03%	0.02%	0.01%	0.01%
5.00E+07	1.15%	0.15%	0.09%	0.05%	0.03%	0.02%	0.01%	0.01%	0.00%	0.00%
1.00E+08	1.44%	0.09%	0.06%	0.03%	0.02%	0.01%	0.00%	0.00%	0.00%	0.00%

The maximum discrepancy is 2.12% at *Re* = 10,000 and *e/D* = 0.01



Problem 8.94

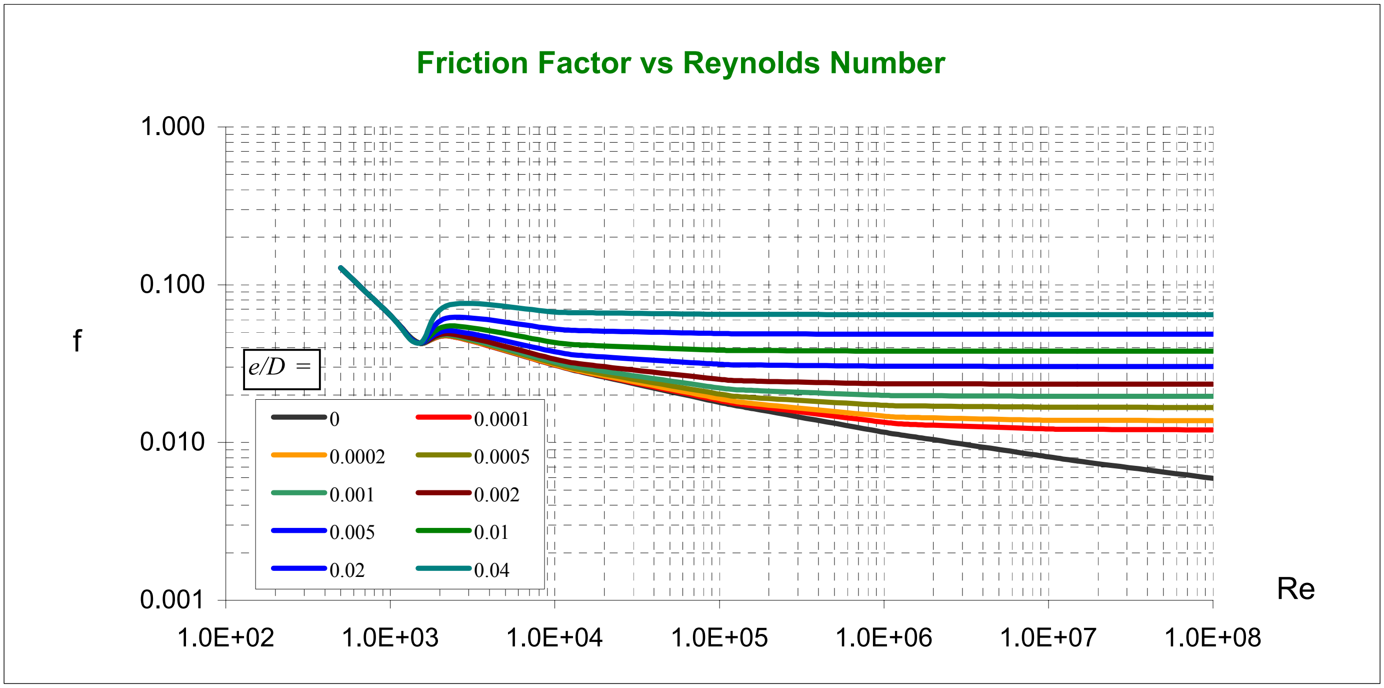
[Difficulty: 3]

8.94 Using Eqs. 8.36 and 8.37, generate the Moody chart of Fig. 8.13.

Solution:

Using the add-in function *Friction factor* from the web site

e/D =	0	0.0001	0.0002	0.0005	0.001	0.002	0.005	0.01	0.02	0.04
Re	***f***									
500	0.1280	0.1280	0.1280	0.1280	0.1280	0.1280	0.1280	0.1280	0.1280	0.1280
1.00E+03	0.0640	0.0640	0.0640	0.0640	0.0640	0.0640	0.0640	0.0640	0.0640	0.0640
1.50E+03	0.0427	0.0427	0.0427	0.0427	0.0427	0.0427	0.0427	0.0427	0.0427	0.0427
2.30E+03	0.0473	0.0474	0.0474	0.0477	0.0481	0.0489	0.0512	0.0549	0.0619	0.0747
1.00E+04	0.0309	0.0310	0.0312	0.0316	0.0324	0.0338	0.0376	0.0431	0.0523	0.0672
1.50E+04	0.0278	0.0280	0.0282	0.0287	0.0296	0.0313	0.0356	0.0415	0.0511	0.0664
1.00E+05	0.0180	0.0185	0.0190	0.0203	0.0222	0.0251	0.0313	0.0385	0.0490	0.0649
1.50E+05	0.0166	0.0172	0.0178	0.0194	0.0214	0.0246	0.0310	0.0383	0.0489	0.0648
1.00E+06	0.0116	0.0134	0.0147	0.0172	0.0199	0.0236	0.0305	0.0380	0.0487	0.0647
1.50E+06	0.0109	0.0130	0.0144	0.0170	0.0198	0.0235	0.0304	0.0379	0.0487	0.0647
1.00E+07	0.0081	0.0122	0.0138	0.0168	0.0197	0.0234	0.0304	0.0379	0.0486	0.0647
1.50E+07	0.0076	0.0121	0.0138	0.0167	0.0197	0.0234	0.0304	0.0379	0.0486	0.0647
1.00E+08	0.0059	0.0120	0.0137	0.0167	0.0196	0.0234	0.0304	0.0379	0.0486	0.0647



Problem 8.95

[Difficulty: 2]

8.95 The Moody diagram gives the Darcy friction factor, f, in terms of Reynolds number and relative roughness. The *Fanning friction factor* for pipe flow is defined as

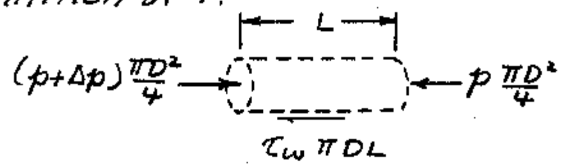
$$f_F = \frac{\tau_w}{\frac{1}{2}\rho \bar{V}^2}$$

where τ_w is the wall shear stress in the pipe. Show that the relation between the Darcy and Fanning friction factors for fully developed pipe flow is given by $f = 4f_F$.

<u>Solution:</u> Consider cylindrical CV containing fluid in pipe; apply force balance, definition of f.

Basic equations: $\sum F_x = 0$

$$\Delta p = f \frac{L}{D} \frac{\rho \bar{V}^2}{2}$$



From the force balance,

$$(p+\Delta p)\frac{\pi D^2}{4} - \tau_w \pi D L - p\frac{\pi D^2}{4} = 0 \quad \text{or} \quad \tau_w = \frac{D}{4}\frac{\Delta p}{L}$$

Substituting,

$$\tau_w = \frac{D}{4L} f \frac{L}{D} \frac{\rho \bar{V}^2}{2} = f \frac{\rho \bar{V}^2}{8}$$

But

$$f_F \equiv \frac{\tau_w}{\frac{1}{2}\rho \bar{V}^2} = \frac{f \rho \bar{V}^2}{8}\frac{2}{\rho \bar{V}^2} = \frac{f}{4}$$ f_F

Problem 8.96

[Difficulty: 3]

8.96 We saw in Section 8.7 that instead of the implicit Colebrook equation (Eq. 8.37) for computing the turbulent friction factor f, an explicit expression that gives reasonable accuracy is

$$\frac{1}{\sqrt{f}} = -1.8\log\left[\left(\frac{e/D}{3.7}\right)^{1.11} + \frac{6.9}{Re}\right]$$

Compare the accuracy of this expression for f with Eq. 8.37 by computing the percentage discrepancy as a function of Re and e/D, for $Re = 10^4$ to 10^8, and $e/D = 0, 0.0001, 0.001, 0.01$, and 0.05. What is the maximum discrepancy for these Re and e/D values? Plot f against Re with e/D as a parameter.

Using the above formula for f_0, and Eq. 8.37 for f_1

e/D =	0	0.0001	0.0002	0.0005	0.001	0.002	0.005	0.01	0.02	0.05
Re	f_0									
1.00E+04	0.0309	0.0310	0.0311	0.0315	0.0322	0.0335	0.0374	0.0430	0.0524	0.0741
2.50E+04	0.0244	0.0245	0.0248	0.0254	0.0265	0.0285	0.0336	0.0401	0.0502	0.0727
5.00E+04	0.0207	0.0210	0.0213	0.0223	0.0237	0.0263	0.0321	0.0391	0.0495	0.0722
7.50E+04	0.0189	0.0193	0.0197	0.0209	0.0226	0.0254	0.0316	0.0387	0.0492	0.0720
1.00E+05	0.0178	0.0183	0.0187	0.0201	0.0220	0.0250	0.0313	0.0385	0.0491	0.0719
2.50E+05	0.0148	0.0156	0.0164	0.0183	0.0207	0.0241	0.0308	0.0382	0.0489	0.0718
5.00E+05	0.0131	0.0143	0.0153	0.0176	0.0202	0.0238	0.0306	0.0381	0.0488	0.0717
7.50E+05	0.0122	0.0137	0.0148	0.0173	0.0200	0.0237	0.0305	0.0381	0.0488	0.0717
1.00E+06	0.0116	0.0133	0.0146	0.0172	0.0199	0.0236	0.0305	0.0380	0.0488	0.0717
5.00E+06	0.0090	0.0123	0.0139	0.0168	0.0197	0.0235	0.0304	0.0380	0.0487	0.0717
1.00E+07	0.0081	0.0122	0.0139	0.0168	0.0197	0.0235	0.0304	0.0380	0.0487	0.0717
5.00E+07	0.0066	0.0120	0.0138	0.0167	0.0197	0.0235	0.0304	0.0380	0.0487	0.0717
1.00E+08	0.0060	0.0120	0.0138	0.0167	0.0197	0.0235	0.0304	0.0380	0.0487	0.0717

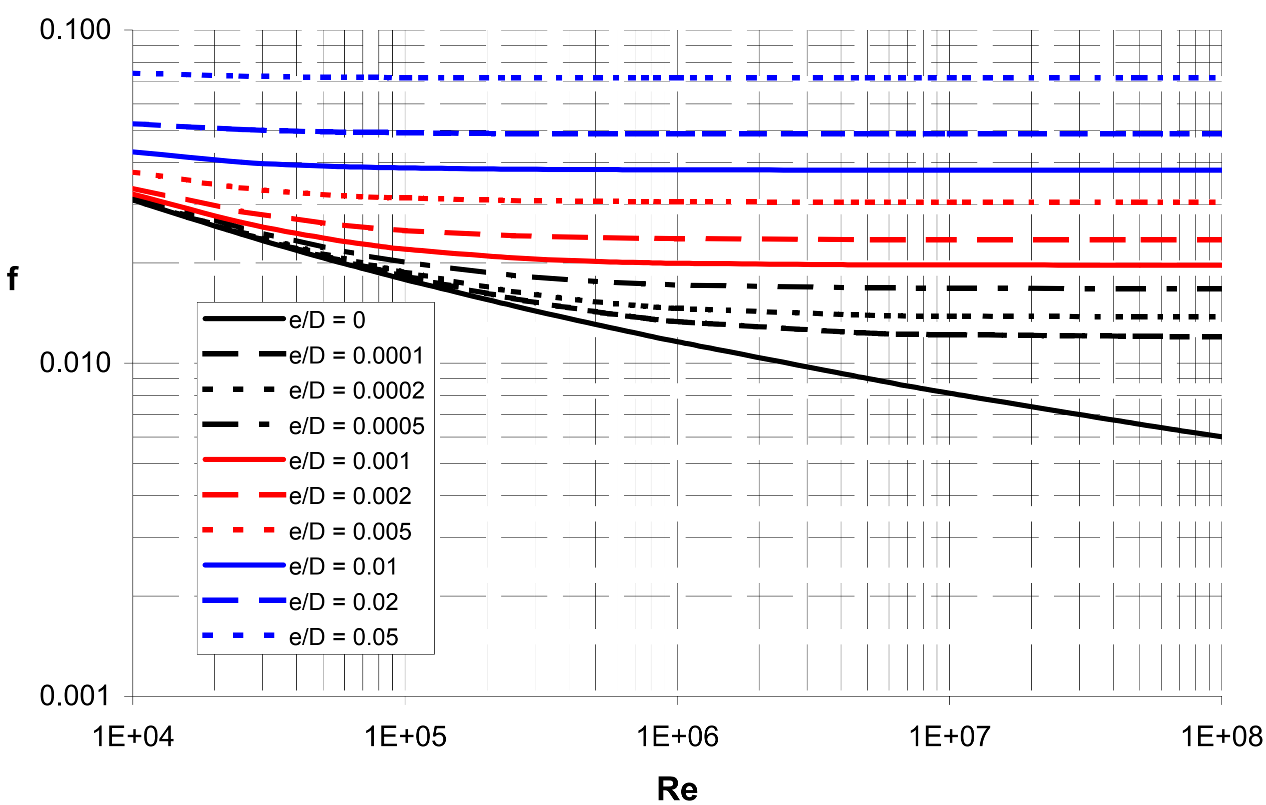
Using the add-in function *Friction factor* from the Web

e/D =	0	0.0001	0.0002	0.0005	0.001	0.002	0.005	0.01	0.02	0.05
Re	f									
1.00E+04	0.0309	0.0310	0.0312	0.0316	0.0324	0.0338	0.0376	0.0431	0.0523	0.0738
2.50E+04	0.0245	0.0248	0.0250	0.0257	0.0268	0.0288	0.0337	0.0402	0.0502	0.0725
5.00E+04	0.0209	0.0212	0.0216	0.0226	0.0240	0.0265	0.0322	0.0391	0.0494	0.0720
7.50E+04	0.0191	0.0196	0.0200	0.0212	0.0228	0.0256	0.0316	0.0387	0.0492	0.0719
1.00E+05	0.0180	0.0185	0.0190	0.0203	0.0222	0.0251	0.0313	0.0385	0.0490	0.0718
2.50E+05	0.0150	0.0158	0.0166	0.0185	0.0208	0.0241	0.0308	0.0381	0.0488	0.0716
5.00E+05	0.0132	0.0144	0.0154	0.0177	0.0202	0.0238	0.0306	0.0380	0.0487	0.0716
7.50E+05	0.0122	0.0138	0.0150	0.0174	0.0200	0.0237	0.0305	0.0380	0.0487	0.0716
1.00E+06	0.0116	0.0134	0.0147	0.0172	0.0199	0.0236	0.0305	0.0380	0.0487	0.0716
5.00E+06	0.0090	0.0123	0.0139	0.0168	0.0197	0.0235	0.0304	0.0379	0.0486	0.0716
1.00E+07	0.0081	0.0122	0.0138	0.0168	0.0197	0.0234	0.0304	0.0379	0.0486	0.0716
5.00E+07	0.0065	0.0120	0.0138	0.0167	0.0196	0.0234	0.0304	0.0379	0.0486	0.0716
1.00E+08	0.0059	0.0120	0.0137	0.0167	0.0196	0.0234	0.0304	0.0379	0.0486	0.0716

The error can now be computed

e/D =	0	0.0001	0.0002	0.0005	0.001	0.002	0.005	0.01	0.02	0.05
Re	Error (%)									
1.00E+04	0.01%	0.15%	0.26%	0.46%	0.64%	0.73%	0.55%	0.19%	0.17%	0.43%
2.50E+04	0.63%	0.88%	1.02%	1.20%	1.22%	1.03%	0.51%	0.11%	0.14%	0.29%
5.00E+04	0.85%	1.19%	1.32%	1.38%	1.21%	0.84%	0.28%	0.00%	0.16%	0.24%
7.50E+04	0.90%	1.30%	1.40%	1.35%	1.07%	0.65%	0.16%	0.06%	0.17%	0.23%
1.00E+05	0.92%	1.34%	**1.42%**	1.28%	0.94%	0.52%	0.09%	0.09%	0.18%	0.22%
2.50E+05	0.84%	1.33%	1.25%	0.85%	0.47%	0.16%	0.07%	0.15%	0.19%	0.21%
5.00E+05	0.70%	1.16%	0.93%	0.48%	0.19%	0.00%	0.13%	0.18%	0.20%	0.20%
7.50E+05	0.59%	0.99%	0.72%	0.30%	0.07%	0.07%	0.16%	0.18%	0.20%	0.20%
1.00E+06	0.50%	0.86%	0.57%	0.20%	0.01%	0.10%	0.17%	0.19%	0.20%	0.20%
5.00E+06	0.07%	0.17%	0.01%	0.11%	0.15%	0.18%	0.19%	0.20%	0.20%	0.20%
1.00E+07	0.35%	0.00%	0.09%	0.15%	0.18%	0.19%	0.20%	0.20%	0.20%	0.20%
5.00E+07	1.02%	0.16%	0.18%	0.19%	0.20%	0.20%	0.20%	0.20%	0.20%	0.20%
1.00E+08	1.31%	0.18%	0.19%	0.20%	0.20%	0.20%	0.20%	0.20%	0.20%	0.20%

The maximum discrepancy is 1.42% at *Re* = 100,000 and *e/D* = 0.0002



Problem 8.97

[Difficulty: 3]

8.97 Water flows at 25 L/s through a gradual contraction, in which the pipe diameter is reduced from 75 mm to 37.5 mm, with a 150° included angle. If the pressure before the contraction is 500 kPa, estimate the pressure after the contraction. Recompute the answer if the included angle is changed to 180° (a sudden contraction).

Given: Flow through gradual contraction

Find: Pressure after contraction; compare to sudden contraction

Solution:

Basic equations $$\left(\frac{p_1}{\rho} + \alpha\cdot\frac{V_1^2}{2} + g\cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha\cdot\frac{V_2^2}{2} + g\cdot z_2\right) = h_{lm} \qquad h_{lm} = K\cdot\frac{V_2^2}{2} \qquad Q = V\cdot A$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) α at 1 and 2 is approximately 1 4) Horizontal

Available data $Q = 25\cdot\frac{L}{s}$ $\quad Q = 0.025\frac{m^3}{s}$ $\quad D_1 = 75\cdot mm$ $\quad D_2 = 37.5\cdot mm$ $\quad p_1 = 500\cdot kPa$ $\quad \rho = 999\cdot\frac{kg}{m^3}$

For an included angle of 150^o and an area ratio $\frac{A_2}{A_1} = \left(\frac{D_2}{D_1}\right)^2 = \left(\frac{37.5}{75}\right)^2 = 0.25$ we find from Table 8.3 $\quad K = 0.35$

Hence the energy equation becomes $$\left(\frac{p_1}{\rho} + \frac{V_1^2}{2}\right) - \left(\frac{p_2}{\rho} + \frac{V_2^2}{2}\right) = K\cdot\frac{V_2^2}{2}$$ with $$V_1 = \frac{4\cdot Q}{\pi\cdot D_1^2} \qquad V_2 = \frac{4\cdot Q}{\pi\cdot D_2^2}$$

$$p_2 = p_1 - \frac{\rho}{2}\cdot\left[(1+K)\cdot V_2^2 - V_1^2\right] = p_2 - \frac{8\cdot\rho\cdot Q^2}{\pi^2}\cdot\left[\frac{(1+K)}{D_2^4} - \frac{1}{D_1^4}\right]$$

$$p_2 = 500\times10^3\cdot\frac{N}{m^2} - \frac{8}{\pi^2}\times999\cdot\frac{kg}{m^3}\times\left(0.025\cdot\frac{m^3}{s}\right)^2\times\left[(1+0.35)\times\frac{1}{(0.0375\cdot m)^4} - \frac{1}{(0.075\cdot m)^4}\right]\times\frac{N\cdot s^2}{kg\cdot m} \qquad p_2 = 170\cdot kPa$$

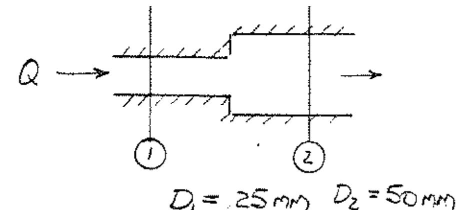
Repeating the above analysis for an included angle of 180^o (sudden contraction) $\quad K = 0.41$

$$p_2 = 500\times10^3\cdot\frac{N}{m^2} - \frac{8}{\pi^2}\times999\cdot\frac{kg}{m^3}\times\left(0.025\cdot\frac{m^3}{s}\right)^2\times\left[(1+0.41)\times\frac{1}{(0.0375\cdot m)^4} - \frac{1}{(0.075\cdot m)^4}\right]\times\frac{N\cdot s^2}{kg\cdot m} \qquad p_2 = 155\cdot kPa$$

Problem 8.98

[Difficulty: 3]

8.98 Water flows through a 25-mm-diameter tube that suddenly enlarges to a diameter of 50 mm. The flow rate through the enlargement is 1.25 Liter/s. Calculate the pressure rise across the enlargement. Compare with the value for frictionless flow.



Solution: Apply energy equation for pipe flow.

Computing equation: $\frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2} + g\cancel{z_1} = \frac{p_2}{\rho} + \alpha_2 \frac{\bar{V}_2^2}{2} + g\cancel{z_2} + h_{l_T}$

$h_{l_T} = K \frac{\bar{V}_1^2}{2}$

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Uniform flow at each section: $\alpha_1 = \alpha_2 = 1$
(4) Horizontal section

Then

$$p_2 - p_1 = \frac{\rho}{2}(\bar{V}_1^2 - \bar{V}_2^2) - \rho h_{l_{T_{12}}}$$

From continuity, $\bar{V}_1 A_1 = \bar{V}_2 A_2$, so $\bar{V}_2 = \bar{V}_1 \frac{A_1}{A_2} = \bar{V}_1 \left(\frac{D_1}{D_2}\right)^2$; $\bar{V}_2^2 = \bar{V}_1^2 \left(\frac{D_1}{D_2}\right)^4$

From Fig. 8.14, at $AR = \left(\frac{D_1}{D_2}\right)^2 = \frac{1}{4}$, $K = 0.56$.

$$\bar{V}_1 = \frac{Q}{A_1} = \frac{4Q}{\pi D_1^2} = \frac{4}{\pi} \times 1.25 \frac{L}{s} \times 10^{-3} \frac{m^3}{L} \times \frac{1}{(25 \times 10^{-3})^2 m^2} = 2.55\ m/s$$

Substituting,

$$p_2 - p_1 = \frac{\rho \bar{V}_1^2}{2}\left[1 - \left(\frac{D_1}{D_2}\right)^4\right] - K\rho\frac{\bar{V}_1^2}{2} = \frac{1}{2}\rho\bar{V}_1^2\left[1 - \left(\frac{D_1}{D_2}\right)^4 - K\right]$$

$$= \frac{1}{2} \times 999 \frac{kg}{m^3} \times (2.55)^2 \frac{m^2}{s^2} \left[1 - \left(\frac{1}{2}\right)^4 - 0.56\right] \frac{N \cdot s^2}{kg \cdot m}$$

$$p_2 - p_1 = 1.22\ kPa$$

Δp_a

For frictionless flow, $K = 0$, and

$$p_2 - p_1 = \frac{1}{2}\rho\bar{V}_1^2\left[1 - \left(\frac{D_1}{D_2}\right)^4\right] = 3.04\ kPa$$

Δp_f

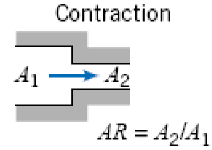
Thus $\frac{\Delta p_{actual}}{\Delta p_{frictionless}} = \frac{1.22}{3.04} = 0.403$ or 40.3%

Ratio

Problem 8.99

[Difficulty: 3]

8.99 Water flows through a 2-in.-diameter tube that suddenly contracts to 1 in. diameter. The pressure drop across the contraction is 0.5 psi. Determine the volume flow rate.



Given: Flow through sudden contraction

Find: Volume flow rate

Solution:

Basic equations
$$\left(\frac{p_1}{\rho} + \alpha\cdot\frac{V_1^2}{2} + g\cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha\cdot\frac{V_2^2}{2} + g\cdot z_2\right) = h_{lm} \qquad h_{lm} = K\cdot\frac{V_2^2}{2} \qquad Q = V\cdot A$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) α at 1 and 2 is approximately 1 4) Horizontal

Hence the energy equation becomes

$$\left(\frac{p_1}{\rho} + \frac{V_1^2}{2}\right) - \left(\frac{p_2}{\rho} + \frac{V_2^2}{2}\right) = K\cdot\frac{V_2^2}{2}$$

From continuity
$$V_1 = V_2\cdot\frac{A_2}{A_1} = V_2\cdot AR$$

Hence
$$\left(\frac{p_1}{\rho} + \frac{V_2^2\cdot AR^2}{2}\right) - \left(\frac{p_2}{\rho} + \frac{V_2^2}{2}\right) = K\cdot\frac{V_2^2}{2}$$

Solving for V_2
$$V_2 = \sqrt{\frac{2\cdot(p_1 - p_2)}{\rho\cdot\left(1 - AR^2 + K\right)}} \qquad AR = \left(\frac{D_2}{D_1}\right)^2 = \left(\frac{1}{2}\right)^2 = 0.25$$

so from Fig. 8.14 $\quad K = 0.4$

Hence
$$V_2 = \sqrt{2 \times 0.5\cdot\frac{lbf}{in^2} \times \left(\frac{12\cdot in}{1\cdot ft}\right)^2 \times \frac{ft^3}{1.94\cdot slug} \times \frac{1}{\left(1 - 0.25^2 + 0.4\right)} \times \frac{slug\cdot ft}{lbf\cdot s^2}} \qquad V_2 = 7.45\cdot\frac{ft}{s}$$

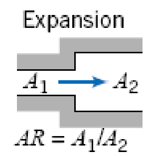
$$Q = V_2\cdot A_2 = \frac{\pi\cdot D_2^2}{4}\cdot V_2$$

$$Q = \frac{\pi}{4} \times \left(\frac{1}{12}\cdot ft\right)^2 \times 7.45\cdot\frac{ft}{s} \qquad Q = 0.0406\cdot\frac{ft^3}{s} \qquad Q = 2.44\cdot\frac{ft^3}{min} \qquad Q = 18.2\cdot gpm$$

Problem 8.100

[Difficulty: 3]

8.100 Air at standard conditions flows through a sudden expansion in a circular duct. The upstream and downstream duct diameters are 75 mm and 225 mm, respectively. The pressure downstream is 5 mm of water higher than that upstream. Determine the average speed of the air approaching the expansion and the volume flow rate.



Given: Flow through sudden expansion

Find: Inlet speed; Volume flow rate

Solution:

Basic equations
$$\left(\frac{p_1}{\rho} + \alpha\cdot\frac{V_1^2}{2} + g\cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha\cdot\frac{V_2^2}{2} + g\cdot z_2\right) = h_{lm} \qquad h_{lm} = K\cdot\frac{V_1^2}{2} \qquad Q = V\cdot A \qquad \Delta p = \rho_{H2O}\cdot g\cdot\Delta h$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) α at 1 and 2 is approximately 1 4) Horizontal

Hence the energy equation becomes

$$\left(\frac{p_1}{\rho} + \frac{V_1^2}{2}\right) - \left(\frac{p_2}{\rho} + \frac{V_2^2}{2}\right) = K\cdot\frac{V_1^2}{2}$$

From continuity
$$V_2 = V_1\cdot\frac{A_1}{A_2} = V_1\cdot AR$$

Hence
$$\left(\frac{p_1}{\rho} + \frac{V_1^2}{2}\right) - \left(\frac{p_2}{\rho} + \frac{V_1^2\cdot AR^2}{2}\right) = K\cdot\frac{V_1^2}{2}$$

Solving for V_1
$$V_1 = \sqrt{\frac{2\cdot(p_2 - p_1)}{\rho\cdot\left(1 - AR^2 - K\right)}} \qquad AR = \left(\frac{D_1}{D_2}\right)^2 = \left(\frac{75}{225}\right)^2 = 0.111$$
so from Fig. 8.14 $\quad K = 0.8$

Also
$$p_2 - p_1 = \rho_{H2O}\cdot g\cdot\Delta h = 1000\cdot\frac{kg}{m^3} \times 9.81\cdot\frac{m}{s^2} \times \frac{5}{1000}\cdot m \times \frac{N\cdot s^2}{kg\cdot m} = 49.1\cdot Pa$$

Hence
$$V_1 = \sqrt{2 \times 49.1\cdot\frac{N}{m^2} \times \frac{m^3}{1.23\cdot kg} \times \frac{1}{\left(1 - 0.111^2 - 0.8\right)} \times \frac{kg\cdot m}{N\cdot s^2}} \qquad V_1 = 20.6\,\frac{m}{s}$$

$$Q = V_1\cdot A_1 = \frac{\pi\cdot D_1^2}{4}\cdot V_1 \qquad Q = \frac{\pi}{4} \times \left(\frac{75}{1000}\cdot m\right)^2 \times 20.6\cdot\frac{m}{s} \qquad Q = 0.0910\cdot\frac{m^3}{s} \qquad Q = 5.46\cdot\frac{m^3}{min}$$

Problem 8.101

[Difficulty: 4]

8.101 In an undergraduate laboratory, you have been assigned the task of developing a crude flow meter for measuring the flow in a 45-mm-diameter water pipe system. You are to install a 22.5-mm-diameter section of pipe and a water manometer to measure the pressure drop at the sudden contraction. Derive an expression for the theoretical calibration constant k in $Q = k\sqrt{\Delta h}$, where Q is the volume flow rate in L/min, and Δh is the manometer deflection in mm. Plot the theoretical calibration curve for a flow rate range of 10 to 50 L/min. Would you expect this to be a practical device for measuring flow rate?

Given: Data on a pipe sudden contraction

Find: Theoretical calibration constant; plot

Solution:

Given data $\quad D_1 = 45 \cdot mm \qquad D_2 = 22.5 \cdot mm$

The governing equations between inlet (1) and exit (2) are

$$\left(\frac{p_1}{\rho} + \alpha_1 \cdot \frac{V_1^2}{2} + g \cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \cdot \frac{V_2^2}{2} + g \cdot z_2\right) = h_l \qquad (8.29)$$

where

$$h_l = K \cdot \frac{V_2^2}{2} \qquad (8.40a)$$

Hence the pressure drop is (assuming $\alpha = 1$)

$$\Delta p = p_1 - p_2 = \rho \cdot \left(\frac{V_2^2}{2} - \frac{V_1^2}{2} + K \cdot \frac{V_2^2}{2}\right)$$

For the sudden contraction

$$V_1 \cdot \frac{\pi}{4} \cdot D_1^2 = V_2 \cdot \frac{\pi}{4} \cdot D_2^2 = Q \qquad \text{or} \qquad V_2 = V_1 \cdot \left(\frac{D_1}{D_2}\right)^2$$

so

$$\Delta p = \frac{\rho \cdot V_1^2}{2} \cdot \left[\left(\frac{D_1}{D_2}\right)^4 (1 + K) - 1\right]$$

For the pressure drop we can use the manometer equation

$$\Delta p = \rho \cdot g \cdot \Delta h$$

Hence

$$\rho \cdot g \cdot \Delta h = \frac{\rho \cdot V_1^2}{2} \cdot \left[\left(\frac{D_1}{D_2}\right)^4 (1 + K) - 1\right]$$

In terms of flow rate Q

$$\rho \cdot g \cdot \Delta h = \frac{\rho}{2} \cdot \frac{Q^2}{\left(\frac{\pi}{4} \cdot D_1^2\right)^2} \cdot \left[\left(\frac{D_1}{D_2}\right)^4 (1 + K) - 1\right]$$

or $$g\cdot\Delta h = \frac{8\cdot Q^2}{\pi^2\cdot D_1^{\,4}}\cdot\left[\left(\frac{D_1}{D_2}\right)^4(1+K)-1\right]$$

Hence for flow rate Q we find $$Q = k\cdot\sqrt{\Delta h}$$

where $$k = \sqrt{\frac{g\cdot\pi^2\cdot D_1^{\,4}}{8\cdot\left[\left(\frac{D_1}{D_2}\right)^4(1+K)-1\right]}}$$

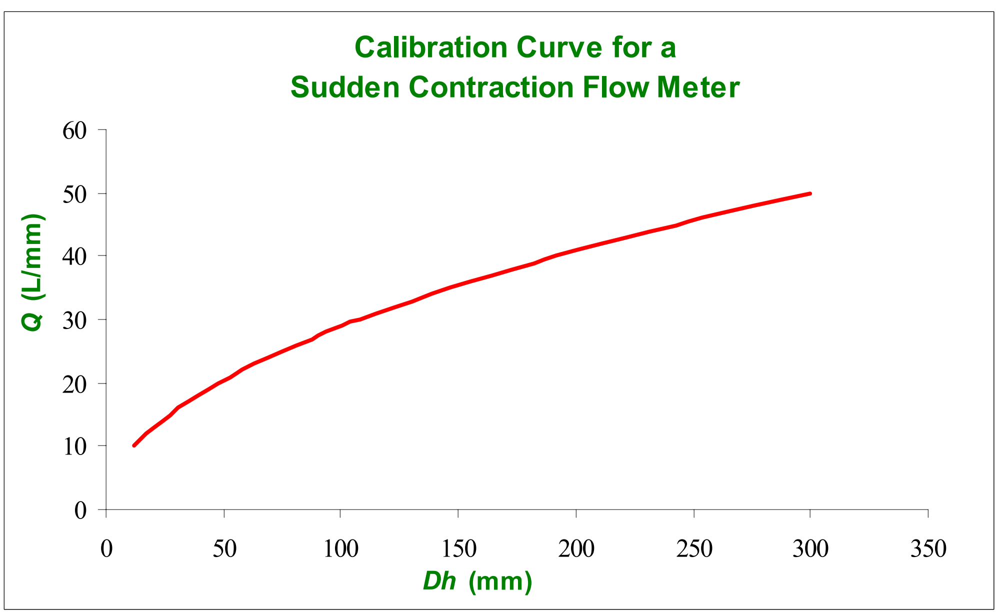
For K, we need the aspect ratio AR $$AR = \left(\frac{D_2}{D_1}\right)^2 \qquad AR = 0.25$$

From Fig. 8.15 $$K = 0.4$$

Using this in the expression for k, with the other given values $$k = \sqrt{\frac{g\cdot\pi^2\cdot D_1^{\,4}}{8\cdot\left[\left(\frac{D_1}{D_2}\right)^4(1+K)-1\right]}} \qquad k = 1.52\times10^{-3}\cdot\frac{m^{\frac{5}{2}}}{s}$$

For Δh in mm and Q in L/min $$k = 2.89\cdot\frac{\frac{L}{min}}{mm^{\frac{1}{2}}}$$

The plot of theoretical Q versus flow rate Δh can be done in *Excel*.

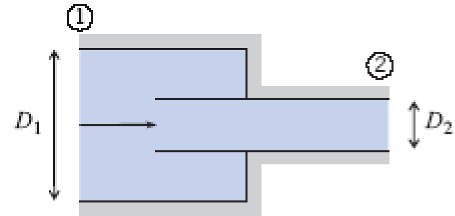


It is a practical device, but is a) Nonlinear and b) has a large energy loss

Problem 8.102

[Difficulty: 3]

8.102 Water flows from a larger pipe, diameter $D_1 = 100$ mm, into a smaller one, diameter $D_2 = 50$ mm, by way of a reentrant device. Find the head loss between points (1) and (2).



Given: Flow through a reentrant device

Find: Head loss

Solution:

Basic equations
$$\left(\frac{p_1}{\rho} + \alpha\cdot\frac{V_1^2}{2} + g\cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha\cdot\frac{V_2^2}{2} + g\cdot z_2\right) = h_{lT} \qquad h_{lT} = h_l + h_{lm} = f\cdot\frac{L}{D}\cdot\frac{V_2^2}{2} + K\cdot\frac{V_2^2}{2} \qquad Q = V\cdot A$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) α at 1 and 2 is approximately 1 4) L << so ignore h_l 5) Reentrant

Available data $\quad D_1 = 100\cdot\text{mm} \quad D_2 = 50\cdot\text{mm} \quad Q = 0.01\cdot\frac{\text{m}^3}{\text{s}}$ and from Table 8.2 $\quad K = 0.78$

$$A_1 = \frac{\pi}{4}\cdot D_1^2 \qquad A_1 = 7.85\times 10^3\ \text{mm}^2 \qquad A_2 = \frac{\pi}{4}\cdot D_2^2 \qquad A_2 = 1.96\times 10^3\ \text{mm}^2$$

Hence between the free surface (Point 1) and the exit (2) the energy equation becomes

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} - \frac{V_2^2}{2} - \frac{p_2}{\rho} = K\cdot\frac{V_2^2}{2}$$

From continuity $\quad Q = V_1\cdot A_1 = V_2\cdot A_2 \quad$ and also $\quad \frac{p_1 - p_2}{\rho} = \frac{\rho\cdot g\cdot h}{\rho} = g\cdot h \quad$ where h is the head loss

Hence
$$g\cdot h + \frac{1}{2}\cdot\left(\frac{Q}{A_1}\right)^2 - \frac{1}{2}\cdot\left(\frac{Q}{A_2}\right)^2 = K\cdot\frac{1}{2}\cdot\left(\frac{Q}{A_2}\right)^2$$

Solving for h
$$h = \frac{\left(\frac{Q}{A_2}\right)^2}{2\cdot g}\cdot\left[1 + K - \left(\frac{A_2}{A_1}\right)^2\right] \qquad h = 2.27\ \text{m}$$

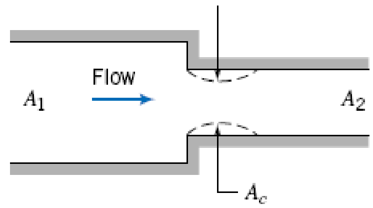
Problem 8.103

[Difficulty: 4]

8.103 Flow through a sudden contraction is shown. The minimum flow area at the vena contracta is given in terms of the area ratio by the contraction coefficient [32],

$$C_c = \frac{A_c}{A_2} = 0.62 + 0.38\left(\frac{A_2}{A_1}\right)^3$$

The loss in a sudden contraction is mostly a result of the vena contracta: The fluid accelerates into the contraction, there is flow separation (as shown by the dashed lines), and the vena contracta acts as a miniature sudden expansion with significant secondary flow losses. Use these assumptions to obtain and plot estimates of the minor loss coefficient for a sudden contraction, and compare with the data presented in Fig. 8.15.



Given: Contraction coefficient for sudden contraction

Find: Expression for minor head loss; compare with Fig. 8.15; plot

Solution:

We analyse the loss at the "sudden expansion" at the vena contracta

The governing CV equations (mass, momentum, and energy) are

$$\frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A} = 0 \qquad (4.12)$$

$$F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u \, \rho \, d\forall + \int_{CS} u \, \rho \vec{V} \cdot d\vec{A} \qquad (4.18a)$$

$$\dot{Q} - \dot{W}_s - \dot{W}_{shear} - \dot{W}_{other} = \frac{\partial}{\partial t}\int_{CV} e \, \rho \, d\forall + \int_{CS}\left(u + pv + \frac{V^2}{2} + gz\right)\rho \vec{V} \cdot d\vec{A} \qquad (4.56)$$

Assume: 1) Steady flow 2) Incompressible flow 3) Uniform flow at each section 4) Horizontal: no body force 5) No shaft work 6) Neglect viscous friction 7) Neglect gravity

The mass equation becomes $$V_c \cdot A_c = V_2 \cdot A_2 \qquad (1)$$

The momentum equation becomes $$p_c \cdot A_2 - p_2 \cdot A_2 = V_c \cdot \left(-\rho \cdot V_c \cdot A_c\right) + V_2 \cdot \left(\rho \cdot V_2 \cdot A_2\right)$$

or (using Eq. 1) $$p_c - p_2 = \rho \cdot V_c \cdot \frac{A_c}{A_2} \cdot \left(V_2 - V_c\right) \qquad (2)$$

The energy equation becomes $$Q_{rate} = \left(u_c + \frac{p_c}{\rho} + V_c^{\,2}\right) \cdot \left(-\rho \cdot V_c \cdot A_c\right) + \left(u_2 + \frac{p_2}{\rho} + V_2^{\,2}\right) \cdot \left(\rho \cdot V_2 \cdot A_2\right)$$

or (using Eq. 1) $$h_{lm} = u_2 - u_c - \frac{Q_{rate}}{m_{rate}} = \frac{V_c^{\,2} - V_2^{\,2}}{2} + \frac{p_c - p_2}{\rho} \qquad (3)$$

Combining Eqs. 2 and 3

$$h_{lm} = \frac{V_c^{\,2} - V_2^{\,2}}{2} + V_c \cdot \frac{A_c}{A_2} \cdot \left(V_2 - V_c\right)$$

$$h_{lm} = \frac{V_c^{\,2}}{2} \cdot \left[1 - \left(\frac{V_2}{V_c}\right)^2\right] + V_c^{\,2} \cdot \frac{A_c}{A_2} \cdot \left[\left(\frac{V_2}{V_c}\right) - 1\right]$$

From Eq. 1

$$C_c = \frac{A_c}{A_2} = \frac{V_2}{V_c}$$

Hence

$$h_{lm} = \frac{V_c^{\,2}}{2} \cdot \left(1 - C_c^{\,2}\right) + V_c^{\,2} \cdot C_c \cdot \left(C_c - 1\right)$$

$$h_{lm} = \frac{V_c^{\,2}}{2} \cdot \left(1 - C_c^{\,2} + 2 \cdot C_c^{\,2} - 2 \cdot C_c\right)$$

$$h_{lm} = \frac{V_c^{\,2}}{2} \cdot \left(1 - C_c\right)^2 \qquad (4)$$

But we have

$$h_{lm} = K \cdot \frac{V_2^{\,2}}{2} = K \cdot \frac{V_c^{\,2}}{2} \cdot \left(\frac{V_2}{V_c}\right)^2 = K \cdot \frac{V_c^{\,2}}{2} \cdot C_c^{\,2} \qquad (5)$$

Hence, comparing Eqs. 4 and 5

$$K = \frac{\left(1 - C_c\right)^2}{C_c^{\,2}}$$

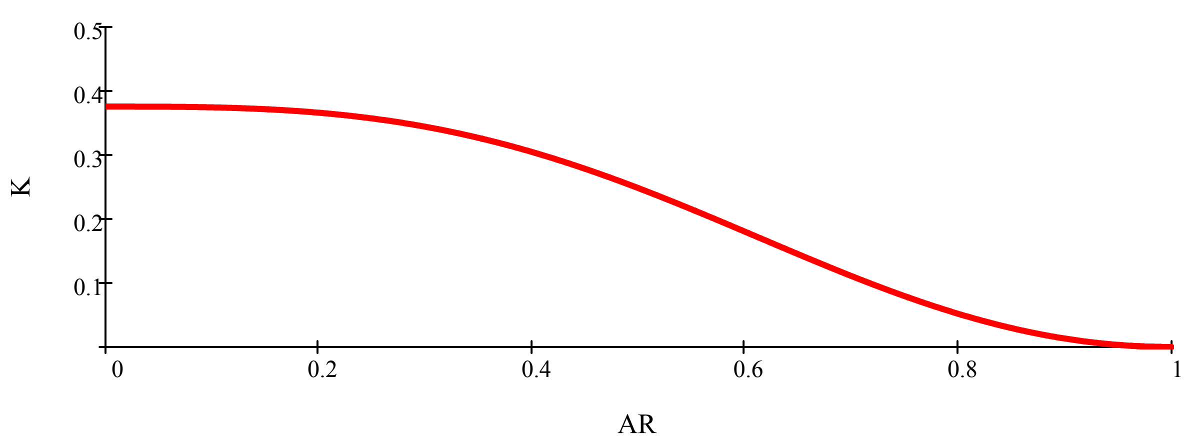
So, finally

$$K = \left(\frac{1}{C_c} - 1\right)^2$$

where

$$C_c = 0.62 + 0.38 \cdot \left(\frac{A_2}{A_1}\right)^3$$

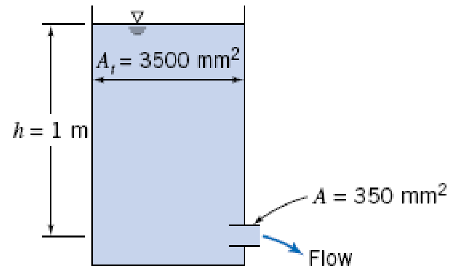
This result,can be plotted in *Excel*. The agreement with Fig. 8.15 is reasonable.



Problem 8.104

[Difficulty: 3]

8.104 Water flows from the tank shown through a very short pipe. Assume the flow is quasi-steady. Estimate the flow rate at the instant shown. How could you improve the flow system if a larger flow rate were desired?



Given: Flow through short pipe

Find: Volume flow rate; How to improve flow rate

Solution:

Basic equations

$$\left(\frac{p_1}{\rho} + \alpha\cdot\frac{V_1^2}{2} + g\cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha\cdot\frac{V_2^2}{2} + g\cdot z_2\right) = h_{lT} \qquad h_{lT} = h_l + h_{lm} = f\cdot\frac{L}{D}\cdot\frac{V_2^2}{2} + K\cdot\frac{V_2^2}{2} \qquad Q = V\cdot A$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) α at 1 and 2 is approximately 1 4) L << so ignore h_l 5) Reentrant

Hence between the free surface (Point 1) and the exit (2) the energy equation becomes

$$\frac{V_1^2}{2} + g\cdot z_1 - \frac{V_2^2}{2} = K\cdot\frac{V_2^2}{2}$$

From continuity $\quad V_1 = V_2\cdot\frac{A_2}{A_1}$

Hence $\quad \frac{V_2^2}{2}\cdot\left(\frac{A_2}{A_1}\right)^2 + g\cdot h - \frac{V_2^2}{2} = K\cdot\frac{V_2^2}{2}$

Solving for V_2 $\quad V_2 = \sqrt{\frac{2\cdot g\cdot h}{\left[1 + K - \left(\frac{A_2}{A_1}\right)^2\right]}}$ $\quad$ and from Table 8.2 $\quad K = 0.78$

Hence $\quad V_2 = \sqrt{2 \times 9.81\cdot\frac{m}{s^2} \times 1\cdot m \times \frac{1}{\left[1 + 0.78 - \left(\frac{350}{3500}\right)^2\right]}}$ $\qquad V_2 = 3.33\,\frac{m}{s}$

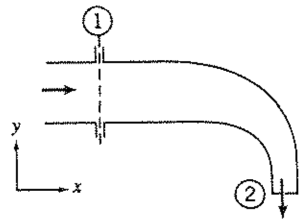
$Q = V_2\cdot A_2 \qquad Q = 3.33\cdot\frac{m}{s} \times 350\cdot mm^2 \times \left(\frac{1\cdot m}{1000\cdot mm}\right)^2 \qquad Q = 1.17 \times 10^{-3}\,\frac{m^3}{s} \qquad Q = 0.070\cdot\frac{m^3}{min}$

The flow rate could be increased by (1) rounding the entrance and/or (2) adding a diffuser (both somewhat expensive)

Problem 8.105

[Difficulty: 3]

8.105 Consider again flow through the elbow analyzed in Example 4.6. Using the given conditions, calculate the minor head loss coefficient for the elbow.



Solution: Apply the energy equation for steady, incompressible pipe flow.

Computing equation: $\left(\frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2} + \cancel{g z_1}\right) - \left(\overset{=0(4)}{\cancel{\frac{p_2}{\rho}}} + \alpha_2 \frac{\bar{V}_2^2}{2} + \cancel{g z_2}\right) = h_{l_T} = \overset{=0}{\cancel{h_l}} + h_{l_m}$

Assumptions: (1) $\alpha_1 = \alpha_2 = 1$

(2) neglect Δz

(3) uniform, incompressible flow so $\bar{V}_1 A_1 = \bar{V}_2 A_2$

(4) use gage pressures

From continuity $\bar{V}_1 = \bar{V}_2 \frac{A_2}{A_1} = 16 \frac{m}{s} \cdot \frac{0.0025\,m^2}{0.01\,m^2} = 4\ m/s$

Then

$$h_{l_m} = \frac{p_{1g}}{\rho} + \frac{\bar{V}_1^2}{2} - \frac{\bar{V}_2^2}{2} = (221-101)10^3 \frac{N}{m^2} \times \frac{m^3}{999\,kg} \times \frac{kg \cdot m}{N \cdot s^2} + \frac{1}{2}\left[(4)^2 - (16)^2\right]\frac{m^2}{s^2}$$

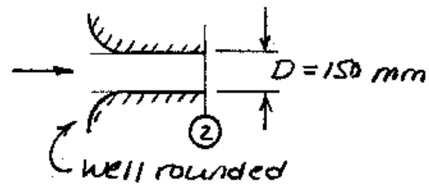
$$h_{l_m} = 0.120\ m^2/s^2$$

But $h_{l_m} = K \frac{\bar{V}_2^2}{2}$; $K = \frac{2 h_{l_m}}{\bar{V}_2^2} = 2 \times 0.120 \frac{m^2}{s^2} \times \frac{s^2}{(16)^2\, m^2} = 9.38 \times 10^{-4}$ $\qquad K$

Problem 8.106

[Difficulty: 3]

8.106 Air flows out of a clean room test chamber through a 150-mm-diameter duct of length L. The original duct had a square edged entrance, but this has been replaced with a well-rounded one. The pressure in the chamber is 2.5 mm of water above ambient. Losses from friction are negligible compared with the entrance and exit losses. Estimate the increase in volume flow rate that results from the change in entrance contour.



Solution: Apply the energy equation for steady, incompressible pipe flow.

Computing equations:

$$\frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \alpha_2 \frac{\bar{V}_2^2}{2} + g z_2 + h_{l_T}$$

$$h_{l_T} = h_l + h_{l_m} \; ; \; h_{l_m} = K_{ent} \frac{\bar{V}_2^2}{2} \; ; \; \Delta p = \rho_{H_2O} g \Delta h$$

($\bar{V}_1 \approx 0$ (1), z terms (2), $h_l \approx 0$ (4))

Assumptions: (1) $\bar{V}_1 \approx 0$ (3) Uniform flow at exit
(2) Neglect elevation changes (4) Neglect frictional losses

Then

$$\frac{\Delta p}{\rho} = \frac{p_1 - p_2}{\rho} = \frac{\bar{V}_2^2}{2} + K_{ent}\frac{\bar{V}_2^2}{2} = \frac{\bar{V}_2^2}{2}(1 + K_{ent}) = \frac{\rho_{H_2O} g \Delta h}{\rho}$$

or

$$\bar{V}_2 = \sqrt{\frac{2(p_1 - p_2)}{\rho(1 + K_{ent})}} = \sqrt{\frac{2 \rho_{H_2O} g \Delta h}{\rho(1 + K_{ent})}}$$

From Table 8.2, $K_{ent} = 0.5$ for square-edged, $K_{ent} = 0.04$ for rounded entrance.

$$\bar{V}_2 = \sqrt{\frac{2}{1.50} \times 999 \frac{kg}{m^3} \times 9.81 \frac{m}{s^2} \times 0.0025\, m \times \frac{m^3}{1.23\, kg}} = 5.15 \text{ m/s}$$

$$\bar{V}_2 \text{(modified)} = \sqrt{\frac{2}{1.04} \times 999 \frac{kg}{m^3} \times 9.81 \frac{m}{s^2} \times 0.0025\, m \times \frac{m^3}{1.23\, kg}} = 6.19 \text{ m/s}$$

Since $Q = \bar{V}A$, then

$$\Delta Q = (\bar{V}_{2m} - \bar{V}_2)A = (6.19 - 5.15)\frac{m}{s} \times \frac{\pi}{4}(0.150)^2 m^2 = 0.0184 \text{ m}^3/\text{s}$$ ΔQ

The percentage improvement is

$$\% = \frac{\Delta Q}{Q} \times 100 = \frac{\bar{V}_{2m} - \bar{V}_2}{\bar{V}_2} \times 100 = \frac{6.19 - 5.15}{5.15} \times 100 = 20.2 \text{ percent}$$

Problem 8.107

[Difficulty: 3]

8.107 A water tank (open to the atmosphere) contains water to a depth of 5 m. A 25-mm-diameter hole is punched in the bottom. Modeling the hole as square-edged, estimate the flow rate (L/s) exiting the tank. If you were to stick a short section of pipe into the hole, by how much would the flow rate change? If instead you were to machine the inside of the hole to give it a rounded edge ($r = 5$ mm), by how much would the flow rate change?

Given: Flow out of water tank

Find: Volume flow rate using hole; Using short pipe section; Using rounded edge

Solution:

Basic equations
$$\left(\frac{p_1}{\rho} + \alpha\cdot\frac{V_1^2}{2} + g\cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha\cdot\frac{V_2^2}{2} + g\cdot z_2\right) = h_{lT} \quad h_{lT} = h_l + h_{lm} = f\cdot\frac{L}{D}\cdot\frac{V_2^2}{2} + K\cdot\frac{V_2^2}{2} \quad Q = V\cdot A$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) α at 1 and 2 is approximately 1 4) V_l << 5) L << so $h_l = 0$

Available data $\quad D = 25\cdot mm \quad r = 5\cdot mm \quad h = 5\cdot m$

Hence for all three cases, between the free surface (Point 1) and the exit (2) the energy equation becomes

$$g\cdot z_1 - \frac{V_2^2}{2} = K\cdot\frac{V_2^2}{2} \qquad \text{and solving for } V_2 \qquad V_2 = \sqrt{\frac{2\cdot g\cdot h}{(1+K)}}$$

From Table 8.2 $\quad K_{hole} = 0.5 \quad$ for a hole (assumed to be square-edged) $\qquad K_{pipe} = 0.78 \quad$ for a short pipe (rentrant)

Also, for a rounded edge $\quad \frac{r}{D} = \frac{5\cdot mm}{25\cdot mm} = 0.2 > 0.15 \quad$ so from Table 8.2 $\quad K_{round} = 0.04$

Hence for the hole $\quad V_2 = \sqrt{2\times 9.81\cdot\frac{m}{s^2}\times 5\cdot m\times\frac{1}{(1+0.5)}} \qquad V_2 = 8.09\,\frac{m}{s}$

$$Q = V_2\cdot A_2 \qquad Q = 8.09\cdot\frac{m}{s}\times\frac{\pi}{4}\times(0.025\,m)^2 \qquad Q = 3.97\times 10^{-3}\,\frac{m^3}{s} \qquad Q = 3.97\cdot\frac{L}{s}$$

Hence for the pipe $\quad V_2 = \sqrt{2\times 9.81\cdot\frac{m}{s^2}\times 5\cdot m\times\frac{1}{(1+0.78)}} \qquad V_2 = 7.42\,\frac{m}{s}$

$$Q = V_2\cdot A_2 \qquad Q = 7.42\cdot\frac{m}{s}\times\frac{\pi}{4}\times(0.025\,m)^2 \qquad Q = 3.64\times 10^{-3}\,\frac{m^3}{s} \qquad Q = 3.64\cdot\frac{L}{s}$$

Hence the change in flow rate is $\quad 3.64 - 3.97 = -0.33\cdot\frac{L}{s} \qquad$ The pipe leads to a LOWER flow rate

Hence for the rounded $\quad V_2 = \sqrt{2\times 9.81\cdot\frac{m}{s^2}\times 5\cdot m\times\frac{1}{(1+0.04)}} \qquad V_2 = 9.71\,\frac{m}{s} \qquad Q = V_2\cdot A_2 \qquad Q = 4.77\cdot\frac{L}{s}$

Hence the change in flow rate is $\quad 4.77 - 3.97 = 0.8\cdot\frac{L}{s} \qquad$ The rounded edge leads to a HIGHER flow rate

Problem 8.108

[Difficulty: 2]

8.108 A conical diffuser is used to expand a pipe flow from a diameter of 100 mm to a diameter of 150 mm. Find the minimum length of the diffuser if we want a loss coefficient (a) $K_{\text{diffuser}} \le 0.2$, (b) $K_{\text{diffuser}} \le 0.35$.

Given: Data on inlet and exit diameters of diffuser

Find: Minimum lengths to satisfy requirements

Solution:

Given data $\quad D_1 = 100 \cdot mm \quad\quad D_2 = 150 \cdot mm$

The governing equations for the diffuser are

$$h_{lm} = K \cdot \frac{V_1^2}{2} = \left(C_{pi} - C_p\right) \cdot \frac{V_1^2}{2} \quad (8.44)$$

and

$$C_{pi} = 1 - \frac{1}{AR^2} \quad (8.42)$$

Combining these we obtain an expression for the loss coefficient *K*

$$K = 1 - \frac{1}{AR^2} - C_p \quad (1)$$

The area ratio *AR* is $\quad AR = \left(\frac{D_2}{D_1}\right)^2 \quad\quad AR = 2.25$

The pressure recovery coefficient C_p is obtained from Eq. 1 above once we select *K*; then, with C_p and *AR* specified, the minimum value of N/R_1 (where *N* is the length and R_1 is the inlet radius) can be read from Fig. 8.15

(a) $\quad K = 0.2 \quad\quad C_p = 1 - \frac{1}{AR^2} - K \quad\quad C_p = 0.602$

From Fig. 8.15 $\quad \frac{N}{R_1} = 5.5 \quad\quad R_1 = \frac{D_1}{2} \quad\quad R_1 = 50 \cdot mm$

$N = 5.5 \cdot R_1 \quad\quad N = 275 \cdot mm$

(b) $\quad K = 0.35 \quad\quad C_p = 1 - \frac{1}{AR^2} - K \quad\quad C_p = 0.452$

From Fig. 8.15 $\quad \frac{N}{R_1} = 3$

$N = 3 \cdot R_1 \quad\quad N = 150 \cdot mm$

Problem 8.109

[Difficulty: 3]

8.109 A conical diffuser of length 6 in. is used to expand a pipe flow from a diameter of 2 in. to a diameter of 3.5 in. For a water flow rate of 750 gal/min, estimate the static pressure rise. What is the approximate value of the loss coefficient?

Given: Data on geometry of conical diffuser; flow rate

Find: Static pressure rise; loss coefficient

Solution:

Basic equations $C_p = \dfrac{p_2 - p_1}{\frac{1}{2}\cdot\rho\cdot V_1^2}$ (8.41) $h_{lm} = K\cdot\dfrac{V_1^2}{2} = \left(C_{pi} - C_p\right)\cdot\dfrac{V_1^2}{2}$ (8.44) $C_{pi} = 1 - \dfrac{1}{AR^2}$ (8.42)

Given data $D_1 = 2\cdot in$ $D_2 = 3.5\cdot in$ $N = 6\cdot in$ (N = length) $Q = 750\cdot gpm$

From Eq. 8.41 $\Delta p = p_2 - p_1 = \frac{1}{2}\cdot\rho\cdot V_1^2\cdot C_p$ (1)

Combining Eqs. 8.44 and 8.42 we obtain an expression for the loss coefficient K $K = 1 - \dfrac{1}{AR^2} - C_p$ (2)

The pressure recovery coefficient C_p for use in Eqs. 1 and 2 above is obtained from Fig. 8.15 once compute AR and the dimensionless length N/R_1 (where R_1 is the inlet radius)

The aspect ratio AR is $AR = \left(\dfrac{D_2}{D_1}\right)^2$ $AR = \left(\dfrac{3.5}{2}\right)^2$ $AR = 3.06$

$R_1 = \dfrac{D_1}{2}$ $R_1 = 1\cdot in$ Hence $\dfrac{N}{R_1} = 6$

From Fig. 8.15, with $AR = 3.06$ and the dimensionless length $N/R_1 = 6$, we find $C_p = 0.6$

To complete the calculations we need V_1 $V_1 = \dfrac{Q}{\frac{\pi}{4}\cdot D_1^2}$ $V_1 = \dfrac{4}{\pi}\times 750\cdot\dfrac{gal}{min}\times\dfrac{1\cdot ft^3}{7.48\cdot gal}\times\dfrac{1\cdot min}{60\cdot s}\times\left(\dfrac{1}{\frac{2}{12}\cdot ft}\right)^2$ $V_1 = 76.6\cdot\dfrac{ft}{s}$

We can now compute the pressure rise and loss coefficient from Eqs. 1 and 2 $\Delta p = \frac{1}{2}\cdot\rho\cdot V_1^2\cdot C_p$

$$\Delta p = \frac{1}{2}\times 1.94\cdot\frac{slug}{ft^3}\times\left(76.6\cdot\frac{ft}{s}\right)^2\times 0.6\times\frac{lbf\cdot s^2}{slug\cdot ft}\times\left(\frac{1\cdot ft}{12\cdot in}\right)^2 \qquad \Delta p = 23.7\cdot psi$$

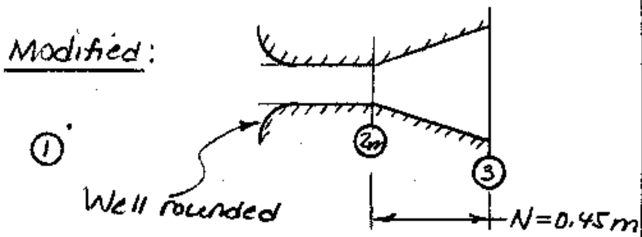
$$K = 1 - \frac{1}{AR^2} - C_p \qquad K = 1 - \frac{1}{3.06^2} - 0.6 \qquad K = 0.293$$

Problem 8.110

[Difficulty: 4]

8.110 Space has been found for a conical diffuser 0.45 m long in the clean room ventilation system described in Problem 8.106. The best diffuser of this size is to be used. Assume that data from Fig. 8.16 may be used. Determine the appropriate diffuser angle and area ratio for this installation and estimate the volume flow rate that will be delivered after it is installed.

Modified:



$h_1 - h_3 = 2.5\text{ mm } H_2O$

Solution: Apply the energy equation for steady, incompressible pipe flow.

Computing equations:

$$\frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \alpha_2 \frac{\bar{V}_2^2}{2} + g z_2 + h_{lT} \quad \text{(or to section 3)}$$

$\bar{V}_1^2 \approx 0\,(1)$

$$h_{lT} = h_l + h_{lm}\,; \quad h_{lm} = K_{ent} + h_{l\,diffuser}\,; \quad \Delta p = \rho_{H_2O}\, g\, \Delta h$$

$h_l \approx 0\,(4)$

From Eq. 8.42, $h_{l\,diffuser} = \frac{\bar{V}_2^2}{2}\left[1 - \frac{1}{AR^2} - C_p\right]$

Assumptions: (1) $\bar{V}_1 \approx 0$
(2) Neglect Δz
(3) Uniform flow at each section
(4) Neglect frictional losses

For the original system, $\frac{p_1 - p_2}{\rho} = \frac{\bar{V}_2^2}{2} + K_{ent}\frac{\bar{V}_2^2}{2} = 1.5\frac{\bar{V}_2^2}{2} = \frac{\rho_{H_2O}\, g\, \Delta h}{\rho}$ $\quad (K_{ent} = 0.5)$

Thus

$$\bar{V}_2 = \sqrt{\frac{2}{1.5}\frac{\rho_{H_2O}\, g\, \Delta h}{\rho}} = \sqrt{\frac{2}{1.5} \times 999\frac{kg}{m^3} \times 9.81\frac{m}{s^2} \times 0.0025\,m \times \frac{m^3}{1.23\,kg}} = 5.15\text{ m/s}$$

For the modified system, $\frac{p_1 - p_3}{\rho} = \frac{\bar{V}_3^2}{2} + K_{ent}\frac{\bar{V}_2^2}{2} + \frac{\bar{V}_2^2}{2}\left[1 - \frac{1}{AR^2} - C_p\right] = \frac{\bar{V}_2^2}{2}\left[1 + K_{ent} - C_p\right]$

since $\bar{V}_3^2 = \bar{V}_2^2 \frac{1}{AR^2}$. Thus the best diffuser has the highest C_p.

From Fig. 8.16, $C_p = f(N/R_1, AR)$. $N/R_1 = 2N/D_1 = 2 \times \frac{0.45\,m}{0.15\,m} = 6$. From the figure, the best diffuser is

$$C_p \approx 0.62 \text{ at } AR \approx 2.7 \text{ and } 2\phi \approx 12 \text{ deg}$$

AR, :

For the modified system,

$$\bar{V}_2 = \sqrt{\frac{2}{1 + K_{ent} - C_p}\frac{\rho_{H_2O}\, g\, \Delta h}{\rho}} = \sqrt{\frac{2}{1 + 0.04 - 0.62} \times 999\frac{kg}{m^3} \times 9.81\frac{m}{s^2} \times 0.0025\,m \times \frac{m^3}{1.23\,kg}} = 9.74\text{ m/s}$$

and

$$Q = \bar{V}_2 A_2 = 9.74\frac{m}{s} \times \frac{\pi}{4}(0.15)^2 m^2 = 0.172\text{ m}^3/\text{s}$$

Q

$$\left\{ \text{The improvement is } \frac{Q_m - Q}{Q} \times 100 = \frac{\bar{V}_m - \bar{V}}{\bar{V}} \times 100 = \frac{9.74 - 5.15}{5.15} \times 100 = 89.1 \text{ percent more} \right\}$$

Problem 8.111

[Difficulty: 4]

8.111 By applying the basic equations to a control volume starting at the expansion and ending downstream, analyze flow through a sudden expansion (assume the inlet pressure p_1 acts on the area A_2 at the expansion). Develop an expression for and plot the minor head loss across the expansion as a function of area ratio, and compare with the data of Fig. 8.15.

Given: Sudden expansion

Find: Expression for minor head loss; compare with Fig. 8.15; plot

Solution:

The governing CV equations (mass, momentum, and energy) are

$$\frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A} = 0 \qquad (4.12)$$

$$F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u \, \rho \, d\forall + \int_{CS} u \, \rho \vec{V} \cdot d\vec{A} \qquad (4.18a)$$

$$\dot{Q} - \dot{W}_s - \dot{W}_{shear} - \dot{W}_{other} = \frac{\partial}{\partial t}\int_{CV} e \, \rho \, d\forall + \int_{CS}\left(u + pv + \frac{V^2}{2} + gz\right)\rho \vec{V} \cdot d\vec{A} \qquad (4.56)$$

Assume: 1) Steady flow 2) Incompressible flow 3) Uniform flow at each section 4) Horizontal: no body force 5) No shaft work 6) Neglect viscous friction 7) Neglect gravity

The mass equation becomes $\quad V_1 \cdot A_1 = V_2 \cdot A_2 \qquad (1)$

The momentum equation becomes $\quad p_1 \cdot A_2 - p_2 \cdot A_2 = V_1 \cdot \left(-\rho \cdot V_1 \cdot A_1\right) + V_2 \cdot \left(\rho \cdot V_2 \cdot A_2\right)$

or (using Eq. 1) $\quad p_1 - p_2 = \rho \cdot V_1 \cdot \frac{A_1}{A_2} \cdot \left(V_2 - V_1\right) \qquad (2)$

The energy equation becomes $\quad Q_{rate} = \left(u_1 + \frac{p_1}{\rho} + V_1^{\,2}\right) \cdot \left(-\rho \cdot V_1 \cdot A_1\right) + \left(u_2 + \frac{p_2}{\rho} + V_2^{\,2}\right) \cdot \left(\rho \cdot V_2 \cdot A_2\right)$

or (using Eq. 1) $\quad h_{lm} = u_2 - u_1 - \frac{Q_{rate}}{m_{rate}} = \frac{V_1^{\,2} - V_2^{\,2}}{2} + \frac{p_1 - p_2}{\rho} \qquad (3)$

Combining Eqs. 2 and 3 $\quad h_{lm} = \frac{V_1^{\,2} - V_2^{\,2}}{2} + V_1 \cdot \frac{A_1}{A_2} \cdot \left(V_2 - V_1\right)$

$$h_{lm} = \frac{V_1^{\,2}}{2} \cdot \left[1 - \left(\frac{V_2}{V_1}\right)^2\right] + V_1^{\,2} \cdot \frac{A_1}{A_2} \cdot \left[\left(\frac{V_2}{V_1}\right) - 1\right]$$

From Eq. 1 $$AR = \frac{A_1}{A_2} = \frac{V_2}{V_1}$$

Hence $$h_{lm} = \frac{V_1^{\;2}}{2}\cdot\left(1 - AR^2\right) + V_1^{\;2}\cdot AR\cdot(AR - 1)$$

$$h_{lm} = \frac{V_1^{\;2}}{2}\cdot\left(1 - AR^2 + 2\cdot AR^2 - 2\cdot AR\right)$$

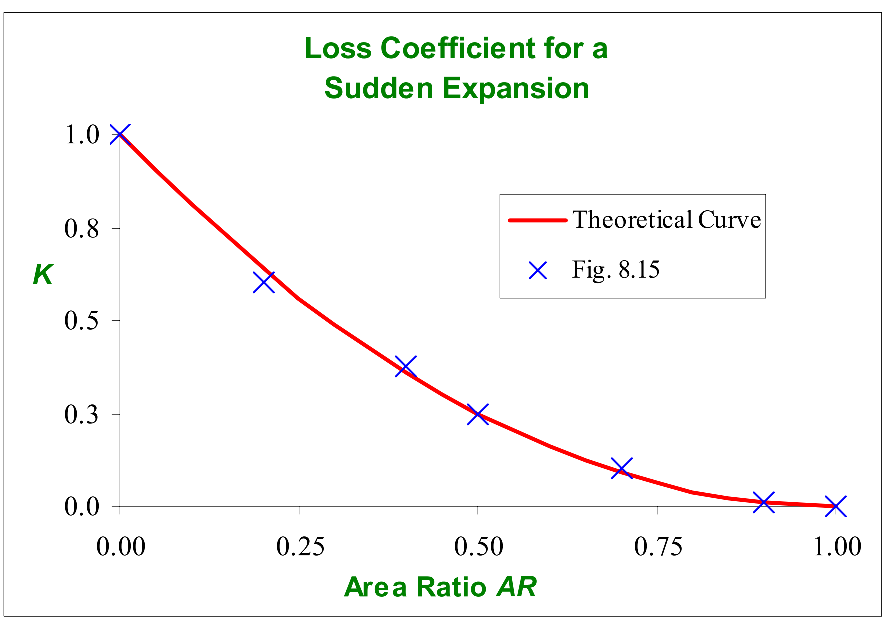
$$h_{lm} = K\cdot\frac{V_1^{\;2}}{2} = (1 - AR)^2\cdot\frac{V_1^{\;2}}{2}$$

Finally $$K = (1 - AR)^2$$

This result, and the curve of Fig. 8.15, are shown below as computed in *Excel*. The agreement is excellent.

AR	K_{CV}	$K_{\text{Fig. 8.15}}$
0.0	1.00	1.00
0.1	0.81	
0.2	0.64	0.60
0.3	0.49	
0.4	0.36	0.38
0.5	0.25	0.25
0.6	0.16	
0.7	0.09	0.10
0.8	0.04	
0.9	0.01	0.01
1.0	0.00	0.00

(Data from Fig. 8.15
is "eyeballed")



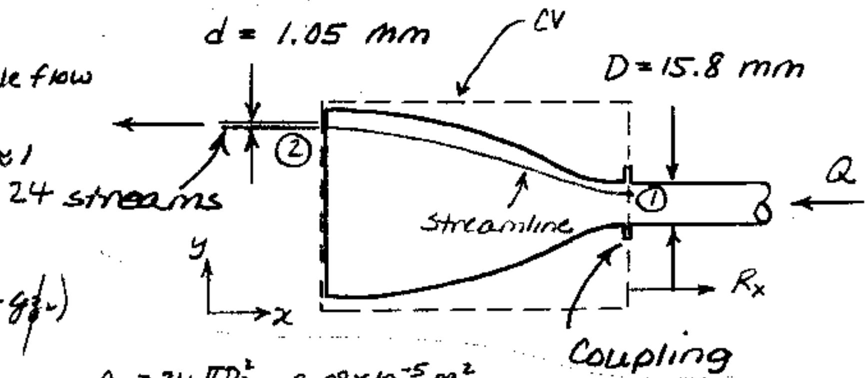
Problem 8.112

[Difficulty: 3]

8.112 Water at 45°C enters a shower head through a circular tube with 15.8 mm inside diameter. The water leaves in 24 streams, each of 1.05 mm diameter. The volume flow rate is 5.67 L/min. Estimate the minimum water pressure needed at the inlet to the shower head. Evaluate the force needed to hold the shower head onto the end of the circular tube. Indicate clearly whether this is a compression or a tension force.

Solution: Apply the energy equation for steady, incompressible pipe flow, and the x component of momentum, using the CV shown.

Assume: (1) Steady flow
(2) Incompressible flow
(3) Neglect changes in z
(4) Uniform flow: $\alpha_1 = \alpha_2 \approx 1$
(5) Use gage pressures



Then

$$\frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2} + g\cancel{z_1} - \Big(\underbrace{\frac{p_2}{\rho}}_{=0\,(gage)} + \alpha_2 \frac{\bar{V}_2^2}{2} + g\cancel{z_2}\Big) = h_{l_T} = \underbrace{\cancel{h_l}}_{=0} + h_{l_m}$$

$A_2 = 24 \frac{\pi D_2^2}{4} = 2.08 \times 10^{-5}\, m^2$

$A_1 = \frac{\pi D_1^2}{4} = 1.96 \times 10^{-4}\, m^2$

$$\bar{V}_1 = \frac{Q}{A_1} = 5.67 \frac{L}{min} \times \frac{1}{1.96 \times 10^{-4}\, m^2} \times \frac{m^3}{1000\, L} \times \frac{min}{60\, s} = 0.487\ m/s$$

$$\bar{V}_2 = \bar{V}_1 \frac{A_1}{A_2} = 0.487 \frac{m}{s} \times \frac{1.96 \times 10^{-4}\, m^2}{2.08 \times 10^{-5}\, m^2} = 4.59\ m/s$$

Use $K = 0.5$, for a square-edged orifice, $\rho = 990\ kg/m^3$ (Table A.8). Then

$$p_1 = \frac{\rho}{2}\left(\bar{V}_2^2 + K \bar{V}_2^2 - \bar{V}_1^2\right) = \frac{\rho}{2}\left[(1+K)\bar{V}_2^2 - \bar{V}_1^2\right]$$

$$p_1 = \frac{1}{2} \times 990 \frac{kg}{m^3}\left[(1+0.5)(4.59)^2 - (0.487)^2\right] \frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m} = 15.5\ kPa\ (gage)$$ p_1

Use momentum to find force:

Basic equation: $F_{S_x} + \underbrace{\cancel{F_{B_x}}}_{=0\,(6)} = \underbrace{\cancel{\frac{\partial}{\partial t}\int_{CV} u \rho\, d\forall}}_{=0\,(1)} + \int_{CS} u \rho \vec{V} \cdot d\vec{A}$

Assume: (6) $F_{B_x} = 0$

Then $R_x - p_{1g} A_1 = u_1\{\overset{\text{Step } ①}{-\rho Q}\} + u_2\{+\rho Q\} = -V_1\{-\rho Q\} + (-V_2)\{+\rho Q\} = \rho Q (V_1 - V_2)$

Step ②: $u_1 = -V_1 \quad u_2 = -V_2$

$$R_x = p_{1g} A_1 + \rho Q (V_1 - V_2) = 15.5 \times 10^3 \frac{N}{m^2} \times 1.96 \times 10^{-4}\, m^2 + 990 \frac{kg}{m^3} \times 5.67 \frac{L}{min} \times (0.487 - 4.59)\frac{m}{s} \times \frac{m^3}{1000\, L} \times \frac{min}{60\, s}$$

$R_x = 2.65\ N$ (in direction shown, i.e., <u>tension</u>) R_x

Problem 8.113

[Difficulty: 2]

8.113 Analyze flow through a sudden expansion to obtain an expression for the upstream average velocity $\overline{V}_1$ in terms of the pressure change $\Delta p = p_2 - p_1$, area ratio AR, fluid density ρ, and loss coefficient K. If the flow were frictionless, would the flow rate indicated by a measured pressure change be higher or lower than a real flow, and why? Conversely, if the flow were frictionless, would a given flow generate a larger or smaller pressure change, and why?

Given: Sudden expansion

Find: Expression for upstream average velocity

Solution:

The basic equation is

$$\left(\frac{p_1}{\rho} + \alpha_1 \cdot \frac{V_1^2}{2} + g \cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \cdot \frac{V_2^2}{2} + g \cdot z_2\right) = h_{lT} \qquad (8.29)$$

$$h_{lT} = h_l + K \cdot \frac{V^2}{2}$$

Assume: 1) Steady flow 2) Incompressible flow 3) $h_l = 0$ 4) $\alpha_1 = \alpha_2 = 1$ 5) Neglect gravity

The mass equation is $V_1 \cdot A_1 = V_2 \cdot A_2$ so $V_2 = V_1 \cdot \frac{A_1}{A_2}$

$$V_2 = AR \cdot V_1 \qquad (1)$$

Equation 8.29 becomes

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} = \frac{p_1}{\rho} + \frac{V_1^2}{2} + K \cdot \frac{V_1^2}{2}$$

or (using Eq. 1)

$$\frac{\Delta p}{\rho} = \frac{p_2 - p_1}{\rho} = \frac{V_1^2}{2} \cdot \left(1 - AR^2 - K\right)$$

Solving for V_1

$$V_1 = \sqrt{\frac{2 \cdot \Delta p}{\rho \cdot \left(1 - AR^2 - K\right)}}$$

If the flow were frictionless, $K = 0$, so

$$V_{inviscid} = \sqrt{\frac{2 \cdot \Delta p}{\rho \cdot \left(1 - AR^2\right)}} < V_1$$

Hence the flow rate indicated by a given Δp would be lower

If the flow were frictionless, $K = 0$, so

$$\Delta p_{invscid} = \frac{V_1^2}{2} \cdot \left(1 - AR^2\right)$$

compared to

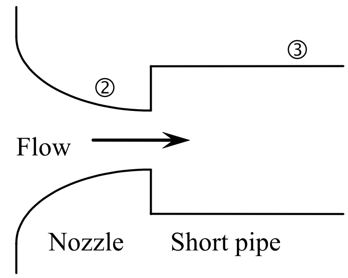
$$\Delta p = \frac{V_1^2}{2} \cdot \left(1 - AR^2 - K\right)$$

Hence a given flow rate would generate a larger Δp for inviscid flow

Problem 8.114

[Difficulty: 4]

8.114 Water discharges to atmosphere from a large reservoir through a moderately rounded horizontal nozzle of 25 mm diameter. The free surface is 2.5 m above the nozzle exit plane. Calculate the change in flow rate when a short section of 50-mm-diameter pipe is attached to the end of the nozzle to form a sudden expansion. Determine the location and estimate the magnitude of the minimum pressure with the sudden expansion in place. If the flow were frictionless (with the sudden expansion in place), would the minimum pressure be higher, lower, or the same? Would the flow rate be higher, lower, or the same?



Given: Flow out of water tank through a nozzle

Find: Change in flow rate when short pipe section is added; Minimum pressure; Effect of frictionless flow

Solution:

Basic equations
$$\left(\frac{p_1}{\rho} + \alpha\cdot\frac{V_1^2}{2} + g\cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha\cdot\frac{V_2^2}{2} + g\cdot z_2\right) = h_{lT} \qquad h_{lT} = h_l + h_{lm} = f\cdot\frac{L}{D}\cdot\frac{V_2^2}{2} + K\cdot\frac{V_2^2}{2} \qquad Q = V\cdot A$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) α at 1 and 2 is approximately 1 4) $V_1 <<$ 5) $L <<$ so $h_l = 0$

Available data $\quad D_2 = 25\cdot mm \quad r = 0.02\cdot D_2 \quad D_3 = 50\cdot mm \quad r = 0.5\cdot mm \quad z_1 = 2.5\cdot m \quad \rho = 999\cdot\frac{kg}{m^3}$

For a rounded edge, we choose the first value from Table 8.2 $\quad K_{nozzle} = 0.28$

Hence for the nozzle case, between the free surface (Point 1) and the exit (2) the energy equation becomes

$$g\cdot z_1 - \frac{V_2^2}{2} = K_{nozzle}\cdot\frac{V_2^2}{2}$$

Solving for V_2
$$V_2 = \sqrt{\frac{2\cdot g\cdot z_1}{\left(1 + K_{nozzle}\right)}}$$

Hence
$$V_2 = \sqrt{2 \times 9.81\cdot\frac{m}{s^2} \times 2.5\cdot m \times \frac{1}{(1 + 0.28)}} \qquad V_2 = 6.19\,\frac{m}{s}$$

$$Q = V_2\cdot A_2 \qquad Q = 6.19\cdot\frac{m}{s} \times \frac{\pi}{4} \times (0.025\cdot m)^2 \qquad Q = 3.04 \times 10^{-3}\,\frac{m^3}{s} \qquad Q = 3.04\,\frac{L}{s}$$

When a small piece of pipe is added the energy equation between the free surface (Point 1) and the exit (3) becomes

$$g\cdot z_1 - \frac{V_3^2}{2} = K_{nozzle}\cdot\frac{V_2^2}{2} + K_e\cdot\frac{V_2^2}{2}$$

From continuity
$$V_3 = V_2\cdot\frac{A_2}{A_3} = V_2\cdot AR$$

Solving for V_2 $$V_2 = \sqrt{\frac{2\cdot g\cdot z_1}{\left(AR^2 + K_{nozzle} + K_e\right)}}$$

We need the AR for the sudden expansion $AR = \frac{A_2}{A_3} = \left(\frac{D_2}{D_3}\right)^2 = \left(\frac{25}{50}\right)^2 = 0.25$ $\quad AR = 0.25$

From Fig. 8.15 for AR = 0.25 $\quad K_e = 0.6$

$$V_2 = \sqrt{\frac{2\cdot g\cdot z_1}{\left(AR^2 + K_{nozzle} + K_e\right)}}$$

Hence $$V_2 = \sqrt{2 \times 9.81\cdot\frac{m}{s^2} \times 2.5\cdot m \times \frac{1}{\left(0.25^2 + 0.28 + 0.6\right)}} \qquad V_2 = 7.21\frac{m}{s}$$

$$Q = V_2\cdot A_2 \qquad Q = 7.21\cdot\frac{m}{s} \times \frac{\pi}{4} \times (0.025\cdot m)^2 \qquad Q = 3.54 \times 10^{-3}\frac{m^3}{s} \qquad Q = 3.54\frac{L}{s}$$

Comparing results we see the flow increases from 3.04 L/s to 3.54 L/s $\quad \frac{\Delta Q}{Q} = \frac{3.54 - 3.04}{3.04} = 16.4\cdot\%$

The flow increases because the effect of the pipe is to allow an exit pressure at the nozzle LESS than atmospheric!

The minimum pressure point will now be at Point 2 (it was atmospheric before adding the small pipe). The energy equation between 1 and 2 is

$$g\cdot z_1 - \left(\frac{p_2}{\rho} + \frac{V_2^2}{2}\right) = K_{nozzle}\cdot\frac{V_2^2}{2}$$

Solving for p_2 $$p_2 = \rho\cdot\left[g\cdot z_1 - \frac{V_2^2}{2}\cdot\left(K_{nozzle} + 1\right)\right]$$

Hence $$p_2 = 999\cdot\frac{kg}{m^3} \times \left[9.81\cdot\frac{m}{s^2} \times 2.5\cdot m - \frac{1}{2} \times \left(7.21\cdot\frac{m}{s}\right)^2 \times (0.28 + 1)\right] \times \frac{lbf\cdot s^2}{slug\cdot ft} \qquad p_2 = -8.736\cdot kPa$$

If the flow were frictionless the the two loss coeffcients would be zero. Instead of

Instead of $$V_2 = \sqrt{\frac{2\cdot g\cdot z_1}{\left(AR^2 + K_{nozzle} + K_e\right)}}$$ we'd have $$V_2 = \sqrt{\frac{2\cdot g\cdot z_1}{AR^2}}$$ which is larger

If V_2 is larger, then p_2, through Bernoulli, would be lower (more negative)

Problem 8.115

[Difficulty: 2]

8.115 Water flows steadily from a large tank through a length of smooth plastic tubing, then discharges to atmosphere. The tubing inside diameter is 3.18 mm, and its length is 15.3 m. Calculate the maximum volume flow rate for which flow in the tubing will remain laminar. Estimate the water level in the tank below which flow will be laminar (for laminar flow, $\alpha = 2$ and $K_{ent} = 1.4$).

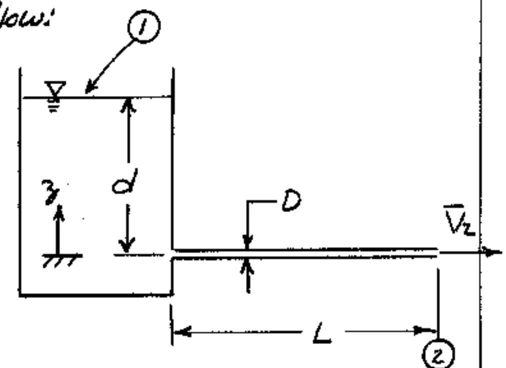
Solution: Assume water at 20°C. From Table A.8, $\rho = 998\ kg/m^3$, $\nu = 1.00\times10^{-6}\ m^2/s$.

$$Re = \frac{\rho \bar{V} D}{\mu} = \frac{\bar{V} D}{\nu} \le 2300\ ;\quad \bar{V}_{max} = \frac{2300\nu}{D} = 2300 \times 1.00\times10^{-6}\frac{m^2}{s}\times\frac{1}{0.00318\ m} = 0.723\ m/s \qquad \bar{V}$$

$$Q = \bar{V}A\ ;\quad A = \frac{\pi D^2}{4} = \frac{\pi}{4}(0.00318)^2 m^2 = 7.94\times10^{-6}\ m^2$$

$$Q = 0.723\frac{m}{s}\times 7.94\times10^{-6}\ m^2 = 5.74\times10^{-6}\frac{m^3}{s}\times 10^3\frac{L}{m^3}\times 60\frac{s}{min} = 0.345\ L/min \qquad Q$$

Apply energy equation for steady, ρ = constant pipe flow:



Computing Equation:

$$\left(\frac{\cancel{p_1}}{\rho} + \alpha_1\frac{\cancel{\bar{V}_1^2}}{2} + gz_1\right) - \left(\frac{\cancel{p_2}}{\rho} + \alpha_2\frac{\bar{V}_2^2}{2} + g\cancel{z_2}\right) = h_{\ell T}$$

($\bar{V}_1^2 \approx 0$, $z_2 = 0$)

$$h_{\ell T} = h_{\ell m} + h_\ell$$

Assumptions: (1) $p_1 = p_2 = p_{atm}$
(2) $\bar{V}_1 \approx 0$
(3) $K_{ent} = 1.4$ (given)

Then $$gd = \alpha_2\frac{\bar{V}_2^2}{2} + K_{ent}\frac{\bar{V}_2^2}{2} + f\frac{L}{D}\frac{\bar{V}_2^2}{2} \quad \text{or} \quad d = \frac{\bar{V}_2^2}{2g}\left(\alpha_2 + K_{ent} + f\frac{L}{D}\right)$$

For laminar flow, $f = \frac{64}{Re} = \frac{64}{2300} = 0.0278$. Substituting

$$d = \frac{1}{2}\times(0.723)^2\frac{m^2}{s^2}\times\frac{s^2}{9.81\ m}\left(2.0 + 1.4 + 0.0278\ \frac{15.3\ m}{0.00318\ m}\right)$$

$$d = 3.65\ m \qquad d$$

Problem 8.116

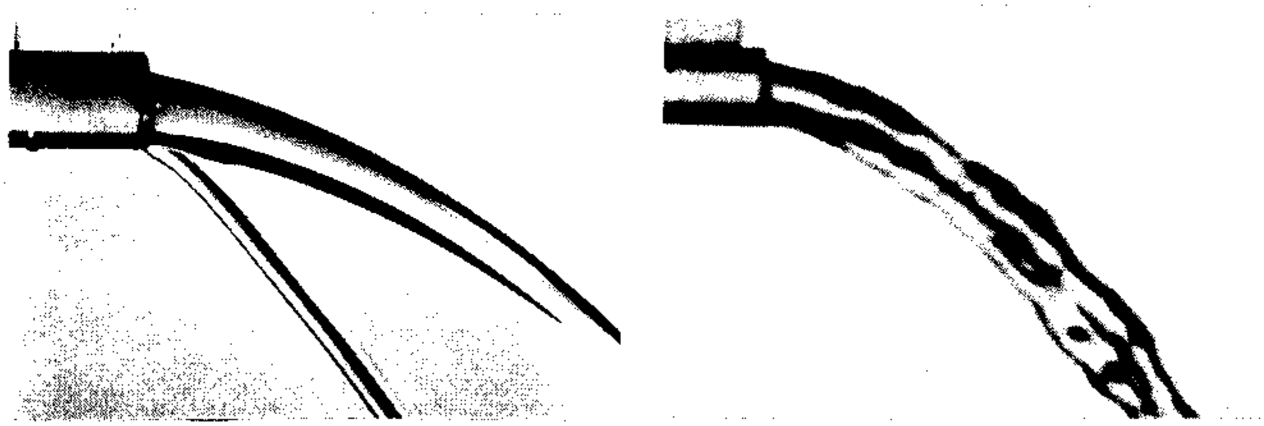
[Difficulty: 4]

8.116 You are asked to compare the behavior of fully developed laminar flow and fully developed turbulent flow in a horizontal pipe under different conditions. For the same flow rate, which will have the larger centerline velocity? Why? If the pipe discharges to atmosphere, what would you expect the trajectory of the discharge stream to look like (for the same flow rate)? Sketch your expectations for each case. For the same flow rate, which flow would give the larger wall shear stress? Why? Sketch the shear stress distribution τ/τ_w as a function of radius for each flow. For the same Reynolds number, which flow would have the larger pressure drop per unit length? Why? For a given imposed pressure differential, which flow would have the larger flow rate? Why?

Discussion: In the following fully developed laminar flow and fully developed turbulent flow in a pipe are compared:

(a) For the same flow rate, laminar flow has the higher maximum velocity, because the turbulent velocity profile is more blunt.

(b) The trajectory of the discharge stream spreads out for laminar flow because of the large variation in velocity across the pipe exit. For turbulent flow the exit profile is more nearly uniform (except for the region adjacent to the wall) and hence the trajectory is more uniform. Since centerline velocity is larger for laminar flow, liquid travels the greatest horizontal distance. Trajectories for the two flow cases are shown below:



(i) Laminar flow (ii) Turbulent flow

(c) For the same flow rate (same mean velocity), turbulent flow has larger wall shear stress because of the larger velocity gradient at the pipe wall. For fully developed flow the pressure force driving the flow is balanced by the shear force at the wall.

(d) Shear stress varies linearly with radius for both flow cases, from its maximum value at the wall to zero at the pipe centerline.

(e) For the same Reynolds number, turbulent flow has a larger pressure drop per unit length because the friction factor is larger.

(f) For a given pressure drop (per unit length), laminar flow has the larger flow rate (larger mean velocity), because it has the smaller friction factor.

The two flow cases are compared in the NCFMF video *Turbulence*, in which R. W. Stewart uses a clever experimental setup to contrast the two flow regimes at constant volume flow rate by varying the liquid viscosity. The trajectories of the liquid streams leaving the end of the pipe are particularly well shown.

Problem 8.117

[Difficulty: 3]

8.117 Estimate the minimum level in the water tank of Problem 8.115 such that the flow will be turbulent.

Given: Data on water flow from a tank/tubing system

Find: Minimum tank level for turbulent flow

Solution:

Basic equations:

$$\left(\frac{p_1}{\rho} + \alpha_1 \cdot \frac{V_1^2}{2} + g \cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \cdot \frac{V_2^2}{2} + g \cdot z_2\right) = h_{lT} = \sum_{major} h_l + \sum_{minor} h_{lm} \quad (8.29)$$

$$Re = \frac{\rho \cdot V \cdot D}{\mu} \qquad h_l = f \cdot \frac{L}{D} \cdot \frac{V^2}{2} \quad (8.34) \qquad h_{lm} = K \cdot \frac{V^2}{2} \quad (8.40a) \qquad h_{lm} = f \cdot \frac{L_e}{D} \cdot \frac{V^2}{2} \quad (8.40b)$$

$$f = \frac{64}{Re} \quad (8.36) \quad \text{(Laminar)} \qquad \frac{1}{\sqrt{f}} = -2.0 \cdot \log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re \cdot \sqrt{f}}\right) \quad (8.37) \quad \text{(Turbulent)}$$

The energy equation (Eq. 8.29) becomes

$$g \cdot d - \alpha \cdot \frac{V^2}{2} = f \cdot \frac{L}{D} \cdot \frac{V^2}{2} + K \cdot \frac{V^2}{2}$$

This can be solved expicitly for height *d*, or solved using *Solver*

Given data:			Tabulated or graphical data:			
L =	15.3	m		ν =	1.00E-06	m^2/s
D =	3.18	mm		ρ =	998	kg/m^3
K_{ent} =	1.4				(Appendix A)	
α =	2					
Computed results:						
Re =	2300	(Transition Re)				
V =	0.723	m/s				
α =	1	(Turbulent)				
f =	0.0473	(Turbulent)				
d =	**6.13**	m	(Vary d to minimize error in energy equation)			
Energy equation:	**Left (m^2/s)**	**Right (m^2/s)**	**Error**			
(Using *Solver*)	59.9	59.9	0.00%			
Note that we used α = 1 (turbulent); using α = 2 (laminar) gives d = 6.16 m						

Problem 8.118

[Difficulty: 2]

8.118 A laboratory experiment is set up to measure pressure drop for flow of water through a smooth tube. The tube diameter is 15.9 mm, and its length is 3.56 m. Flow enters the tube from a reservoir through a square-edged entrance. Calculate the volume flow rate needed to obtain turbulent flow in the tube. Evaluate the reservoir height differential required to obtain turbulent flow in the tube.

Solution:

Flow will be turbulent for $Re_D > 2300$

$$Re_D = \frac{\rho \bar{V} D}{\mu} = \frac{\bar{V} D}{\nu} = \frac{Q D}{A \nu} = \frac{Q D}{\nu}\frac{4}{\pi D^2} = \frac{4Q}{\pi \nu D} \quad \text{so} \quad Q = \frac{\pi \nu D\, Re}{4}$$

Assume $T = 20°C$, $\nu = 1.00 \times 10^{-6}\ m^2/s$ (Table A.8)

For $Re = 2300$,

$$Q = \frac{\pi}{4} \times 1.0 \times 10^{-6}\frac{m^2}{s} \times 15.9 \times 10^{-3}\ m \times 2300 = 2.87 \times 10^{-5}\ m^3/s \qquad Q$$

Basic equations: $$\left(\cancel{\frac{p_1}{\rho}} + \alpha_1 \cancel{\frac{\bar{V}_1^2}{2}} + g z_1\right) - \left(\cancel{\frac{p_2}{\rho}} + \alpha_2 \cancel{\frac{\bar{V}_2^2}{2}} + g z_2\right) = h_{l_T} \qquad (8.29)$$

$$h_{l_T} = h_l + h_{lm} \qquad h_l = f\frac{L}{D}\frac{\bar{V}^2}{2}, \quad h_{lm} = K\frac{\bar{V}^2}{2}$$

Assumptions: (1) $p_1 = p_2 = p_{atm}$ (2) $\bar{V}_1 = \bar{V}_2 = 0$
(3) $K_{ent} = 0.5$ (Table 8.2), $K_{exit} = 1.0$

Then,
$$z_1 - z_2 = \frac{\bar{V}^2}{2g}\left[f\frac{L}{D} + K_{ent} + K_{exit}\right] \qquad (1)$$

$$\bar{V} = \frac{Q}{A} = \frac{4Q}{\pi D^2} = \frac{4}{\pi} \times 2.87 \times 10^{-5}\frac{m}{s} \times \frac{1}{(15.9 \times 10^{-3}\ m)^2} = 0.145\ m/s$$

For turbulent flow in a smooth pipe at $Re = 2300$,
$f = 0.05$ (Fig 8.13)

From Eq. 1

$$d = z_1 - z_2 = \frac{(0.145)^2}{2}\frac{m^2}{s^2} \times \frac{s^2}{9.81\ m}\left[0.05 \times \frac{3.56 \times 10^3}{15.9} + 0.5 + 1.0\right]$$

$$d = 0.0136\ m \text{ or } 13.6\ mm \qquad d$$

Problem 8.119

[Difficulty: 3]

8.119 A benchtop experiment consists of a reservoir with a 500-mm-long horizontal tube of diameter 7.5 mm attached to its base. The tube exits to a sink. A flow of water at 10°C is to be generated such that the Reynolds number is 10,000. What is the flow rate? If the entrance to the tube is square-edged, how deep should the reservoir be? If the entrance to the tube is well-rounded, how deep should the reservoir be?

Given: Data on water flow from a tank/tubing system

Find: Minimum tank level for turbulent flow

Solution:

Basic equations:

$$\left(\frac{p_1}{\rho} + \alpha_1 \cdot \frac{V_1^2}{2} + g \cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \cdot \frac{V_2^2}{2} + g \cdot z_2\right) = h_{lT} = \sum_{major} h_l + \sum_{minor} h_{lm} \quad (8.29)$$

$$Re = \frac{\rho \cdot V \cdot D}{\mu} \qquad h_l = f \cdot \frac{L}{D} \cdot \frac{V^2}{2} \quad (8.34) \qquad h_{lm} = K \cdot \frac{V^2}{2} \quad (8.40a) \qquad h_{lm} = f \cdot \frac{L_e}{D} \cdot \frac{V^2}{2} \quad (8.40b)$$

$$\frac{1}{\sqrt{f}} = -2.0 \cdot \log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re \cdot \sqrt{f}}\right) \quad (8.37) \quad \text{(Turbulent)}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) α at 1 and 2 is approximately 1 4) Velocity at free surface is <<

The available data is $D = 7.5 \cdot mm$ $\quad L = 500 \cdot mm$ $\quad$ From Table A.8 at 10°C $\quad \rho = 1000 \frac{kg}{m^3}$ $\quad \mu = 1.3 \cdot 10^{-3} \cdot \frac{N \cdot s}{m^2}$

$Re = 10000$ $\quad K_{ent} = 0.5$ $\quad$ (Table 8.2) $\quad K_{exit} = 1$

From $\quad Re = \frac{\rho \cdot V \cdot D}{\mu}$ $\quad Re = \frac{\rho \cdot Q \cdot D}{\frac{\pi}{4} \cdot D^2}$ $\quad$ or $\quad Q = \frac{\pi \cdot \mu \cdot D \cdot Re}{4 \cdot \rho}$ $\quad Q = 7.66 \times 10^{-5} \frac{m^3}{s}$ $\quad Q = 0.0766 \cdot \frac{l}{s}$

Hence $\quad V = \frac{Q}{\left(\frac{\pi \cdot D^2}{4}\right)}$ $\quad V = 1.73 \frac{m}{s}$

Assuming a smooth tube $\quad \frac{1}{\sqrt{f}} = -2 \cdot \log\left(\frac{2.51}{Re \cdot \sqrt{f}}\right)$ $\quad$ so $\quad f = 0.0309$

The energy equation (Eq. 8.29) becomes $\quad g \cdot d = f \cdot \frac{L}{D} \cdot \frac{V^2}{2} + K_{ent} \cdot \frac{V^2}{2} + K_{exit} \cdot \frac{V^2}{2}$

Solving for d $\quad d = \frac{V^2}{2 \cdot g} \cdot \left(f \cdot \frac{L}{D} + K_{ent} + K_{exit}\right)$ $\quad d = 545 \cdot mm$

FOR $r > 0.15D$) $\quad K_{ent} = 0.04$ $\quad$ (Table 8.2) $\quad d = 475 \cdot mm$

Problem 8.120

[Difficulty: 4]

8.120 As discussed in Problem 8.52, the applied pressure difference, Δp, and corresponding volume flow rate, Q, for laminar flow in a tube can be compared to the applied DC voltage V across, and current I through, an electrical resistor, respectively. Investigate whether or not this analogy is valid for turbulent flow by plotting the "resistance" $\Delta p/Q$ as a function of Q for turbulent flow of kerosene (at 40°C) in a tube 250 mm long with inside diameter 7.5 mm.

Given: Data on a tube

Find: "Resistance" of tube for flow of kerosine; plot

Solution:

The basic equations for turbulent flow are

$$\left(\frac{p_1}{\rho} + \alpha_1 \cdot \frac{V_1^2}{2} + g \cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \cdot \frac{V_2^2}{2} + g \cdot z_2\right) = h_l \qquad (8.29)$$

$$h_l = f \cdot \frac{L}{D} \cdot \frac{V^2}{2} \qquad (8.34) \qquad\qquad \frac{1}{\sqrt{f}} = -2.0 \cdot \log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re \cdot \sqrt{f}}\right) \qquad (8.37)$$

The given data is $\quad L = 250 \cdot mm \qquad D = 7.5 \cdot mm$

From Fig. A.2 and Table A.2 $\quad \mu = 1.1 \times 10^{-3} \cdot \frac{N \cdot s}{m^2} \qquad \rho = 0.82 \times 990 \cdot \frac{kg}{m^3} = 812 \cdot \frac{kg}{m^3}$ (Kerosene)

For an electrical resistor $\quad V = R \cdot I \qquad (1)$

Simplifying Eqs. 8.29 and 8.34 for a horizontal, constant-area pipe

$$\frac{p_1 - p_2}{\rho} = f \cdot \frac{L}{D} \cdot \frac{V^2}{2} = f \cdot \frac{L}{D} \cdot \frac{\left(\frac{Q}{\frac{\pi}{4} \cdot D^2}\right)^2}{2} \qquad \text{or} \qquad \Delta p = \frac{8 \cdot \rho \cdot f \cdot L}{\pi^2 \cdot D^5} \cdot Q^2 \qquad (2)$$

By analogy, current I is represented by flow rate Q, and voltage V by pressure drop Δp. Comparing Eqs. (1) and (2), the "resistance" of the tube is

$$R = \frac{\Delta p}{Q} = \frac{8 \cdot \rho \cdot f \cdot L \cdot Q}{\pi^2 \cdot D^5}$$

The "resistance" of a tube is not constant, but is proportional to the "current" Q! Actually, the dependence is not quite linear, because f decreases slightly (and nonlinearly) with Q. The analogy fails!

The analogy is hence invalid for $\quad Re > 2300 \quad$ or $\quad \frac{\rho \cdot V \cdot D}{\mu} > 2300$

Writing this constraint in terms of flow rate $\dfrac{\rho\cdot\dfrac{Q}{\dfrac{\pi}{4}\cdot D^2}\cdot D}{\mu} > 2300$ or $Q > \dfrac{2300\cdot\mu\cdot\pi\cdot D}{4\cdot\rho}$

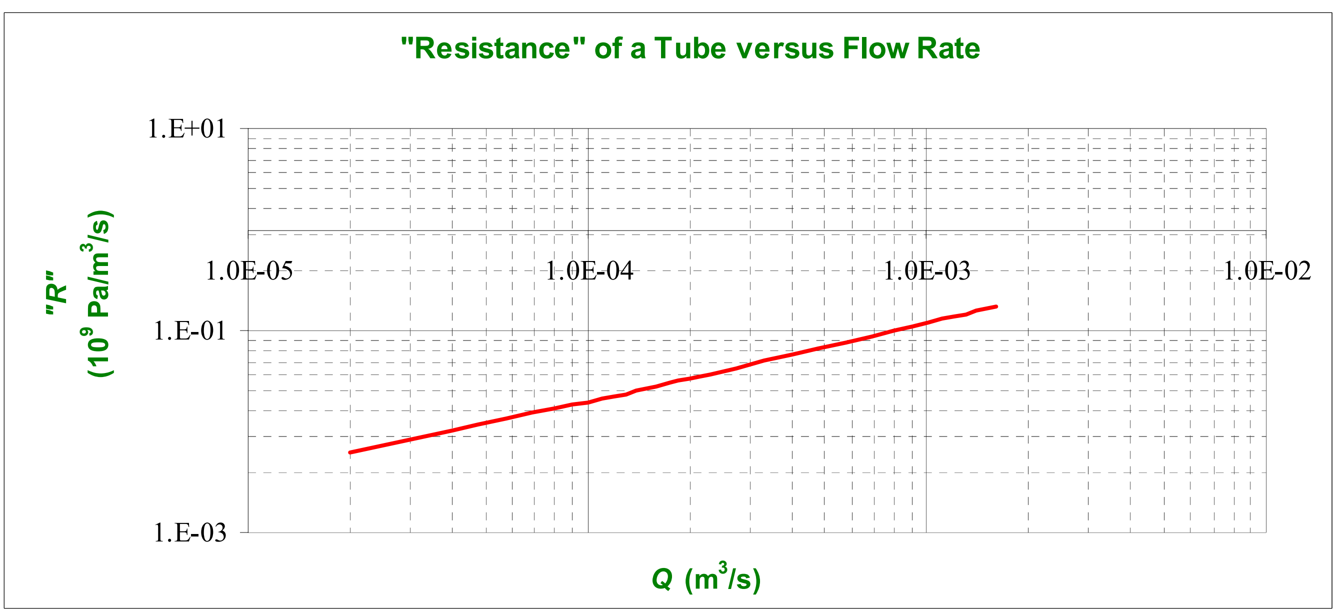
Flow rate above which analogy fails $Q = 1.84 \times 10^{-5}\,\dfrac{m^3}{s}$

The plot of "resistance" versus flow rate cab be done in *Excel*.

Given data:				Tabulated or graphical data:		
L =	250	mm		μ =	1.01E-03	$N.s/m^2$
D =	7.5	mm		SG_{ker} =	0.82	
				ρ_w =	990	kg/m^3
				ρ =	812	kg/m^3
					(Appendix A)	
Computed results:						

Q (m^3/s)	V (m/s)	Re	f	"R" (10^9 $Pa/m^3/s$)
2.0E-05	0.5	2729	0.0448	0.0062
4.0E-05	0.9	5458	0.0365	0.0101
8.0E-05	1.8	10916	0.0302	0.0167
1.2E-04	2.7	16374	0.0272	0.0226
1.6E-04	3.6	21832	0.0253	0.0281
2.0E-04	4.5	27290	0.0240	0.0333
4.0E-04	9.1	54580	0.0205	0.0568
8.0E-04	18.1	109161	0.0177	0.0980
1.2E-03	27.2	163741	0.0163	0.1353
1.6E-03	36.2	218322	0.0154	0.1705

The "resistance" is not constant; the analogy is invalid for turbulent flow



Problem 8.121

[Difficulty: 3]

8.121 Plot the required reservoir depth of water to create flow in a smooth tube of diameter 10 mm and length 100 m, for a flow rate range of 1 L/min through 10 L/min.

Given: Data on tube geometry

Find: Plot of reservoir depth as a function of flow rate

Solution:

Basic equations:

$$\left(\frac{p_1}{\rho}+\alpha_1\cdot\frac{V_1^2}{2}+g\cdot z_1\right)-\left(\frac{p_2}{\rho}+\alpha_2\cdot\frac{V_2^2}{2}+g\cdot z_2\right)=h_{lT}=\sum_{\text{major}} h_l+\sum_{\text{minor}} h_{lm} \quad (8.29)$$

$$Re=\frac{\rho\cdot V\cdot D}{\mu} \qquad h_l=f\cdot\frac{L}{D}\cdot\frac{V^2}{2} \quad (8.34) \qquad h_{lm}=K\cdot\frac{V^2}{2} \quad (8.40a) \qquad h_{lm}=f\cdot\frac{L_e}{D}\cdot\frac{V^2}{2} \quad (8.40b)$$

$$f=\frac{64}{Re} \quad (8.36) \quad \text{(Laminar)} \qquad \frac{1}{\sqrt{f}}=-2.0\cdot\log\left(\frac{\frac{e}{D}}{3.7}+\frac{2.51}{Re\cdot\sqrt{f}}\right) \quad (8.37) \quad \text{(Turbulent)}$$

The energy equation (Eq. 8.29) becomes

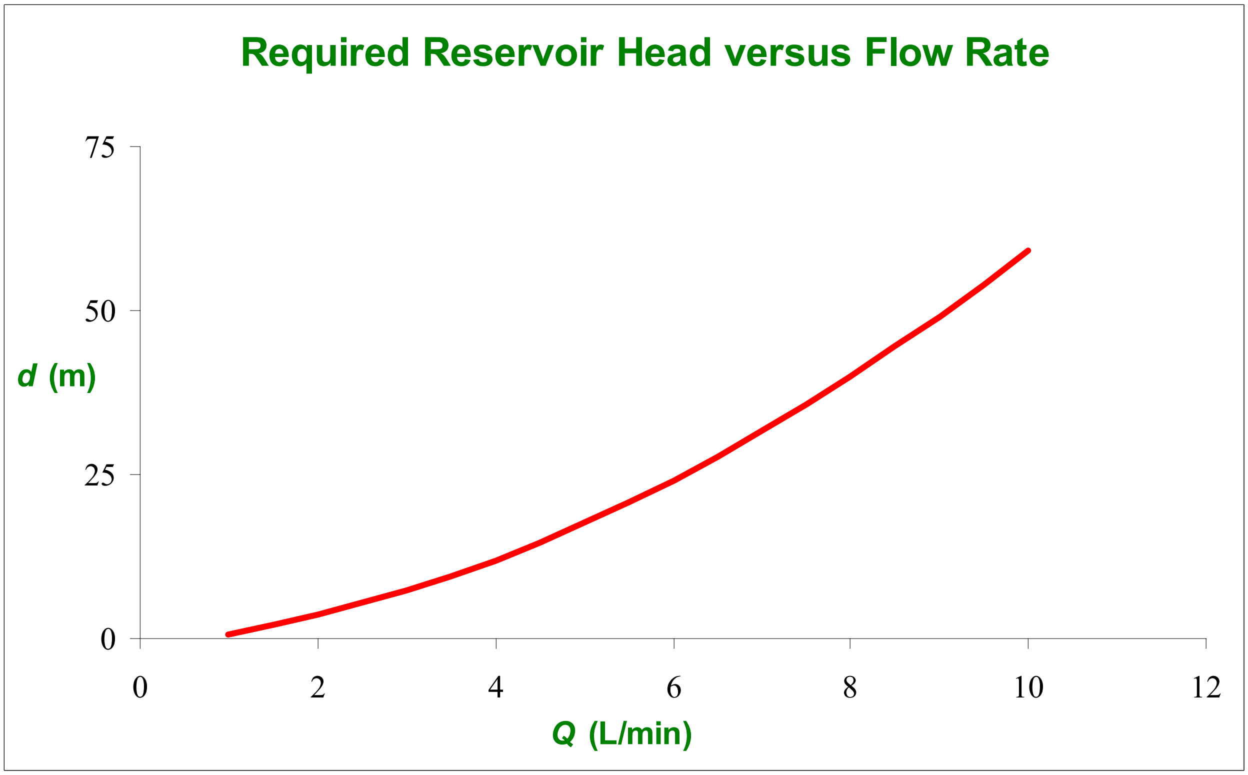
$$g\cdot d-\alpha\cdot\frac{V^2}{2}=f\cdot\frac{L}{D}\cdot\frac{V^2}{2}+K\cdot\frac{V^2}{2}$$

This can be solved expicitly for height *d*, or solved using *Solver*

$$d=\frac{V^2}{2\cdot g}\cdot\left(\alpha+f\cdot\frac{L}{D}+K\right)$$

In *Excel*:

Given data:				Tabulated or graphical data:			
L =	100	m		μ =	1.01E-03	N.s/m^2	
D =	10	mm		ρ =	998	kg/m^3	
α =	1	(All flows turbulent)			(Table A.8)		
				K_{ent} =	0.5	(Square-edged)	
					(Table 8.2)		
Computed results:							
***Q* (L/min)**	***V* (m/s)**	***Re***	***f***	***d* (m)**			
1	0.2	2.1E+03	0.0305	0.704			
2	0.4	4.2E+03	0.0394	3.63			
3	0.6	6.3E+03	0.0350	7.27			
4	0.8	8.4E+03	0.0324	11.9			
5	1.1	1.0E+04	0.0305	17.6			
6	1.3	1.3E+04	0.0291	24.2			
7	1.5	1.5E+04	0.0280	31.6			
8	1.7	1.7E+04	0.0270	39.9			
9	1.9	1.9E+04	0.0263	49.1			
10	2.1	2.1E+04	0.0256	59.1			


Required Reservoir Head versus Flow Rate
75
50
25
0
d (m)
0
2
4
6
8
10
12
Q (L/min)

Problem 8.122

[Difficulty: 3]

8.122 Oil with kinematic viscosity $\nu = 7.5 \times 10^{-4}\ \mathrm{ft^2/s}$ flows at 45 gpm in a 100-ft-long horizontal drawn-tubing pipe of 1 in. diameter. By what percentage ratio will the energy loss increase if the same flow rate is maintained while the pipe diameter is reduced to 0.75 in.?

Given: Flow of oil in a pipe

Find: Percentage change in loss if diameter is reduced

Solution:

Basic equations $\quad h_l = f\cdot\frac{L}{D}\cdot\frac{V^2}{2} \quad f = \frac{64}{Re}$ Laminar $\quad \frac{1}{\sqrt{f}} = -2.0\cdot\log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re\cdot\sqrt{f}}\right)$ Turbulent

Available data $\quad \nu = 7.5\cdot10^{-4}\cdot\frac{ft^2}{s} \quad L = 100\cdot ft \quad D = 1\cdot in \quad Q = 45\cdot gpm \quad Q = 0.100\,\frac{ft^3}{s}$

Here $\quad V = \frac{Q}{A} = \frac{4\cdot Q}{\pi\cdot D^2} \quad V = \frac{4}{\pi}\times 0.1\cdot\frac{ft^3}{s}\times\left(\frac{12}{1}\cdot\frac{1}{ft}\right)^2 \quad V = 18.3\cdot\frac{ft}{s}$

Then $\quad Re = \frac{V\cdot D}{\nu} \quad Re = 18.3\cdot\frac{ft}{s}\times\frac{1}{12}\cdot ft\times\frac{s}{7.5\times10^{-4}\cdot ft^2} \quad Re = 2033$

The flow is LAMINAR $\quad h_l = f\cdot\frac{L}{D}\cdot\frac{V^2}{2} \quad h_l = \frac{64}{Re}\cdot\frac{L}{D}\cdot\frac{V^2}{2} \quad h_l = \frac{64}{2033}\times\frac{100}{\frac{1}{12}}\times\frac{\left(18.3\,\frac{ft}{s}\right)^2}{2} \quad h_l = 6326\cdot\frac{ft^2}{s^2}$

When the diameter is reduced to $\quad D = 0.75\cdot in$

$$V = \frac{Q}{A} = \frac{4\cdot Q}{\pi\cdot D^2} \quad V = \frac{4}{\pi}\times 0.1\cdot\frac{ft^3}{s}\times\left(\frac{12}{0.75}\cdot\frac{1}{ft}\right)^2 \quad V = 32.6\cdot\frac{ft}{s}$$

$$Re = \frac{V\cdot D}{\nu} \quad Re = 32.6\cdot\frac{ft}{s}\times\frac{0.75}{12}\cdot ft\times\frac{s}{7.5\times10^{-4}\cdot ft^2} \quad Re = 2717$$

The flow is TURBULENT For drawn tubing, from Table 8.1 $\quad e = 0.000005\cdot ft$

Given $\quad \frac{1}{\sqrt{f}} = -2.0\cdot\log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re\cdot\sqrt{f}}\right) \quad f = 0.0449$

$$h_l = f\cdot\frac{L}{D}\cdot\frac{V^2}{2} \quad h_l = .0449\times\frac{100}{\frac{0.75}{12}}\times\frac{\left(32.6\,\frac{ft}{s}\right)^2}{2} \quad h_l = 3.82\times10^4\cdot\frac{ft^2}{s^2}$$

The increase in loss is $\quad \frac{3.82\times10^4 - 6326}{6326} = 504\cdot\%$

This is a HUGH increase! The main increase is because the diameter reduction causes the velocity to increase; the loss goes as V^2, and $1/D$, so it increases very rapidly

Problem 8.123

[Difficulty: 4]

8.123 A water system is used in a laboratory to study flow in a smooth pipe. The water is at 10°C. To obtain a reasonable range, the maximum Reynolds number in the pipe must be 100,000. The system is supplied from an overhead constant-head tank. The pipe system consists of a square-edged entrance, two 45° standard elbows, two 90° standard elbows, and a fully open gate valve. The pipe diameter is 7.5 mm, and the total length of pipe is 1 m. Calculate the minimum height of the supply tank above the pipe system discharge to reach the desired Reynolds number. If a pressurized chamber is used instead of the reservoir, what will be the required pressure?

Given: Data on water system

Find: Minimum tank height; equivalent pressure

Solution:

Basic equations:

$$\left(\frac{p_1}{\rho}+\alpha_1\cdot\frac{V_1^2}{2}+g\cdot z_1\right)-\left(\frac{p_2}{\rho}+\alpha_2\cdot\frac{V_2^2}{2}+g\cdot z_2\right)=h_{lT}=\sum_{major}h_l+\sum_{minor}h_{lm} \quad (8.29)$$

$$Re=\frac{\rho\cdot V\cdot D}{\mu} \qquad h_l=f\cdot\frac{L}{D}\cdot\frac{V^2}{2} \quad (8.34) \qquad h_{lm}=K\cdot\frac{V^2}{2} \quad (8.40a) \qquad h_{lm}=f\cdot\frac{L_e}{D}\cdot\frac{V^2}{2} \quad (8.40b)$$

$$\frac{1}{\sqrt{f}}=-2.0\cdot\log\left(\frac{\frac{e}{D}}{3.7}+\frac{2.51}{Re\cdot\sqrt{f}}\right) \quad (8.37) \qquad \text{(Turbulent)}$$

Available data $\quad D = 7.5\cdot mm \quad L = 1\cdot m \quad Re = 100000$ and so $f = 0.0180$ at this Re

From Section 8.7 $\quad K_{ent} = 0.5 \quad L_{elbow45} = 16\cdot D \quad L_{elbow90} = 30\cdot D \quad L_{GV} = 8\cdot D$

$L_{elbow45} = 0.12\,m \quad L_{elbow90} = 0.225\,m \quad L_{GV} = 0.06\,m$

From Table A.8 at 10°C $\quad \rho = 1000\frac{kg}{m^3} \quad \mu = 1.3\cdot 10^{-3}\cdot\frac{N\cdot s}{m^2}$

Then $\quad Q = \frac{\pi\cdot\mu\cdot D\cdot Re}{4\cdot\rho} \quad Q = 7.66\times 10^{-4}\frac{m^3}{s} \quad Q = 0.766\frac{l}{s} \quad V = \frac{Q}{\left(\frac{\pi\cdot D^2}{4}\right)} \quad V = 17.3\frac{m}{s}$

The energy equation becomes

$$d-\frac{V^2}{2\cdot g}=\frac{V^2}{2\cdot g}\cdot\left(f\cdot\frac{L}{D}+2\cdot f\cdot\frac{L_{elbow90}}{D}+2\cdot f\cdot\frac{L_{elbow45}}{D}+f\cdot\frac{L_{GV}}{D}\right)$$

Hence

$$d=\frac{V^2}{2\cdot g}\cdot\left(1+f\cdot\frac{L}{D}+2\cdot f\cdot\frac{L_{elbow90}}{D}+2\cdot f\cdot\frac{L_{elbow45}}{D}+f\cdot\frac{L_{GV}}{D}\right) \qquad d = 79.6\cdot m$$

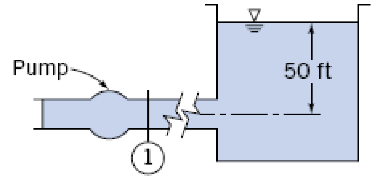
Unrealistic!

IF INSTEAD the reservoir was pressurized $\quad \Delta p = \rho\cdot g\cdot d \quad \Delta p = 781\cdot kPa \quad$ which is feasible

Problem 8.124

[Difficulty: 2]

8.124 Water from a pump flows through a 9-in.-diameter commercial steel pipe for a distance of 4 miles from the pump discharge to a reservoir open to the atmosphere. The level of the water in the reservoir is 50 ft above the pump discharge, and the average speed of the water in the pipe is 10 ft/s. Calculate the pressure at the pump discharge.



Given: Flow from pump to reservoir

Find: Pressure at pump discharge

Solution:

Basic equations
$$\left(\frac{p_1}{\rho} + \alpha\cdot\frac{V_1^2}{2} + g\cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha\cdot\frac{V_2^2}{2} + g\cdot z_2\right) = h_{lT} \qquad h_{lT} = h_l + h_{lm} = f\cdot\frac{L}{D}\cdot\frac{V_1^2}{2} + K_{exit}\cdot\frac{V_1^2}{2}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) α at 1 and 2 is approximately 1 4) V_2 <<

Hence the energy equation between Point 1 and the free surface (Point 2) becomes

$$\left(\frac{p_1}{\rho} + \frac{V^2}{2}\right) - \left(g\cdot z_2\right) = f\cdot\frac{L}{D}\cdot\frac{V^2}{2} + K_{exit}\cdot\frac{V^2}{2}$$

Solving for p_1
$$p_1 = \rho\cdot\left(g\cdot z_2 - \frac{V^2}{2} + f\cdot\frac{L}{D}\cdot\frac{V^2}{2} + K_{exit}\cdot\frac{V^2}{2}\right)$$

From Table A.7 (68°F) $\rho = 1.94\cdot\frac{slug}{ft^3}$ $\qquad \nu = 1.08\times 10^{-5}\cdot\frac{ft^2}{s}$

$$Re = \frac{V\cdot D}{\nu} \qquad Re = 10\cdot\frac{ft}{s}\times\frac{9}{12}\cdot ft\times\frac{s}{1.08\times 10^{-5}\cdot ft^2} \qquad Re = 6.94\times 10^5 \qquad \text{Turbulent}$$

For commercial steel pipe $e = 0.00015\cdot ft$ (Table 8.1) so $\frac{e}{D} = 0.000200$

Flow is turbulent: Given
$$\frac{1}{\sqrt{f}} = -2.0\cdot\log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re\cdot\sqrt{f}}\right) \qquad f = 0.0150$$

For the exit $K_{exit} = 1.0$ so we find $p_1 = \rho\cdot\left(g\cdot z_2 + f\cdot\frac{L}{D}\cdot\frac{V^2}{2}\right)$

$$p_1 = 1.94\cdot\frac{slug}{ft^3}\times\left[32.2\cdot\frac{ft}{s^2}\times 50\cdot ft + .0150\times\frac{4\cdot mile}{0.75\cdot ft}\times\frac{5280\cdot ft}{1mile}\times\frac{1}{2}\times\left(10\cdot\frac{ft}{s}\right)^2\right]\times\frac{lbf\cdot s^2}{slug\cdot ft} \qquad p_1 = 4.41\times 10^4\cdot\frac{lbf}{ft^2} \qquad p_1 = 306\cdot psi$$

Problem 8.125

[Difficulty: 3]

8.125 Water is to flow by gravity from one reservoir to a lower one through a straight, inclined galvanized iron pipe. The pipe diameter is 50 mm, and the total length is 250 m. Each reservoir is open to the atmosphere. Plot the required elevation difference Δz as a function of flow rate Q, for Q ranging from 0 to 0.01 m³/s. Estimate the fraction of Δz due to minor losses.

Given: Data on reservoir/pipe system

Find: Plot elevation as a function of flow rate; fraction due to minor losses

Solution:

$L =$ 250 m

$D =$ 50 mm

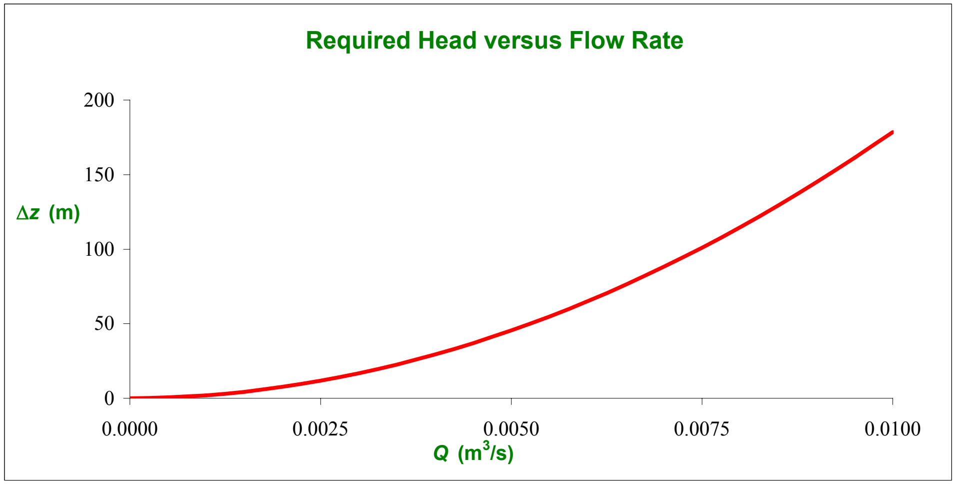
$e/D =$ 0.003

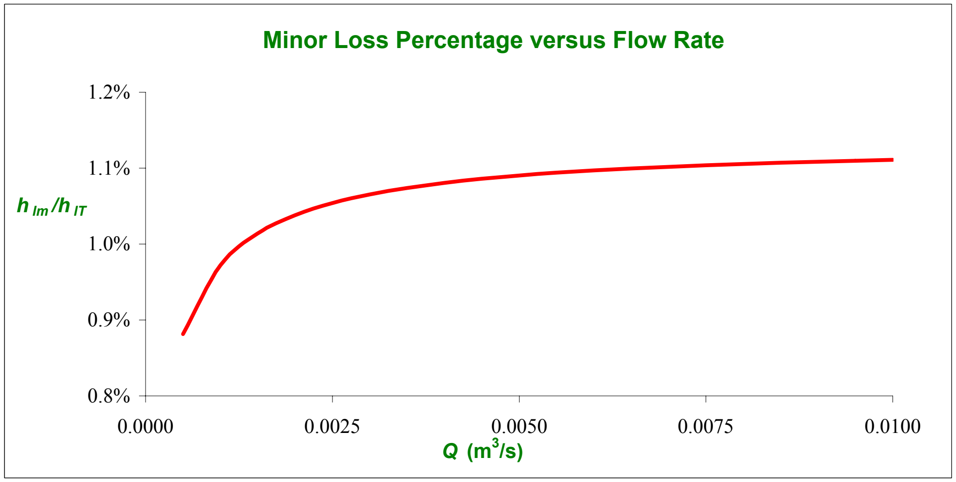
$K_{ent} =$ 0.5

$K_{exit} =$ 1.0

$\nu =$ 1.01E-06 m²/s

Q (m³/s)	V (m/s)	Re	f	Δz (m)	h_{lm}/h_{lT}
0.0000	0.000	0.00E+00		0.000	
0.0005	0.255	1.26E+04	0.0337	0.562	0.882%
0.0010	0.509	2.52E+04	0.0306	2.04	0.972%
0.0015	0.764	3.78E+04	0.0293	4.40	1.01%
0.0020	1.02	5.04E+04	0.0286	7.64	1.04%
0.0025	1.27	6.30E+04	0.0282	11.8	1.05%
0.0030	1.53	7.56E+04	0.0279	16.7	1.07%
0.0035	1.78	8.82E+04	0.0276	22.6	1.07%
0.0040	2.04	1.01E+05	0.0275	29.4	1.08%
0.0045	2.29	1.13E+05	0.0273	37.0	1.09%
0.0050	2.55	1.26E+05	0.0272	45.5	1.09%
0.0055	2.80	1.39E+05	0.0271	54.8	1.09%
0.0060	3.06	1.51E+05	0.0270	65.1	1.10%
0.0065	3.31	1.64E+05	0.0270	76.2	1.10%
0.0070	3.57	1.76E+05	0.0269	88.2	1.10%
0.0075	3.82	1.89E+05	0.0269	101	1.10%
0.0080	4.07	2.02E+05	0.0268	115	1.11%
0.0085	4.33	2.14E+05	0.0268	129	1.11%
0.0090	4.58	2.27E+05	0.0268	145	1.11%
0.0095	4.84	2.40E+05	0.0267	161	1.11%
0.0100	5.09	2.52E+05	0.0267	179	1.11%

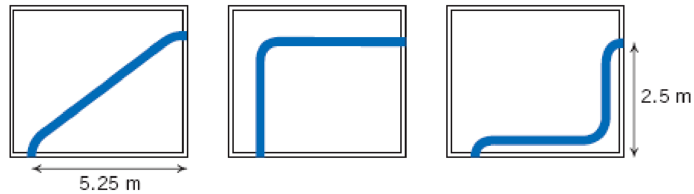




Problem 8.126

[Difficulty: 3]

8.126 A 5-cm-diameter potable water line is to be run through a maintenance room in a commercial building. Three possible layouts for the water line are proposed, as shown. Which is the best option, based on minimizing losses? Assume galvanized iron, and a flow rate of 350 L/min.



(a) Two miter bends (b) A standard elbow (c) Three standard elbows

Given: Flow through three different layouts

Find: Which has minimum loss

Solution:

Basic equations
$$\left(\frac{p_1}{\rho} + \alpha\cdot\frac{V_1^2}{2} + g\cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha\cdot\frac{V_2^2}{2} + g\cdot z_2\right) = h_l\, h_{lT} = h_l + h_{lm} = f\cdot\frac{L}{D}\cdot\frac{V^2}{2} + \sum_{\text{Minor}}\left(f\cdot\frac{L_e}{D}\cdot\frac{V^2}{2}\right)$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) α is approximately 1 4) Ignore additional length of elbows

For a flow rate of $Q = 350\cdot\frac{L}{min}$ $\quad V = \frac{Q}{A} = \frac{4\cdot Q}{\pi\cdot D^2}$ $\quad V = \frac{4}{\pi}\times 350\cdot\frac{L}{min}\times\frac{0.001\cdot m^3}{1\cdot L}\times\frac{1\cdot min}{60\cdot s}\times\left(\frac{1}{0.05\cdot m}\right)^2$ $\quad V = 2.97\frac{m}{s}$

For water at 20ºC $\quad \nu = 1.01\times 10^{-6}\cdot\frac{m^2}{s}$ $\quad Re = \frac{V\cdot D}{\nu}$ $\quad Re = 2.97\cdot\frac{m}{s}\times 0.05\cdot m\times\frac{s}{1.01\times 10^{-6}\cdot m^2}$ $\quad Re = 1.47\times 10^5$

Flow is turbulent. From Table 8.1 $\quad e = 0.15\cdot mm$ $\quad \frac{e}{D} = 6.56\times 10^{-4}$

Given $\quad \frac{1}{\sqrt{f}} = -2.0\cdot\log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re\cdot\sqrt{f}}\right)$ $\quad f = 0.0201$

For Case (a) $\quad L = \sqrt{5.25^2 + 2.5^2}\cdot m$ $\quad L = 5.81\,m$ $\quad$ Two 45º miter bends (Fig. 8.16), for each $\frac{L_e}{D} = 13$

Hence the energy equation is $\quad \frac{p_1}{\rho} - \frac{p_2}{\rho} = f\cdot\frac{L}{D}\cdot\frac{V^2}{2} + 2\cdot f\cdot\frac{L_e}{D}\cdot\frac{V^2}{2}$

Solving for Δp $\quad \Delta p = p_1 - p_2 = \rho\cdot f\cdot\frac{V^2}{2}\cdot\left(\frac{L}{D} + 2\cdot\frac{L_e}{D}\right)$

$$\Delta p = 1000\cdot\frac{kg}{m^3}\times .0201\times\left(2.97\cdot\frac{m}{s}\right)^2\times\left(\frac{5.81}{0.05} + 2\cdot 13\right)\times\frac{N\cdot s^2}{kg\cdot m} \qquad \Delta p = 25.2\cdot kPa$$

For Case (b) $\quad L = (5.25 + 2.5)\cdot m$ $\quad L = 7.75\,m$ $\quad$ One standard 90º elbow (Table 8.4) $\quad \frac{L_e}{D} = 30$

Hence the energy equation is $\quad \frac{p_1}{\rho} - \frac{p_2}{\rho} = f\cdot\frac{L}{D}\cdot\frac{V^2}{2} + f\cdot\frac{L_e}{D}\cdot\frac{V^2}{2}$

Solving for Δp $\quad \Delta p = p_1 - p_2 = \rho \cdot f \cdot \frac{V^2}{2} \cdot \left(\frac{L}{D} + \frac{L_e}{D} \right)$

$$\Delta p = 1000 \cdot \frac{kg}{m^3} \times .0201 \times \left(2.97 \cdot \frac{m}{s} \right)^2 \times \left(\frac{7.75}{0.05} + 30 \right) \times \frac{N \cdot s^2}{kg \cdot m} \qquad \Delta p = 32.8 \cdot kPa$$

For Case (c) $\quad L = (5.25 + 2.5) \cdot m \qquad L = 7.75\,m \qquad$ Three standard 90º elbows, for each $\frac{L_e}{D} = 30$

Hence the energy equation is $\quad \frac{p_1}{\rho} - \frac{p_2}{\rho} = f \cdot \frac{L}{D} \cdot \frac{V^2}{2} + 3 \cdot f \cdot \frac{L_e}{D} \cdot \frac{V^2}{2}$

Solving for Δp $\quad \Delta p = p_1 - p_2 = \rho \cdot f \cdot \frac{V^2}{2} \cdot \left(\frac{L}{D} + 3 \cdot \frac{L_e}{D} \right)$

$$\Delta p = 1000 \cdot \frac{kg}{m^3} \times .0201 \times \left(2.97 \cdot \frac{m}{s} \right)^2 \times \left(\frac{7.75}{0.05} + 3 \times 30 \right) \times \frac{N \cdot s^2}{kg \cdot m} \qquad \Delta p = 43.4 \cdot kPa$$

Hence we conclude Case (a) is the best and Case (c) is the worst

Problem 8.127

[Difficulty: 2]

8.127 In an air-conditioning installation, a flow rate of 1750 cfm of air at 50°F is required. A smooth sheet metal duct of rectangular section (0.75 ft by 2.5 ft) is to be used. Determine the pressure drop (inches of water) for a 1000-ft horizontal duct section.

Given: Flow through rectangular duct

Find: Pressure drop

Solution:

Basic equations

$$\left(\frac{p_1}{\rho} + \alpha\cdot\frac{V_1^2}{2} + g\cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha\cdot\frac{V_2^2}{2} + g\cdot z_2\right) = h_{lT} \qquad h_{lT} = h_l + h_{lm} = f\cdot\frac{L}{D}\cdot\frac{V^2}{2} + \sum_{\text{Minor}}\left(f\cdot\frac{L_e}{D}\cdot\frac{V^2}{2}\right)$$

$$D_h = \frac{4\cdot a\cdot b}{2\cdot(a+b)}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) α is approximately 1

Available data $\quad Q = 1750\cdot\text{cfm} \qquad L = 1000\cdot\text{ft} \qquad b = 2.5\cdot\text{ft} \qquad a = 0.75\cdot\text{ft}$

At 50°F, from Table A.9 $\quad \rho = 0.00242\cdot\frac{\text{slug}}{\text{ft}^3} \qquad \mu = 3.69\cdot10^{-7}\cdot\frac{\text{lbf}\cdot\text{s}}{\text{ft}^2} \qquad \rho_w = 1.94\cdot\frac{\text{slug}}{\text{ft}^3}$

Hence $\quad V = \frac{Q}{a\cdot b} \qquad V = 15.6\cdot\frac{\text{ft}}{\text{s}}$ and $\quad D_h = \frac{4\cdot a\cdot b}{2\cdot(a+b)} \qquad D_h = 1.15\cdot\text{ft}$

$$Re = \frac{\rho\cdot V\cdot D_h}{\mu} \qquad Re = 1.18\times10^5$$

For a smooth duct $\quad \frac{1}{\sqrt{f}} = -2\cdot\log\left(\frac{2.51}{Re\cdot\sqrt{f}}\right)$ so $\quad f = 0.017$

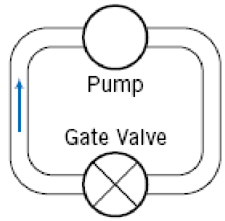
Hence $\quad \Delta p = f\cdot\frac{L}{D_h}\cdot\rho\cdot\frac{V^2}{2} \qquad \Delta p = 0.031\cdot\text{psi}$

or, in in water $\quad h = \frac{\Delta p}{\rho_w\cdot g} \qquad h = 0.848\cdot\text{in}$

Problem 8.128

[Difficulty: 3]

8.128 A system for testing variable-output pumps consists of the pump, four standard elbows, and an open gate valve forming a closed circuit as shown. The circuit is to absorb the energy added by the pump. The tubing is 75-mm-diameter cast iron, and the total length of the circuit is 20-m. Plot the pressure difference required from the pump for water flow rates Q ranging from 0.01 m^3/s to 0.06 m^3/s.



Given: Data on circuit

Find: Plot pressure difference for a range of flow rates

Solution:

Basic equations:

$$\left(\frac{p_1}{\rho}+\alpha_1\cdot\frac{V_1^{\ 2}}{2}+g\cdot z_1\right)-\left(\frac{p_2}{\rho}+\alpha_2\cdot\frac{V_2^{\ 2}}{2}+g\cdot z_2\right)=h_{lT}=\sum_{major} h_l+\sum_{minor} h_{lm} \quad (8.29)$$

$$Re=\frac{\rho\cdot V\cdot D}{\mu} \qquad h_l=f\cdot\frac{L}{D}\cdot\frac{V^2}{2} \quad (8.34) \qquad h_{lm}=K\cdot\frac{V^2}{2} \quad (8.40a) \qquad h_{lm}=f\cdot\frac{L_e}{D}\cdot\frac{V^2}{2} \quad (8.40b)$$

$$f=\frac{64}{Re} \quad (8.36) \quad \text{(Laminar)} \qquad \frac{1}{\sqrt{f}}=-2.0\cdot\log\left(\frac{\frac{e}{D}}{3.7}+\frac{2.51}{Re\cdot\sqrt{f}}\right) \quad (8.37) \quad \text{(Turbulent)}$$

The energy equation (Eq. 8.29) becomes for the circuit (1 = pump inlet, 2 = pump outlet)

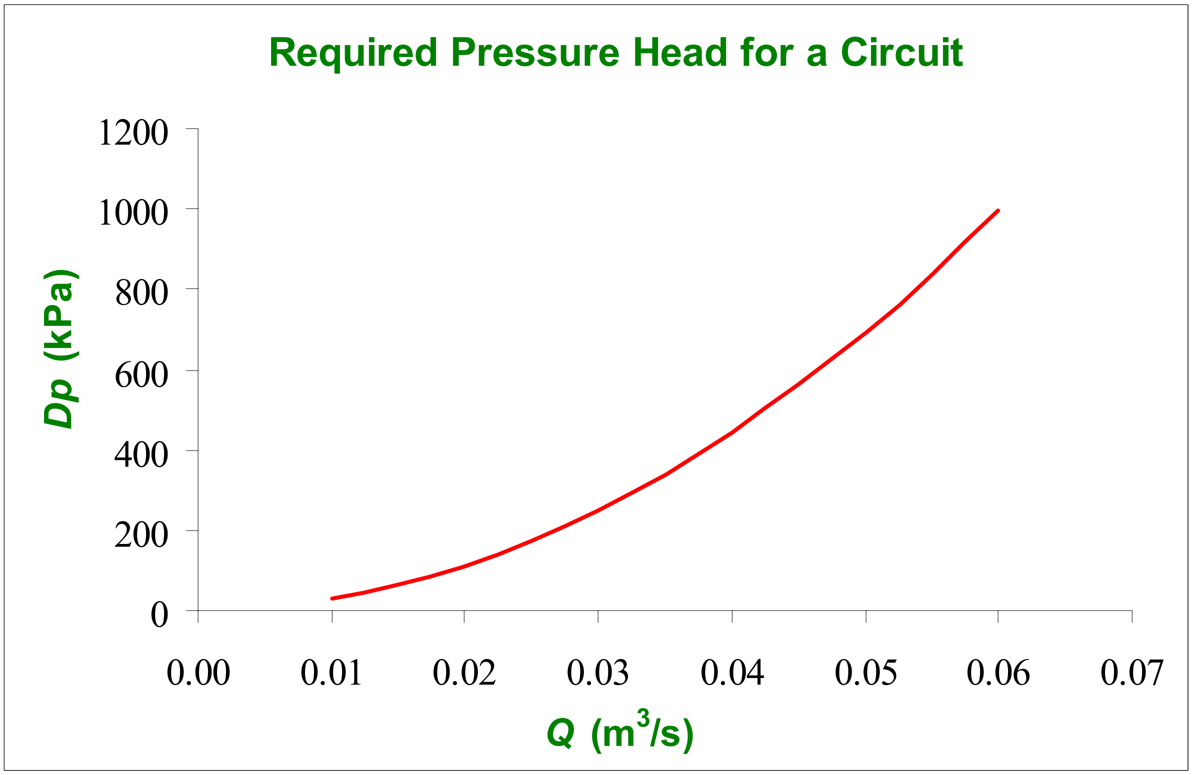
$$\frac{p_1-p_2}{\rho}=f\cdot\frac{L}{D}\cdot\frac{V^2}{2}+4\cdot f\cdot L_{elbow}\cdot\frac{V^2}{2}+f\cdot L_{valve}\cdot\frac{V^2}{2} \quad \text{or} \quad \Delta p=\rho\cdot f\cdot\frac{V^2}{2}\cdot\left(\frac{L}{D}+4\cdot\frac{L_{elbow}}{D}+\frac{L_{valve}}{D}\right)$$

In *Excel*:

Given data:				Tabulated or graphical data:		
L =	20	m		e =	0.26	mm
D =	75	mm			(Table 8.1)	
				μ =	1.00E-03	N.s/m^2
				ρ =	999	kg/m^3
					(Appendix A)	
				Gate valve L_e/D =	8	
				Elbow L_e/D =	30	
					(Table 8.4)	

Computed results:

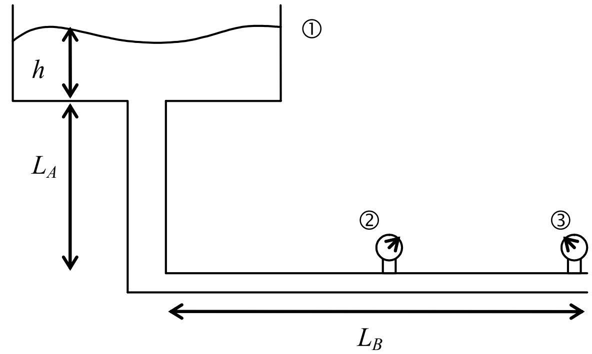
Q (m^3/s)	V (m/s)	Re	f	Δp (kPa)
0.010	2.26	1.70E+05	0.0280	28.3
0.015	3.40	2.54E+05	0.0277	63.1
0.020	4.53	3.39E+05	0.0276	112
0.025	5.66	4.24E+05	0.0276	174
0.030	6.79	5.09E+05	0.0275	250
0.035	7.92	5.94E+05	0.0275	340
0.040	9.05	6.78E+05	0.0274	444
0.045	10.2	7.63E+05	0.0274	561
0.050	11.3	8.48E+05	0.0274	692
0.055	12.4	9.33E+05	0.0274	837
0.060	13.6	1.02E+06	0.0274	996


Required Pressure Head for a Circuit
1200
1000
800
600
400
200
0
Dp (kPa)
0.00
0.01
0.02
0.03
0.04
0.05
0.06
0.07
Q (m3/s)

Problem 8.129

[Difficulty: 3]

8.129 A pipe friction experiment is to be designed, using water, to reach a Reynolds number of 100,000. The system will use 5-cm smooth PVC pipe from a constant-head tank to the flow bench and 20 m of smooth 2.5-cm PVC line mounted horizontally for the test section. The water level in the constant-head tank is 0.5 m above the entrance to the 5-cm PVC line. Determine the required average speed of water in the 2.5-cm pipe. Estimate the feasibility of using a constant-head tank. Calculate the pressure difference expected between taps 5 m apart in the horizontal test section.



Given: Pipe friction experiment

Find: Required average speed; Estimate feasibility of constant head tank; Pressure drop over 5 m

Solution:

Basic equations
$$\left(\frac{p_1}{\rho} + \alpha\cdot\frac{V_1^2}{2} + g\cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha\cdot\frac{V_2^2}{2} + g\cdot z_2\right) = h_{lT} \qquad h_{lT} = h_A + h_B = f_A\cdot\frac{L_A}{D_A}\cdot\frac{V_A^2}{2} + f_B\cdot\frac{L_B}{D_B}\cdot\frac{V_B^2}{2}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) α is approximately 1 4) Ignore minor losses

We wish to have $Re_B = 10^5$

Hence, from $Re_B = \frac{V_B\cdot D_B}{\nu}$ $\quad V_B = \frac{Re_B\cdot\nu}{D_B}$ and for water at 20ºC $\nu = 1.01\times10^{-6}\cdot\frac{m^2}{s}$

$$V_B = 10^5\times1.01\times10^{-6}\cdot\frac{m^2}{s}\times\frac{1}{0.025\cdot m} \qquad V_B = 4.04\,\frac{m}{s}$$

We will also need $V_A = V_B\cdot\left(\frac{D_B}{D_A}\right)^2$

$$V_A = 4.04\cdot\frac{m}{s}\times\left(\frac{2.5}{5}\right)^2 \qquad V_A = 1.01\,\frac{m}{s}$$

$$Re_A = \frac{V_A\cdot D_A}{\nu} \qquad Re_A = 1.01\cdot\frac{m}{s}\times0.05\cdot m\times\frac{s}{1.01\times10^{-6}\cdot m^2} \qquad Re_A = 5\times10^4$$

Both tubes have turbulent flow

For PVC pipe (from Googling!) $e = 0.0015\cdot mm$

For tube A Given
$$\frac{1}{\sqrt{f_A}} = -2.0\cdot\log\left(\frac{\frac{e}{D_A}}{3.7} + \frac{2.51}{Re_A\cdot\sqrt{f_A}}\right) \qquad f_A = 0.0210$$

For tube B Given
$$\frac{1}{\sqrt{f_B}} = -2.0\cdot\log\left(\frac{\frac{e}{D_B}}{3.7} + \frac{2.51}{Re_B\cdot\sqrt{f_B}}\right) \qquad f_B = 0.0183$$

Applying the energy equation between Points 1 and 3

$$g\cdot\left(L_A + h\right) - \frac{V_B^{\ 2}}{2} = f_A\cdot\frac{L_A}{D_A}\cdot\frac{V_A^{\ 2}}{2} + f_B\cdot\frac{L_B}{D_B}\cdot\frac{V_B^{\ 2}}{2}$$

Solving for L_A

$$L_A = \frac{\frac{V_B^{\ 2}}{2}\cdot\left(1 + f_B\cdot\frac{L_B}{D_B}\right) - g\cdot h}{\left(g - \frac{f_A}{D_A}\cdot\frac{V_A^{\ 2}}{2}\right)}$$

$$L_A = \frac{\frac{1}{2}\times\left(4.04\cdot\frac{m}{s}\right)^2\times\left(1 + 0.0183\times\frac{20}{0.025}\right) - 9.81\cdot\frac{m}{s^2}\times 0.5\cdot m}{9.81\cdot\frac{m}{s^2} - \frac{0.0210}{2}\times\frac{1}{0.05\cdot m}\times\left(1.01\cdot\frac{m}{s}\right)^2}$$

$$L_A = 12.8\,m$$

Most ceilings are about 3.5 m or 4 m, so this height is IMPRACTICAL

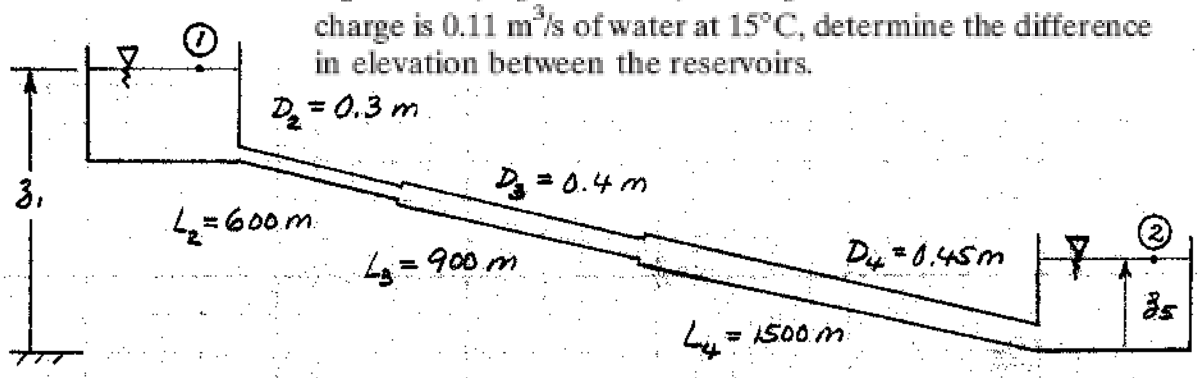
Applying the energy equation between Points 2 and 3

$$\left(\frac{p_2}{\rho} + \frac{V_B^{\ 2}}{2}\right) - \left(\frac{p_3}{\rho} + \frac{V_B^{\ 2}}{2}\right) = f_B\cdot\frac{L}{D_B}\cdot\frac{V_B^{\ 2}}{2} \qquad \text{or} \qquad \Delta p = \rho\cdot f_B\cdot\frac{L}{D_B}\cdot\frac{V_B^{\ 2}}{2}$$

$$\Delta p = 1000\cdot\frac{kg}{m^3}\times\frac{0.0183}{2}\times\frac{5\cdot m}{0.025\cdot m}\times\left(4.04\cdot\frac{m}{s}\right)^2\times\frac{N\cdot s^2}{kg\cdot m}$$

$$\Delta p = 29.9\cdot kPa$$

8.130 Two reservoirs are connected by three clean cast-iron pipes in series, $L_1 = 600$ m, $D_1 = 0.3$ m, $L_2 = 900$ m, $D_2 = 0.4$ m, $L_3 = 1500$ m, and $D_3 = 0.45$ m. When the discharge is 0.11 m³/s of water at 15°C, determine the difference in elevation between the reservoirs.



Solution: Apply the energy equation for steady, incompressible flow that is uniform at each section.

Basic equations: $\frac{p_1}{\rho} + \frac{\bar{V}_1^2}{2} + g z_1 = \frac{p_5}{\rho} + \frac{\bar{V}_5^2}{2} + g z_5 + h_{l_T}$ (1) (2) (1) (2)

$h_{l_T} = \sum f \frac{L}{D} \frac{\bar{V}^2}{2} + h_{l_m}$; $h_{l_m} = K_{ent} \frac{\bar{V}_2^2}{2} + \sum h_{l_{exp}} + K_{exit} \frac{\bar{V}_4^2}{2}$ ($\sum h_{l_{exp}} \approx 0$ (3))

Assumptions: (1) $p_1 = p_5 = p_{atm}$
(2) $\bar{V}_1 = \bar{V}_5 \simeq 0$
(3) Neglect $h_{l_{exp}}$ at pipe joints (note all minor losses are probably small due to long lengths of straight pipe sections, but we will check)

For non-smooth pipe, $f = f(Re, e/D)$. $\mu = 1.1 \times 10^{-3}$ N·s/m², from Table A.8.

Section ②: $e/D_2 = 0.26$ mm/300 mm $= 0.00087$ (for cast iron, $e = 0.26$ mm, Table 8.1)

$$\bar{V}_2 = \frac{Q}{A_2} = 0.11 \frac{m^3}{s} \times \frac{4}{\pi} \frac{1}{(0.3)^2 m^2} = 1.56 \text{ m/s}$$

$$Re_2 = \frac{\rho \bar{V}_2 D_2}{\mu} = 999 \frac{kg}{m^3} \times 1.56 \frac{m}{s} \times 0.3 m \times \frac{m^2}{1.14 \times 10^{-3} N \cdot s} \times \frac{N \cdot s^2}{kg \cdot m} = 4.10 \times 10^5$$

From Fig 8.13, $f_2 = 0.020$

Section ③: $e/D_3 = 0.00065$

$$\bar{V}_3 = \frac{Q}{A_3} = 0.11 \frac{m^3}{s} \times \frac{4}{\pi} \frac{1}{(0.4)^2 m^2} = 0.875 \text{ m/s}$$

$$Re_3 = \frac{\rho \bar{V}_3 D_3}{\mu} = 999 \frac{kg}{m^3} \times 0.875 \frac{m}{s} \times 0.4 m \times \frac{m^2}{1.14 \times 10^{-3} N \cdot s} \times \frac{N \cdot s^2}{kg \cdot m} = 3.07 \times 10^5$$

From Fig. 8.13, $f_3 = 0.019$

Section ④: $e/D_4 = 0.00058$

Problem 8.130

$$\bar{V}_4 = \frac{Q}{A_4} = 0.11\,\frac{m^3}{s} \times \frac{4}{\pi}\,\frac{1}{(0.45)^2 m^2} = 0.692\ m/s$$

$$Re_4 = \frac{\rho \bar{V}_4 D_4}{\mu} = 999\,\frac{kg}{m^3} \times 0.692\,\frac{m}{s} \times 0.45\,m \times \frac{m^2}{1.14\times 10^{-3} N\cdot s} \times \frac{N\cdot s^2}{kg\cdot m} = 2.73\times 10^5$$

From Fig. 8.13, $f_4 = 0.0185$

Then $\sum f \frac{L}{D}\frac{\bar{V}^2}{2} = 0.020 \times \frac{600\,m}{0.3\,m} \times \frac{1}{2}(1.56)^2 \frac{m^2}{s^2} + 0.019 \times \frac{900\,m}{0.4\,m} \times \frac{1}{2}(0.875)^2 \frac{m^2}{s^2}$

$$+ 0.0185 \times \frac{1500\,m}{0.45\,m} \times \frac{1}{2}(0.692)^2 \frac{m^2}{s^2} = 79.8\ m^2/s^2$$

The minor loss coefficients are $K_{ent} = 0.5$ (Table 8.2) and $K_{exit} = 1.0$. Thus,

$$h_{\ell m} = K_{ent}\frac{\bar{V}_2^2}{2} + K_{exit}\frac{\bar{V}_4^2}{2}$$

$$h_{\ell m} = 0.5 \times \frac{1}{2} \times (1.56)^2 \frac{m^2}{s^2} + 1.0 \times \frac{1}{2} \times (0.692)^2 \frac{m^2}{s^2} = 0.848\ m^2/s^2$$

Therefore minor losses are roughly 1 percent of the frictional losses, so they may be neglected. Thus from the energy equation

$$z_1 - z_5 = \sum f \frac{L}{D}\frac{\bar{V}^2}{2g} = 79.8\,\frac{m^2}{s^2} \times \frac{s^2}{9.81\,m} = 8.13\ m$$

$z_1 - z_5$

Problem 8.131

[Difficulty; 3]

8.131 Consider flow of standard air at 1250 ft^3/min. Compare the pressure drop per unit length of a round duct with that for rectangular ducts of aspect ratio 1, 2, and 3. Assume that all ducts are smooth, with cross-sectional areas of 1 ft^2.

Given: Same flow rate in various ducts

Find: Pressure drops of each compared to round duct

Solution:

Basic equations
$$\left(\frac{p_1}{\rho} + \alpha\cdot\frac{V_1^{\,2}}{2} + g\cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha\cdot\frac{V_2^{\,2}}{2} + g\cdot z_2\right) = h_l \qquad D_h = \frac{4\cdot A}{P_w} \qquad e = 0 \qquad \text{(Smooth)}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) α is approximately 1 4) Ignore minor losses

The energy equation simplifies to

$$\Delta p = p_1 - p_2 = \rho\cdot f\cdot\frac{L}{D_h}\cdot\frac{V^2}{2} \qquad \text{or} \qquad \frac{\Delta p}{L} = \rho\cdot\frac{f}{D_h}\cdot\frac{V^2}{2}$$

But we have
$$V = \frac{Q}{A} \qquad V = 1250\cdot\frac{ft^3}{min}\times\frac{1\cdot min}{60\cdot s}\times\frac{1}{1\cdot ft^2} \qquad V = 20.8\cdot\frac{ft}{s}$$

From Table A.9
$$\nu = 1.62\times10^{-4}\cdot\frac{ft^2}{s} \qquad \rho = 0.00234\cdot\frac{slug}{ft^3} \qquad \text{at } 68^oF$$

Hence
$$Re = \frac{V\cdot D_h}{\nu} \qquad Re = 20.8\cdot\frac{ft}{s}\times\frac{s}{1.62\times10^{-4}\cdot ft^2}\times D_h = 1.284\times10^5\cdot D_h \qquad (D_h \text{ in ft})$$

For a round duct
$$D_h = D = \sqrt{\frac{4\cdot A}{\pi}} \qquad D_h = \sqrt{\frac{4}{\pi}\times1\cdot ft^2} \qquad D_h = 1.13\cdot ft$$

For a rectangular duct
$$D_h = \frac{4\cdot A}{P_w} = \frac{4\cdot b\cdot h}{2\cdot(b+h)} = \frac{2\cdot h\cdot ar}{1+ar} \qquad \text{where} \qquad ar = \frac{b}{h}$$

But
$$h = \frac{b}{ar} \qquad \text{so} \qquad h^2 = \frac{b\cdot h}{ar} = \frac{A}{ar} \qquad \text{or} \qquad h = \sqrt{\frac{A}{ar}} \qquad \text{and} \qquad D_h = \frac{2\cdot\sqrt{ar}}{1+ar}\cdot\sqrt{A}$$

The results are:

Round
$$D_h = 1.13\cdot ft \qquad Re = 1.284\times10^5\cdot\frac{1}{ft}\cdot D_h \qquad Re = 1.45\times10^5$$

Given
$$\frac{1}{\sqrt{f}} = -2.0\cdot\log\left(\frac{\frac{e}{D_h}}{3.7} + \frac{2.51}{Re\cdot\sqrt{f}}\right) \qquad f = 0.0167 \qquad \frac{\Delta p}{L} = \rho\cdot\frac{f}{D_h}\cdot\frac{V^2}{2} \qquad \frac{\Delta p}{L} = 7.51\times10^{-3}\cdot\frac{lbf}{ft^3}$$

$ar = 1$
$$D_h = \frac{2\cdot\sqrt{ar}}{1+ar}\cdot\sqrt{A} \qquad D_h = 1\cdot ft \qquad Re = 1.284\times10^5\cdot\frac{1}{ft}\cdot D_h \qquad Re = 1.28\times10^5$$

Given $\quad \dfrac{1}{\sqrt{f}} = -2.0\cdot\log\left(\dfrac{\dfrac{e}{D_h}}{3.7} + \dfrac{2.51}{Re\cdot\sqrt{f}}\right) \qquad f = 0.0171 \qquad \dfrac{\Delta p}{L} = \rho\cdot\dfrac{f}{D_h}\cdot\dfrac{V^2}{2} \qquad \dfrac{\Delta p}{L} = 8.68\times10^{-3}\cdot\dfrac{lbf}{ft^3}$

Hence the square duct experiences a percentage increase in pressure drop of $\quad \dfrac{8.68\times10^{-3} - 7.51\times10^{-3}}{7.51\times10^{-3}} = 15.6\cdot\%$

$ar = 2 \qquad D_h = \dfrac{2\cdot\sqrt{ar}}{1+ar}\cdot\sqrt{A} \qquad D_h = 0.943\cdot ft \qquad Re = 1.284\times10^{5}\cdot\dfrac{1}{ft}\cdot D_h \qquad Re = 1.21\times10^{5}$

Given $\quad \dfrac{1}{\sqrt{f}} = -2.0\cdot\log\left(\dfrac{\dfrac{e}{D_h}}{3.7} + \dfrac{2.51}{Re\cdot\sqrt{f}}\right) \qquad f = 0.0173 \qquad \dfrac{\Delta p}{L} = \rho\cdot\dfrac{f}{D_h}\cdot\dfrac{V^2}{2} \qquad \dfrac{\Delta p}{L} = 9.32\times10^{-3}\cdot\dfrac{lbf}{ft^3}$

Hence the 2 x 1 duct experiences a percentage increase in pressure drop of $\quad \dfrac{9.32\times10^{-3} - 7.51\times10^{-3}}{7.51\times10^{-3}} = 24.1\cdot\%$

$ar = 3 \qquad D_h = \dfrac{2\cdot\sqrt{ar}}{1+ar}\cdot\sqrt{A} \qquad D_h = 0.866\cdot ft \qquad Re = 1.284\times10^{5}\cdot\dfrac{1}{ft}\cdot D_h \qquad Re = 1.11\times10^{5}$

Given $\quad \dfrac{1}{\sqrt{f}} = -2.0\cdot\log\left(\dfrac{\dfrac{e}{D_h}}{3.7} + \dfrac{2.51}{Re\cdot\sqrt{f}}\right) \qquad f = 0.0176 \qquad \dfrac{\Delta p}{L} = \rho\cdot\dfrac{f}{D_h}\cdot\dfrac{V^2}{2} \qquad \dfrac{\Delta p}{L} = 0.01\cdot\dfrac{lbf}{ft^3}$

Hence the 3 x 1 duct experiences a percentage increase in pressure drop of $\quad \dfrac{0.01 - 7.51\times10^{-3}}{7.51\times10^{-3}} = 33.2\cdot\%$

Note that f varies only about 7%; the large change in $\Delta p/L$ is primarily due to the $1/D_h$ factor

Problem 8.132

[Difficulty: 4]

8.132 Data were obtained from measurements on a vertical section of old, corroded, galvanized iron pipe of 50 mm inside diameter. At one section the pressure was $p_1 = 750$ kPa (gage); at a second section, 40 m lower, the pressure was $p_2 = 250$ kPa (gage). The volume flow rate of water was 0.015 m³/s. Estimate the relative roughness of the pipe. What percentage savings in pumping power would result if the pipe were restored to its new, clean relative roughness?

Given: Flow down corroded iron pipe

Find: Pipe roughness; Power savings with new pipe

Solution:

Basic equations
$$\left(\frac{p_1}{\rho} + \alpha\cdot\frac{V_1^2}{2} + g\cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha\cdot\frac{V_2^2}{2} + g\cdot z_2\right) = h_l \qquad h_l = f\cdot\frac{L}{D}\cdot\frac{V^2}{2}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) α is approximately 1 4) No minor losses

Available data
$$D = 50\cdot mm \qquad \Delta z = 40\cdot m \qquad L = \Delta z \qquad p_1 = 750\cdot kPa \qquad p_2 = 250\cdot kPa \qquad Q = 0.015\cdot\frac{m^3}{s} \qquad \rho = 999\cdot\frac{kg}{m^3}$$

Hence the energy equation becomes

$$\left(\frac{p_1}{\rho} + g\cdot z_1\right) - \left(\frac{p_2}{\rho} + g\cdot z_2\right) = f\cdot\frac{L}{D}\cdot\frac{V^2}{2}$$

Here
$$V = \frac{Q}{A} = \frac{4\cdot Q}{\pi\cdot D^2} \qquad V = \frac{4}{\pi}\times 0.015\frac{m^3}{s}\times\frac{1}{(0.05\cdot m)^2} \qquad V = 7.64\frac{m}{s}$$

In this problem we can compute directly f and Re, and hence obtain e/D

Solving for f
$$f = \frac{2\cdot D}{L\cdot V^2}\cdot\left(\frac{p_1 - p_2}{\rho} + g\left(z_1 - z_2\right)\right)$$

$$f = 2\times\frac{0.05}{40}\times\left(\frac{s}{7.64\cdot m}\right)^2\times\left[(750 - 250)\times 10^3\cdot\frac{N}{m^2}\times\frac{m^3}{1000\cdot kg}\times\frac{kg\cdot m}{s^2\cdot N} + 9.81\cdot\frac{m}{s^2}\times 40\cdot m\right] \qquad f = 0.0382$$

From Table A.8 (20ºF) $\nu = 1.01\times 10^{-6}\cdot\frac{m^2}{s}$ $\qquad Re = \frac{V\cdot D}{\nu} \qquad Re = 7.64\cdot\frac{m}{s}\times 0.05\cdot m\times\frac{s}{1.01\times 10^{-6}\cdot m^2} \qquad Re = 3.78\times 10^5$

Flow is turbulent:
$$\frac{1}{\sqrt{f}} = -2.0\cdot\log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re\cdot\sqrt{f}}\right)$$

Solving for e $\quad e = 3.7 \cdot D \cdot \left(10^{-\frac{1}{2 \cdot \sqrt{f}}} - \frac{2.51}{Re \cdot \sqrt{f}} \right) \qquad e = 0.507\,mm \qquad \frac{e}{D} = 0.0101$

New pipe (Table 8.1) $\quad e = 0.15 \cdot mm \qquad \frac{e}{D} = 0.003$

Given $\quad \frac{1}{\sqrt{f}} = -2.0 \cdot \log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re \cdot \sqrt{f}} \right)$

$f = 0.0326$

In this problem $\quad \Delta p = p_1 - p_2 = \rho \cdot \left[g \cdot (z_2 - z_1) + f \cdot \frac{L}{D} \cdot \frac{V^2}{2} \right]$

Hence $\quad \Delta p_{new} = 1000 \cdot \frac{kg}{m^3} \times \left[9.81 \cdot \frac{m}{s^2} \times (-40 \cdot m) + \frac{0.0326}{2} \times \frac{40}{0.05} \times \left(7.64 \cdot \frac{m}{s} \right)^2 \right] \times \frac{N \cdot s^2}{kg \cdot m} \qquad \Delta p_{new} = 369 \cdot kPa$

$\Delta p_{old} = p_1 - p_2 \qquad \Delta p_{old} = 500\,kPa$

Compared to $\Delta p_{old} = 500 \cdot kPa$ we find $\quad \frac{\Delta p_{old} - \Delta p_{new}}{\Delta p_{old}} = 26.3 \cdot \%$

As power is ΔpQ and Q is constant, the power reduction is the same as the above percentage!

Problem 8.133

[Difficulty: 3]

8.133 Water, at volume flow rate $Q = 0.75\ \text{ft}^3/\text{s}$, is delivered by a fire hose and nozzle assembly. The hose ($L = 250$ ft, $D = 3$ in., and $e/D = 0.004$) is made up of four 60-ft sections joined by couplings. The entrance is square-edged; the minor loss coefficient for each coupling is $K_c = 0.5$, based on mean velocity through the hose. The nozzle loss coefficient is $K_n = 0.02$, based on velocity in the exit jet, of $D_2 = 1$ in. diameter. Estimate the supply pressure required at this flow rate.

Given: Flow through fire hose and nozzle

Find: Supply pressure

Solution:

Basic equations

$$\left(\frac{p_1}{\rho} + \alpha\cdot\frac{V_1^2}{2} + g\cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha\cdot\frac{V_2^2}{2} + g\cdot z_2\right) = h_{lT} \qquad h_{lT} = h_l + h_{lm} = f\cdot\frac{L}{D}\cdot\frac{V^2}{2} + \sum_{\text{Minor}}\left(K\cdot\frac{V^2}{2}\right)$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) α is approximately 1 4) $p_2 = p_{atm}$ so $p_2 = 0$ gage

Hence the energy equation between Point 1 at the supply and the nozzle exit (Point n); let the velocity in the hose be V

$$\frac{p_1}{\rho} - \frac{V_n^2}{2} = f\cdot\frac{L}{D}\cdot\frac{V^2}{2} + \left(K_e + 4\cdot K_c\right)\cdot\frac{V^2}{2} + K_n\cdot\frac{V_n^2}{2}$$

From continuity $\quad V_n = \left(\frac{D}{D_2}\right)^2\cdot V \quad$ and $\quad V = \frac{Q}{A} = \frac{4\cdot Q}{\pi\cdot D^2} \qquad V = \frac{4}{\pi}\times 0.75\cdot\frac{\text{ft}^3}{\text{s}}\times\frac{1}{\left(\frac{1}{4}\cdot\text{ft}\right)^2} \qquad V = 15.3\cdot\frac{\text{ft}}{\text{s}}$

Solving for p_1 $\quad p_1 = \frac{\rho\cdot V^2}{2}\cdot\left[f\cdot\frac{L}{D} + K_e + 4\cdot K_c + \left(\frac{D}{D_2}\right)^4\cdot\left(1 + K_n\right)\right]$

From Table A.7 (68°F) $\quad \rho = 1.94\cdot\frac{\text{slug}}{\text{ft}^3} \qquad \nu = 1.08\times 10^{-5}\cdot\frac{\text{ft}^2}{\text{s}}$

$$Re = \frac{V\cdot D}{\nu} \qquad Re = 15.3\cdot\frac{\text{ft}}{\text{s}}\times\frac{3}{12}\cdot\text{ft}\times\frac{\text{s}}{1.08\times 10^{-5}\cdot\text{ft}^2} \qquad Re = 3.54\times 10^5 \qquad \text{Turbulent}$$

For the hose $\quad \frac{e}{D} = 0.004$

Flow is turbulent: $\quad$ Given $\quad \frac{1}{\sqrt{f}} = -2.0\cdot\log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re\cdot\sqrt{f}}\right) \qquad f = 0.0287$

$$p_1 = \frac{1}{2}\times 1.94\cdot\frac{\text{slug}}{\text{ft}^3}\times\left(15.3\cdot\frac{\text{ft}}{\text{s}}\right)^2\times\left[0.0287\times\frac{250}{\frac{1}{4}} + 0.5 + 4\times 0.5 + \left(\frac{3}{1}\right)^4\times(1 + 0.02)\right]\times\frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}}$$

$$p_1 = 2.58\times 10^4\cdot\frac{\text{lbf}}{\text{ft}^2} \qquad p_1 = 179\cdot\text{psi}$$

Problem 8.134

[Difficulty: 4]

8.134 Flow in a tube may alternate between laminar and turbulent states for Reynolds numbers in the transition zone. Design a bench-top experiment consisting of a constant-head cylindrical transparent plastic tank with depth graduations, and a length of plastic tubing (assumed smooth) attached at the base of the tank through which the water flows to a measuring container. Select tank and tubing dimensions so that the system is compact, but will operate in the transition zone range. Design the experiment so that you can easily increase the tank head from a low range (laminar flow) through transition to turbulent flow, and vice versa. (Write instructions for students on recognizing when the flow is laminar or turbulent.) Generate plots (on the same graph) of tank depth against Reynolds number, assuming laminar or turbulent flow.

Given: Proposal for bench top experiment

Find: Design it; Plot tank depth versus Re

Solution:

Basic equations:

$$\left(\frac{p_1}{\rho} + \alpha_1 \cdot \frac{V_1^2}{2} + g \cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \cdot \frac{V_2^2}{2} + g \cdot z_2\right) = h_{lT} = \sum_{\text{major}} h_l + \sum_{\text{minor}} h_{lm} \qquad (8.29)$$

$$Re = \frac{\rho \cdot V \cdot D}{\mu} \qquad h_l = f \cdot \frac{L}{D} \cdot \frac{V^2}{2} \quad (8.34) \qquad h_{lm} = K \cdot \frac{V^2}{2} \quad (8.40a) \qquad h_{lm} = f \cdot \frac{L_e}{D} \cdot \frac{V^2}{2} \quad (8.40b)$$

$$f = \frac{64}{Re} \quad (8.36) \quad \text{(Laminar)} \qquad \frac{1}{\sqrt{f}} = -2.0 \cdot \log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re \cdot \sqrt{f}}\right) \quad (8.37) \quad \text{(Turbulent)}$$

The energy equation (Eq. 8.29) becomes

$$g \cdot H - \alpha \cdot \frac{V^2}{2} = f \cdot \frac{L}{D} \cdot \frac{V^2}{2} + K \cdot \frac{V^2}{2}$$

This can be solved explicity for reservoir height *H*

$$H = \frac{V^2}{2 \cdot g} \cdot \left(\alpha + f \cdot \frac{L}{D} + K\right)$$

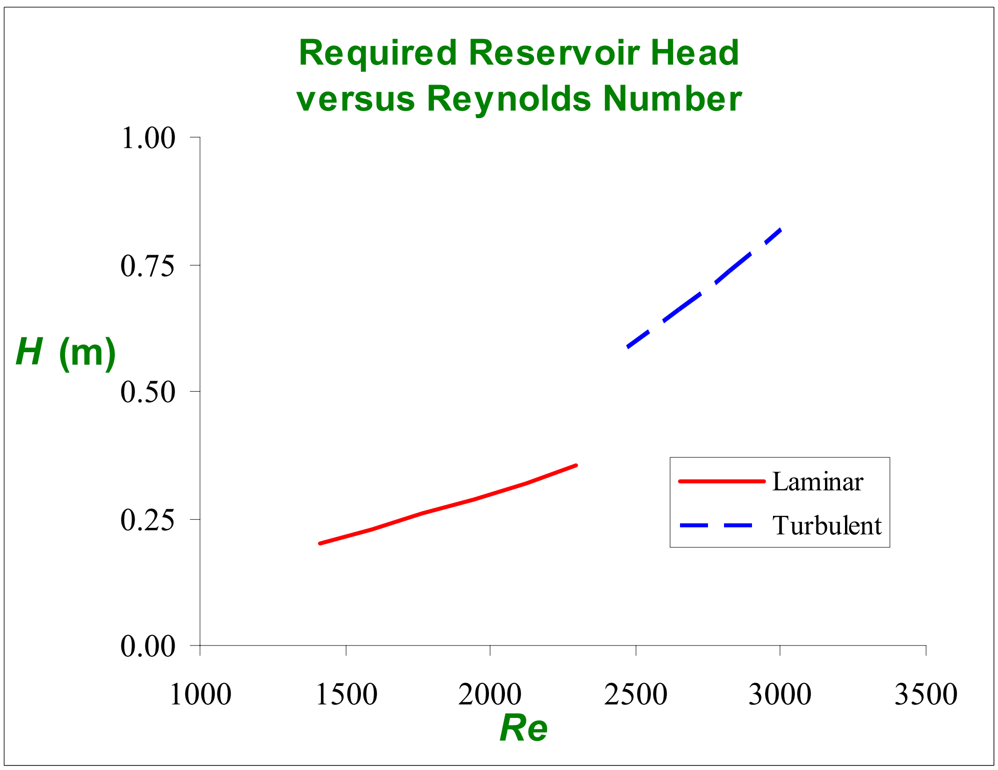
In *Excel*:

Choose data:				Tabulated or graphical data:			
L =	1.0	m		μ =	1.00E-03	N.s/m^2	
D =	3.0	mm		ρ =	999	kg/m^3	
e =	0.0	mm			(Appendix A)		
α =	2	(Laminar)		K_{ent} =	0.5	(Square-edged)	
=	1	(Turbulent)			(Table 8.2)		

Computed results:

Q (L/min)	V (m/s)	Re	Regime	f	H (m)
0.200	0.472	1413	Laminar	0.0453	0.199
0.225	0.531	1590	Laminar	0.0403	0.228
0.250	0.589	1767	Laminar	0.0362	0.258
0.275	0.648	1943	Laminar	0.0329	0.289
0.300	0.707	2120	Laminar	0.0302	0.320
0.325	0.766	2297	Laminar	0.0279	0.353
0.350	0.825	2473	Turbulent	0.0462	0.587
0.375	0.884	2650	Turbulent	0.0452	0.660
0.400	0.943	2827	Turbulent	0.0443	0.738
0.425	1.002	3003	Turbulent	0.0435	0.819
0.450	1.061	3180	Turbulent	0.0428	0.904

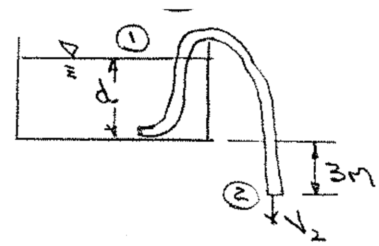
The flow rates are realistic, and could easily be measured using a tank/timer system
The head required is also realistic for a small-scale laboratory experiment
Around Re = 2300 the flow may oscillate between laminar and turbulent:
Once turbulence is triggered (when H > 0.353 m), the resistance to flow increases requiring H >0.587 m to maintain; hence the flow reverts to laminar, only to trip over again to turbulent! This behavior will be visible: the exit flow will switch back and forth between smooth (laminar) and chaotic (turbulent)



Problem 8.135

[Difficulty: 3]

8.135 A small swimming pool is drained using a garden hose. The hose has 20 mm inside diameter, a roughness height of 0.2 mm, and is 30 m long. The free end of the hose is located 3 m below the elevation of the bottom of the pool. The average velocity at the hose discharge is 1.2 m/s. Estimate the depth of the water in the swimming pool. If the flow were inviscid, what would be the velocity?



Solution:

Apply the energy equation for steady incompressible flow between sections ① and ②

Basic equations: $\left(\cancel{\frac{p_1}{\rho}} + \cancel{\alpha_1 \frac{\bar{V}_1^2}{2}} + g z_1\right) - \left(\cancel{\frac{p_2}{\rho}} + \alpha_2 \frac{\bar{V}_2^2}{2} + g z_2\right) = h_{l_T}$ (8.29)

$z_1 - z_2 = d(2)$

$h_{l_T} = h_l + h_{l_m}$; $h_l = f \frac{L}{D} \frac{\bar{V}^2}{2}$; $h_{l_m} = K_{ent} \frac{\bar{V}^2}{2}$

Assumptions: (1) $p_1 = p_2 = p_{atm}$

(2) $\bar{V}_1 = 0$, $\alpha_2 = 1.0$

(3) square edged entrance

Then

$$z_1 - z_2 = d + 3m = f \frac{L}{D} \frac{\bar{V}^2}{2g} + K_{ent} \frac{\bar{V}^2}{2g} + \frac{\bar{V}_2^2}{2g} = \frac{\bar{V}^2}{2g}\left[f \frac{L}{D} + K_{ent} + 1\right]$$

$$\therefore d = \frac{\bar{V}^2}{2g}\left[f \frac{L}{D} + K_{ent} + 1\right] - 3m \qquad (1)$$

For square edged entrance (Table 8.2) $K_{ent} = 0.5$

$Re = \frac{D\bar{V}}{\nu} = 0.020\,m \times 1.2\frac{m}{s} \times \frac{s}{1.00 \times 10^{-6}\,m^2} = 2.4 \times 10^4$ {assume T = 20°C, Table A.8}

$e/D = 0.2/20 = 0.01$. From Fig. 8.13, $f = 0.04$

Then from Eq. 1

$$d = \frac{(1.2)^2}{2}\frac{m^2}{s^2} \times \frac{s^2}{9.81\,m}\left[0.04 \times \frac{30}{0.02} + 0.5 + 1\right] - 3m = 1.51\,m \qquad d$$

For frictionless flow, $h_{l_T} = f \frac{L}{D}\frac{\bar{V}^2}{2} + K_{ent}\frac{\bar{V}^2}{2} = 0$ and Eq. 1 gives $d = \frac{\bar{V}^2}{2g} - 3m$

and $\bar{V} = \left[2g(d + 3m)\right]^{1/2} = \left[2 \times 9.81\frac{m}{s^2}(1.51 + 3)m\right]^{1/2}$

$\bar{V} = 9.41\,m/s$ $\qquad \bar{V}_{inviscid}$

Problem 8.136

[Difficulty: 3]

8.136 When you drink you beverage with a straw, you need to overcome both gravity and friction in the straw. Estimate the fraction of the total effort you put into quenching your thirst of each factor, making suitable assumptions about the liquid and straw properties, and your drinking rate (for example, how long it would take you to drink a 12-oz drink if you drank it all in one go (quite a feat with a straw). Is the flow laminar or turbulent? (Ignore minor losses.)

Given: Drinking of a beverage

Find: Fraction of effort of drinking of friction and gravity

Solution:

Basic equations $$\left(\frac{p_1}{\rho} + \alpha\cdot\frac{V_1^2}{2} + g\cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha\cdot\frac{V_2^2}{2} + g\cdot z_2\right) = h_l \qquad h_l = f\cdot\frac{L}{D}\cdot\frac{V^2}{2}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) α is approximately 1 4) No minor losses

Hence the energy equation becomes, between the bottom of the straw (Point 1) and top (Point 2)

$$g\cdot z_1 - \left(\frac{p_2}{\rho} + g\cdot z_2\right) = f\cdot\frac{L}{D}\cdot\frac{V^2}{2}$$ where p_2 is the gage pressure in the mouth

The negative gage pressure the mouth must create is therefore due to two parts

$$p_{grav} = -\rho\cdot g\cdot(z_2 - z_1) \qquad p_{fric} = -\rho\cdot f\cdot\frac{L}{D}\cdot\frac{V^2}{2}$$

Assuming a person can drink 12 fluid ounces in 5 s $\quad Q = \dfrac{\frac{12}{128}\cdot gal}{5\cdot s} \times \dfrac{1\cdot ft^3}{7.48\cdot gal} \qquad Q = 2.51\times 10^{-3}\cdot\dfrac{ft^3}{s}$

Assuming a straw is 6 in long diameter 0.2 in, with roughness $e = 5\times 10^{-5}$ in (from Googling!)

$$V = \frac{4\cdot Q}{\pi\cdot D^2} \qquad V = \frac{4}{\pi}\times 2.51\times 10^{-3}\,\frac{ft^3}{s}\times\left(\frac{1}{0.2\cdot in}\times\frac{12\cdot in}{1\cdot ft}\right)^2 \qquad V = 11.5\cdot\frac{ft}{s}$$

From Table A.7 (68oF) $\nu = 1.08\times 10^{-5}\cdot\dfrac{ft^2}{s}$ (for water, but close enough)

$$Re = \frac{V\cdot D}{\nu} \qquad Re = 11.5\cdot\frac{ft}{s}\times\frac{0.2}{12}\cdot ft\times\frac{s}{1.08\times 10^{-5}\,ft^2} \qquad Re = 1.775\times 10^4$$

Given $$\frac{1}{\sqrt{f}} = -2.0\cdot\log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re\cdot\sqrt{f}}\right) \qquad f = 0.0272$$

Then $$p_{grav} = -1.94\cdot\frac{slug}{ft^3}\times 32.2\cdot\frac{ft}{s^2}\times\frac{1}{2}\cdot ft\times\frac{lbf\cdot s^2}{slug\cdot ft} \qquad p_{grav} = -31.2\cdot\frac{lbf}{ft^2} \qquad p_{grav} = -0.217\cdot psi$$

and $$p_{fric} = -1.94\cdot\frac{slug}{ft^3}\times 0.0272\times\frac{6}{0.2}\times\frac{1}{2}\times\left(11.5\cdot\frac{ft}{s}\right)^2\times\frac{lbf\cdot s^2}{slug\cdot ft} \qquad p_{fric} = -105\cdot\frac{lbf}{ft^2} \qquad p_{fric} = -0.727\cdot psi$$

Hence the fraction due to friction is $\dfrac{p_{fric}}{p_{fric}+p_{grav}} = 77\cdot\%$ and gravity is $\dfrac{p_{grav}}{p_{fric}+p_{grav}} = 23\cdot\%$

These results will vary depending on assumptions, but it seems friction is significant!

Problem 8.137

[Difficulty: 4]

8.137 The hose in Problem 8.135 is replaced with a larger-diameter hose, diameter 25 mm (same length and roughness). Assuming a pool depth of 1.5 m, what will be the new average velocity and flow rate?

Given: Draining of swimming pool with larger hose

Find: Flow rate and average velocity

Solution:

Basic equations $\left(\frac{p_1}{\rho} + \alpha\cdot\frac{V_1^2}{2} + g\cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha\cdot\frac{V_2^2}{2} + g\cdot z_2\right) = h_l$ $\quad h_l = f\cdot\frac{L}{D}\cdot\frac{V^2}{2}$ $\quad h_{lm} = K_{ent}\cdot\frac{V^2}{2}$

Assumptions: 1) Steady flow 2) Incompressible flow 3) α is approximately 1 4) No minor losses

Available data $D = 25\cdot mm$ $\quad L = 30\cdot m$ $\quad e = 0.2\cdot mm$ $\quad h = 3\cdot m$ $\quad \Delta z = 1.5\cdot m$ $\quad K_{ent} = 0.5$ $\quad \nu = 1\cdot 10^{-6}\cdot\frac{m^2}{s}$

Hence the energy equation becomes $\quad g\cdot(\Delta z + h) = f\cdot\frac{L}{D}\cdot\frac{V^2}{2} + K_{ent}\cdot\frac{V^2}{2} + \frac{V^2}{2}$

Solving for V $\quad V = \sqrt{\dfrac{2\cdot g\cdot(\Delta z + h)}{f\cdot\frac{L}{D} + K_{ent} + 1}}$ $\quad$ (1)

We also have $\quad Re = \frac{V\cdot D}{\nu}$ $\quad$ (2) $\quad$ In addition $\quad \frac{1}{\sqrt{f}} = -2.0\cdot\log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re\cdot\sqrt{f}}\right)$ $\quad$ (3)

Equations 1, 2 and 3 form a set of simultaneous equations for V, Re and f, which we can solve iteratively

Make a guess for f $\quad f = 0.1$ $\quad$ then $\quad V = \sqrt{\dfrac{2\cdot g\cdot(\Delta z + h)}{f\cdot\frac{L}{D} + K_{ent} + 1}}$ $\quad V = 0.852\,\frac{m}{s}$ $\quad Re = \frac{V\cdot D}{\nu}$ $\quad Re = 2.13\times 10^4$

using Eq 3, at this Re $\quad f = 0.0382$

Then, repeating $\quad V = \sqrt{\dfrac{2\cdot g\cdot(\Delta z + h)}{f\cdot\frac{L}{D} + K_{ent} + 1}}$ $\quad V = 1.37\,\frac{m}{s}$ $\quad Re = \frac{V\cdot D}{\nu}$ $\quad Re = 3.41\times 10^4$

using Eq 3, at this Re $\quad f = 0.0371$

Then, repeating $\quad V = \sqrt{\dfrac{2\cdot g\cdot(\Delta z + h)}{f\cdot\frac{L}{D} + K_{ent} + 1}}$ $\quad V = 1.38\,\frac{m}{s}$ $\quad Re = \frac{V\cdot D}{\nu}$ $\quad Re = 3.46\times 10^4$

Using Eq 3, at this Re $\quad f = 0.0371$ $\quad$ which is the same as before, so we have convergence

The flow rate is then $\quad Q = V\cdot\frac{\pi\cdot D^2}{4}$ $\quad Q = 6.79\times 10^{-4}\,\frac{m^3}{s}$ $\quad Q = 0.679\,\frac{l}{s}$

Note that we could use *Excel*'s *Solver* for this problem

Problem 8.138

[Difficulty: 4]

8.138 What flow rate (gpm) will be produced in a 75-mm-diameter water pipe for which there is a pressure drop of 425 kPa over a 200-m length? The pipe roughness is 2.5 mm. The water is at 0°C.

Given: Flow in horizontal pipe

Find: Flow rate

Solution:

Basic equations
$$\left(\frac{p_1}{\rho} + \alpha\cdot\frac{V_1^{\,2}}{2} + g\cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha\cdot\frac{V_2^{\,2}}{2} + g\cdot z_2\right) = h_l \qquad h_l = f\cdot\frac{L}{D}\cdot\frac{V^2}{2}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) α is approximately 1 4) No minor losses

Available data $L = 200\cdot m$ $\quad D = 75\cdot mm$ $\quad e = 2.5\cdot mm$ $\quad \Delta p = 425\cdot kPa$ $\quad \rho = 1000\cdot\frac{kg}{m^3}$ $\quad \mu = 1.76\cdot 10^{-3}\cdot\frac{N\cdot s}{m^2}$

Hence the energy equation becomes

$$\frac{p_1}{\rho} - \frac{p_2}{\rho} = \frac{\Delta p}{\rho} = f\cdot\frac{L}{D}\cdot\frac{V^2}{2}$$

Solving for V $\quad V = \sqrt{\frac{2\cdot D\cdot\Delta p}{L\cdot\rho\cdot f}}$ $\quad V = \frac{k}{\sqrt{f}}$ (1) $\quad k = \sqrt{\frac{2\cdot D\cdot\Delta p}{L\cdot\rho}}$ $\quad k = 0.565\,\frac{m}{s}$

We also have $\quad Re = \frac{\rho\cdot V\cdot D}{\mu}$ or $\quad Re = c\cdot V$ (2) where $\quad c = \frac{\rho\cdot D}{\mu}$ $\quad c = 4.26\times 10^4\,\frac{s}{m}$

In addition $\quad \frac{1}{\sqrt{f}} = -2.0\cdot\log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re\cdot\sqrt{f}}\right)$ (3)

Equations 1, 2 and 3 form a set of simultaneous equations for V, Re and f

Make a guess for f $\quad f = 0.1$ then $\quad V = \frac{k}{\sqrt{f}}$ $\quad V = 5.86\cdot\frac{ft}{s}$ $\quad Re = c\cdot V$ $\quad Re = 7.61\times 10^4$

Given $\quad \frac{1}{\sqrt{f}} = -2.0\cdot\log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re\cdot\sqrt{f}}\right)$ $\quad f = 0.0573$ $\quad V = \frac{k}{\sqrt{f}}$ $\quad V = 7.74\cdot\frac{ft}{s}$ $\quad Re = c\cdot V$ $\quad Re = 1.01\times 10^5$

Given $\quad \frac{1}{\sqrt{f}} = -2.0\cdot\log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re\cdot\sqrt{f}}\right)$ $\quad f = 0.0573$ $\quad V = \frac{k}{\sqrt{f}}$ $\quad V = \blacksquare\cdot\frac{ft}{s}$ $\quad Re = c\cdot V$ $\quad Re = 1.01\times 10^5$

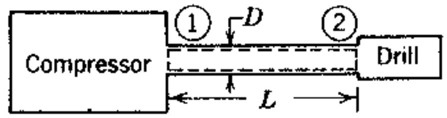
The flow rate is then $\quad Q = V\cdot\frac{\pi\cdot D^2}{4}$ $\quad Q = 0.0104\,\frac{m^3}{s}$ $\quad Q = 10.42\,\frac{1}{s}$ $\quad Q = 165\cdot gpm$

Note that we could use *Excel*'s *Solver* for this problem

Problem 8.139

[Difficulty: 2]

8.139 A compressed air drill requires 0.25 kg/s of air at 650 kPa (gage) at the drill. The hose from the air compressor to the drill is 40 mm inside diameter. The maximum compressor discharge gage pressure is 670 kPa; air leaves the compressor at 40°C. Neglect changes in density and any effects of hose curvature. Calculate the longest hose that may be used.



Solution:

Computing equation: $\left(\frac{p_1}{\rho}+\alpha_1\frac{\bar{V}_1^2}{2}+gz_1\right)-\left(\frac{p_2}{\rho}+\alpha_2\frac{\bar{V}_2^2}{2}+gz_2\right)=h_{l_T}=h_l+h_{l_m}$ (8.29)

where $h_l = f\frac{L}{D}\frac{\bar{V}^2}{2}$ $\quad h_{l_m}=K\frac{\bar{V}^2}{2}$

For $\rho = c$, then $\bar{V}_1=\bar{V}_2$, since $A_1=A_2$. Since p_1 and p_2 are given, neglect minor losses. Assume $\alpha_1=\alpha_2$ and neglect elevation changes. Then Eq. 8.29 can be written as

$$\frac{p_1-p_2}{\rho}=f\frac{L}{D}\frac{\bar{V}^2}{2} \quad \text{or} \quad L=\frac{(p_1-p_2)}{\rho}\frac{2D}{f\bar{V}^2}$$

The density is

$$\rho=\rho_1=\frac{p_1}{RT_1}=7.91\times10^5\,\frac{N}{m^2}\times\frac{kg\cdot K}{287\,N\cdot m}\times\frac{1}{313\,K}=8.81\,kg/m^3$$

From continuity

$$\bar{V}=\frac{\dot{m}}{\rho A}=\frac{4\dot{m}}{\pi\rho D^2}=\frac{4}{\pi}\times0.25\,\frac{kg}{sec}\times\frac{m^3}{8.81\,kg}\times\frac{1}{(0.04)^2\,m^2}=22.6\,m/sec$$

For air at 40°C, $\mu=1.91\times10^{-5}$ kg/m·s (Table A.10), so

$$Re=\frac{\rho\bar{V}D}{\mu}=8.81\,\frac{kg}{m^3}\times22.6\,\frac{m}{sec}\times0.04\,m\times\frac{m\cdot sec}{1.91\times10^{-5}\,kg}=4.17\times10^5$$

Assume smooth pipe; then from Fig. 8.13, $f=0.0134$

Substituting gives

$$L=\frac{(p_1-p_2)}{\rho}\frac{2D}{f\bar{V}^2}$$

$$=20\times10^3\,\frac{N}{m^2}\times2\times0.04\,m\times\frac{m^3}{8.81\,kg}\times\frac{1}{0.0134}\times\frac{sec^2}{(22.6)^2\,m^2}\times\frac{kg\cdot m}{N\cdot sec^2}$$

$L = 26.5\ m$ ← L

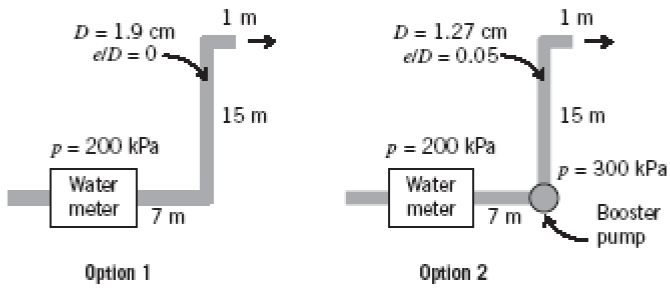
Problem 8.140 [Difficulty: 4]

8.140 You recently bought a house and want to improve the flow rate of water on your top floor. The poor flow rate is due to three reasons: The city water pressure at the water meter is poor ($p = 200$ kPa gage); the piping has a small diameter ($D = 1.27$ cm) and has been crudded up, increasing its roughness ($e/D = 0.05$); and the top floor of the house is 15 m higher than the water meter. You are considering two options to improve the flow rate: Option 1 is replacing all the piping after the water meter with new smooth piping with a diameter of 1.9 cm; and option 2 is installing a booster pump while keeping the original pipes. The booster pump has an outlet pressure of 300 kPa. Which option would be more effective? Neglect minor losses.

Given: Two potential solutions to improve flowrate.

Find: Which solution provides higher flowrate

Solution:



Basic equations: $$\left(\frac{p_1}{\rho}+\alpha_1\frac{\overline{V}_1^2}{2}+gz_1\right)-\left(\frac{p_2}{\rho}+\alpha_2\frac{\overline{V}_2^2}{2}+gz_2\right)=h_{l_T}$$

$$h_{l_T}=h_l+h_{l_m}=f\frac{L}{D}\frac{\overline{V}^2}{2}+K\frac{\overline{V}^2}{2} \qquad \frac{1}{\sqrt{f}}=-2.0\log\left(\frac{e/D}{3.7}+\frac{2.51}{\text{Re}\sqrt{f}}\right) \qquad Q=VA$$

Assumptions: 1) Steady flow 2) Incompressible 3) Neglect minor losses 4) $\alpha_1\frac{\overline{V}_1^2}{2}=\alpha_2\frac{\overline{V}_2^2}{2}$

Option 1: let $z_1 = 0$ $\quad p_2 = p_{atm} = 0$ kPa gage

Given data $\quad p_1 = 200$ kPa gage $\quad D = 0.019$ m $\quad z_2 = 15$ m $\quad \frac{e}{D} = 0$ $\quad L = 23$ m

The energy equation becomes: $$\frac{p_1}{\rho}-gz_2=f\frac{L}{D}\frac{\overline{V}^2}{2}$$

Solving for V: $$\overline{V}=\sqrt{\frac{2\cdot D\cdot\left(\frac{p_1}{\rho}-g\cdot z_2\right)}{f\cdot L}} \qquad \overline{V}=\frac{k}{\sqrt{f}} \quad (1)$$

$$k = \sqrt{\frac{2 \cdot D \cdot \left(\frac{p_1}{\rho} - g \cdot z_2\right)}{L}} = \sqrt{2 \times 0.019\,\text{m} \times \left(200{,}000\frac{\text{N}}{\text{m}^2} \times \frac{\text{m}^3}{999\,\text{kg}} \times \frac{\text{kg}\cdot\text{m}}{\text{N}\cdot\text{s}^2} - 9.81\frac{\text{m}}{\text{s}^2} \times 15\,\text{m}\right) \times \frac{1}{23\,\text{m}}}$$

$$k = 0.296\frac{\text{m}}{\text{s}}$$

We also have $\mathrm{Re} = \frac{\rho \cdot \bar{V} \cdot D}{\mu}$ or $\mathrm{Re} = c \cdot \bar{V}$ (2) where $c = \frac{\rho \cdot D}{\mu}$

Assuming water at 20°C (ρ = 999 kg/m^3, μ = 1 x 10^{-3} kg/(m s)): $c = 999\frac{\text{kg}}{\text{m}^3} \times 0.019\,\text{m} \times \frac{\text{m}\cdot\text{s}}{1\times 10^{-3}\,\text{kg}} = 18981\frac{\text{s}}{\text{m}}$

In addition: $\frac{1}{\sqrt{f}} = -2.0\log\left(\frac{2.51}{\mathrm{Re}\sqrt{f}}\right)$ (3)

Equations 1, 2 and 3 form a set of simultaneous equations for $\bar{V}$, Re and f

Make a guess for f $f = 0.015$ then $\bar{V} = \frac{k}{\sqrt{f}} = 2.42\frac{\text{m}}{\text{s}}$ $\mathrm{Re} = c \cdot \bar{V} = 4.59 \times 10^4$

Given $\frac{1}{\sqrt{f}} = -2.0\log\left(\frac{2.51}{\mathrm{Re}\sqrt{f}}\right)$ $f = 0.0213$ $\bar{V} = \frac{k}{\sqrt{f}} = 2.03\frac{\text{m}}{\text{s}}$ $\mathrm{Re} = c \cdot \bar{V} = 3.85 \times 10^4$

Given $\frac{1}{\sqrt{f}} = -2.0\log\left(\frac{2.51}{\mathrm{Re}\sqrt{f}}\right)$ $f = 0.0222$ $\bar{V} = \frac{k}{\sqrt{f}} = 1.99\frac{\text{m}}{\text{s}}$ $\mathrm{Re} = c \cdot \bar{V} = 3.77 \times 10^4$

Given $\frac{1}{\sqrt{f}} = -2.0\log\left(\frac{2.51}{\mathrm{Re}\sqrt{f}}\right)$ $f = 0.0223$ $\bar{V} = \frac{k}{\sqrt{f}} = 1.98\frac{\text{m}}{\text{s}}$ $\mathrm{Re} = c \cdot \bar{V} = 3.76 \times 10^4$

The flowrate is then: $Q_1 = \frac{\pi}{4} \times (0.019)^2\,\text{m}^2 \times 1.98\frac{\text{m}}{\text{s}} = 5.61 \times 10^{-4}\,\frac{\text{m}^3}{\text{s}}$

Option 2: let $z_1 = 0$ $p_2 = p_{atm} = 0\,\text{kPa gage}$

Given data $p_1 = 300\,\text{kPa gage}$ $D = 0.0127\,\text{m}$ $z_2 = 15\,\text{m}$ $\frac{e}{D} = 0.05$ $L = 16\,\text{m}$

The analysis for Option 2 is identical to Option 1:

The energy equation becomes: $\frac{p_1}{\rho} - gz_2 = f\frac{L}{D}\frac{\bar{V}^2}{2}$

Solving for V: $\qquad \overline{V} = \sqrt{\dfrac{2 \cdot D \cdot \left(\dfrac{p_1}{\rho} - g \cdot z_2 \right)}{f \cdot L}} \qquad \overline{V} = \dfrac{k}{\sqrt{f}}$ (4)

$$k = \sqrt{\frac{2 \cdot D \cdot \left(\frac{p_1}{\rho} - g \cdot z_2 \right)}{L}} = \sqrt{2 \times 0.0127\,\text{m} \times \left(300{,}000 \frac{\text{N}}{\text{m}^2} \times \frac{\text{m}^3}{999\,\text{kg}} \times \frac{\text{kg} \cdot \text{m}}{\text{N} \cdot \text{s}^2} - 9.81 \frac{\text{m}}{\text{s}^2} \times 15\,\text{m} \right) \times \frac{1}{16\,\text{m}}}$$

$$k = 0.493 \frac{\text{m}}{\text{s}}$$

We also have $\quad \text{Re} = \dfrac{\rho \cdot \overline{V} \cdot D}{\mu} \quad$ or $\quad \text{Re} = c \cdot \overline{V}$ (5) $\quad$ where $\quad c = \dfrac{\rho \cdot D}{\mu}$

$$c = 999 \frac{\text{kg}}{\text{m}^3} \times 0.0127\,\text{m} \times \frac{\text{m} \cdot \text{s}}{1 \times 10^{-3}\ \text{kg}} = 12687.3 \frac{\text{s}}{\text{m}}$$

In addition: $\quad \dfrac{1}{\sqrt{f}} = -2.0 \log\left(\dfrac{e/D}{3.7} + \dfrac{2.51}{\text{Re}\sqrt{f}} \right) = -2.0 \log\left(\dfrac{0.05}{3.7} + \dfrac{2.51}{\text{Re}\sqrt{f}} \right)$ (6)

Equations 4, 5 and 6 form a set of simultaneous equations for $\overline{V}, \text{Re and } f$

Make a guess for f $\quad f = 0.07 \quad$ then $\quad \overline{V} = \dfrac{k}{\sqrt{f}} = 1.86 \dfrac{\text{m}}{\text{s}} \quad \text{Re} = c \cdot \overline{V} = 2.36 \times 10^4$

Given $\quad \dfrac{1}{\sqrt{f}} = -2.0 \log\left(\dfrac{0.05}{3.7} + \dfrac{2.51}{\text{Re}\sqrt{f}} \right) \quad f = 0.0725 \quad \overline{V} = \dfrac{k}{\sqrt{f}} = 1.83 \dfrac{\text{m}}{\text{s}} \quad \text{Re} = c \cdot \overline{V} = 2.32 \times 10^4$

Given $\quad \dfrac{1}{\sqrt{f}} = -2.0 \log\left(\dfrac{0.05}{3.7} + \dfrac{2.51}{\text{Re}\sqrt{f}} \right) \quad f = 0.0725 \quad \overline{V} = \dfrac{k}{\sqrt{f}} = 1.83 \dfrac{\text{m}}{\text{s}} \quad \text{Re} = c \cdot \overline{V} = 2.32 \times 10^4$

The flowrate is then: $\quad Q_2 = \dfrac{\pi}{4} \times (0.0127)^2\,\text{m}^2 \times 1.83 \dfrac{\text{m}}{\text{s}} = 2.32 \times 10^{-4}\ \dfrac{\text{m}^3}{\text{s}} \quad$ Option 1 is 2.42 times more effective!

This problem can also be solved explicitly:

The energy equation becomes: $\quad \dfrac{p_1}{\rho} - gz_2 = f \dfrac{L}{D} \dfrac{\overline{V}^2}{2}$

or: $\quad \sqrt{f} = \dfrac{1}{\overline{V}} \sqrt{\dfrac{2D\left(\dfrac{p_1}{\rho} - gz_2 \right)}{L}}$

Plugging this into the Colebrook equation:

$$\frac{1}{\sqrt{f}} = \overline{V}\sqrt{\frac{L}{2D\left(\frac{p_1}{\rho} - gz_2\right)}} = -2.0\log\left(\frac{e/D}{3.7} + \frac{2.51\mu}{\rho\overline{V}D}\left(\overline{V}\sqrt{\frac{L}{2D\left(\frac{p_1}{\rho} - gz_2\right)}}\right)\right)$$

Noting that the $\overline{V}$ s on the right hand side cancel provides:

$$\overline{V} = -2.0\log\left(\frac{e/D}{3.7} + \frac{2.51\mu}{\rho D}\sqrt{\frac{L}{2D\left(\frac{p_1}{\rho} - gz_2\right)}}\right)\left(\sqrt{\frac{2D\left(\frac{p_1}{\rho} - gz_2\right)}{L}}\right)$$

Assuming water at 20°C (ρ = 999 kg/m^3, μ = 1 x 10^{-3} kg/(m s)) gives the remaining information needed to perform the calculation.

For Option 1:

$$\overline{V} = -2.0\log\left(\begin{array}{l} 2.51\times 1\times 10^{-3}\,\frac{\text{kg}}{\text{m}\cdot\text{s}}\times\frac{\text{m}^3}{999\,\text{kg}}\times\frac{1}{0.019\,\text{m}} \\ \times\sqrt{\frac{23\,\text{m}}{2\times 0.019\,\text{m}}\times\frac{1}{\left(200{,}000\frac{\text{N}}{\text{m}^2}\times\frac{\text{m}^3}{999\,\text{kg}}\times\frac{\text{kg}\cdot\text{m}}{\text{N}\cdot\text{s}^2} - 9.81\frac{\text{m}}{\text{s}^2}\times 15\,\text{m}\right)}}\end{array}\right)$$

$$\times\left(\sqrt{\frac{2\times 0.019\,\text{m}}{23\,\text{m}}\times\left(200{,}000\frac{\text{N}}{\text{m}^2}\times\frac{\text{m}^3}{999\,\text{kg}}\times\frac{\text{kg}\cdot\text{m}}{\text{N}\cdot\text{s}^2} - 9.81\frac{\text{m}}{\text{s}^2}\times 15\,\text{m}\right)}\right)$$

$$\overline{V} = 1.98\,\frac{\text{m}}{\text{s}}$$

and:

$$Q_1 = \frac{\pi}{4}\times(0.019)^2\,\text{m}^2\times 1.98\frac{\text{m}}{\text{s}} = 5.61\times 10^{-4}\,\frac{\text{m}^3}{\text{s}}$$

Option 2: let $z_1 = 0 \qquad p_2 = p_{atm} = 0\,\text{kPa gage}$

Given data $\quad p_1 = 300\,\text{kPa gage} \qquad D = 0.0127\,\text{m} \qquad z_2 = 15\,\text{m} \qquad \frac{e}{D} = 0.05 \qquad L = 16\,\text{m}$

The analysis for Option 2 results in the same equations as used in Option 1 once again giving:

$$\overline{V} = -2.0\log\left(\frac{e/D}{3.7} + \frac{2.51\mu}{\rho D}\sqrt{\frac{L}{2D\left(\frac{p_1}{\rho} - gz_2\right)}}\right)\left(\sqrt{\frac{2D\left(\frac{p_1}{\rho} - gz_2\right)}{L}}\right)$$

$$\bar{V} = -2.0 \log\left(\begin{array}{l} \dfrac{0.05}{3.7} + 2.51 \times 1 \times 10^{-3} \dfrac{\text{kg}}{\text{m}\cdot\text{s}} \times \dfrac{\text{m}^3}{999\,\text{kg}} \times \dfrac{1}{0.0127\,\text{m}} \\ \times \sqrt{\dfrac{16\,\text{m}}{2 \times 0.0127\,\text{m}} \times \dfrac{1}{\left(300{,}000 \dfrac{\text{N}}{\text{m}^2} \times \dfrac{\text{m}^3}{999\,\text{kg}} \times \dfrac{\text{kg}\cdot\text{m}}{\text{N}\cdot\text{s}^2} - 9.81 \dfrac{\text{m}}{\text{s}^2} \times 15\,\text{m}\right)}} \end{array} \right)$$

$$\times \left(\sqrt{\frac{2 \times 0.0127\,\text{m}}{16\,\text{m}} \times \left(300{,}000 \frac{\text{N}}{\text{m}^2} \times \frac{\text{m}^3}{999\,\text{kg}} \times \frac{\text{kg}\cdot\text{m}}{\text{N}\cdot\text{s}^2} - 9.81 \frac{\text{m}}{\text{s}^2} \times 15\,\text{m}\right)} \right)$$

$$\bar{V} = 1.83 \frac{\text{m}}{\text{s}}$$

The flowrate is then: $$Q_2 = \frac{\pi}{4} \times (0.0127)^2\,\text{m}^2 \times 1.83 \frac{\text{m}}{\text{s}} = \quad 2.32 \times 10^{-4} \frac{\text{m}^3}{\text{s}}$$

Problem 8.141

[Difficulty: 3]

8.141 The students of Phi Gamma Delta are putting a kiddy pool on a porch attached to the second story of their house and plan to fill it with water from a garden hose. The kiddy pool has a diameter of 5 ft, and is 2.5 ft deep. The porch is 18 ft above the faucet. The garden hose is very smooth on the inside, has a length of 50 ft, and a diameter of 5/8 in. If the water pressure at the faucet is 60 psi, how long will it take to fill the pool? Neglect minor losses.

Given:

Kiddy pool on a porch.

Find:

Time to fill pool with a hose.

Solution:

Basic equations:

$$\left(\frac{p_1}{\rho}+\alpha_1\frac{\overline{V}_1^2}{2}+gz_1\right)-\left(\frac{p_2}{\rho}+\alpha_2\frac{\overline{V}_2^2}{2}+gz_2\right)=h_{l_T}$$

$$h_{l_T}=h_l+h_{l_m}=f\frac{L}{D}\frac{\overline{V}^2}{2}+K\frac{\overline{V}^2}{2} \qquad \frac{1}{\sqrt{f}}=-2.0\log\left(\frac{e/D}{3.7}+\frac{2.51}{\mathrm{Re}\sqrt{f}}\right) \qquad Q=VA$$

Assumptions: 1) Steady flow 2) Incompressible 3) Neglect minor losses 4) $\alpha_1\frac{\overline{V}_1^2}{2}=\alpha_2\frac{\overline{V}_2^2}{2}$

Given data $\quad p_1=60\,\frac{\mathrm{lbf}}{\mathrm{in}^2}\,\mathrm{gage} \qquad z_1=0\,\mathrm{ft} \qquad D=0.625\,\mathrm{in} \qquad p_2=0\,\frac{\mathrm{lbf}}{\mathrm{in}^2}\,\mathrm{gage}$

$z_2=20.5\,\mathrm{ft} \qquad \frac{e}{D}=0 \qquad L=50\,\mathrm{ft}$

The energy equation becomes: $\quad \frac{p_1}{\rho}-gz_2=f\frac{L}{D}\frac{\overline{V}^2}{2}$

Solving for V: $\quad \overline{V}=\sqrt{\frac{2\cdot D\cdot\left(\frac{p_1}{\rho}-g\cdot z_2\right)}{f\cdot L}} \qquad \overline{V}=\frac{k}{\sqrt{f}} \qquad (1)$

$$k=\sqrt{\frac{2\cdot D\cdot\left(\frac{p_1}{\rho}-g\cdot z_2\right)}{L}}=\sqrt{2\times0.625\,\mathrm{in}\times\frac{\mathrm{ft}}{12\,\mathrm{in}}\times\left(60\frac{\mathrm{lbf}}{\mathrm{in}^2}\times\frac{144\,\mathrm{in}^2}{\mathrm{ft}^2}\times\frac{\mathrm{ft}^3}{1.94\,\mathrm{slug}}\times\frac{\mathrm{slug}\cdot\mathrm{ft}}{\mathrm{lbf}\cdot\mathrm{s}^2}-32.2\frac{\mathrm{ft}}{\mathrm{s}^2}\times20.5\,\mathrm{ft}\right)\times\frac{1}{50\,\mathrm{ft}}}$$

$$k=2.81\frac{\mathrm{ft}}{\mathrm{s}}$$

We also have $\mathrm{Re} = \dfrac{\rho \cdot \overline{V} \cdot D}{\mu}$ or $\mathrm{Re} = c \cdot \overline{V}$ (2) where $c = \dfrac{\rho \cdot D}{\mu}$

Assuming water at 68°F (ρ = 1.94 slug/ft^3, μ = 2.1 x 10^{-5} lbf·s/ft^2):

$$c = 1.94\,\frac{\text{slug}}{\text{ft}^3} \times 0.625\,\text{in} \times \frac{\text{ft}}{12\,\text{in}} \times \frac{\text{ft}^2}{2.1\times 10^{-5}\ \text{lbf}\cdot\text{s}} \times \frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}} = 4811.5\,\frac{\text{s}}{\text{ft}}$$

In addition: $$\frac{1}{\sqrt{f}} = -2.0\log\left(\frac{2.51}{\mathrm{Re}\sqrt{f}}\right) \quad (3)$$

Equations 1, 2 and 3 form a set of simultaneous equations for $\overline{V}, \mathrm{Re}$ and f

Make a guess for f $\quad f = 0.015$ then $\quad \overline{V} = \dfrac{k}{\sqrt{f}} = 22.94\,\dfrac{\text{ft}}{\text{s}}$ $\quad \mathrm{Re} = c \cdot \overline{V} = 1.1\times 10^5$

Given $\dfrac{1}{\sqrt{f}} = -2.0\log\left(\dfrac{2.51}{\mathrm{Re}\sqrt{f}}\right)$ $\quad f = 0.0177$ $\quad \overline{V} = \dfrac{k}{\sqrt{f}} = 21.12\,\dfrac{\text{ft}}{\text{s}}$ $\quad \mathrm{Re} = c \cdot \overline{V} = 1.02\times 10^5$

Given $\dfrac{1}{\sqrt{f}} = -2.0\log\left(\dfrac{2.51}{\mathrm{Re}\sqrt{f}}\right)$ $\quad f = 0.0179$ $\quad \overline{V} = \dfrac{k}{\sqrt{f}} = 21.0\,\dfrac{\text{ft}}{\text{s}}$ $\quad \mathrm{Re} = c \cdot \overline{V} = 1.01\times 10^5$

The flowrate is then:

$$Q = \frac{\pi}{4} \times (0.625)^2\,\text{in}^2 \times \frac{\text{ft}^2}{144\,\text{in}^2} \times 21.0\,\frac{\text{ft}}{\text{s}} = 0.0447\,\frac{\text{ft}^3}{s}$$

$$Volume_{pool} = 2.5\,\text{ft} \times \frac{\pi}{4} \times (5)^2\,\text{ft}^2 = 49.1\,\text{ft}^3$$

$$time = \frac{Volume_{pool}}{Q} = \frac{49.1\,\text{ft}^3}{0.0447\,\dfrac{\text{ft}^3}{\text{s}}} = 1097\,\text{s} = \quad 18.3\text{ min.}$$

This problem can also be solved explicitly in the following manner:

The energy equation becomes: $$\frac{p_1}{\rho} - gz_2 = f\,\frac{L}{D}\frac{\overline{V}^2}{2}$$

or: $$\sqrt{f} = \frac{1}{\overline{V}}\sqrt{\frac{2D\left(\dfrac{p_1}{\rho} - gz_2\right)}{L}}$$

Plugging this into the Colebrook equation:

$$\frac{1}{\sqrt{f}} = \overline{V}\sqrt{\frac{L}{2D\left(\dfrac{p_1}{\rho} - gz_2\right)}} = -2.0\log\left(\frac{2.51\mu}{\rho\overline{V}D}\left(\overline{V}\sqrt{\frac{L}{2D\left(\dfrac{p_1}{\rho} - gz_2\right)}}\right)\right)$$

Noting that the $\overline{V}$ s on the right hand side cancel provides:

$$\overline{V} = -2.0\log\left(\frac{2.51\mu}{\rho D}\sqrt{\frac{L}{2D\left(\frac{p_1}{\rho}-gz_2\right)}}\right)\left(\sqrt{\frac{2D\left(\frac{p_1}{\rho}-gz_2\right)}{L}}\right)$$

Assuming water at 68°F (ρ = 1.94 slug/ft^3, μ = 2 x 10^{-5} lbf s/ft^2) and g = 32.2 ft/s^2 gives the remaining information needed to perform the calculation finding:

$$\overline{V} = -2.0\log\left(\begin{array}{l} 2.51\times 2.1\times 10^{-5}\,\frac{\text{lbf}\cdot\text{s}}{\text{ft}^2}\times\frac{\text{ft}^3}{1.94\,\text{slug}}\times\frac{\text{slug}\cdot\text{ft}}{\text{lbf}\cdot\text{s}^2}\times\frac{1}{0.625\,\text{in}}\times\frac{12\,\text{in}}{\text{ft}} \\ \times\sqrt{\frac{50\,\text{ft}}{2\times 0.625\,\text{in}}\times\frac{12\,\text{in}}{\text{ft}}\times\frac{1}{\left(60\frac{\text{lbf}}{\text{in}^2}\times\frac{144\,\text{in}^2}{\text{ft}^2}\times\frac{\text{ft}^3}{1.94\,\text{slug}}\times\frac{\text{slug}\cdot\text{ft}}{\text{lbf}\cdot\text{s}^2}-32.2\frac{\text{ft}}{\text{s}^2}\times 20.5\,\text{ft}\right)}}\end{array}\right)$$

$$\times\left(\sqrt{\frac{2\times 0.625\,\text{in}}{50\,\text{ft}}\times\frac{\text{ft}}{12\,\text{in}}\times\left(60\frac{\text{lbf}}{\text{in}^2}\times\frac{144\,\text{in}^2}{\text{ft}^2}\times\frac{\text{ft}^3}{1.94\,\text{slug}}\times\frac{\text{slug}\cdot\text{ft}}{\text{lbf}\cdot\text{s}^2}-32.2\frac{\text{ft}}{\text{s}^2}\times 20.5\,\text{ft}\right)}\right)$$

$$\overline{V} = 21.0\,\frac{\text{ft}}{\text{s}}$$

and:

$$Q = \frac{\pi}{4}\times(0.052)^2\,\text{ft}^2\times 21.0\frac{\text{ft}}{\text{s}} = 0.0447\,\frac{\text{ft}^3}{s}$$

$$Volume_{pool} = 2.5\,\text{ft}\times\frac{\pi}{4}\times(5)^2\,\text{ft}^2 = 49.1\,\text{ft}^3$$

$$time = \frac{Volume_{pool}}{Q} = \frac{49.1\,\text{ft}^3}{0.0447\,\frac{\text{ft}^3}{\text{s}}} = 1097\,\text{s} = \quad 18.3\text{ min.}$$

Problem 8.142

[Difficulty: 3]

8.142 Gasoline flows in a long, underground pipeline at a constant temperature of 15°C. Two pumping stations at the same elevation are located 13 km apart. The pressure drop between the stations is 1.4 MPa. The pipeline is made from 0.6-m-diameter pipe. Although the pipe is made from commercial steel, age and corrosion have raised the pipe roughness to approximately that for galvanized iron. Compute the volume flow rate.

Solution: Apply the energy equation for steady, incompressible flow that is uniform at each section.

Basic equation: $\frac{p_1}{\rho} + \overbrace{\frac{\bar{V}_1^2}{2}}^{=0(1)} + \overbrace{g z_1}^{=0(2)} = \frac{p_2}{\rho} + \overbrace{\frac{\bar{V}_2^2}{2}}^{=0(1)} + \overbrace{g z_2}^{=0(2)} + h_{l_T}$; $h_{l_T} = f\frac{L}{D}\frac{\bar{V}^2}{2} + \overbrace{h_{l_m}}^{=0(1)}$

Assumptions: (1) Constant area pipe, so $\bar{V}_1 = \bar{V}_2$, $h_{l_m} = 0$
(2) Level, so $z_1 = z_2$

Thus

$$\frac{p_1 - p_2}{\rho} = f\frac{L}{D}\frac{\bar{V}^2}{2} \quad \text{or} \quad \bar{V} = \left[\frac{2D(p_1 - p_2)}{\rho f L}\right]^{\frac{1}{2}}$$

But $f = f(Re, e/D)$, and the Reynolds number is not known. Therefore iteration is required. Choose f in the fully-rough zone. From Table 8.1, $e = 0.15\ mm$; $e/D = 0.00025$. Then from Fig. 8.13, $f \simeq 0.014$. {From Eq. 8.37, using Excel's solver, $f = 0.014$}. Then,

$$\bar{V} = \left[2 \times 0.6\ m \times 1.4\times10^6 \frac{N}{m^2} \times \frac{m^3}{(0.72)1000\ kg} \times \frac{1}{0.014} \times \frac{1}{13\times10^3\ m} \times \frac{kg\cdot m}{N\cdot s^2}\right]^{\frac{1}{2}}$$

{SG = 0.72, Table A.2}

$\bar{V} = 3.58\ m/s$

Now compute Re and check on guess for f. Choose $\mu \simeq 5\times10^{-4}\ N\cdot s/m^2$ (Fig. A.2).*

$$Re = \frac{\rho \bar{V} D}{\mu} = (0.72)1000 \frac{kg}{m^3} \times 3.58 \frac{m}{s} \times 0.6\ m \times \frac{m^2}{5\times10^{-4}\ N\cdot s} \times \frac{N\cdot s^2}{kg\cdot m} = 3.09\times10^6$$

Checking on Fig. 8.13, flow is essentially in the fully-rough zone, and initial guess for f was okay. Thus

$$Q = \bar{V}A = 3.58 \frac{m}{s} \times \frac{\pi}{4}(0.6)^2\ m^2 = 1.01\ m^3/s$$

Q

* Note gasoline is between heptane and octane.

Problem 8.143

[Difficulty: 3]

8.143 Water flows steadily in a horizontal 125-mm-diameter cast-iron pipe. The pipe is 150 m long and the pressure drop between sections ① and ② is 150 kPa. Find the volume flow rate through the pipe.

Solution: Apply the energy equation for steady, incompressible pipe flow.

Computing equation:
$$\left(\frac{p_1}{\rho}+\alpha_1\frac{\bar{V}_1^2}{2}+gz_1\right)-\left(\frac{p_2}{\rho}+\alpha_2\frac{\bar{V}_2^2}{2}+gz_2\right)+h_{lT}$$
$$h_{lT}=h_l+h_{lm}=f\frac{L}{D}\frac{\bar{V}^2}{2}+K\frac{\bar{V}^2}{2}$$

Assumptions: (1) Fully developed flow: $\alpha_1\bar{V}_1^2=\alpha_2\bar{V}_2^2$
(2) Horizontal: $z_1=z_2$
(3) Constant area, so $K=0$

Then
$$\frac{\Delta p}{\rho}=h_{lT}=f\frac{L}{D}\frac{\bar{V}^2}{2}\quad\text{so}\quad \bar{V}=\sqrt{\frac{2\Delta p D}{\rho f L}}$$

Since flow rate (hence Re and f) are unknown, must iterate. Guess a trial value of f in the fully rough zone. From Table 8.1, $e=0.26$ mm

Then $e/D = \frac{0.26}{125}=0.0021$. Then from Eq. 8.37*, $f=0.0237$ for $Re \geq 6\times10^5$

$$\bar{V}=\left[2\times150\times10^3\frac{N}{m^2}\times0.125\,m\times\frac{m^3}{999\,kg}\times\frac{1}{0.0237}\times\frac{1}{150\,m}\times\frac{kg\cdot m}{N\cdot s^2}\right]^{1/2}=3.25\,m/s$$

and, checking Re, with $\nu=1.14\times10^{-6}\,m^2/s$ at $T=15°C$ (Table A.8),
$$Re=\frac{\bar{V}D}{\nu}=3.25\frac{m}{s}\times0.125\,m\times\frac{s}{1.14\times10^{-6}\,m^2}=3.56\times10^5$$

The friction factor at this Re is still $f=0.0242$ (2% error), so convergence is ok.
$$Q=\bar{V}A=3.25\frac{m}{s}\times\frac{\pi}{4}\times(0.125\,m)^2=0.0399\,m^3/s \qquad Q$$

Using $f=0.0242$, $\bar{V}=3.22\,m/s$ and $Q=0.0395\,m^3/s$

* Value of $f=0.237$ obtained using Excel's Solver (or Goal Seek)

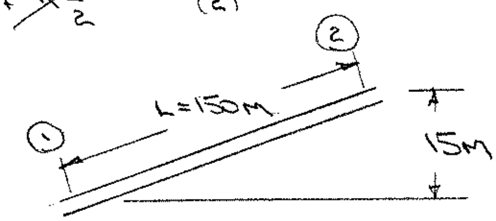
8.144 Water flows steadily in a 125-mm-diameter cast-iron pipe 150 m long. The pressure drop between sections ① and ② is 150 kPa, and section ② is located 15 m above section ①. Find the volume flow rate.

Solution: Apply the energy equation for steady, incompressible pipe flow

Computing equation:

$$\left(\frac{p_1}{\rho} + \cancel{\alpha_1\frac{\bar{V}_1^2}{2}} + gz_1\right) - \left(\frac{p_2}{\rho} + \cancel{\alpha_2\frac{\bar{V}_2^2}{2}} + gz_2\right) = h_{l_T} \qquad (1)$$

$$h_{l_T} = h_l + h_{l_m} = f\frac{L}{D}\frac{\bar{V}^2}{2} + \cancel{K\frac{\bar{V}^2}{2}} \approx 0 \,(4) \qquad (2)$$



Assumptions: (1) $\bar{V}_1 = \bar{V}_2$ from continuity

(2) $\alpha_1 = \alpha_2$

(3) $z_2 - z_1 = 15\text{ m}$

(4) neglect minor losses

For cast iron pipe with $D = 125\text{ mm}$ $\frac{e}{D} = 0.0021$ ($e = 0.26\text{ mm}$, Table 8.1)

Since $f = f(Re)$ and $\bar{V}$ is unknown, iteration will be required

From Eqs (1) and (2)

$$\left(\frac{p_1}{\rho} + gz_1\right) - \left(\frac{p_2}{\rho} + gz_2\right) = f\frac{L}{D}\frac{\bar{V}^2}{2}$$

Then

$$f\bar{V}^2 = \frac{2D}{L}\left[\frac{(p_1 - p_2)}{\rho} + g(z_1 - z_2)\right]$$

$$f\bar{V}^2 = 2 \times \frac{0.125\text{ m}}{150\text{ m}}\left[150\times10^3\,\frac{\text{N}}{\text{m}^2} \times \frac{\text{m}^3}{999\text{ kg}} \times \frac{\text{kg}\cdot\text{m}}{\text{N}\cdot\text{s}^2}\right] + 9.81\,\frac{\text{m}}{\text{s}^2} \times (-15\text{ m})$$

$$f\bar{V}^2 = 0.005\ \text{m}^2/\text{s}^2$$

Assume flow in fully rough region, $f = 0.0237$, then $\bar{V} = 0.46\text{ m/s}$

Check Re. Assume $T = 15°\text{C}$, $\nu = 1.14\times10^{-6}\ \text{m}^2/\text{s}$ (Table A.8)

Then $Re = \frac{D\bar{V}}{\nu} = 0.125\text{ m} \times 0.46\,\frac{\text{m}}{\text{s}} \times \frac{\text{s}}{1.14\times10^{-6}\text{ m}^2} = 50{,}400$

From Eq. 8.37 with $Re = 50{,}400$, $e/D = 0.0021$, then using Excel's Solver (or Goal Seek)

$$f = 0.0267 \text{ and } \bar{V} = 0.433\text{ m/s}$$

With this value of $\bar{V}$, $Re = 47{,}500$, $f = 0.0268$, $\bar{V} = 0.432\text{ m/s}$

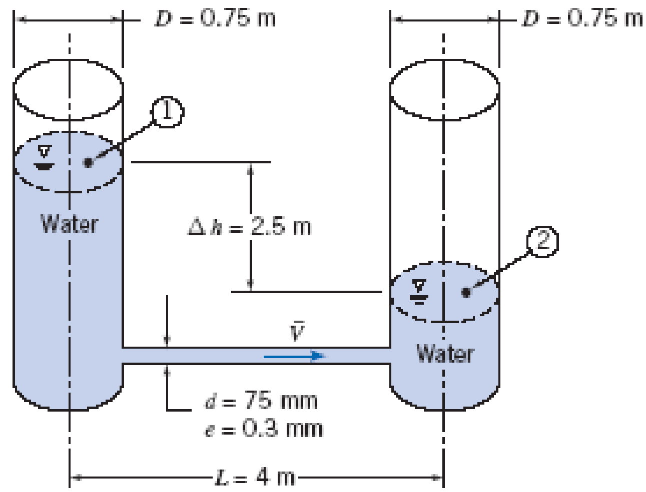
Then

$$Q = A\bar{V} = \frac{\pi D^2}{4}\bar{V} = \frac{\pi}{4}(0.125\text{ m})^2 \times 0.432\,\frac{\text{m}}{\text{s}} = 0.0053\text{ m}^3/\text{s} \qquad Q$$

Problem 8.145

[Difficulty: 3]

8.145 Two open standpipes of equal diameter are connected by a straight tube, as shown. Water flows by gravity from one standpipe to the other. For the instant shown, estimate the rate of change of water level in the left standpipe.



Solution: Apply the energy equation for quasi-steady, incompressible pipe flow.

Computing equation:

$$\left(\frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2} + g z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \frac{\bar{V}_2^2}{2} + g z_2\right) = h_{\ell T};\quad h_{\ell T} = h_\ell + h_{\ell m} = \left[f\frac{(L-D)}{d} + K_{ent} + K_{exit}\right]\frac{\bar{V}^2}{2}$$

$z_2 = 0$

From continuity, $A_1 V_1 = A_p \bar{V}$

Assumptions: (1) Neglect unsteady effects
(2) Incompressible flow
(3) $p_1 = p_2 = p_{atm}$
(4) $\bar{V}_1 = \bar{V}_2$ since diameters are equal

Then $g\Delta h = h_{\ell T} = \left[f\frac{(L-D)}{d} + K_{ent} + K_{exit}\right]\frac{\bar{V}^2}{2}$

Flow rate (hence speed) is unknown, so assume flow is in fully rough zone.

$\frac{e}{D} = \frac{0.3}{75} = 0.004$, so $f \approx 0.0285$ from Eq. 8.37 (using Excel's Solver or Goal Seek)

From Table 8.2, $K_{ent} = 0.5$; from Fig. 8.15, $K_{exit} = 1$. Then

$$\bar{V} = \left[\frac{2g\Delta h}{f\frac{(L-D)}{d} + K_{ent} + K_{exit}}\right]^{\frac{1}{2}} = \left[\frac{2 \times 9.81\frac{m}{s^2} \times 2.5\,m}{0.028\left(\frac{4-0.75}{0.075}\right) + 0.5 + 1.0}\right]^{\frac{1}{2}} = 4.23\ m/s$$

Check Re and f. For water at 20°C, $\nu = 1.00 \times 10^{-6}\ m^2/s$ (Table A.8)

$$Re = \frac{\bar{V}d}{\nu} = 4.23\frac{m}{s} \times 0.075\,m \times \frac{s}{1.00 \times 10^{-6}\,m^2} = 3.18 \times 10^5$$

From Equation 8.37, $f \approx 0.0288$, so this is satisfactory agreement. (~1%)

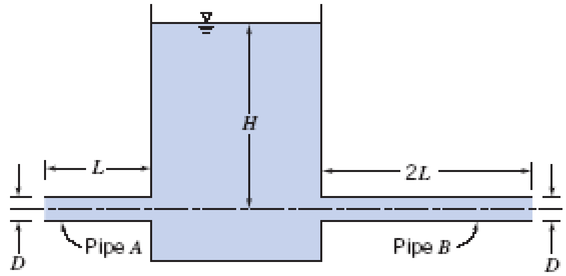
$$V_1 = \frac{A_p}{A_1}V_p = \left(\frac{d}{D}\right)^2 V_p = \left(\frac{0.075}{0.75}\right)^2 \times 4.23\frac{m}{s} = 0.0423\ m/s \text{ (down)}$$

The water level in the left tank falls at about 42.3 mm/s

Problem 8.146

[Difficulty: 4]

8.146 Two galvanized iron pipes of diameter D are connected to a large water reservoir, as shown. Pipe A has length L and pipe B has length $2L$. Both pipes discharge to atmosphere. Which pipe will pass the larger flow rate? Justify (without calculating the flow rate in each pipe). Compute the flow rates if $H = 10$ m, $D = 50$ mm, and $L = 10$ m.



Given: Flow from large reservoir

Find: Flow rates in two pipes

Solution:

Basic equations
$$\left(\frac{p_1}{\rho} + \alpha\cdot\frac{V_1^2}{2} + g\cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha\cdot\frac{V_2^2}{2} + g\cdot z_2\right) = h_l \qquad h_l = f\cdot\frac{L}{D}\cdot\frac{V^2}{2} \qquad h_{lm} = K_{ent}\cdot\frac{V^2}{2}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) α is approximately 1

Available data $\quad D = 50\cdot mm \qquad H = 10\cdot m \qquad L = 10\cdot m$

$e = 0.15\cdot mm$ (Table 8.1) $\qquad \nu = 1\cdot 10^{-6}\cdot\frac{m^2}{s}$ (Table A.8) $\qquad K_{ent} = 0.5$

The energy equation becomes

$$g\cdot(z_1 - z_2) - \frac{1}{2}\cdot V_2^{\,2} = f\cdot\frac{L}{D}\cdot\frac{V_2^{\,2}}{2} + K_{ent}\cdot\frac{V_2^{\,2}}{2} \quad \text{and} \quad V_2 = V \qquad z_1 - z_2 = H$$

Solving for V
$$V = \sqrt{\frac{2\cdot g\cdot H}{f\cdot\frac{L}{D} + K_{ent} + 1}} \qquad (1)$$

We also have
$$\frac{1}{\sqrt{f}} = -2.0\cdot\log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re\cdot\sqrt{f}}\right) \qquad (2) \qquad\qquad Re = \frac{V\cdot D}{\nu} \qquad (3)$$

We must solve Eqs. 1, 2 and 3 iteratively.

Make a guess for V $\quad V = 1\cdot\frac{m}{s}$ $\quad$ Then $\quad Re = \frac{V\cdot D}{\nu} \qquad Re = 5.00\times 10^4$

and
$$\frac{1}{\sqrt{f}} = -2.0\cdot\log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re\cdot\sqrt{f}}\right) \qquad f = 0.0286$$

Then
$$V = \sqrt{\frac{2\cdot g\cdot H}{f\cdot\frac{L}{D} + K_{ent} + 1}} \qquad V = 5.21\,\frac{m}{s}$$

Repeating $\quad Re = \frac{V \cdot D}{\nu} \quad Re = 2.61 \times 10^5$

and $\quad \frac{1}{\sqrt{f}} = -2.0 \cdot \log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re \cdot \sqrt{f}}\right) \quad f = 0.0267$

Then $\quad V = \sqrt{\frac{2 \cdot g \cdot H}{f \cdot \frac{L}{D} + K_{ent} + 1}} \quad V = 5.36 \frac{m}{s}$

Repeating $\quad Re = \frac{V \cdot D}{\nu} \quad Re = 2.68 \times 10^5$

and $\quad \frac{1}{\sqrt{f}} = -2.0 \cdot \log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re \cdot \sqrt{f}}\right) \quad f = 0.0267$

Then $\quad V = \sqrt{\frac{2 \cdot g \cdot H}{f \cdot \frac{L}{D} + K_{ent} + 1}} \quad V = 5.36 \frac{m}{s}$

Hence $\quad Q = \frac{\pi}{4} \cdot D^2 \cdot V \quad Q = 0.0105 \frac{m^3}{s} \quad Q = 10.5 \cdot \frac{l}{s}$

We repeat the analysis for the second pipe, using 2L instead of L:

Make a guess for V $\quad V = 1 \cdot \frac{m}{s}$ $\quad$ Then $\quad Re = \frac{V \cdot D}{\nu} \quad Re = 5.00 \times 10^4$

and $\quad \frac{1}{\sqrt{f}} = -2.0 \cdot \log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re \cdot \sqrt{f}}\right) \quad f = 0.0286$

Then $\quad V = \sqrt{\frac{2 \cdot g \cdot H}{f \cdot \frac{2 \cdot L}{D} + K_{ent} + 1}} \quad V = 3.89 \frac{m}{s}$

Repeating $\quad Re = \frac{V \cdot D}{\nu} \quad Re = 1.95 \times 10^5$

and $\quad \frac{1}{\sqrt{f}} = -2.0 \cdot \log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re \cdot \sqrt{f}}\right) \quad f = 0.0269$

Then $\quad V = \sqrt{\frac{2 \cdot g \cdot H}{f \cdot \frac{2 \cdot L}{D} + K_{ent} + 1}} \quad V = 4.00 \frac{m}{s}$

Repeating $\quad Re = \frac{V \cdot D}{\nu} \qquad Re = 2.00 \times 10^5$

and $\quad \frac{1}{\sqrt{f}} = -2.0 \cdot \log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re \cdot \sqrt{f}}\right) \qquad f = 0.0268$

Then $\quad V = \sqrt{\frac{2 \cdot g \cdot H}{f \cdot \frac{2 \cdot L}{D} + K_{ent} + 1}} \qquad V = 4.00 \frac{m}{s}$

Hence $\quad Q = \frac{\pi}{4} \cdot D^2 \cdot V \qquad Q = 7.861 \times 10^{-3} \frac{m^3}{s} \qquad Q = 7.86 \cdot \frac{l}{s}$

As expected, the flow is considerably less in the longer pipe.

Problem 8.147

[Difficulty: 3]

8.147 Galvanized iron drainpipes of diameter 50 mm are located at the four corners of a building, but three of them become clogged with debris. Find the rate of downpour (cm/min) at which the single functioning drainpipe can no longer drain the roof. The building roof area is 500 m^2, and the height is 5 m. Assume the drainpipes are the same height as the building, and that both ends are open to atmosphere. Ignore minor losses.

Given: Galvanized drainpipe

Find: Maximum downpour it can handle

Solution:

Basic equations
$$\left(\frac{p_1}{\rho}+\alpha\cdot\frac{V_1^2}{2}+g\cdot z_1\right)-\left(\frac{p_2}{\rho}+\alpha\cdot\frac{V_2^2}{2}+g\cdot z_2\right)=h_l \qquad h_l = f\cdot\frac{L}{D}\cdot\frac{V^2}{2}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) α is approximately 1 4) No minor losses

Available data $D = 50\cdot mm$ $\quad e = 0.15\cdot mm$ (Table 8.1) $\quad h = L$ $\quad$ From Table A.7 (20ºC) $\quad \nu = 1.01\times 10^{-6}\cdot\frac{m^2}{s}$

The energy equation becomes
$$g\cdot z_1 - g\cdot z_2 = g\cdot(z_1 - z_2) = g\cdot h = f\cdot\frac{L}{D}\cdot\frac{V^2}{2}$$

Solving for V
$$V=\sqrt{\frac{2\cdot D\cdot g\cdot h}{L\cdot f}}=\sqrt{\frac{2\cdot D\cdot g}{f}} \qquad V=\frac{k}{\sqrt{f}} \qquad (1)$$

$$k=\sqrt{2\cdot D\cdot g} \qquad k = \sqrt{2\times 0.05\cdot m\times 9.81\cdot\frac{m}{s^2}} \qquad k = 0.99\,\frac{m}{s}$$

We also have
$$Re=\frac{V\cdot D}{\nu} \quad \text{or} \quad Re = c\cdot V \quad (2) \quad \text{where} \quad c=\frac{D}{\nu}$$

$$c = 0.05\cdot m\times\frac{s}{1.01\times 10^{-6}\cdot m^2} \qquad c = 4.95\times 10^4\cdot\frac{s}{m}$$

In addition
$$\frac{1}{\sqrt{f}}=-2.0\cdot\log\left(\frac{\frac{e}{D}}{3.7}+\frac{2.51}{Re\cdot\sqrt{f}}\right) \qquad (3)$$

Equations 1, 2 and 3 form a set of simultaneous equations for V, Re and f

Make a guess for f $\quad f = 0.01 \quad$ then $\quad V=\frac{k}{\sqrt{f}} \quad V = 9.90\,\frac{m}{s} \quad Re = c\cdot V \quad Re = 4.9\times 10^5$

Given
$$\frac{1}{\sqrt{f}}=-2.0\cdot\log\left(\frac{\frac{e}{D}}{3.7}+\frac{2.51}{Re\cdot\sqrt{f}}\right)$$

$f = 0.0264 \qquad V=\frac{k}{\sqrt{f}} \qquad V = 6.09\,\frac{m}{s} \qquad Re = c\cdot V \qquad Re = 3.01\times 10^5$

Given $\frac{1}{\sqrt{f}} = -2.0 \cdot \log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re \cdot \sqrt{f}}\right)$

$f = 0.0266$ $\quad V = \frac{k}{\sqrt{f}}$ $\quad V = 6.07\frac{m}{s}$ $\quad Re = c \cdot V$ $\quad Re = 3.00 \times 10^5$

Given $\frac{1}{\sqrt{f}} = -2.0 \cdot \log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re \cdot \sqrt{f}}\right)$

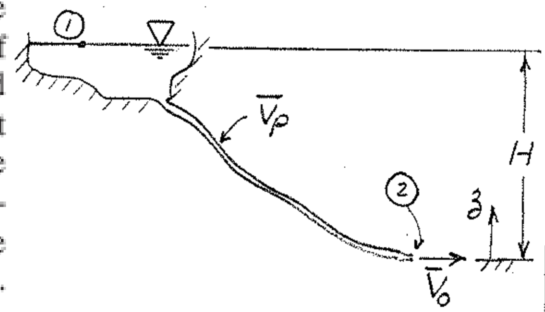
$f = 0.0266$ $\quad V = \frac{k}{\sqrt{f}}$ $\quad V = 6.07\frac{m}{s}$ $\quad Re = c \cdot V$ $\quad Re = 3.00 \times 10^5$

The flow rate is then $\quad Q = V \cdot \frac{\pi \cdot D^2}{4}$ $\quad Q = 0.0119 \cdot \frac{m^3}{s}$

The downpour rate is then $\quad \frac{Q}{A_{roof}} = \frac{0.0119 \cdot \frac{m^3}{s}}{500 \cdot m^2} \times \frac{100 \cdot cm}{1 \cdot m} \times \frac{60 \cdot s}{1 \cdot min} = 0.143 \cdot \frac{cm}{min}$ $\quad$ The drain can handle 0.143 cm/min

Note that we could use *Excel*'s *Solver* for this problem

8.148 A mining engineer plans to do hydraulic mining with a high-speed jet of water. A lake is located $H = 300$ m above the mine site. Water will be delivered through $L = 900$ m of fire hose; the hose has inside diameter $D = 75$ mm and relative roughness $e/D = 0.01$. Couplings, with equivalent length $L_e = 20\,D$, are located every 10 m along the hose. The nozzle outlet diameter is $d = 25$ mm. Its minor loss coefficient is $K = 0.02$ based on outlet velocity. Estimate the maximum outlet velocity that this system could deliver. Determine the maximum force exerted on a rock face by this water jet.



Solution: Apply the energy equation for steady, incompressible pipe flow.

Computing equation: $$\left(\frac{p_1}{\rho}^{=0(1)} + \alpha_1 \frac{\bar{V}_1^2}{2}^{=0(2)} + g z_1\right) - \left(\frac{p_2}{\rho}^{=0(3)} + \alpha_2 \frac{\bar{V}_2^2}{2} + g z_2\right) = h_{\ell T}$$

Assume: (1) $p_1 = 0$; (2) $\bar{V}_1 = 0$; (3) $p_2 = 0$; (4) $\alpha_2 = 1$; (5) $z_2 = 0$; (6) Fully-rough zone

Then $$gH = h_{\ell T} + \frac{\bar{V}_2^2}{2} = f\frac{L}{D}\frac{\bar{V}_p^2}{2} + f \times 90\frac{L_e}{D}\frac{\bar{V}_p^2}{2} + K\frac{\bar{V}_o^2}{2} + \frac{\bar{V}_2^2}{2}$$

From continuity $\bar{V}_p A_p = \bar{V}_o A_o$; $\bar{V}_2 = \bar{V}_o \frac{A_o}{A_2}$; $\bar{V}_2^2 = \bar{V}_o^2\left(\frac{A_o}{A_2}\right)^2 = \bar{V}_o^2\left(\frac{d}{D}\right)^4$

Substituting, $$gH = \left[f\left(\frac{L}{D} + 90\frac{L_e}{D}\right)\left(\frac{d}{D}\right)^4 + 1 + K\right]\frac{\bar{V}_o^2}{2}$$

$$\bar{V}_o = \left[\frac{2gH}{f\left(\frac{L}{D} + 90\frac{L_e}{D}\right)\left(\frac{d}{D}\right)^4 + 1 + K}\right]^{1/2}$$; in fully-rough zone ($\frac{e}{D} = 0.01$), $f = 0.038^*$ (Eq. 8.37)

$$\bar{V}_o = \left[2 \times 9.81\frac{m}{s^2} \times 300\,m \times \frac{1}{0.038\left(\frac{900\,m}{0.075\,m} + 90\,(20)\right)\left(\frac{0.025}{0.075}\right)^4 + 1 + 0.02}\right]^{1/2} = 28.0\ m/s\ (est.)$$

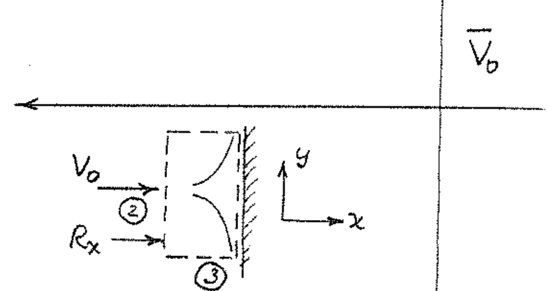
Check for fully-rough flow zone:

$$Re = \frac{\bar{V}_p D}{\nu};\quad \bar{V}_p = \bar{V}_o\left(\frac{d}{D}\right)^4 = 28.0\frac{m}{s}\left(\frac{1}{3}\right)^4 = 0.346\ m/s$$ {Assume $T = 20°C$}

$$Re = 0.346\frac{m}{sec} \times 0.075\,m \times \frac{1}{1\times10^{-6}\,m^2} = 2.60\times10^4;$$ at $\frac{e}{D} = 0.01$, $f = 0.040$ (Eq. 8.37)

The new estimate is

$$\bar{V}_o = \sqrt{\frac{0.038}{0.040}}\,\bar{V}_o\,(est) = \sqrt{\frac{0.038}{0.040}}\,28.0\frac{m}{s} = 27.3\ m/s$$



Apply momentum to find force: CV is shown.

$$F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$$

Assumptions: (1) No pressure forces
(2) $F_{B_x} = 0$
(3) Steady flow

(4) Uniform flow at each cross-section

Then

$$R_x = u_2\{-\rho \bar{V}_0 A_0\} + u_3\{+\rho \bar{V}_0 A_0\}$$

$$u_2 = \bar{V}_0 \qquad u_3 = 0$$

$$R_x = -\rho \bar{V}_0^2 A_0$$

The force <u>on</u> the rock face is

$$K_x = -R_x = \rho \bar{V}_0^2 A_0$$

$$= 999 \frac{kg}{m^3} \times (27.3)^2 \frac{m^2}{s^2} \times \frac{\pi}{4}(0.025)^2 m^2 \times \frac{N \cdot s^2}{kg \cdot m}$$

$$K_x = 365\ N \text{ (to right)} \qquad K_x$$

* Values of f obtained from Eq. 8.37 using Excel's Solver (or Goal Seek)

Problem 8.149

[Difficulty: 3]

8.149 Investigate the effect of tube roughness on flow rate by computing the flow generated by a pressure difference $\Delta p = 100$ kPa applied to a length $L = 100$ m of tubing, with diameter $D = 25$ mm. Plot the flow rate against tube relative roughness e/D for e/D ranging from 0 to 0.05 (this could be replicated experimentally by progressively roughening the tube surface). Is it possible that this tubing could be roughened so much that the flow could be slowed to a laminar flow rate?

Given: Flow in a tube

Find: Effect of tube roughness on flow rate; Plot

Solution:

Governing equations:

$$\left(\frac{p_1}{\rho} + \alpha_1\cdot\frac{V_1^2}{2} + g\cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2\cdot\frac{V_2^2}{2} + g\cdot z_2\right) = h_{lT} = \sum_{\text{major}} h_l + \sum_{\text{minor}} h_{lm} \quad (8.29)$$

$$Re = \frac{\rho\cdot V\cdot D}{\mu} \qquad h_l = f\cdot\frac{L}{D}\cdot\frac{V^2}{2} \quad (8.34) \qquad h_{lm} = K\cdot\frac{V^2}{2} \quad (8.40a) \qquad h_{lm} = f\cdot\frac{L_e}{D}\cdot\frac{V^2}{2} \quad (8.40b)$$

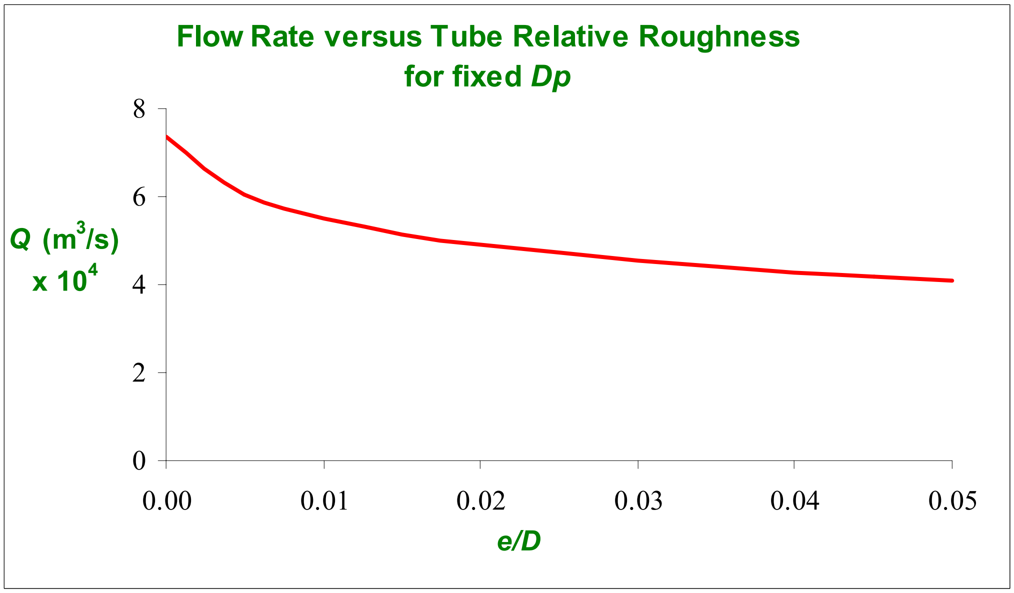
$$f = \frac{64}{Re} \quad (8.36) \quad \text{(Laminar)} \qquad \frac{1}{\sqrt{f}} = -2.0\cdot\log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re\cdot\sqrt{f}}\right) \quad (8.37) \quad \text{(Turbulent)}$$

The energy equation (Eq. 8.29) becomes for flow in a tube

$$p_1 - p_2 = \Delta p = \rho\cdot f\cdot\frac{L}{D}\cdot\frac{V^2}{2}$$

This cannot be solved explicitly for velocity V, (and hence flow rate Q) because f depends on V; solution for a given relative roughness e/D requires iteration (or use of *Solver*)

Given data:				Tabulated or graphical data:			
Δp =	100	kPa		μ =	1.00E-03	N.s/m^2	
D =	25	mm		ρ =	999	kg/m^3	
L =	100	m			(Water - Appendix A)		
Computed results:							
e/D	***V* (m/s)**	***Q* (m^3/s) x 10^4**	***Re***	**Regime**	***f***	**Δp (kPa)**	**Error**
0.000	1.50	7.35	37408	Turbulent	0.0223	100	0.0%
0.005	1.23	6.03	30670	Turbulent	0.0332	100	0.0%
0.010	1.12	5.49	27953	Turbulent	0.0400	100	0.0%
0.015	1.05	5.15	26221	Turbulent	0.0454	100	0.0%
0.020	0.999	4.90	24947	Turbulent	0.0502	100	0.0%
0.025	0.959	4.71	23939	Turbulent	0.0545	100	0.0%
0.030	0.925	4.54	23105	Turbulent	0.0585	100	0.0%
0.035	0.897	4.40	22396	Turbulent	0.0623	100	0.0%
0.040	0.872	4.28	21774	Turbulent	0.0659	100	0.0%
0.045	0.850	4.17	21224	Turbulent	0.0693	100	0.0%
0.050	0.830	4.07	20730	Turbulent	0.0727	100	0.0%



It is not possible to roughen the tube sufficiently to slow the flow down to a laminar flow for this Δp. Even a relative roughness of 0.5 (a physical impossibility!) would not work.

Problem 8.150

[Difficulty: 3]

8.150 Investigate the effect of tube length on water flow rate by computing the flow generated by a pressure difference $\Delta p = 100$ kPa applied to a length L of smooth tubing, of diameter $D = 25$ mm. Plot the flow rate against tube length for flow ranging from low speed laminar to fully turbulent.

Given: Flow in a tube

Find: Effect of tube length on flow rate; Plot

Solution:

Governing equations:

$$\left(\frac{p_1}{\rho} + \alpha_1 \cdot \frac{V_1^{\ 2}}{2} + g \cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \cdot \frac{V_2^{\ 2}}{2} + g \cdot z_2\right) = h_{lT} = \sum_{major} h_l + \sum_{minor} h_{lm} \quad (8.29)$$

$$Re = \frac{\rho \cdot V \cdot D}{\mu} \qquad h_l = f \cdot \frac{L}{D} \cdot \frac{V^2}{2} \quad (8.34) \qquad h_{lm} = K \cdot \frac{V^2}{2} \quad (8.40a) \qquad h_{lm} = f \cdot \frac{L_e}{D} \cdot \frac{V^2}{2} \quad (8.40b)$$

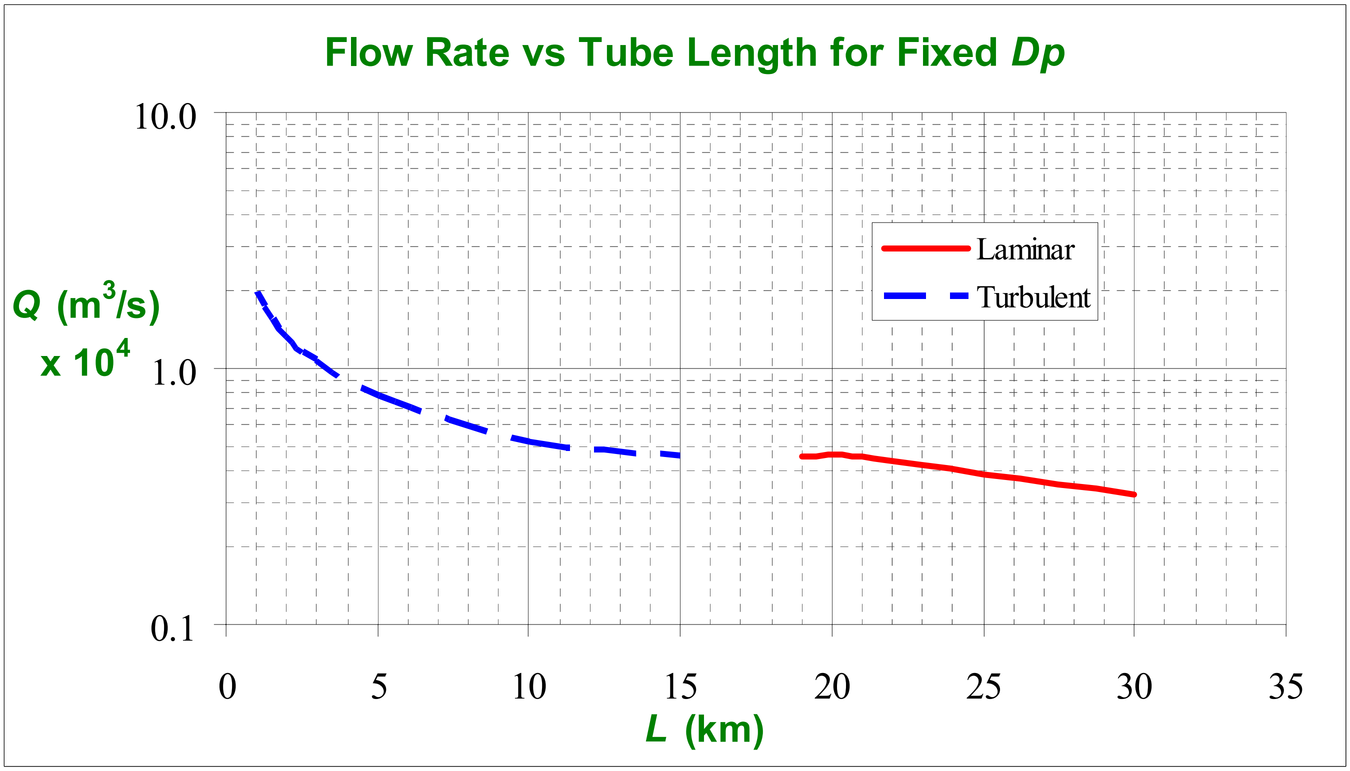
$$f = \frac{64}{Re} \quad (8.36) \quad \text{(Laminar)} \qquad \frac{1}{\sqrt{f}} = -2.0 \cdot \log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re \cdot \sqrt{f}}\right) \quad (8.37) \quad \text{(Turbulent)}$$

The energy equation (Eq. 8.29) becomes for flow in a tube

$$p_1 - p_2 = \Delta p = \rho \cdot f \cdot \frac{L}{D} \cdot \frac{V^2}{2}$$

This cannot be solved explicitly for velocity V, (and hence flow rate Q) because f depends on V; solution for a given L requires iteration (or use of *Solver*)

Given data:				Tabulated or graphical data:			
Δp =	100	m		μ =	1.00E-03	N.s/m^2	
D =	25	mm		ρ =	999	kg/m^3	
					(Water - Appendix A)		
Computed results:							
L (km)	V (m/s)	Q (m^3/s) x 10^4	Re	Regime	f	Δp (kPa)	Error
1.0	0.40	1.98	10063	Turbulent	0.0308	100	0.0%
1.5	0.319	1.56	7962	Turbulent	0.0328	100	0.0%
2.0	0.270	1.32	6739	Turbulent	0.0344	100	0.0%
2.5	0.237	1.16	5919	Turbulent	0.0356	100	0.0%
5.0	0.158	0.776	3948	Turbulent	0.0401	100	0.0%
10	0.105	0.516	2623	Turbulent	0.0454	100	0.0%
15	0.092	0.452	2300	Turbulent	0.0473	120	20.2%
19	0.092	0.452	2300	Laminar	0.0278	90	10.4%
21	0.092	0.452	2300	Laminar	0.0278	99	1.0%
25	0.078	0.383	1951	Laminar	0.0328	100	0.0%
30	0.065	0.320	1626	Laminar	0.0394	100	0.0%



The "critical" length of tube is between 15 and 20 km. For this range, the fluid is making a transition between laminar and turbulent flow, and is quite unstable. In this range the flow oscillates between laminar and turbulent; no consistent solution is found (i.e., an Re corresponding to turbulent flow needs an f assuming laminar to produce the Δp required, and vice versa!) More realistic numbers (e.g., tube length) are obtained for a fluid such as SAE 10W oil (The graph will remain the same except for scale)

Problem 8.151

[Difficulty: 5]

8.151 For the pipe flow into a reservoir of Example 8.5 consider the effect of pipe roughness on flow rate, assuming the pressure of the pump is maintained at 153 kPa. Plot the flow rate against pipe roughness ranging from smooth ($e = 0$) to very rough ($e = 3.75$ mm). Also consider the effect of pipe length (again assuming the pump always produces 153 kPa) for smooth pipe. Plot the flow rate against pipe length for $L = 100$ m through $L = 1000$ m.

Given: Flow from a reservoir

Find: Effect of pipe roughness and pipe length on flow rate; Plot

Solution:

Governing equations:
$$\left(\frac{p_1}{\rho} + \alpha_1\cdot\frac{V_1^2}{2} + g\cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2\cdot\frac{V_2^2}{2} + g\cdot z_2\right) = h_{lT} = \sum_{major} h_l + \sum_{minor} h_{lm} \quad (8.29)$$

$$Re = \frac{\rho\cdot V\cdot D}{\mu} \qquad h_l = f\cdot\frac{L}{D}\cdot\frac{V^2}{2} \quad (8.34) \qquad h_{lm} = K\cdot\frac{V^2}{2} \quad (8.40a) \qquad h_{lm} = f\cdot\frac{L_e}{D}\cdot\frac{V^2}{2} \quad (8.40b)$$

$$f = \frac{64}{Re} \quad (8.36) \quad \text{(Laminar)} \qquad \frac{1}{\sqrt{f}} = -2.0\cdot\log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re\cdot\sqrt{f}}\right) \quad (8.37) \quad \text{(Turbulent)}$$

The energy equation (Eq. 8.29) becomes for this flow (see Example 8.5)

$$p_{pump} = \Delta p = \rho\cdot\left(g\cdot d + f\cdot\frac{L}{D}\cdot\frac{V^2}{2}\right)$$

We need to solve this for velocity V, (and hence flow rate Q) as a function of roughness e, then length L. This cannot be solved explicitly for velocity V, (and hence flow rate Q) because f depends on V; solution for a given relative roughness e/D or length L requires iteration (or use of *Solver*)

Given data:				Tabulated or graphical data:			
Δp =	153	kPa		μ =	1.00E-03	N.s/m^2	
D =	75	mm		ρ =	999	kg/m^3	
L =	100	m			(Water - Appendix A)		

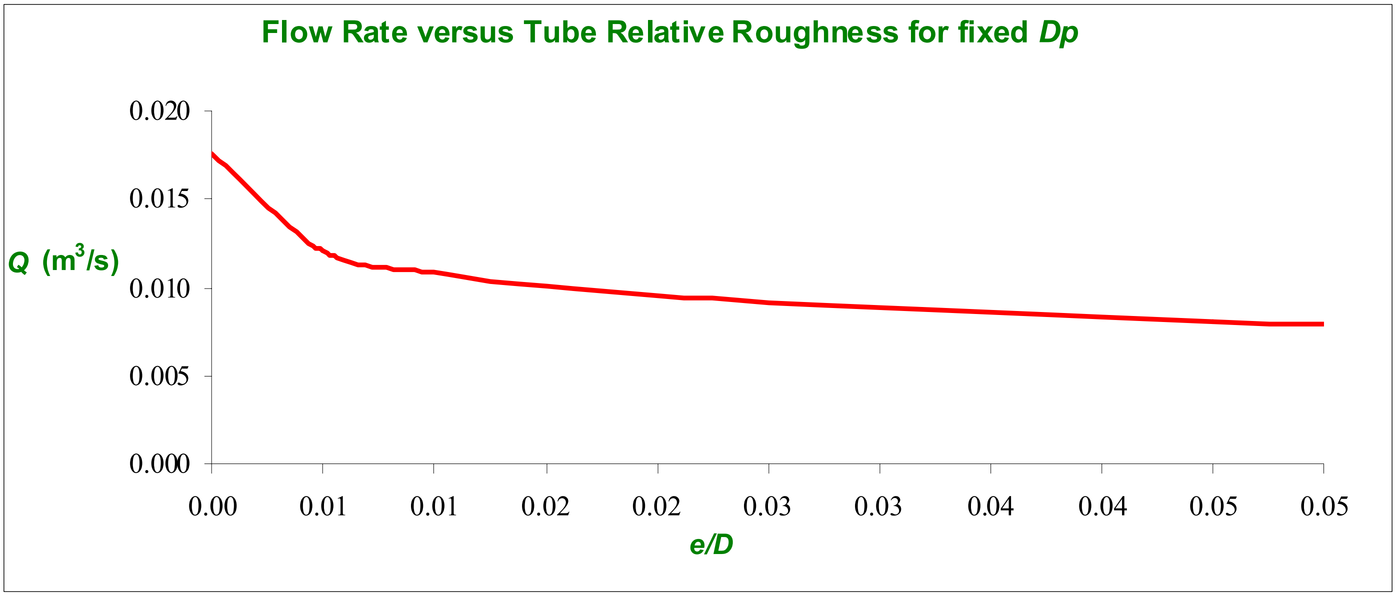
Computed results:

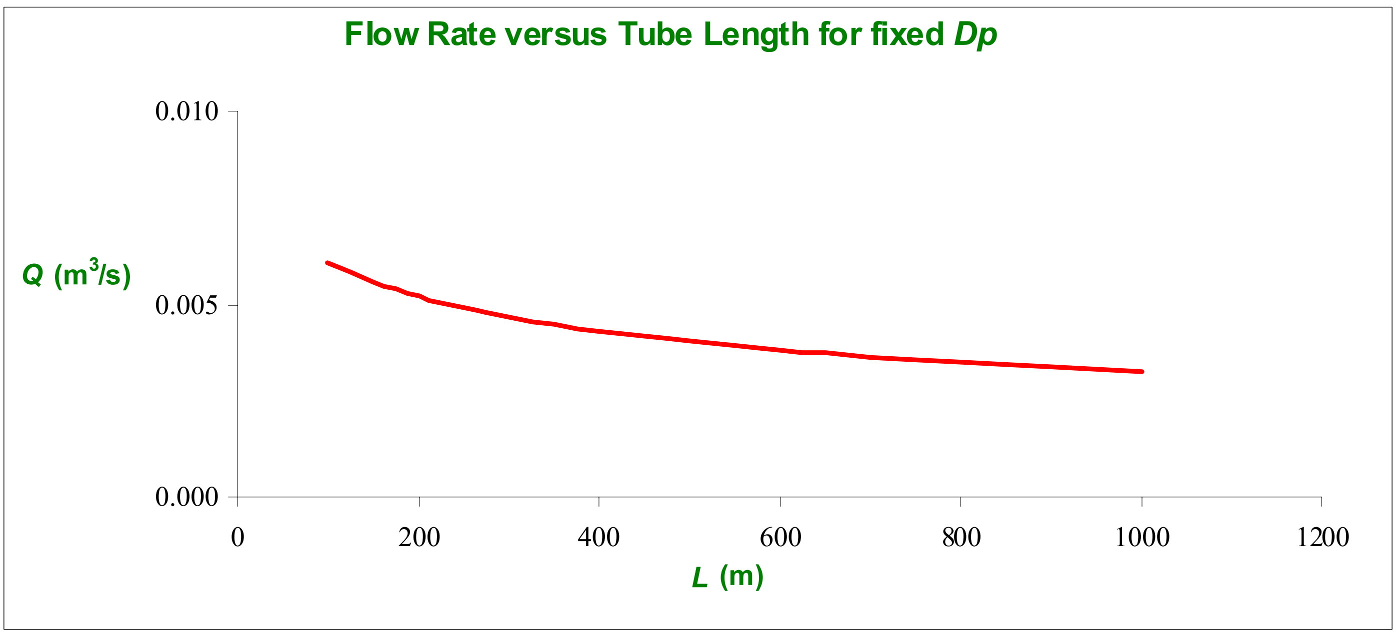
e/D	V (m/s)	Q (m^3/s)	Re	Regime	f	Δp (kPa)	Error
0.000	3.98	0.0176	2.98E+05	Turbulent	0.0145	153	0.0%
0.005	2.73	0.0121	2.05E+05	Turbulent	0.0308	153	0.0%
0.010	2.45	0.0108	1.84E+05	Turbulent	0.0382	153	0.0%
0.015	2.29	0.0101	1.71E+05	Turbulent	0.0440	153	0.0%
0.020	2.168	0.00958	1.62E+05	Turbulent	0.0489	153	0.0%
0.025	2.076	0.00917	1.56E+05	Turbulent	0.0533	153	0.0%
0.030	2.001	0.00884	1.50E+05	Turbulent	0.0574	153	0.0%
0.035	1.937	0.00856	1.45E+05	Turbulent	0.0612	153	0.0%
0.040	1.882	0.00832	1.41E+05	Turbulent	0.0649	153	0.0%
0.045	1.833	0.00810	1.37E+05	Turbulent	0.0683	153	0.0%
0.050	1.790	0.00791	1.34E+05	Turbulent	0.0717	153	0.0%

Computed results:

L (m)	V (m/s)	Q (m^3/s)	Re	Regime	f	Δp (kPa)	Error
100	1.37	0.00606	1.03E+05	Turbulent	0.1219	153	0.0%
200	1.175	0.00519	8.80E+04	Turbulent	0.0833	153	0.0%
300	1.056	0.00467	7.92E+04	Turbulent	0.0686	153	0.0%
400	0.975	0.00431	7.30E+04	Turbulent	0.0604	153	0.0%
500	0.913	0.004036	6.84E+04	Turbulent	0.0551	153	0.0%
600	0.865	0.003821	6.48E+04	Turbulent	0.0512	153	0.0%
700	0.825	0.003645	6.18E+04	Turbulent	0.0482	153	0.0%
800	0.791	0.003496	5.93E+04	Turbulent	0.0459	153	0.0%
900	0.762	0.003368	5.71E+04	Turbulent	0.0439	153	0.0%
1000	0.737	0.003257	5.52E+04	Turbulent	0.0423	153	0.0%

It is not possible to roughen the tube sufficiently to slow the flow down to a laminar flow for this Δp.


Flow Rate versus Tube Relative Roughness for fixed Dp
Q (m3/s)
0.020
0.015
0.010
0.005
0.000
0.00
0.01
0.01
0.02
0.02
0.03
0.03
0.04
0.04
0.05
0.05
e/D


Flow Rate versus Tube Length for fixed Dp
Q (m3/s)
0.010
0.005
0.000
0
200
400
600
800
1000
1200
L (m)

Problem 8.152

[Difficulty: 4]

8.152 Water for a fire protection system is supplied from a water tower through a 150-mm cast-iron pipe. A pressure gage at a fire hydrant indicates 600 kPa when no water is flowing. The total pipe length between the elevated tank and the hydrant is 200 m. Determine the height of the water tower above the hydrant. Calculate the maximum volume flow rate that can be achieved when the system is flushed by opening the hydrant wide (assume minor losses are 10 percent of major losses at this condition). When a fire hose is attached to the hydrant, the volume flow rate is 0.75 m^3/min. Determine the reading of the pressure gage at this flow condition.

Given: System for fire protection

Find: Height of water tower; Maximum flow rate; Pressure gage reading

Solution:

Governing equations:

$$\left(\frac{p_1}{\rho} + \alpha_1 \cdot \frac{V_1^2}{2} + g \cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \cdot \frac{V_2^2}{2} + g \cdot z_2\right) = h_{lT} = \sum_{major} h_l + \sum_{minor} h_{lm} \quad (8.29)$$

$$Re = \frac{\rho \cdot V \cdot D}{\mu} \qquad h_l = f \cdot \frac{L}{D} \cdot \frac{V^2}{2} \quad (8.34) \qquad h_{lm} = 0.1 \cdot h_l \qquad h_{lm} = f \cdot \frac{L_e}{D} \cdot \frac{V^2}{2} \quad (8.40b)$$

$$f = \frac{64}{Re} \quad (8.36) \quad \text{(Laminar)} \qquad \frac{1}{\sqrt{f}} = -2.0 \cdot \log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re \cdot \sqrt{f}}\right) \quad (8.37) \quad \text{(Turbulent)}$$

For no flow the energy equation (Eq. 8.29) applied between the water tower free surface (state 1; height *H*) and pressure gage is

$$g \cdot H = \frac{p_2}{\rho} \qquad \text{or} \qquad H = \frac{p_2}{\rho \cdot g} \quad (1)$$

The energy equation (Eq. 8.29) becomes, for maximum flow (and $\alpha = 1$)

$$g \cdot H - \frac{V^2}{2} = h_{lT} = (1 + 0.1) \cdot h_l \qquad \text{or} \qquad g \cdot H = \frac{V^2}{2} \cdot \left(1 + 1.1 \cdot f \cdot \frac{L}{D}\right) \quad (2)$$

This can be solved for *V* (and hence *Q*) by iterating, or by using *Solver*

The energy equation (Eq. 8.29) becomes, for restricted flow

$$g \cdot H - \frac{p_2}{\rho} + \frac{V^2}{2} = h_{lT} = (1 + 0.1) \cdot h_l \qquad p_2 = \rho \cdot g \cdot H - \rho \cdot \frac{V^2}{2} \cdot \left(1 + 1.1 \cdot \rho \cdot f \cdot \frac{L}{D}\right) \quad (3)$$

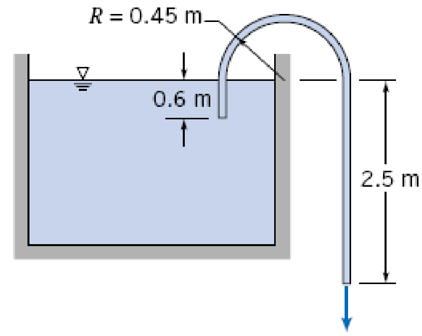
The results in Excel are shown below:

Given data:				Tabulated or graphical data:						
p_2 =	600	kPa		e =	0.26	mm				
	(Closed)				(Table 8.1)					
D =	150	mm		μ =	1.00E-03	N.s/m^2				
L =	200	m		ρ =	999	kg/m^3				
Q =	0.75	m^3/min			(Water - Appendix A)					
	(Open)									
Computed results:										
Closed:			Fully open:				Partially open:			
H =	**61.2**	m	V =	5.91	m/s		Q =	0.75	m^3/min	
	(Eq. 1)		Re =	8.85E+05			V =	0.71	m/s	
			f =	0.0228			Re =	1.06E+05		
							f =	0.0243		
			Eq. 2, solved by varying V using *Solver*:				p_2 =	**591**	kPa	
			Left (m^2/s)	**Right (m^2/s)**	**Error**			(Eq. 3)		
			601	601	0%					
			Q =	**0.104**	m^3/s					

Problem 8.153

[Difficulty: 4]

8.153 The siphon shown is fabricated from 50-mm-i.d. drawn aluminum tubing. The liquid is water at 15°C. Compute the volume flow rate through the siphon. Estimate the minimum pressure inside the tube.



Given: Syphon system

Find: Flow rate; Minimum pressure

Solution:

Basic equations

$$\left(\frac{p_1}{\rho} + \alpha\cdot\frac{V_1^2}{2} + g\cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha\cdot\frac{V_2^2}{2} + g\cdot z_2\right) = h_{lT} \qquad h_{lT} = f\cdot\frac{L}{D}\cdot\frac{V^2}{2} + h_{lm}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) α is approximately 1

Hence the energy equation applied between the tank free surface (Point 1) and the tube exit (Point 2, z = 0) becomes

$$g\cdot z_1 - \frac{V_2^2}{2} = g\cdot z_1 - \frac{V^2}{2} = f\cdot\frac{L}{D}\cdot\frac{V^2}{2} + K_{ent}\cdot\frac{V^2}{2} + f\cdot\frac{L_e}{D}\cdot\frac{V^2}{2}$$

From Table 8.2 for reentrant entrance $\quad K_{ent} = 0.78$

For the bend $\quad \frac{R}{D} = 9 \quad$ so from Fig. 8.16 $\quad \frac{L_e}{D} = 28 \quad$ for a 90° bend so for a 180° bend $\quad \frac{L_e}{D} = 56$

Solving for V

$$V = \sqrt{\frac{2\cdot g\cdot h}{\left[1 + K_{ent} + f\cdot\left(\frac{L}{D} + \frac{L_e}{D}\right)\right]}} \qquad (1) \qquad \text{and} \qquad h = 2.5\cdot m$$

The two lengths are $\quad L_e = 56\cdot D \quad L_e = 2.8\,m \quad L = (0.6 + \pi\cdot 0.45 + 2.5)\cdot m \quad L = 4.51\,m$

We also have $\quad Re = \frac{V\cdot D}{\nu} \quad$ or $\quad Re = c\cdot V \quad (2) \quad$ where $\quad c = \frac{D}{\nu}$

From Table A.7 (15°C) $\quad \nu = 1.14\times 10^{-6}\cdot\frac{m^2}{s} \quad c = 0.05\cdot m\times\frac{s}{1.14\times 10^{-6}\cdot m^2} \quad c = 4.39\times 10^{4}\cdot\frac{s}{m}$

In addition

$$\frac{1}{\sqrt{f}} = -2.0\cdot\log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re\cdot\sqrt{f}}\right) \qquad (3) \qquad e = 0.0015\cdot mm \quad \text{(Table 8.1)}$$

Equations 1, 2 and 3 form a set of simultaneous equations for V, Re and f

Make a guess for f $\quad f = 0.01 \quad$ then

$$V = \sqrt{\frac{2 \cdot g \cdot h}{\left[1 + K_{ent} + f \cdot \left(\frac{L}{D} + \frac{L_e}{D}\right)\right]}} \qquad V = 3.89 \frac{m}{s} \qquad Re = c \cdot V \qquad Re = 1.71 \times 10^5$$

Given $\quad \frac{1}{\sqrt{f}} = -2.0 \cdot \log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re \cdot \sqrt{f}}\right) \qquad f = 0.0164$

$$V = \sqrt{\frac{2 \cdot g \cdot h}{\left[1 + K_{ent} + f \cdot \left(\frac{L}{D} + \frac{L_e}{D}\right)\right]}} \qquad V = 3.43 \frac{m}{s} \qquad Re = c \cdot V \qquad Re = 1.50 \times 10^5$$

Given $\quad \frac{1}{\sqrt{f}} = -2.0 \cdot \log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re \cdot \sqrt{f}}\right) \qquad f = 0.0168$

$$V = \sqrt{\frac{2 \cdot g \cdot h}{\left[1 + K_{ent} + f \cdot \left(\frac{L}{D} + \frac{L_e}{D}\right)\right]}} \qquad V = 3.40 \frac{m}{s} \qquad Re = c \cdot V \qquad Re = 1.49 \times 10^5$$

Given $\quad \frac{1}{\sqrt{f}} = -2.0 \cdot \log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re \cdot \sqrt{f}}\right) \qquad f = 0.0168$

$$V = \sqrt{\frac{2 \cdot g \cdot h}{\left[1 + K_{ent} + f \cdot \left(\frac{L}{D} + \frac{L_e}{D}\right)\right]}} \qquad V = 3.40 \frac{m}{s} \qquad Re = c \cdot V \qquad Re = 1.49 \times 10^5$$

Note that we could use *Excel*'s *Solver* for this problem.

The flow rate is then $\qquad Q = \frac{\pi \cdot D^2}{4} \cdot V \qquad Q = 6.68 \times 10^{-3} \frac{m^3}{s}$

The minimum pressure occurs at the top of the curve (Point 3). Applying the energy equation between Points 1 and 3

$$g \cdot z_1 - \left(\frac{p_3}{\rho} + \frac{V_3^2}{2} + g \cdot z_3\right) = g \cdot z_1 - \left(\frac{p_3}{\rho} + \frac{V^2}{2} + g \cdot z_3\right) = f \cdot \frac{L}{D} \cdot \frac{V^2}{2} + K_{ent} \cdot \frac{V^2}{2} + f \cdot \frac{L_e}{D} \cdot \frac{V^2}{2}$$

where we have $\frac{L_e}{D} = 28$ for the first 90º of the bend, and $\quad L = \left(0.6 + \frac{\pi \times 0.45}{2}\right) \cdot m \qquad L = 1.31\,m$

$$p_3 = \rho \cdot \left[g \cdot (z_1 - z_3) - \frac{V^2}{2} \cdot \left[1 + K_{ent} + f \cdot \left(\frac{L}{D} + \frac{L_e}{D}\right)\right]\right]$$

$$p_3 = 1000 \cdot \frac{kg}{m^3} \times \left[9.81 \cdot \frac{m}{s^2} \times (-0.45 \cdot m) - \frac{\left(3.4 \cdot \frac{m}{s}\right)^2}{2} \cdot \left[1 + 0.78 + 0.0168 \cdot \left(\frac{1.31}{0.05} + 28\right)\right]\right] \times \frac{N \cdot s^2}{kg \cdot m} \qquad p_3 = -20.0 \cdot kPa$$

Problem 8.154

[Difficulty: 4]

8.154 A large open water tank has a horizontal cast iron drainpipe of diameter $D = 1$ in. and length $L = 2$ ft attached at its base. If the depth of water is $h = 3$ ft, find the flow rate (gpm) if the pipe entrance is (a) reentrant, (b) square-edged, and (c) rounded ($r = 0.2$ in.).

Given: Tank with drainpipe

Find: Flow rate for rentrant, square-edged, and rounded entrances

Solution:

Basic equations $$\left(\frac{p_1}{\rho} + \alpha\cdot\frac{V_1^{\,2}}{2} + g\cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha\cdot\frac{V_2^{\,2}}{2} + g\cdot z_2\right) = h_{lT} \qquad h_{lT} = f\cdot\frac{L}{D}\cdot\frac{V^2}{2} + K_{ent}\cdot\frac{V^2}{2}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) α is approximately 1

Available data $D = 1\cdot in$ $\quad L = 2\cdot ft$ $\quad e = 0.00085\cdot ft$ (Table 8.1) $\quad h = 3\cdot ft$ $\quad r = 0.2\cdot in$

Hence the energy equation applied between the tank free surface (Point 1) and the pipe exit (Point 2, z = 0) becomes

$$g\cdot z_1 - \frac{V_2^{\,2}}{2} = g\cdot z_1 - \frac{V^2}{2} = f\cdot\frac{L}{D}\cdot\frac{V^2}{2} + K_{ent}\cdot\frac{V^2}{2}$$

Solving for V $$V = \sqrt{\frac{2\cdot g\cdot h}{\left(1 + K_{ent} + f\cdot\frac{L}{D}\right)}} \qquad (1)$$

We also have $Re = \frac{V\cdot D}{\nu}$ or $Re = c\cdot V$ (2) where $c = \frac{D}{\nu}$

From Table A.7 (20ºC) $\nu = 1.01\times10^{-6}\cdot\frac{m^2}{s}$ $\quad \nu = 1.09\times10^{-5}\,\frac{ft^2}{s}$ $\quad c = \frac{D}{\nu}$ $\quad c = 7665\cdot\frac{s}{ft}$

In addition $$\frac{1}{\sqrt{f}} = -2.0\cdot\log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re\cdot\sqrt{f}}\right) \qquad (3)$$

Equations 1, 2 and 3 form a set of simultaneous equations for V, Re and f

For a reentrant entrance, from Table 8.2 $K_{ent} = 0.78$

Make a guess for f $f = 0.01$ then $V = \sqrt{\frac{2\cdot g\cdot h}{\left(1 + K_{ent} + f\cdot\frac{L}{D}\right)}}$ $\quad V = 2.98\,\frac{m}{s}$

$Re = c\cdot V$ $\quad Re = 7.49\times10^{4}$

Given $$\frac{1}{\sqrt{f}} = -2.0\cdot\log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re\cdot\sqrt{f}}\right) \qquad f = 0.0389$$

$$V = \sqrt{\frac{2\cdot g\cdot h}{\left(1 + K_{ent} + f\cdot\frac{L}{D}\right)}} \qquad V = 2.57\frac{m}{s} \qquad Re = c\cdot V \qquad Re = 6.46\times 10^{4}$$

Given $\frac{1}{\sqrt{f}} = -2.0\cdot\log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re\cdot\sqrt{f}}\right)$ $\qquad f = 0.0391$

$$V = \sqrt{\frac{2\cdot g\cdot h}{\left(1 + K_{ent} + f\cdot\frac{L}{D}\right)}} \qquad V = 2.57\frac{m}{s} \qquad Re = c\cdot V \qquad Re = 6.46\times 10^{4}$$

Given $\frac{1}{\sqrt{f}} = -2.0\cdot\log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re\cdot\sqrt{f}}\right)$ $\qquad f = 0.0391$

$$V = \sqrt{\frac{2\cdot g\cdot h}{\left(1 + K_{ent} + f\cdot\frac{L}{D}\right)}} \qquad V = 2.57\frac{m}{s} \qquad Re = c\cdot V \qquad Re = 6.46\times 10^{4}$$

Note that we could use *Excel*'s *Solver* for this problem

The flow rate is then $\qquad Q = V\cdot\frac{\pi\cdot D^2}{4} \qquad Q = 0.0460\frac{ft^3}{s} \qquad Q = 20.6\cdot gpm$

For a square-edged entrance, from Table 8.2 $\qquad K_{ent} = 0.5$

Make a guess for f $\qquad f = 0.01$ then $\qquad V = \sqrt{\frac{2\cdot g\cdot h}{\left(1 + K_{ent} + f\cdot\frac{L}{D}\right)}} \qquad V = 3.21\frac{m}{s} \qquad Re = c\cdot V \qquad Re = 8.07\times 10^{4}$

Given $\frac{1}{\sqrt{f}} = -2.0\cdot\log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re\cdot\sqrt{f}}\right)$ $\qquad f = 0.0389$

$$V = \sqrt{\frac{2\cdot g\cdot h}{\left(1 + K_{ent} + f\cdot\frac{L}{D}\right)}} \qquad V = 2.71\frac{m}{s} \qquad Re = c\cdot V \qquad Re = 6.83\times 10^{4}$$

Given $\frac{1}{\sqrt{f}} = -2.0\cdot\log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re\cdot\sqrt{f}}\right)$ $\qquad f = 0.0390$

$$V = \sqrt{\frac{2\cdot g\cdot h}{\left(1 + K_{ent} + f\cdot\frac{L}{D}\right)}} \qquad V = 2.71\frac{m}{s} \qquad Re = c\cdot V \qquad Re = 6.82\times 10^{4}$$

The flow rate is then $\qquad Q = V\cdot\frac{\pi\cdot D^2}{4} \qquad Q = 0.0485\frac{ft^3}{s} \qquad Q = 21.8\cdot gpm$

For a rounded entrance, from Table 8.2 $\qquad \frac{r}{D} = 0.2 \qquad K_{ent} = 0.04$

Make a guess for f $f = 0.01$ then $V = \sqrt{\frac{2\cdot g\cdot h}{\left(1 + K_{ent} + f\cdot\frac{L}{D}\right)}}$

$V = 3.74\,\frac{m}{s}$ $Re = c\cdot V$ $Re = 9.41\times 10^4$

Given $\frac{1}{\sqrt{f}} = -2.0\cdot\log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re\cdot\sqrt{f}}\right)$ $f = 0.0388$

$V = \sqrt{\frac{2\cdot g\cdot h}{\left(1 + K_{ent} + f\cdot\frac{L}{D}\right)}}$ $V = 3.02\,\frac{m}{s}$ $Re = c\cdot V$ $Re = 7.59\times 10^4$

Given $\frac{1}{\sqrt{f}} = -2.0\cdot\log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re\cdot\sqrt{f}}\right)$ $f = 0.0389$

$V = \sqrt{\frac{2\cdot g\cdot h}{\left(1 + K_{ent} + f\cdot\frac{L}{D}\right)}}$ $V = 3.01\,\frac{m}{s}$ $Re = c\cdot V$ $Re = 7.58\times 10^4$

Given $\frac{1}{\sqrt{f}} = -2.0\cdot\log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re\cdot\sqrt{f}}\right)$ $f = 0.0389$

$V = \sqrt{\frac{2\cdot g\cdot h}{\left(1 + K_{ent} + f\cdot\frac{L}{D}\right)}}$ $V = 3.01\,\frac{m}{s}$ $Re = c\cdot V$ $Re = 7.58\times 10^4$

Note that we could use *Excel*'s *Solver* for this problem

The flow rate is then $Q = V\cdot\frac{\pi\cdot D^2}{4}$ $Q = 0.0539\,\frac{ft^3}{s}$ $Q = 24.2\cdot gpm$

In summary: Renentrant: $Q = 20.6\cdot gpm$ Square-edged: $Q = 21.8\cdot gpm$ Rounded: $Q = 24.2\cdot gpm$

Problem 8.155

[Difficulty: 4]

8.155 Repeat Problem 8.154, except now the pipe is vertical, as shown.

Given: Tank with drainpipe

Find: Flow rate for rentrant, square-edged, and rounded entrances

Solution:

Basic equations
$$\left(\frac{p_1}{\rho} + \alpha\cdot\frac{V_1^{\,2}}{2} + g\cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha\cdot\frac{V_2^{\,2}}{2} + g\cdot z_2\right) = h_{lT} \qquad h_{lT} = f\cdot\frac{L}{D}\cdot\frac{V^2}{2} + K_{ent}\cdot\frac{V^2}{2}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) α is approximately 1

Available data $D = 1\cdot in$ $L = 2\cdot ft$ $e = 0.00085\cdot ft$ (Table 8.1) $h = 3\cdot ft$ $r = 0.2\cdot in$

Hence the energy equation applied between the tank free surface (Point 1) and the pipe exit (Point 2, z = 0) becomes

$$g\cdot z_1 - \frac{V_2^{\,2}}{2} = g\cdot z_1 - \frac{V^2}{2} = f\cdot\frac{L}{D}\cdot\frac{V^2}{2} + K_{ent}\cdot\frac{V^2}{2}$$

Solving for V
$$V = \sqrt{\frac{2\cdot g\cdot H}{\left(1 + K_{ent} + f\cdot\frac{L}{D}\right)}} \qquad (1)$$

where now we have $H = h + L$ $H = 5\,ft$

We also have $Re = \frac{V\cdot D}{\nu}$ or $Re = c\cdot V$ (2) where $c = \frac{D}{\nu}$

From Table A.7 (20°C) $\nu = 1.01\times 10^{-6}\cdot\frac{m^2}{s}$ $\nu = 1.09\times 10^{-5}\,\frac{ft^2}{s}$ $c = \frac{D}{\nu}$ $c = 7665\cdot\frac{s}{ft}$

In addition
$$\frac{1}{\sqrt{f}} = -2.0\cdot\log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re\cdot\sqrt{f}}\right) \qquad (3)$$

Equations 1, 2 and 3 form a set of simultaneous equations for V, Re and f

For a reentrant entrance, from Table 8.2 $K_{ent} = 0.78$

Make a guess for f $f = 0.01$ then $V = \sqrt{\frac{2\cdot g\cdot H}{\left(1 + K_{ent} + f\cdot\frac{L}{D}\right)}}$ $V = 3.85\,\frac{m}{s}$

$Re = c\cdot V$ $Re = 9.67\times 10^4$

Given
$$\frac{1}{\sqrt{f}} = -2.0\cdot\log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re\cdot\sqrt{f}}\right) \qquad f = 0.0388$$

$$V = \sqrt{\frac{2 \cdot g \cdot H}{\left(1 + K_{ent} + f \cdot \frac{L}{D}\right)}} \qquad V = 3.32 \frac{m}{s} \qquad Re = c \cdot V \qquad Re = 8.35 \times 10^4$$

Given $$\frac{1}{\sqrt{f}} = -2.0 \cdot \log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re \cdot \sqrt{f}}\right) \qquad f = 0.0389$$

$$V = \sqrt{\frac{2 \cdot g \cdot H}{\left(1 + K_{ent} + f \cdot \frac{L}{D}\right)}} \qquad V = 3.32 \frac{m}{s} \qquad Re = c \cdot V \qquad Re = 8.35 \times 10^4$$

Note that we could use *Excel*'s *Solver* for this problem

The flow rate is then $Q = V \cdot \frac{\pi \cdot D^2}{4}$ $\qquad Q = 0.0594 \frac{ft^3}{s}$ $\qquad Q = 26.7 \cdot gpm$

For a square-edged entrance, from Table 8.2 $\qquad K_{ent} = 0.5$

Make a guess for f $\qquad f = 0.01$ $\qquad$ then $V = \sqrt{\frac{2 \cdot g \cdot H}{\left(1 + K_{ent} + f \cdot \frac{L}{D}\right)}}$ $\qquad V = 4.14 \frac{m}{s}$ $\qquad Re = c \cdot V$ $\qquad Re = 1.04 \times 10^5$

Given $$\frac{1}{\sqrt{f}} = -2.0 \cdot \log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re \cdot \sqrt{f}}\right) \qquad f = 0.0387$$

$$V = \sqrt{\frac{2 \cdot g \cdot H}{\left(1 + K_{ent} + f \cdot \frac{L}{D}\right)}} \qquad V = 3.51 \frac{m}{s} \qquad Re = c \cdot V \qquad Re = 8.82 \times 10^4$$

Given $$\frac{1}{\sqrt{f}} = -2.0 \cdot \log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re \cdot \sqrt{f}}\right) \qquad f = 0.0388$$

$$V = \sqrt{\frac{2 \cdot g \cdot H}{\left(1 + K_{ent} + f \cdot \frac{L}{D}\right)}} \qquad V = 3.51 \frac{m}{s} \qquad Re = c \cdot V \qquad Re = 8.82 \times 10^4$$

The flow rate is then $Q = V \cdot \frac{\pi \cdot D^2}{4}$ $\qquad Q = 0.0627 \frac{ft^3}{s}$ $\qquad Q = 28.2 \cdot gpm$

For a rounded entrance, from Table 8.2 $\frac{r}{D} = 0.2$ $\qquad K_{ent} = 0.04$

Make a guess for f $\qquad f = 0.01$ $\qquad$ then $V = \sqrt{\frac{2 \cdot g \cdot H}{\left(1 + K_{ent} + f \cdot \frac{L}{D}\right)}}$

$$V = 4.83 \frac{m}{s} \qquad Re = c \cdot V \qquad Re = 1.22 \times 10^5$$

Given $$\frac{1}{\sqrt{f}} = -2.0 \cdot \log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re \cdot \sqrt{f}}\right)$$ $f = 0.0386$

$$V = \sqrt{\frac{2 \cdot g \cdot H}{\left(1 + K_{ent} + f \cdot \frac{L}{D}\right)}}$$ $V = 3.90 \frac{m}{s}$ $Re = c \cdot V$ $Re = 9.80 \times 10^4$

Given $$\frac{1}{\sqrt{f}} = -2.0 \cdot \log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re \cdot \sqrt{f}}\right)$$ $f = 0.0388$

$$V = \sqrt{\frac{2 \cdot g \cdot H}{\left(1 + K_{ent} + f \cdot \frac{L}{D}\right)}}$$ $V = 3.89 \frac{m}{s}$ $Re = c \cdot V$ $Re = 9.80 \times 10^4$

Given $$\frac{1}{\sqrt{f}} = -2.0 \cdot \log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re \cdot \sqrt{f}}\right)$$ $f = 0.0388$

$$V = \sqrt{\frac{2 \cdot g \cdot H}{\left(1 + K_{ent} + f \cdot \frac{L}{D}\right)}}$$ $V = 3.89 \frac{m}{s}$ $Re = c \cdot V$ $Re = 9.80 \times 10^4$

Note that we could use *Excel*'s *Solver* for this problem

The flow rate is then $Q = V \cdot \frac{\pi \cdot D^2}{4}$ $Q = 0.0697 \frac{ft^3}{s}$ $Q = 31.3 \cdot gpm$

In summary: Renentrant: $Q = 26.7 \cdot gpm$ Square-edged: $Q = 28.2 \cdot gpm$ Rounded: $Q = 31.3 \cdot gpm$

Problem 8.156

[Difficulty: 5]

8.156 A tank containing 30 m^3 of kerosene is to be emptied by a gravity feed using a drain hose of diameter 15 mm, roughness 0.2 mm, and length 1 m. The top of the tank is open to the atmosphere and the hose exits to an open chamber. If the kerosene level is initially 10 m above the drain exit, estimate (by assuming steady flow) the initial drainage rate. Estimate the flow rate when the kerosene level is down to 5 m, and then down to 1 m. Based on these three estimates, make a rough estimate of the time it took to drain to the 1-m level.

Given: Tank with drain hose

Find: Flow rate at different instants; Estimate of drain time

Solution:

Basic equations
$$\left(\frac{p_1}{\rho} + \alpha\cdot\frac{V_1^2}{2} + g\cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha\cdot\frac{V_2^2}{2} + g\cdot z_2\right) = h_l \qquad h_l = f\cdot\frac{L}{D}\cdot\frac{V^2}{2}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) α is approximately 1 4) Ignore minor loss at entrance (L >>; verify later)

Available data $\quad L = 1\cdot m \quad D = 15\cdot mm \quad e = 0.2\cdot mm \quad Vol = 30\cdot m^3$

Hence the energy equation applied between the tank free surface (Point 1) and the hose exit (Point 2, z = 0) becomes

$$g\cdot z_1 - \frac{V_2^2}{2} = g\cdot z_1 - \frac{V^2}{2} = f\cdot\frac{L}{D}\cdot\frac{V^2}{2}$$

Solving for V
$$V = \sqrt{\frac{2\cdot g\cdot h}{\left(1 + f\cdot\frac{L}{D}\right)}} \qquad (1) \qquad \text{and} \qquad h = 10\cdot m \qquad \text{initially}$$

We also have
$$Re = \frac{V\cdot D}{\nu} \quad \text{or} \quad Re = c\cdot V \quad (2) \qquad \text{where} \qquad c = \frac{D}{\nu}$$

From Fig. A.2 (20ºC)
$$\nu = 1.8\times 10^{-6}\cdot\frac{m^2}{s} \qquad c = \frac{D}{\nu} \qquad c = 8333\,\frac{s}{m}$$

In addition
$$\frac{1}{\sqrt{f}} = -2.0\cdot\log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re\cdot\sqrt{f}}\right) \qquad (3)$$

Equations 1, 2 and 3 form a set of simultaneous equations for V, Re and f

Make a guess for f $\quad f = 0.01 \quad$ then

$$V = \sqrt{\frac{2\cdot g\cdot h}{\left(1 + f\cdot\frac{L}{D}\right)}} \qquad V = 10.8\,\frac{m}{s} \qquad Re = c\cdot V \qquad Re = 9.04\times 10^4$$

Given $\quad \dfrac{1}{\sqrt{f}} = -2.0\cdot\log\left(\dfrac{\frac{e}{D}}{3.7} + \dfrac{2.51}{Re\cdot\sqrt{f}}\right) \quad f = 0.0427 \quad V = \sqrt{\dfrac{2\cdot g\cdot h}{\left(1 + f\cdot\frac{L}{D}\right)}} \quad V = 7.14\,\dfrac{m}{s} \quad Re = c\cdot V \quad Re = 5.95\times 10^{4}$

Given $\quad \dfrac{1}{\sqrt{f}} = -2.0\cdot\log\left(\dfrac{\frac{e}{D}}{3.7} + \dfrac{2.51}{Re\cdot\sqrt{f}}\right) \quad f = 0.0427 \quad V = \sqrt{\dfrac{2\cdot g\cdot h}{\left(1 + f\cdot\frac{L}{D}\right)}} \quad V = 7.14\,\dfrac{m}{s} \quad Re = c\cdot V \quad Re = 5.95\times 10^{4}$

Note that we could use *Excel*'s *Solver* for this problem $\qquad$ Note: $\quad f\cdot\dfrac{L}{D} = 2.8 \quad K_e = 0.5 \quad h_{lm} < h_l$

The flow rate is then $\quad Q = V\cdot\dfrac{\pi\cdot D^2}{4} \quad Q = 1.26\times 10^{-3}\,\dfrac{m^3}{s} \quad Q = 1.26\cdot\dfrac{l}{s}$

Next we recompute everything for $\quad h = 5\cdot m$

Given $\quad \dfrac{1}{\sqrt{f}} = -2.0\cdot\log\left(\dfrac{\frac{e}{D}}{3.7} + \dfrac{2.51}{Re\cdot\sqrt{f}}\right) \quad f = 0.0430 \quad V = \sqrt{\dfrac{2\cdot g\cdot h}{\left(1 + f\cdot\frac{L}{D}\right)}} \quad V = 5.04\,\dfrac{m}{s} \quad Re = c\cdot V \quad Re = 4.20\times 10^{4}$

Given $\quad \dfrac{1}{\sqrt{f}} = -2.0\cdot\log\left(\dfrac{\frac{e}{D}}{3.7} + \dfrac{2.51}{Re\cdot\sqrt{f}}\right) \quad f = 0.0430 \quad V = \sqrt{\dfrac{2\cdot g\cdot h}{\left(1 + f\cdot\frac{L}{D}\right)}} \quad V = 5.04\,\dfrac{m}{s} \quad Re = c\cdot V \quad Re = 4.20\times 10^{4}$

The flow rate is then $\quad Q = V\cdot\dfrac{\pi\cdot D^2}{4} \quad Q = 8.9\times 10^{-4}\,\dfrac{m^3}{s} \quad Q = 0.890\cdot\dfrac{l}{s}$

Next we recompute everything for $\quad h = 1\cdot m$

Given $\quad \dfrac{1}{\sqrt{f}} = -2.0\cdot\log\left(\dfrac{\frac{e}{D}}{3.7} + \dfrac{2.51}{Re\cdot\sqrt{f}}\right) \quad f = 0.0444 \quad V = \sqrt{\dfrac{2\cdot g\cdot h}{\left(1 + f\cdot\frac{L}{D}\right)}} \quad V = 2.23\,\dfrac{m}{s} \quad Re = c\cdot V \quad Re = 1.85\times 10^{4}$

Given $\quad \dfrac{1}{\sqrt{f}} = -2.0\cdot\log\left(\dfrac{\frac{e}{D}}{3.7} + \dfrac{2.51}{Re\cdot\sqrt{f}}\right) \quad f = 0.0444 \quad V = \sqrt{\dfrac{2\cdot g\cdot h}{\left(1 + f\cdot\frac{L}{D}\right)}} \quad V = 2.23\,\dfrac{m}{s} \quad Re = c\cdot V \quad Re = 1.85\times 10^{4}$

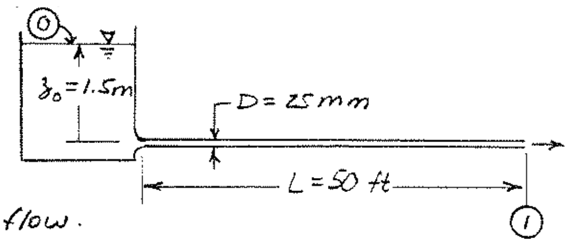
The flow rate is then $\quad Q = V\cdot\dfrac{\pi\cdot D^2}{4} \quad Q = 3.93\times 10^{-4}\,\dfrac{m^3}{s} \quad Q = 0.393\cdot\dfrac{l}{s}$

Initially we have dQ/dt = -1.26L/s, then -.890 L/s, then -0.393 L/s. These occur at h = 10 m, 5 m and 1 m. The corresponding volumes in the tank are then Q = 30,000 L, 15,000 L, and 3,000 L^3. Using *Excel* we can fit a power trendline to the dQ/dt versus Q data to find, approximately

$\dfrac{dQ}{dt} = -0.00683\cdot Q^{\frac{1}{2}}$ where dQ/dt is in L/s and t is s. Solving this with initial condition Q = -1.26 L/s when t = 0 gives

$t = 293\cdot\left(\sqrt{30} - \sqrt{Q}\right)$ Hence, when Q = 3000 L (h = 1 m) $\quad t = 293\cdot\left(\sqrt{30000} - \sqrt{3000}\right)\cdot s \quad t = 3.47\times 10^{4}\,s \quad t = 9.64\cdot hr$

8.157 Consider again the Roman water supply discussed in Example 8.10. Assume that the 50 ft length of horizontal constant-diameter pipe required by law has been installed. The relative roughness of the pipe is 0.01. Estimate the flow rate of water delivered by the pipe under the inlet conditions of the example. What would be the effect of adding the same diffuser to the end of the 50 ft pipe?



Solution: Apply the energy equation for steady, incompressible pipe flow.

Computing equation:

$$\frac{p_0}{\rho} + \alpha_0\frac{\bar{V}_0^2}{2} + gz_0 = \frac{p_1}{\rho} + \alpha_1\frac{\bar{V}_1^2}{2} + gz_1 + h_{l_T}\,; \quad h_{l_T} = \left(f\frac{L}{D} + K_{ent}\right)\frac{\bar{V}_1^2}{2}$$

Assumptions: (1) $p_0 = p_1 = p_{atm}$ (3) $\alpha_1 \approx 1$
(2) $\bar{V}_1 \approx 0$ (4) $K_{ent} = 0.04$

Then

$$gz_0 = \frac{\bar{V}_1^2}{2} + \left(f\frac{L}{D} + K_{ent}\right)\frac{\bar{V}_1^2}{2} \quad \text{or} \quad \bar{V}_1 = \sqrt{\frac{2gz_0}{1 + f\frac{L}{D} + K}}$$

For $e/D = 0.01$, $f = 0.038$ from Eq. 8.37*, so

$$\bar{V}_1 = \left[2 \times 9.81\frac{m}{s^2} \times 1.5\,m \times \frac{1}{1 + 0.038 \times 50\,ft \times \frac{1}{25\,mm} \times 304.8\frac{mm}{ft} + 0.04}\right]^{1/2} = 1.10\,m/s$$

Checking, assuming $T = 20°C$,

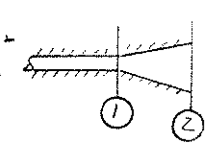
$$Re = \frac{\bar{V}D}{\nu} = 1.10\frac{m}{sec} \times 0.025\,m \times \frac{sec}{1.0\times10^{-6}\,m^2} = 2.75\times10^4\,; \text{ from Eq. 8.37*}, f \approx 0.040, \text{ so}$$

$$\bar{V}_1 = \left[2 \times 9.81\frac{m}{s^2} \times 1.5\,m \times \frac{1}{1 + 0.040 \times 50\,ft \times \frac{1}{25\,mm} \times 304.8\frac{mm}{ft} + 0.04}\right]^{1/2} = 1.08\,m/s$$

$$Q = \bar{V}_1 A = 1.08\frac{m}{s} \times \frac{\pi}{4}(0.025)^2\,m^2 = 5.30\times10^{-4}\,m^3/s \quad \text{(no diffuser)}$$ Q

The diffuser would increase head loss by $K_{diffuser} = 0.3$ (see Example 8.10), but would reduce $\bar{V}_2$ to $\frac{1}{2}\bar{V}_1$. The energy equation would be

$$gz_0 = \frac{\bar{V}_2^2}{2} + \left(f\frac{L}{D} + K_{ent} + K_{diff}\right)\frac{\bar{V}_1^2}{2} = \left(\frac{1}{4} + f\frac{L}{D} + K_{ent} + K_d\right)\frac{\bar{V}_1^2}{2}$$



Thus

$$\bar{V}_1 = \sqrt{\frac{2gz_0}{0.25 + f\frac{L}{D} + K_{ent} + K_{diff}}}$$

$$\bar{V}_1 = \left[2 \times 9.81\frac{m}{s^2} \times 1.5\,m \times \frac{1}{0.25 + 0.040 \times 50\,ft \times \frac{1}{25\,mm} \times 304.8\frac{mm}{ft} + 0.04 + 0.3}\right]^{1/2} = 1.09\,m/s$$

and

$$Q = \bar{V}_1 A = 1.09\frac{m}{s} \times \frac{\pi}{4}(0.025)^2\,m^2 = 5.35\times10^{-4}\,m^3/s \quad \text{(with diffuser)}$$ Q

{The diffuser increases flow rate only slightly (~1 percent), because loss is dominated by fL/D.}

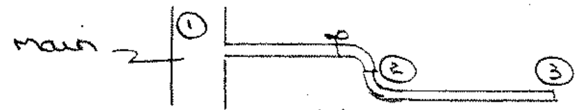
* Values of f obtained using Excel's Solver (or Goal Seek)

Problem 8.158

[Difficulty: 4] Part 1/2

8.158 You are watering your lawn with an *old* hose. Because lime deposits have built up over the years, the 0.75-in.-i.d. hose now has an average roughness height of 0.022 in. One 50-ft length of the hose, attached to your spigot, delivers 15 gpm of water (60°F). Compute the pressure at the spigot, in psi. Estimate the delivery if two 50-ft lengths of the hose are connected. Assume that the pressure at the spigot varies with flow rate and the water main pressure remains constant at 50 psig.

Solution:



Apply the energy equation for steady, incompressible flow between the spigot ② and the hose discharge ③

Basic equations: $\left(\frac{p_2}{\rho}+\alpha_2\frac{\bar{V}_2^2}{2}+gz_2\right)-\left(\frac{p_3}{\rho}+\alpha_3\frac{\bar{V}_3^2}{2}+gz_3\right)=h_{\ell_T}$ (8.29)

$$h_{\ell_T}=h_\ell+h_{\ell_m}\,,\quad h_\ell=f\frac{L}{D}\frac{\bar{V}^2}{2}$$

Assumptions: (1) $p_3=p_{atm}$ (4) Turbulent flow so $\Delta p_{1\to 2}\propto Q^2$

(2) $\bar{V}_2=\bar{V}_3$, $\alpha_2=\alpha_3=1.0$

(3) $z_2=z_3$

Then $p_2=\rho f\frac{L}{D}\frac{\bar{V}^2}{2}$ ——— (1)

$$\bar{V}=\frac{Q}{A}=\frac{4Q}{\pi D^2}=\frac{4}{\pi}\times 15\frac{gal}{min}\times\frac{min}{60\,s}\times\frac{ft^3}{7.48\,gal}\times\left(\frac{12}{0.75}\right)^2 ft^2=10.9\ ft/s$$

$$Re=\frac{D\bar{V}}{\nu}=\frac{0.75}{12}ft\times 10.9\frac{ft}{s}\times\frac{s}{1.21\times10^{-5}\,ft^2}=5.63\times10^4\quad\{\nu\text{ from Table A.7}\}$$

$$e/D=0.022/0.75=0.0293$$

From Eq. 8.37 $f=0.056^*$. From Eq. 1,

$$p_2=1.94\frac{slug}{ft^3}\times 0.056\times\frac{50\,ft}{0.75\,in}\times\frac{12\,in}{ft}\times\frac{1}{2}\times(10.9)^2\frac{ft^2}{s^2}\times\frac{lbf\cdot s^2}{slug\cdot ft}\times\frac{ft^2}{144\,in^2}$$

$p_2=35.9$ psi gage. $\longleftarrow p_2$

The pressure drop from the main ① to the spigot ② is proportional to the square of the flow rate. Obtain the loss coefficient using the energy equation between ① and ②.

$$\left(\frac{p_1}{\rho}+\alpha_1\frac{\bar{V}_1^2}{2}+gz_1\right)-\left(\frac{p_2}{\rho}+\alpha_2\frac{\bar{V}_2^2}{2}+gz_2\right)=K\frac{\bar{V}_2^2}{2}$$

Assumptions: (4) $\bar{V}_1=0$ (5) $z_1=z_2$

$$p_1-p_2=\rho\left[K\frac{\bar{V}_2^2}{2}+\frac{\bar{V}_2^2}{2}\right]=\rho\frac{\bar{V}_2^2}{2}\left[K+1\right]$$

$$K=\frac{\Delta p}{\frac{1}{2}\rho\bar{V}_2^2}-1=2(50-35.9)144\frac{lbf}{ft^2}\times\frac{ft^3}{1.94\,slug\times(10.9)^2 ft^2}\times\frac{s^2}{}\times\frac{slug\cdot ft}{lbf\cdot s^2}-1$$

$$K=16.6$$

* Value of f obtained using Excel's Solver (or Goal Seek).

To find the delivery with two hoses, again apply the energy equation from the main ① to the end ④ of the second hose

$$\left(\frac{p_1}{\rho} + \alpha_1\frac{\bar{V}_1^2}{2} + gz_1\right) - \left(\frac{p_4}{\rho} + \alpha_4\frac{\bar{V}_4^2}{2} + gz_4\right) = f\frac{L_{24}}{D}\frac{\bar{V}_4^2}{2} + K\frac{\bar{V}_4^2}{2}$$

$p_4 = p_{atm}$, $z_1 = z_4$, $\bar{V}_1 = 0$, $\alpha_4 = 1$.

$$\frac{p_1 - p_{atm}}{\rho} = \frac{p_{1g}}{\rho} = \frac{\bar{V}_4^2}{2}\left(f\frac{L_{24}}{D} + K + 1\right) \quad \text{and}$$

$$\bar{V}_4 = \left[\frac{2 p_{1g}}{\rho\left(2f\frac{L}{D} + K + 1\right)}\right]^{1/2}$$

Delivery will be reduced somewhat with two lengths of hose, but f will not change much. Assume $f \approx 0.056$ and check

$$\bar{V}_4 = \left[2 \times 50\,\frac{lbf}{in^2} \times 144\,\frac{in^2}{ft^2} \times \frac{ft^3}{1.94\,slug} \times \frac{1}{\left(0.056 \times \frac{100\,ft}{0.75\,in} \times \frac{12\,in}{ft} + 16.6 + 1\right)} \times \frac{slug \cdot ft}{lbf \cdot s^2}\right]^{1/2}$$

$\bar{V}_4 = 8.32$ ft/s.

Checking,

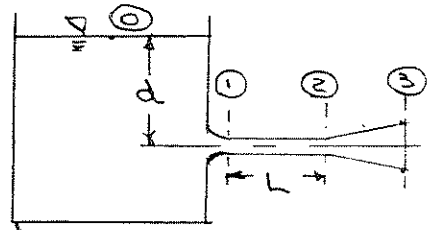
$$Re = \frac{D\bar{V}}{\nu} = \frac{0.75}{12}\,ft \times 8.32\,\frac{ft}{s} \times \frac{s}{1.21\times10^{-5}\,ft^2} = 4.30 \times 10^4, \quad \text{so } f \approx 0.056$$

Thus with two hoses,

$$Q = \bar{V}A = 8.32\,\frac{ft}{s} \times \frac{\pi}{4}\times\left(\frac{0.75}{12}\right)^2 ft^2 \times 7.48\,\frac{gal}{ft^3} \times 60\,\frac{s}{min} = 11.5\ \text{gpm} \qquad Q$$

{Similar calculations could be performed using any desired number of hose lengths.}

8.159 In Example 8.10 we found that the flow rate from a water main could be increased (by as much as 33 percent) by attaching a diffuser to the outlet of the nozzle installed into the water main. We read that the Roman water commissioner required that the tube attached to the nozzle of each customer's pipe be the same diameter for at least 50 feet from the public water main. Was the commissioner overly conservative? Using the data of the problem, estimate the length of pipe (with $e/D = 0.01$) at which the system of pipe and diffuser would give a flow rate equal to that with the nozzle alone. Plot the volume flow ratio Q/Q_i as a function of L/D, where L is the length of pipe between the nozzle and the diffuser, Q_i is the volume flow rate for the nozzle alone, and Q is the actual volume flow rate with the pipe inserted between nozzle and diffuser.



Solution:

Apply the energy equation for steady, incompressible flow between the water surface and the diffuser discharge.

Basic equations: $$\left(\frac{p_0}{\rho} + \alpha_0\frac{\bar{V}_0^2}{2} + g z_0\right) - \left(\frac{p_3}{\rho} + \alpha_3\frac{\bar{V}_3^2}{2} + g z_3\right) = h_{l_T} \quad (8.29)$$

$$h_{l_T} = h_l + h_{l_m} \; ; \; h_l = f\frac{L}{D}\frac{\bar{V}^2}{2} \; ; \; h_{l_m} = (K_{ent} + K_{diff})\frac{\bar{V}_2^2}{2}$$

Assumptions: (1) $p_0 = p_3 = p_{atm}$
(2) $\bar{V}_0 = 0$, $\alpha_3 = 1.0$
(3) water @ 20°C, $\nu = 1.00\times10^{-6}\ m^2/s$

Then,
$$g(z_0 - z_3) = gd = f\frac{L}{D}\frac{\bar{V}^2}{2} + (K_{ent} + K_{diff})\frac{\bar{V}_2^2}{2} + \frac{\bar{V}_3^2}{2}$$

From continuity $A_2\bar{V}_2 = A_3\bar{V}_3$ $\therefore$ $\bar{V}_3 = \dfrac{\bar{V}_2}{AR}$

$$gd = f\frac{L}{D}\frac{\bar{V}_2^2}{2} + \left(K_{ent} + K_{diff} + \frac{1}{AR^2}\right)\frac{\bar{V}_2^2}{2} = f\frac{L}{D}\frac{\bar{V}_2^2}{2} + 0.59\frac{\bar{V}_2^2}{2} \quad \text{--- (1)}$$

and
$$L = \frac{D}{f}\left[\frac{2gd}{\bar{V}_2^2} - 0.590\right] \quad \text{--- (2)}$$

For $\bar{V}_1 = 5.32\ m/s$, $Re = \dfrac{D\bar{V}}{\nu} = 0.025\,m \times 5.32\,\dfrac{m}{s} \times \dfrac{s}{1.00\times10^{-6}\,m^2} = 1.33\times10^5$

with $e/D = 0.01$, $f = 0.038$ (Fig. 8.13) and

$$L = \frac{0.025\,m}{0.038}\left[2\times9.81\,\frac{m}{s^2}\times1.5\,m\times\frac{s^2}{(5.32)^2\,m^2} - 0.590\right] = 0.296\,m \quad L$$

$(0.971\ ft,\ L/D = 11.8)$

This is significantly less than the 50 ft required by the water commissioner. He was extremely conservative.

Note that $Q/Q_i = \bar{V}/\bar{V}_i$ where $\bar{V}_i = 5.32\ m/s$

Increasing L reduces $\bar{V}$

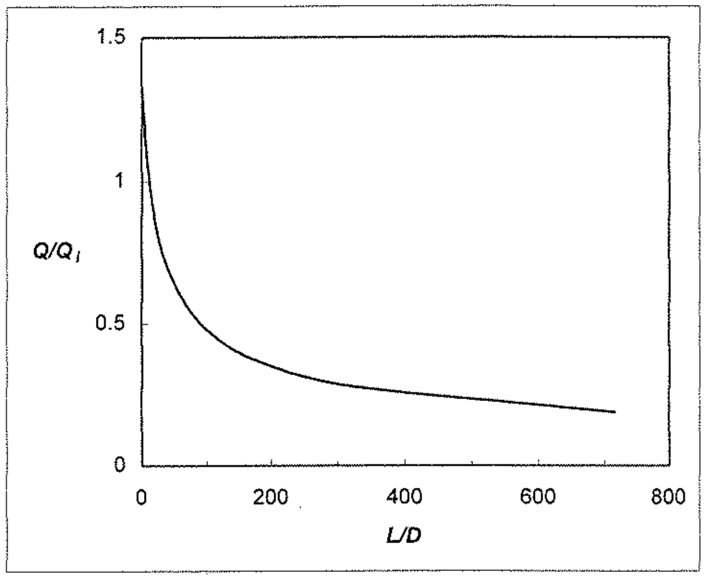
With $L = 0$ $Q/Q_i = 1.33$

$L = 0.296$ m ($L/D = 11.8$) $Q/Q_i = 1.00$

As L is increased $\bar{V}_2$ (and hence Re) will decrease; the friction factor will increase slightly from 0.038.

The plot of Q/Q_i ($\bar{V}/\bar{V}_i$) is best done by assuming values of $\bar{V}_i$ and solving Eq. 2 for L.

V_{avg}	Re	f	L/D	V/V_i
(m/s)	(---)	(---)	(---)	(---)
7.06	1.77E+05	0.0382	0.0	1.33
5.32	1.33E+05	0.0384	11.7	1.00
5.0	1.25E+05	0.0384	15.3	0.940
4.5	1.13E+05	0.0384	22.5	0.846
4.0	1.00E+05	0.0385	32.4	0.752
3.5	8.75E+04	0.0386	47.0	0.658
3.0	7.50E+04	0.0387	69.3	0.564
2.5	6.25E+04	0.0388	106	0.470
2.0	5.00E+04	0.0391	173	0.376
1.5	3.75E+04	0.0394	317	0.282
1.0	2.50E+04	0.0402	718	0.188



Problem 8.160

[Difficulty: 5]

8.160 Your boss, from the "old school," claims that for pipe flow the flow rate, $Q \propto \sqrt{\Delta p}$, where Δp is the pressure difference driving the flow. You dispute this, so perform some calculations. You take a 1-in.-diameter commercial steel pipe and assume an initial flow rate of 1.25 gal/min of water. You then increase the applied pressure in equal increments and compute the new flow rates so you can plot Q versus Δp, as computed by you and your boss. Plot the two curves on the same graph. Was your boss right?

Applying the energy equation between inlet and exit:

$$\frac{\Delta p}{\rho} = f\frac{L}{D}\frac{V^2}{2} \quad \text{or} \quad \frac{\Delta p}{L} = \frac{\rho f}{D}\frac{V^2}{2}$$

"Old school":

$$\frac{\Delta p}{L} = \left(\frac{\Delta p}{L}\right)_0 \left(\frac{Q_0}{Q}\right)^2$$

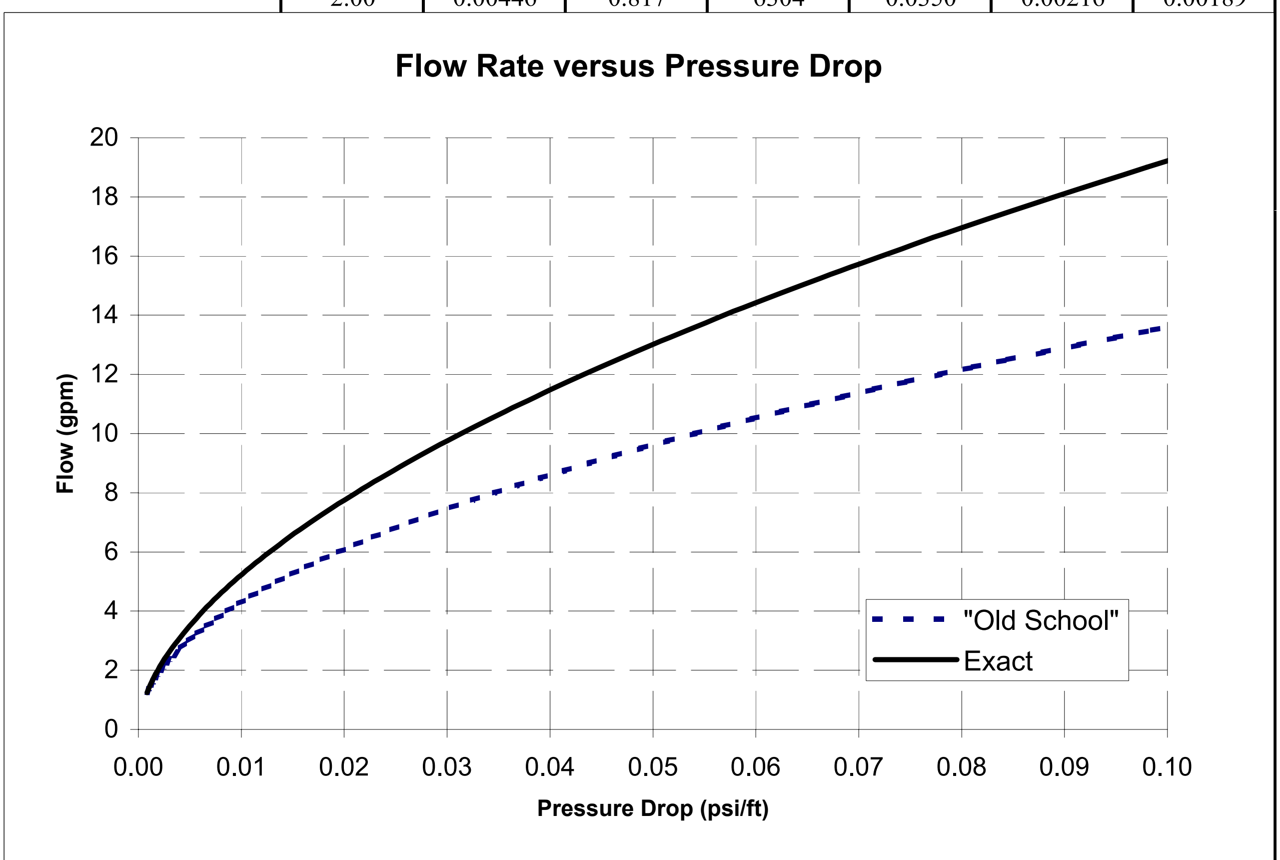
$D =$ 1 in

$e =$ 0.00015 ft

$\nu =$ 1.08E-05 ft^2/s

$\rho =$ 1.94 slug/ft^3

Q (gpm)	Q (ft^3/s)	V (ft/s)	Re	f	Δp (old school) (psi)	Δp (psi/ft)
1.25	0.00279	0.511	3940	0.0401	0.00085	0.00085
1.50	0.00334	0.613	4728	0.0380	0.00122	0.00115
1.75	0.00390	0.715	5516	0.0364	0.00166	0.00150
2.00	0.00446	0.817	6304	0.0350	0.00216	0.00189
8.75	0.01950	3.575	27582	0.0240	0.04142	0.02477
9.00	0.02005	3.677	28370	0.0238	0.04382	0.02604



Your boss was wrong!

Problem 8.161

[Difficulty: 5]

8.161 For Problem 8.146, what would the diameter of the pipe of length $2L$ need to be to generate the same flow as the pipe of length L?

Given: Flow from large reservoir

Find: Diameter for flow rates in two pipes to be same

Solution:

Basic equations $\left(\frac{p_1}{\rho} + \alpha\cdot\frac{V_1^2}{2} + g\cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha\cdot\frac{V_2^2}{2} + g\cdot z_2\right) = h_l \qquad h_l = f\cdot\frac{L}{D}\cdot\frac{V^2}{2} \qquad h_{lm} = K_{ent}\cdot\frac{V^2}{2}$

Assumptions: 1) Steady flow 2) Incompressible flow 3) α is approximately 1

Available data $D = 50\cdot mm \qquad H = 10\cdot m \qquad L = 10\cdot m$

$e = 0.15\cdot mm$ (Table 8.1) $\nu = 1\cdot10^{-6}\cdot\frac{m^2}{s}$ (Table A.8) $K_{ent} = 0.5$

For the pipe of length L the energy equation becomes

$$g\cdot(z_1 - z_2) - \frac{1}{2}\cdot V_2^2 = f\cdot\frac{L}{D}\cdot\frac{V_2^2}{2} + K_{ent}\cdot\frac{V_2^2}{2} \quad \text{and} \quad V_2 = V \qquad z_1 - z_2 = H$$

Solving for V $\qquad V = \sqrt{\frac{2\cdot g\cdot H}{f\cdot\frac{L}{D} + K_{ent} + 1}} \qquad (1)$

We also have $\qquad \frac{1}{\sqrt{f}} = -2.0\cdot\log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re\cdot\sqrt{f}}\right) \qquad (2) \qquad Re = \frac{V\cdot D}{\nu} \qquad (3)$

We must solve Eqs. 1, 2 and 3 iteratively.

Make a guess for V $\qquad V = 1\cdot\frac{m}{s}$ Then $\qquad Re = \frac{V\cdot D}{\nu} \qquad Re = 5.00\times10^4$

and $\qquad \frac{1}{\sqrt{f}} = -2.0\cdot\log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re\cdot\sqrt{f}}\right) \qquad f = 0.0286$

Then $\qquad V = \sqrt{\frac{2\cdot g\cdot H}{f\cdot\frac{L}{D} + K_{ent} + 1}} \qquad V = 5.21\,\frac{m}{s}$

Repeating $\qquad Re = \frac{V\cdot D}{\nu} \qquad Re = 2.61\times10^5$

and $\quad \dfrac{1}{\sqrt{f}} = -2.0\cdot\log\left(\dfrac{\frac{e}{D}}{3.7} + \dfrac{2.51}{Re\cdot\sqrt{f}}\right) \qquad f = 0.0267$

Then $\quad V = \sqrt{\dfrac{2\cdot g\cdot H}{f\cdot\frac{L}{D} + K_{ent} + 1}} \qquad V = 5.36\,\dfrac{m}{s}$

Repeating $\quad Re = \dfrac{V\cdot D}{\nu} \qquad Re = 2.68\times 10^5$

and $\quad \dfrac{1}{\sqrt{f}} = -2.0\cdot\log\left(\dfrac{\frac{e}{D}}{3.7} + \dfrac{2.51}{Re\cdot\sqrt{f}}\right) \qquad f = 0.0267$

Then $\quad V = \sqrt{\dfrac{2\cdot g\cdot H}{f\cdot\frac{L}{D} + K_{ent} + 1}} \qquad V = 5.36\,\dfrac{m}{s}$

Hence $\quad Q = \dfrac{\pi}{4}\cdot D^2\cdot V \qquad Q = 0.0105\,\dfrac{m^3}{s} \qquad Q = 10.5\cdot\dfrac{l}{s}$

This is the flow rate we require in the second pipe (of length 2L)

For the pipe of length 2L the energy equation becomes

$$g\cdot\left(z_1 - z_2\right) - \frac{1}{2}\cdot V_2^{\,2} = f\cdot\frac{2\cdot L}{D}\cdot\frac{V_2^{\,2}}{2} + K_{ent}\cdot\frac{V_2^{\,2}}{2} \quad \text{and} \quad V_2 = V \qquad z_1 - z_2 = H \qquad Q = 0.0105\,\frac{m^3}{s}$$

Hence $\quad H = \dfrac{V^2}{2\cdot g}\cdot\left(f\cdot\dfrac{2\cdot L}{D} + K_{ent} + 1\right) \qquad (4)$

We must make a guess for D (larger than the other pipe) $\quad D = 0.06\cdot m \quad$ Then we have $\quad V = \dfrac{Q}{\frac{\pi}{4}\cdot D^2} \qquad V = 3.72\,\dfrac{m}{s}$

Then $\quad Re = \dfrac{V\cdot D}{\nu} \qquad Re = 2.23\times 10^5$

and $\quad \dfrac{1}{\sqrt{f}} = -2.0\cdot\log\left(\dfrac{\frac{e}{D}}{3.7} + \dfrac{2.51}{Re\cdot\sqrt{f}}\right) \qquad f = 0.0256$

Using Eq 4 to find H $\quad H_{iterate} = \dfrac{V^2}{2\cdot g}\cdot\left(f\cdot\dfrac{2\cdot L}{D} + K_{ent} + 1\right) \qquad H_{iterate} = 7.07\,m \qquad$ But $\qquad H = 10\,m$

Hence the diameter is too large: Only a head of $\quad H_{iterate} = 7.07\,m \quad$ would be needed to generate the flow. We make D smaller

Try $\quad D = 0.055 \cdot m \quad$ Then we have $\quad V = \dfrac{Q}{\frac{\pi}{4} \cdot D^2} \quad V = 4.43 \dfrac{m}{s}$

Then $\quad Re = \dfrac{V \cdot D}{\nu} \quad Re = 2.43 \times 10^5$

and $\quad \dfrac{1}{\sqrt{f}} = -2.0 \cdot \log\left(\dfrac{\frac{e}{D}}{3.7} + \dfrac{2.51}{Re \cdot \sqrt{f}}\right) \quad f = 0.0261$

Using Eq 4 to find H $\quad H_{iterate} = \dfrac{V^2}{2 \cdot g} \cdot \left(f \cdot \dfrac{2 \cdot L}{D} + K_{ent} + 1\right) \quad H_{iterate} = 10.97\, m \quad$ But $\quad H = 10\, m$

Hence the diameter is too small: A head of $H_{iterate} = 10.97\, m$ would be needed. We make D slightly larger

Try $\quad D = 0.056 \cdot m \quad$ Then we have $\quad V = \dfrac{Q}{\frac{\pi}{4} \cdot D^2} \quad V = 4.27 \dfrac{m}{s}$

Then $\quad Re = \dfrac{V \cdot D}{\nu} \quad Re = 2.39 \times 10^5$

and $\quad \dfrac{1}{\sqrt{f}} = -2.0 \cdot \log\left(\dfrac{\frac{e}{D}}{3.7} + \dfrac{2.51}{Re \cdot \sqrt{f}}\right) \quad f = 0.0260$

Using Eq 4 to find H $\quad H_{iterate} = \dfrac{V^2}{2 \cdot g} \cdot \left(f \cdot \dfrac{2 \cdot L}{D} + K_{ent} + 1\right) \quad H_{iterate} = 10.02\, m \quad$ But $\quad H = 10\, m$

Hence the diameter is too large A head of $H_{iterate} = 10.02\, m$ would be needed. We can make D smaller

Try $\quad D = 0.05602 \cdot m \quad$ Then we have $\quad V = \dfrac{Q}{\frac{\pi}{4} \cdot D^2} \quad V = 4.27 \dfrac{m}{s}$

Then $\quad Re = \dfrac{V \cdot D}{\nu} \quad Re = 2.39 \times 10^5$

and $\quad \dfrac{1}{\sqrt{f}} = -2.0 \cdot \log\left(\dfrac{\frac{e}{D}}{3.7} + \dfrac{2.51}{Re \cdot \sqrt{f}}\right) \quad f = 0.0260$

Using Eq 4 to find H $\quad H_{iterate} = \dfrac{V^2}{2 \cdot g} \cdot \left(f \cdot \dfrac{2 \cdot L}{D} + K_{ent} + 1\right) \quad H_{iterate} = 10\, m \quad$ But $\quad H = 10\, m$

Hence we have $\quad D = 0.05602\, m \quad D = 56.02 \cdot mm \quad V = 4.27 \dfrac{m}{s}$

Check $\quad Q = 0.0105 \dfrac{m^3}{s} \quad \dfrac{\pi}{4} \cdot D^2 \cdot V = 0.0105 \dfrac{m^3}{s}$

Problem 8.162

[Difficulty: 3]

8.162 A hydraulic press is powered by a remote high-pressure pump. The gage pressure at the pump outlet is 3000 psi, whereas the pressure required for the press is 2750 psi (gage), at a flow rate of 0.02 ft^3/s. The press and pump are connected by 165 ft of smooth, drawn steel tubing. The fluid is SAE 10W oil at 100°F. Determine the minimum tubing diameter that may be used.

Given: Hydraulic press system

Find: Minimum required diameter of tubing

Solution:

Basic equations $$\left(\frac{p_1}{\rho} + \alpha\cdot\frac{V_1^2}{2} + g\cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha\cdot\frac{V_2^2}{2} + g\cdot z_2\right) = h_l \qquad h_l = f\cdot\frac{L}{D}\cdot\frac{V_2^2}{2}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) α at 1 and 2 is approximately 1 4) Ignore minor losses

The flow rate is low and it's oil, so try assuming laminar flow. Then, from Eq. 8.13c

$$\Delta p = \frac{128\cdot\mu\cdot Q\cdot L}{\pi\cdot D^4} \qquad \text{or} \qquad D = \left(\frac{128\cdot\mu\cdot Q\cdot L}{\pi\cdot\Delta p}\right)^{\frac{1}{4}}$$

For SAE 10W oil at 100°F (Fig. A.2, 38°C) $$\mu = 3.5\times 10^{-2}\cdot\frac{N\cdot s}{m^2}\times\frac{0.0209\frac{lbf\cdot s}{ft^2}}{1\cdot\frac{N\cdot s}{m^2}} \qquad \mu = 7.32\times 10^{-4}\cdot\frac{lbf\cdot s}{ft^2}$$

Hence $$D = \left[\frac{128}{\pi}\times 7.32\times 10^{-4}\frac{lbf\cdot s}{ft^2}\times 0.02\cdot\frac{ft^3}{s}\times 165\cdot ft\times\frac{in^2}{(3000-2750)\cdot lbf}\times\left(\frac{1\cdot ft}{12\cdot in}\right)^2\right]^{\frac{1}{4}} \qquad D = 0.0407\cdot ft \qquad D = 0.488\,in$$

Check Re to assure flow is laminar $$V = \frac{Q}{A} = \frac{4\cdot Q}{\pi\cdot D^2} \qquad V = \frac{4}{\pi}\times 0.02\cdot\frac{ft^3}{s}\times\left(\frac{12}{0.488}\cdot\frac{1}{ft}\right)^2 \qquad V = 15.4\frac{ft}{s}$$

From Table A.2 $SG_{oil} = 0.92$ so $$Re = \frac{SG_{oil}\cdot\rho_{H2O}\cdot V\cdot D}{\mu}$$

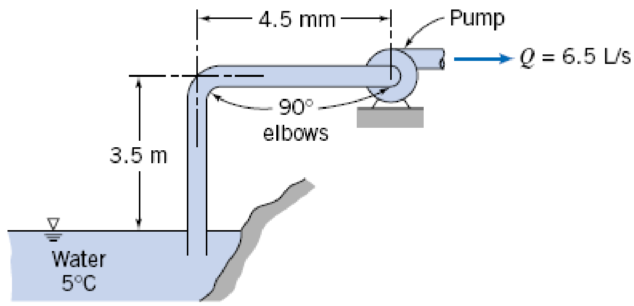
$$Re = 0.92\times 1.94\frac{slug}{ft^3}\times 15.4\frac{ft}{s}\times\frac{0.488}{12}\cdot ft\times\frac{ft^2}{7.32\times 10^{-4}\,lbf\cdot s}\times\frac{lbf\cdot s^2}{slug\cdot ft} \qquad Re = 1527$$

Hence the flow is laminar, Re < 2300. The minimum diameter is 0.488 in, so 0.5 in ID tube should be chosen

Problem 8.163

[Difficulty: 4]

8.163 A pump is located 4.5 m to one side of, and 3.5 m above a reservoir. The pump is designed for a flow rate of 6 L/s. For satisfactory operation, the static pressure at the pump inlet must not be lower than −6 m of water gage. Determine the smallest standard commercial steel pipe that will give the required performance.



Given: Flow out of reservoir by pump

Find: Smallest pipe needed

Solution:

Basic equations
$$\left(\frac{p_1}{\rho} + \alpha\cdot\frac{V_1^2}{2} + g\cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha\cdot\frac{V_2^2}{2} + g\cdot z_2\right) = h_{lT} \qquad h_{lT} = h_l + h_{lm} = f\cdot\frac{L}{D}\cdot\frac{V_2^2}{2} + K_{ent}\cdot\frac{V_2^2}{2} + f\cdot\frac{L_e}{D}\cdot\frac{V_2^2}{2}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) α at 1 and 2 is approximately 1 4) V_1 <<

Hence for flow between the free surface (Point 1) and the pump inlet (2) the energy equation becomes

$$-\frac{p_2}{\rho} - g\cdot z_2 - \frac{V_2^2}{2} = -\frac{p_2}{\rho} - g\cdot z_2 - \frac{V^2}{2} = f\cdot\frac{L}{D}\cdot\frac{V^2}{2} + K_{ent}\cdot\frac{V^2}{2} + f\cdot\frac{L_e}{D}\cdot\frac{V^2}{2} \text{ and } \qquad p = \rho\cdot g\cdot h$$

Solving for $h_2 = p_2/\rho g$ $\qquad h_2 = -z_2 - \frac{V^2}{2\cdot g}\cdot\left[f\cdot\left(\frac{L}{D} + \frac{L_e}{D}\right) + K_{ent}\right]$ $\qquad$ (1)

From Table 8.2 $\quad K_{ent} = 0.78 \quad$ for rentrant, and from Table 8.4 two standard elbows lead to $\quad \frac{L_e}{D} = 2\times 30 = 60$

We also have $\quad e = 0.046\cdot mm$ (Table 8.1) $\quad \nu = 1.51\times 10^{-6}\cdot\frac{m^2}{s}$ (Table A.8)

and we are given $\quad Q = 6\cdot\frac{L}{s} \qquad Q = 6\times 10^{-3}\frac{m^3}{s} \qquad z_2 = 3.5\cdot m \quad L = (3.5 + 4.5)\cdot m \quad L = 8\,m \quad h_2 = -6\cdot m$

Equation 1 is tricky because D is unknown, so V is unknown (even though Q is known), L/D and L_e/D are unknown, and Re and hence f are unknown! We COULD set up *Excel* to solve Eq 1, the Reynolds number, and f, simultaneously by varying D, but here we try guesses:

$$D = 2.5\cdot cm \quad V = \frac{4\cdot Q}{\pi\cdot D^2} \qquad V = 12.2\frac{m}{s} \qquad Re = \frac{V\cdot D}{\nu} \qquad Re = 2.02\times 10^5$$

Given $\quad \frac{1}{\sqrt{f}} = -2.0\cdot\log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re\cdot\sqrt{f}}\right) \qquad f = 0.0238$

$$h_2 = -z_2 - \frac{V^2}{2\cdot g}\cdot\left[f\cdot\left(\frac{L}{D} + \frac{L_e}{D}\right) + K_{ent}\right] \quad h_2 = -78.45\,m \quad \text{but we need -6 m!}$$

$D = 5 \cdot cm \quad V = \dfrac{4 \cdot Q}{\pi \cdot D^2} \qquad V = 3.06 \dfrac{m}{s} \qquad Re = \dfrac{V \cdot D}{\nu} \qquad Re = 1.01 \times 10^5$

Given $\qquad \dfrac{1}{\sqrt{f}} = -2.0 \cdot \log\left(\dfrac{\frac{e}{D}}{3.7} + \dfrac{2.51}{Re \cdot \sqrt{f}}\right) \qquad f = 0.0219$

$h_2 = -z_2 - \dfrac{V^2}{2 \cdot g} \cdot \left[f \cdot \left(\dfrac{L}{D} + \dfrac{L_e}{D}\right) + K_{ent}\right] \quad h_2 = -6.16\,m$ $\qquad$ but we need -6 m!

$D = 5.1 \cdot cm \quad V = \dfrac{4 \cdot Q}{\pi \cdot D^2} \qquad V = 2.94 \dfrac{m}{s} \qquad Re = \dfrac{V \cdot D}{\nu} \qquad Re = 9.92 \times 10^4$

Given $\qquad \dfrac{1}{\sqrt{f}} = -2.0 \cdot \log\left(\dfrac{\frac{e}{D}}{3.7} + \dfrac{2.51}{Re \cdot \sqrt{f}}\right) \qquad f = 0.0219$

$h_2 = -z_2 - \dfrac{V^2}{2 \cdot g} \cdot \left[f \cdot \left(\dfrac{L}{D} + \dfrac{L_e}{D}\right) + K_{ent}\right] \quad h_2 = -5.93\,m$

To within 1%, we can use 5-5.1 cm tubing; this corresponds to standard 2 in pipe.

Problem 8.164

[Difficulty: 4]

8.164 Determine the minimum size smooth rectangular duct with an aspect ratio of 3 that will pass 1 m^3/s of 10°C air with a head loss of 25 mm of water per 100 m of duct.

Given: Flow of air in rectangular duct

Find: Minimum required size

Solution:

Basic equations

$$\left(\frac{p_1}{\rho} + \alpha \cdot \frac{V_1^2}{2} + g \cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha \cdot \frac{V_2^2}{2} + g \cdot z_2\right) = h_l \qquad h_l = f \cdot \frac{L}{D_h} \cdot \frac{V^2}{2} \qquad D_h = \frac{4 \cdot A}{P_w}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) α at 1 and 2 is approximately 1 4) Ignore minor losses

Available data

$$Q = 1 \cdot \frac{m^3}{s} \qquad L = 100 \cdot m \qquad \Delta h = 25 \cdot mm \qquad ar = 3 \qquad e = 0 \cdot m$$

$$\rho_{H2O} = 999 \cdot \frac{kg}{m^3} \qquad \rho = 1.25 \cdot \frac{kg}{m^3} \qquad \nu = 1.41 \cdot 10^{-5} \cdot \frac{m^2}{s} \qquad \text{(Table A.10)}$$

Hence for flow between the inlet (Point 1) and the exit (2) the energy equation becomes

$$\frac{p_1}{\rho} - \frac{p_2}{\rho} = \frac{\Delta p}{\rho} = f \cdot \frac{L}{D_h} \cdot \frac{V^2}{2}$$

and $\quad \Delta p = \rho_{H2O} \cdot g \cdot \Delta h \qquad \Delta p = 245\ \text{Pa}$

For a rectangular duct $\quad D_h = \frac{4 \cdot b \cdot h}{2 \cdot (b + h)} = \frac{2 \cdot h^2 \cdot ar}{h \cdot (1 + ar)} = \frac{2 \cdot h \cdot ar}{1 + ar} \qquad$ and also $\quad A = b \cdot h = h^2 \cdot \frac{b}{h} = h^2 \cdot ar$

Hence

$$\Delta p = \rho \cdot f \cdot L \cdot \frac{V^2}{2} \cdot \frac{(1 + ar)}{2 \cdot h \cdot ar} = \rho \cdot f \cdot L \cdot \frac{Q^2}{2 \cdot A^2} \cdot \frac{(1 + ar)}{2 \cdot h \cdot ar} = \frac{\rho \cdot f \cdot L \cdot Q^2}{4} \cdot \frac{(1 + ar)}{ar^3} \cdot \frac{1}{h^5}$$

Solving for h

$$h = \left[\frac{\rho \cdot f \cdot L \cdot Q^2}{4 \cdot \Delta p} \cdot \frac{(1 + ar)}{ar^3}\right]^{\frac{1}{5}} \qquad (1)$$

Equation 1 is tricky because h is unknown, so D_h is unknown, hence V is unknown (even though Q is known), and Re and hence f are unknown! We COULD set up *Excel* to solve Eq 1, the Reynolds number, and f, simmultaneously by varying h, but here we try guesses:

$$f = 0.01 \qquad h = \left[\frac{\rho \cdot f \cdot L \cdot Q^2}{4 \cdot \Delta p} \cdot \frac{(1 + ar)}{ar^3}\right]^{\frac{1}{5}} \qquad h = 0.180\ \text{m} \qquad V = \frac{Q}{h^2 \cdot ar} \qquad V = 10.3\ \frac{m}{s}$$

$$D_h = \frac{2 \cdot h \cdot ar}{1 + ar} \qquad D_h = 0.270\,\text{m} \qquad Re = \frac{V \cdot D_h}{\nu} \qquad Re = 1.97 \times 10^5$$

Given $$\frac{1}{\sqrt{f}} = -2.0 \cdot \log\left(\frac{\frac{e}{D_h}}{3.7} + \frac{2.51}{Re \cdot \sqrt{f}}\right) \qquad f = 0.0157$$

$$h = \left[\frac{\rho \cdot f \cdot L \cdot Q^2}{4 \cdot \Delta p} \cdot \frac{(1 + ar)}{ar^3}\right]^{\frac{1}{5}} \qquad h = 0.197\,\text{m} \qquad V = \frac{Q}{h^2 \cdot ar} \qquad V = 8.59\,\frac{\text{m}}{\text{s}}$$

$$D_h = \frac{2 \cdot h \cdot ar}{1 + ar} \qquad D_h = 0.295\,\text{m} \qquad Re = \frac{V \cdot D_h}{\nu} \qquad Re = 1.8 \times 10^5$$

Given $$\frac{1}{\sqrt{f}} = -2.0 \cdot \log\left(\frac{\frac{e}{D_h}}{3.7} + \frac{2.51}{Re \cdot \sqrt{f}}\right) \qquad f = 0.0160$$

$$h = \left[\frac{\rho \cdot f \cdot L \cdot Q^2}{4 \cdot \Delta p} \cdot \frac{(1 + ar)}{ar^3}\right]^{\frac{1}{5}} \qquad h = 0.198\,\text{m} \qquad V = \frac{Q}{h^2 \cdot ar} \qquad V = 8.53\,\frac{\text{m}}{\text{s}}$$

$$D_h = \frac{2 \cdot h \cdot ar}{1 + ar} \qquad D_h = 0.297\,\text{m} \qquad Re = \frac{V \cdot D_h}{\nu} \qquad Re = 1.79 \times 10^5$$

Given $$\frac{1}{\sqrt{f}} = -2.0 \cdot \log\left(\frac{\frac{e}{D_h}}{3.7} + \frac{2.51}{Re \cdot \sqrt{f}}\right) \qquad f = 0.0160$$

Hence $$h = 0.198\,\text{m} \qquad h = 198\,\text{mm} \qquad b = 2 \cdot h \qquad b = 395 \cdot \text{mm} \qquad V = \frac{Q}{h^2 \cdot ar} \qquad V = 8.53\,\frac{\text{m}}{\text{s}}$$

$$D_h = \frac{2 \cdot h \cdot ar}{1 + ar} \qquad D_h = 0.297\,\text{m} \qquad Re = \frac{V \cdot D_h}{\nu} \qquad Re = 1.79 \times 10^5$$

In this process h and f have converged to a solution. The minimum dimensions are 198 mm by 395 mm

Problem 8.165

[Difficulty: 4]

8.165 A new industrial plant requires a water flow rate of 5.7 m^3/min. The gage pressure in the water main, located in the street 50 m from the plant, is 800 kPa. The supply line will require installation of 4 elbows in a total length of 65 m. The gage pressure required in the plant is 500 kPa. What size galvanized iron line should be installed?

Solution: Apply the energy equation for steady, incompressible flow that is uniform at each section ($\alpha \approx 1$).

Basic equation: $$\frac{p_1}{\rho} + \frac{\bar{V}_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \frac{\bar{V}_2^2}{2} + g z_2 + f\frac{L}{D}\frac{\bar{V}^2}{2} + h_{\ell m}$$

(2) and (3) above the velocity and elevation terms.

Assumptions: (1) $p_1 - p_2 = 300\ \text{kPa} = \Delta p$

(2) Fully developed flow in constant-area pipe, $\bar{V}_1 = \bar{V}_2 = \bar{V}$

(3) $z_1 = z_2$

(4) $h_{\ell m} = 4\left(\frac{L_e}{D}\right)_{elbow}\frac{\bar{V}^2}{2} = 120\frac{\bar{V}^2}{2}$ ($\frac{L_e}{D} = 30$, from Table 8.5)

Then

$$\frac{\Delta p}{\rho} = f\left(\frac{L}{D}+120\right)\frac{\bar{V}^2}{2} \quad \text{or} \quad \Delta p = \rho f\left(\frac{L}{D}+120\right)\frac{\bar{V}^2}{2}$$

Since D is unknown, iteration is required. The calculating equations are:

$$\bar{V} = \frac{Q}{A} = \frac{4Q}{\pi D^2} = \frac{4}{\pi}\times 5.7\frac{m^3}{min}\times\frac{1}{D^2 m^2}\times\frac{min}{60\,s} = \frac{0.121}{D^2}\ (m/s)$$

$$Re = \frac{\bar{V}D}{\nu} = \frac{4Q}{\pi\nu D} = \frac{4}{\pi}\times 5.7\frac{m^3}{min}\times\frac{s}{1.14\times10^{-6}\,m^2}\times\frac{1}{D\,m}\times\frac{min}{60\,s} = \frac{1.06\times10^5}{D} \quad (T = 15°C)$$

$e = 0.15$ mm (Table 8.1), f from Eq. 8.37*, $L = 65$ m. D from Table 8.5.

D (nom.)	D (m)	$\bar{V}$ (m/s)	Re (–)	e/D (–)	f (–)	L/D (–)	Δp (kPa)
3	0.0779	19.9	1.36×10^6	0.0019	0.024	834	4530
5	0.128	7.39	8.29×10^5	0.0012	0.021	508	360
6	0.154	5.10	6.89×10^5	0.001	0.020	422	141

Pipe friction calculations are accurate only within about ± 10 percent. Line resistance (and consequently Δp) will increase with age.

Recommend installation of 6 in. (nominal) line. D

* Values of f obtained using Excel's Solver (or Goal Seek)

Problem 8.166

[Difficulty: 4]

8.166 Air at 40°F flows in a horizontal square cross-section duct made from commercial steel. The duct is 1000 ft long. What size (length of a side) duct is required to convey 1500 cfm of air with a pressure drop of 0.75 in. H_2O?

Given: Flow of air in square duct

Find: Minimum required size

Solution:

Basic equations
$$\left(\frac{p_1}{\rho} + \alpha \cdot \frac{V_1^{\,2}}{2} + g \cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha \cdot \frac{V_2^{\,2}}{2} + g \cdot z_2\right) = h_l \quad h_l = f \cdot \frac{L}{D_h} \cdot \frac{V^2}{2} \qquad D_h = \frac{4 \cdot A}{P_w}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) α at 1 and 2 is approximately 1 4) Ignore minor losses

Available data $Q = 1500 \cdot \text{cfm}$ $\quad L = 1000 \cdot \text{ft}$ $\quad e = 0.00015 \cdot \text{ft}$ (Table 8.1) $\quad \Delta h = 0.75 \cdot \text{in}$

$\rho_{H2O} = 1.94 \cdot \frac{\text{slug}}{\text{ft}^3}$ $\quad \nu = 1.47 \cdot 10^{-4} \cdot \frac{\text{ft}^2}{\text{s}}$ $\quad \rho = 0.00247 \cdot \frac{\text{slug}}{\text{ft}^3}$ (Table A.9)

Hence for flow between the inlet (Point 1) and the exit (2) the energy equation becomes

$$\frac{p_1}{\rho} - \frac{p_2}{\rho} = \frac{\Delta p}{\rho} = f \cdot \frac{L}{D_h} \cdot \frac{V^2}{2} \quad \text{and} \quad \Delta p = \rho_{H2O} \cdot g \cdot \Delta h \qquad \Delta p = 3.90 \frac{\text{lbf}}{\text{ft}^2} \qquad \Delta p = 0.0271 \cdot \text{psi}$$

For a square duct $D_h = \frac{4 \cdot h \cdot h}{2 \cdot (h + h)} = h$ $\quad$ and also $\quad A = h \cdot h = h^2$

Hence
$$\Delta p = \rho \cdot f \cdot L \cdot \frac{V^2}{2 \cdot h} = \rho \cdot f \cdot L \cdot \frac{Q^2}{2 \cdot h \cdot A^2} = \frac{\rho \cdot f \cdot L \cdot Q^2}{2 \cdot h^5}$$

Solving for h
$$h = \left(\frac{\rho \cdot f \cdot L \cdot Q^2}{2 \cdot \Delta p}\right)^{\frac{1}{5}} \qquad (1)$$

Equation 1 is tricky because h is unknown, so D_h is unknown, hence V is unknown (even though Q is known), and Re and hence f are unknown! We COULD set up *Excel* to solve Eq 1, the Reynolds number, and f, simmultaneously by varying h, but here we try guesses:

$f = 0.01$ $\qquad h = \left(\frac{\rho \cdot f \cdot L \cdot Q^2}{2 \cdot \Delta p}\right)^{\frac{1}{5}}$ $\qquad h = 1.15 \cdot \text{ft}$ $\qquad V = \frac{Q}{h^2}$ $\qquad V = 19.0 \cdot \frac{\text{ft}}{\text{s}}$

$D_h = h$ $\qquad D_h = 1.15 \cdot \text{ft}$ $\qquad Re = \frac{V \cdot D_h}{\nu}$ $\qquad Re = 1.48 \times 10^5$

Given $\frac{1}{\sqrt{f}} = -2.0 \cdot \log\left(\frac{\frac{e}{D_h}}{3.7} + \frac{2.51}{Re \cdot \sqrt{f}}\right)$ $f = 0.0174$

$h = \left(\frac{\rho \cdot f \cdot L \cdot Q^2}{2 \cdot \Delta p}\right)^{\frac{1}{5}}$ $h = 1.28 \cdot ft$ $V = \frac{Q}{h^2}$ $V = 15.2 \cdot \frac{ft}{s}$

$D_h = h$ $D_h = 1.28 \cdot ft$ $Re = \frac{V \cdot D_h}{\nu}$ $Re = 1.33 \times 10^5$

Given $\frac{1}{\sqrt{f}} = -2.0 \cdot \log\left(\frac{\frac{e}{D_h}}{3.7} + \frac{2.51}{Re \cdot \sqrt{f}}\right)$ $f = 0.0177$

$h = \left(\frac{\rho \cdot f \cdot L \cdot Q^2}{2 \cdot \Delta p}\right)^{\frac{1}{5}}$ $h = 1.28 \cdot ft$ $V = \frac{Q}{h^2}$ $V = 15.1 \cdot \frac{ft}{s}$

$D_h = h$ $D_h = 1.28 \cdot ft$ $Re = \frac{V \cdot D_h}{\nu}$ $Re = 1.32 \times 10^5$

Given $\frac{1}{\sqrt{f}} = -2.0 \cdot \log\left(\frac{\frac{e}{D_h}}{3.7} + \frac{2.51}{Re \cdot \sqrt{f}}\right)$ $f = 0.0177$

Hence $h = 1.28 \cdot ft$ $D_h = h$ $D_h = 1.28 \cdot ft$ $V = \frac{Q}{h^2}$ $V = 15.1 \cdot \frac{ft}{s}$

$Re = \frac{V \cdot D_h}{\nu}$ $Re = 1.32 \times 10^5$

In this process h and f have converged to a solution. The minimum dimensions are 1.28 ft square (15.4 in square)

Problem 8.167

[Difficulty: 3]

8.167 Investigate the effect of tube diameter on water flow rate by computing the flow generated by a pressure difference, $\Delta p = 100$ kPa, applied to a length $L = 100$ m of smooth tubing. Plot the flow rate against tube diameter for a range that includes laminar and turbulent flow.

Given:
Flow in a tube

Find:
Effect of diameter; Plot flow rate versus diameter

Solution:

Basic equations:

$$\left(\frac{p_1}{\rho} + \alpha_1 \cdot \frac{V_1^{\,2}}{2} + g \cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \cdot \frac{V_2^{\,2}}{2} + g \cdot z_2\right) = h_l \qquad (8.29)$$

$$Re = \frac{\rho \cdot V \cdot D}{\mu} \qquad h_l = f \cdot \frac{L}{D} \cdot \frac{V^2}{2} \qquad (8.34)$$

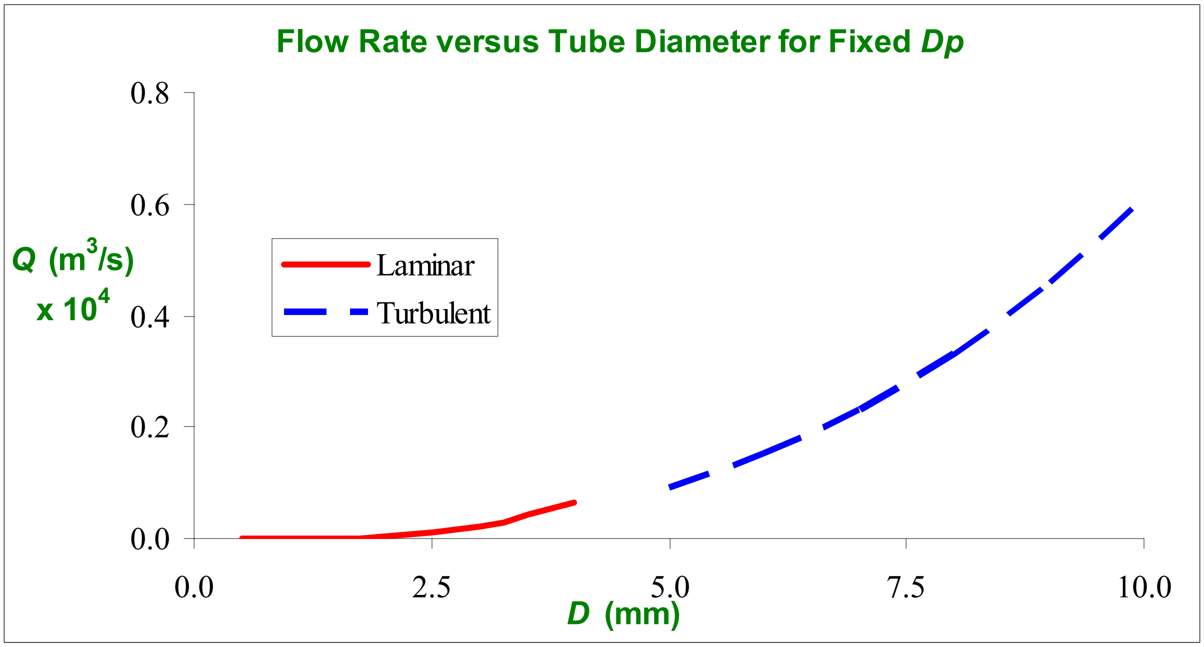
$$f = \frac{64}{Re} \qquad (8.36) \qquad \text{(Laminar)} \qquad \frac{1}{\sqrt{f}} = -2.0 \cdot \log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re \cdot \sqrt{f}}\right) \qquad (8.37) \qquad \text{(Turbulent)}$$

The energy equation (Eq. 8.29) becomes for flow in a tube

$$p_1 - p_2 = \Delta p = \rho \cdot f \cdot \frac{L}{D} \cdot \frac{V^2}{2}$$

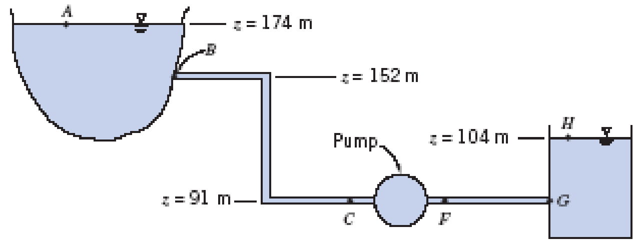
This cannot be solved explicitly for velocity V (and hence flow rate Q), because f depends on V; solution for a given diameter D requires iteration (or use of *Solver*)

Given data:				Tabulated or graphical data:			
Δp =	100	kPa		μ =	1.00E-03	N.s/m^2	
L =	100	m		ρ =	999	kg/m^3	
					(Water - Appendix A)		
Computed results:							
***D* (mm)**	***V* (m/s)**	***Q* (m^3/s) x 10^4**	***Re***	**Regime**	***f***	**Δp (kPa)**	**Error**
0.5	0.00781	0.0000153	4	Laminar	16.4	100	0.0%
1.0	0.0312	0.000245	31	Laminar	2.05	100	0.0%
2.0	0.125	0.00393	250	Laminar	0.256	100	0.0%
3.0	0.281	0.0199	843	Laminar	0.0759	100	0.0%
4.0	0.500	0.0628	1998	Laminar	0.0320	100	0.0%
5.0	0.460	0.0904	2300	Turbulent	0.0473	100	0.2%
6.0	0.530	0.150	3177	Turbulent	0.0428	100	0.0%
7.0	0.596	0.229	4169	Turbulent	0.0394	100	0.0%
8.0	0.659	0.331	5270	Turbulent	0.0368	100	0.0%
9.0	0.720	0.458	6474	Turbulent	0.0348	100	0.0%
10.0	0.778	0.611	7776	Turbulent	0.0330	100	0.0%



Flow Rate versus Tube Diameter for Fixed *Dp*
Q (m3/s) x 104
0.8
0.6
0.4
0.2
0.0
Laminar
Turbulent
0.0
2.5
5.0
7.5
10.0
D (mm)

8.169 A large reservoir supplies water for a community. A portion of the water supply system is shown. Water is pumped from the reservoir to a large storage tank before being sent on to the water treatment facility. The system is designed to provide 1310 L/s of water at 20°C. From B to C the system consists of a square-edged entrance, 760 m of pipe, three gate valves, four 45° elbows, and two 90° elbows. Gage pressure at C is 197 kPa. The system between F and G contains 760 m of pipe, two gate valves, and four 90° elbows. All pipe is 508 mm diameter, cast iron. Calculate the average velocity of water in the pipe, the gage pressure at section F, the power input to the pump (its efficiency is 80 percent), and the wall shear stress in section FG.



Solution:

Since $Q = A\bar{V}$, $\bar{V} = \frac{Q}{A} = \frac{4Q}{\pi D^2} = \frac{4}{\pi} \times 1310\frac{L}{s} \times \frac{1}{(0.508m)^2} \times 10^{-3}\frac{m^3}{L} = 6.46\, m/s$ $\bar{V}$

To determine the pressure at point F, apply the energy equation for steady, incompressible flow between F and G.

Basic equation: $\left(\frac{p_F}{\rho} + \alpha\frac{\bar{V}_F^2}{2} + gz_F\right) - \left(\cancel{\frac{p_H}{\rho}} + \cancel{\alpha\frac{\bar{V}_H^2}{2}} + gz_H\right) = h_{l_T}$ (8.29)

$h_{l_T} = h_l + h_{l_m}$, $h_l = f\frac{L}{D}\frac{\bar{V}^2}{2}$, $h_{l_m} = \frac{\bar{V}^2}{2}\sum f\frac{L_e}{D} + \frac{\bar{V}^2}{2}K_{exit}$

Assume: (1) $\bar{V}_H = 0$ (large storage tank) (2) $p_H = p_{atm}$
(3) $\alpha_F \approx 1.0$.

Then $\frac{p_F}{\rho} = h_{l_T} + g(z_H - z_F) - \frac{\bar{V}_F^2}{2}$

$\frac{p_F}{\rho} = h_{l_{F-G}} + 2h_{l_{m_{gv}}} + 4h_{l_{m_{90°el}}} + h_{l_{m_{exit}}} + g(z_H - z_F) - \frac{\bar{V}_F^2}{2}$

$\frac{p_F}{\rho} = f\frac{L}{D}\frac{\bar{V}^2}{2} + 2f\left(\frac{L_e}{D}\right)_{gv}\frac{\bar{V}^2}{2} + 4f\left(\frac{L_e}{D}\right)_{90°el}\frac{\bar{V}^2}{2} + K_{exit}\frac{\bar{V}^2}{2} + g(z_H - z_F) - \frac{\bar{V}_F^2}{2}$ ----(1)

From Table 8.4 $\left(\frac{L_e}{D}\right)_{gv} = 8$, $\left(\frac{L_e}{D}\right)_{90°el} = 30$; also $K_{exit} = 1$

$Re = \frac{D\bar{V}}{\nu} = 0.508m \times 6.46\frac{m}{s} \times \frac{s}{1.00\times10^{-6}m^2} = 3.28\times10^6$ (ν from Table A.8)

From Table 8.1, $e = 0.26\,mm$ $\therefore e/D = 0.00051$

From Eq. 8.37, $f = 0.017$ (using Excel's Solver [or Goal Seek])

From Eq (1)

$\frac{p_F}{\rho} = f\frac{\bar{V}^2}{2}\left[\frac{L}{D} + 2\left(\frac{L_e}{D}\right)_{gv} + 4\left(\frac{L_e}{D}\right)_{90°el}\right] + g(z_H - z_F)$

$$\frac{p_F}{\rho} = f\frac{\bar{V}^2}{2}\left[\frac{760}{0.508} + 2(8) + 4(30)\right] + g(z_H - z_F) = f\frac{\bar{V}^2}{2}(1630) + g(z_H - z_F)$$

$$p_F = \rho\left[1630\, f\frac{\bar{V}^2}{2} + g(z_H - z_F)\right]$$

$$= 999\frac{kg}{m^3}\left[\frac{1630}{2} \times 0.017 \times (6.46)^2\frac{m^2}{s^2} + 9.81\frac{m}{s^2}(104-91)m\right] \times \frac{N \cdot s^2}{kg \cdot m}$$

$p_F = 705$ kPa (gage) ——— p_F

For fully developed flow in a pipe $\tau = \frac{r}{2}\frac{\partial p}{\partial x}$ (8.15)

At the pipe centerline, $\tau = 0$ ——— $\tau_{\mathbb{C}}$

To determine the power input to the fluid apply the energy equation across the pump. Assuming 100% efficiency

$$\frac{\dot{W}_{pump}}{\dot{m}} = \left(\frac{p}{\rho} + \frac{\bar{V}^2}{2} + gz\right)_{discharge} - \left(\frac{p}{\rho} + \frac{\bar{V}^2}{2} + gz\right)_{suction} \quad (8.47)$$

$$\dot{W}_{pump} = \left(\frac{p_F}{\rho} - \frac{p_C}{\rho}\right)\rho A\bar{V} = (p_F - p_C)Q$$

$$\dot{W}_{pump} = (705 - 197) \times 10^3\frac{N}{m^2} \times 1310\frac{L}{s} \times 10^{-3}\frac{m^3}{L} = 6.65 \times 10^5\frac{N \cdot m}{s}$$

The actual pump input, $\dot{W}_{pump})_{act} = \dot{W}_{pump})_{ideal}/\eta$

$$\dot{W}_{pump})_{actual} = 8.32 \times 10^5\, N\cdot m/s = 832\text{ kW}$$ ——— $\dot{W}_{actual}$

From Eq. 8.15 $\tau_w = \frac{R}{2}\frac{\partial p}{\partial x}$

Along the pipe from F to G $\frac{\Delta p}{\rho} = f\frac{L}{D}\frac{\bar{V}^2}{2}$

$$\therefore \frac{\partial p}{\partial x} = \frac{\Delta p}{L} = \rho\frac{f}{D}\frac{\bar{V}^2}{2} = 999\frac{kg}{m^3} \times \frac{0.017}{0.508\, m} \times \frac{1}{2} \times (6.46)^2\frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m}$$

$$\frac{\partial p}{\partial x} = 698\ N/m^2/m$$

$$\therefore \tau_w = \frac{R}{2}\frac{\partial p}{\partial x} = \frac{0.254\, m}{2} \times 698\frac{N}{m^3} = 88.6\ N/m^2$$ ——— τ_w

Problem 8.170

[Difficulty: 3]

8.170 An air-pipe friction experiment consists of a smooth brass tube with 63.5 mm inside diameter; the distance between pressure taps is 1.52 m. The pressure drop is indicated by a manometer filled with Meriam red oil. The centerline velocity U is measured with a pitot cylinder. At one flow condition, $U = 23.1$ m/s and the pressure drop is 12.3 mm of oil. For this condition, evaluate the Reynolds number based on average flow velocity. Calculate the friction factor and compare with the value obtained from Eq. 8.37 (use $n = 7$ in the power-law velocity profile).

Solution:

Apply the energy equation for steady, incompressible flow along the pipe

Basic equation: $\left(\frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2} + g z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \frac{\bar{V}_2^2}{2} + g z_2\right) = h_{l_T}$ (8.29)

$h_l = f \frac{L}{D} \frac{\bar{V}^2}{2}$

Computing equation: $\frac{\bar{V}}{U} = \frac{2n^2}{(n+1)(2n+1)}$ (8.24)

Assumptions: (1) power law profile, $n = 7$

(2) $\alpha_1 = \alpha_2$, $z_1 = z_2$

(3) air at $T = 15^\circ C$ $\nu = 1.46 \times 10^{-5}\ m^2/s$ (Table A.10)

From Eq. 8.24 with $n = 7$

$$\frac{\bar{V}}{U} = \frac{2(7)^2}{(8)(15)} = 0.817$$

$$Re_{\bar{V}} = \frac{D\bar{V}}{\nu} = 0.0635\,m \times 0.817 \times 23.1\frac{m}{s} \times \frac{s}{1.45\times10^{-5}\,m^2} = 8.26 \times 10^4 \quad Re_{\bar{V}}$$

From Eq 8.29 $\Delta p / \rho = f \frac{L}{D} \frac{\bar{V}^2}{2}$

and

$$f = \frac{\Delta p}{\rho \frac{L}{D} \frac{\bar{V}^2}{2}} = \frac{\rho_{oil}\, g\, \Delta h}{\rho_{air} \frac{L}{D} \frac{\bar{V}^2}{2}} = \frac{2\, \rho_{H_2O}\, SG_{oil}\, g\, \Delta h}{\rho_{air} \frac{L}{D} \bar{V}^2} \qquad \left\{ \begin{matrix} SG = 0.827 \\ \text{Table A.1} \end{matrix} \right\}$$

$$f = 2 \times \frac{10^3}{1.23} \times 0.827 \times 9.81\frac{m}{s^2} \times 0.0123\,m \times \frac{0.0635\,m}{1.52\,m} \times \frac{s^2}{(0.817\times 23.1)^2\, m^2}$$

$$f = 0.0190 \quad f$$

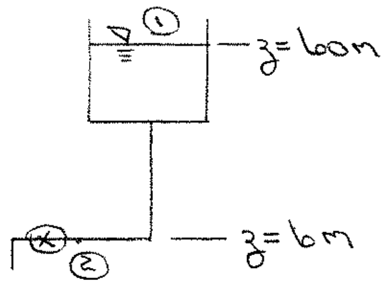
From Eq. 8.37 at $Re = 8.26 \times 10^4$ for smooth tube, $f = 0.0187$

The value of f is obtained using Excel's Solver (or Goal Seek)

Problem 8.171

8.171 Oil has been flowing from a large tank on a hill to a tanker at the wharf. The compartment in the tanker is nearly full and an operator is in the process of stopping the flow. A valve on the wharf is closed at a rate such that 1 MPa is maintained in the line immediately upstream of the valve. Assume:

Length of line from tank to valve	3 km
Inside diameter of line	200 mm
Elevation of oil surface in tank	60 m
Elevation of valve on wharf	6 m
Instantaneous flow rate	2.5 m^3/min
Head loss in line (exclusive of valve being closed) at this rate of flow	23 m of oil
Specific gravity of oil	0.88



Calculate the initial instantaneous rate of change of volume flow rate.

Solution: For unsteady flow with friction, we modify the unsteady Bernoulli equation (Eq. 6.21) to include a head loss term.

Computing equation: $$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2 + \int_1^2 \frac{\partial V_s}{\partial t} ds + h_\ell$$

Assume: (1) $V_1 \simeq 0$ (2) $p_1 = p_{atm}$ (3) ρ = constant

Then
$$\int_1^2 \frac{\partial V_s}{\partial t} ds = \frac{p_1 - p_2}{\rho} + g(z_1 - z_2) - h_\ell - \frac{V_2^2}{2}$$

If we neglect velocity in the tank except for small region near the inlet to the pipe, then

$$\int_1^2 \frac{\partial V_s}{\partial t} ds \approx \int_0^L \frac{\partial V_s}{\partial t} ds$$. Since $V_s = V_2$ everywhere, then

$$\int_0^L \frac{\partial V_s}{\partial t} ds = L \frac{dV_2}{dt}$$ and

$$\frac{dV_2}{dt} = \frac{1}{L}\left[\frac{p_1 - p_2}{\rho} + g(z_1 - z_2) - h_\ell - \frac{V_2^2}{2}\right], \quad V_2 = \frac{Q}{A} = \frac{4Q}{\pi D^2}$$

Note $h_\ell = h_\ell(V)$ and hence this result can only be used to obtain the initial instantaneous rate of change of flow velocity

$$\therefore \left.\frac{dV_2}{dt}\right)_{initial} = \frac{1}{3\times10^3\,m}\left[10^6 \frac{N}{m^2} \times \frac{m^3}{999\,kg} \times \frac{1}{0.88}\, \frac{kg \cdot m}{N \cdot s^2} + 9.81\frac{m}{s^2} \times 54m - 23m \times 9.81\frac{m}{s^2} - \frac{1}{2}\left\{\frac{4}{\pi} \times 2.5\frac{m^3}{min} \times \frac{1}{(0.2m)^2} \times \frac{1\,min}{60s}\right\}^2\right]$$

$$\left.\frac{dV_2}{dt}\right)_{initial} = -0.278 \text{ m/s/s}$$

The instantaneous rate of change of volume flow rate is

$$dQ/dt = \frac{d}{dt}(AV) = A\frac{dV}{dt} = \frac{\pi D^2}{4}\frac{dV}{dt}$$

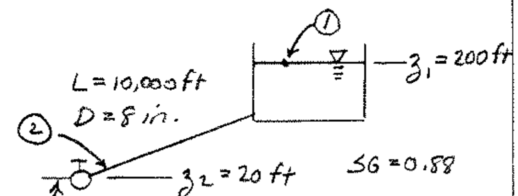
$$dQ/dt = \frac{\pi}{4}(0.2m)^2 \times \left(-0.278 \frac{m/s}{s} \times \frac{60s}{min}\right) = -0.524 \text{ m}^3\text{/s/min}$$ $\longleftarrow$ dQ/dt

Problem 8.172

[Difficulty: 5] Part 1/2

8.172 Problem 8.171 describes a situation in which flow in a long pipeline from a hilltop tank is slowed gradually to avoid a large pressure rise. Expand this analysis to predict and plot the closing schedule (valve loss coefficient versus time) needed to maintain the maximum pressure at the valve at or below a given value throughout the process of stopping the flow from the tank.

Solution: Apply the unsteady Bernoulli equation with a head loss term added.



$Q_0 = 1.5\ ft^3/s$

Line loss = 75 ft of oil

Computing equation:

$$\overset{\approx 0}{\frac{p_1}{\rho}} + \overset{\approx 0(1)}{\frac{V_1^2}{2}} + g z_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + g z_2 + \int_1^2 \frac{\partial V}{\partial t} ds + h_{\ell T}$$

Assume: (1) $V_1 \approx 0$ (3) ρ = constant

(2) $p_1 = p_{atm}$

At the initial condition, $V = \frac{Q}{A} = \frac{4Q}{\pi D^2} = \frac{4}{\pi} \times \left(\frac{12}{8}\right)^2 \frac{1}{ft^2} \times 1.5 \frac{ft^3}{s} = 4.30\ ft/s$

$$H_{\ell T} = 75\ ft = \frac{h_{\ell T}}{g} = f \frac{L}{D} \frac{V^2}{2g}\ ; \quad f\frac{L}{D} = H_{\ell T} \frac{2g}{V^2} = 2 \times 75\ ft \times 32.2 \frac{ft}{s^2} \times \frac{s^2}{(4.30)^2 ft^2} = 261$$

Neglecting velocity in tank, $\int_1^2 \frac{\partial V}{\partial t} ds \approx \int_0^L \frac{\partial V}{\partial t} ds = \frac{dV}{dt} L$

Thus $\frac{dV}{dt} = \frac{1}{L}\left[-\frac{p_2}{\rho} + g(z_1 - z_2) - f\frac{L}{D}\frac{V^2}{2} - \frac{V^2}{2}\right]$

Substituting values,

$$\frac{dV}{dt} = \frac{1}{10{,}000\ ft}\left[-150 \frac{lbf}{in.^2} \times \frac{ft^3}{(0.88)1.94\ slug} \times 144 \frac{in.^2}{ft^2} \times \frac{slug \cdot ft}{lbf \cdot s^2} + 32.2 \frac{ft}{s^2}(200 - 20)\ ft - (261 + 1)\frac{V^2}{2}\right]$$

$$\frac{dV}{dt} = -0.686 \frac{ft}{s^2} - 0.0131 V^2 = -(a^2 + b^2 V^2); \quad a = \sqrt{0.686} = 0.828\ ; V \text{ in ft/s}$$

$$b = \sqrt{0.0131} = 0.114$$

Separating variables and integrating

$$\int_{V_0}^{V} \frac{dV}{a^2 + b^2 t} = \frac{1}{ab} \tan^{-1} \frac{bV}{a}\Big]_{V_0}^{V} = \frac{1}{ab}\left[\tan^{-1}\frac{bV}{a} - \tan^{-1}\frac{bV_0}{a}\right] = -\int_0^t dt = -t$$

Thus

$$\tan^{-1}\frac{bV}{a} = -abt + \tan^{-1}\frac{bV_0}{a} \quad \text{or} \quad V = \frac{a}{b}\tan\left[\tan^{-1}\frac{bV_0}{a} - abt\right]$$ $V(t)$

The pressure must drop across the valve:

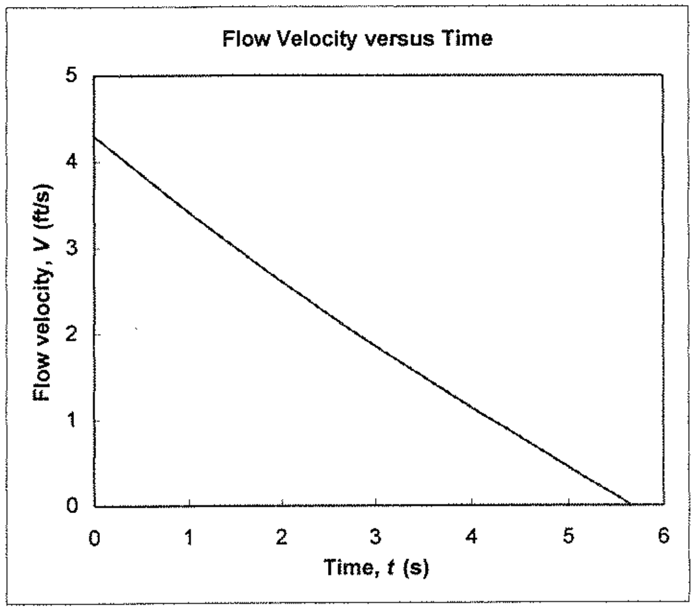
$$\frac{p_2}{\rho} + \cancel{\frac{V_2^2}{2}} + \cancel{g z_2} - \left(\frac{p_3}{\rho} + \cancel{\frac{V_3^2}{2}} + \cancel{g z_3}\right) = h_{\ell T} = K_v \frac{V^2}{2} \quad \text{or} \quad K_v = \frac{2(p_2 - p_3)}{\rho V^2} \approx \frac{2 p_2}{\rho V^2}$$

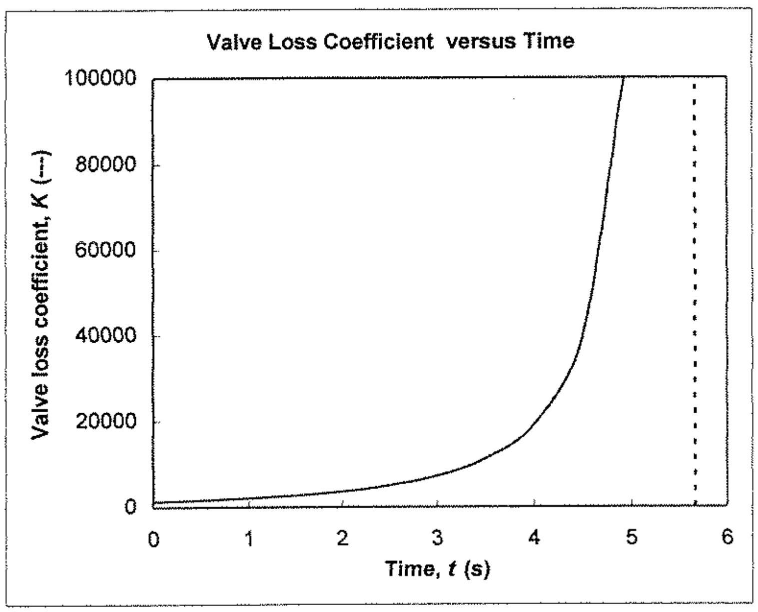
$$\text{At } t = 0,\ K_v = 2 \times 150 \frac{lbf}{in.^2} \times \frac{ft^3}{(0.88)1.94\ slug} \times \frac{s^2}{(4.3)^2 ft^2} \times 144 \frac{in.^2}{ft^2} \times \frac{slug \cdot ft}{lbf \cdot s^2} = 1{,}370 \quad (t = 0)$$ $K_v(0)$

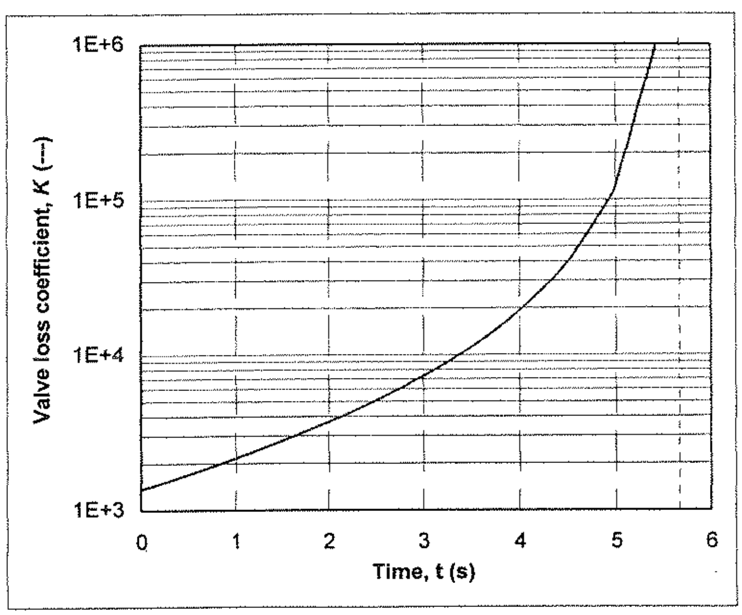
Calculations and plots are shown on the spreadsheet, next page.

Problem 8.172

	a =	0.828	(---)
	b =	0.114	(---)
	V_0 =	4.3	(ft/s)
t (s)	***V*** (ft/s)	***K*** (---)	
0.0	4.300	1.37E+3	
0.5	3.849	1.71E+3	
1.0	3.421	2.16E+3	
1.5	3.011	2.79E+3	
2.0	2.616	3.70E+3	
2.5	2.235	5.07E+3	
3.0	1.865	7.28E+3	
3.5	1.504	1.12E+4	
4.0	1.150	1.92E+4	
4.5	0.801	3.95E+4	
4.93	0.503	1.00E+5	
5.0	0.455	1.22E+5	
5.431	0.159	1.00E+6	
5.5	0.112	2.03E+6	
5.663	0.000	2.92E+13	







Problem 8.173

[Difficulty: 2]

8.173 A pump draws water at a steady flow rate of 25 lbm/s through a piping system. The pressure on the suction side of the pump is −2.5 psig. The pump outlet pressure is 50 psig. The inlet pipe diameter is 3 in.; the outlet pipe diameter is 2 in. The pump efficiency is 70 percent. Calculate the power required to drive the pump.

Given: Flow through water pump

Find: Power required

Solution:

Basic equations
$$h_{pump} = \left(\frac{p_d}{\rho} + \frac{V_d^2}{2} + g\cdot z_d\right) - \left(\frac{p_s}{\rho} + \frac{V_s^2}{2} + g\cdot z_s\right) \qquad V = \frac{Q}{A} = \frac{4\cdot Q}{\pi\cdot D^2}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) Uniform flow

Hence for the inlet
$$V_s = \frac{4}{\pi} \times 25\cdot\frac{\text{lbm}}{\text{s}} \times \frac{1\cdot\text{slug}}{32.2\cdot\text{lbm}} \times \frac{\text{ft}^3}{1.94\cdot\text{slug}} \times \left(\frac{12}{3}\cdot\frac{1}{\text{ft}}\right)^2 \qquad V_s = 8.15\cdot\frac{\text{ft}}{\text{s}} \qquad p_s = -2.5\cdot\text{psi}$$

For the outlet
$$V_d = \frac{4}{\pi} \times 25\cdot\frac{\text{lbm}}{\text{s}} \times \frac{1\cdot\text{slug}}{32.2\cdot\text{lbm}} \times \frac{\text{ft}^3}{1.94\cdot\text{slug}} \times \left(\frac{12}{2}\cdot\frac{1}{\text{ft}}\right)^2 \qquad V_d = 18.3\cdot\frac{\text{ft}}{\text{s}} \qquad p_d = 50\cdot\text{psi}$$

Then
$$h_{pump} = \frac{p_d - p_s}{\rho} + \frac{V_d^2 - V_s^2}{2} \qquad \text{and} \qquad W_{pump} = m_{pump}\cdot h_{pump}$$

$$W_{pump} = m_{pump}\cdot\left(\frac{p_d - p_s}{\rho} + \frac{V_d^2 - V_s^2}{2}\right)$$

Note that the software cannot render a dot, so the power is W_{pump} and mass flow rate is m_{pump}!

$$W_{pump} = 25\cdot\frac{\text{lbm}}{\text{s}} \times \frac{1\cdot\text{slug}}{32.2\cdot\text{lbm}} \times \left[(50 - -2.5)\cdot\frac{\text{lbf}}{\text{in}^2} \times \left(\frac{12\cdot\text{in}}{1\cdot\text{ft}}\right)^2 \times \frac{\text{ft}^3}{1.94\cdot\text{slug}} + \frac{1}{2} \times \left(18.3^2 - 8.15^2\right)\cdot\left(\frac{\text{ft}}{\text{s}}\right)^2 \times \frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}}\right] \times \frac{1\cdot\text{hp}}{550\cdot\frac{\text{ft}\cdot\text{lbf}}{\text{s}}}$$

$W_{pump} = 5.69\cdot\text{hp}$ For an efficiency of $\eta = 70\cdot\%$ $\qquad W_{required} = \frac{W_{pump}}{\eta} \qquad W_{required} = 8.13\cdot\text{hp}$

Problem 8.174

[Difficulty: 1]

8.174 The pressure rise across a water pump is 35 psi when the volume flow rate is 500 gpm. If the pump efficiency is 80 percent, determine the power input to the pump.

Given: Flow through water pump

Find: Power required

Solution:

Basic equations

$$h_{pump} = \left(\frac{p_d}{\rho} + \frac{V_d^2}{2} + g\cdot z_d\right) - \left(\frac{p_s}{\rho} + \frac{V_s^2}{2} + g\cdot z_s\right)$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) Uniform flow

In this case we assume $D_s = D_d$ so $V_s = V_d$

The available data is $\Delta p = 35\cdot psi$ $Q = 500\cdot gpm$ $\eta = 80\cdot\%$

Then $h_{pump} = \frac{p_d - p_s}{\rho} = \frac{\Delta p}{\rho}$ and $W_{pump} = m_{pump}\cdot h_{pump}$

$$W_{pump} = m_{pump}\cdot\frac{\Delta p}{\rho} = \rho\cdot Q\cdot\frac{\Delta p}{\rho}$$

$W_{pump} = Q\cdot\Delta p$ $W_{pump} = 5615\frac{ft\cdot lbf}{s}$ $W_{pump} = 10.2\cdot hp$

Note that the software cannot render a dot, so the power is W_{pump} and mass flow rate is m_{pump}!

For an efficiency of $\eta = 80\%$ $W_{required} = \frac{W_{pump}}{\eta}$ $W_{required} = 12.8\cdot hp$

Problem 8.175

[Difficulty: 3]

8.175 A 125-mm-diameter pipeline conveying water at 10°C contains 50 m of straight galvanized pipe, 5 fully open gate valves, 1 fully open angle valve, 7 standard 90° elbows, 1 square-edged entrance from a reservoir, and 1 free discharge. The entrance conditions are $p_1 = 150$ kPa and $z_1 = 15$ m, and exit conditions are $p_2 = 0$ kPa and $z_2 = 30$ m. A centrifugal pump is installed in the line to move the water. What pressure rise must the pump deliver so that the volume flow rate will be $Q = 50$ L/s?

Given: Flow in pipeline with pump

Find: Pump pressure Δp

Solution:

Basic equations

$$\left(\frac{p_1}{\rho} + \alpha\cdot\frac{V_1^{\,2}}{2} + g\cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha\cdot\frac{V_2^{\,2}}{2} + g\cdot z_2\right) + \frac{\Delta p_{pump}}{\rho} = h_{lT}$$

$$h_l = f\cdot\frac{L}{D}\cdot\frac{V^2}{2} \qquad h_{lm} = f\cdot\frac{L_e}{D}\cdot\frac{V^2}{2} \qquad h_{lm} = K\cdot\frac{V^2}{2}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) α is approximately 1

Available data $L = 50\cdot m$ $\quad D = 125\cdot mm$ $\quad Q = 50\cdot\frac{l}{s}$ $\quad e = 0.15\cdot mm$

$p_1 = 150\cdot kPa$ $\quad p_2 = 0\cdot kPa$ $\quad z_1 = 15\cdot m$ $\quad z_2 = 30\cdot m$

From Section 8.8 $K_{ent} = 0.5$ $\quad L_{elbow90} = 30\cdot D$ $\quad L_{elbow90} = 3.75\,m$ $\quad L_{GV} = 8\cdot D$ $\quad L_{GV} = 1\,m$

$L_{AV} = 150\cdot D$ $\quad L_{AV} = 18.75\,m$ $\quad \rho = 1000\frac{kg}{m^3}$ $\quad \mu = 1.3\cdot 10^{-3}\cdot\frac{N\cdot s}{m^2}$ (Table A.8)

Hence $V = \dfrac{Q}{\left(\frac{\pi\cdot D^2}{4}\right)}$ $\quad V = 4.07\frac{m}{s}$ $\quad Re = \dfrac{\rho\cdot V\cdot D}{\mu}$ $\quad Re = 3.918\times 10^5$

and Given $\quad \dfrac{1}{\sqrt{f}} = -2\cdot\log\left(\dfrac{\frac{e}{D}}{3.7} + \dfrac{2.51}{Re\cdot\sqrt{f}}\right)$ $\quad f = 0.0212$

The loss is then

$$h_{lT} = \frac{V^2}{2}\cdot\left(f\cdot\frac{L}{D} + 7\cdot f\cdot\frac{L_{elbow90}}{D} + 5\cdot f\cdot\frac{L_{GV}}{D} + f\cdot\frac{L_{AV}}{D} + K_{ent}\right) \qquad h_{lT} = 145\frac{m^2}{s^2}$$

The energy equation becomes

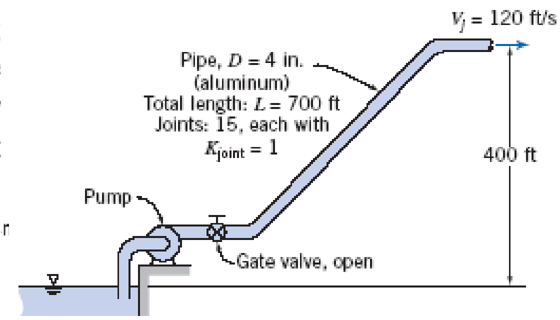
$$\frac{p_1 - p_2}{\rho} + g\cdot(z_1 - z_2) - \frac{V^2}{2} + \frac{\Delta p_{pump}}{\rho} = h_{lT}$$

$$\Delta p_{pump} = \rho\cdot h_{lT} + \rho\cdot g\cdot(z_2 - z_1) + \rho\cdot\frac{V^2}{2} + (p_2 - p_1) \qquad \Delta p_{pump} = 150\cdot kPa$$

Problem 8.176

[Difficulty: 3]

8.176 Cooling water is pumped from a reservoir to rock drills on a construction job using the pipe system shown. The flow rate must be 600 gpm and water must leave the spray nozzle at 120 ft/s. Calculate the minimum pressure needed at the pump outlet. Estimate the required power input if the pump efficiency is 70 percent.



Solution:

Computing equations: $\left(\cancel{\frac{p_1}{\rho}} + \cancel{\alpha_1\frac{\bar{V}_1^2}{2}} + \cancel{g z_1}\right) - \left(\cancel{\frac{p_2}{\rho}} + \alpha_2\frac{\bar{V}_2^2}{2} + g z_2\right) + \Delta h_{pump} = h_{l_T}$ (8.48)

$$h_{l_T} = h_l + h_{l_m}, \quad h_l = f\frac{L}{D}\frac{\bar{V}^2}{2}, \quad h_{l_m} = \frac{\bar{V}^2}{2}\left(\Sigma K + \Sigma f\left(\frac{L_e}{D}\right)\right)$$

Assumptions: (1) $\bar{V}_1 = 0$ (2) $\alpha_2 = \alpha_3 = 1$ (3) $p_1 = p_2 = p_{atm}$

Then

$$\Delta h_{pump} = g z_2 + \frac{\bar{V}_2^2}{2} + f\frac{L}{D}\frac{\bar{V}^2}{2} + \frac{\bar{V}^2}{2}\left[K_{ent} + f\left(\frac{L_e}{D}\right)_{GV} + 2f\left(\frac{L_e}{D}\right)_{45^\circ el} + 15K_j\right] \quad (1)$$

$$\bar{V} = \frac{Q}{A} = \frac{4Q}{\pi D^2} = \frac{4}{\pi} \times 600\frac{gal}{min} \times \frac{ft^3}{7.48\,gal} \times \frac{min}{60\,sec} \times \frac{1}{(0.333\,ft)^2} = 15.3\,ft/s$$

$$Re = \frac{D\bar{V}}{\nu} = \frac{1}{3}ft \times 15.3\frac{ft}{s} \times \frac{s}{1.24\times10^{-5}\,ft^2} = 4.11\times10^5 \quad \{\nu \text{ at } T = 60^\circ F, \text{ Table A.7}\}$$

Table 8.1, $e = 5\times10^{-6}$ ft (drawn tubing) $\therefore e/D = 1.5\times10^{-5}$

From Fig. 8.13, $f = 0.0135$

From Table 8.1, $K_{ent} = 0.78$

From Table 8.4, $L_e/D)_{g.v} = 8$, $L_e/D)_{90^\circ el} = 30$, $L_e/D)_{45^\circ el} = 16$

Then from Eq. (1)

$$\Delta h_{pump} = 32.2\frac{ft}{s^2} \times 400\,ft + \frac{1}{2}(120)^2\frac{ft^2}{s^2} + 0.0135 \times \frac{700}{0.333} \times \frac{1}{2} \times (15.3)^2\frac{ft^2}{s^2}$$
$$+ \frac{1}{2}(15.3)^2\frac{ft^2}{s^2}\left[0.78 + 0.0135(30) + 2\times0.0135(16) + 15(1)\right]$$

$$\Delta h_{pump} = 2.53\times10^4\,ft^2/s^2$$

The theoretical power input to the pump is given by $\dot{W}_{pump} = \dot{m}\,\Delta h_{pump}$

From the definition of efficiency, $\eta = \dot{W}_{theor}/\dot{W}_{act}$, then

$$\dot{W}_{act} = \frac{\dot{m}\,\Delta h_{pump}}{\eta} = \frac{\rho Q\,\Delta h_{pump}}{\eta}$$

$$\dot{W}_{act} = \frac{1.94}{0.7}\frac{slug}{ft^3} \times 600\frac{gal}{min} \times \frac{ft^3}{7.48\,gal} \times \frac{min}{60\,s} \times 2.53\times10^4\frac{ft^2}{s^2} \times \frac{lbf\cdot s^2}{ft\cdot slug} \times \frac{hp\cdot s}{550\,ft\cdot lbf} = \underline{170\,hp} \quad \dot{W}_{in}$$

The discharge pressure from the pump is obtained by applying Eq. 8.48 between sections ① and ③ neglecting losses in the inlet section, elevation change, and kinetic energy at ③

$$p_3 - p_A = \rho\,\Delta h_{pump} = 1.94\frac{slug}{ft^3} \times 2.53\times10^4\frac{ft^2}{s^2} \times \frac{lbf\cdot s^2}{ft\cdot s} \times \frac{ft^2}{144\,in^2} = \underline{341\,psi} \quad p_3$$

Problem 8.177

8.177 You are asked to size a pump for installation in the water supply system of the Willis Tower (formerly the Sears Tower) in Chicago. The system requires 100 gpm of water pumped to a reservoir at the top of the tower 340 m above the street. City water pressure at the street-level pump inlet is 400 kPa (gage). Piping is to be commercial steel. Determine the minimum diameter required to keep the average water velocity below 3.5 m/s in the pipe. Calculate the pressure rise required across the pump. Estimate the minimum power needed to drive the pump.

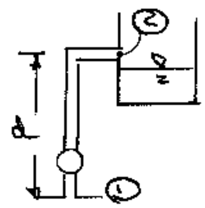
Solution:

$$\bar{V} = \frac{Q}{A} = \frac{4Q}{\pi D^2} \quad \therefore D = \left[\frac{4Q}{\pi \bar{V}}\right]^{1/2} = \left[\frac{4}{\pi} \times 100\frac{gal}{min} \times \frac{ft^3}{7.48\,gal} \times \frac{min}{60\,s} \times \left(\frac{0.305\,m}{ft}\right)^3 \times \frac{s}{3.5\,m}\right]^{1/2}$$

$$D = 0.048\,m = 48\,mm \longleftarrow D$$

Computing equations:
$$\left(\frac{p_1}{\rho} + \cancel{\alpha_1 \frac{\bar{V}_1^2}{2}} + gz_1\right) - \left(\cancel{\frac{p_2}{\rho}}^{\,0(2)} + \cancel{\alpha_2 \frac{\bar{V}_2^2}{2}}^{\,(1)} + gz_2\right) + \Delta h_{pump} = h_{\ell_T} \quad (8.48)$$

$$h_{\ell_T} = h_\ell + \cancel{h_{\ell_m}}^{\,(3)}, \quad h_\ell = f\frac{L}{D}\frac{\bar{V}^2}{2}$$



Assume: (1) $\alpha_1 = \alpha_2$ (2) $p_2 = p_{atm}$

(3) minor losses are negligible.

(4) water at 20°C ($\nu = 1\times10^{-6}\,m^2/s$, Table A.8).

Then

$$\Delta h_{pump} = h_\ell - \frac{p_1}{\rho} + g(z_2 - z_1) = f\frac{L}{D}\frac{\bar{V}^2}{2} - \frac{p_1}{\rho} + gd \quad \text{.......(1)}$$

From Table 8.1, $e = 0.046\,mm$ $\therefore e/D = 0.046/48 = 0.00096$

$$Re = \frac{D\bar{V}}{\nu} = 0.048\,m \times 3.5\frac{m}{s} \times \frac{s}{1\times10^{-6}\,m^2} = 1.68\times10^5$$

From Fig. 8.13, $f = 0.021$. Then from Eq. 1

$$\Delta h_{pump} = 0.021 \times \frac{340}{0.048} \times \frac{1}{2}(3.5)^2\frac{m^2}{s^2} - 400\times10^3\frac{N}{m^2} \times \frac{m^3}{998\,kg} \times \frac{kg\cdot m}{N\cdot s^2} + 9.81\frac{m}{s^2} \times 340\,m$$

$$\Delta h_{pump} = 3{,}850\ m^2/s^2 \quad \text{(This is head added to fluid).}$$

$$\Delta h_{pump} = \frac{\dot{W}_{pump}}{\dot{m}} = \left(\frac{p}{\rho} + \frac{\bar{V}^2}{2} + gz\right)_{discharge} - \left(\frac{p}{\rho} + \frac{\bar{V}^2}{2} + gz\right)_{suction} \quad (8.47)$$

Assume: (5) $\bar{V}_{disch} = \bar{V}_{suct}$; $z_{discharge} \approx z_{suction}$

$$\Delta p = \rho\,\Delta h = 998\frac{kg}{m^3} \times 3850\frac{m^2}{s^2} \times \frac{N\cdot s^2}{kg\cdot m} = 3840\,kPa \quad \Delta p$$

Also from Eq. 8.47

$$\dot{W}_{pump} = \dot{m}\,\Delta h_{pump} = \rho Q\,\Delta h_{pump}$$

$$\dot{W}_{pump} = 998\frac{kg}{m^3} \times 100\frac{gal}{min} \times \frac{ft^3}{7.48\,gal} \times \left(\frac{0.305\,m}{ft}\right)^3 \times \frac{min}{60\,s} \times 3850\frac{m^2}{s^2} \times \frac{N\cdot s^2}{kg\cdot m}$$

$$\dot{W}_{pump} = 24.3\,kW \ (32.6\,hp) \longleftarrow \dot{W}_{pump}$$

Problem 8.178

[Difficulty: 3]

8.178 Air conditioning on a university campus is provided by chilled water (10°C) pumped through a main supply pipe. The pipe makes a loop 5 km in length. The pipe diameter is 0.75 m and the material is steel. The maximum design volume flow rate is 0.65 m^3/s. The circulating pump is driven by an electric motor. The efficiencies of pump and motor are $\eta_p = 85$ percent and $\eta_m = 85$ percent, respectively. Electricity cost is 14¢/(kW·hr). Determine (a) the pressure drop, (b) the rate of energy addition to the water, and (c) the daily cost of electrical energy for pumping.

Given: Flow in air conditioning system

Find: Pressure drop; cost

Solution:

Basic equations

$$\left(\frac{p_1}{\rho} + \alpha \cdot \frac{V_1^{\,2}}{2} + g \cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha \cdot \frac{V_2^{\,2}}{2} + g \cdot z_2\right) = h_l \qquad h_l = f \cdot \frac{L}{D} \cdot \frac{V^2}{2}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) α is approximately 1

Available data

$$L = 5 \cdot km \qquad D = 0.75 \cdot m \qquad e = 0.046 \cdot mm \qquad Q = 0.65 \cdot \frac{m^3}{s} \qquad \eta_p = 85 \cdot \% \qquad \eta_m = 85 \cdot \%$$

$$\rho = 1000 \frac{kg}{m^3} \qquad \mu = 1.3 \cdot 10^{-3} \cdot \frac{N \cdot s}{m^2} \quad \text{(Table A.8)} \qquad cost = \frac{0.14}{kW \cdot hr} \quad \text{(dollars)}$$

Then

$$V = \frac{Q}{\left(\frac{\pi \cdot D^2}{4}\right)} \qquad V = 1.47 \frac{m}{s} \qquad Re = \frac{\rho \cdot V \cdot D}{\mu} \qquad Re = 8.49 \times 10^5$$

so Given

$$\frac{1}{\sqrt{f}} = -2 \cdot \log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re \cdot \sqrt{f}}\right) \qquad f = 0.0131$$

The energy equation becomes

$$\Delta p = f \cdot \frac{L}{D} \cdot \rho \cdot \frac{V^2}{2} \qquad \Delta p = 94.4 \cdot kPa$$

and

$$W_{pump} = Q \cdot \Delta p \qquad W_{pump} = 61.3 \cdot kW$$

The required power is

$$Power = \frac{W_{pump}}{\eta_p \cdot \eta_m} \qquad Power = 84.9 \cdot kW$$

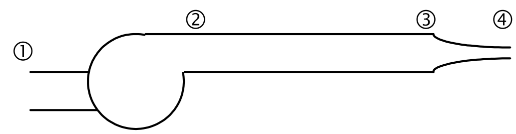
The daily cost is then

$$C = cost \cdot Power \cdot day \qquad C = 285 \quad \text{dollars}$$

Problem 8.179

[Difficulty: 4]

8.179 A fire nozzle is supplied through 100 m of 3.5-cm-diameter, smooth, rubber-lined hose. Water from a hydrant is supplied to a booster pump on board the pumper truck at 350 kPa (gage). At design conditions, the pressure at the nozzle inlet is 700 kPa (gage), and the pressure drop along the hose is 750 kPa per 100 m of length. Determine (a) the design flow rate, (b) the nozzle exit velocity, assuming no losses in the nozzle, and (c) the power required to drive the booster pump, if its efficiency is 70 percent.



Given: Fire nozzle/pump system

Find: Design flow rate; nozzle exit velocity; pump power needed

Solution:

Basic equations
$$\left(\frac{p_2}{\rho} + \alpha\cdot\frac{V_2^2}{2} + g\cdot z_2\right) - \left(\frac{p_3}{\rho} + \alpha\cdot\frac{V_3^2}{2} + g\cdot z_3\right) = h_l \qquad h_l = f\cdot\frac{L}{D}\cdot\frac{V_2^2}{2} \quad \text{for the hose}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) α at 2 and 3 is approximately 1 4) No minor loss

$$\frac{p_3}{\rho} + \frac{V_3^2}{2} + g\cdot z_3 = \frac{p_4}{\rho} + \frac{V_4^2}{2} + g\cdot z_4 \quad \text{for the nozzle (assuming Bernoulli applies)}$$

$$\left(\frac{p_2}{\rho} + \alpha\cdot\frac{V_2^2}{2} + g\cdot z_2\right) - \left(\frac{p_1}{\rho} + \alpha\cdot\frac{V_1^2}{2} + g\cdot z_1\right) = h_{pump} \quad \text{for the pump}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) α at 1 and 2 is approximately 1 4) No minor loss

Hence for the hose $\quad \frac{\Delta p}{\rho} = \frac{p_2 - p_3}{\rho} = f\cdot\frac{L}{D}\cdot\frac{V^2}{2} \quad$ or $\quad V = \sqrt{\frac{2\cdot\Delta p\cdot D}{\rho\cdot f\cdot L}}$

We need to iterate to solve this for V because f is unknown until Re is known. This can be done using *Excel*'s Solver, but here:

$\Delta p = 750\cdot kPa \qquad L = 100\cdot m \qquad e = 0 \qquad D = 3.5\cdot cm \qquad \rho = 1000\cdot\frac{kg}{m^3} \qquad \nu = 1.01\times 10^{-6}\cdot\frac{m^2}{s}$

Make a guess for f $\quad f = 0.01 \qquad V = \sqrt{\frac{2\cdot\Delta p\cdot D}{\rho\cdot f\cdot L}} \qquad V = 7.25\,\frac{m}{s} \qquad Re = \frac{V\cdot D}{\nu} \qquad Re = 2.51\times 10^5$

Given $\quad \frac{1}{\sqrt{f}} = -2.0\cdot\log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re\cdot\sqrt{f}}\right) \qquad f = 0.0150$

$V = \sqrt{\frac{2\cdot\Delta p\cdot D}{\rho\cdot f\cdot L}} \qquad V = 5.92\,\frac{m}{s} \qquad Re = \frac{V\cdot D}{\nu} \qquad Re = 2.05\times 10^5$

Given $\quad \frac{1}{\sqrt{f}} = -2.0\cdot\log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re\cdot\sqrt{f}}\right) \qquad f = 0.0156$

$$V = \sqrt{\frac{2\cdot\Delta p\cdot D}{\rho\cdot f\cdot L}} \qquad V = 5.81\,\frac{m}{s} \qquad Re = \frac{V\cdot D}{\nu} \qquad Re = 2.01\times 10^{5}$$

Given $$\frac{1}{\sqrt{f}} = -2.0\cdot\log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re\cdot\sqrt{f}}\right) \qquad f = 0.0156$$

$$V = \sqrt{\frac{2\cdot\Delta p\cdot D}{\rho\cdot f\cdot L}} \qquad V = 5.80\,\frac{m}{s} \qquad Re = \frac{V\cdot D}{\nu} \qquad Re = 2.01\times 10^{5}$$

$$Q = V\cdot A = \frac{\pi\cdot D^2}{4}\cdot V \qquad Q = \frac{\pi}{4}\times(0.035\cdot m)^2\times 5.80\cdot\frac{m}{s} \qquad Q = 5.58\times 10^{-3}\,\frac{m^3}{s} \qquad Q = 0.335\cdot\frac{m^3}{min}$$

For the nozzle $$\frac{p_3}{\rho} + \frac{V_3{}^2}{2} + g\cdot z_3 = \frac{p_4}{\rho} + \frac{V_4{}^2}{2} + g\cdot z_4$$ so $$V_4 = \sqrt{\frac{2\cdot(p_3 - p_4)}{\rho} + V_3{}^2}$$

$$V_4 = \sqrt{2\times 700\times 10^3\cdot\frac{N}{m^2}\times\frac{m^3}{1000\cdot kg}\times\frac{kg\cdot m}{s^2\cdot N} + \left(5.80\cdot\frac{m}{s}\right)^2} \qquad V_4 = 37.9\,\frac{m}{s}$$

For the pump $$\left(\frac{p_2}{\rho} + \alpha\cdot\frac{V_2{}^2}{2} + g\cdot z_2\right) - \left(\frac{p_1}{\rho} + \alpha\cdot\frac{V_1{}^2}{2} + g\cdot z_1\right) = h_{pump}$$ so $$h_{pump} = \frac{p_2 - p_1}{\rho}$$

$$p_1 = 350\cdot kPa \qquad p_2 = 700\cdot kPa + 750\cdot kPa \qquad p_2 = 1450\cdot kPa$$

The pump power is $P_{pump} = m_{pump}\cdot h_{pump}$ $\quad$ P_{pump} and m_{pump} are pump power and mass flow rate (software can't do a dot!)

$$P_{pump} = \rho\cdot Q\cdot\frac{(p_2 - p_1)}{\rho} = Q\cdot(p_2 - p_1) \qquad P_{pump} = 5.58\times 10^{-3}\cdot\frac{m^3}{s}\times(1450 - 350)\times 10^3\cdot\frac{N}{m^2} \qquad P_{pump} = 6.14\cdot kW$$

$$P_{required} = \frac{P_{pump}}{\eta} \qquad P_{required} = \frac{6.14\cdot kW}{70\cdot\%} \qquad P_{required} = 8.77\cdot kW$$

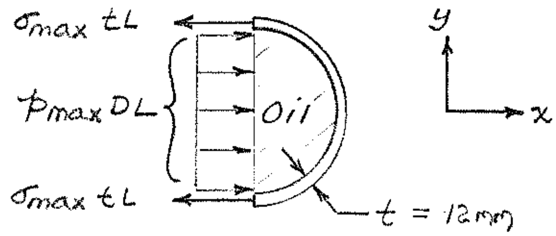
Problem 8.180

[Difficulty: 4] Part 1/2

8.180 Heavy crude oil (SG = 0.925 and $\nu = 1.0 \times 10^{-4}\,\text{m}^2/\text{s}$) is pumped through a pipeline laid on flat ground. The line is made from steel pipe with 600 mm i.d. and has a wall thickness of 12 mm. The allowable tensile stress in the pipe wall is limited to 275 MPa by corrosion considerations. It is important to keep the oil under pressure to ensure that gases remain in solution. The minimum recommended pressure is 500 kPa. The pipeline carries a flow of 400,000 barrels (in the petroleum industry, a "barrel" is 42 gal) per day. Determine the maximum spacing between pumping stations. Compute the power added to the oil at each pumping station.

Solution: First find the maximum pressure allowable in pipe. Consider a free body diagram of a segment of length, L:

Basic equation: $\Sigma F_x = 0$



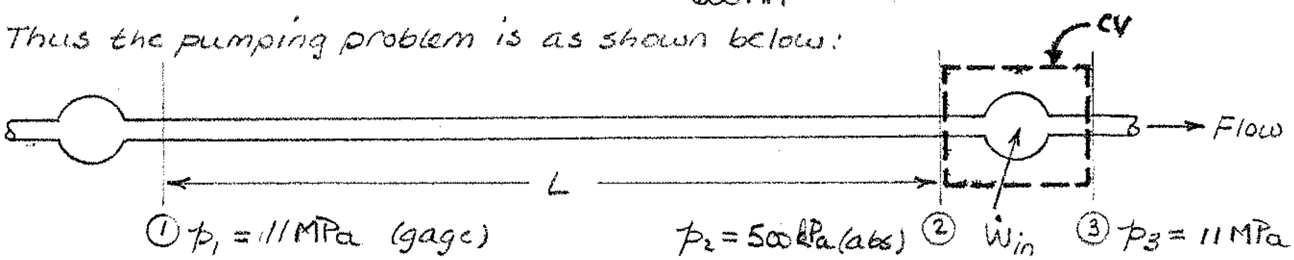
Assumption: Neglect hydrostatic pressure variation, and atmospheric pressure

Then

$$\Sigma F_x = p_{max} DL - 2\sigma_{max} tL = 0$$

$$p_{max} = 2\sigma_{max}\frac{t}{D} = 2 \times 275\,\text{MPa} \times \frac{12\,\text{mm}}{600\,\text{mm}} = 11\,\text{MPa (gage)}$$

Thus the pumping problem is as shown below:



To find L, apply the energy equation for steady, incompressible flow that is uniform at each section.

Basic equation: $$\frac{p_1}{\rho} + \frac{\bar{V}_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \frac{\bar{V}_2^2}{2} + g z_2 + h_{l_T}\,; \quad h_{l_T} = f\frac{L}{D}\frac{\bar{V}^2}{2} + \underbrace{h_{l_m}}_{=0(3)}$$

Assumptions: (1) $\bar{V}_1 = \bar{V}_2$

(2) $z_1 = z_2$ (level)

(3) $h_{l_m} = 0$, since straight, constant area pipe

Then

$$f\frac{L}{D}\frac{\bar{V}^2}{2} = \frac{p_1 - p_2}{\rho} \quad \text{or} \quad L = \frac{D}{f}\left(\frac{p_1 - p_2}{\rho}\right)\frac{2}{\bar{V}^2} \quad \text{--------- (1)}$$

$$\bar{V} = \frac{Q}{A} = 4\times10^5\,\frac{\text{bbl}}{\text{day}} \times \frac{\text{day}}{24\,\text{hr}} \times \frac{\text{hr}}{3600\,\text{s}} \times 42\,\frac{\text{gal}}{\text{bbl}} \times 4\,\frac{\text{qt}}{\text{gal}} \times 9.46\times10^{-4}\,\frac{\text{m}^3}{\text{qt}} \times \frac{4}{\pi}\,\frac{1}{(0.6\,\text{m})^2} = 2.60\,\text{m/s}$$

$f = f(Re, e/D)$. From Table 8.1, $e = 0.046\,\text{mm}$, so $e/D = 7.7\times10^{-5}$. Reynolds number is

$$Re = \frac{\rho\bar{V}D}{\mu} = \frac{\bar{V}D}{\nu} = 2.6\,\frac{\text{m}}{\text{s}} \times 0.6\,\text{m} \times \frac{\text{s}}{1.0\times10^{-4}\,\text{m}^2} = 1.56\times10^4$$

From Eq 8.37, $f = 0.0277$ (Using Excel's Solver or Goal Seek)

Thus, substituting into Eq. 1

$$L = \frac{0.6\,m}{0.0277}\left[11\times10^6\,\frac{N}{m^2} - (500-101)\times10^3\,\frac{N}{m^2}\right]\times\frac{m^3}{(0.925)\,999\,kg}\times 2\times\frac{s^2}{(2.16)^2\,m^2}\times\frac{kg\cdot m}{N\cdot s^2}$$

$$L = 72.8\ km$$

L

To find pump power delivered to the oil, apply the energy equation to the CV shown, between sections ② and ③

$$\left(\frac{p}{\rho}+\alpha\frac{\bar{V}^2}{2}+gz\right)_{discharge} - \left(\frac{p}{\rho}+\alpha\frac{\bar{V}^2}{2}+gz\right)_{suction} = \frac{\dot{W}_{pump}}{\dot{m}} = \Delta h_{pump} \qquad (8.45)$$

Since $\bar{V}$ = constant and elevation change is small, this reduces to

$$\Delta h_{pump} = \frac{p_3 - p_2}{\rho}$$

$$= \left[11\times10^6 - (500-101)\times10^3\right]\frac{N}{m^2}\times\frac{m^3}{(0.925)\,999\,kg}\times\frac{kg\cdot m}{N\cdot s^2}$$

$$\Delta h_{pump} = 1.15\times10^4\ m^2/s^2$$

The mass flow rate is

$$\dot{m} = \rho Q = (0.925)\,999\,\frac{kg}{m^3}\times 400{,}000\,\frac{bbl}{day}\times 42\,\frac{gal}{bbl}\times 9.46\times10^{-4}\,\frac{m^3}{qt}\times 4\,\frac{qt}{gal}\times\frac{day}{24\,hr}\times\frac{hr}{3600\,s}$$

$$\dot{m} = 680\ kg/s$$

The power added to the oil is

$$\dot{W}_{pump} = \dot{m}\,\Delta h_{pump}$$

$$= 680\,\frac{kg}{s}\times 1.15\times10^4\,\frac{m^2}{s^2}\times\frac{N\cdot s^2}{kg\cdot m}$$

$$\dot{W}_{pump} = 7{,}730\ kW$$

$\dot{W}_{pum}$

Note pump efficiency does not affect the power that must be added to the oil.

Problem 8.181

[Difficulty: 2]

8.181 The volume flow rate through a water fountain on a college campus is 0.075 m^3/s. Each water stream can rise to a height of 10 m. Estimate the daily cost to operate the fountain. Assume that the pump motor efficiency is 85 percent, the pump efficiency is 85 percent, and the cost of electricity is 14¢/(kW · hr).

Given: Flow in water fountain

Find: Daily cost

Solution:

Basic equations $\quad W_{pump} = Q \cdot \Delta p \qquad \Delta p = \rho \cdot g \cdot H$

Assumptions: 1) Steady flow 2) Incompressible flow 3) α is approximately 1

Available data $\quad Q = 0.075 \cdot \frac{m^3}{s} \qquad H = 10 \cdot m \qquad \eta_p = 85 \cdot \% \qquad \eta_m = 85 \cdot \%$

$\rho = 999 \cdot \frac{kg}{m^3} \qquad Cost = \frac{0.14}{kW \cdot hr}$ (dollars)

Hence $\quad \Delta p = \rho \cdot g \cdot H \qquad \Delta p = 98 \cdot kPa$

$W_{pump} = Q \cdot \Delta p \qquad W_{pump} = 7.35 \cdot kW$

$Power = \frac{W_{pump}}{\eta_p \cdot \eta_m} \qquad Power = 10.2 \cdot kW$

$C = Cost \cdot Power \cdot day \qquad C = 34.17$ (dollars)

Problem 8.182

[Difficulty: 3]

8.182 Petroleum products are transported over long distances by pipeline, e.g., the Alaskan pipeline (see Example 8.6). Estimate the energy needed to pump a typical petroleum product, expressed as a fraction of the throughput energy carried by the pipeline. State and critique your assumptions clearly.

Solution: From Example Problem 8.6, for the Alaskan pipeline, $Q = 1.6\times10^6$ bpd.

Thus $Q = 1.6\times10^6 \frac{bbl}{day}\times 42\frac{gal}{bbl}\times\frac{ft^3}{7.48\,gal}\times\frac{day}{24\,hr}\times\frac{hr}{3600\,s} = 104\,ft^3/s$

and

$$\dot{m} = \rho Q = SG\,\rho_{H_2O} Q = 0.93\times 1.94\frac{slug}{ft^3}\times 104\frac{ft^3}{s} = 188\,slug/s$$

The energy content of a typical petroleum product is about 18,000 Btu/lbm, so the throughput energy is

$$\dot{E} = e\dot{m} = 18{,}000\frac{Btu}{lbm}\times 188\frac{slug}{s}\times 32.2\frac{lbm}{slug} = 1.09\times10^8\,Btu/s$$

$\dot{E}$

From Example Problem 8.6, each pumping station requires 36,800 hp, and they are located $L = 120$ mi apart.

The entire pipeline is about 750 mi long. Thus there must be $N = 750/120$ or about $N = 7$ pumping stations. Thus the total energy required to pump must be

$$\mathcal{P} = N\dot{W} = 7\,stations\times 36{,}800\frac{hp}{station} = 258{,}000\,hp$$

$\mathcal{P}$

Expressed as a fraction of throughput energy

$$\frac{\mathcal{P}}{\dot{E}} = 258{,}000\,hp\times\frac{s}{1.09\times10^8\,Btu}\times 2545\frac{Btu}{hp\cdot hr}\times\frac{hr}{3600\,s} = 1.67\times10^{-3} \text{ or } 0.00167$$

Thus about 0.167% of energy is used for transporting petroleum.

The assumptions outlined above appear reasonable. The computed result is probably accurate within ±10%.

A more universal metric would be energy per unit mass and distance, e.g., energy per ton-mile of transport.

$$\frac{E}{M\cdot L} = \frac{E/t}{M/t\cdot L} = \frac{\mathcal{P}}{\dot{m}L} = 36{,}800\,hp\times\frac{s}{188\,slug}\times\frac{1}{120\,mi}\times 2545\frac{Btu}{hp\cdot hr}\times\frac{slug}{32.2\,lbm}\times 2000\frac{lbm}{ton}\times\frac{hr}{3600\,s}$$

Thus

$$e = \frac{\mathcal{P}}{\dot{m}L} = 71.6\,Btu/ton\cdot mi$$

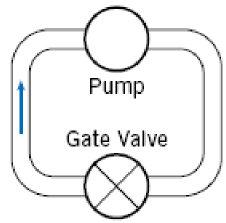
e

This specific metric allows direct comparison with other modes of transport.

Problem 8.183

[Difficulty: 4]

8.183 The pump testing system of Problem 8.128 is run with a pump that generates a pressure difference given by $\Delta p = 750 - 15 \times 10^4 Q^2$ where Δp is in kPa, and the generated flow rate is Q m^3/s. Find the water flow rate, pressure difference, and power supplied to the pump if it is 70 percent efficient.



Given: Flow in a pump testing system

Find: Flow rate; Pressure difference; Power

Solution:

Governing equations:

$$\left(\frac{p_1}{\rho} + \alpha_1 \cdot \frac{V_1^2}{2} + g \cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \cdot \frac{V_2^2}{2} + g \cdot z_2\right) = h_{lT} = \sum_{major} h_l + \sum_{minor} h_{lm} \quad (8.29)$$

$$Re = \frac{\rho \cdot V \cdot D}{\mu} \qquad h_l = f \cdot \frac{L}{D} \cdot \frac{V^2}{2} \quad (8.34) \qquad h_{lm} = f \cdot \frac{L_e}{D} \cdot \frac{V^2}{2} \quad (8.40b)$$

$$f = \frac{64}{Re} \quad (8.36) \quad \text{(Laminar)} \qquad \frac{1}{\sqrt{f}} = -2.0 \cdot \log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re \cdot \sqrt{f}}\right) \quad (8.37) \quad \text{(Turbulent)}$$

The energy equation (Eq. 8.29) becomes for the circuit (1 = pump outlet, 2 = pump inlet)

$$\frac{p_1 - p_2}{\rho} = f \cdot \frac{L}{D} \cdot \frac{V^2}{2} + 4 \cdot f \cdot L_{elbow} \cdot \frac{V^2}{2} + f \cdot L_{valve} \cdot \frac{V^2}{2}$$

or

$$\Delta p = \rho \cdot f \cdot \frac{V^2}{2} \cdot \left(\frac{L}{D} + 4 \cdot \frac{L_{elbow}}{D} + \frac{L_{valve}}{D}\right) \quad (1)$$

This must be matched to the pump characteristic equation; at steady state, the pressure generated by the pump just equals that lost to friction in the circuit

$$\Delta p = 750 - 15 \times 10^4 \cdot Q^2 \quad (2)$$

Finally, the power supplied to the pump, efficiency η, is

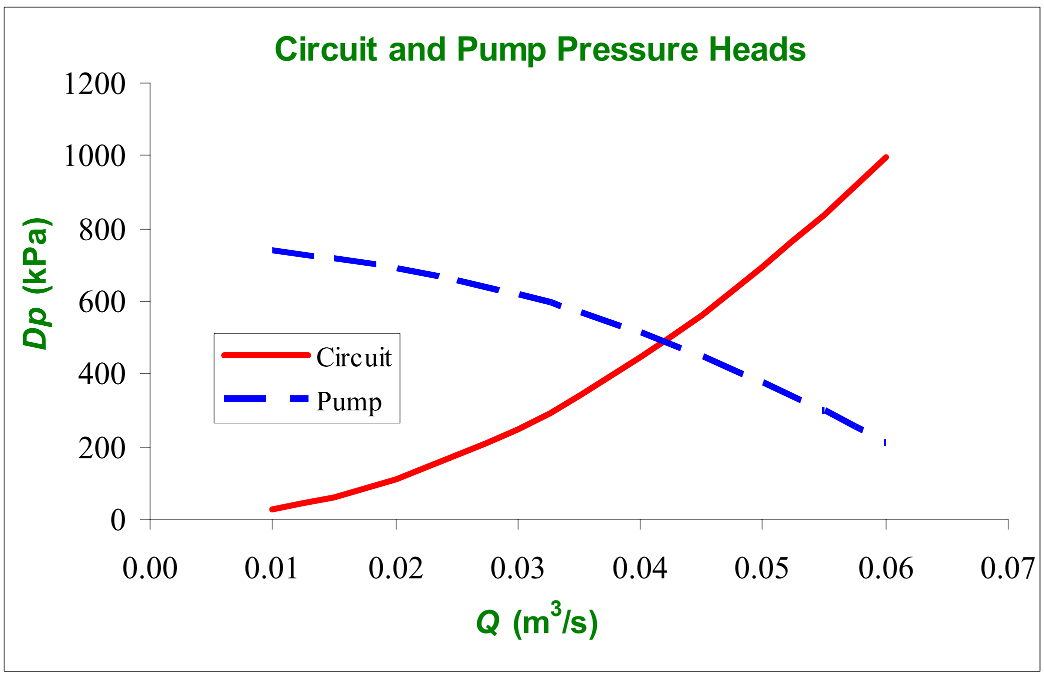
$$Power = \frac{Q \cdot \Delta p}{\eta} \quad (3)$$

In *Excel*:

Given data:				Tabulated or graphical data:			
L =	20	m			e =	0.26	mm
D =	75	mm				(Table 8.1)	
η_{pump} =	70%				μ =	1.00E-03	N.s/m^2
					ρ =	999	kg/m^3
						(Appendix A)	
					Gate valve L_e/D =	8	
					Elbow L_e/D =	30	
						(Table 8.4)	

Computed results:

Q (m^3/s)	V (m/s)	Re	f	Δp (kPa) (Eq 1)	Δp (kPa) (Eq 2)		
0.010	2.26	1.70E+05	0.0280	28.3	735		
0.015	3.40	2.54E+05	0.0277	63.1	716		
0.020	4.53	3.39E+05	0.0276	112	690		
0.025	5.66	4.24E+05	0.0276	174	656		
0.030	6.79	5.09E+05	0.0275	250	615		
0.035	7.92	5.94E+05	0.0275	340	566		
0.040	9.05	6.78E+05	0.0274	444	510		
0.045	10.2	7.63E+05	0.0274	561	446		
0.050	11.3	8.48E+05	0.0274	692	375		
0.055	12.4	9.33E+05	0.0274	837	296		
0.060	13.6	1.02E+06	0.0274	996	210		
						Error	
0.0419	9.48	7.11E+05	0.0274	487	487	0	Using *Solver*!
Power =	29.1	kW	(Eq. 3)				



Problem 8.184 Equations

[Difficulty: 4]

8.184 A water pump can generate a pressure difference Δp (psi) given by $\Delta p = 145 - 0.1\,Q^2$, where the flow rate is Q ft^3/s. It supplies a pipe of diameter 20 in., roughness 0.5 in., and length 2500 ft. Find the flow rate, pressure difference, and the power supplied to the pump if it is 70 percent efficient. If the pipe were replaced with one of roughness 0.25 in., how much would the flow increase, and what would the required power be?

Given: Pump/pipe system

Find: Flow rate, pressure drop, and power supplied; Effect of roughness

Solution:

$$Re = \frac{\rho \cdot V \cdot D}{\mu} \qquad \left(\frac{p_1}{\rho} + \alpha_1 \cdot \frac{V_1^2}{2} + g \cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \cdot \frac{V_2^2}{2} + g \cdot z_2\right) = h_{lT} - \Delta h_{pump} \qquad h_{lT} = f \cdot \frac{L}{D} \cdot \frac{V^2}{2}$$

$$f = \frac{64}{Re} \quad \text{(Laminar)} \qquad \frac{1}{\sqrt{f}} = -2.0 \cdot \log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re \cdot \sqrt{f}}\right) \quad \text{(Turbulent)}$$

The energy equation becomes for the system (1 = pipe inlet, 2 = pipe outlet)

$$\Delta h_{pump} = f \cdot \frac{L}{D} \cdot \frac{V^2}{2} \qquad \text{or} \qquad \Delta p_{pump} = \rho \cdot f \cdot \frac{L}{D} \cdot \frac{V^2}{2} \qquad (1)$$

This must be matched to the pump characteristic equation; at steady state, the pressure generated by the pump just equals that lost to friction in the circuit

$$\Delta p_{pump} = 145 - 0.1 \cdot Q^2 \qquad (2)$$

Finally, the power supplied to the pump, efficiency η, is

$$Power = \frac{Q \cdot \Delta p}{\eta} \qquad (3)$$

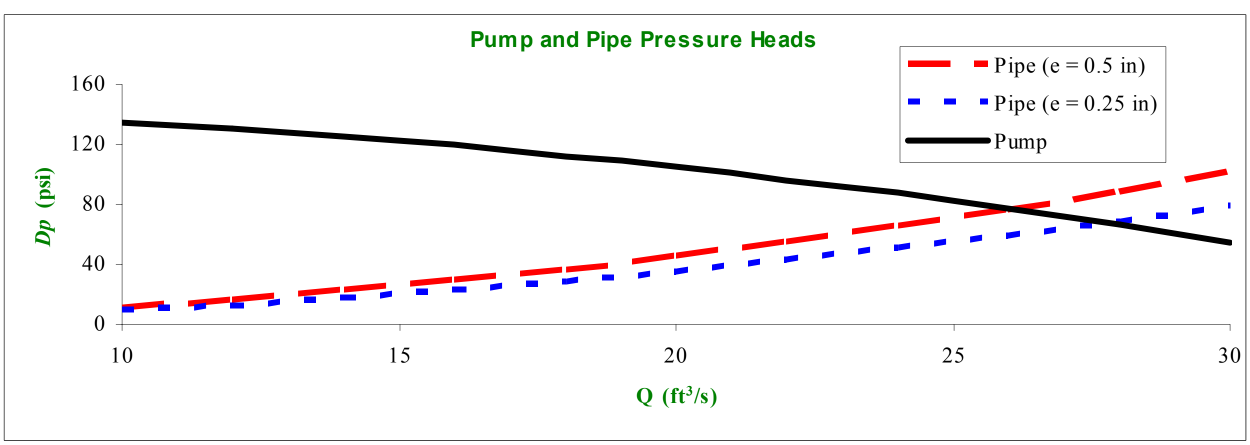
In *Excel*:

Tabulated or graphical data:					Given data:		
	μ =	2.10E-05	lbf·s/ft^2		L =	2500	ft
	ρ =	1.94	slug/ft^3		D =	20	in
		(Appendix A)			η_{pump} =	70%	

Computed results:		e =	0.5	in		
Q (ft^3/s)	V (ft/s)	Re	f	Δp (psi) (Eq 1)	Δp (psi) (Eq 2)	
10	4.58	7.06E+05	0.0531	11.3	135.0	
12	5.50	8.47E+05	0.0531	16.2	130.6	
14	6.42	9.88E+05	0.0531	22.1	125.4	
16	7.33	1.13E+06	0.0531	28.9	119.4	
18	8.25	1.27E+06	0.0531	36.5	112.6	
20	9.17	1.41E+06	0.0531	45.1	105.0	
22	10.08	1.55E+06	0.0531	54.6	96.6	
24	11.00	1.69E+06	0.0531	64.9	87.4	
26	11.92	1.83E+06	0.0531	76.2	77.4	
28	12.83	1.98E+06	0.0531	88.4	66.6	
30	13.75	2.12E+06	0.0531	101.4	55.0	
						Error
26.1	12.0	1.84E+06	0.0531	76.8	76.8	0.00	Using *Solver* !
Power =	750	hp	(Eq. 3)			

Repeating, with smoother pipe

Computed results:		e =	0.25	in		
Q (ft^3/s)	V (ft/s)	Re	f	Δp (psi) (Eq 1)	Δp (psi) (Eq 2)	
10	4.58	7.06E+05	0.0410	8.71	135.0	
12	5.50	8.47E+05	0.0410	12.5	130.6	
14	6.42	9.88E+05	0.0410	17.1	125.4	
16	7.33	1.13E+06	0.0410	22.3	119.4	
18	8.25	1.27E+06	0.0410	28.2	112.6	
20	9.17	1.41E+06	0.0410	34.8	105.0	
22	10.08	1.55E+06	0.0410	42.1	96.6	
24	11.00	1.69E+06	0.0410	50.1	87.4	
26	11.92	1.83E+06	0.0410	58.8	77.4	
28	12.83	1.98E+06	0.0410	68.2	66.6	
30	13.75	2.12E+06	0.0410	78.3	55.0	
						Error
27.8	12.8	1.97E+06	0.0410	67.4	67.4	0.00	Using *Solver* !
Power =	702	hp	(Eq. 3)			



Problem 8.185

[Difficulty: 3]

8.185 A square cross-section duct (0.35 m × 0.35 m × 175 m) is used to convey air ($\rho = 1.1$ kg/m^3) into a clean room in an electronics manufacturing facility. The air is supplied by a fan and passes through a filter installed in the duct. The duct friction factor is $f = 0.003$, the filter has a loss coefficient of $K = 3$. The fan performance is given by $\Delta p = 2250 - 250Q - 150Q^2$, where Δp (Pa) is the pressure generated by the fan at flow rate Q (m^3/s). Determine the flow rate delivered to the room.

Given: Fan/duct system

Find: Flow rate

Solution:

$$\left(\frac{p_1}{\rho} + \alpha_1 \cdot \frac{V_1^2}{2} + g \cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \cdot \frac{V_2^2}{2} + g \cdot z_2\right) = h_{lT} - \Delta h_{fan} \qquad h_{lT} = f \cdot \frac{L}{D_h} \cdot \frac{V^2}{2} + K \cdot \frac{V^2}{2}$$

The energy equation becomes for the system (1 = duct inlet, 2 = duct outlet)

$$\Delta h_{fan} = f \cdot \frac{L}{D_h} \cdot \frac{V^2}{2} + K \cdot \frac{V^2}{2}$$

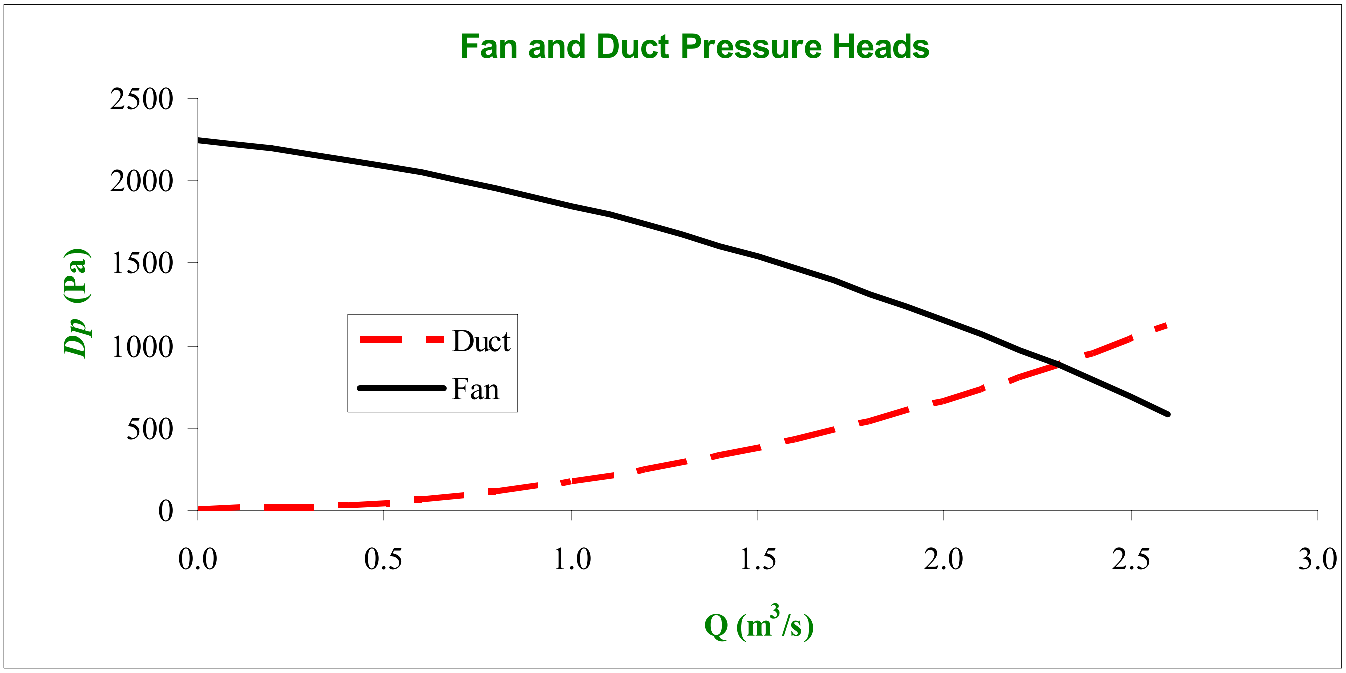
or $$\Delta p_{pump} = \frac{\rho \cdot V^2}{2} \cdot \left(f \cdot \frac{L}{D_h} + K\right) \quad (1) \qquad \text{where} \quad D_h = \frac{4 \cdot A}{P_w} = \frac{4 \cdot h^2}{4 \cdot h} = h$$

This must be matched to the fan characteristic equation; at steady state, the pressure generated by the fan just equals that lost to friction in the circuit

$$\Delta p_{fan} = 1020 - 25 \cdot Q - 30 \cdot Q^2 \quad (2)$$

In *Excel*:

Given data:			Computed results:						
L =	175	m	*Q* (m^3/s)	*V* (m/s)	**Δ*p* (Pa) (Eq 1)**	**Δ*p* (Pa) (Eq 2)**			
D$_h$ =	0.35	m	0.00	0.00	0	2250			
K =	3		0.20	1.63	7	2194			
f =	0.003		0.40	3.27	26	2126			
ρ =	1.1	kg/m^3	0.60	4.90	59	2046			
			0.80	6.53	106	1954			
			1.00	8.16	165	1850			
			1.20	9.80	238	1734			
			1.40	11.43	323	1606			
			1.60	13.06	422	1466			
			1.80	14.69	534	1314			
			2.00	16.33	660	1150			
			2.20	17.96	798	974			
			2.40	19.59	950	786			
			2.60	21.22	1115	586			
							Error		
			2.31	18.82	877	877	0.00	Using *Solver*!	


Fan and Duct Pressure Heads
2500
2000
1500
1000
500
0
Dp (Pa)
Duct
Fan
0.0
0.5
1.0
1.5
2.0
2.5
3.0
Q (m3/s)

Problem 8.186

8.186 The head versus capacity curve for a certain fan may be approximated by the equation $H = 30 - 10^{-7}Q^2$, where H is the output static head in inches of water and Q is the air flow rate in ft^3/min. The fan outlet dimensions are 8×16 in. Determine the air flow rate delivered by the fan into a 200 ft straight length of 8×16 in. rectangular duct.

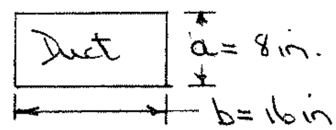
Solution:

Basic equation:

$$\left(\frac{p_1}{\rho g} + \alpha_1 \cancel{\frac{\bar{V}_1^2}{2g}} + \cancel{z_1}\right) - \left(\frac{p_2}{\rho g} + \alpha_2 \cancel{\frac{\bar{V}_2^2}{2g}} + \cancel{z_2}\right) = h_{l_T} \qquad (8.30)$$

$$h_{l_T} = f\frac{L}{D_h}\frac{\bar{V}^2}{2g} + h_{l_m} \; ; \quad D_h = \frac{4A}{P_w}$$

Assumptions: (1) $\bar{V}_1 = \bar{V}_2$, $\alpha_1 = \alpha_2 \simeq 1$
(2) $z_1 = z_2$
(3) $h_{l_m} = 0$



$$A = ab = \frac{8}{12}\,ft \times \frac{16}{12}\,ft = 0.889\ ft^2$$

$$D_h = \frac{4A}{P_w} = \frac{4A}{2(a+b)} = \frac{2 \times 0.889\ ft^2}{(2/3 + 4/3)\ ft} = 0.889\ ft$$

From Eq. 8.30 $\quad \Delta p = f\frac{L}{D}\rho_{air}\frac{\bar{V}^2}{2} = f\frac{L}{D}\frac{\rho}{2}\frac{Q^2}{A^2} = \gamma_{H_2O} H_{duct}$

where H_{duct} is the pressure drop in head of water.

$$H_{duct} = \frac{f L \rho_{air} Q^2}{2\gamma_{H_2O} D_h A^2} = \frac{f}{2} \times \frac{200\ ft}{0.889\ ft} \times 0.00238\frac{slug}{ft^3} \times \frac{ft^3}{62.4\ lbf} \times \frac{1}{(0.889)^2\ ft^4} \times \left[Q\frac{ft^3}{min} \times \frac{min}{60\ s}\right]^2 \times \frac{12\ in}{ft}$$

$H_{duct} = 1.81 \times 10^{-5} f Q^2$ (where H is in in. H_2O) .... (1)

For a smooth duct, $f = f(Re)$

$Re = \frac{\bar{V}D_h}{\nu} = \frac{D_h Q}{\nu A}$. For $T = 68°F$, from Table A.9, $\nu = 1.62 \times 10^{-4}\ ft^2/s$

$$Re = \frac{0.889\ ft}{0.889\ ft^2} \times \frac{1}{1.62 \times 10^{-4}}\frac{s}{ft^2} \times Q\frac{ft^3}{min} \times \frac{min}{60\ s} = 103\ Q$$

To determine the air flow rate delivered we need to determine the operating point of the fan.

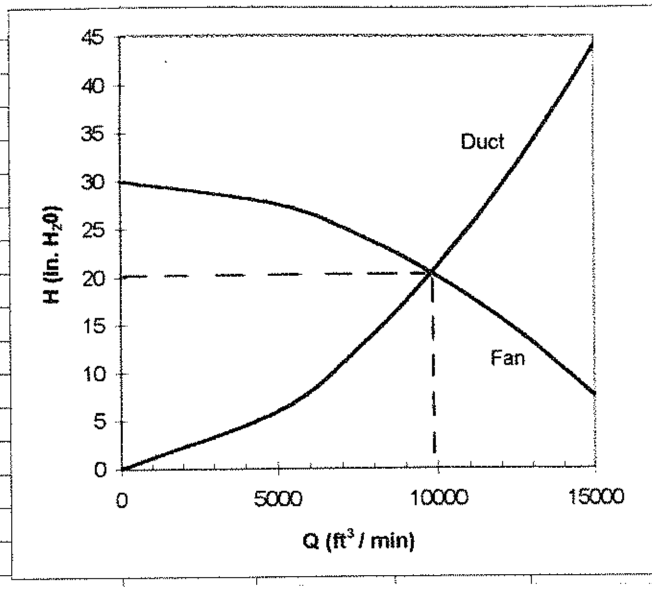
The operating point is at the intersection of the
- fan head capacity curve, and the
- system curve (the head loss in the duct)

This is shown on the plot below.

Note that the friction factor f is determined from the Colebrook equation (8.37a) using Eq. 8.37b for the initial estimate of f.

Problem 8.186

Q	Re	H (fan)	f_0	f	H (duct)
(ft^3/min)	(-)	(in.H_2O)	(-)	(-)	(in.H_2O)
0		30			0
5,000	5.15E+05	27.5	0.0130	0.0131	5.9
7,500	7.73E+05	24.4	0.0121	0.0122	12.4
10,000	1.03E+06	20.0	0.0115	0.0116	21.0
12,500	1.29E+06	14.4	0.0111	0.0112	31.6
15,000	1.55E+06	7.5	0.0108	0.0108	44.1



Operating point

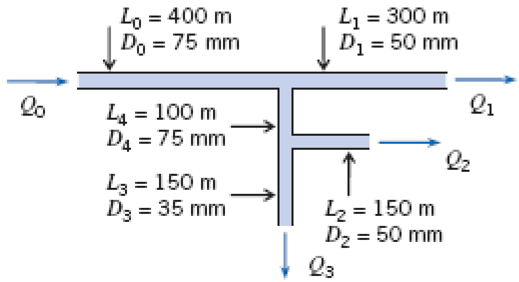
$Q \approx 10{,}000\ ft^3/min$

$H = 20\ in.\ H_2O$

Problem 8.187

[Difficulty: 5]

8.187 The water pipe system shown is constructed from galvanized iron pipe. Minor losses may be neglected. The inlet is at 400 kPa (gage), and all exits are at atmospheric pressure. Find the flow rates Q_0, Q_1, Q_2, Q_3, and Q_4.



Given: Pipe system

Find: Flow in each branch

Solution:

Governing equations:

$$\left(\frac{p_1}{\rho} + \alpha_1\cdot\frac{V_1^2}{2} + g\cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2\cdot\frac{V_2^2}{2} + g\cdot z_2\right) = h_l \quad (8.29) \qquad h_{lT} = f\cdot\frac{L}{D}\cdot\frac{V^2}{2} \quad (8.34)$$

$$f = \frac{64}{Re} \quad \text{(Laminar)} \quad (8.36) \qquad \frac{1}{f^{0.5}} = -2.0\cdot\log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re\cdot f^{0.5}}\right) \quad \text{(Turbulent)} \quad (8.37)$$

The energy equation (Eq. 8.29) can be simplified to $\Delta p = \rho\cdot f\cdot\frac{L}{D}\cdot\frac{V^2}{2}$

This can be written for each pipe section

In addition we have the following contraints

$$Q_0 = Q_1 + Q_4 \quad (1) \qquad Q_4 = Q_2 + Q_3 \quad (2)$$

$$\Delta p = \Delta p_0 + \Delta p_1 \quad (3) \qquad \Delta p = \Delta p_0 + \Delta p_4 + \Delta p_2 \quad (4)$$

$$\Delta p_2 = \Delta p_3 \quad (5)$$

We have 5 unknown flow rates (or, equivalently, velocities) and five equations

In *Excel*:

Pipe Data:

Pipe	*L* (m)	*D* (mm)	*e* (mm)
0	400	75	0.15
1	300	50	0.15
2	150	50	0.15
3	150	35	0.15
4	100	75	0.15

Fluid Properties:

ρ =	999	kg/m^3
μ =	0.001	N.s/m^2

Available Head:

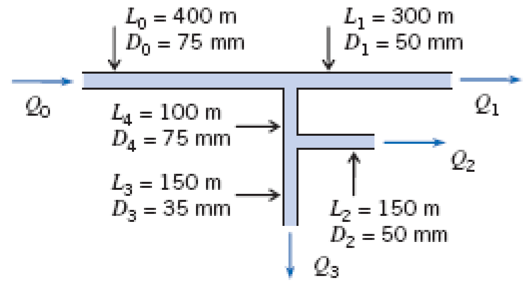
Δp =	400	kPa

Flows:		Q_0 (m^3/s)	Q_1 (m^3/s)	Q_2 (m^3/s)	Q_3 (m^3/s)	Q_4 (m^3/s)
		0.00812	0.00286	0.00379	0.00147	0.00526
		V_0 (m/s)	V_1 (m/s)	V_2 (m/s)	V_3 (m/s)	V_4 (m/s)
		1.84	1.458	1.932	1.526	1.19
		Re_0	Re_1	Re_2	Re_3	Re_4
		1.38E+05	7.28E+04	9.65E+04	5.34E+04	8.92E+04
		f_0	f_1	f_2	f_3	f_4
		0.0247	0.0279	0.0275	0.0309	0.0253
Heads:		Δp_0 (kPa)	Δp_1 (kPa)	Δp_2 (kPa)	Δp_3 (kPa)	Δp_4 (kPa)
		222.3	177.7	153.9	153.9	23.9
Constraints:		(1) $Q_0 = Q_1 + Q_4$			(2) $Q_4 = Q_2 + Q_3$	
		0.00%			**0.00%**	
		(3) $\Delta p = \Delta p_0 + \Delta p_1$			(4) $\Delta p = \Delta p_0 + \Delta p_4 + \Delta p_2$	
		0.01%			**0.01%**	
		(5) $\Delta p_2 = \Delta p_3$				
		0.00%				
		Error:	**0.03%**	Vary Q_0, Q_1, Q_2, Q_3 and Q_4		
				using *Solver* to minimize total error		

Problem 8.188

[Difficulty: 5]

8.188 Find flow rates Q_0, Q_1, Q_2, and Q_4 if pipe 3 becomes blocked.



Given: Pipe system

Find: Flow in each branch if pipe 3 is blocked

Solution:

Governing equations:
$$\left(\frac{p_1}{\rho} + \alpha_1\cdot\frac{V_1^{\,2}}{2} + g\cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2\cdot\frac{V_2^{\,2}}{2} + g\cdot z_2\right) = h_l \qquad (8.29)$$

$$h_{lT} = f\cdot\frac{L}{D}\cdot\frac{V^2}{2} \qquad (8.34)$$

$$f = \frac{64}{Re} \quad \text{(Laminar)} \qquad (8.36)$$

$$\frac{1}{f^{0.5}} = -2.0\cdot\log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re\cdot f^{0.5}}\right) \quad \text{(Turbulent)} \qquad (8.37)$$

The energy equation (Eq. 8.29) can be simplified to
$$\Delta p = \rho\cdot f\cdot\frac{L}{D}\cdot\frac{V^2}{2}$$

This can be written for each pipe section

In addition we have the following contraints

$$Q_0 = Q_1 + Q_4 \qquad (1) \qquad\qquad Q_4 = Q_2 \qquad (2)$$

$$\Delta p = \Delta p_0 + \Delta p_1 \qquad (3) \qquad\qquad \Delta p = \Delta p_0 + \Delta p_4 + \Delta p_2 \qquad (4)$$

We have 4 unknown flow rates (or, equivalently, velocities) and four equations

In *Excel*:

Pipe Data:

Pipe	*L* (m)	*D* (mm)	*e* (mm)
0	400	75	0.15
1	300	50	0.15
2	150	50	0.15
3	150	35	0.15
4	100	75	0.15

Fluid Properties:

ρ =	999	kg/m^3
μ =	0.001	N.s/m^2

Available Head:

Δp =	400	kPa

Flows:		Q_0 (m^3/s)	Q_1 (m^3/s)	Q_2 (m^3/s)	Q_3 (m^3/s)	Q_4 (m^3/s)
		0.00744	0.00314	0.00430	NA	0.00430
		V_0 (m/s)	V_1 (m/s)	V_2 (m/s)	V_3 (m/s)	V_4 (m/s)
		1.68	1.599	2.190	NA	0.97
		Re_0	Re_1	Re_2	Re_3	Re_4
		1.26E+05	7.99E+04	1.09E+05	NA	7.29E+04
		f_0	f_1	f_2	f_3	f_4
		0.0248	0.0278	0.0274	NA	0.0257
Heads:		Δp_0 (kPa)	Δp_1 (kPa)	Δp_2 (kPa)	Δp_3 (kPa)	Δp_4 (kPa)
		187.2	212.8	196.6	NA	16.2
Constraints:		(1) $Q_0 = Q_1 + Q_4$			(2) $Q_4 = Q_2$	
		0.00%			**0.00%**	
		(3) $\Delta p = \Delta p_0 + \Delta p_1$			(4) $\Delta p = \Delta p_0 + \Delta p_4 + \Delta p_2$	
		0.00%			**0.00%**	
		Error:	**0.00%**	Vary Q_0, Q_1, Q_2, and Q_4		
				using *Solver* to minimize total error		

Problem 8.189

[Difficulty: 5]

8.189 A cast-iron pipe system consists of a 500-ft section of water pipe, after which the flow branches into two 300-ft sections. The two branches then meet in a final 250-ft section. Minor losses may be neglected. All sections are 1.5-in. diameter, except one of the two branches, which is 1-in. diameter. If the applied pressure across the system is 100 psi, find the overall flow rate and the flow rates in each of the two branches.

Given: Water pipe system

Find: Flow rates

Solution:

Basic equations
$$\left(\frac{p_1}{\rho}+\alpha_1\cdot\frac{V_1^2}{2}+g\cdot z_1\right)-\left(\frac{p_2}{\rho}+\alpha_2\cdot\frac{V_2^2}{2}+g\cdot z_2\right)=h_l \qquad h_{lT}=f\cdot\frac{L}{D}\cdot\frac{V^2}{2}$$

$$f=\frac{64}{Re} \quad \text{(Laminar)} \qquad \frac{1}{\sqrt{f}}=-2.0\cdot\log\left(\frac{\frac{e}{D}}{3.7}+\frac{2.51}{Re\cdot\sqrt{f}}\right) \quad \text{(Turbulent)}$$

The energy equation can be simplified to $\Delta p=\rho\cdot f\cdot\frac{L}{D}\cdot\frac{V^2}{2}$

This can be written for each pipe section

Pipe A (first section) $$\Delta p_A=\rho\cdot f_A\cdot\frac{L_A}{D_A}\cdot\frac{V_A^2}{2} \qquad (1)$$

Pipe B (1.5 in branch) $$\Delta p_B=\rho\cdot f_B\cdot\frac{L_B}{D_B}\cdot\frac{V_B^2}{2} \qquad (2)$$

Pipe C (1 in branch) $$\Delta p_C=\rho\cdot f_C\cdot\frac{L_C}{D_C}\cdot\frac{V_C^2}{2} \qquad (3)$$

Pipe D (last section) $$\Delta p_D=\rho\cdot f_D\cdot\frac{L_D}{D_D}\cdot\frac{V_D^2}{2} \qquad (4)$$

In addition we have the following contraints

$$Q_A=Q_D \qquad (5)$$

$$Q_A=Q_B+Q_C \qquad (6)$$

$$\Delta p=\Delta p_A+\Delta p_B+\Delta p_D \qquad (7)$$

$$\Delta p_B=\Delta p_C \qquad (8)$$

We have 4 unknown flow rates (or velocities) and four equations (5 - 8); Eqs 1 - 4 relate pressure drops to flow rates (velocities)

In *Excel*:

The workbook for Example Problem 8.11 is modified for use in this problem

Pipe Data:

Pipe	L (ft)	D (in)	e (ft)
A	500	1.5	0.00085
B	300	1.5	0.00085
C	300	1	0.00085
D	250	1.5	0.00085

Fluid Properties:

ρ = 1.94 slug/ft^3

μ = 2.10E-05 lbf·s/ft^2

Available Head:

Δp = 100 psi

Flows:

Q_A (ft^3/s)	Q_B (ft^3/s)	Q_C (ft^3/s)	Q_D (ft^3/s)
0.0942	0.0704	0.0238	0.0942

V_A (ft/s)	V_B (ft/s)	V_C (ft/s)	V_D (ft/s)
7.68	5.74	4.36	7.68

Re_A	Re_B	Re_C	Re_D
8.87E+04	6.63E+04	3.36E+04	8.87E+04

f_A	f_B	f_C	f_D
0.0343	0.0345	0.0399	0.0343

Heads:

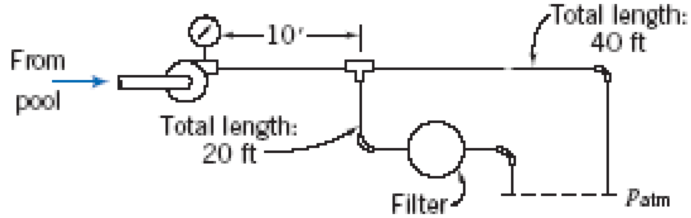
Δp_A (psi)	Δp_B (psi)	Δp_C (psi)	Δp_D (psi)
54.5	18.4	18.4	27.2

Constraints:

(5) $Q_A = Q_D$	(6) $Q_A = Q_B + Q_C$
0.00%	**0.00%**
(7) $\Delta p = \Delta p_A + \Delta p_B + \Delta p_D$	(8) $\Delta p_B = \Delta p_C$
0.00%	**0.00%**

Error: **0.00%** Vary Q_A, Q_B, Q_C, and Q_D using *Solver* to minimize total error

8.190 A swimming pool has a partial-flow filtration system. Water at 75°F is pumped from the pool through the system shown. The pump delivers 30 gpm. The pipe is nominal 3/4-in. PVC (i.d. = 0.824 in.). The pressure loss through the filter is approximately $\Delta p = 0.6\,Q^2$, where Δp is in psi and Q is in gpm. Determine the pump pressure and the flow rate through each branch of the system.



Solution: Apply the energy equation for steady, incompressible pipe flow.

Computing equation: $\frac{p_1}{\rho} + \alpha_1\frac{\bar{V}_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \alpha_2\frac{\bar{V}_2^2}{2} + gz_2 + h_{l_T}$; $h_{l_T} = \left[f\left(\frac{L}{D} + \frac{L_e}{D}\right) + K\right]\frac{\bar{V}^2}{2}$

Assumptions: (1) $\alpha_1\bar{V}_1^2 = \alpha_2\bar{V}_2^2$; (2) $z_1 = z_2$; (3) $h_{l_m} = 0$ for $1 \to 2$; (4) Ignore "tee" at ②

The flow rate is $Q_{12} = 30$ gpm $(0.0668\ \text{ft}^3/\text{sec})$, so $\bar{V} = \frac{Q}{A} = 18.0$ ft/sec. Then

$$Re = \frac{\bar{V}D}{\nu} = 18.0\,\frac{\text{ft}}{\text{sec}} \times \left(\frac{0.824}{12}\right)\text{ft} \times \frac{\text{sec}}{1.0\times10^{-5}\,\text{ft}^2} = 1.24\times10^5,\ \text{so } f = 0.017$$

$$\Delta p_{12} = f\frac{L}{D}\rho\frac{\bar{V}^2}{2} = 0.017 \times \frac{10\text{ ft}}{0.824\text{ in.}} \times \frac{1}{2} \times 1.94\,\frac{\text{slug}}{\text{ft}^3} \times (18.0)^2\frac{\text{ft}^2}{\text{sec}^2} \times \frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}} \times \frac{\text{ft}}{12\text{ in.}} = 5.40\text{ psi}$$

Branch flow rates are unknown, but flow split must produce the same drop in each branch. Solve by iteration to obtain

$Q_{23} = 5.2$ gpm; $\bar{V}_{23} = 3.12$ ft/s; $Re = 2.15\times10^4$, and $f = 0.025$*

$$\Delta p_{23} = f\left(\frac{L}{D} + 2\frac{L_e}{D}\right)\rho\frac{\bar{V}^2}{2} + 0.6Q^2$$

$$\Delta p_{23} = 0.025\left[\frac{240}{0.824} + 2(30)\right]\frac{1}{2} \times 1.94\,\frac{\text{slug}}{\text{ft}^3} \times (3.12)^2\frac{\text{ft}^2}{\text{s}^2} \times \frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}} \times \frac{\text{ft}^2}{144\text{ in.}^2} + 0.6(5.2)^2\frac{\text{lbf}}{\text{in.}^2} = 16.8\text{ psi}$$

$Q_{24} = 24.8$ gpm; $\bar{V}_{24} = 14.9$ ft/s; $Re = 1.03\times10^5$, and $f = 0.018$

$$\Delta p_{24} = f\left(\frac{L}{D} + \frac{L_e}{D}\right)\rho\frac{\bar{V}^2}{2} = 0.018\left(\frac{480}{0.824} + 30\right)\frac{1}{2} \times 1.94\,\frac{\text{slug}}{\text{ft}^3} \times (14.9)^2\frac{\text{ft}^2}{\text{s}^2} \times \frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}} \times \frac{\text{ft}^2}{144\text{ in.}^2} = 16.5\text{ psi}$$

The pump outlet pressure is

$\Delta p_{pump} = \Delta p_{12} + \Delta p_{23} = (5.4 + 16.8)\text{ psi} = 22.2\text{ psi}$ Δp

The branch flow rates are

$Q_{23} \approx 5.2$ gpm Q_{23}

$Q_{24} \approx 24.8$ gpm Q_{24}

* Value of f obtained from Eq 8.37 using Excel's solver (or Goal Seek)

8.191 Why does the shower temperature change when a toilet is flushed? Sketch pressure curves for the hot and cold water supply systems to explain what happens.

Discussion: Assume the pressure in the water main servicing the dwelling remains constant. The hot and cold water flow rates reaching the shower are controlled by valve(s) in the shower. Assuming a water heater temperature of 140°F, a cold water temperature of 60°F, and a shower water temperature of 100°F, the hot and cold flow rates must be equal. The two water streams mix before reaching the shower head, then spray out into the shower itself at 100°F.

Supply curves and system curves for the hot and cold water streams are shown below. Diagram *a* is the cold water system and diagram *b* is the hot water system. The numerical values are representative of an actual system.

In general the supply curves for the hot and cold streams are not the same. The difference is caused by the two systems having different pipe lengths and different fittings.

Each stream operates at the flow rate where the curves intersect. An equal flow split is accomplished by adjusting the shower valves to vary their resistances.

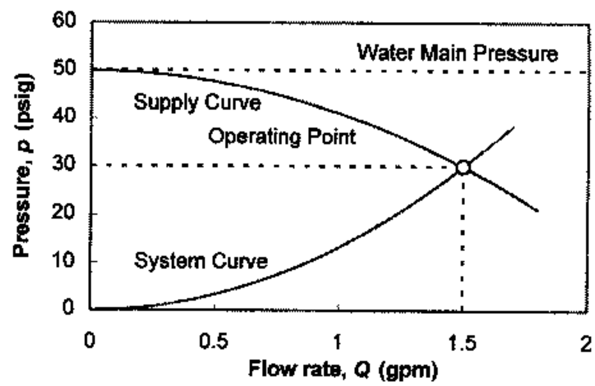
Flushing the toilet temporarily increases the flow rate of cold water to the bathroom. This reduces the cold water supply pressure reaching the shower. The system curves do not change because the valve settings stay the same. Therefore the flow rate of cold water must decrease to again match the supply and system curves (diagram *c*).

When the flow rate of cold water decreases the shower temperature increases, as experience testifies!

Problem 8.191

(a) Cold water system:

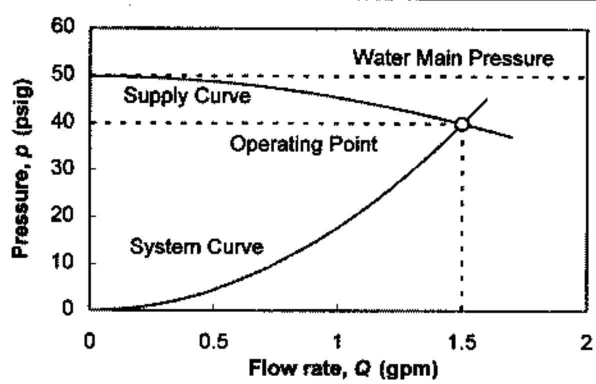
Q (gpm)	System Curve p (psig)	Supply Curve p (psig)
0	0.00	50.0
0.2	0.53	49.6
0.4	2.13	48.6
0.6	4.80	46.8
0.8	8.53	44.3
1.0	13.3	41.1
1.2	19.2	37.2
1.4	26.1	32.6
1.6	34.1	27.2
1.7	38.5	24.3
1.8		21.2



(a) Cold water system curves

(b) Hot water system:

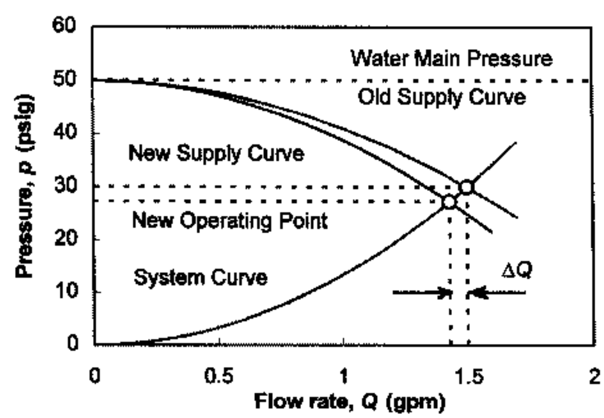
Q (gpm)	System Curve p (psig)	Supply Curve p (psig)
0	0.00	50.0
0.2	0.71	49.8
0.4	2.84	49.3
0.6	6.40	48.4
0.8	11.38	47.2
1.0	17.78	45.6
1.2	25.60	43.6
1.4	34.84	41.3
1.6	45.51	38.6
1.7		37.2
1.8		



(b) Hot water system curves

(c) Cold water system: toilet flush

Q (gpm)	System Curve p (psig)	Old Supply Curve p (psig)	New Supply Curve p (psig)
0	0.00	50.0	50.0
0.2	0.53	49.6	49.6
0.4	2.13	48.6	48.2
0.6	4.80	46.8	46.0
0.8	8.53	44.3	42.9
1.0	13.3	41.1	38.9
1.2	19.2	37.2	34.0
1.4	26.1	32.6	28.2
1.430	27.27	31.8	27.28
1.6	34.1	27.2	21.6
1.7	38.5	24.3	
1.8			



(c) Cold water system curves: toilet flush

Problem 8.192

[Difficulty: 2]

8.192 A square-edged orifice with corner taps and a water manometer are used to meter compressed air. The following data are given:

Inside diameter of air line	150 mm
Orifice plate diameter	100 mm
Upstream pressure	600 kPa
Temperature of air	25°C
Manometer deflection	750 mm H_2O

Calculate the volume flow rate in the line, expressed in cubic meters per hour.

Solution: Apply analysis of section 8-10; data from Fig. 8.23 apply

Computing equation: $\dot{m}_{actual} = K A_t \sqrt{2\rho(p_1 - p_2)}$ (8.56)

Since $\dot{m} = \rho Q$, then

$$Q = \frac{\dot{m}}{\rho} = K A_t \sqrt{\frac{2(p_1-p_2)}{\rho}}$$

$$p_1 - p_2 = 750 \text{ mm } H_2O = \rho g \Delta h_{H_2O} = 999 \frac{kg}{m^3} \times 9.81 \frac{m}{s^2} \times 0.75 m \times \frac{N \cdot s^2}{kg \cdot m} = 7.35 kPa$$

For this small Δp, the assumption of incompressible flow is certainly valid

$$\rho = \frac{p_1}{RT} = 701 \times 10^3 \frac{N}{m^2} \times \frac{kg \cdot K}{287 J} \times \frac{1}{298 K} \times \frac{J}{N \cdot m} = 8.20 \, kg/m^3$$

The flow coefficient $K = K(Re_{D_1}, \frac{D_t}{D_1})$

Assume $Re > 2 \times 10^5$. For $\beta = \frac{D_t}{D_1} = \frac{2}{3}$, from Fig. 8.20, $K = 0.675$

$$Q = K A_t \sqrt{\frac{2(p_1-p_2)}{\rho}} = 0.675 \frac{\pi}{4}(0.1 m)^2 \left[2 \times 7350 \frac{N}{m^2} \times \frac{m^3}{8.20 kg} \times \frac{kg \cdot m}{N \cdot s^2} \right]^{1/2}$$

$$Q = 0.224 \, m^3/s$$ Q

Check Re. At $T = 25°C$ $\mu = 1.84 \times 10^{-5} \, N \cdot s/m^2$ (Table A.10)

$$Re = \frac{\rho D V}{\mu} = \frac{\rho D Q}{\mu A} = \frac{\rho D Q}{\mu} \frac{4}{\pi D^2} = \frac{4 \rho Q}{\pi \mu D}$$

$$Re = \frac{4}{\pi} \times 8.2 \frac{kg}{m^3} \times 0.224 \frac{m^3}{s} \times \frac{1}{1.84 \times 10^{-5}} \frac{m^2}{N \cdot s} \times \frac{1}{0.15 m} \times \frac{N \cdot s^2}{kg \cdot m}$$

$Re = 8.47 \times 10^5$ ✓ assumption is valid

Problem 8.193

[Difficulty: 2]

8.193 Water at 65°C flows through a 75-mm-diameter orifice installed in a 150-mm-i.d. pipe. The flow rate is 20 L/s. Determine the pressure difference between the corner taps.

Given: Flow through an orifice

Find: Pressure drop

Solution:

Basic equation $$m_{actual} = K \cdot A_t \cdot \sqrt{2 \cdot \rho \cdot (p_1 - p_2)} = K \cdot A_t \cdot \sqrt{2 \cdot \rho \cdot \Delta p}$$

Note that m_{actual} is mass flow rate (the software cannot render a dot!)

For the flow coefficient $$K = K\left(Re_{D1}, \frac{D_t}{D_1}\right)$$

At 65°C,(Table A.8) $\rho = 980 \cdot \frac{kg}{m^3}$ $\quad \nu = 4.40 \times 10^{-7} \cdot \frac{m^2}{s}$

$$V = \frac{Q}{A} \qquad V = \frac{4}{\pi} \times \frac{1}{(0.15 \cdot m)^2} \times 20 \cdot \frac{L}{s} \times \frac{0.001 \cdot m^3}{1 \cdot L} \qquad V = 1.13 \frac{m}{s}$$

$$Re_{D1} = \frac{V \cdot D}{\nu} \qquad Re_{D1} = 1.13 \cdot \frac{m}{s} \times 0.15 \cdot m \times \frac{s}{4.40 \times 10^{-7} \cdot m^2} \qquad Re_{D1} = 3.85 \times 10^5$$

$$\beta = \frac{D_t}{D_1} \qquad \beta = \frac{75}{150} \qquad \beta = 0.5$$

From Fig. 8.20 $K = 0.624$

Then $$\Delta p = \left(\frac{m_{actual}}{K \cdot A_t}\right)^2 \cdot \frac{1}{2 \cdot \rho} = \left(\frac{\rho \cdot Q}{K \cdot A_t}\right)^2 \cdot \frac{1}{2 \cdot \rho} = \frac{\rho}{2} \cdot \left(\frac{Q}{K \cdot A_t}\right)^2$$

$$\Delta p = \frac{1}{2} \times 980 \cdot \frac{kg}{m^3} \times \left[20 \cdot \frac{L}{s} \times \frac{0.001 \cdot m^3}{1 \cdot L} \times \frac{1}{0.624} \times \frac{4}{\pi} \times \frac{1}{(0.075 \cdot m)^2}\right]^2 \qquad \Delta p = 25.8 \cdot kPa$$

Problem 8.194

[Difficulty: 3]

8.194 A smooth 200-m pipe, 100 mm diameter connects two reservoirs (the entrance and exit of the pipe are sharp-edged). At the midpoint of the pipe is an orifice plate with diameter 40 mm. If the water levels in the reservoirs differ by 30 m, estimate the pressure differential indicated by the orifice plate and the flow rate.

Given: Reservoir-pipe system

Find: Orifice plate pressure difference; Flow rate

Solution:

Basic equations:

$$\left(\frac{p_1}{\rho} + \alpha_1 \cdot \frac{V_1^{\,2}}{2} + g \cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \cdot \frac{V_2^{\,2}}{2} + g \cdot z_2\right) = h_{lT} = h_l + \Sigma h_{lm} \quad (8.29)$$

$$h_l = f \cdot \frac{L}{D} \cdot \frac{V^2}{2} \quad (8.34) \qquad h_{lm} = K \cdot \frac{V^2}{2} \quad (8.40a)$$

$$f = \frac{64}{Re} \quad \text{(Laminar)} \quad (8.36) \qquad \frac{1}{f^{0.5}} = -2.0 \cdot \log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re \cdot f^{0.5}}\right) \quad \text{(Turbulent)} \quad (8.37)$$

There are three minor losses: at the entrance; at the orifice plate; at the exit. For each $h_{lm} = K \cdot \frac{V^2}{2}$

The energy equation (Eq. 8.29) becomes ($\alpha = 1$)

$$g \cdot \Delta H = \frac{V^2}{2} \cdot \left(f \cdot \frac{L}{D} + K_{ent} + K_{orifice} + K_{exit}\right) \quad (1)$$

(ΔH is the difference in reservoir heights)

This cannot be solved for V (and hence Q) because f depends on V; we can solve by manually iterating, or by using *Solver*

The tricky part to this problem is that the orifice loss coefficient $K_{orifice}$ is given in Fig. 8.23 as a percentage of pressure differential Δp across the orifice, which is unknown until V is known!

The mass flow rate is given by

$$m_{rate} = K \cdot A_t \cdot \sqrt{2 \cdot \rho \cdot \Delta p} \quad (2)$$

where K is the orifice flow coefficient, A_t is the orifice area, and Δp is the pressure drop across the orifice

Equations 1 and 2 form a set for solving for TWO unknowns: the pressure drop Δp across the orifice (leading to a value for $K_{orifice}$) and the velocity V. The easiest way to do this is by using *Solver*

In *Excel*:

Given data:				Tabulated or graphical data:				
ΔH =	30	m		K_{ent} =	0.50	(Fig. 8.14)		
L =	200	m		K_{exit} =	1.00	(Fig. 8.14)		
D =	100	mm		Loss at orifice =	80%	(Fig. 8.23)		
D_t =	40	mm		μ =	0.001	N.s/m^2		
β =	0.40			ρ =	999	kg/m^3		
					(Water - Appendix A)			
Computed results:								
Orifice loss coefficient:			Flow system:			Orifice pressure drop		
K =	0.61			V =	2.25	m/s	Δp =	265	kPa
	(Fig. 8.20			Q =	0.0176	m^3/s		
	Assuming high Re)			Re =	2.24E+05			
				f =	0.0153			
	Eq. 1, solved by varying V AND Δp, using *Solver*:							
		Left (m^2/s)	**Right (m^2/s)**	**Error**		Procedure using *Solver*:		
		294	293	0.5%		a) Guess at V and Δp		
						b) Compute error in Eq. 1		
	Eq. 2 and $m_{rate} = \rho Q$ compared, varying V AND Δp					c) Compute error in mass flow rate		
		(From Q)	**(From Eq. 2)**	**Error**		d) Minimize total error		
m_{rate} (kg/s) =		17.6	17.6	0.0%		e) Minimize total error by varying V and Δp		
			Total Error	0.5%				

Problem 8.195

[Difficulty: 2]

8.195 A venturi meter with a 3-in.-diameter throat is placed in a 6-in.-diameter line carrying water at 75°F. The pressure drop between the upstream tap and the venturi throat is 12 in. of mercury. Compute the rate of flow.

Given: Flow through a venturi meter (NOTE: Throat is obviously 3 in not 30 in!)

Find: Flow rate

Solution:

Basic equation $$m_{actual} = \frac{C \cdot A_t}{\sqrt{1-\beta^4}} \cdot \sqrt{2 \cdot \rho \cdot (p_1 - p_2)} = \frac{C \cdot A_t}{\sqrt{1-\beta^4}} \cdot \sqrt{2 \cdot \rho \cdot \Delta p}$$

Note that m_{actual} is mass flow rate (the software cannot render a dot!)

For $Re_{D1} > 2 \times 10^5$, $0.980 < C < 0.995$. Assume C = 0.99, then check Re

$$\beta = \frac{D_t}{D_1} \qquad \beta = \frac{3}{6} \qquad \beta = 0.5$$

Also $$\Delta p = \rho_{Hg} \cdot g \cdot \Delta h = SG_{Hg} \cdot \rho \cdot g \cdot \Delta h$$

Then $$Q = \frac{m_{actual}}{\rho} = \frac{C \cdot A_t}{\rho \cdot \sqrt{1-\beta^4}} \cdot \sqrt{2 \cdot \rho \cdot \Delta p} = \frac{\pi \cdot C \cdot D_t^2}{4 \cdot \rho \cdot \sqrt{1-\beta^4}} \cdot \sqrt{2 \cdot \rho \cdot SG_{Hg} \cdot \rho \cdot g \cdot \Delta h} = \frac{\pi \cdot C \cdot D_t^2}{4 \cdot \sqrt{1-\beta^4}} \cdot \sqrt{2 \cdot SG_{Hg} \cdot g \cdot \Delta h}$$

$$Q = \frac{\pi}{4 \times \sqrt{1-0.5^4}} \times 0.99 \times \left(\frac{1}{4} \cdot ft\right)^2 \times \sqrt{2 \times 13.6 \times 32.2 \cdot \frac{ft}{s^2} \times 1 \cdot ft} \qquad Q = 1.49 \cdot \frac{ft^3}{s}$$

Hence $$V = \frac{Q}{A} = \frac{4 \cdot Q}{\pi \cdot D_1^2} \qquad V = \frac{4}{\pi} \times \frac{1}{\left(\frac{1}{2} \cdot ft\right)^2} \times 1.49 \cdot \frac{ft^3}{s} \qquad V = 7.59 \cdot \frac{ft}{s}$$

At 75°F,(Table A.7) $$\nu = 9.96 \times 10^{-6} \cdot \frac{ft^2}{s}$$

$$Re_{D1} = \frac{V \cdot D_1}{\nu} \qquad Re_{D1} = 7.59 \cdot \frac{ft}{s} \times \frac{1}{2} \cdot ft \times \frac{s}{9.96 \times 10^{-6} \cdot ft^2} \qquad Re_{D1} = 3.81 \times 10^5$$

Thus $Re_{D1} > 2 \times 10^5$. The volume flow rate is $$Q = 1.49 \cdot \frac{ft^3}{s}$$

Problem 8.196

[Difficulty: 2]

8.196 Consider a horizontal 2 in. × 1 in. venturi with water flow. For a differential pressure of 25 psi, calculate the volume flow rate (gpm).

Given: Flow through an venturi meter

Find: Flow rate

Solution:

Basic equation $$m_{actual} = \frac{C \cdot A_t}{\sqrt{1-\beta^4}} \cdot \sqrt{2 \cdot \rho \cdot \left(p_1 - p_2\right)} = \frac{C \cdot A_t}{\sqrt{1-\beta^4}} \cdot \sqrt{2 \cdot \rho \cdot \Delta p}$$

Note that m_{actual} is mass flow rate (the software cannot render a dot!)

For $Re_{D1} > 2 \times 10^5$, $0.980 < C < 0.995$. Assume $C = 0.99$, then check Re

Available data $D_1 = 2 \cdot in$ $\quad D_t = 1 \cdot in$ $\quad \Delta p = 25 \cdot psi$ $\quad \rho = 1.94 \cdot \frac{slug}{ft^3}$

$\beta = \frac{D_t}{D_1}$ $\quad \beta = 0.5$ $\quad$ and assume $\quad C = 0.99$

Then $$Q = \frac{m_{actual}}{\rho} = \frac{C \cdot A_t}{\rho \cdot \sqrt{1-\beta^4}} \cdot \sqrt{2 \cdot \rho \cdot \Delta p}$$

$$Q = \frac{\pi \cdot C \cdot D_t^{\,2}}{4 \cdot \sqrt{1-\beta^4}} \cdot \sqrt{\frac{2 \cdot \Delta p}{\rho}} \qquad Q = 0.340 \frac{ft^3}{s} \qquad Q = 152 \cdot gpm$$

Hence $V = \frac{Q}{A}$ $\quad V = \frac{4 \cdot Q}{\pi \cdot D_1^{\,2}}$ $\quad V = 15.6 \cdot \frac{ft}{s}$

At 68°F(Table A.7) $\nu = 1.08 \cdot 10^{-5} \cdot \frac{ft^2}{s}$ $\quad Re_{D1} = \frac{V \cdot D_1}{\nu}$ $\quad Re_{D1} = 2.403 \times 10^5$

Thus $Re_{D1} > 2 \times 10^5$. The volume flow rate is $\quad Q = 152 \cdot gpm$

8.197 Gasoline flows through a 2 × 1 in. venturi meter. The differential pressure is 380 mm of mercury. Find the volume flow rate.

Solution: Apply the analysis of Section 8-10.3.

Computing equations:

$$\dot{m}_{actual} = \frac{C A_t}{\sqrt{1-\beta^4}} \sqrt{2\rho(p_1 - p_2)} \qquad (8.52)$$

$$C = 0.99 \quad \text{for } Re_{D_1} > 2 \times 10^5$$

For the manometer, $\Delta p = \rho_{Hg} g \Delta h = SG_{Hg} \rho_{H_2O} g \Delta h$

Then

$$Q = \frac{\dot{m}}{\rho} = \frac{C A_t}{\sqrt{1-\beta^4}} \sqrt{\frac{2\Delta p}{\rho}} = \frac{C A_t}{\sqrt{1-\beta^4}} \sqrt{\frac{2 SG_{Hg} \rho_{H_2O} g \Delta h}{SG_{gas} \rho_{H_2O}}} = \frac{C A_t}{\sqrt{1-\beta^4}} \sqrt{\frac{2 SG_{Hg} g \Delta h}{SG_{gas}}}$$

$$Q = \frac{0.99}{\sqrt{1-(0.5)^4}} \frac{\pi (0.0254)^2 m^2}{4} \sqrt{2 \times \frac{13.6}{0.73} \times 9.81 \frac{m}{s^2} \times 0.38\, m} = 0.00611\ m^3/s$$

Now check Reynolds number:

$$\bar{V}_1 = \frac{Q}{A_1} = 0.00611 \frac{m^3}{s} \times \frac{4}{\pi (0.0508)^2 m^2} = 3.01\ m/s$$

Assume viscosity midway between octane and heptane at 20°C. From Fig. A.1,

$$\mu \approx 5.0 \times 10^{-4}\ N \cdot s / m^2$$

$$Re_{D_1} = \frac{\rho \bar{V}_1 D_1}{\mu} = (0.73)\, 1000 \frac{kg}{m^3} \times 3.01 \frac{m}{s} \times 0.0508\, m \times \frac{m^2}{5.0 \times 10^{-4} N \cdot s} \times \frac{N \cdot s^2}{kg \cdot m} = 2.23 \times 10^5$$

Thus assumption that $C = 0.99$ is okay

$$Q = 0.00611\ m^3/s \qquad Q$$

Problem 8.198

[Difficulty: 3]

8.198 Air flows through the venturi meter described in Problem 8.195. Assume that the upstream pressure is 60 psi, and that the temperature is everywhere constant at 68°F. Determine the maximum possible mass flow rate of air for which the assumption of incompressible flow is a valid engineering approximation. Compute the corresponding differential pressure reading on a mercury manometer.

Given: Flow through a venturi meter

Find: Maximum flow rate for incompressible flow; Pressure reading

Solution:

Basic equation
$$m_{actual} = \frac{C\cdot A_t}{\sqrt{1-\beta^4}}\cdot\sqrt{2\cdot\rho\cdot(p_1 - p_2)} = \frac{C\cdot A_t}{\sqrt{1-\beta^4}}\cdot\sqrt{2\cdot\rho\cdot\Delta p}$$

Note that m_{actual} is mass flow rate (the software cannot render a dot!)

Assumptions: 1) Neglect density change 2) Use ideal gas equation for density

Then
$$\rho = \frac{p}{R_{air}\cdot T} \qquad \rho = 60\cdot\frac{lbf}{in^2}\times\left(\frac{12\cdot in}{1\cdot ft}\right)^2\times\frac{lbm\cdot R}{53.33\cdot ft\cdot lbf}\times\frac{1\cdot slug}{32.2\cdot lbm}\cdot\frac{1}{(68+460)\cdot R} \qquad \rho = 9.53\times10^{-3}\cdot\frac{slug}{ft^3}$$

For incompressible flow V must be less than about 100 m/s or 330 ft/s at the throat. Hence

$$m_{actual} = \rho\cdot V_2\cdot A_2 \qquad m_{actual} = 9.53\times10^{-3}\frac{slug}{ft^3}\times330\cdot\frac{ft}{s}\times\frac{\pi}{4}\times\left(\frac{1}{4}\cdot ft\right)^2 \qquad m_{actual} = 0.154\cdot\frac{slug}{s}$$

$$\beta = \frac{D_t}{D_1} \qquad \beta = \frac{3}{6} \qquad \beta = 0.5$$

Also
$$\Delta p = \rho_{Hg}\cdot g\cdot\Delta h \qquad \Delta h = \frac{\Delta p}{\rho_{Hg}\cdot g}$$

and in addition
$$\Delta p = \frac{1}{2\cdot\rho}\cdot\left(\frac{m_{actual}}{C\cdot A_t}\right)^2\cdot\left(1-\beta^4\right) \quad \text{so} \quad \Delta h = \frac{\left(1-\beta^4\right)}{2\cdot\rho\cdot\rho_{Hg}\cdot g}\cdot\left(\frac{m_{actual}}{C\cdot A_t}\right)^2$$

For $Re_{D1} > 2\times10^5$, $0.980 < C < 0.995$. Assume C = 0.99, then check Re

$$\Delta h = \frac{\left(1-0.5^4\right)}{2}\times\frac{ft^3}{9.53\times10^{-3}\,slug}\times\frac{ft^3}{13.6\cdot1.94\cdot slug}\times\frac{s^2}{32.2\cdot ft}\times\left[0.154\,\frac{slug}{s}\times\frac{1}{0.99}\times\frac{4}{\pi}\times\left(\frac{4}{1\cdot ft}\right)^2\right]^2 \qquad \Delta h = 6.98\cdot in$$

Hence
$$V = \frac{Q}{A} = \frac{4\cdot m_{actual}}{\pi\cdot\rho\cdot D_1^2} \qquad V = \frac{4}{\pi}\times\frac{ft^3}{9.53\times10^{-3}\,slug}\times\frac{1}{\left(\frac{1}{2}\cdot ft\right)^2}\times0.154\,\frac{slug}{s} \qquad V = 82.3\cdot\frac{ft}{s}$$

At 68°F,(Table A.7)
$$\nu = 1.08\times10^{-5}\cdot\frac{ft^2}{s}$$

$$Re_{D1} = \frac{V\cdot D_1}{\nu} \qquad Re_{D1} = 82.3\cdot\frac{ft}{s}\times\frac{1}{2}\cdot ft\times\frac{s}{1.08\times10^{-5}\cdot ft^2} \qquad Re_{D1} = 3.81\times10^6$$

Thus $Re_{D1} > 2\times10^5$. The mass flow rate is $m_{actual} = 0.154\cdot\frac{slug}{s}$ and pressure $\Delta h = 6.98\cdot in$ Hg

Problem 8.199

[Difficulty: 3]

8.199 Air flow rate in a test of an internal combustion engine is to be measured using a flow nozzle installed in a plenum. The engine displacement is 1.6 liters, and its maximum operating speed is 6000 rpm. To avoid loading the engine, the maximum pressure drop across the nozzle should not exceed 0.25 m of water. The manometer can be read to ±0.5 mm of water. Determine the flow nozzle diameter that should be specified. Find the minimum rate of air flow that can be metered to ±2 percent using this setup.

Solution: Apply computing equation for flow nozzle.

Computing equation: $\dot{m} = KA_t\sqrt{2\rho(p_1 - p_2)}$ (8.54)

Assumptions: (1) $K = 0.97$ (Section 8-10.2b.)
(2) $\beta = 0$ (nozzle inlet is from atmosphere)
(3) Four-stroke cycle engine with 100 percent volumetric efficiency ($\forall$/rev = displacement/2)
(4) Standard air

Then

$$\dot{m} = \rho Q = 1.23\,\frac{kg}{m^3} \times \frac{1.6\,\ell}{2\,rev} \times 6000\,\frac{rev}{min} \times \frac{m^3}{1000\,\ell} \times \frac{min}{60\,s} = 0.0984\ kg/s$$

Solving for A_t,

$$A_t = \frac{\dot{m}}{K\sqrt{2\rho\Delta p}} = \frac{\dot{m}}{K\sqrt{2\rho\rho_{H_2O}g\Delta h}}$$

$$A_t = 0.0984\,\frac{kg}{s} \times \frac{1}{0.97}\left[\frac{1}{2} \times \frac{m^3}{1.23\,kg} \times \frac{m^3}{999\,kg} \times \frac{s^2}{9.81\,m} \times \frac{1}{0.25\,m}\right]^{\frac{1}{2}} = 1.31 \times 10^{-3}\,m^2$$

$$A_t = \frac{\pi D_t^2}{4}\ ;\ D_t = \sqrt{\frac{4A_t}{\pi}} = 40.8\ mm$$

D_t

The allowable error is ± 2 percent, or ± 0.02. As discussed in Appendix E, the square-root relationship halves the experimental uncertainty. Thus

$$e = \pm 0.02 \text{ when } e_{\Delta h} = \pm 0.04\ ;\ \Delta h_{min} = \frac{\pm 0.5\ mm}{\pm 0.04} = 12.5\ mm$$

$$\dot{m}_{min} \simeq \dot{m}\sqrt{\frac{\Delta h_{min}}{\Delta h}} = 0.0984\,\frac{kg}{s}\sqrt{\frac{12.5\,mm}{250\,mm}} = 0.0220\ kg/s$$

$\dot{m}_{min}$

The air flow rate could be measured with ±2 percent accuracy down to about

$$\omega = 6000\ rpm\ \frac{0.0220}{0.0984} = 1340\ rpm$$

with this setup.

Problem 8.200

[Difficulty: 3]

8.200 Water at 10°C flows steadily through a venturi. The pressure upstream from the throat is 200 kPa (gage). The throat diameter is 50 mm; the upstream diameter is 100 mm. Estimate the maximum flow rate this device can handle without cavitation.

Given: Flow through venturi

Find: Maximum flow rate before cavitation

Solution:

Basic equation $$m_{actual} = \frac{C \cdot A_t}{\sqrt{1-\beta^4}} \cdot \sqrt{2\cdot\rho\cdot(p_1 - p_2)} = \frac{C \cdot A_t}{\sqrt{1-\beta^4}} \cdot \sqrt{2\cdot\rho\cdot\Delta p}$$ Note that m_{actual} is mass flow rate (the software cannot render a dot!)

For $Re_{D1} > 2 \times 10^5$, $0.980 < C < 0.995$. Assume $C = 0.99$, then check Re

Available data $D_1 = 100\cdot mm$ $\quad D_t = 50\cdot mm$ $\quad p_{1g} = 200\cdot kPa$ $\quad C = 0.99$ (Asumption - verify later)

$p_{atm} = 101\cdot kPa$ $\quad p_v = 1.23\cdot kPa$ Steam tables - saturation pressure at 10°C

$\rho = 1000\cdot\frac{kg}{m^3}$ $\quad \nu = 1.3\cdot 10^{-6}\cdot\frac{m^2}{s}$ (Table A.8)

Then $A_t = \frac{\pi\cdot D_t^2}{4}$ $\quad A_t = 1963\cdot mm^2$ $\quad A_1 = \frac{\pi\cdot D_1^2}{4}$ $\quad A_1 = 7854\cdot mm^2$

$\beta = \frac{D_t}{D_1}$ $\quad \beta = 0.5$

$p_1 = p_{atm} + p_{1g}$ $\quad p_1 = 301\cdot kPa$

The smallest allowable throat pressure is the saturation pressure $\quad p_t = p_v$ $\quad p_t = 1.23\, kPa$

Hence the largest Δp is $\quad \Delta p = p_1 - p_t$ $\quad \Delta p = 300\cdot kPa$

Then $$m_{rate} = \frac{C\cdot A_t}{\sqrt{1-\beta^4}}\cdot\sqrt{2\cdot\rho\cdot\Delta p}$$ $m_{rate} = 49.2\frac{kg}{s}$

$Q = \frac{m_{rate}}{\rho}$ $\quad Q = 0.0492\frac{m^3}{s}$

$V_1 = \frac{Q}{A_1}$ $\quad V_1 = 6.26\frac{m}{s}$

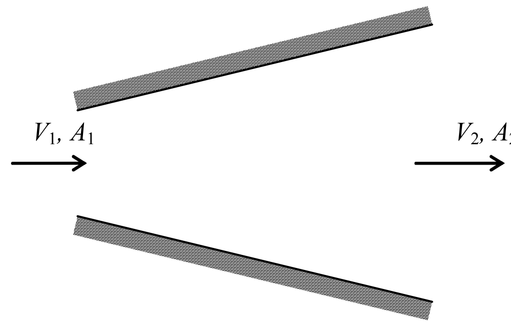
Check the Re $\quad Re_1 = \frac{V_1\cdot D_1}{\nu}$ $\quad Re_1 = 4.81\times 10^5$

Thus $Re_{D1} > 2 \times 10^5$. The volume flow rate is $\quad Q = 0.0492\frac{m^3}{s}$ $\quad Q = 49.2\frac{L}{s}$

Problem 8.201

[Difficulty: 1]

8.201 Derive Eq. 8.42, the pressure loss coefficient for a diffuser assuming ideal (frictionless) flow.



Given: Flow through a diffuser

Find: Derivation of Eq. 8.42

Solution:

Basic equations $\quad C_p = \dfrac{p_2 - p_1}{\frac{1}{2}\cdot\rho\cdot V_1^2} \qquad \dfrac{p_1}{\rho} + \dfrac{V_1^2}{2} + g\cdot z_1 = \dfrac{p_2}{\rho} + \dfrac{V_2^2}{2} + g\cdot z_2 \qquad Q = V\cdot A$

Assumptions: 1) All the assumptions of the Bernoulli equation 2) Horizontal flow 3) No flow separation

From Bernoulli $\quad \dfrac{p_2 - p_1}{\rho} = \dfrac{V_1^2}{2} - \dfrac{V_2^2}{2} = \dfrac{V_1^2}{2} - \dfrac{V_1^2}{2}\cdot\left(\dfrac{A_1}{A_2}\right)^2 \quad$ using continuity

Hence $\quad C_p = \dfrac{p_2 - p_1}{\frac{1}{2}\cdot\rho\cdot V_1^2} = \dfrac{1}{\frac{1}{2}\cdot V_1^2}\cdot\left[\dfrac{V_1^2}{2} - \dfrac{V_1^2}{2}\cdot\left(\dfrac{A_1}{A_2}\right)^2\right] = 1 - \left(\dfrac{A_1}{A_2}\right)^2$

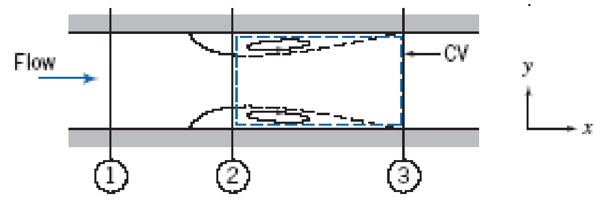
Finally $\quad C_p = 1 - \dfrac{1}{AR^2} \quad$ which is Eq. 8.42.

This result is not realistic as a real diffuser is very likely to have flow separation

8.202 Consider a flow nozzle installation in a pipe. Apply the basic equations to the control volume indicated, to show that the permanent head loss across the meter can be expressed, in dimensionless form, as the head loss coefficient,

$$C_l = \frac{p_1 - p_3}{p_1 - p_2} = \frac{1 - A_2/A_1}{1 + A_2/A_1}$$

Plot C_l as a function of diameter ratio, D_2/D_1.



Solution: Apply the Bernoulli, continuity, momentum and energy equations, using the CV shown.

Basic equations:

$$\frac{p_1}{\rho} + \frac{\bar{V}_1^2}{2} + \overset{(4)}{\cancel{g z_1}} = \frac{p_2}{\rho} + \frac{\bar{V}_2^2}{2} + \overset{(4)}{\cancel{g z_2}}$$

$$0 = \overset{=0(1)}{\cancel{\frac{\partial}{\partial t}}} \int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$

$$F_{S_x} + \overset{=0(5)}{\cancel{F_{B_x}}} = \overset{=0(1)}{\cancel{\frac{\partial}{\partial t}}} \int_{CV} u \rho \, d\forall + \int_{CS} u \rho \vec{V} \cdot d\vec{A}$$

$$\dot{Q} + \overset{=0(6)}{\cancel{\dot{W}_s}} = \overset{=0(1)}{\cancel{\frac{\partial}{\partial t}}} \int_{CV} e \rho \, d\forall + \int_{CS} \left(u + \frac{\bar{V}^2}{2} + \overset{(4)}{\cancel{g z}} + \frac{p}{\rho}\right) \rho \vec{V} \cdot d\vec{A}$$

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) No friction between ① and ②
(4) Neglect elevation terms
(5) $F_{B_x} = 0$
(6) $\dot{W}_s = 0$
(7) Uniform flow at each section

From continuity,

$$Q = \bar{V}_1 A_1 = \bar{V}_2 A_2 = \bar{V}_3 A_3$$

Apply Bernoulli along a streamline from ① to ②, noting $A_1 = A_3$,

$$\frac{p_1 - p_2}{\rho} = \frac{\bar{V}_2^2 - \bar{V}_1^2}{2} = \frac{\bar{V}_2^2}{2}\left[1 - \left(\frac{A_2}{A_1}\right)^2\right] = \frac{\bar{V}_2^2}{2}\left[1 - \left(\frac{A_2}{A_3}\right)^2\right]$$

From momentum, and using continuity,

$$F_{S_x} = p_2 A_1 - p_3 A_3 = \bar{V}_2 \{-|\rho \bar{V}_2 A_2|\} + \bar{V}_3 \{+|\rho \bar{V}_3 A_3|\} = (\bar{V}_3 - \bar{V}_2) \rho \bar{V}_3 A_3$$

or

$$\frac{p_3 - p_2}{\rho} = \bar{V}_3 (\bar{V}_2 - \bar{V}_3) = \bar{V}_2 \frac{A_2}{A_3}\left[\bar{V}_2 - \bar{V}_2 \frac{A_2}{A_3}\right] = \bar{V}_2^2 \frac{A_2}{A_3}\left(1 - \frac{A_2}{A_3}\right)$$

From energy,

$$\dot{Q} = \left(u_2 + \frac{\bar{V}_2^2}{2} + \frac{p_2}{\rho}\right)\{-|\rho \bar{V}_2 A_2|\} + \left(u_3 + \frac{\bar{V}_3^2}{2} + \frac{p_3}{\rho}\right)\{|\rho \bar{V}_3 A_3|\}$$

or $h_{l_{23}} = u_3 - u_2 - \frac{\dot{Q}}{\dot{m}} = \frac{\bar{V}_2^2 - \bar{V}_3^2}{2} - \frac{p_3 - p_2}{\rho} = \frac{\bar{V}_2^2}{2}\left[1 - \left(\frac{A_2}{A_3}\right)^2\right] - \frac{p_3 - p_2}{\rho}$

But $h_{l_{12}} \simeq 0$ by assumption (3), so $h_{l_{13}} \simeq h_{l_{23}}$ and using momentum

$$h_{l_{13}} \simeq \frac{\bar{V}_2^2}{2}\left[1 - \left(\frac{A_2}{A_3}\right)^2\right] - \bar{V}_2^2 \frac{A_2}{A_3}\left(1 - \frac{A_2}{A_3}\right)$$

After a little algebra, this may be written

$$h_{l_{13}} \simeq \frac{\bar{V}_2^2}{2}\left(1 - \frac{A_2}{A_3}\right)^2$$

Dividing by $(p_1 - p_2)/\rho$, a loss coefficient is derived as

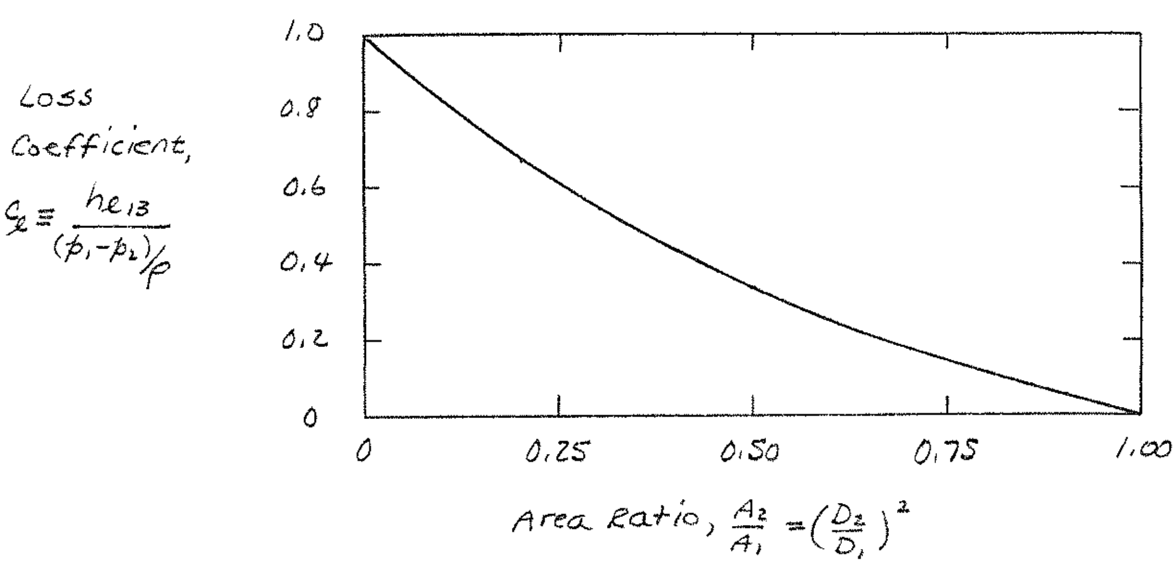
$$C_l = \frac{h_{l_{13}}}{(p_1 - p_2)/\rho} = \frac{\frac{\bar{V}_2^2}{2}\left(1 - \frac{A_2}{A_3}\right)^2}{\frac{\bar{V}_2^2}{2}\left[1 - \left(\frac{A_2}{A_3}\right)^2\right]} = \frac{\left(1 - A_2/A_3\right)^2}{\left[1 - \left(A_2/A_3\right)^2\right]}$$

But $1 - \left(\frac{A_2}{A_3}\right)^2 = \left(1 + \frac{A_2}{A_3}\right)\left(1 - \frac{A_2}{A_3}\right)$, so

$$C_l = \frac{h_{l_{13}}}{(p_1 - p_2)/\rho} = \frac{1 - A_2/A_3}{1 + A_2/A_3}$$

C_l

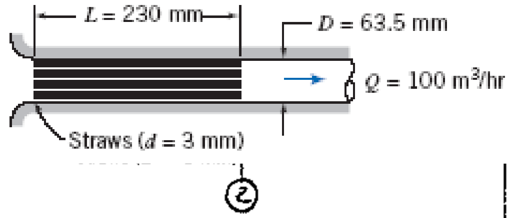
Plotting:



Problem 8.203

[Difficulty: 3]

8.203 Drinking straws are to be used to improve the air flow in a pipe-flow experiment. Packing a section of the air pipe with drinking straws to form a "laminar flow element" might allow the air flow rate to be measured directly, and simultaneously would act as a flow straightener. To evaluate this idea, determine (a) the Reynolds number for flow in each drinking straw, (b) the friction factor for flow in each straw, and (c) the gage pressure at the exit from the drinking straws. (For laminar flow in a tube, the entrance loss coefficient is $K_{ent} = 1.4$ and $\alpha = 2.0$.) Comment on the utility of this idea.



Solution: Apply energy equation for steady, incompressible pipe flow.

Computing equation:

$$\frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \alpha_2 \frac{\bar{V}_2^2}{2} + g z_2 + h_{l_T} \qquad (\bar{V}_1 \approx 0 \ (1))$$

$$h_{l_T} = h_l + h_{l_m} = f \frac{L}{D}\frac{\bar{V}^2}{2} + K_{ent}\frac{\bar{V}^2}{2} = \left(f\frac{L}{D} + K_{ent}\right)\frac{\bar{V}^2}{2}$$

Assumptions: (1) Flow from atmosphere; $p_1 = p_{atm}$, $\bar{V}_1 \approx 0$
(2) Horizontal
(3) Neglect thickness of straws

Then

$$\bar{V}_2 = \frac{Q}{A} = 100\,\frac{m^3}{hr} \times \frac{4}{\pi\,(0.0635)^2\,m^2} \times \frac{hr}{3600\,s} = 8.77\ m/s$$

$$Re_d = \frac{\bar{V}_2 d}{\nu} = 8.77\,\frac{m}{sec} \times 0.003\,m \times \frac{sec}{1.46 \times 10^{-5}\,m^2} = 1800$$ Re_d

For laminar flow,

$$f = \frac{64}{Re} = \frac{64}{1800} = 0.0356$$ f

The gage pressure at ② is

$$p_{2g} = -\frac{\rho \bar{V}_2^2}{2}\left(\alpha_2 + K_{ent} + f\frac{L}{D}\right)$$

$$= -\frac{1}{2} \times 1.23\,\frac{kg}{m^3} \times (8.77)^2\,\frac{m^2}{s^2}\left(2.0 + 1.4 + 0.0356 \times \frac{230\,mm}{3\,mm}\right)\frac{N \cdot s^2}{kg \cdot m}$$

$$p_{2g} = -290\ N/m^2\ (gage)$$ p_{2g}

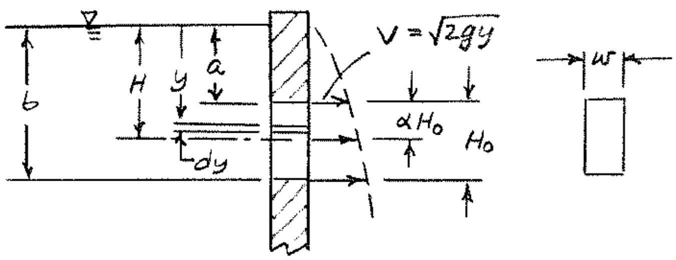
This pressure drop is equivalent to

$$\Delta h = \frac{\Delta p}{\rho_{H_2O}\, g} = 290\,\frac{N}{m^2} \times \frac{m^3}{999\,kg} \times \frac{s^2}{9.81\,m} \times \frac{kg \cdot m}{N \cdot s^2} = 29.6\ mm\ H_2O$$

Comments: (1) This pressure drop is large enough to measure readily. The straws could be used as a flowmeter.
(2) Straws would eliminate any swirl from the flow.

8.204 In some western states, water for mining and irrigation was sold by the "miner's inch," the rate at which water flows through an opening in a vertical plank of 1 in.^{2} area, up to 4 in. tall, under a head of 6 to 9 in. Develop an equation to predict the flow rate through such an orifice. Specify clearly the aspect ratio of the opening, thickness of the plank, and datum level for measurement of head (top, bottom, or middle of the opening). Show that the unit of measure varies from 38.4 (in Colorado) to 50 (in Arizona, Idaho, Nevada, and Utah) miner's inches equal to 1 ft^3/s.

Analysis: The geometry of the opening in a vertical plank is shown. The analysis includes the effect on flow speed of the variation in water depth vertically across the opening.



Aspect ratio, $ar = \frac{H_0}{w}$; area, $A = wH_0 = 1\ \text{in.}^2$

$\alpha = 0, \frac{1}{2}, 1$

$$Q_{geom} = \int V\,dA = \int_a^b \sqrt{2gy}\; w\,dy = w\sqrt{2g}\left[\frac{2}{3}y^{3/2}\right]_a^b = \frac{2}{3}wa\sqrt{2ga}\left[\left(\frac{b}{a}\right)^{3/2} - 1\right]$$

For $ar = 1$, $\alpha = 0$, $a = H = 9$ in., $b = 10.0$ in., $w = 1.0$ in.

$$Q_{geom} = \frac{2}{3} \times 1.0\ \text{in.} \times 9\ \text{in.}\left[2 \times 32.2\,\frac{\text{ft}}{\text{s}^2} \times 9\ \text{in.} \times \frac{\text{ft}}{12\ \text{in.}}\right]^{\frac{1}{2}}\left[\left(\frac{10}{9}\right)^{3/2} - 1\right]\frac{\text{ft}^2}{144\ \text{in.}^2} = 0.0496\ \frac{\text{ft}^3}{\text{s}}$$

$Q_{actual} = 0.6\,Q_{geom} = 0.0297\ \text{ft}^3/\text{s}$; thus $1/0.0297 = 33.6\ MI = 1\ cfs$ MI

Numerical results are presented in the spread sheet on the next page.

Discussion: All results assume a *vena contracta* in the liquid jet leaving the opening, reducing the effective flow area to 60 percent of the geometric area of the opening.

The calculated unit of measure varies from 31.3 to 52.4 miner's inch per cubic foot of water flow per second. This range encompasses the 38.4 and 50 values given in the problem statement.

Trends may be summarized as follows. The largest flow rate occurs when datum H is measured to the top of the opening in the vertical plank. This gives the deepest submergence and thus the highest flow speeds through the opening.

When $ar = 1$, the opening is square; when $ar = 16$, the opening is 4 inches tall and 1/4 inch wide. Increasing ar from 1 to 16 increases the flow rate through the opening when H is measured to the top of the opening, because it increases the submergence of the lower portion of the opening, thus increasing the flow speeds. When H is measured to the center of the opening ar has almost no effect on flow rate. When H is measured to the bottom of the opening, increasing ar reduces the flow rate. For this case, the depth of the opening decreases as ar becomes larger.

Plank thickness does not affect calculated flow rates since a *vena contracta* is assumed. In this flow model, water separates from the interior edges of the opening in the vertical plank. Only if the plank were several inches thick might the stream reattach and affect the flow rate.

The actual relationship between Q_{flow} and Q_{geom} might be a weak function of aspect ratio. The flow separates from all four edges of the opening in the vertical plank. At large ar, contraction on the narrow ends of the stream has a relatively small effect on flow area. As ar approaches 1 the effect becomes more pronounced, but would need to be measured experimentally. Assuming a constant 60 percent area fraction certainly gives reasonable trends.

Computation of "Miner's Inch" in Engineering Units:

a =	depth to top of opening	(in.)
ar =	aspect ratio of opening	(---)
A =	area of opening	1 in.^{2}
b =	depth to bottom of opening	(in.)
H =	nominal head	(in.)
H_0 =	height of opening	(in.)
MI =	"miner's inch"	(mixed)
Q =	volume flow rate	(ft^3/s)
w =	width of opening	(in.)

Assume $Q_{flow} = 0.6 \times Q_{geometric}$ to account for contraction of the stream leaving the opening.

(*a*) Measure *H* to top of opening:

H	ar	H_0	a	b	w	Q_{geom}	Q_{flow}	MI/cfs
9	1	1.00	9.00	10.0	1.00	0.0496	0.0297	33.6
9	2	1.41	9.00	10.4	0.707	0.0501	0.0301	33.3
9	4	2.00	9.00	11.0	0.500	0.0509	0.0305	32.8
9	8	2.83	9.00	11.8	0.354	0.0519	0.0311	32.1
9	16	4.00	9.00	13.0	0.250	0.0533	0.0320	31.3
6	1	1.00	6.00	7.00	1.00	0.0410	0.0246	40.6
6	2	1.41	6.00	7.41	0.707	0.0416	0.0250	40.0
6	4	2.00	6.00	8.00	0.500	0.0425	0.0255	39.2
6	8	2.83	6.00	8.83	0.354	0.0437	0.0262	38.1
6	16	4.00	6.00	10.0	0.250	0.0454	0.0272	36.7

(*b*) Measure *H* to middle of opening:

H	ar	H_0	a	b	w	Q_{geom}	Q_{flow}	MI/cfs
9	1	1.00	8.50	9.50	1.00	0.0483	0.0290	34.5
9	2	1.41	8.29	9.71	0.707	0.0483	0.0290	34.5
9	4	2.00	8.00	10.0	0.500	0.0482	0.0289	34.6
9	8	2.83	7.59	10.4	0.354	0.0482	0.0289	34.6
9	16	4.00	7.00	11.0	0.250	0.0482	0.0289	34.6
6	1	1.00	5.50	6.50	1.00	0.0394	0.0236	42.3
6	2	1.41	5.29	6.71	0.707	0.0394	0.0236	42.3
6	4	2.00	5.00	7.00	0.500	0.0394	0.0236	42.3
6	8	2.83	4.59	7.41	0.354	0.0393	0.0236	42.4
6	16	4.00	4.00	8.00	0.250	0.0392	0.0235	42.5

(*c*) Measure *H* to bottom of opening:

H	ar	H_0	a	b	w	Q_{geom}	Q_{flow}	MI/cfs
9	1	1.00	8.00	9.00	1.00	0.0469	0.0281	35.5
9	2	1.41	7.59	9.00	0.707	0.0463	0.0278	36.0
9	4	2.00	7.00	9.00	0.500	0.0455	0.0273	36.7
9	8	2.83	6.17	9.00	0.354	0.0442	0.0265	37.7
9	16	4.00	5.00	9.00	0.250	0.0424	0.0254	39.3
6	1	1.00	5.00	6.00	1.00	0.0377	0.0226	44.2
6	2	1.41	4.59	6.00	0.707	0.0370	0.0222	45.1
6	4	2.00	4.00	6.00	0.500	0.0359	0.0215	46.4
6	8	2.83	3.17	6.00	0.354	0.0343	0.0206	48.6
6	16	4.00	2.00	6.00	0.250	0.0318	0.0191	52.4

Problem 8.205

8.205 The volume flow rate in a circular duct may be measured by "pitot traverse," i.e., by measuring the velocity in each of several area segments across the duct, then summing. Comment on the way such a traverse should be set up. Quantify and plot the expected error in measurement of flow rate as a function of the number of radial locations used in the traverse.

Solution: First divide the duct cross section into segments of equal area. Then measure velocity at the mean area of each segment.

Assume flow is turbulent, and that the velocity profile is well represented by the 1/7-power profile. From Eq. 8.24 the ratio of average flow velocity to centerline velocity is 0.817.

Distinguish two cases, depending on whether velocity is measured at the centerline.

Case 1: Measure velocity at the duct centerline, plus at (k - 1) other locations.

For $k = 1$, the sole measurement is at the duct centerline. This measures the centerline velocity U, which is 1/0.817 = 1.22 times the average flow velocity $\bar{\ }$. Thus the volume flow rate estimated by this 1-point measurement is 22 percent larger than the true value.

For $k = 2$, the duct is divided into two segments of equal area. The centerline velocity is measured and assigned the half of the duct area surrounding the centerline. The second measurement point is located at the midpoint of the remaining half of the duct area. Thus this point is located at the radius that encloses 3/4 of the duct area, or $r_2/R = (3/4)^{½} = 0.866$, as shown on the attached spreadsheet. The velocity ratio at this point is $\bar{u}/U = 0.92$. Averaging the segmental flow rates gives (1.22 + 0.92)/2 = 1.07. Thus the volume flow rate estimated by this 2-point measurement is 7 percent high.

For $k = 3$, the duct is divided into three portions of equal area. The centerline velocity is measured and assigned the one-third of the duct area surrounding the centerline. The second measurement point is located at the midpoint of the second one-third of the duct area. This point is located at the radius that encloses half the duct area, or at $r_2/R = (1/2)^{½} = 0.707$. The third measurement point is located at the midpoint of the third one-third of the duct area. This point is located at the radius enclosing 5/6 of the duct area, or at $r_3/R = (5/6)^{½} = 0.913$.

Results of calculations for $k = 4$ and 5 are also given on the spreadsheet.

Case 2: Measure velocity at k locations, not including the centerline.

For $k = 1$, the radius is chosen at half the duct area. Thus $r_1/R = (1/2)^{½} = 0.707$, $\bar{u}/U =$ 0.839, and $\bar{u}/\bar{\ } = 1.03$, or about 3 percent too high, as shown on the spreadsheet.

For $k = 2$, the duct is divided into two equal areas. The first measurement is made at the midpoint of the inner area, where the radius includes one fourth of the total area. The second is made at the midpoint of the outer area, where the radius includes three fourths of the total duct area. The results are shown; the flow rate estimate is high by about 1.4 percent.

For $k = 3$, the duct is divided into three equal areas. The first measurement is made at the midpoint of the inner 1/3 of the duct area, where the radius includes 1/6 of the total area. The second is made at the midpoint of the second 1/3 of the duct area, where the radius includes 1/2 of the total duct area. The third is made at the midpoint of the third 1/3 of the duct area, where the radius includes 5/6 of the total duct area. The results are shown; the flow rate estimate is high by about 0.9 percent.

Results of calculations for $k = 4$ and 5 also are given on the spreadsheet.

Remarkably, Case 2 gives less than 2 percent error for any number of locations.

Problem 8.205

$V_{bar}/U =$ 0.817 $\quad n =$ 7 $\quad k =$ Number of measurement points

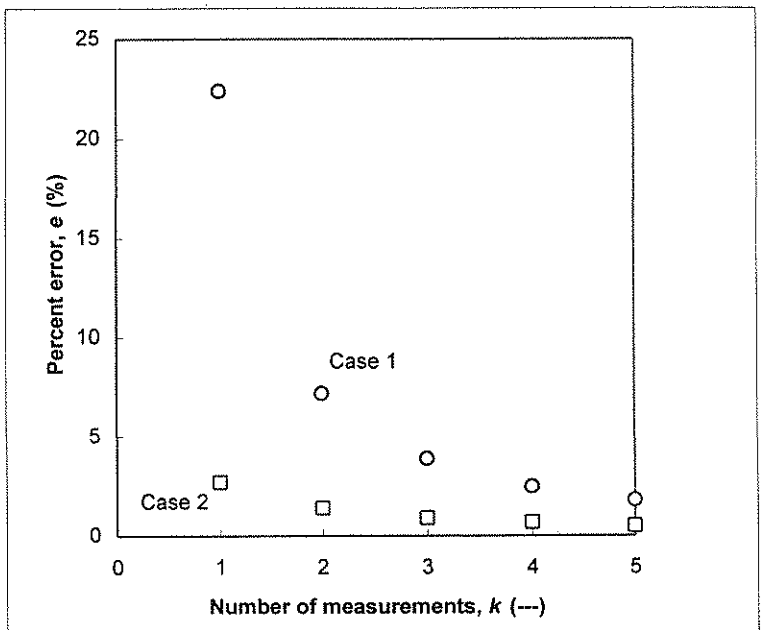
Case 1: Measure at centerline plus at (k - 1) other locations

k	i	r_i/R	u/U	u/V_{bar}	(%) Error
1	1	0.000	1.000	1.22	22.4
2	1	0.000	1.000	1.22	
	2	0.866	0.750	0.92	
				1.07	7.2
3	1	0.000	1.000	1.22	
	2	0.707	0.839	1.03	
	3	0.913	0.706	0.864	
				1.04	3.9
4	1	0.000	1.000	1.22	
	2	0.612	0.873	1.07	
	3	0.791	0.800	0.98	
	4	0.935	0.676	0.828	
				1.03	2.5
5	1	0.000	1.000	1.22	
	2	0.548	0.893	1.09	
	3	0.707	0.839	1.03	
	4	0.837	0.772	0.945	
	5	0.949	0.654	0.801	
				1.02	1.8

Case 2: Measure at k locations not including the centerline

k	i	r_i/R	u/U	u/V_{bar}	(%) Error
1	1	0.707	0.839	1.03	2.7
2	1	0.500	0.906	1.11	
	2	0.866	0.750	0.92	
				1.01	1.4
3	1	0.408	0.928	1.14	
	2	0.707	0.839	1.03	
	3	0.913	0.706	0.86	
				1.01	0.9
4	1	0.354	0.940	1.15	
	2	0.612	0.873	1.07	
	3	0.791	0.800	0.98	
	4	0.935	0.676	0.83	
				1.01	0.7
5	1	0.316	0.947	1.16	
	2	0.548	0.893	1.09	
	3	0.707	0.839	1.03	
	4	0.837	0.772	0.95	
	5	0.949	0.654	0.80	
				1.01	0.5

	Case 1	Case 2
k	e (%)	e (%)
1	22.4	2.7
2	7.2	1.4
3	3.9	0.9
4	2.5	0.7
5	1.8	0.5



Problem 9.1

[Difficulty: 2]

9.1 The roof of a minivan is approximated as a horizontal flat plate. Plot the length of the laminar boundary layer as a function of minivan speed, V, as the minivan accelerates from 10 mph to 90 mph.

Given: Minivan traveling at various speeds

Find: Plot of boundary layer length as function of speed

Solution:

Governing equations:

The critical Reynolds number for transition to turbulence is

$$Re_{crit} = \rho V L_{crit}/\mu = 500000$$

The critical length is then

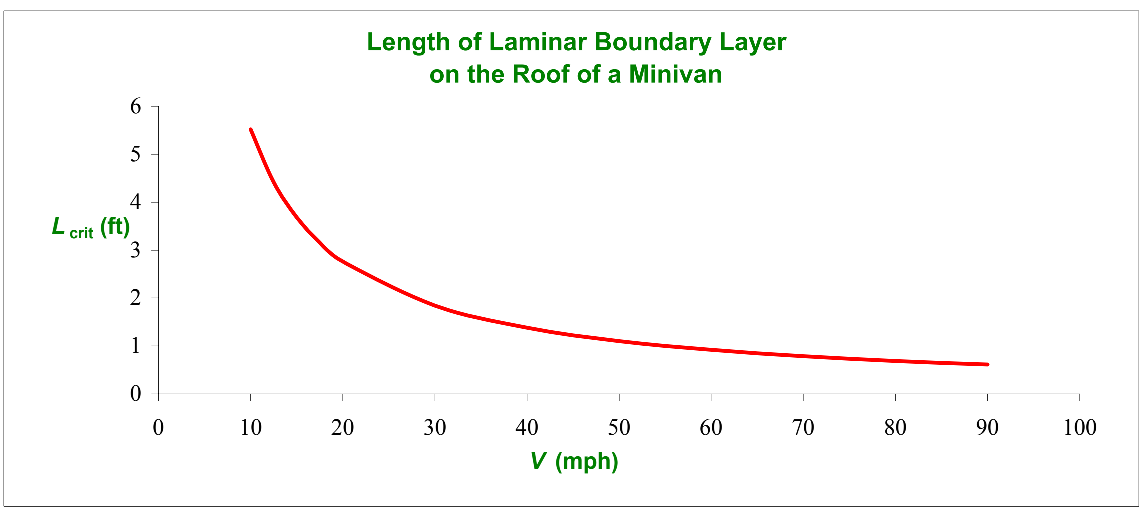
$$L_{crit} = 500000\mu/V\rho$$

Tabulated or graphical data:

$\mu =$ 3.79E-07 lbf.s/ft^2
$\rho =$ 0.00234 slug/ft^3
(Table A.9, 68^{o}F)

Computed results:

V (mph)	L_{crit} (ft)
10	5.52
13	4.42
15	3.68
18	3.16
20	2.76
30	1.84
40	1.38
50	1.10
60	0.920
70	0.789
80	0.690
90	0.614



Problem 9.2

[Difficulty: 2]

9.2 A model of a river towboat is to be tested at 1:18 scale. The boat is designed to travel at 3.5 m/s in fresh water at 10°C. Estimate the distance from the bow where transition occurs. Where should transition be stimulated on the model towboat?

Given: Model of riverboat

Find: Distance at which transition occurs

Solution:

Basic equation $Re_x = \frac{\rho \cdot U \cdot x}{\mu} = \frac{U \cdot x}{\nu}$ and transition occurs at about $Re_x = 5 \times 10^5$

For water at 10ºC $\nu = 1.30 \times 10^{-6} \cdot \frac{m^2}{s}$ (Table A.8) and we are given $U = 3.5 \cdot \frac{m}{s}$

Hence $x_p = \frac{\nu \cdot Re_x}{U}$ $x_p = 0.186\,m$ $x_p = 18.6 \cdot cm$

For the model $x_m = \frac{x_p}{18}$ $x_m = 0.0103\,m$ $x_m = 10.3 \cdot mm$

Problem 9.3

[Difficulty: 3]

9.3 The takeoff speed of a Boeing 757 is 160 mph. At approximately what distance will the boundary layer on the wings become turbulent? If it cruises at 530 mph at 33,000 ft, at approximately what distance will the boundary layer on the wings become turbulent?

Given:
Boeing 757

Find:
Point at which BL transition occurs during takeoff and at cruise

Solution:

Basic equation $\quad Re_x = \frac{\rho \cdot U \cdot x}{\mu} = \frac{U \cdot x}{\nu}$ $\quad$ and transition occurs at about $\quad Re_x = 5 \times 10^5$

For air at 68°F $\quad \nu = 1.62 \times 10^{-4} \cdot \frac{ft^2}{s}$ $\quad$ (Table A.9) $\quad$ and we are given $\quad U = 160 \cdot \frac{mi}{hr} = 234.7 \cdot \frac{ft}{s}$

Hence $\quad x_p = \frac{\nu \cdot Re_x}{U}$ $\quad x_p = 0.345 \cdot ft$ $\quad x_p = 4.14 \cdot in$

At 33,000 ft $\quad T = 401.9 \cdot R$ $\quad$ (Intepolating from Table A.3) $\quad T = -57.8 \cdot °F$

We need to estimate ν or μ at this temperature. From Appendix A-3

$$\mu = \frac{b \cdot \sqrt{T}}{1 + \frac{S}{T}} \qquad b = 1.458 \times 10^{-6} \cdot \frac{kg}{m \cdot s \cdot K^{\frac{1}{2}}} \qquad S = 110.4 \cdot K$$

Hence $\quad \mu = \frac{b \cdot \sqrt{T}}{1 + \frac{S}{T}}$ $\quad \mu = 1.458 \times 10^{-5} \cdot \frac{N \cdot s}{m^2}$ $\quad \mu = 3.045 \times 10^{-7} \cdot \frac{lbf \cdot s}{ft^2}$

For air at 10,000 m (Table A.3)

$$\frac{\rho}{\rho_{SL}} = 0.3376 \qquad \rho_{SL} = 0.002377 \cdot \frac{slug}{ft^3} \qquad \rho = 0.3376 \cdot \rho_{SL} \qquad \rho = 8.025 \times 10^{-4} \cdot \frac{slug}{ft^3}$$

$\nu = \frac{\mu}{\rho}$ $\quad \nu = 3.79 \times 10^{-4} \cdot \frac{ft^2}{s}$ $\quad$ and we are given $\quad U = 530 \cdot \frac{mi}{hr}$

Hence $\quad x_p = \frac{\nu \cdot Re_x}{U}$ $\quad x_p = 0.244 \cdot ft$ $\quad x_p = 2.93 \cdot in$

Problem 9.4

[Difficulty: 2]

9.4 A student is to design an experiment involving dragging a sphere through a tank of fluid to illustrate (a) "creeping flow" ($Re_D < 1$) and (b) flow for which the boundary layer becomes turbulent ($Re_D \approx 2.5 \times 10^5$). She proposes to use a smooth sphere of diameter 1 cm in SAE 10 oil at room temperature. Is this realistic for both cases? If either case is unrealistic, select an alternative reasonable sphere diameter and common fluid for that case.

Given: Experiment with 1 cm diameter sphere in SAE 10 oil

Find: Reasonableness of two flow extremes

Solution:

Basic equation $Re_D = \frac{\rho \cdot U \cdot D}{\mu} = \frac{U \cdot D}{\nu}$ and transition occurs at about $Re_D = 2.5 \times 10^5$

For SAE 10 $\nu = 1.1 \times 10^{-4} \cdot \frac{m^2}{s}$ (Fig. A.3 at 20°C) and $D = 1 \cdot cm$

For $Re_D = 1$ we find $U = \frac{\nu \cdot Re_D}{D}$ $U = 0.011 \cdot \frac{m}{s}$ $U = 1.10 \cdot \frac{cm}{s}$ which is reasonable

For $Re_D = 2.5 \times 10^5$ $U = \frac{\nu \cdot Re_D}{D}$ $U = 2750 \frac{m}{s}$ which is much too high!

Note that for $Re_D = 2.5 \times 10^5$ we need to increase the sphere diameter D by a factor of about 1000, or reduce the viscosity ν by the same factor, or some combination of these. One possible solution is

For water $\nu = 1.01 \times 10^{-6} \cdot \frac{m^2}{s}$ (Table A.8 at 20°C) and $D = 10 \cdot cm$

For $Re_D = 2.5 \times 10^5$ we find $U = \frac{\nu \cdot Re_D}{D}$ $U = 2.52 \cdot \frac{m}{s}$ which is reasonable

Hence one solution is to use a 10 cm diameter sphere in a water tank.

Problem 9.5

[Difficulty: 2]

9.5 For flow around a sphere the boundary layer becomes turbulent around $Re_D \approx 2.5 \times 10^5$. Find the speeds at which (a) an American golf ball ($D = 1.68$ in.), (b) a British golf ball ($D = 41.1$mm), and (c) a soccer ball ($D = 8.75$ in.) develop turbulent boundary layers. Assume standard atmospheric conditions.

Given: Flow around American and British golf balls, and soccer ball

Find: Speed at which boundary layer becomes turbulent

Solution:

Basic equation $\quad Re_D = \frac{\rho \cdot U \cdot D}{\mu} = \frac{U \cdot D}{\nu}$ and transition occurs at about $\quad Re_D = 2.5 \times 10^5$

For air $\quad \nu = 1.62 \times 10^{-4} \cdot \frac{ft^2}{s}$ (Table A.9)

For the American golf ball $\quad D = 1.68 \cdot in \quad$ Hence $\quad U = \frac{\nu \cdot Re_D}{D} \quad U = 289 \cdot \frac{ft}{s} \quad U = 197 \cdot mph \quad U = 88.2 \frac{m}{s}$

For the British golf ball $\quad D = 41.1 \cdot mm \quad$ Hence $\quad U = \frac{\nu \cdot Re_D}{D} \quad U = 300 \cdot \frac{ft}{s} \quad U = 205 \cdot mph \quad U = 91.5 \frac{m}{s}$

For soccer ball $\quad D = 8.75 \cdot in \quad$ Hence $\quad U = \frac{\nu \cdot Re_D}{D} \quad U = 55.5 \cdot \frac{ft}{s} \quad U = 37.9 \cdot mph \quad U = 16.9 \frac{m}{s}$

Problem 9.6

[Difficulty: 2]

9.6 1 m × 2 m sheet of plywood is attached to the roof of your vehicle after being purchased at the hardware store. At what speed (in kilometers per hour, in 20°C air) will the boundary layer first start becoming turbulent? At what speed is about 90 percent of the boundary layer turbulent?

Given: Sheet of plywood attached to the roof of a car

Find: Speed at which boundary layer becomes turbulent; Speed at which 90% is turbulent

Solution:

Basic equation $Re_x = \frac{\rho \cdot U \cdot x}{\mu} = \frac{U \cdot x}{\nu}$ and transition occurs at about $Re_x = 5 \times 10^5$

For air $\nu = 1.50 \times 10^{-5} \cdot \frac{m^2}{s}$ (Table A.10)

Now if we assume that we orient the plywood such that the longer dimension is parallel to the motion of the car, we can say: $x = 2 \cdot m$

Hence $U = \frac{\nu \cdot Re_x}{x}$ $U = 3.8 \cdot \frac{m}{s}$ $U = 13.50 \cdot \frac{km}{hr}$

When 90% of the boundary layer is turbulent $x = 0.1 \times 2 \cdot m$ Hence $U = \frac{\nu \cdot Re_x}{x}$ $U = 37.5 \cdot \frac{m}{s}$ $U = 135.0 \cdot \frac{km}{hr}$

Problem 9.7

[Difficulty: 2]

9.7 Plot on one graph the length of the laminar boundary layer on a flat plate, as a function of freestream velocity, for (a) water and standard air at (b) sea level and (c) 10 km altitude. Use log-log axes, and compute data for the boundary-layer length ranging from 0.01 m to 10 m.

Given: Laminar boundary layer (air & water)

Find: Plot of boundary layer length as function of speed (at various altitudes for air)

Solution:

Governing equations:

The critical Reynolds number for transition to turbulence is

$$Re_{crit} = UL_{crit}/\mu = 500000$$

The critical length is then

$$L_{crit} = 500000\mu/U\rho$$

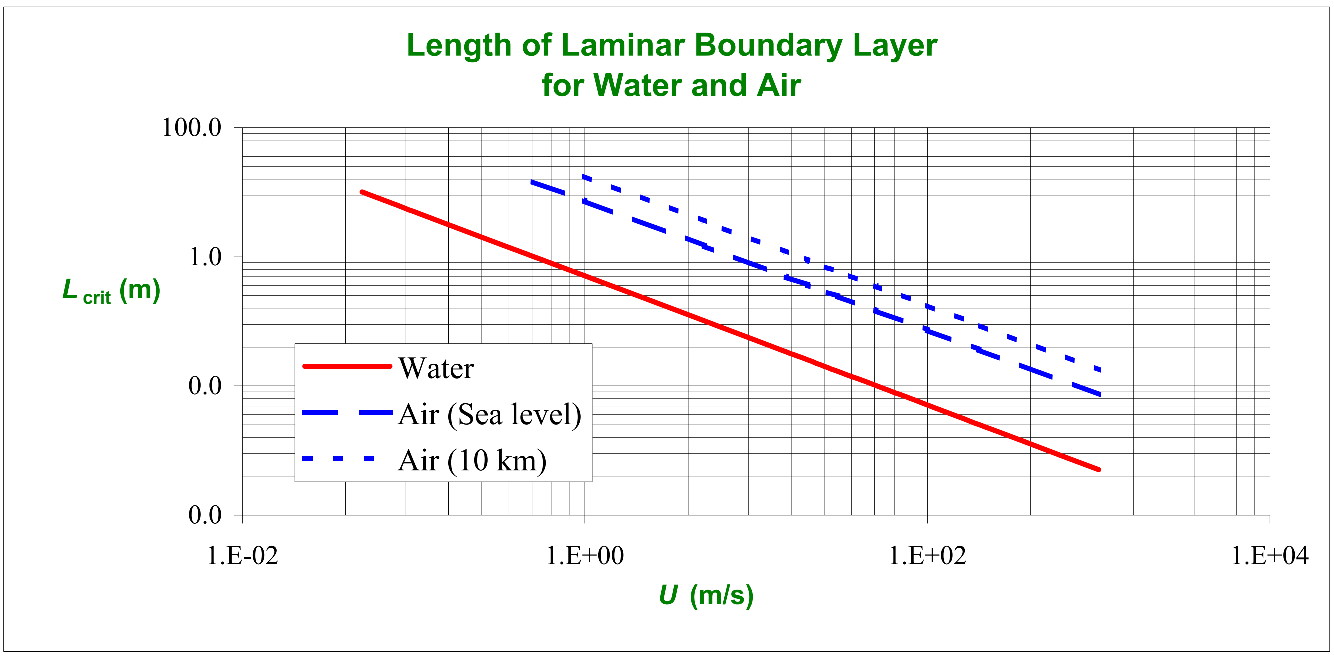
For air at sea level and 10 km, we can use tabulated data for density ρ from Table A.3.
For the viscosity μ, use the Sutherland correlation (Eq. A.1)

$$\mu = bT^{1/2}/(1+S/T)$$
b = 1.46E-06 kg/m.s.K$^{1/2}$
S = 110.4 K

Air (sea level, T = 288.2 K):			Air (10 km, T = 223.3 K):			Water (20^{o}C):		
ρ =	1.225	kg/m^3	ρ =	0.414	kg/m^3	ρ =	998	slug/ft^3
	(Table A.3)			(Table A.3)		μ =	1.01E-03	N.s/m^2
μ =	1.79E-05	N.s/m^2	μ =	1.46E-05	N.s/m^2		(Table A.8)	
	(Sutherland)			(Sutherland)				

Computed results:

U (m/s)	Water L_{crit} (m)	Air (Sea level) L_{crit} (m)	Air (10 km) L_{crit} (m)
0.05	10.12	146.09	352.53
0.10	5.06	73.05	176.26
0.5	1.01	14.61	35.25
1.0	0.506	7.30	17.63
5.0	0.101	1.46	3.53
15	0.0337	0.487	1.18
20	0.0253	0.365	0.881
25	0.0202	0.292	0.705
30	0.0169	0.243	0.588
50	0.0101	0.146	0.353
100	0.00506	0.0730	0.176
200	0.00253	0.0365	0.0881
1000	0.00051	0.0073	0.0176



Problem 9.8

[Difficulty: 2]

9.8 The extent of the laminar boundary layer on the surface of an aircraft or missile varies with altitude. For a given speed, will the laminar boundary-layer length increase or decrease with altitude? Why? Plot the ratio of laminar boundary-layer length at altitude z, to boundary-layer length at sea level, as a function of z, up to altitude $z = 30$ km, for a standard atmosphere.

Given: Aircraft or missile at various altitudes

Find: Plot of boundary layer length as function of altitude

Solution:

Governing equations:

The critical Reynolds number for transition to turbulence is

$$Re_{crit} = \rho U L_{crit}/\mu = 500000$$

The critical length is then

$$L_{crit} = 500000\mu/U\rho$$

Let L_0 be the length at sea level (density ρ_0 and viscosity μ_0). Then

$$L_{crit}/L_0 = (\mu/\mu_0)/(\rho/\rho_0)$$

The viscosity of air increases with temperature so generally decreases with elevation;
the density also decreases with elevation, but much more rapidly.
Hence we expect that the length ratio increases with elevation

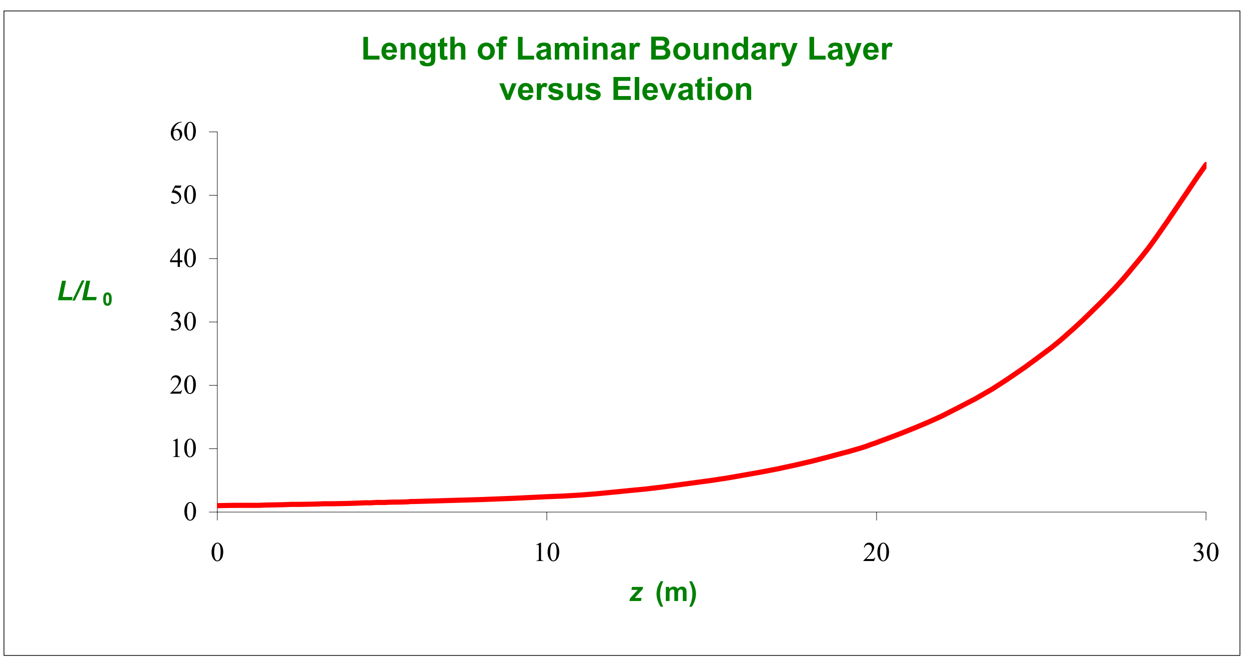
For the density ρ, we use data from Table A.3.
For the viscosity μ, we use the Sutherland correlation (Eq. A.1)

$$\mu = bT^{1/2}/(1+S/T)$$

$b =$ 1.46E-06 kg/m.s.K$^{1/2}$
$S =$ 110.4 K

Computed results:

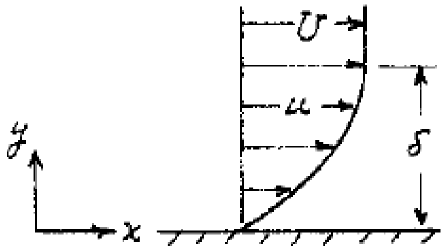
z (km)	T (K)	ρ/ρ_0	μ/μ_0	L_{crit}/L_0
0.0	288.2	1.0000	1.000	1.000
0.5	284.9	0.9529	0.991	1.04
1.0	281.7	0.9075	0.982	1.08
1.5	278.4	0.8638	0.973	1.13
2.0	275.2	0.8217	0.965	1.17
2.5	271.9	0.7812	0.955	1.22
3.0	268.7	0.7423	0.947	1.28
3.5	265.4	0.7048	0.937	1.33
4.0	262.2	0.6689	0.928	1.39
4.5	258.9	0.6343	0.919	1.45
5.0	255.7	0.6012	0.910	1.51
6.0	249.2	0.5389	0.891	1.65
7.0	242.7	0.4817	0.872	1.81
8.0	236.2	0.4292	0.853	1.99
9.0	229.7	0.3813	0.834	2.19
10.0	223.3	0.3376	0.815	2.41
11.0	216.8	0.2978	0.795	2.67
12.0	216.7	0.2546	0.795	3.12
13.0	216.7	0.2176	0.795	3.65
14.0	216.7	0.1860	0.795	4.27
15.0	216.7	0.1590	0.795	5.00
16.0	216.7	0.1359	0.795	5.85
17.0	216.7	0.1162	0.795	6.84
18.0	216.7	0.0993	0.795	8.00
19.0	216.7	0.0849	0.795	9.36
20.0	216.7	0.0726	0.795	10.9
22.0	218.6	0.0527	0.800	15.2
24.0	220.6	0.0383	0.806	21.0
26.0	222.5	0.0280	0.812	29.0
28.0	224.5	0.0205	0.818	40.0
30.0	226.5	0.0150	0.824	54.8



Problem 9.9

[Difficulty: 2]

9.9 The most general sinusoidal velocity profile for laminar boundary-layer flow on a flat plate is $u = A \sin(By) + C$. State three boundary conditions applicable to the laminar boundary-layer velocity profile. Evaluate constants A, B, and C.



Given: Sinusoidal velocity profile for laminar boundary layer:

$$u = A \cdot \sin(B \cdot y) + C$$

Find: (a) Three boundary conditions applicable to this profile
(b) Constants A, B, and C.

Solution: For the boundary layer, the following conditions apply:

$u = 0$ at $y = 0$ (no slip condition)

$u = U$ at $y = \delta$ (continuity with freestream)

$\frac{\partial}{\partial y} u = 0$ at $y = \delta$ (no shear stress at freestream)

Applying these boundary conditions:

(1) $u(0) = A \cdot \sin(0) + C = 0 \qquad C = 0$

(2) $u(\delta) = A \cdot \sin(B \cdot \delta) = U$

(3) $\frac{\partial}{\partial y} u = A \cdot B \cdot \cos(B \cdot y)$ Thus: $\frac{\partial}{\partial y} u(\delta) = A \cdot B \cdot \cos(B \cdot \delta) = 0$ Therefore: $B \cdot \delta = \frac{\pi}{2}$ or $B = \frac{\pi}{2 \cdot \delta}$

Back into (2): $A \cdot \sin\left(\frac{\pi}{2\delta} \cdot \delta\right) = U$ Therefore: $A = U$

So the expression for the velocity profile is: $$u = U \cdot \sin\left(\frac{\pi}{2} \cdot \frac{y}{\delta}\right)$$

Problem 9.10

[Difficulty: 2]

9.10 Velocity profiles in laminar boundary layers often are approximated by the equations

Linear: $\dfrac{u}{U} = \dfrac{y}{\delta}$

Sinusoidal: $\dfrac{u}{U} = \sin\left(\dfrac{\pi}{2}\dfrac{y}{\delta}\right)$

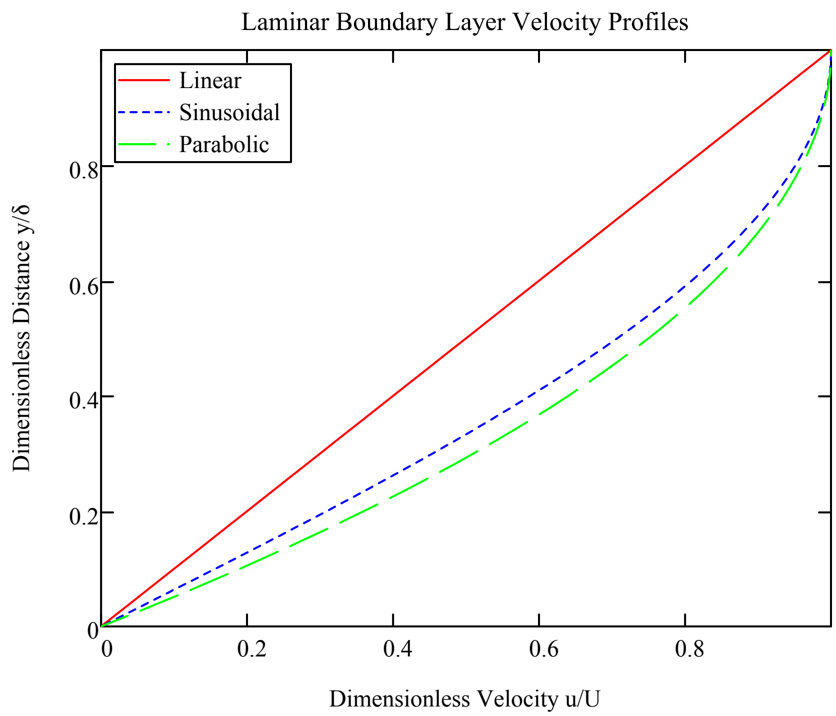
Parabolic: $\dfrac{u}{U} = 2\left(\dfrac{y}{\delta}\right) - \left(\dfrac{y}{\delta}\right)^2$

Compare the shapes of these velocity profiles by plotting y/δ (on the ordinate) versus u/U (on the abscissa).

Given: Linear, sinusoidal, and parabolic velocity profiles

Find: Plots of y/δ vs u/U for all three profiles

Solution: Here are the profiles:



Problem 9.11

[Difficulty: 2]

9.11 An approximation for the velocity profile in a laminar boundary layer is

$$\frac{u}{U} = \frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$$

Does this expression satisfy boundary conditions applicable to the laminar boundary-layer velocity profile? Evaluate δ^*/δ and θ/δ.

Given: Laminar boundary layer profile

Find: If it satisfies BC's; Evaluate δ^*/δ and θ/δ

Solution:

The boundary layer equation is $\quad \frac{u}{U} = \frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$ for which $u = U$ at $y = \delta$

The BC's are $\quad u(0) = 0 \qquad \left.\frac{du}{dy}\right|_{y=\delta} = 0$

At $y = 0$ $\quad \frac{u}{U} = \frac{3}{2}(0) - \frac{1}{2}(0)^3 = 0$

At $y = \delta$ $\quad \frac{du}{dy} = U\left.\left(\frac{3}{2}\frac{1}{\delta} - \frac{3}{2}\frac{y^2}{\delta^3}\right)\right|_{y=\delta} = U\left(\frac{3}{2}\frac{1}{\delta} - \frac{3}{2}\frac{\delta^2}{\delta^3}\right) = 0$

For δ^*: $\quad \delta^* = \int_0^\infty \left(1 - \frac{u}{U}\right)dy = \int_0^\delta \left(1 - \frac{u}{U}\right)dy$

Then $\quad \frac{\delta^*}{\delta} = \frac{1}{\delta}\int_0^\delta \left(1 - \frac{u}{U}\right)dy = \int_0^1 \left(1 - \frac{u}{U}\right)d\left(\frac{y}{\delta}\right) = \int_0^1 \left(1 - \frac{u}{U}\right)d\eta$

with $\quad \frac{u}{U} = \frac{3}{2}\eta - \frac{1}{2}\eta^3$

Hence $\quad \frac{\delta^*}{\delta} = \int_0^1 \left(1 - \frac{u}{U}\right)d\eta = \int_0^1 \left(1 - \frac{3}{2}\eta + \frac{1}{2}\eta^3\right)d\eta = \left[\eta - \frac{3}{4}\eta^2 + \frac{1}{8}\eta^4\right]_0^1 = \frac{3}{8} = 0.375$

For θ: $\quad \theta = \int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right)dy = \int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right)dy$

Then $\quad \frac{\theta}{\delta} = \frac{1}{\delta}\int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right)dy = \int_0^1 \frac{u}{U}\left(1 - \frac{u}{U}\right)d\left(\frac{y}{\delta}\right) = \int_0^1 \frac{u}{U}\left(1 - \frac{u}{U}\right)d\eta$

Hence $\quad \frac{\theta}{\delta} = \int_0^1 \frac{u}{U}\left(1 - \frac{u}{U}\right)d\eta = \int_0^1 \left(\frac{3}{2}\eta - \frac{1}{2}\eta^3\right)\left(1 - \frac{3}{2}\eta + \frac{1}{2}\eta^3\right)d\eta = \int_0^1 \left(\frac{3}{2}\eta - \frac{9}{4}\eta^2 - \frac{1}{2}\eta^3 + \frac{3}{2}\eta^4 - \frac{1}{4}\eta^6\right)d\eta$

$$\frac{\theta}{\delta} = \left[\frac{3}{4}\eta^2 - \frac{3}{4}\eta^3 - \frac{1}{8}\eta^4 + \frac{3}{10}\eta^5 - \frac{1}{28}\eta^7\right]_0^1 = \frac{39}{280} = 0.139$$

Problem 9.12

[Difficulty: 2]

9.12 An approximation for the velocity profile in a laminar boundary layer is

$$\frac{u}{U} = 2\frac{y}{\delta} - 2\left(\frac{y}{\delta}\right)^3 + \left(\frac{y}{\delta}\right)^4$$

Does this expression satisfy boundary conditions applicable to the laminar boundary-layer velocity profile? Evaluate δ^*/δ and θ/δ.

Given:

Laminar boundary layer profile

Find:

If it satisfies BC's; Evaluate δ^*/δ and θ/δ

Solution:

The boundary layer equation is $\frac{u}{U} = 2\frac{y}{\delta} - 2\left(\frac{y}{\delta}\right)^3 + \left(\frac{y}{\delta}\right)^4$ for which $u = U$ at $y = \delta$

The BC's are $u(0) = 0 \qquad \left.\frac{du}{dy}\right|_{y=\delta} = 0$

At $y = 0$ $\qquad \frac{u}{U} = 2(0) - 2(0)^3 + (0)^4 = 0$

At $y = \delta$ $\qquad \frac{du}{dy} = U\left(2\frac{1}{\delta} - 6\frac{y^2}{\delta^3} + 4\frac{y^3}{\delta^4}\right)\Bigg|_{y=\delta} = U\left(2\frac{1}{\delta} - 6\frac{\delta^2}{\delta^3} + 4\frac{\delta^3}{\delta^4}\right) = 0$

For δ^*: $\qquad \delta^* = \int_0^\infty \left(1 - \frac{u}{U}\right)dy = \int_0^\delta \left(1 - \frac{u}{U}\right)dy$

Then $\qquad \frac{\delta^*}{\delta} = \frac{1}{\delta}\int_0^\delta \left(1 - \frac{u}{U}\right)dy = \int_0^1 \left(1 - \frac{u}{U}\right)d\left(\frac{y}{\delta}\right) = \int_0^1 \left(1 - \frac{u}{U}\right)d\eta$

with $\qquad \frac{u}{U} = 2\eta - 2\eta^3 + \eta^4$

Hence $\qquad \frac{\delta^*}{\delta} = \int_0^1 \left(1 - \frac{u}{U}\right)d\eta = \int_0^1 \left(1 - 2\eta + 2\eta^3 - \eta^4\right)d\eta = \left[\eta - \eta^2 + \frac{1}{2}\eta^4 - \frac{1}{5}\eta^5\right]_0^1 = \frac{3}{10} = 0.3$

For θ: $\qquad \theta = \int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right)dy = \int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right)dy$

Then $\qquad \frac{\theta}{\delta} = \frac{1}{\delta}\int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right)dy = \int_0^1 \frac{u}{U}\left(1 - \frac{u}{U}\right)d\left(\frac{y}{\delta}\right) = \int_0^1 \frac{u}{U}\left(1 - \frac{u}{U}\right)d\eta$

Hence

$$\frac{\theta}{\delta} = \int_0^1 \frac{u}{U}\left(1 - \frac{u}{U}\right)d\eta = \int_0^1 \left(2\eta - \eta^3 + \eta^4\right)\left(1 - 2\eta + \eta^3 - \eta^4\right)d\eta = \int_0^1 \left(2\eta - 4\eta^2 - 2\eta^3 + 9\eta^4 - 4\eta^5 - 4\eta^6 + 4\eta^7 - \eta^8\right)d\eta$$

$$\frac{\theta}{\delta} = \left[\eta^2 - \frac{4}{3}\eta^3 - \frac{1}{2}\eta^4 + \frac{9}{5}\eta^5 - \frac{4}{7}\eta^7 + \frac{1}{2}\eta^8 - \frac{1}{9}\eta^9\right]_0^1 = \frac{37}{315} = 0.117$$

Problem 9.13

[Difficulty: 3]

9.13 A simplistic laminar boundary-layer model is

$$\frac{u}{U} = \sqrt{2}\frac{y}{\delta} \qquad 0 < y \le \frac{\delta}{2}$$

$$\frac{u}{U} = (2-\sqrt{2})\frac{y}{\delta} + (\sqrt{2}-1) \qquad \frac{\delta}{2} < y \le \delta$$

Does this expression satisfy boundary conditions applicable to the laminar boundary-layer velocity profile? Evaluate δ^*/δ and θ/δ.

Given: Laminar boundary layer profile

Find: If it satisfies BC's; Evaluate δ^*/δ and θ/δ

Solution:

The boundary layer equation is $\quad \dfrac{u}{U} = \sqrt{2}\dfrac{y}{\delta} \qquad 0 < y < \dfrac{\delta}{2}$

$$\frac{u}{U} = \left(2-\sqrt{2}\right)\frac{y}{\delta} + \left(\sqrt{2}-1\right) \qquad \frac{\delta}{2} < y < \delta$$ for which $u = U$ at $y = \delta$

The BC's are $\quad u(0) = 0 \qquad \left.\dfrac{du}{dy}\right|_{y=\delta} = 0$

At $y = 0$ $\quad \dfrac{u}{U} = \sqrt{2}(0) = 0$

At $y = \delta$ $\quad \dfrac{du}{dy} = U\left[\left(2-\sqrt{2}\right)\dfrac{1}{\delta}\right]_{y=\delta} \neq 0$ so it fails the outer BC.

This simplistic distribution is a piecewise linear profile: The first half of the layer has velocity gradient $\sqrt{2}\dfrac{U}{\delta} = 1.414\dfrac{U}{\delta}$, and the second half has velocity gradient $\left(2-\sqrt{2}\right)\dfrac{U}{\delta} = 0.586\dfrac{U}{\delta}$. At $y = \delta$, we make another transition to zero velocity gradient.

For δ^*: $\quad \delta^* = \displaystyle\int_0^\infty \left(1-\frac{u}{U}\right)dy = \int_0^\delta \left(1-\frac{u}{U}\right)dy$

Then $\quad \dfrac{\delta^*}{\delta} = \dfrac{1}{\delta}\displaystyle\int_0^\delta \left(1-\frac{u}{U}\right)dy = \int_0^1 \left(1-\frac{u}{U}\right)d\left(\frac{y}{\delta}\right) = \int_0^1 \left(1-\frac{u}{U}\right)d\eta$

with $\quad \dfrac{u}{U} = \sqrt{2}\eta \qquad 0 < \eta < \dfrac{1}{2}$

$$\frac{u}{U} = \left(2-\sqrt{2}\right)\eta + \left(\sqrt{2}-1\right) \qquad \frac{1}{2} < \eta < 1$$

Hence

$$\frac{\delta^*}{\delta} = \int_0^1 \left(1-\frac{u}{U}\right)d\eta = \int_0^{1/2} \left(1-\sqrt{2}\eta\right)d\eta + \int_{1/2}^1 \left[1-\left(2-\sqrt{2}\right)\eta - \left(\sqrt{2}-1\right)\right]d\eta = \left[\frac{1}{2\sqrt{2}}\left(\sqrt{2}\eta-1\right)^2\right]_0^{1/2} + \left[\frac{1}{2}(\eta-1)^2\left(\sqrt{2}-2\right)\right]_{1/2}^1$$

$$\frac{\delta *}{\delta} = \left[\frac{1}{2} - \frac{\sqrt{2}}{8}\right] + \left[\frac{1}{4} - \frac{\sqrt{2}}{8}\right] = \frac{3}{4} - \frac{\sqrt{2}}{4} = 0.396$$

For θ:

$$\theta = \int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right)dy = \int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right)dy$$

Then

$$\frac{\theta}{\delta} = \frac{1}{\delta}\int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right)dy = \int_0^1 \frac{u}{U}\left(1 - \frac{u}{U}\right)d\left(\frac{y}{\delta}\right) = \int_0^1 \frac{u}{U}\left(1 - \frac{u}{U}\right)d\eta$$

Hence, after a LOT of work

$$\frac{\theta}{\delta} = \int_0^1 \frac{u}{U}\left(1 - \frac{u}{U}\right)d\eta = \int_0^{1/2} \sqrt{2}\eta\left(1 - \sqrt{2}\eta\right)d\eta + \int_{1/2}^1 \left[\left(\left(2 - \sqrt{2}\right)\eta + \left(\sqrt{2} - 1\right)\right)\left(1 - \left(2 - \sqrt{2}\right)\eta - \left(\sqrt{2} - 1\right)\right)\right]d\eta$$

$$\frac{\theta}{\delta} = \left[\sqrt{2}\eta^2\left(\frac{\sqrt{2}\eta}{3} - \frac{1}{2}\right)\right]_0^{1/2} + \left[\left(\frac{1}{3}\left(\sqrt{2} - 2\right)\left(\eta - 1\right) - \frac{1}{2}\right)\left(\sqrt{2} - 2\right)\left(\eta - 1\right)^2\right]_{1/2}^1 = \frac{\sqrt{2}}{8} - \frac{1}{12} + \frac{\sqrt{2}}{24} = \frac{\sqrt{2}}{6} - \frac{1}{12} = 0.152$$

Problem 9.14

[Difficulty: 2]

9.14 The velocity profile in a turbulent boundary layer often is approximated by the $\frac{1}{7}$-power-law equation

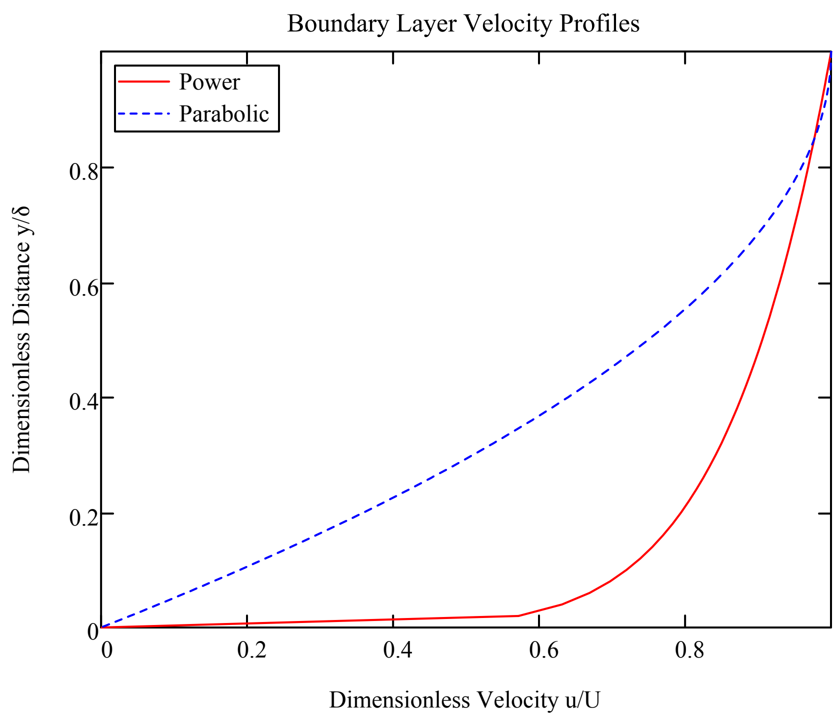
$$\frac{u}{U} = \left(\frac{y}{\delta}\right)^{1/7}$$

Compare the shape of this profile with the parabolic laminar boundary-layer velocity profile (Problem 9.10) by plotting y/δ (on the ordinate) versus u/U (on the abscissa) for both profiles.

Given: Power law velocity profiles

Find: Plots of y/δ vs u/U for this profile and the parabolic profile of Problem 9.10

Solution: Here are the profiles:



Note that the power law profile gives and infinite value of du/dy as y approaches zero:

$$\frac{du}{dy} = \frac{U}{\delta}\frac{d(u/U)}{d(y/\delta)} = \frac{U}{7\delta}\left(\frac{y}{\delta}\right)^{-\frac{6}{7}} \to \infty \quad as \quad y \to 0$$

Problem 9.15

[Difficulty: 2]

9.15 Evaluate θ/δ for each of the laminar boundary-layer velocity profiles given in Problem 9.10.

Linear: $\dfrac{u}{U} = \dfrac{y}{\delta}$

Sinusoidal: $\dfrac{u}{U} = \sin\left(\dfrac{\pi}{2}\dfrac{y}{\delta}\right)$

Parabolic: $\dfrac{u}{U} = 2\left(\dfrac{y}{\delta}\right) - \left(\dfrac{y}{\delta}\right)^2$

Given: Linear, sinusoidal, and parabolic velocity profiles

Find: The momentum thickness expressed as θ/δ for each profile

Solution: We will apply the definition of the momentum thickness to each profile.

Governing Equation:

$$\theta = \int_0^\delta \frac{u}{U}\cdot\left(1 - \frac{u}{U}\right) dy \qquad \text{(Definition of momentum thickness)}$$

If we divide both sides of the equation by δ, we get: $\dfrac{\theta}{\delta} = \dfrac{1}{\delta}\cdot\displaystyle\int_0^\delta \frac{u}{U}\cdot\left(1 - \frac{u}{U}\right) dy$ However, we can change the variable of integration to $\eta = y/\delta$, resulting in: $d\eta = \dfrac{1}{\delta}\cdot dy$ Therefore: $\dfrac{\theta}{\delta} = \displaystyle\int_0^1 \frac{u}{U}\cdot\left(1 - \frac{u}{U}\right) d\eta$

For the linear profile: $\dfrac{u}{U} = \eta$ Into the momentum thickness:

$$\frac{\theta}{\delta} = \int_0^1 \eta\cdot(1-\eta)\, d\eta = \int_0^1 \left(\eta - \eta^2\right) d\eta$$

Evaluating this integral: $\dfrac{\theta}{\delta} = \dfrac{1}{2} - \dfrac{1}{3} = \dfrac{1}{6}$ $\qquad \dfrac{\theta}{\delta} = 0.1667$

For the sinusoidal profile: $\dfrac{u}{U} = \sin\left(\dfrac{\pi}{2}\cdot\eta\right)$ Into the momentum thickness:

$$\frac{\theta}{\delta} = \int_0^1 \sin\left(\frac{\pi}{2}\cdot\eta\right)\cdot\left(1 - \sin\left(\frac{\pi}{2}\cdot\eta\right)\right) d\eta = \int_0^1 \left[\sin\left(\frac{\pi}{2}\cdot\eta\right) - \left(\sin\left(\frac{\pi}{2}\cdot\eta\right)\right)^2\right] d\eta$$

Evaluating this integral: $\dfrac{\theta}{\delta} = \dfrac{2}{\pi} - \dfrac{2}{\pi}\cdot\dfrac{\pi}{4}$ $\qquad \dfrac{\theta}{\delta} = 0.1366$

For the parabolic profile: $\dfrac{u}{U} = 2\cdot\eta - \eta^2$ Into the momentum thickness:

$$\frac{\theta}{\delta} = \int_0^1 \left(2\cdot\eta - \eta^2\right)\cdot\left[1 - \left(2\cdot\eta - \eta^2\right)\right] d\eta = \int_0^1 \left(2\cdot\eta - 5\cdot\eta^2 + 4\cdot\eta^3 - \eta^4\right) d\eta$$

Evaluating this integral: $\dfrac{\theta}{\delta} = 1 - \dfrac{5}{3} + 1 - \dfrac{1}{5} = \dfrac{2}{15}$ $\qquad \dfrac{\theta}{\delta} = 0.1333$

Problem 9.16

[Difficulty: 2]

9.16 Evaluate δ^*/δ for each of the laminar boundary-layer velocity profiles given in Problem 9.10.

Linear: $\dfrac{u}{U} = \dfrac{y}{\delta}$

Sinusoidal: $\dfrac{u}{U} = \sin\left(\dfrac{\pi}{2}\dfrac{y}{\delta}\right)$

Parabolic: $\dfrac{u}{U} = 2\left(\dfrac{y}{\delta}\right) - \left(\dfrac{y}{\delta}\right)^2$

Given: Linear, sinusoidal, and parabolic velocity profiles

Find: The displacement thickness expressed as δ^*/δ for each profile

Solution: We will apply the definition of the displacement thickness to each profile.

Governing Equation:

$$\delta_{disp} = \int_0^{infinity} \left(1 - \frac{u}{U}\right) dy = \int_0^{\delta} \left(1 - \frac{u}{U}\right) dy \qquad \text{(Definition of displacement thickness)}$$

If we divide both sides of the equation by δ, we get: $\dfrac{\delta_{disp}}{\delta} = \dfrac{1}{\delta}\cdot\displaystyle\int_0^{\delta}\left(1 - \frac{u}{U}\right) dy$ However, we can change the variable of integration to $\eta = y/\delta$, resulting in: $d\eta = \dfrac{1}{\delta}\cdot dy$ Therefore: $\dfrac{\delta_{disp}}{\delta} = \displaystyle\int_0^1 \left(1 - \frac{u}{U}\right) d\eta$

For the linear profile: $\dfrac{u}{U} = \eta$ Into the displacement thickness:

$$\frac{\delta_{disp}}{\delta} = \int_0^1 (1 - \eta)\, d\eta \qquad \text{Evaluating this integral:} \quad \frac{\delta_{disp}}{\delta} = 1 - \frac{1}{2} = \frac{1}{2} \qquad \frac{\delta_{disp}}{\delta} = 0.5000$$

For the sinusoidal profile: $\dfrac{u}{U} = \sin\left(\dfrac{\pi}{2}\cdot\eta\right)$ Into the displacement thickness:

$$\frac{\delta_{disp}}{\delta} = \int_0^1 \left(1 - \sin\left(\frac{\pi}{2}\cdot\eta\right)\right) d\eta \qquad \text{Evaluating this integral:} \quad \frac{\delta_{disp}}{\delta} = 1 - \frac{2}{\pi} \qquad \frac{\delta_{disp}}{\delta} = 0.3634$$

For the parabolic profile: $\dfrac{u}{U} = 2\cdot\eta - \eta^2$ Into the displacement thickness:

$$\frac{\delta_{disp}}{\delta} = \int_0^1 \left[1 - \left(2\cdot\eta - \eta^2\right)\right] d\eta = \int_0^1 \left(1 - 2\cdot\eta + \eta^2\right) d\eta$$

Evaluating this integral: $\dfrac{\delta_{disp}}{\delta} = 1 - 1 + \dfrac{1}{3} = \dfrac{1}{3}$ $\qquad \dfrac{\delta_{disp}}{\delta} = 0.3333$

Problem 9.17

[Difficulty: 2]

9.17 Evaluate δ^*/δ and θ/δ for the turbulent $\frac{1}{7}$-power-law velocity profile given in Problem 9.14. Compare with ratios for the parabolic laminar boundary-layer velocity profile given in Problem 9.10.

$$\frac{u}{U} = \left(\frac{y}{\delta}\right)^{1/7} \qquad \text{Parabolic:} \quad \frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$$

Given: Power law and parabolic velocity profiles

Find: The displacement and momentum thicknesses expressed as δ^*/δ and θ/δ for each profile

Solution: We will apply the definition of the displacement and momentum thickness to each profile.

Governing Equations:

$$\delta_{disp} = \int_0^{infinity} \left(1 - \frac{u}{U}\right) dy = \int_0^{\delta} \left(1 - \frac{u}{U}\right) dy \qquad \text{(Definition of displacement thickness)}$$

$$\theta = \int_0^{\delta} \frac{u}{U} \cdot \left(1 - \frac{u}{U}\right) dy \qquad \text{(Definition of momentum thickness)}$$

If we divide both sides of the equations by δ, we get: $\dfrac{\delta_{disp}}{\delta} = \dfrac{1}{\delta} \cdot \displaystyle\int_0^{\delta} \left(1 - \frac{u}{U}\right) dy \qquad \dfrac{\theta}{\delta} = \dfrac{1}{\delta} \cdot \displaystyle\int_0^{\delta} \frac{u}{U} \cdot \left(1 - \frac{u}{U}\right) dy$

However, we can change the variable of integration to $\eta = y/\delta$, resulting in: $d\eta = \dfrac{1}{\delta} \cdot dy$ Therefore:

$$\frac{\delta_{disp}}{\delta} = \int_0^1 \left(1 - \frac{u}{U}\right) d\eta \qquad \frac{\theta}{\delta} = \int_0^1 \frac{u}{U} \cdot \left(1 - \frac{u}{U}\right) d\eta$$

For the power law profile: $\dfrac{u}{U} = \eta^{\frac{1}{7}}$ Into the displacement thickness: $\dfrac{\delta_{disp}}{\delta} = \displaystyle\int_0^1 \left(1 - \eta^{\frac{1}{7}}\right) d\eta$

Evaluating this integral: $\dfrac{\delta_{disp}}{\delta} = 1 - \dfrac{7}{8} = \dfrac{1}{8}$ $\qquad \dfrac{\delta_{disp}}{\delta} = 0.1250$

Into the momentum thickness: $\dfrac{\theta}{\delta} = \displaystyle\int_0^1 \eta^{\frac{1}{7}} \cdot \left(1 - \eta^{\frac{1}{7}}\right) d\eta = \int_0^1 \left(\eta^{\frac{1}{7}} - \eta^{\frac{2}{7}}\right) d\eta$ Evaluating this integral: $\dfrac{\theta}{\delta} = \dfrac{7}{8} - \dfrac{7}{9} = \dfrac{7}{72}$

$$\frac{\theta}{\delta} = 0.0972$$

For the parabolic profile: $\dfrac{u}{U} = 2 \cdot \eta - \eta^2$

Into the displacement thickness: $\dfrac{\delta_{disp}}{\delta} = \displaystyle\int_0^1 \left[1 - \left(2 \cdot \eta - \eta^2\right)\right] d\eta = \int_0^1 \left(1 - 2 \cdot \eta + \eta^2\right) d\eta$

Evaluating this integral: $\frac{\delta_{disp}}{\delta} = 1 - 1 + \frac{1}{3} = \frac{1}{3}$ $\frac{\delta_{disp}}{\delta} = 0.3333$

Into the momentum thickness: $\frac{\theta}{\delta} = \int_0^1 \left(2\cdot\eta - \eta^2\right)\cdot\left[1 - \left(2\cdot\eta - \eta^2\right)\right] d\eta = \int_0^1 \left(2\cdot\eta - 5\cdot\eta^2 + 4\cdot\eta^3 - \eta^4\right) d\eta$

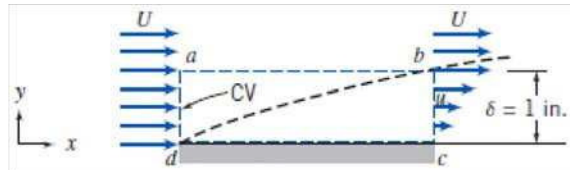
Evaluating this integral: $\frac{\theta}{\delta} = 1 - \frac{5}{3} + 1 - \frac{1}{5} = \frac{2}{15}$ $\frac{\theta}{\delta} = 0.1333$

Profile	δ_{disp}	θ
Power Law	$0.1250\cdot\delta$	$0.0972\cdot\delta$
Parabolic	$0.3333\cdot\delta$	$0.1333\cdot\delta$

Problem 9.18

[Difficulty: 3]

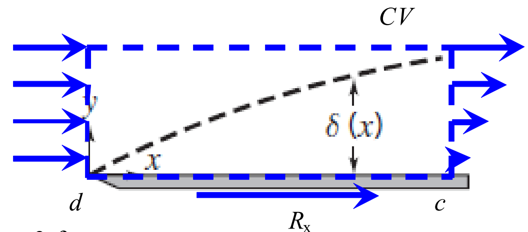
9.18 A fluid, with density $\rho = 1.5$ slug/ft^3, flows at $U = 10$ ft/s over a flat plate 10 ft long and 3 ft wide. At the trailing edge, the boundary-layer thickness is $\delta = 1$ in. Assume the velocity profile is linear, as shown, and that the flow is two-dimensional (flow conditions are independent of z). Using control volume $abcd$, shown by the dashed lines, compute the mass flow rate across surface ab. Determine the drag force on the upper surface of the plate. Explain how this (viscous) drag can be computed from the given data even though we do not know the fluid viscosity (see Problem 9.41).



Given: Data on fluid and boundary layer geometry

Find: Mass flow rate across ab; Drag

Solution:



The given data is $\rho = 1.5\cdot\dfrac{\text{slug}}{\text{ft}^3}$ $\quad U = 10\cdot\dfrac{\text{ft}}{\text{s}}$ $\quad L = 10\cdot\text{ft}$ $\quad \delta = 1\cdot\text{in}$ $\quad b = 3\cdot\text{ft}$

Governing equations:

Mass
$$\frac{\partial}{\partial t}\int_{CV}\rho\, d\forall + \int_{CS}\rho\vec{V}\cdot d\vec{A} = 0 \qquad (4.12)$$

Momentum
$$\vec{F} = \vec{F}_S + \vec{F}_B = \frac{\partial}{\partial t}\int_{CV}\vec{V}\,\rho\, d\forall + \int_{CS}\vec{V}\,\rho\vec{V}\cdot d\vec{A} \qquad (4.17a)$$

Assumptions: (1) Steady flow (2) No pressure force (3) No body force in x direction (4) Uniform flow at a

Applying these to the CV $abcd$

Mass
$$(-\rho\cdot U\cdot b\cdot\delta) + \int_0^{\delta}\rho\cdot u\cdot b\, dy + m_{ab} = 0$$

For the boundary layer
$$\frac{u}{U} = \frac{y}{\delta} = \eta \qquad\qquad \frac{dy}{\delta} = d\eta$$

Hence
$$m_{ab} = \rho\cdot U\cdot b\cdot\delta - \int_0^1\rho\cdot U\cdot\eta\cdot\delta\, dy = \rho\cdot U\cdot b\cdot\delta - \frac{1}{2}\cdot\rho\cdot U\cdot b\cdot\delta$$

$$m_{ab} = \frac{1}{2}\cdot\rho\cdot U\cdot b\cdot\delta \qquad\qquad m_{ab} = 1.875\cdot\frac{\text{slug}}{\text{s}}$$

Momentum
$$R_x = U\cdot(-\rho\cdot U\cdot\delta) + m_{ab}\cdot u_{ab} + \int_0^{\delta}u\cdot\rho\cdot u\cdot b\, dy$$

Note that $\quad u_{ab} = U \quad$ and $\quad \displaystyle\int_0^{\delta}u\cdot\rho\cdot u\cdot b\, dy = \int_0^1\rho\cdot U^2\cdot b\cdot\delta\cdot\eta^2\, d\eta$

$$R_x = -\rho\cdot U^2\cdot b\cdot\delta + \frac{1}{2}\cdot\rho\cdot U\cdot b\cdot\delta\cdot U + \int_0^1\rho\cdot U^2\cdot b\cdot\delta\cdot\eta^2\, dy$$

$$R_x = -\rho\cdot U^2\cdot b\cdot\delta + \frac{1}{2}\cdot\rho\cdot U^2\cdot\delta + \frac{1}{3}\cdot\rho\cdot U^2\cdot\delta \qquad R_x = -\frac{1}{6}\cdot\rho\cdot U^2\cdot b\cdot\delta \qquad R_x = -6.25\cdot\text{lbf}$$

We are able to compute the boundary layer drag even though we do not know the viscosity because it is the viscosity that creates the boundary layer in the first place

Problem 9.19

[Difficulty: 3]

9.19 The flat plate of Problem 9.18 is turned so that the 3-ft side is parallel to the flow (the width becomes 10 ft). Should we expect that the drag increases or decreases? Why? The trailing edge boundary-layer thickness is now $\delta = 0.6$ in. Assume again that the velocity profile is linear and that the flow is two-dimensional (flow conditions are independent of z). Repeat the analysis of Problem 9.18.

Given: Data on fluid and boundary layer geometry

Find: Mass flow rate across *ab*; Drag; Compare to Problem 9.18

Solution:

The given data is $\rho = 1.5\cdot\frac{\text{slug}}{\text{ft}^3}$ $\quad U = 10\cdot\frac{\text{ft}}{\text{s}}$ $\quad L = 3\cdot\text{ft}$ $\quad \delta = 0.6\cdot\text{in}$ $\quad b = 10\cdot\text{ft}$

Governing equations:

Mass
$$\frac{\partial}{\partial t}\int_{CV}\rho\, d\forall + \int_{CS}\rho\vec{V}\cdot d\vec{A} = 0 \qquad (4.12)$$

Momentum
$$\vec{F} = \vec{F}_S + \vec{F}_B = \frac{\partial}{\partial t}\int_{CV}\vec{V}\,\rho\, d\forall + \int_{CS}\vec{V}\,\rho\vec{V}\cdot d\vec{A} \qquad (4.17a)$$

Assumptions: (1) Steady flow (2) No pressure force (3) No body force in x direction (4) Uniform flow at a

Applying these to the CV *abcd*

Mass
$$(-\rho\cdot U\cdot b\cdot\delta) + \int_0^{\delta}\rho\cdot u\cdot b\, dy + m_{ab} = 0$$

For the boundary layer
$$\frac{u}{U} = \frac{y}{\delta} = \eta \qquad \frac{dy}{\delta} = d\eta$$

Hence
$$m_{ab} = \rho\cdot U\cdot b\cdot\delta - \int_0^1 \rho\cdot U\cdot\eta\cdot\delta\, dy = \rho\cdot U\cdot b\cdot\delta - \frac{1}{2}\cdot\rho\cdot U\cdot b\cdot\delta$$

$$m_{ab} = \frac{1}{2}\cdot\rho\cdot U\cdot b\cdot\delta \qquad m_{ab} = 3.75\cdot\frac{\text{slug}}{\text{s}}$$

Momentum
$$R_x = U\cdot(-\rho\cdot U\cdot\delta) + m_{ab}\cdot u_{ab} + \int_0^{\delta} u\cdot\rho\cdot u\cdot b\, dy$$

Note that $u_{ab} = U$ and $\int_0^{\delta} u\cdot\rho\cdot u\cdot b\, dy = \int_0^1 \rho\cdot U^2\cdot b\cdot\delta\cdot\eta^2\, d\eta$

$$R_x = -\rho\cdot U^2\cdot b\cdot\delta + \frac{1}{2}\cdot\rho\cdot U\cdot b\cdot\delta\cdot U + \int_0^1 \rho\cdot U^2\cdot b\cdot\delta\cdot\eta^2\, dy$$

$$R_x = -\rho\cdot U^2\cdot b\cdot\delta + \frac{1}{2}\cdot\rho\cdot U^2\cdot\delta + \frac{1}{3}\cdot\rho\cdot U^2\cdot\delta$$

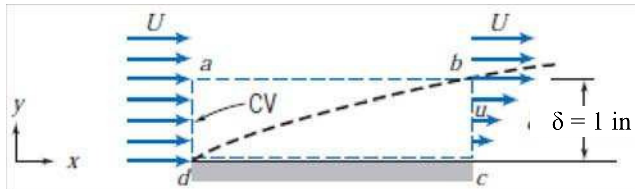
$$R_x = -\frac{1}{6}\cdot\rho\cdot U^2\cdot b\cdot\delta \qquad R_x = -12.50\cdot\text{lbf}$$

We should expect the drag to be larger than for Problem 9.18 because the viscous friction is mostly concentrated near the leading edge (which is only 3 ft wide in Problem 9.18 but 10 ft here). The reason viscous stress is highest at the front region is that the boundary layer is very small (δ <<) so $\tau = \mu du/dy \sim \mu U/\delta$ >>

Problem 9.20

[Difficulty: 3]

9.20 Solve Problem 9.18 again with the velocity profile at section bc given by the parabolic expression from Problem 9.10.



Given: Flow over a flat plate with parabolic laminar boundary layer profile

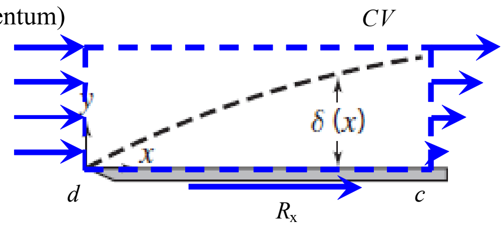
Find: (a) Mass flow rate across ab
(b) x component (and direction) of force needed to hold the plate in place

Solution: We will apply the continuity and x-momentum equations to this system.

Governing Equations:

$$\frac{\partial}{\partial t}\int_{CV}\rho d\forall + \int_{CS}\rho\vec{V}\cdot d\vec{A} = 0 \qquad \text{(Continuity)}$$

$$\frac{\partial}{\partial t}\int_{CV}\rho u d\forall + \int_{CS}u\rho\vec{V}\cdot d\vec{A} = F_{sx} + F_{bx} \qquad \text{(x- Momentum)}$$



Assumptions:
(1) Steady flow
(2) No net pressure forces
(3) No body forces in the x-direction
(4) Uniform flow at da

From the assumptions, the continuity equation becomes: $-\rho\cdot U\cdot b\cdot\delta + \int_0^\delta \rho\cdot u\cdot b\,dy + m_{ab} = 0$ The integral can be written as:

$$\int_0^\delta \rho\cdot u\cdot b\,dy = \rho\cdot b\cdot\int_0^\delta u\,dy = \rho\cdot U\cdot b\cdot\delta\cdot\int_0^1\left(2\cdot\eta - \eta^2\right)d\eta \quad \text{where} \quad \eta = \frac{y}{\delta} \quad \text{This integral is equal to:} \quad \rho\cdot U\cdot b\cdot\delta\cdot\left(1 - \frac{1}{3}\right) = \frac{2}{3}\cdot\rho\cdot U\cdot b\cdot\delta$$

Solving continuity for the mass flux through ab we get: $m_{ab} = \rho\cdot U\cdot b\cdot\delta - \frac{2}{3}\cdot\rho\cdot U\cdot b\cdot\delta = \frac{1}{3}\cdot\rho\cdot U\cdot b\cdot\delta$ Substituting known values:

$$m_{ab} = \frac{1}{3}\times 1.5\cdot\frac{slug}{ft^3}\times 10\cdot\frac{ft}{s}\times 3.0\cdot ft\times 1\cdot in\times\frac{ft}{12\cdot in} \qquad m_{ab} = 1.250\cdot\frac{slug}{s}$$

From the assumptions, the momentum equation becomes: $R_x = u_{da}\cdot(-\rho\cdot U\cdot b\cdot\delta) + u_{ab}\cdot m_{ab} + \int_0^\delta u\cdot\rho\cdot u\cdot b\,dy$ where $u_{da} = u_{ab} = U$

Thus: $R_x = -\rho\cdot U^2\cdot b\cdot\delta + \frac{1}{3}\cdot\rho\cdot U^2\cdot b\cdot\delta + \int_0^\delta u\cdot\rho\cdot u\cdot b\,dy = -\frac{2}{3}\cdot\rho\cdot U^2\cdot b\cdot\delta + \int_0^\delta u\cdot\rho\cdot u\cdot b\,dy$ The integral can be written as:

$$\int_0^\delta u\cdot\rho\cdot u\cdot b\,dy = \rho\cdot b\cdot\int_0^\delta u^2\,dy = \rho\cdot U^2\cdot b\cdot\delta\cdot\int_0^1\left(2\cdot\eta - \eta^2\right)^2 d\eta = \rho\cdot U^2\cdot b\cdot\delta\cdot\int_0^1\left(4\cdot\eta^2 - 4\cdot\eta^3 + \eta^4\right)d\eta \quad \text{This integral is equal to:}$$

$$\rho\cdot U^2\cdot b\cdot\delta\cdot\left(\frac{4}{3} - 1 + \frac{1}{5}\right) = \frac{8}{15}\cdot\rho\cdot U\cdot b\cdot\delta \quad \text{Therefore the force on the plate is:} \quad R_x = \left(\frac{8}{15} - \frac{2}{3}\right)\cdot\rho\cdot U^2\cdot b\cdot\delta = -\frac{2}{15}\cdot\rho\cdot U^2\cdot b\cdot\delta$$

Substituting known values: $R_x = -\frac{2}{15}\times 1.5\cdot\frac{slug}{ft^3}\times\left(10\cdot\frac{ft}{s}\right)^2\times 3.0\cdot ft\times 1\cdot in\times\frac{ft}{12\cdot in}\times\frac{lbf\cdot s^2}{slug\cdot ft}$

$R_x = -5.00\cdot lbf$
(to the left)

This force must be applied to the control volume by the plate.

Problem 9.21

[Difficulty: 2]

9.21 The test section of a low-speed wind tunnel is 5 ft long, preceded by a nozzle with a diffuser at the outlet. The tunnel cross section is 1 ft × 1 ft. The wind tunnel is to operate with 100°F air and have a design velocity of 160 ft/s in the test section. A potential problem with such a wind tunnel is boundary-layer blockage. The boundary-layer displacement thickness reduces the effective cross-sectional area (the test area, in which we have uniform flow); in addition, the uniform flow will be accelerated. If these effects are pronounced, we end up with a smaller useful test cross section with a velocity somewhat higher than anticipated. If the boundary layer thickness is 0.4 in. at the entrance and 1 in. at the exit, and the boundary layer velocity profile is given by $u/U = (y/\delta)^{1/7}$, estimate the displacement thickness at the end of the test section and the percentage change in the uniform velocity between the inlet and outlet.

Given: Data on wind tunnel and boundary layers

Find: Displacement thickness at exit; Percent change in uniform velocity through test section

Solution:

The solution involves using mass conservation in the inviscid core, allowing for the fact that as the boundary layer grows it reduces the size of the core. One approach would be to integrate the 1/7 law velocity profile to compute the mass flow in the boundary layer; an easier approach is to simply use the displacement thickness!

Basic equations

$$\frac{\partial}{\partial t}\int_{CV}\rho\, dV + \int_{CS}\rho \vec{V}\cdot d\vec{A} = 0 \quad (4.12) \qquad \delta_{disp} = \int_0^{\delta}\left(1 - \frac{u}{U}\right) dy$$

Assumptions: 1) Steady flow 2) Incompressible 3) No friction outside boundary layer 4) Flow along streamline 5) Horizontal

For this flow $\rho\cdot U\cdot A = const$ and $\frac{u}{U} = \left(\frac{y}{\delta}\right)^{\frac{1}{7}}$

The design data is $U_{design} = 160\cdot\frac{ft}{s}$ $\quad w = 1\cdot ft \quad h = 1\cdot ft \quad A_{design} = w\cdot h \quad A_{design} = 1\cdot ft^2$

The volume flow rate is $Q = U_{design}\cdot A_{design} \qquad Q = 160\cdot\frac{ft^3}{s}$

We also have $\delta_{in} = 0.4\cdot in \qquad \delta_{exit} = 1\cdot in$

Hence

$$\delta_{disp} = \int_0^{\delta}\left(1 - \frac{u}{U}\right) dy = \int_0^{\delta}\left[1 - \left(\frac{y}{\delta}\right)^{\frac{1}{7}}\right] dy = \delta\cdot\int_0^1\left(1 - \eta^{\frac{1}{7}}\right) d\eta \quad \text{where} \quad \eta = \frac{y}{\delta} \qquad \delta_{disp} = \frac{\delta}{8}$$

Hence at the inlet and exit

$$\delta_{dispin} = \frac{\delta_{in}}{8} \qquad \delta_{dispin} = 0.05\cdot in \qquad \delta_{dispexit} = \frac{\delta_{exit}}{8} \qquad \delta_{dispexit} = 0.125\cdot in$$

Hence the areas are $A_{in} = \left(w - 2\cdot\delta_{dispin}\right)\cdot\left(h - 2\cdot\delta_{dispin}\right)$ $\quad A_{in} = 0.9834\cdot ft^2$

$$A_{exit} = \left(w - 2\cdot\delta_{dispexit}\right)\cdot\left(h - 2\cdot\delta_{dispexit}\right) \qquad A_{exit} = 0.9588\cdot ft^2$$

Applying mass conservation between "design" conditions and the inlet

$$\left(-\rho\cdot U_{design}\cdot A_{design}\right) + \left(\rho\cdot U_{in}\cdot A_{in}\right) = 0$$

or $\quad U_{in} = U_{design}\cdot\dfrac{A_{design}}{A_{in}} \qquad U_{in} = 162.7\cdot\dfrac{ft}{s}$

Also $\quad U_{exit} = U_{design}\cdot\dfrac{A_{design}}{A_{exit}} \qquad U_{exit} = 166.9\cdot\dfrac{ft}{s}$

The percent change in uniform velocity is then $\dfrac{U_{exit} - U_{in}}{U_{in}} = 2.57\cdot\%$ The exit displacement thickness is $\delta_{dispexit} = 0.125\cdot in$

Problem 9.22

[Difficulty: 2]

9.22 Air flows in a horizontal cylindrical duct of diameter $D = 100$ mm. At a section a few meters from the entrance, the turbulent boundary layer is of thickness $\delta_1 = 5.25$ mm, and the velocity in the inviscid central core is $U_1 = 12.5$ m/s. Farther downstream the boundary layer is of thickness $\delta_2 = 24$ mm. The velocity profile in the boundary layer is approximated well by the $\frac{1}{7}$-power expression. Find the velocity, U_2, in the inviscid central core at the second section, and the pressure drop between the two sections.

Given: Data on boundary layer in a cylindrical duct

Find: Velocity U_2 in the inviscid core at location 2; Pressure drop

Solution:

The solution involves using mass conservation in the inviscid core, allowing for the fact that as the boundary layer grows it reduces the size of the core. One approach would be to integrate the 1/7 law velocity profile to compute the mass flow in the boundary layer; an easier approach is to simply use the displacement thickness!

The given or available data (from Appendix A) is

$\rho = 1.23 \cdot \frac{kg}{m^3}$ $\quad U_1 = 12.5 \cdot \frac{m}{s}$ $\quad D = 100 \cdot mm$ $\quad \delta_1 = 5.25 \cdot mm$ $\quad \delta_2 = 24 \cdot mm$

Governing equations:

Mass $\quad \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A} = 0 \quad$ (4.12)

Bernoulli $\quad \frac{p}{\rho} + \frac{V^2}{2} + g \cdot z = \text{constant} \quad$ (4.24)

The displacement thicknesses can be computed from boundary layer thicknesses using Eq. 9.1

$$\delta_{disp} = \int_0^{\delta} \left(1 - \frac{u}{U}\right) dy = \delta \cdot \int_0^1 \left(1 - \eta^{\frac{1}{7}}\right) d\eta = \frac{\delta}{8}$$

Hence at locations 1 and 2 $\quad \delta_{disp1} = \frac{\delta_1}{8} \quad \delta_{disp1} = 0.656 \cdot mm \quad \delta_{disp2} = \frac{\delta_2}{8} \quad \delta_{disp2} = 3 \cdot mm$

Applying mass conservation at locations 1 and 2 $\quad (-\rho \cdot U_1 \cdot A_1) + (\rho \cdot U_2 \cdot A_2) = 0 \quad$ or $\quad U_2 = U_1 \cdot \frac{A_1}{A_2}$

The two areas are given by the duct cross section area minus the displacement boundary layer

$A_1 = \frac{\pi}{4} \cdot (D - 2 \cdot \delta_{disp1})^2 \quad A_1 = 7.65 \times 10^{-3} m^2 \quad A_2 = \frac{\pi}{4} \cdot (D - 2 \cdot \delta_{disp2})^2 \quad A_2 = 6.94 \times 10^{-3} m^2$

Hence $\quad U_2 = U_1 \cdot \frac{A_1}{A_2} \quad U_2 = 13.8 \frac{m}{s}$

For the pressure drop we can apply Bernoulli to locations 1 and 2 to find $\quad p_1 - p_2 = \Delta p = \frac{\rho}{2} \cdot \left(U_2^2 - U_1^2\right) \quad \Delta p = 20.6 \, Pa$

Problem 9.23

[Difficulty: 2]

9.23 Laboratory wind tunnels have test sections 25 cm square and 50 cm long. With nominal air speed $U_1 = 25$ m/s at the test section inlet, turbulent boundary layers form on the top, bottom, and side walls of the tunnel. The boundary-layer thickness is $\delta_1 = 20$ mm at the inlet and $\delta_2 = 30$ mm at the outlet from the test section. The boundary-layer velocity profiles are of power-law form, with $u/U = (y/\delta)^{1/7}$. Evaluate the freestream velocity, U_2, at the exit from the wind-tunnel test section. Determine the change in static pressure along the test section.

Given: Data on wind tunnel and boundary layers

Find: Uniform velocity at exit; Change in static pressure through the test section

Solution:

Basic equations

$$\frac{\partial}{\partial t}\int_{CV}\rho\, d\mathcal{V} + \int_{CS}\rho\vec{V}\cdot d\vec{A} = 0 \quad (4.12) \qquad \delta_{disp} = \int_0^{\delta}\left(1-\frac{u}{U}\right)dy \qquad \frac{p}{\rho}+\frac{V^2}{2}+g\cdot z = \text{const}$$

Assumptions: 1) Steady flow 2) Incompressible 3) No friction outside boundary layer 4) Flow along streamline 5) Horizontal

For this flow $\rho\cdot U\cdot A = \text{const}$ and $\frac{u}{U} = \left(\frac{y}{\delta}\right)^{\frac{1}{7}}$

The given data is $U_1 = 25\cdot\frac{m}{s}$ $\quad h = 25\cdot cm$ $\quad A = h^2$ $\quad A = 625\cdot cm^2$

We also have $\delta_1 = 20\cdot mm$ $\quad \delta_2 = 30\cdot mm$

Hence $$\delta_{disp} = \int_0^{\delta}\left(1-\frac{u}{U}\right)dy = \int_0^{\delta}\left[1-\left(\frac{y}{\delta}\right)^{\frac{1}{7}}\right]dy = \delta\cdot\int_0^1\left(1-\eta^{\frac{1}{7}}\right)d\eta$$ where $\eta = \frac{y}{\delta}$ $\quad \delta_{disp} = \frac{\delta}{8}$

Hence at the inlet and exit

$\delta_{disp1} = \frac{\delta_1}{8}$ $\quad \delta_{disp1} = 2.5\cdot mm$ $\quad \delta_{disp2} = \frac{\delta_2}{8}$ $\quad \delta_{disp2} = 3.75\cdot mm$

Hence the areas are $A_1 = \left(h - 2\cdot\delta_{disp1}\right)^2$ $\quad A_1 = 600\cdot cm^2$

$A_2 = \left(h - 2\cdot\delta_{disp2}\right)^2$ $\quad A_2 = 588\cdot cm^2$

Applying mass conservation between Points 1 and 2

$$\left(-\rho\cdot U_1\cdot A_1\right) + \left(\rho\cdot U_2\cdot A_2\right) = 0 \quad \text{or} \quad U_2 = U_1\cdot\frac{A_1}{A_2} \qquad U_2 = 25.52\,\frac{m}{s}$$

The pressure change is found from Bernoulli $\quad \frac{p_1}{\rho}+\frac{U_1^2}{2} = \frac{p_2}{\rho}+\frac{U_2^2}{2}$ $\quad$ with $\quad \rho = 1.21\cdot\frac{kg}{m^3}$

Hence $\quad \Delta p = \frac{\rho}{2}\cdot\left(U_1^2 - U_2^2\right)$ $\quad \Delta p = -15.8\,\text{Pa}$ $\quad$ The pressure drops slightly through the test section

Problem 9.24

[Difficulty: 2]

9.24 The square test section of a small laboratory wind tunnel has sides of width $W = 40$ cm. At one measurement location, the turbulent boundary layers on the tunnel walls are $\delta_1 = 1$ cm thick. The velocity profile is approximated well by the $\frac{1}{7}$-power expression. At this location, the freestream air speed is $U_1 = 20$ m/s, and the static pressure is $p_1 = -250$ Pa (gage). At a second measurement location downstream, the boundary layer thickness is $\delta_2 = 1.3$ cm. Evaluate the air speed in the freestream in the second section. Calculate the difference in static pressure from section ① to section ②.

Given: Data on wind tunnel and boundary layers

Find: Uniform velocity at Point 2; Change in static pressure through the test section

Solution:

Basic equations

$$\frac{\partial}{\partial t}\int_{CV}\rho\, d\forall + \int_{CS}\rho\vec{V}\cdot d\vec{A} = 0 \qquad (4.12) \qquad \delta_{disp} = \int_0^\delta \left(1 - \frac{u}{U}\right) dy \qquad \frac{p}{\rho} + \frac{V^2}{2} + g\cdot z = \text{const}$$

Assumptions: 1) Steady flow 2) Incompressible 3) No friction outside boundary layer 4) Flow along streamline 5) Horizontal

For this flow $\rho\cdot U\cdot A = \text{const}$ and $\frac{u}{U} = \left(\frac{y}{\delta}\right)^{\frac{1}{7}}$

The given data is $U_1 = 20\cdot\frac{m}{s}$ $\quad W = 40\cdot cm$ $\quad A = W^2$ $\quad A = 0.1600\cdot m^2$

We also have $\delta_1 = 1\cdot cm$ $\quad \delta_2 = 1.3\cdot cm$

Hence $$\delta_{disp} = \int_0^\delta \left(1 - \frac{u}{U}\right) dy = \int_0^\delta \left[1 - \left(\frac{y}{\delta}\right)^{\frac{1}{7}}\right] dy = \delta\cdot\int_0^1 \left(1 - \eta^{\frac{1}{7}}\right) d\eta \qquad \text{where} \qquad \eta = \frac{y}{\delta} \qquad \delta_{disp} = \frac{\delta}{8}$$

Hence at the inlet and exit

$\delta_{disp1} = \frac{\delta_1}{8}$ $\quad \delta_{disp1} = 0.125\cdot cm$ $\quad \delta_{disp2} = \frac{\delta_2}{8}$ $\quad \delta_{disp2} = 0.1625\cdot cm$

Hence the areas are $A_1 = \left(W - 2\cdot\delta_{disp1}\right)^2$ $\quad A_1 = 0.1580\cdot m^2$

$A_2 = \left(W - 2\cdot\delta_{disp2}\right)^2$ $\quad A_2 = 0.1574\cdot m^2$

Applying mass conservation between Points 1 and 2

$$\left(-\rho\cdot U_1\cdot A_1\right) + \left(\rho\cdot U_2\cdot A_2\right) = 0 \qquad \text{or} \qquad U_2 = U_1\cdot\frac{A_1}{A_2} \qquad U_2 = 20.1\cdot\frac{m}{s}$$

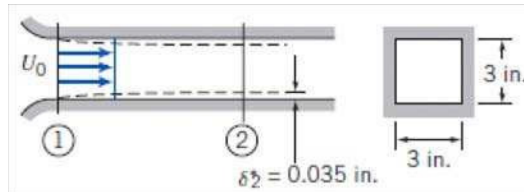
The pressure change is found from Bernoulli $\frac{p_1}{\rho} + \frac{U_1^2}{2} = \frac{p_2}{\rho} + \frac{U_2^2}{2}$ with $\rho = 1.21\cdot\frac{kg}{m^3}$

Hence $$\Delta p = \frac{\rho}{2}\cdot\left(U_1^2 - U_2^2\right) \qquad \Delta p = -2.66\times 10^{-4}\cdot psi \qquad \Delta p = -1.835\cdot Pa$$

Problem 9.25

[Difficulty: 2]

9.25 Air flows in the entrance region of a square duct, as shown. The velocity is uniform, $U_0 = 100$ ft/s, and the duct is 3 in. square. At a section 1 ft downstream from the entrance, the displacement thickness, δ^*, on each wall measures 0.035 in. Determine the pressure change between sections ① and ②.



Given: Data on wind tunnel and boundary layers

Find: Pressure change between points 1 and 2

Solution:

Basic equations

$$\frac{\partial}{\partial t}\int_{CV}\rho\, d\forall + \int_{CS}\rho\vec{V}\cdot d\vec{A} = 0 \qquad (4.12) \qquad \frac{p}{\rho} + \frac{V^2}{2} + g\cdot z = \text{const}$$

Assumptions: 1) Steady flow 2) Incompressible 3) No friction outside boundary layer 4) Flow along streamline 5) Horizontal

For this flow $\quad \rho\cdot U\cdot A = \text{const}$

The given data is $\quad U_0 = 100\cdot\frac{ft}{s} \qquad U_1 = U_0 \qquad h = 3\cdot in \qquad A_1 = h^2 \qquad A_1 = 9\cdot in^2$

We also have $\quad \delta_{disp2} = 0.035\cdot in$

Hence at the Point 2 $\quad A_2 = \left(h - 2\cdot\delta_{disp2}\right)^2 \qquad A_2 = 8.58\cdot in^2$

Applying mass conservation between Points 1 and 2

$$\left(-\rho\cdot U_1\cdot A_1\right) + \left(\rho\cdot U_2\cdot A_2\right) = 0 \qquad \text{or} \qquad U_2 = U_1\cdot\frac{A_1}{A_2} \qquad U_2 = 105\cdot\frac{ft}{s}$$

The pressure change is found from Bernoulli $\quad \frac{p_1}{\rho} + \frac{U_1^2}{2} = \frac{p_2}{\rho} + \frac{U_2^2}{2}$ with $\quad \rho = 0.00234\cdot\frac{slug}{ft^3}$

Hence $\quad \Delta p = \frac{\rho}{2}\cdot\left(U_1^2 - U_2^2\right) \qquad \Delta p = -8.05\times 10^{-3}\cdot psi \qquad \Delta p = -1.16\cdot\frac{lbf}{ft^2}$

The pressure drops by a small amount as the air accelerates

Problem 9.26

[Difficulty: 3]

9.26 Flow of 68°F air develops in a flat horizontal duct following a well-rounded entrance section. The duct height is $H = 1$ ft. Turbulent boundary layers grow on the duct walls, but the flow is not yet fully developed. Assume that the velocity profile in each boundary layer is $u/U = (y/\delta)^{1/7}$. The inlet flow is uniform at $V = 40$ ft/s at section ①. At section ②, the boundary-layer thickness on each wall of the channel is $\delta_2 = 4$ in. Show that, for this flow, $\delta^* = \delta/8$. Evaluate the static gage pressure at section ①. Find the average wall shear stress between the entrance and section ②, located at $L = 20$ ft.

Given: Developing flow of air in flat horizontal duct. Assume 1/7-power law velocity profile in boundary layer.

Find:
(a) Displacement thickness is 1/8 times boundary layer thickness
(b) Static gage pressure at section 1.
(c) Average wall shear stress between entrance and section 2.

Solution: We will apply the continuity and x-momentum equations to this problem.

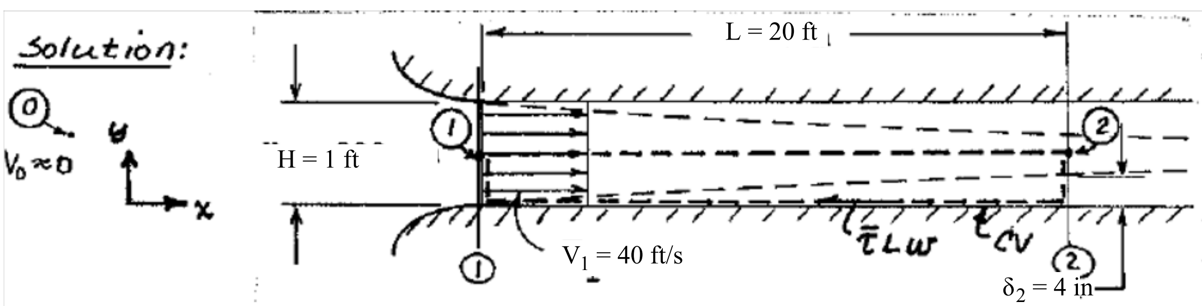
Governing Equations:

$$\delta_{disp} = \int_0^{infinity} \left(1 - \frac{u}{U}\right) dy = \int_0^{\delta} \left(1 - \frac{u}{U}\right) dy \qquad \text{(Definition of displacement thickness)}$$

$$\frac{\partial}{\partial t}\int_{CV} \rho d\forall + \int_{CS} \rho \vec{V}\cdot d\vec{A} = 0 \qquad \text{(Continuity)}$$

$$\frac{\partial}{\partial t}\int_{CV} \rho u d\forall + \int_{CS} u\rho \vec{V}\cdot d\vec{A} = F_{sx} + F_{bx} \qquad \text{(x- Momentum)}$$

Assumptions:
(1) Steady, incompressible flow
(2) No body forces in the x-direction
(3) No viscous forces outside boundary layer
(4) Boundary layers only grow on horizontal walls



If we divide both sides of the displacement thickness definition by δ, we get: $\dfrac{\delta_{disp}}{\delta} = \dfrac{1}{\delta}\cdot\displaystyle\int_0^{\delta}\left(1 - \frac{u}{U}\right) dy$

However, we can change the variable of integration to $\eta = y/\delta$, resulting in: $d\eta = \dfrac{1}{\delta}\cdot dy$ Therefore: $\dfrac{\delta_{disp}}{\delta} = \displaystyle\int_0^1 \left(1 - \frac{u}{U}\right) d\eta$

For the power law profile: $\dfrac{u}{U} = \eta^{\frac{1}{7}}$ Into the displacement thickness: $\dfrac{\delta_{disp}}{\delta} = \displaystyle\int_0^1 \left(1 - \eta^{\frac{1}{7}}\right) d\eta$

Evaluating this integral: $\dfrac{\delta_{disp}}{\delta} = 1 - \dfrac{7}{8} = \dfrac{1}{8}$

$$\frac{\delta_{disp}}{\delta} = \frac{1}{8}$$

After applying the assumptions from above, continuity reduces to: $V_1 \cdot A_1 = V_2 \cdot A_2$ or $V_1 \cdot w \cdot H = V_2 \cdot w \cdot \left(H - 2 \cdot \delta_{disp2}\right)$

Solving for the velocity at 2: $V_2 = V_1 \cdot \dfrac{H}{H - 2 \cdot \delta_{disp2}} = V_1 \cdot \dfrac{H}{H - \dfrac{\delta_2}{4}}$ Substituting known values:

$$V_2 = 40 \cdot \frac{ft}{s} \times 1 \cdot ft \times \left(\frac{1}{1 - \frac{1}{4} \times \frac{4}{12}}\right) \cdot \frac{1}{ft} \qquad V_2 = 43.6 \cdot \frac{ft}{s}$$

From Bernoulli equation, since z = constant: $\dfrac{p_0}{\rho} = \dfrac{p}{\rho} + \dfrac{V^2}{2}$ along a streamline. Therefore:

$$p_{1g} = p_1 - p_0 = -\frac{1}{2} \cdot \rho \cdot V_1^2 \qquad p_{1g} = -\frac{1}{2} \times 0.00234 \cdot \frac{slug}{ft^3} \times \left(40 \cdot \frac{ft}{s}\right)^2 \times \frac{lbf \cdot s^2}{slug \cdot ft} \times \left(\frac{ft}{12 \cdot in}\right)^2 \qquad p_{1g} = -0.01300 \cdot psi$$

$$p_{2g} = p_2 - p_0 = -\frac{1}{2} \cdot \rho \cdot V_2^2 \qquad p_{2g} = -\frac{1}{2} \times 0.00234 \cdot \frac{slug}{ft^3} \times \left(43.6 \cdot \frac{ft}{s}\right)^2 \times \frac{lbf \cdot s^2}{slug \cdot ft} \times \left(\frac{ft}{12 \cdot in}\right)^2 \qquad p_{2g} = -0.01545 \cdot psi$$

Now if we apply the momentum equation to the control volume (considering the assumptions shown): $F_{sx} = \int_{CS} u \rho \vec{V} \cdot d\vec{A}$

$$\left(p_1 - p_2\right) \cdot w \cdot \frac{H}{2} - \tau \cdot w \cdot L = V_1\left(-\rho \cdot V_1 \cdot \frac{H}{2} \cdot w\right) + \int_0^{\delta_2} u \cdot \rho \cdot u \cdot w \, dy + V_2 \cdot \left[\rho \cdot V_2 \cdot \left(\frac{H}{2} - \delta_2\right) \cdot w\right]$$

The integral is equal to: $\rho \cdot w \cdot \displaystyle\int_0^{\delta_2} u^2 \, dy = \rho \cdot V_2^2 \cdot \delta_2 \cdot w \cdot \int_0^1 \eta^{\frac{2}{7}} \, d\eta = \rho \cdot V_2^2 \cdot \frac{7}{9} \cdot \delta_2 \cdot w$ Therefore the momentum equation becomes:

$\left(p_1 - p_2\right) \cdot w \cdot \dfrac{H}{2} - \tau \cdot w \cdot L = -\rho \cdot V_1^2 \cdot \dfrac{H}{2} \cdot w + \rho \cdot V_2^2 \cdot \left(\dfrac{H}{2} - \dfrac{2}{9} \cdot \delta_2\right) \cdot w$ Simplifying and solving for the shear stress we get:

$\tau = \dfrac{1}{L} \cdot \left[\left(p_1 - p_2\right) \cdot \dfrac{H}{2} + \rho \cdot \left[V_1^2 \cdot \dfrac{H}{2} - V_2^2 \cdot \left(\dfrac{H}{2} - \dfrac{2}{9} \cdot \delta_2\right)\right]\right]$ Substituting in known values we get:

$$\tau = \frac{1}{20 \cdot ft} \cdot \left[[(-0.01328) - (-0.01578)] \cdot \frac{lbf}{in^2} \cdot \frac{1 \cdot ft}{2} + 0.00234 \cdot \frac{slug}{ft^3} \cdot \left[\left(40 \cdot \frac{ft}{s}\right)^2 \cdot \frac{1 \cdot ft}{2} - \left(43.6 \cdot \frac{ft}{s}\right)^2 \cdot \left(\frac{1}{2} - \frac{2}{9} \cdot \frac{4}{12}\right) \cdot ft\right] \cdot \frac{lbf \cdot s^2}{slug \cdot ft} \cdot \left(\frac{ft}{12 \cdot in}\right)^2\right]$$

$$\tau = 5.46 \times 10^{-5} \cdot psi$$

Problem 9.27

[Difficulty: 3]

9.27 A laboratory wind tunnel has a square test section with sides of width $W = 1$ ft and length $L = 2$ ft. When the freestream air speed at the test section entrance is $U_1 = 80$ ft/s, the head loss from the atmosphere is 0.3 in. H_2O. Turbulent boundary layers form on the top, bottom, and side walls of the test section. Measurements show the boundary-layer thicknesses are $\delta_1 = 0.8$ in at the entrance and $\delta_2 = 1$ in at the outlet of the test section. The velocity profiles are of $\frac{1}{7}$-power form. Evaluate the freestream air speed at the outlet from the test section. Determine the static pressures at the test section inlet and outlet.

Given: Air flow in laboratory wind tunnel test section.

Find: (a) Freestream speed at exit
(b) Pressure at exit

Solution: We will apply the continuity and Bernoulli equations to this problem.

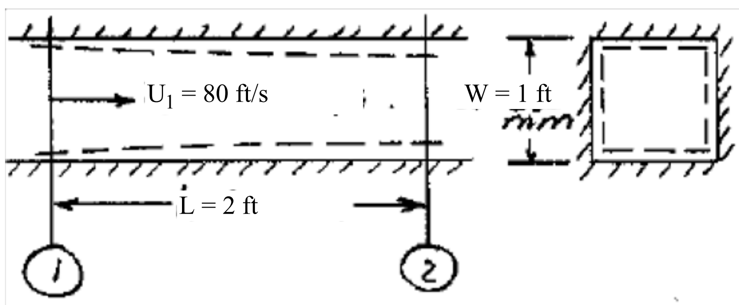
Governing Equations:

$$\delta_{disp} = \int_0^{infinity} \left(1 - \frac{u}{U}\right) dy = \int_0^{\delta} \left(1 - \frac{u}{U}\right) dy$$ (Definition of displacement thickness)

$$\frac{\partial}{\partial t}\int_{CV} \rho d\mathit{V\!\!\!\!-} + \int_{CS} \rho \vec{V} \cdot d\vec{A} = 0$$ (Continuity)

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = const$$ (Bernoulli)

Assumptions:
(1) Steady, incompressible flow
(2) No body forces in the x-direction
(3) No viscous forces outside boundary layer
(4) Streamline exists between stations 1 and 2
(5) Uniform flow outside boundary layer
(6) Boundary layer is the same on all walls
(7) Neglect corner effects
(8) Constant elevation between 1 and 2



If we divide both sides of the displacement thickness definition by δ, we get:

$$\frac{\delta_{disp}}{\delta} = \frac{1}{\delta} \cdot \int_0^{\delta} \left(1 - \frac{u}{U}\right) dy$$

However, we can change the variable of integration to η = y/δ, resulting in:

$$d\eta = \frac{1}{\delta} \cdot dy$$

Therefore: $$\frac{\delta_{disp}}{\delta} = \int_0^1 \left(1 - \frac{u}{U}\right) d\eta$$ For the power law profile: $$\frac{u}{U} = \eta^{\frac{1}{7}}$$ Into the displacement thickness:

$$\frac{\delta_{disp}}{\delta} = \int_0^1 \left(1 - \eta^{\frac{1}{7}}\right) d\eta$$ Evaluating this integral: $$\frac{\delta_{disp}}{\delta} = 1 - \frac{7}{8} = \frac{1}{8}$$ So the displacement thicknesses are:

$$\delta_{disp1} = \frac{1}{8} \times 0.8 \cdot in \qquad \delta_{disp1} = 0.100 \cdot in \qquad \delta_{disp2} = \frac{1}{8} \times 1 \cdot in \qquad \delta_{disp2} = 0.125 \cdot in$$

After applying the assumptions from above, continuity reduces to: $U_1 \cdot A_1 = U_2 \cdot A_2$ or $U_1 \cdot \left(W - 2 \cdot \delta_{disp1}\right)^2 = U_2 \cdot \left(W - 2 \cdot \delta_{disp2}\right)^2$

Solving for the speed at 2: $U_2 = U_1 \cdot \left(\dfrac{W - 2 \cdot \delta_{disp1}}{W - 2 \cdot \delta_{disp2}}\right)^2$ Substituting known values: $U_2 = 80 \cdot \dfrac{ft}{s} \times \left(\dfrac{1 - 2 \times 0.100}{1 - 2 \times 0.125}\right)^2$

$$U_2 = 91.0 \cdot \frac{ft}{s}$$

From Bernoulli equation, since z = constant: $\dfrac{p_1}{\rho} + \dfrac{U_1^{\,2}}{2} = \dfrac{p_2}{\rho} + \dfrac{U_2^{\,2}}{2}$ along a streamline. Therefore:

$$\Delta p_{12} = \frac{\rho}{2} \cdot \left(U_1^{\,2} - U_2^{\,2}\right) \qquad \Delta p_{12} = \frac{1}{2} \times 0.00239 \cdot \frac{slug}{ft^3} \times \left(80^2 - 91^2\right) \frac{ft^2}{s^2} \times \frac{lbf \cdot s^2}{slug \cdot ft} \times \left(\frac{ft}{12 \cdot in}\right)^2 \qquad \Delta p_{12} = -0.01561 \cdot psi$$

From ambient to station 1 we see a loss at the tunnel entrance:

$$\left(\frac{p_0}{\rho} + \frac{U_0^{\,2}}{2}\right) - \left(\frac{p_1}{\rho} + \frac{U_1^{\,2}}{2}\right) = h_{lT}$$ Since $U_0 = 0$ and $p_0 = p_{atm} = 0$ we can solve for the pressure at 1:

$p_1 = -\rho \cdot h_{lT} + \dfrac{1}{2} \cdot \rho \cdot U_1^{\,2}$ where $\rho h_{lT} = -\dfrac{0.3}{12} \cdot ft \times 1.94 \cdot \dfrac{slug}{ft^3} \times 32.2 \cdot \dfrac{ft}{s^2} \times \dfrac{lbf \cdot s^2}{slug \cdot ft} \times \left(\dfrac{ft}{12 \cdot in}\right)^2$ $\quad \rho h_{lT} = -0.01085 \cdot psi$

Therefore: $p_1 = -0.01085 \cdot \dfrac{lbf}{in^2} - \dfrac{1}{2} \times 0.00239 \cdot \dfrac{slug}{ft^3} \times \left(80 \cdot \dfrac{ft}{s}\right)^2 \times \dfrac{lbf \cdot s^2}{slug \cdot ft} \times \left(\dfrac{ft}{12 \cdot in}\right)^2 = -0.0640 \cdot psi$ So the pressure at 2 is:

$p_2 = p_1 + \Delta p_{12}$ $\quad p_2 = -0.0640 \cdot psi - 0.01561 \cdot psi = -0.0796 \cdot psi$ Since the pressure drop can be expressed as $p_2 = \rho \cdot g \cdot h_2$

it follows that: $h_2 = \dfrac{p_2}{\rho \cdot g}$ So in terms of water height: $h_2 = 0.0796 \cdot \dfrac{lbf}{in^2} \times \left(\dfrac{12 \cdot in}{ft}\right)^2 \times \dfrac{ft^3}{1.94 \cdot slug} \times \dfrac{s^2}{32.2 \cdot ft} \times \dfrac{slug \cdot ft}{lbf \cdot s^2} \times \dfrac{12 \cdot in}{ft}$

$$p_2 = -0.0796 \cdot psi$$

$$h_2 = 2.20 \cdot in$$

Problem 9.28

[Difficulty: 3]

9.28 Flow of air develops in a horizontal cylindrical duct, of diameter $D = 15$ in., following a well-rounded entrance. A turbulent boundary grows on the duct wall, but the flow is not yet fully developed. Assume that the velocity profile in the boundary layer is $u/U = (y/\delta)^{1/7}$. The inlet flow is $U = 50$ ft/s at section ①. At section ②, the boundary-layer thickness is $\delta_2 = 4$ in. Evaluate the static gage pressure at section ②, located at $L = 20$ ft. Find the average wall shear stress.

Given: Data on fluid and boundary layer geometry

Find: Gage pressure at location 2; average wall stress

Solution:

The solution involves using mass conservation in the inviscid core, allowing for the fact that as the boundary layer grows it reduces the size of the core. One approach would be to integrate the 1/7 law velocity profile to compute the mass flow in the boundary layer; an easier approach is to simply use the displacement thickness!

The average wall stress can be estimated using the momentum equation for a CV

The given and available (from Appendix A) data is

$$\rho = 0.00234 \cdot \frac{\text{slug}}{\text{ft}^3} \qquad U_1 = 50 \cdot \frac{\text{ft}}{\text{s}} \qquad L = 20 \cdot \text{ft} \qquad D = 15 \cdot \text{in} \qquad \delta_2 = 4 \cdot \text{in}$$

Governing equations:

Mass
$$\frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A} = 0 \qquad (4.12)$$

Momentum
$$\vec{F} = \vec{F}_S + \vec{F}_B = \frac{\partial}{\partial t}\int_{CV} \vec{V} \rho \, d\forall + \int_{CS} \vec{V} \rho \vec{V} \cdot d\vec{A} \qquad (4.17a)$$

Bernoulli
$$\frac{p}{\rho} + \frac{V^2}{2} + g \cdot z = \text{constant} \qquad (4.24)$$

Assumptions: (1) Steady flow (2) No pressure force (3) No body force in *x* direction

The displacement thickness at location 2 can be computed from boundary layer thickness using Eq. 9.1

$$\delta_{disp2} = \int_0^{\delta_2} \left(1 - \frac{u}{U}\right) dy = \delta_2 \cdot \int_0^1 \left(1 - \eta^{\frac{1}{7}}\right) d\eta = \frac{\delta_2}{8}$$

Hence
$$\delta_{disp2} = \frac{\delta_2}{8} \qquad \delta_{disp2} = 0.500 \cdot \text{in}$$

Applying mass conservation at locations 1 and 2 $\quad \left(-\rho \cdot U_1 \cdot A_1\right) + \left(\rho \cdot U_2 \cdot A_2\right) = 0 \quad$ or $\quad U_2 = U_1 \cdot \frac{A_1}{A_2}$

$$A_1 = \frac{\pi}{4} \cdot D^2 \qquad A_1 = 1.227 \cdot \text{ft}^2$$

The area at location 2 is given by the duct cross section area minus the displacement boundary layer

$$A_2 = \frac{\pi}{4} \cdot \left(D - 2 \cdot \delta_{disp2}\right)^2 \qquad A_2 = 1.069 \cdot \text{ft}^2$$

Hence $$U_2 = U_1 \cdot \frac{A_1}{A_2}$$ $$U_2 = 57.4 \cdot \frac{ft}{s}$$

For the pressure change we can apply Bernoulli to locations 1 and 2 to find

$$p_1 - p_2 = \Delta p = \frac{\rho}{2} \cdot \left(U_2^{\ 2} - U_1^{\ 2} \right)$$ $$\Delta p = 6.46 \times 10^{-3} \cdot psi$$ $$p_2 = -\Delta p$$

Hence $$p_2(gage) = p_1(gage) - \Delta p$$ $$p_2 = -6.46 \times 10^{-3} \cdot psi$$

For the average wall shear stress we use the momentum equation, simplified for this problem

$$\Delta p \cdot A_1 - \tau \cdot \pi \cdot D \cdot L = -\rho \cdot U_1^{\ 2} \cdot A_1 + \rho \cdot U_2^{\ 2} \cdot \frac{\pi}{4} \cdot \left(D - 2 \cdot \delta_2 \right)^2 + \rho \cdot \int_{\frac{D}{2} - \delta_2}^{\frac{D}{2}} 2 \cdot \pi \cdot r \cdot u^2 \, dr$$

where $$u(r) = U_2 \cdot \left(\frac{y}{\delta_2} \right)^{\frac{1}{7}}$$ and $$r = \frac{D}{2} - y$$ $$dr = -dy$$

The integral is $$\rho \cdot \int_{\frac{D}{2} - \delta_2}^{\frac{D}{2}} 2 \cdot \pi \cdot r \cdot u^2 \, dr = -2 \cdot \pi \cdot \rho \cdot U_2^{\ 2} \cdot \int_{\delta_2}^{0} \left(\frac{D}{2} - y \right) \cdot \left(\frac{y}{\delta_2} \right)^{\frac{2}{7}} dy$$

$$\rho \cdot \int_{\frac{D}{2} - \delta_2}^{\frac{D}{2}} 2 \cdot \pi \cdot r \cdot u^2 \, dr = 7 \cdot \pi \cdot \rho \cdot U_2^{\ 2} \cdot \delta_2 \cdot \left(\frac{D}{9} - \frac{\delta_2}{8} \right)$$

Hence $$\tau = \frac{\Delta p \cdot A_1 + \rho \cdot U_1^{\ 2} \cdot A_1 - \rho \cdot U_2^{\ 2} \cdot \frac{\pi}{4} \cdot \left(D - 2 \cdot \delta_2 \right)^2 - 7 \cdot \pi \cdot \rho \cdot U_2^{\ 2} \cdot \delta_2 \cdot \left(\frac{D}{9} - \frac{\delta_2}{8} \right)}{\pi \cdot D \cdot L}$$

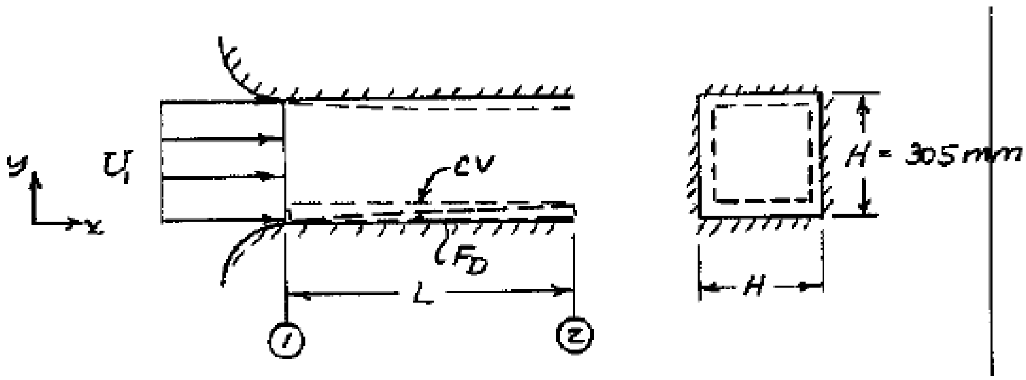
$$\tau = 6.767 \times 10^{-5} \cdot psi$$

Problem 9.29

[Difficulty: 5]

9.29 Air flows into the inlet contraction section of a wind tunnel in an undergraduate laboratory. From the inlet the air enters the test section, which is square in cross-section with side dimensions of 305 mm. The test section is 609 mm long. At one operating condition air leaves the contraction at 50.2 m/s with negligible boundary-layer thickness. Measurements show that boundary layers at the downstream end of the test section are 20.3 mm thick. Evaluate the displacement thickness of the boundary layers at the downstream end of the wind tunnel test section. Calculate the change in static pressure along the wind tunnel test section. Estimate the approximate total drag force caused by skin friction on each wall of the wind tunnel.

Given: Air flow in laboratory wind tunnel test section.



Find:
(a) Displacement thickness at station 2
(b) Pressure drop between 1 and 2
(c) Total drag force caused by friction on each wall

Solution: We will apply the continuity, x-momentum, and Bernoulli equations to this problem.

Governing Equations:

$$\delta_{disp} = \int_0^{infinity} \left(1 - \frac{u}{U}\right) dy = \int_0^{\delta} \left(1 - \frac{u}{U}\right) dy$$ (Definition of displacement thickness)

$$\frac{\partial}{\partial t}\int_{CV} \rho dV\!\!\!/ + \int_{CS} \rho \vec{V} \cdot d\vec{A} = 0$$ (Continuity)

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = const$$ (Bernoulli)

$$\frac{\partial}{\partial t}\int_{CV} \rho u dV\!\!\!/ + \int_{CS} u \rho \vec{V} \cdot d\vec{A} = F_{sx} + F_{bx}$$ (x- Momentum)

Assumptions:
(1) Steady, incompressible flow
(2) No body forces in the x-direction
(3) No viscous forces outside boundary layer
(4) Streamline exists between stations 1 and 2
(5) Uniform flow outside boundary layer
(6) Boundary layer is the same on all walls
(7) Neglect corner effects
(8) Constant elevation between 1 and 2

If we divide both sides of the displacement thickness definition by δ, we get: $\frac{\delta_{disp}}{\delta} = \frac{1}{\delta} \cdot \int_0^{\delta} \left(1 - \frac{u}{U}\right) dy$

However, we can change the variable of integration to $\eta = y/\delta$, resulting in: $d\eta = \frac{1}{\delta} \cdot dy$ Therefore: $\frac{\delta_{disp}}{\delta} = \int_0^1 \left(1 - \frac{u}{U}\right) d\eta$

If we assume the power law profile (turbulent BL): $\frac{u}{U} = \eta^{\frac{1}{7}}$ Into the displacement thickness: $\frac{\delta_{disp}}{\delta} = \int_0^1 \left(1 - \eta^{\frac{1}{7}}\right) d\eta$

Evaluating this integral: $\frac{\delta_{disp}}{\delta} = 1 - \frac{7}{8} = \frac{1}{8}$ So the displacement thickness is: $\delta_{disp2} = \frac{1}{8} \times 20.3 \cdot mm$ $\delta_{disp2} = 2.54 \cdot mm$

After applying the assumptions from above, continuity reduces to: $U_1 \cdot A_1 = U_2 \cdot A_2$ or $U_1 \cdot H^2 = U_2 \cdot \left(H - 2 \cdot \delta_{disp2}\right)^2$

Solving for the speed at 2: $U_2 = U_1 \cdot \left(\frac{H}{H - 2 \cdot \delta_{disp2}}\right)^2$ Substituting known values: $U_2 = 50.2 \cdot \frac{m}{s} \times \left(\frac{305}{305 - 2 \times 2.54}\right)^2$

$$U_2 = 51.9 \frac{m}{s}$$

From Bernoulli equation, since z = constant: $\frac{p_1}{\rho} + \frac{U_1^2}{2} = \frac{p_2}{\rho} + \frac{U_2^2}{2}$ along a streamline. Therefore:

$$\Delta p_{12} = \frac{\rho}{2} \cdot \left(U_1^2 - U_2^2\right) \quad \Delta p_{12} = \frac{1}{2} \times 1.23 \cdot \frac{kg}{m^3} \times \left(50.2^2 - 51.9^2\right) \frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m} \quad \Delta p_{12} = -106.7\ Pa$$

To determine the drag on the walls, we choose the control volume shown above and apply the x-momentum equation.

From the assumptions, the equation reduces to: $\int_{CS} u \rho \vec{V} \cdot d\vec{A} = F_{sx}$ Applying this to the control volume:

$p_1 \cdot H \cdot \delta_2 - F_D - p_2 \cdot H \cdot \delta_2 = U_1 \cdot \left(-\rho \cdot U_1 \cdot H \cdot \delta_2\right) + U_{avg} \cdot m_{top} + \int_0^{\delta_2} u \cdot \rho \cdot u \cdot H\, dy$ The mass flow rate through the top of the CV can be determined using the continuity equation across the control volume: $m_{top} = m_1 - m_2 = \rho \cdot U_1 \cdot H \cdot \delta_2 - \int_0^{\delta_2} \rho \cdot u \cdot H\, dy$

This integral can be evaluated using the power law profile: $\int_0^{\delta_2} \rho \cdot u \cdot H\, dy = \rho \cdot U_2 \cdot H \cdot \delta_2 \cdot \int_0^1 \eta^{\frac{1}{7}} d\eta = \frac{7}{8} \cdot \rho \cdot U_2 \cdot H \cdot \delta_2$ Therefore:

$m_{top} = \rho \cdot H \cdot \delta_2 \cdot \left(U_1 - \frac{7}{8} \cdot U_2\right)$ The average speed can be approximated as the mean of the speeds at 1 and 2: $U_{avg} = \frac{U_1 + U_2}{2}$

Finally the integral in the momentum equation may also be evaluated using the power law profile:

$\int_0^{\delta_2} u \cdot \rho \cdot u \cdot H\, dy = \rho \cdot U_2^2 \cdot H \cdot \delta_2 \cdot \int_0^1 \eta^{\frac{2}{7}} d\eta = \frac{7}{9} \cdot \rho \cdot U_2^2 \cdot H \cdot \delta_2$ Thus, the momentum equation may be rewritten as:

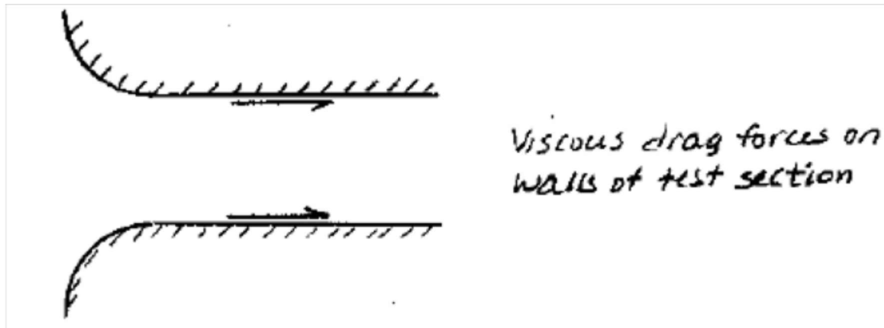
$p_1 \cdot H \cdot \delta_2 - F_D - p_2 \cdot H \cdot \delta_2 = U_1 \cdot \left(-\rho \cdot U_1 \cdot H \cdot \delta_2\right) + \frac{U_1 + U_2}{2} \cdot \rho \cdot H \cdot \delta_2 \cdot \left(U_1 - \frac{7}{8} \cdot U_2\right) + \frac{7}{9} \cdot \rho \cdot U_2^2 \cdot H \cdot \delta_2$ Solving for the drag force:

$$F_D = \left[\left(p_1 - p_2\right) + \rho \cdot \left[U_1^{\,2} - \left(\frac{U_1 + U_2}{2}\right) \cdot \left(U_1 - \frac{7}{8} \cdot U_2\right) - \frac{7}{9} \cdot U_2^{\,2}\right]\right] \cdot H \cdot \delta_2$$ Substituting in all known values yields:

$$F_D = \left[106.7 \cdot \frac{N}{m^2} + 1.23 \cdot \frac{kg}{m^3} \times \left[\left(50.2 \cdot \frac{m}{s}\right)^2 - \frac{(50.2 + 51.9)}{2} \times \left(50.2 - \frac{7}{8} \cdot 51.9\right) \cdot \frac{m^2}{s^2} - \frac{7}{9} \cdot \left(51.9 \cdot \frac{m}{s}\right)^2\right] \times \frac{N \cdot s^2}{kg \cdot m}\right] \times 0.305 \cdot m \times 0.020$$

$$F_D = 2.04\,N$$

The viscous drag force acts on the CV in the direction shown. The viscous drag force on the wall of the test section is equal and opposite:



Problem 9.30

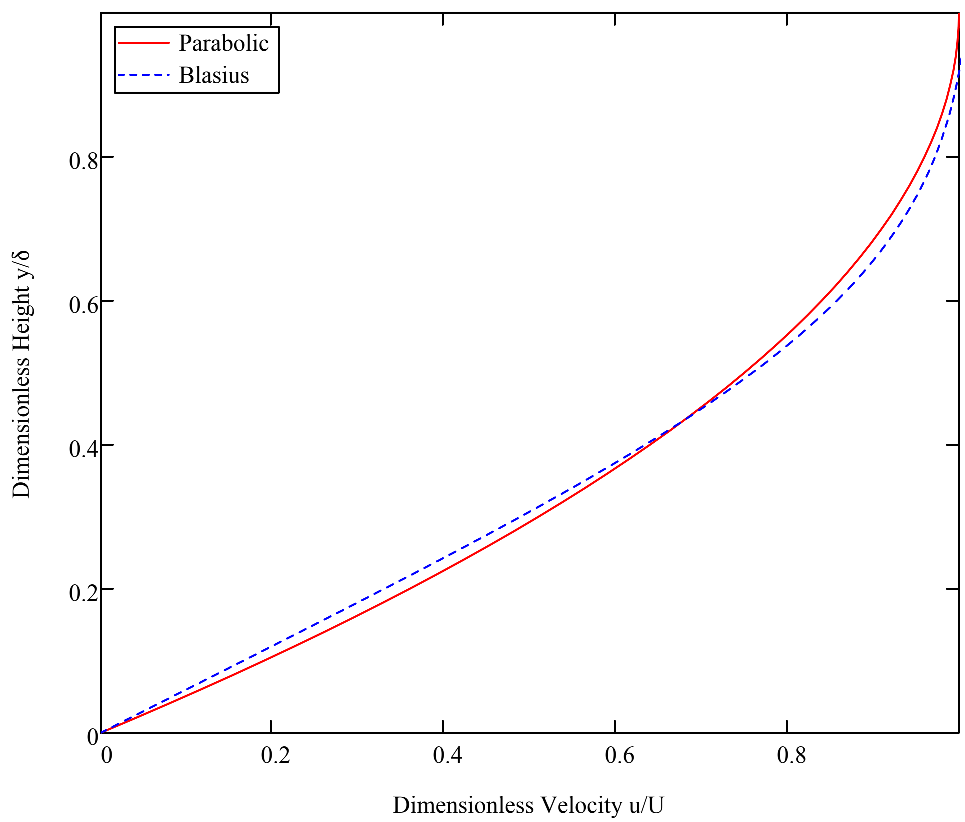
[Difficulty: 2]

*9.30 Using numerical results for the Blasius exact solution for laminar boundary-layer flow on a flat plate, plot the dimensionless velocity profile, u/U (on the abscissa), versus dimensionless distance from the surface, y/δ (on the ordinate). Compare with the approximate parabolic velocity profile of Problem 9.10.

Given: Blasius exact solution for laminar boundary layer flow

Find: Plot and compare to parabolic velocity profile: $\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$

Solution: The Blasius solution is given in Table 9.1; it is plotted below.



Problem 9.31

[Difficulty: 3]

*9.31 Using numerical results obtained by Blasius (Table 9.1), evaluate the distribution of shear stress in a laminar boundary layer on a flat plate. Plot τ/τ_w versus y/δ. Compare with results derived from the approximate sinusoidal velocity profile given in Problem 9.10.

Given: Blasius exact solution for laminar boundary layer flow

Find: (a) Evaluate shear stress distribution
(b) Plot τ/τ_w versus y/δ
(c) Compare with results from sinusoidal velocity profile: $\frac{u}{U} = \sin\left(\frac{\pi}{2}\cdot\frac{y}{\delta}\right)$

Solution: We will apply the shear stress definition to both velocity profiles.

Governing Equation: $\tau = \mu\cdot\frac{\partial}{\partial y}u$ (Shear stress in Newtonian fluid)

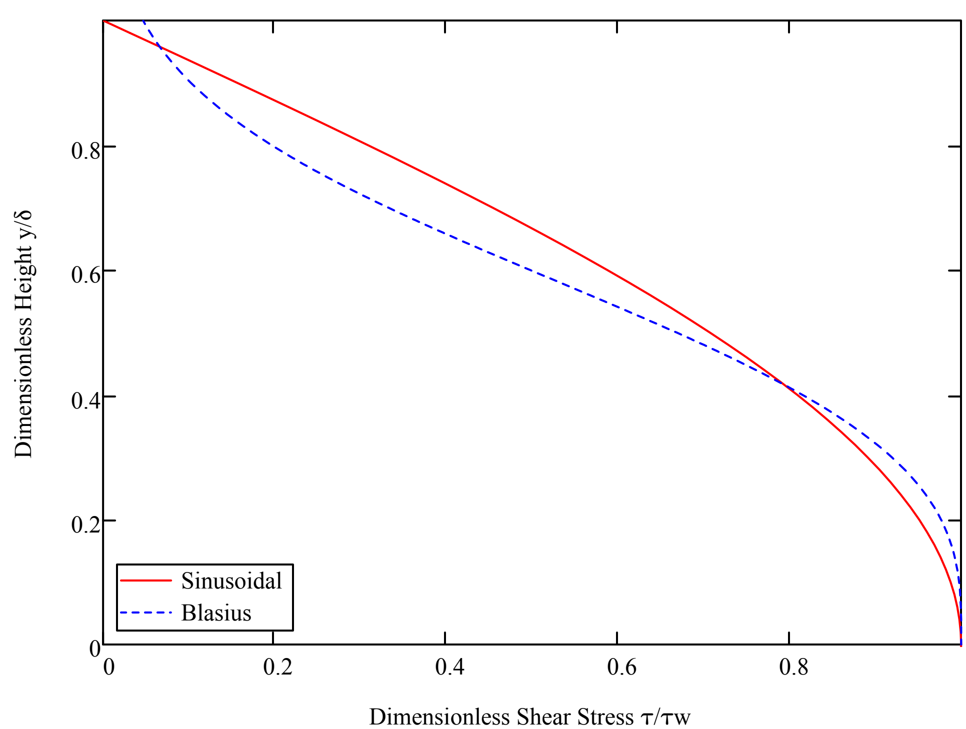
For Blasius: $u = U\cdot f'(\eta)$ and $\eta = y\cdot\sqrt{\frac{U}{\nu\cdot x}}$ The shear stress is: $\tau = \mu\cdot\frac{\partial}{\partial y}(U\cdot f'(\eta)) = U\cdot\mu\cdot(f''(\eta))\cdot\left(\frac{\partial}{\partial y}\eta\right) = U\cdot\mu\cdot f''(\eta)\cdot\sqrt{\frac{U}{\nu\cdot x}}$

Therefore: $\frac{\tau}{\rho\cdot U^2} = \frac{\mu}{\rho\cdot U}\cdot f''(\eta)\cdot\sqrt{\frac{U}{\nu\cdot x}} = \frac{f''(\eta)}{\sqrt{Re_x}}$ τ is proportional to $f''(\eta)$

From the above equation: $\frac{\tau}{\tau_w} = \frac{f''(\eta)}{f''(0)} = \frac{f''(\eta)}{0.33206}$ Since $y = \delta$ at $\eta = 5$ it follows that $\frac{y}{\delta} = \frac{\eta}{5}$

For the sinusoidal profile: $\tau = \frac{\mu\cdot U}{\delta}\cdot\left[\frac{d}{d\left(\frac{y}{\delta}\right)}\left(\frac{u}{U}\right)\right] = \frac{\mu\cdot U}{\delta}\cdot\frac{\pi}{2}\cdot\cos\left(\frac{\pi}{2}\cdot\frac{y}{\delta}\right)$ $\tau_w = \frac{\mu\cdot U}{\delta}\cdot\frac{\pi}{2}$ Thus: $\frac{\tau}{\tau_w} = \cos\left(\frac{\pi}{2}\cdot\frac{y}{\delta}\right)$

Both profiles are plotted here:



Problem 9.32

[Difficulty: 3]

***9.32** Using numerical results obtained by Blasius (Table 9.1), evaluate the distribution of shear stress in a laminar boundary layer on a flat plate. Plot τ/τ_w versus y/δ. Compare with results derived from the approximate parabolic velocity profile given in Problem 9.10.

Given: Blasius exact solution for laminar boundary layer flow

Find:
(a) Evaluate shear stress distribution
(b) Plot $\tau/\tau w$ versus y/δ
(c) Compare with results from sinusoidal velocity profile: $\frac{u}{U} = 2\cdot\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2$

Solution: We will apply the shear stress definition to both velocity profiles.

Governing Equation:

$$\tau = \mu\cdot\frac{\partial}{\partial y}u \qquad \text{(Shear stress in Newtonian fluid)}$$

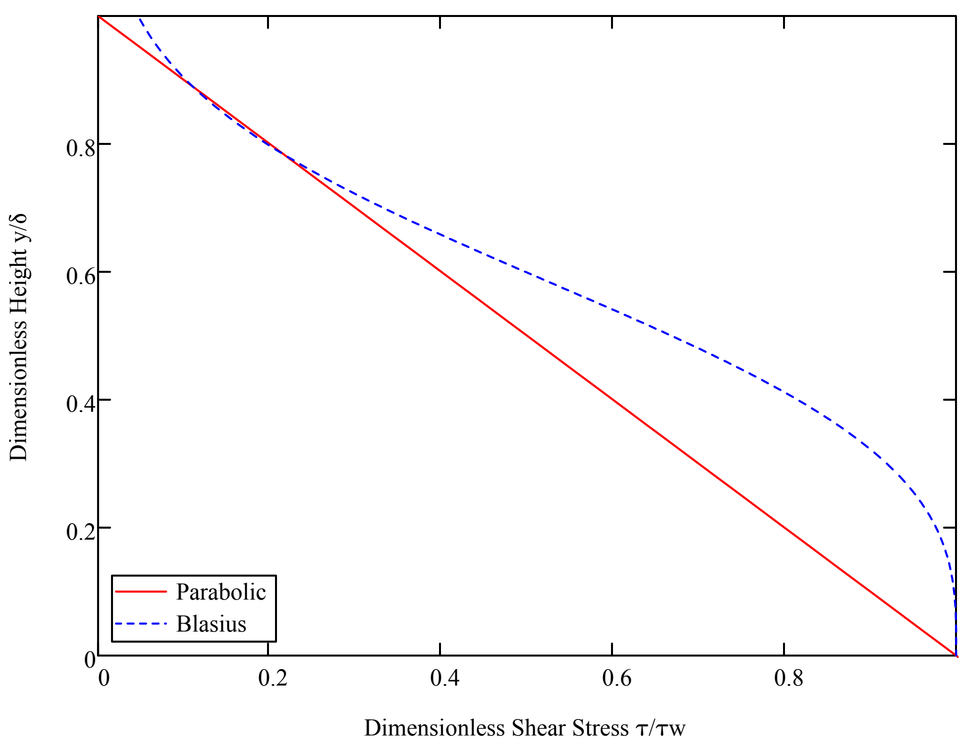
For Blasius: $u = U\cdot f'(\eta)$ and $\eta = y\cdot\sqrt{\frac{U}{\nu\cdot x}}$ The shear stress is: $\tau = \mu\cdot\frac{\partial}{\partial y}(U\cdot f'(\eta)) = U\cdot\mu\cdot(f''(\eta))\cdot\left(\frac{\partial}{\partial y}\eta\right) = U\cdot\mu\cdot f''(\eta)\cdot\sqrt{\frac{U}{\nu\cdot x}}$

Therefore: $\frac{\tau}{\rho\cdot U^2} = \frac{\mu}{\rho\cdot U}\cdot f''(\eta)\cdot\sqrt{\frac{U}{\nu\cdot x}} = \frac{f''(\eta)}{\sqrt{Re_x}}$ $\qquad \tau$ is proportional to $f''(\eta)$

From the above equation: $\frac{\tau}{\tau_w} = \frac{f''(\eta)}{f''(0)} = \frac{f''(\eta)}{0.33206}$ Since $y = \delta$ at $\eta = 5$ it follows that $\frac{y}{\delta} = \frac{\eta}{5}$

For the parabolic profile: $\tau = \frac{\mu\cdot U}{\delta}\cdot\left[\frac{d}{d\left(\frac{y}{\delta}\right)}\left(\frac{u}{U}\right)\right] = \frac{\mu\cdot U}{\delta}\cdot\left(2 - 2\cdot\frac{y}{\delta}\right)$ $\qquad \tau_w = \frac{\mu\cdot U}{\delta}\cdot 2$ $\qquad$ Thus: $\frac{\tau}{\tau_w} = 1 - \frac{y}{\delta}$

Both profiles are plotted here:



Problem 9.33

[Difficulty: 3]

*9.33 Using numerical results obtained by Blasius (Table 9.1), evaluate the vertical component of velocity in a laminar boundary layer on a flat plate. Plot v/U versus y/δ for $Re_x = 10^5$.

Given: Blasius exact solution for laminar boundary layer flow

Find: Plot v/U versus y/δ for $Re_x = 10^5$

Solution: We will apply the stream function definition to the Blasius solution.

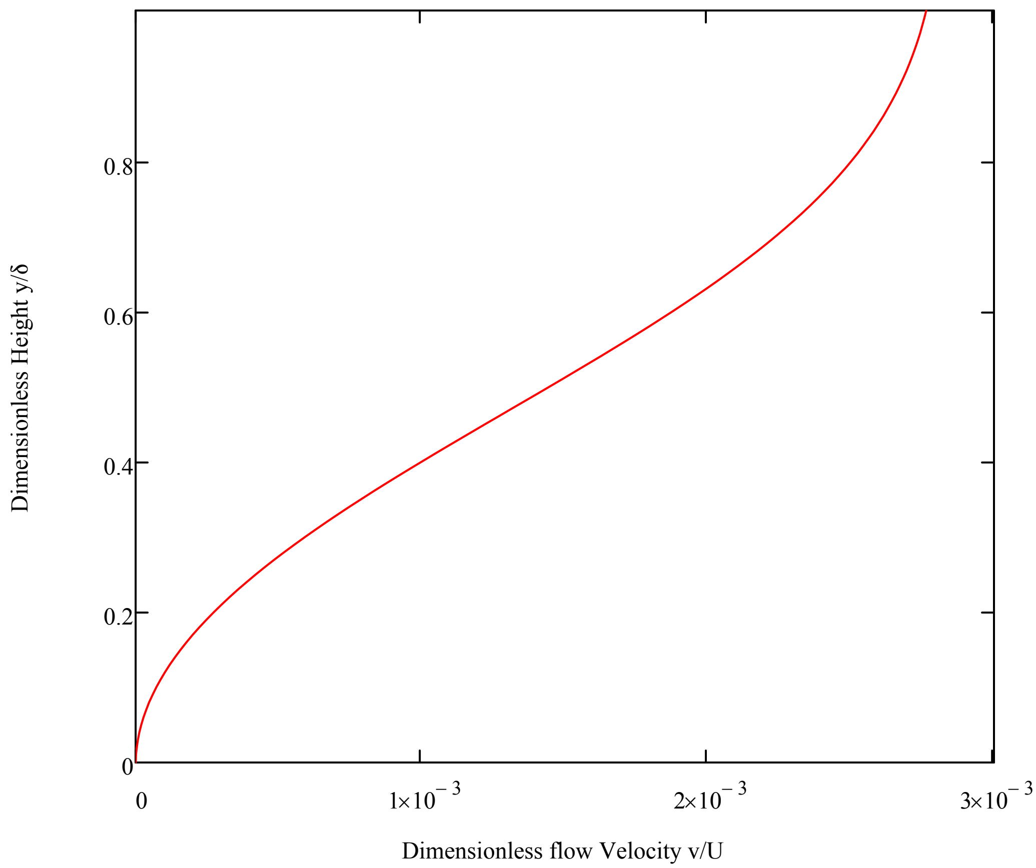
For Blasius: $u = U \cdot f'(\eta)$ and $\eta = y \cdot \sqrt{\frac{U}{\nu \cdot x}}$ The stream function is: $\psi = \sqrt{U \cdot \nu \cdot x} \cdot f(\eta)$

From the stream function: $v = -\frac{\partial}{\partial x}\psi = -\left[\frac{1}{2} \cdot \sqrt{\frac{\nu \cdot U}{x}} \cdot f(\eta) + \sqrt{\nu \cdot U \cdot x} \cdot \left(\frac{d}{d\eta} f\right) \cdot \left(\frac{\partial}{\partial x}\eta\right)\right]$ But $\frac{\partial}{\partial x}\eta = -\frac{1}{2} \cdot \frac{y}{x} \cdot \sqrt{\frac{U}{\nu \cdot x}} = -\frac{1}{2} \cdot \frac{\eta}{x}$

Thus $v = -\left[\frac{1}{2} \cdot \sqrt{\frac{\nu \cdot U}{x}} \cdot f(\eta) + \sqrt{\nu \cdot U \cdot x} \cdot \left(\frac{d}{d\eta} f\right) \cdot \left(-\frac{1}{2} \cdot \frac{\eta}{x}\right)\right] = \frac{1}{2} \cdot \sqrt{\frac{\nu \cdot U}{x}} \cdot (\eta \cdot f'(\eta) - f(\eta))$ and $\frac{v}{U} = \frac{1}{2} \cdot \sqrt{\frac{\nu}{U \cdot x}} \cdot (\eta \cdot f'(\eta) - f(\eta))$

$$\frac{v}{U} = \frac{\eta \cdot f'(\eta) - f(\eta)}{2\sqrt{Re_x}}$$

Since $y = \delta$ at $\eta = 5$ it follows that $\frac{y}{\delta} = \frac{\eta}{5}$ Plotting v/U as a function of y/δ:



Problem 9.34

[Difficulty: 3]

*9.34 Verify that the y component of velocity for the Blasius solution to the Prandtl boundary-layer equations is given by Eq. 9.10. Obtain an algebraic expression for the x component of the acceleration of a fluid particle in the laminar boundary layer. Plot a_x versus η to determine the maximum x component of acceleration at a given x.

Given: Blasius exact solution for laminar boundary layer flow

Find: (a) Prove that the y component of velocity in the solution is given by Eq. 9.10.
(b) Algebraic expression for the x component of a fluid particle in the BL
(c) Plot a_x vs η to determine the maximum x component of acceleration at a given x

Solution: We will apply the stream function definition to the Blasius solution.

For Blasius: $u = U\cdot f'(\eta)$ and $\eta = y\cdot\sqrt{\frac{U}{\nu\cdot x}}$ The stream function is: $\psi = \sqrt{U\cdot\nu\cdot x}\cdot f(\eta)$

From the stream function: $v = -\frac{\partial}{\partial x}\psi = -\left[\frac{1}{2}\cdot\sqrt{\frac{\nu\cdot U}{x}}\cdot f(\eta) + \sqrt{\nu\cdot U\cdot x}\cdot\left(\frac{d}{d\eta}f\right)\cdot\left(\frac{\partial}{\partial x}\eta\right)\right]$ But $\frac{\partial}{\partial x}\eta = -\frac{1}{2}\cdot\frac{y}{x}\cdot\sqrt{\frac{U}{\nu\cdot x}} = -\frac{1}{2}\cdot\frac{\eta}{x}$

Thus $v = -\left[\frac{1}{2}\cdot\sqrt{\frac{\nu\cdot U}{x}}\cdot f(\eta) + \sqrt{\nu\cdot U\cdot x}\cdot\left(\frac{d}{d\eta}f\right)\cdot\left(-\frac{1}{2}\cdot\frac{\eta}{x}\right)\right] = \frac{1}{2}\cdot\sqrt{\frac{\nu\cdot U}{x}}\cdot(\eta\cdot f'(\eta) - f(\eta))$ which is Eq. 9.10.

$$v = \frac{1}{2}\cdot\sqrt{\frac{\nu\cdot U}{x}}\cdot(\eta\cdot f'(\eta) - f(\eta))$$

The acceleration in the x-direction is given by: $a_x = u\cdot\frac{\partial}{\partial x}u + v\cdot\frac{\partial}{\partial y}u$ where $u = U\cdot f'(\eta)$ Evaluating the partial derivatives:

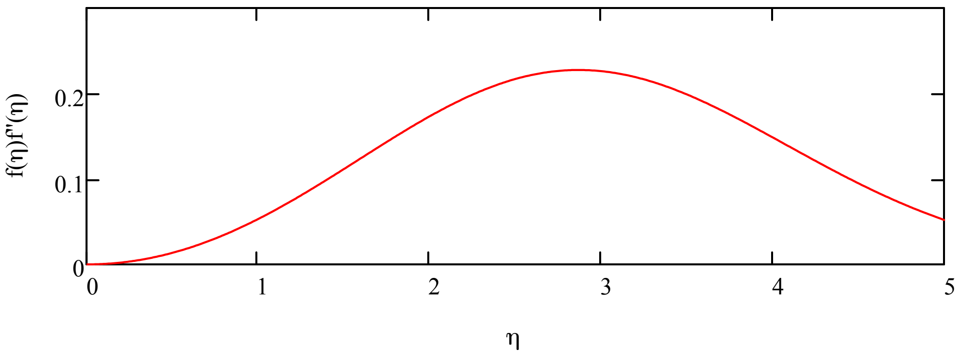
$\frac{\partial}{\partial x}u = U\cdot\frac{d}{d\eta}f'(\eta)\cdot\left(\frac{d}{dx}\eta\right) = U\cdot f''(\eta)\cdot\left(-\frac{\eta}{2\cdot x}\right) = -\frac{1}{2}\cdot\frac{\eta\cdot U\cdot f''(\eta)}{x}$ $\quad\frac{\partial}{\partial y}u = U\cdot\frac{d}{d\eta}f'(\eta)\cdot\left(\frac{d}{dy}\eta\right) = U\cdot f''(\eta)\cdot\sqrt{\frac{U}{\nu\cdot x}}$ Therefore:

$$a_x = U\cdot f'(\eta)\cdot\left(-\frac{1}{2}\cdot\frac{\eta\cdot U\cdot f''}{x}\right) + \left[\frac{1}{2}\cdot\sqrt{\frac{\nu\cdot U}{x}}\cdot(\eta\cdot f' - f)\right]\cdot\left(U\cdot f''\cdot\sqrt{\frac{U}{\nu\cdot x}}\right) = -\frac{1}{2}\cdot\frac{U^2}{x}\cdot\eta\cdot f'\cdot f'' + \frac{1}{2}\cdot\frac{U^2}{x}\cdot(\eta\cdot f'\cdot f'' - f\cdot f'')$$

Simplifying yields:

$$a_x = -\frac{U^2}{2x}\cdot f(\eta)\cdot f''(\eta)$$

If we plot $f(\eta)f''(\eta)$ as a function of η:



The maximum value of this function is 0.23 at η of approximately 3.

$$a_{xmax} = -0.115\cdot\frac{U^2}{x}$$

Problem 9.35

[Difficulty: 4]

*9.35 Numerical results of the Blasius solution to the Prandtl boundary-layer equations are presented in Table 9.1. Consider steady, incompressible flow of standard air over a flat plate at freestream speed $U = 5$ m/s. At $x = 20$ cm, estimate the distance from the surface at which $u = 0.95\,U$. Evaluate the slope of the streamline through this point. Obtain an algebraic expression for the local skin friction, $\tau_w(x)$. Obtain an algebraic expression for the total skin friction drag force on the plate. Evaluate the momentum thickness at $L = 1$ m.

Given: Blasius solution for laminar boundary layer

Find: Point at which u = 0.95U; Slope of streamline; expression for skin friction coefficient and total drag; Momentum thickness

Solution:

Basic equation: Use results of Blasius solution (Table 9.1 on the web), and $\eta = y\cdot\sqrt{\dfrac{\nu\cdot x}{U}}$

$$f' = \frac{u}{U} = 0.9130 \qquad \text{at} \qquad \eta = 3.5$$

$$f' = \frac{u}{U} = 0.9555 \qquad \text{at} \qquad \eta = 4.0$$

Hence by linear interpolation, when $f' = 0.95$ $\qquad \eta = 3.5 + \dfrac{(4 - 3.5)}{(0.9555 - 0.9130)}\cdot(0.95 - 0.9130 \qquad \eta = 3.94$

From Table A.10 at 20oC $\qquad \nu = 1.50\times 10^{-5}\cdot\dfrac{m^2}{s}$ $\qquad$ and $\qquad U = 5\cdot\dfrac{m}{s} \qquad x = 20\cdot cm$

Hence $\qquad y = \eta\cdot\sqrt{\dfrac{\nu\cdot x}{U}} \qquad y = 0.305\cdot cm$

The streamline slope is given by $\qquad \dfrac{dy}{dx} = \dfrac{v}{u}$ $\qquad$ where $\qquad u = U\cdot f'$ $\qquad$ and $\qquad v = \dfrac{1}{2}\cdot\sqrt{\dfrac{\nu\cdot U}{x}}\cdot(\eta\cdot f' - f)$

$$\frac{dy}{dx} = \frac{1}{2}\cdot\sqrt{\frac{\nu\cdot U}{x}}\cdot(\eta\cdot f' - f)\cdot\frac{1}{U\cdot f'} = \frac{1}{2}\cdot\sqrt{\frac{\nu}{U\cdot x}}\cdot\frac{(\eta\cdot f' - f)}{f'} = \frac{1}{2\cdot\sqrt{Re_x}}\cdot\frac{(\eta\cdot f' - f)}{f'}$$

We have $\qquad Re_x = \dfrac{U\cdot x}{\nu} \qquad Re_x = 6.67\times 10^4$

From the Blasius solution (Table 9.1 on the web)

$$f = 1.8377 \qquad \text{at} \qquad \eta = 3.5$$

$$f = 2.3057 \qquad \text{at} \qquad \eta = 4.0$$

Hence by linear interpolation $\qquad f = 1.8377 + \dfrac{(2.3057 - 1.8377)}{(4.0 - 3.5)}\cdot(3.94 - 3.5) \qquad f = 2.25$

$$\frac{dy}{dx} = \frac{1}{2\cdot\sqrt{Re_x}}\cdot\frac{(\eta\cdot f' - f)}{f'} = 0.00326$$

The shear stress is $\qquad \tau_w = \mu\cdot\left(\dfrac{\partial}{\partial y}u + \dfrac{\partial}{\partial x}v\right) = \mu\cdot\dfrac{\partial}{\partial y}u$ $\quad$ at y = 0 (v = 0 at the wall for all x, so the derivative is zero there)

$\tau_w = \mu\cdot U\cdot\sqrt{\dfrac{U}{\nu\cdot x}}\cdot\dfrac{d^2f}{d\eta^2}$ $\qquad$ and at $\eta = 0$ $\qquad \dfrac{d^2f}{d\eta^2} = 0.3321$ $\qquad$ (from Table 9.1)

$$\tau_w = 0.3321 \cdot U \cdot \sqrt{\frac{\rho \cdot U \cdot \mu}{x}} \qquad \tau_w = 0.3321 \cdot \rho \cdot U^2 \cdot \sqrt{\frac{\mu}{\rho \cdot U \cdot x}} = 0.3321 \cdot \frac{\rho \cdot U^2}{\sqrt{Re_x}}$$

The friction drag is $F_D = \int \tau_w \, dA = \int_0^L \tau_w \cdot b \, dx$ where b is the plate width

$$F_D = \int_0^L 0.3321 \cdot \frac{\rho \cdot U^2}{\sqrt{Re_x}} \cdot b \, dx = 0.3321 \cdot \rho \cdot U^2 \cdot \sqrt{\frac{\nu}{U}} \cdot \int_0^L \frac{1}{x^{\frac{1}{2}}} \, dx$$

$$F_D = 0.3321 \cdot \rho \cdot U^2 \cdot \sqrt{\frac{\nu}{U}} \cdot b \cdot 2 \cdot L^{\frac{1}{2}} \qquad F_D = \rho \cdot U^2 \cdot b \cdot L \cdot \frac{0.6642}{\sqrt{Re_L}}$$

For the momentum integral $\frac{\tau_w}{\rho \cdot U^2} = \frac{d\theta}{dx}$ or $d\theta = \frac{\tau_w}{\rho \cdot U^2} \cdot dx$

$$\theta_L = \frac{1}{\rho \cdot U^2} \cdot \int_0^L \tau_w \, dx = \frac{1}{\rho \cdot U^2} \cdot \frac{F_D}{b} = \frac{0.6642 \cdot L}{\sqrt{Re_L}}$$

We have $L = 1 \cdot m$ $\quad Re_L = \frac{U \cdot L}{\nu}$ $\quad Re_L = 3.33 \times 10^5$

$$\theta_L = \frac{0.6642 \cdot L}{\sqrt{Re_L}} \qquad \theta_L = 0.115 \cdot cm$$

Problem 9.36

[Difficulty: 2]

*9.36 Consider flow of air over a flat plate. On one graph, plot the laminar boundary-layer thickness as a function of distance along the plate (up to transition) for freestream speeds $U = 1$ m/s, 2 m/s, 3 m/s, 4 m/s, 5 m/s, and 10 m/s.

Given: Data on flow over flat plate

Find: Plot of laminar thickness at various speeds

Solution:

Given or available data: $\nu = 1.5 \times 10^{-5} \cdot \frac{m^2}{s}$ (from Table A.10 at 20°C)

Governing Equations: $\frac{\delta}{x} = \frac{5.48}{\sqrt{Re_x}}$ (9.21) and $Re_x = \frac{U \cdot x}{\nu}$ so $\delta = 5.48 \cdot \sqrt{\frac{\nu \cdot x}{U}}$

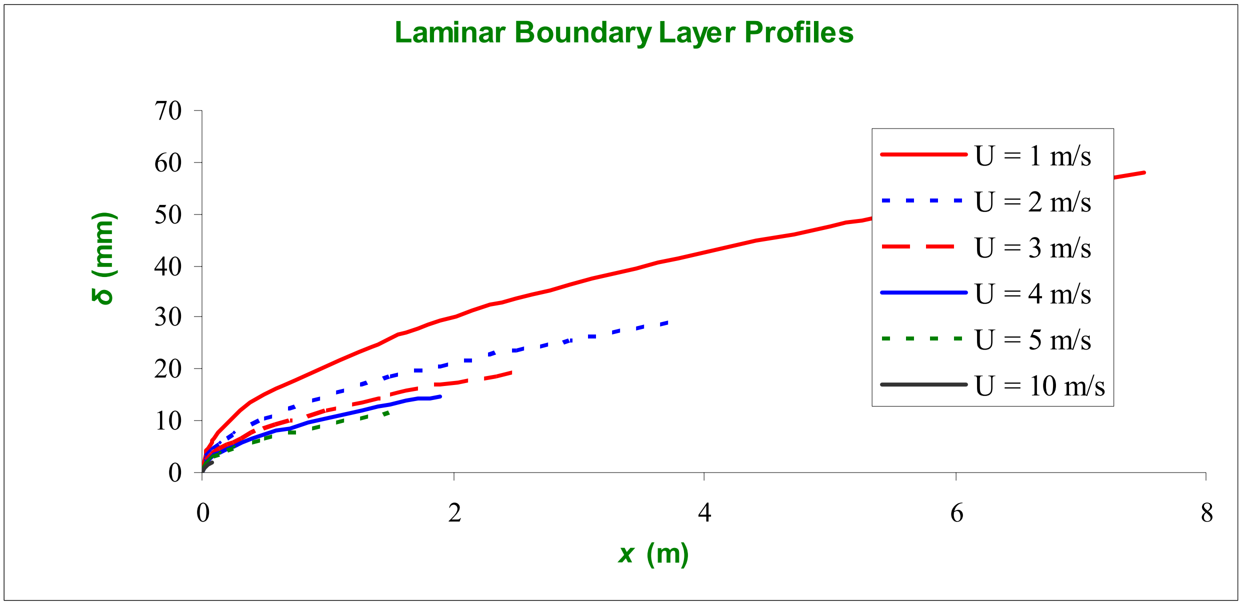
The critical Reynolds number is $Re_{crit} = 500000$

Hence, for velocity U the critical length x_{crit} is $x_{crit} = 500000 \cdot \frac{\nu}{U}$

The calculations and plot were generated in Excel and are shown below:

***U* (m/s)**	1	2	3	4	5	10
***x*crit (m)**	7.5	3.8	2.5	1.9	1.5	0.75

***x* (m)**	***δ* (mm)**	***δ* (mm)**	***δ* (mm)**	***δ* (mm)**	***δ* (mm)**	***δ* (mm)**
0.000	0.00	0.00	0.00	0.00	0.00	0.00
0.025	3.36	2.37	1.94	1.68	1.50	1.06
0.050	4.75	3.36	2.74	2.37	2.12	1.50
0.075	5.81	4.11	3.36	2.91	2.60	1.84
0.100	6.71	4.75	3.87	3.36	3.00	
0.2	9.49	6.71	5.48	4.75	4.24	
0.5	15.01	10.61	8.66	7.50	6.71	
1.5	25.99	18.38	15.01	13.00	11.62	
1.9	29.26	20.69	16.89	14.63		
2.5	33.56	23.73	19.37			
3.8	41.37	29.26				
5.0	47.46					
6.0	51.99					
7.5	58.12					



Laminar Boundary Layer Profiles
70
60
50
40
30
20
10
0
δ (mm)
0
2
4
6
8
x (m)
U = 1 m/s
U = 2 m/s
U = 3 m/s
U = 4 m/s
U = 5 m/s
U = 10 m/s

Problem 9.37 [Difficulty: 5]

***9.37** The Blasius exact solution involves solving a nonlinear equation, Eq. 9.11, with initial and boundary conditions given by Eq. 9.12. Set up an *Excel* workbook to obtain a numerical solution of this system. The workbook should consist of columns for η, f, f', and f''. The rows should consist of values of these, with a suitable step size for η (e.g., for 1000 rows the step size for η would be 0.01 to generate data through $\eta = 10$, to go a little beyond the data in Table 9.1). The values of f and f' for the first row are zero (from the initial conditions, Eq. 9.12); a guess value is needed for f'' (try 0.5). Subsequent row values for f, f', and f'' can be obtained from previous row values using the Euler method of Section 5.5 for approximating first derivatives (and Eq. 9.11). Finally, a solution can be found by using *Excel's Goal Seek* or *Solver* functions to vary the initial value of f'' until $f' = 1$ for large η (e.g., $\eta = 10$, boundary condition of Eq. 9.12). Plot the results. Note: Because the Euler method is relatively crude, the results will agree with Blasius' only to within about 1%.

Given: Blasius nonlinear equation

Find: Blasius solution using Excel

Solution:

The equation to be solved is

$$2\frac{d^3 f}{d\eta^3} + f\frac{d^2 f}{d\eta^2} = 0 \tag{9.11}$$

The boundary conditions are

$$f = 0 \quad \text{and} \quad \frac{df}{d\eta} = 0 \quad \text{at} \quad \eta = 0$$

$$f' = \frac{df}{d\eta} = 1 \quad \text{at} \quad \eta \to \infty \tag{9.12}$$

Recall that these somewhat abstract variables are related to physically meaningful variables:

$$\frac{u}{U} = f'$$

and

$$\eta = y\sqrt{\frac{U}{\nu x}} \propto \frac{y}{\delta}$$

Using Euler's numerical method

$$f_{n+1} \approx f_n + \Delta\eta f_n' \tag{1}$$

$$f_{n+1}' \approx f_n' + \Delta\eta f_n'' \tag{2}$$

$$f_{n+1}'' \approx f_n'' + \Delta\eta f_n'''$$

In these equations, the subscripts refer to the n^{th} discrete value of the variables, and $\Delta\eta = 10/N$ is the step size for η (N is the total number of steps).

But from Eq. 9.11

$$f''' = -\frac{1}{2} f f''$$

so the last of the three equations is

$$f''_{n+1} \approx f''_n + \Delta\eta\left(-\frac{1}{2} f_n f''_n\right) \tag{3}$$

Equations 1 through 3 form a complete set for computing f, f', f''. All we need is the starting condition for each. From Eqs. 9.12

$$f_0 = 0 \quad \text{and} \quad f'_0 = 0$$

We do NOT have a starting condition for f''! Instead we must choose (using *Solver*) f''_0 so that the last condition of Eqs. 9.12 is met:

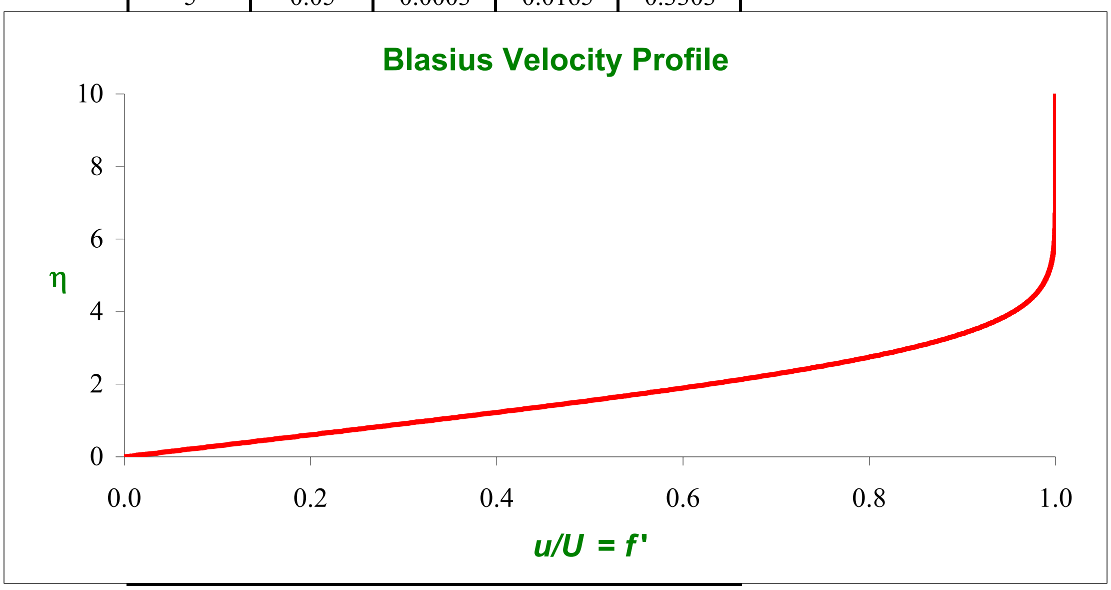
$$f'_N = 1$$

Computations (only the first few lines of 1000 are shown):

$\Delta\eta =$ 0.01

Make a guess for the first *f*"; use Solver to vary it until *f*'N = 1

Count	η	*f*	*f'*	*f''*
0	0.00	0.0000	0.0000	0.3303
1	0.01	0.0000	0.0033	0.3303
2	0.02	0.0000	0.0066	0.3303
3	0.03	0.0001	0.0099	0.3303
4	0.04	0.0002	0.0132	0.3303
5	0.05	0.0003	0.0165	0.3303



Problem 9.38

[Difficulty: 2]

*9.38 A thin flat plate, $L = 9$ in. long and $b = 3$ ft wide, is installed in a water tunnel as a splitter. The freestream speed is $U = 5$ ft/s, and the velocity profile in the boundary layer is approximated as parabolic. Plot δ, δ^*, and τ_w versus x/L for the plate.

Given: Parabolic solution for laminar boundary layer

Find: Plot of δ, δ^*, and τ_w versus x/L

Solution:

Given or available data: $\nu = 1.08 \times 10^{-5} \cdot \frac{ft^2}{s}$ (From Table A.8 at 68°F) $L = 9 \cdot in$ $U = 5 \cdot \frac{ft}{s}$

Basic equations:

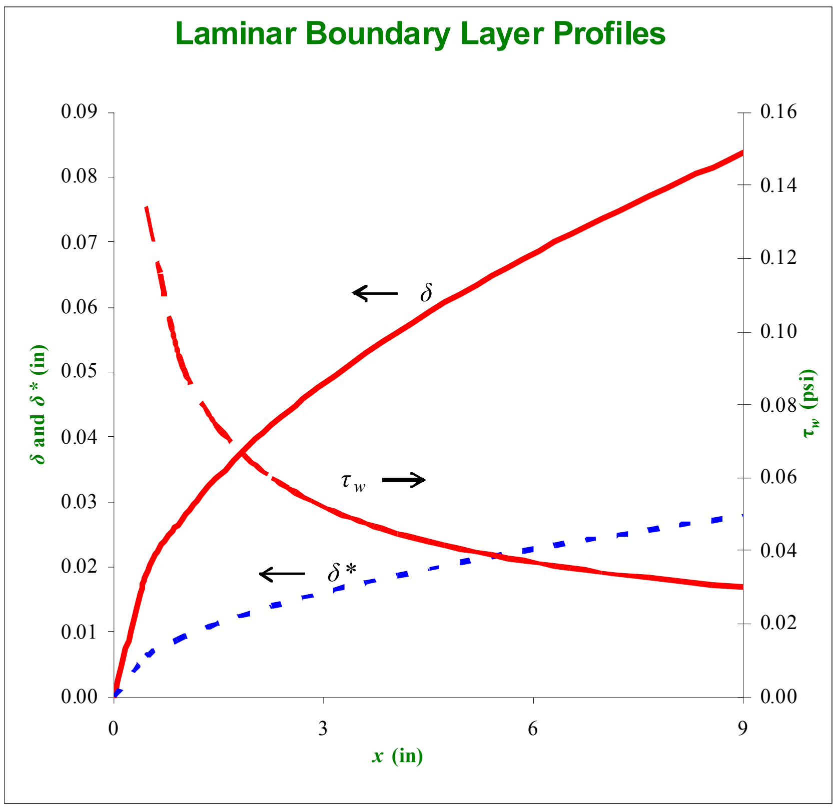
$$\frac{u}{U} = 2 \cdot \left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2 \qquad \frac{\delta}{x} = \frac{5.48}{\sqrt{Re_x}} \qquad c_f = \frac{\tau_w}{\frac{1}{2} \cdot \rho \cdot U^2} = \frac{0.730}{\sqrt{Re_x}}$$

Hence:

$$\delta^* = \int_0^\delta \left(1 - \frac{u}{U}\right) dy = \delta \int_0^1 \left(1 - \frac{u}{U}\right) d\left(\frac{y}{\delta}\right) = \delta \int_0^1 \left(1 - 2 + \eta^2\right) d\eta = \delta \left[\eta - \eta^2 + \frac{1}{3}\eta^3\right]_0^1 = \frac{\delta}{3}$$

The computed results are from Excel, shown below:

x (in)	Re_x	δ (in)	δ^* (in)	τ_w (psi)
0.00	0.00.E+00	0.000	0.000	
0.45	1.74.E+04	0.019	0.006	0.1344
0.90	3.47.E+04	0.026	0.009	0.0950
1.35	5.21.E+04	0.032	0.011	0.0776
1.80	6.94.E+04	0.037	0.012	0.0672
2.25	8.68.E+04	0.042	0.014	0.0601
2.70	1.04.E+05	0.046	0.015	0.0548
3.15	1.22.E+05	0.050	0.017	0.0508
3.60	1.39.E+05	0.053	0.018	0.0475
4.05	1.56.E+05	0.056	0.019	0.0448
4.50	1.74.E+05	0.059	0.020	0.0425
4.95	1.91.E+05	0.062	0.021	0.0405
5.40	2.08.E+05	0.065	0.022	0.0388
5.85	2.26.E+05	0.067	0.022	0.0373
6.30	2.43.E+05	0.070	0.023	0.0359
6.75	2.60.E+05	0.072	0.024	0.0347
7.20	2.78.E+05	0.075	0.025	0.0336
7.65	2.95.E+05	0.077	0.026	0.0326
8.10	3.13.E+05	0.079	0.026	0.0317
8.55	3.30.E+05	0.082	0.027	0.0308
9.00	3.47.E+05	0.084	**0.028**	**0.0300**



Problem 9.39

[Difficulty: 2]

9.39 Consider flow over the splitter plate of Problem 9.38. Show algebraically that the total drag force on one side of the splitter plate may be written $F_D = \rho U^2 \theta_L b$. Evaluate $\theta_L b$ and the total drag for the given conditions.

Given: Parabolic solution for laminar boundary layer

Find: Derivation of F_D; Evaluate F_D and θ_L

Solution:

Basic equations:

$$\frac{u}{U} = 2\cdot\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2 \qquad \frac{\delta}{x} = \frac{5.48}{\sqrt{Re_x}} \qquad \frac{\tau_w}{\rho} = \frac{d}{dx}(U^2\theta) + \delta^* U \frac{dU}{dx}$$

$$L = 9\cdot in \qquad b = 3\cdot ft \qquad U = 5\cdot\frac{ft}{s} \qquad \rho = 1.94\cdot\frac{slug}{ft^3}$$

Assumptions:

1) Flat plate so $\frac{\partial}{\partial x}p = 0$, and U = const
2) δ is a function of x only
3) Incompressible
4) Steady flow

The momentum integral equation then simplifies to $\frac{\tau_w}{\rho} = \frac{d}{dx}\left(U^2\cdot\theta\right)$ where $\theta = \int_0^\delta \frac{u}{U}\cdot\left(1 - \frac{u}{U}\right) dy$

For U = const $\tau_w = \rho\cdot U^2\cdot\frac{d\theta}{dx}$

The drag force is then $F_D = \int \tau_w\, dA = \int_0^L \tau_w\cdot b\, dx = \int_0^L \rho\cdot U^2\cdot\frac{d\theta}{dx}\cdot b\, dx = \rho\cdot U^2\cdot b\cdot\int_0^{\theta_L} 1\, d\theta$ $\qquad F_D = \rho\cdot U^2\cdot b\cdot\theta_L$

For the given profile $\frac{\theta}{\delta} = \int_0^1 \frac{u}{U}\cdot\left(1 - \frac{u}{U}\right) d\eta = \int_0^1 \left(2\cdot\eta - \eta^2\right)\cdot\left(1 - 2\cdot\eta + \eta^2\right) d\eta = \int_0^1 \left(2\cdot\eta - 5\cdot\eta^2 + 4\cdot\eta^3 - \eta^4\right) d\eta = \frac{2}{15}$

$$\theta = \frac{2}{15}\cdot\delta$$

From Table A.7 at 68°F $\nu = 1.08\times 10^{-5}\cdot\frac{ft^2}{s}$ $\qquad Re_L = \frac{U\cdot L}{\nu}$ $\qquad Re_L = 3.47\times 10^5$

$$\delta_L = L\cdot\frac{5.48}{\sqrt{Re_L}} \qquad \delta_L = 0.0837\cdot in$$

$$\theta_L = \frac{2}{15}\cdot\delta_L \qquad \theta_L = 0.01116\cdot in$$

$$F_D = \rho\cdot U^2\cdot b\cdot\theta_L \qquad F_D = 0.1353\cdot lbf$$

Problem 9.40

[Difficulty: 3]

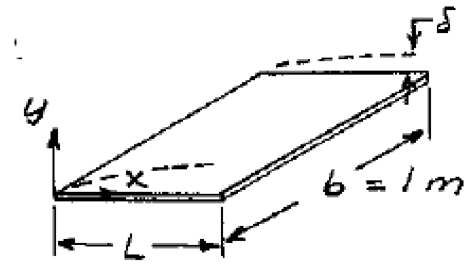
9.40 A thin flat plate is installed in a water tunnel as a splitter. The plate is 0.3 m long and 1 m wide. The freestream speed is 1.6 m/s. Laminar boundary layers form on both sides of the plate. The boundary-layer velocity profile is approximated as parabolic. Determine the total viscous drag force on the plate assuming that pressure drag is negligible.

Given: Thin flat plate installed in a water tunnel. Laminar BL's with parabolic profiles form on both sides of the plate.

$$L = 0.3\cdot m \quad b = 1\cdot m \quad U = 1.6\cdot\frac{m}{s} \quad \nu = 1\times 10^{-6}\cdot\frac{m^2}{s} \qquad \frac{u}{U} = 2\cdot\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$$

Find: Total viscous drag force acting on the plate.

Solution: We will determine the drag force from the shear stress at the wall



First we will check the Reynolds number of the flow: $Re_L = \frac{U\cdot L}{\nu} = 4.8\times 10^5$

Therefore the flow is laminar throughout.

The viscous drag for the two sides of the plate is: $F_D = 2\cdot\int_0^L \tau_w\cdot b\,dx$ The wall shear stress τ_w is:

$\tau_w = \mu\cdot\left(\frac{\partial}{\partial y}u\right)$ at y = 0, which for the parabolic profile yields: $\tau_w = \mu\cdot U\cdot\left(\frac{2}{\delta} - \frac{2\times 0}{\delta^2}\right) = \frac{2\cdot\mu\cdot U}{\delta}$ The BL thickness δ is:

$\delta = 5.48\cdot\sqrt{\frac{\nu}{U}}\cdot x^{\frac{1}{2}}$ Therefore: $F_D = 2\cdot b\cdot\int_0^L \frac{2\cdot\mu\cdot U}{5.48\cdot\sqrt{\frac{\nu}{U}}\cdot x^{\frac{1}{2}}}\,dx = \frac{4}{5.48}\cdot b\cdot\mu\cdot U\cdot\sqrt{\frac{U}{\nu}}\cdot\int_0^L x^{-\frac{1}{2}}\,dx$

Evaluating this integral: $F_D = \frac{8\cdot b\cdot\mu\cdot U}{5.48}\cdot\sqrt{\frac{U\cdot L}{\nu}}$ $\qquad F_D = 1.617\,N$

Problem 9.41

[Difficulty: 2]

9.41 In Problems 9.18 and 9.19 the drag on the upper surface of a flat plate, with flow (fluid density ρ = 1.5 slug/ft^3) at freestream speed U = 10 ft/s, was determined from momentum flux calculations. The drag was determined for the plate with its long edge (10 ft) and its short edge (3 ft) parallel to the flow. If the fluid viscosity $\mu = 4 \times 10^{-4}$ lbf·s/ft^2, compute the drag using boundary-layer calculations.

Given: Data on fluid and plate geometry

Find: Drag at both orientations using boundary layer equation

Solution:

The given data is $\rho = 1.5 \cdot \frac{\text{slug}}{\text{ft}^3}$ $\quad \mu = 0.0004 \cdot \frac{\text{lbf}\cdot\text{s}}{\text{ft}^2}$ $\quad U = 10 \cdot \frac{\text{ft}}{\text{s}}$ $\quad L = 10 \cdot \text{ft}$ $\quad b = 3 \cdot \text{ft}$

First determine the nature of the boundary layer $\quad Re_L = \frac{\rho \cdot U \cdot L}{\mu} \quad Re_L = 3.75 \times 10^5$

The maximum Reynolds number is less than the critical value of 5 x 10^5

Hence:

Governing equations: $\quad c_f = \frac{\tau_w}{\frac{1}{2} \cdot \rho \cdot U^2}$ (9.22) $\quad c_f = \frac{0.730}{\sqrt{Re_x}}$ (9.23)

The drag (one side) is $\quad F_D = \int_0^L \tau_w \cdot b \, dx$

Using Eqs. 9.22 and 9.23 $\quad F_D = \frac{1}{2} \cdot \rho \cdot U^2 \cdot b \cdot \int_0^L \frac{0.73}{\sqrt{\frac{\rho \cdot U \cdot x}{\mu}}} \, dx$

$F_D = 0.73 \cdot b \cdot \sqrt{\mu \cdot L \cdot \rho \cdot U^3} \quad F_D = 5.36 \cdot \text{lbf}$ (Compare to 6.25 lbf for Problem 9.18)

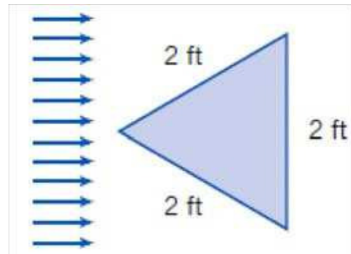
Repeating for $\quad L = 3 \cdot \text{ft} \quad b = 10 \cdot \text{ft}$

$F_D = 0.73 \cdot b \cdot \sqrt{\mu \cdot L \cdot \rho \cdot U^3} \quad F_D = 9.79 \cdot \text{lbf}$ (Compare to 12.5 lbf for Problem 9.19)

Problem 9.42

[Difficulty: 3]

9.42 Assume laminar boundary-layer flow to estimate the drag on the plate shown when it is placed parallel to a 15 ft/s air flow. The air is 70°F and 1 atm.



Given: Triangular plate

Find: Drag

Solution:

Basic equations:

$$c_f = \frac{\tau_w}{\frac{1}{2}\cdot\rho\cdot U^2} \qquad c_f = \frac{0.730}{\sqrt{Re_x}}$$

$$L = 2\cdot ft\cdot\frac{\sqrt{3}}{2} \qquad L = 1.732\cdot ft \qquad W = 2\cdot ft \qquad U = 15\cdot\frac{ft}{s}$$

Assumptions: (1) Parabolic boundary layer profile
(2) Boundary layer thickness is based on distance from leading edge (the "point" of the triangle).

From Table A.9 at 70°F $\quad \nu = 1.63\times 10^{-4}\cdot\frac{ft^2}{s} \qquad \rho = 0.00233\cdot\frac{slug}{ft^3}$

First determine the nature of the boundary layer $\quad Re_L = \frac{U\cdot L}{\nu} \qquad Re_L = 2\times 10^5 \quad$ so definitely laminar

The drag (one side) is $\quad F_D = \int \tau_w\, dA \qquad F_D = \int_0^L \tau_w\cdot w(x)\, dx \qquad w(x) = W\cdot\frac{x}{L}$

We also have $\quad \tau_w = c_f\cdot\frac{1}{2}\cdot\rho\cdot U^2 = \frac{1}{2}\cdot\rho\cdot U^2\cdot\frac{0.730}{\sqrt{Re_x}}$

Hence

$$F_D = \frac{1}{2}\cdot\rho\cdot U^2\cdot\frac{W}{L}\cdot\int_0^L \frac{0.730\cdot x}{\sqrt{\frac{U\cdot x}{\nu}}}\, dx = \frac{0.730}{2}\cdot\rho\cdot U^{\frac{3}{2}}\cdot\frac{W}{L}\cdot\sqrt{\nu}\cdot\int_0^L x^{\frac{1}{2}}\, dx$$

The integral is $\quad \int_0^L x^{\frac{1}{2}}\, dx = \frac{2}{3}\cdot L^{\frac{3}{2}} \quad$ so $\quad F_D = 0.243\cdot\rho\cdot W\cdot\sqrt{\nu\cdot L\cdot U^3} \qquad F_D = 1.11\times 10^{-3}\cdot lbf$

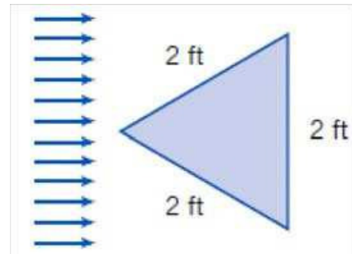
Note: For two-sided solution $\qquad 2\cdot F_D = 2.21\times 10^{-3}\cdot lbf$

Problem 9.43

[Difficulty: 3]

9.43 Assume laminar boundary-layer flow to estimate the drag on the plate shown when it is placed parallel to a 15 ft/s air flow, except that the base rather than the tip faces the flow. Would you expect this to be larger than, the same as, or lower than the drag for Problem 9.42?

Plate is reversed from this!



Given: Triangular plate

Find: Drag

Solution:

Basic equations: $c_f = \dfrac{\tau_w}{\frac{1}{2}\cdot\rho\cdot U^2}$ $\qquad c_f = \dfrac{0.730}{\sqrt{Re_x}}$

$L = 2\cdot ft\cdot\dfrac{\sqrt{3}}{2}$ $\qquad L = 1.732\cdot ft$ $\qquad W = 2\cdot ft$ $\qquad U = 15\cdot\dfrac{ft}{s}$

From Table A.9 at 70ºF $\quad \nu = 1.63\times 10^{-4}\cdot\dfrac{ft^2}{s}$ $\qquad \rho = 0.00233\cdot\dfrac{slug}{ft^3}$

First determine the nature of the boundary layer $\quad Re_L = \dfrac{U\cdot L}{\nu}$ $\qquad Re_L = 2\times 10^5$ $\qquad$ so definitely laminar

The drag (one side) is $\quad F_D = \displaystyle\int \tau_w\, dA$ $\qquad F_D = \displaystyle\int_0^L \tau_w\cdot w(x)\, dx$ $\qquad w(x) = W\cdot\left(1 - \dfrac{x}{L}\right)$

We also have $\quad \tau_w = c_f\cdot\dfrac{1}{2}\cdot\rho\cdot U^2 = \dfrac{1}{2}\cdot\rho\cdot U^2\cdot\dfrac{0.730}{\sqrt{Re_x}}$

Hence

$$F_D = \frac{1}{2}\cdot\rho\cdot U^2\cdot W\cdot\int_0^L \frac{0.730\cdot\left(1 - \frac{x}{L}\right)}{\sqrt{\frac{U\cdot x}{\nu}}}\, dx = \frac{0.730}{2}\cdot\rho\cdot U^{\frac{3}{2}}\cdot W\cdot\sqrt{\nu}\cdot\int_0^L \left(x^{-\frac{1}{2}} - \frac{x^{\frac{1}{2}}}{L}\right) dx$$

The integral is

$$\int_0^L \left(x^{-\frac{1}{2}} - \frac{x^{\frac{1}{2}}}{L}\right) dx = 2\cdot L^{\frac{1}{2}} - \frac{2}{3}\cdot\frac{L^{\frac{3}{2}}}{L} = \frac{4}{3}\cdot\sqrt{L}$$

$F_D = 0.487\cdot\rho\cdot W\cdot\sqrt{\nu\cdot L\cdot U^3}$ $\qquad F_D = 2.22\times 10^{-3}\cdot lbf$

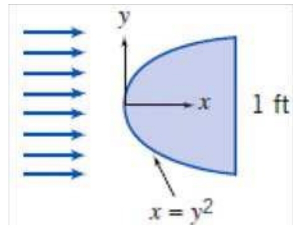
Note: For two-sided solution $\qquad 2\cdot F_D = 4.43\times 10^{-3}\cdot lbf$

The drag is much higher (twice as much) compared to Problem 9.42. This is because τ_w is largest near the leading edge and falls off rapidly; in this problem the widest area is also at the front

Problem 9.44

[Difficulty: 3]

9.44 Assume laminar boundary-layer flow to estimate the drag on the plate shown when it is placed parallel to a 15 ft/s air flow. The air is at 70°F and 1 atm. (Note that the shape is given by $x = y^2$, where x and y are in feet.)



Given: Parabolic plate

Find: Drag

Solution:

Basic equations: $c_f = \dfrac{\tau_w}{\frac{1}{2}\cdot\rho\cdot U^2}$ $\qquad c_f = \dfrac{0.730}{\sqrt{Re_x}}$

$W = 1\cdot ft \qquad L = \dfrac{\left(\frac{W}{2}\right)^2}{1\cdot ft} \qquad L = 0.25\cdot ft \qquad U = 15\cdot\dfrac{ft}{s}$

Note: "y" is the equation of the upper and lower surfaces, so y = W/2 at x = L

From Table A.9 at 70ºF $\quad \nu = 1.63\times 10^{-4}\cdot\dfrac{ft^2}{s} \qquad \rho = 0.00233\cdot\dfrac{slug}{ft^3}$

First determine the nature of the boundary layer $\quad Re_L = \dfrac{U\cdot L}{\nu} \qquad Re_L = 2.3\times 10^4 \qquad$ so just laminar

The drag (one side) is $\quad F_D = \displaystyle\int \tau_w\, dA \qquad F_D = \int_0^L \tau_w\cdot w(x)\, dx \qquad w(x) = W\cdot\sqrt{\frac{x}{L}}$

We also have $\quad \tau_w = c_f\cdot\dfrac{1}{2}\cdot\rho\cdot U^2 = \dfrac{1}{2}\cdot\rho\cdot U^2\cdot\dfrac{0.730}{\sqrt{Re_x}}$

Hence $\quad F_D = \dfrac{1}{2}\cdot\rho\cdot U^2\cdot W\cdot\displaystyle\int_0^L \frac{0.730\cdot\sqrt{\frac{x}{L}}}{\sqrt{\frac{U\cdot x}{\nu}}}\, dx = \frac{0.730}{2}\cdot\rho\cdot U^{\frac{3}{2}}\cdot W\cdot\sqrt{\frac{\nu}{L}}\cdot\int_0^L 1\, dx$

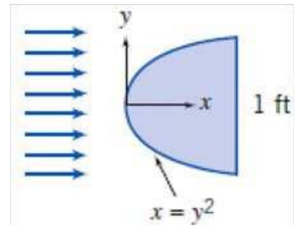
$F_D = 0.365\cdot\rho\cdot W\cdot\sqrt{\nu\cdot L\cdot U^3} \qquad F_D = 3.15\times 10^{-4}\cdot lbf$

Note: For two-sided solution $\quad 2\cdot F_D = 6.31\times 10^{-4}\cdot lbf$

Problem 9.45

[Difficulty: 4]

9.45 Assume laminar boundary-layer flow to estimate the drag on the plate shown when it is placed parallel to a 15 ft/s flow, except that the base rather than the tip faces the flow. Would you expect this to be large than, the same as, or lower than the drag for Problem 9.44?



Note: Plate is now reversed!

Given: Parabolic plate

Find: Drag

Solution:

Basic equations:

$$c_f = \frac{\tau_w}{\frac{1}{2}\cdot\rho\cdot U^2} \qquad c_f = \frac{0.730}{\sqrt{Re_x}}$$

$$W = 1\cdot ft \qquad L = \frac{\left(\frac{W}{2}\right)^2}{1\cdot ft} \qquad L = 0.25\cdot ft \qquad U = 15\cdot\frac{ft}{s}$$

Note: "y" is the equation of the upper and lower surfaces, so y = W/2 at x = 0

From Table A.10 at 70ºF $\quad \nu = 1.63\times 10^{-4}\cdot\frac{ft^2}{s} \qquad \rho = 0.00234\cdot\frac{slug}{ft^3}$

First determine the nature of the boundary layer $\quad Re_L = \frac{U\cdot L}{\nu} \qquad Re_L = 2.3\times 10^4 \qquad$ so just laminar

The drag (one side) is $\quad F_D = \int \tau_w\, dA \qquad F_D = \int_0^L \tau_w\cdot w(x)\, dx \qquad w(x) = W\cdot\sqrt{1-\frac{x}{L}}$

We also have $\quad \tau_w = c_f\cdot\frac{1}{2}\cdot\rho\cdot U^2 = \frac{1}{2}\cdot\rho\cdot U^2\cdot\frac{0.730}{\sqrt{Re_x}}$

Hence

$$F_D = \frac{1}{2}\cdot\rho\cdot U^2\cdot W\cdot\int_0^L \frac{0.730\cdot\sqrt{1-\frac{x}{L}}}{\sqrt{\frac{U\cdot x}{\nu}}}\,dx = \frac{0.730}{2}\cdot\rho\cdot U^{\frac{3}{2}}\cdot W\cdot\sqrt{\nu}\cdot\int_0^L\sqrt{\frac{1}{x}-\frac{1}{L}}\,dx$$

The tricky integral is (this might be easier to do numerically!)

$$\int\sqrt{\frac{1}{x}-\frac{1}{L}}\,dx = \sqrt{x-\frac{x^2}{L}} - \frac{i}{2}\cdot\sqrt{L}\cdot\ln\left(\frac{\sqrt{-L-x}-\sqrt{x}}{\sqrt{-L-x}+\sqrt{x}}\right) \quad \text{so} \quad \int_0^L\sqrt{\frac{1}{x}-\frac{1}{L}}\,dx = 0.434\cdot\sqrt{m}$$

$$F_D = \frac{0.730}{2}\cdot\rho\cdot U^{\frac{3}{2}}\cdot W\cdot\sqrt{\nu}\cdot\int_0^L\sqrt{\frac{1}{x}-\frac{1}{L}}\,dx \qquad F_D = 4.98\times 10^{-4}\cdot lbf$$

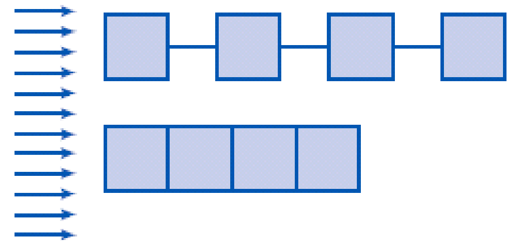
Note: For two-sided solution $\quad 2\cdot F_D = 9.95\times 10^{-4}\cdot lbf$

The drag is much higher compared to Problem 9.44. This is because τ_w is largest near the leading edge and falls off rapidly; in this problem the widest area is also at the front

Problem 9.46

[Difficulty: 3]

9.46 Assume laminar boundary-layer flow to estimate the drag on four square plates (each 3 in. × 3 in.) placed parallel to a 3 ft/s water flow, for the two configurations shown. Before calculating, which configuration do you expect to experience the lower drag? Assume that the plates attached with string are far enough apart for wake effects to be negligible and that the water is at 70°F.



Given: Pattern of flat plates

Find: Drag on separate and composite plates

Solution:

Basic equations:

$$c_f = \frac{\tau_w}{\frac{1}{2}\cdot\rho\cdot U^2} \qquad c_f = \frac{0.730}{\sqrt{Re_x}}$$

Assumption: Parabolic boundary layer profile

For separate plates $L = 3\cdot in$ $\quad W = 3\cdot in$ $\quad U = 3\cdot\frac{ft}{s}$ $\quad$ From Table A.7 at 70ºF $\nu = 1.06\times 10^{-5}\cdot\frac{ft^2}{s}$ $\quad \rho = 1.93\cdot\frac{slug}{ft^3}$

First determine the nature of the boundary layer $\quad Re_L = \frac{U\cdot L}{\nu} \quad Re_L = 7.08\times 10^4$ $\quad$ so definitely laminar

The drag (one side) is $\quad F_D = \int \tau_w\,dA \qquad F_D = \int_0^L \tau_w\cdot W\,dx$

We also have $\quad \tau_w = c_f\cdot\frac{1}{2}\cdot\rho\cdot U^2 = \frac{1}{2}\cdot\rho\cdot U^2\cdot\frac{0.730}{\sqrt{Re_x}}$

Hence

$$F_D = \frac{1}{2}\cdot\rho\cdot U^2\cdot W\cdot\int_0^L \frac{0.730}{\sqrt{\frac{U\cdot x}{\nu}}}\,dx = \frac{0.730}{2}\cdot\rho\cdot U^{\frac{3}{2}}\cdot W\cdot\sqrt{\nu}\cdot\int_0^L x^{-\frac{1}{2}}\,dx$$

The integral is $\quad \int_0^L x^{-\frac{1}{2}}\,dx = 2\cdot L^{\frac{1}{2}}$ $\quad$ so $\quad F_D = 0.730\cdot\rho\cdot W\cdot\sqrt{\nu\cdot L\cdot U^3}$ $\qquad F_D = 0.0030\cdot lbf$

This is the drag on one plate. The total drag is then $\quad F_{Total} = 4\cdot F_D \qquad F_{Total} = 0.0119\cdot lbf$

For both sides: $\quad 2\cdot F_{Total} = 0.0238\cdot lbf$

For the composite plate $\quad L = 4\times 3\cdot in \quad L = 1.00\cdot ft \quad Re_L = \frac{U\cdot L}{\nu} = 2.83\times 10^5$ $\quad$ so still laminar

$$F_{Composite} = 0.730\cdot\rho\cdot W\cdot\sqrt{\nu\cdot L\cdot U^3} \qquad F_{Composite} = 0.0060\cdot lbf$$

For both sides: $\quad 2\cdot F_{Composite} = 0.0119\cdot lbf$

The drag is much lower on the composite compared to the separate plates. This is because τ_w is largest near the leading edges and falls off rapidly; in this problem the separate plates experience leading edges four times!

Problem 9.47

[Difficulty: 3]

9.47 The velocity profile in a laminar boundary-layer flow at zero pressure gradient is approximated by the linear expression given in Problem 9.10. Use the momentum integral equation with this profile to obtain expressions for δ/x and C_f.

Given: Laminar boundary layer flow with linear velocity profile: $\frac{u}{U} = \frac{y}{\delta} = \eta$

Find: Expressions for δ/x and C_f using the momentum integral equation

Solution: We will apply the momentum integral equation

Governing Equations:

$$\frac{\tau_w}{\rho} = \frac{d}{dx}\left(U^2\cdot\theta\right) + \delta_{disp}\cdot U\cdot\left(\frac{d}{dx}U\right) \qquad \text{(Momentum integral equation)}$$

$$C_f = \frac{\tau_w}{\frac{1}{2}\cdot\rho\cdot U^2} \qquad \text{(Skin friction coefficient)}$$

Assumptions:
(1) Zero pressure gradient, so U is constant and dp/dx = 0
(2) δ is a function of x only, and $\delta = 0$ at x = 0
(3) Incompressible flow

Applying the assumptions to the momentum integral equation yields: $\tau_w = \rho\cdot U^2\cdot\left(\frac{d}{dx}\theta\right) = \rho\cdot U^2\cdot\left[\frac{d}{dx}\left[\delta\cdot\int_0^1 \frac{u}{U}\cdot\left(1 - \frac{u}{U}\right)d\eta\right]\right]$

Substituting for the velocity profile: $\tau_w = \rho\cdot U^2\cdot\left[\frac{d}{dx}\left[\delta\cdot\int_0^1 \left(\eta - \eta^2\right)d\eta\right]\right] = \rho\cdot U^2\cdot\frac{1}{6}\cdot\left(\frac{d}{dx}\delta\right)$

Now the wall shear stress is also: $\tau_w = \mu\cdot\left(\frac{\partial}{\partial y}u\right)$ at y = 0 Substituting the velocity profile: $\tau_w = \frac{\mu\cdot U}{\delta}$

Setting both expressions for the wall shear stress equal: $\rho\cdot U^2\cdot\frac{1}{6}\cdot\left(\frac{d}{dx}\delta\right) = \frac{\mu\cdot U}{\delta}$ Separating variables: $\delta\cdot d\delta = \frac{6\cdot\mu}{\rho\cdot U}\cdot dx$

Integrating this expression: $\frac{\delta^2}{2} = \frac{6\cdot\mu}{\rho\cdot U}\cdot x + C$ However, we know that C = 0 since $\delta = 0$ when x = 0. Therefore: $\frac{\delta^2}{2} = \frac{6\cdot\mu}{\rho\cdot U}\cdot x$

Solving for the boundary layer thickness: $\delta = \sqrt{\frac{12\cdot\mu}{\rho\cdot U}\cdot x}$ or $\frac{\delta}{x} = \sqrt{\frac{12\cdot\mu}{\rho\cdot U\cdot x}}$

$$\frac{\delta}{x} = \frac{3.46}{\sqrt{Re_x}}$$

From the definition for skin friction coefficient: $C_f = \frac{\tau_w}{\frac{1}{2}\cdot\rho\cdot U^2} = \frac{\mu\cdot U}{\delta}\cdot\frac{2}{\rho\cdot U^2} = \frac{2\cdot\mu}{\rho\cdot U\cdot\delta} = 2\cdot\frac{\mu}{\rho\cdot U\cdot x}\cdot\frac{x}{\delta} = \frac{2}{Re_x}\cdot\frac{\sqrt{Re_x}}{3.46}$

Upon simplification: $$C_f = \frac{0.577}{\sqrt{Re_x}}$$

Problem 9.48

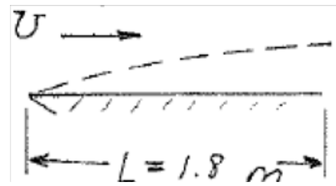
[Difficulty: 2]

9.48 A horizontal surface, with length $L = 1.8$ m and width $b = 0.9$ m, is immersed in a stream of standard air flowing at $U = 3.2$ m/s. Assume a laminar boundary layer forms and approximate the velocity profile as sinusoidal. Plot δ, δ^*, and τ_w versus x/L for the plate.

Given: Horizontal surface immersed in a stream of standard air. Laminar BL with sinusoidal profile forms.

$$L = 1.8\cdot\text{m} \quad b = 0.9\cdot\text{m} \quad U = 3.2\cdot\frac{\text{m}}{\text{s}} \quad \nu = 1.46\times 10^{-5}\cdot\frac{\text{m}^2}{\text{s}} \quad \frac{u}{U} = \sin\left(\frac{\pi}{2}\cdot\frac{y}{\delta}\right)$$

Find: Plot δ, δ^*, and τ_w versus x/L for the plate



Solution: We will determine the drag force from the shear stress at the wall

Governing Equations:

$$\tau_w = \rho\cdot U^2\cdot\left(\frac{d}{dx}\theta\right) = \mu\cdot\left(\frac{\partial}{\partial y}u\right) \text{ at } y = 0 \qquad \text{(Wall shear stress)}$$

$$\frac{\delta_{disp}}{\delta} = \int_0^1 \left(1 - \frac{u}{U}\right) d\eta \qquad \text{(Displacement thickness)}$$

$$\frac{\theta}{\delta} = \int_0^1 \frac{u}{U}\cdot\left(1 - \frac{u}{U}\right) d\eta \qquad \text{(Momentum thickness)}$$

For the sinusoidal velocity profile: $\frac{\theta}{\delta} = \int_0^1 \sin\left(\frac{\pi}{2}\cdot\eta\right)\cdot\left(1 - \sin\left(\frac{\pi}{2}\cdot\eta\right)\right) d\eta = \int_0^1 \left[\sin\left(\frac{\pi}{2}\cdot\eta\right) - \left(\sin\left(\frac{\pi}{2}\cdot\eta\right)\right)^2\right] d\eta$

Evaluating this integral: $\frac{\theta}{\delta} = \frac{4-\pi}{2\cdot\pi} = 0.1366$ Therefore it follows that $\frac{d}{dx}\theta = \frac{d}{d\delta}\theta\cdot\left(\frac{\partial}{\partial x}\delta\right) = \frac{4-\pi}{2\cdot\pi}\cdot\left(\frac{\partial}{\partial x}\delta\right)$

To determine the wall shear stress: $\tau_w = \mu\cdot U\cdot\cos\left(\frac{\pi}{2}\cdot\frac{0}{\delta}\right)\cdot\frac{\pi}{2\cdot\delta} = \frac{\pi\cdot\mu\cdot U}{2\cdot\delta} = \rho\cdot U^2\cdot\frac{4-\pi}{2\cdot\pi}\cdot\left(\frac{\partial}{\partial x}\delta\right)$

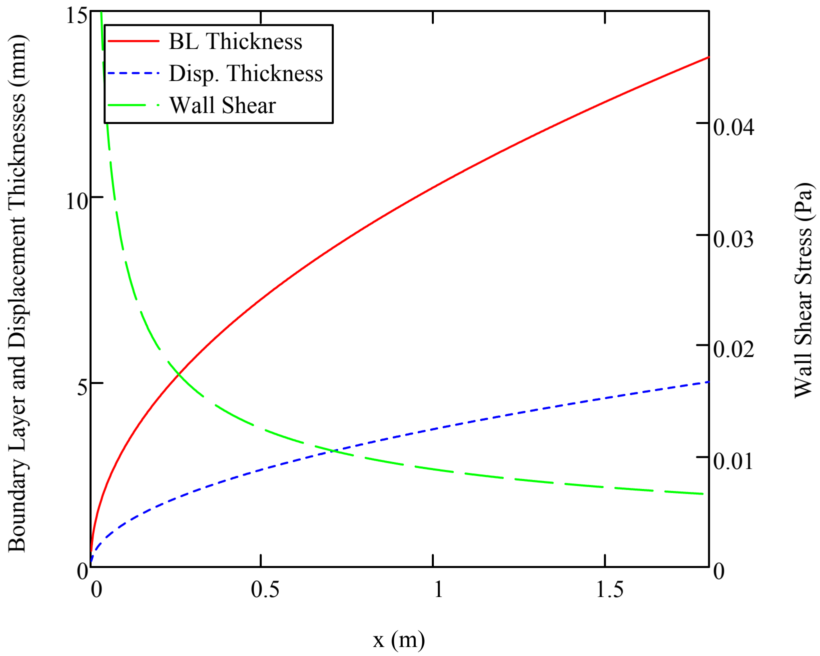
Separating variables yields: $\frac{\pi\cdot\mu\cdot U}{2\cdot\rho\cdot U^2}\cdot\frac{2\cdot\pi}{4-\pi}\cdot dx = \delta\cdot d\delta$ or $\delta\cdot d\delta = \frac{\pi^2}{4-\pi}\cdot\frac{\mu}{\rho\cdot U}\cdot dx$ Integrating yields: $\frac{\delta^2}{2} = \frac{\pi^2}{4-\pi}\cdot\frac{\mu}{\rho\cdot U}\cdot x$

Solving this expression for δ/x: $\frac{\delta}{x} = \sqrt{\frac{\pi^2}{4-\pi}}\cdot\sqrt{\frac{\mu}{\rho\cdot U\cdot x}}$ $\qquad \frac{\delta}{x} = \frac{4.80}{\sqrt{Re_x}}$

Also, $\frac{\delta_{disp}}{\delta} = \int_0^1 \left(1 - \sin\left(\frac{\pi}{2}\cdot\eta\right)\right) d\eta$ Evaluating this integral: $\frac{\delta_{disp}}{\delta} = 1 - \frac{2}{\pi} = 0.363$ $\qquad \frac{\delta_{disp}}{\delta} = 0.363$

The Reynolds number is related to x through: $Re_x = 2.19\times 10^5\cdot x$ where x is measured in meters.

Plots of δ, δ_{disp} and τ_w as functions of x are shown on the next page.


BL Thickness
Disp. Thickness
Wall Shear
Boundary Layer and Displacement Thicknesses (mm)
Wall Shear Stress (Pa)
x (m)
15
10
5
0
0.04
0.03
0.02
0.01
0
0
0.5
1
1.5

Problem 9.49

[Difficulty: 3]

9.49 Water at 10°C flows over a flat plate at a speed of 0.8 m/s. The plate is 0.35 m long and 1 m wide. The boundary layer on each surface of the plate is laminar. Assume that the velocity profile may be approximated as linear. Determine the drag force on the plate.

Given: Water flow over flat plate

Find: Drag on plate for linear boundary layer

Solution:

Basic equations: $F_D = 2\cdot\int \tau_w\, dA$ $\quad \tau_w = \mu\cdot\frac{du}{dy}$ at y = 0, and also $\tau_w = \rho\cdot U^2\cdot\frac{d\delta}{dx}\cdot\int_0^1 \frac{u}{U}\cdot\left(1-\frac{u}{U}\right) d\eta$

$L = 0.35\cdot m \qquad W = 1\cdot m \qquad U = 0.8\cdot\frac{m}{s}$

From Table A.8 at 10°C $\quad \nu = 1.30\times 10^{-6}\cdot\frac{m^2}{s} \qquad \rho = 1000\cdot\frac{kg}{m^3}$

First determine the nature of the boundary layer $\quad Re_L = \frac{U\cdot L}{\nu} \qquad Re_L = 2.15\times 10^5$ so laminar

The velocity profile is $\quad u = U\cdot\frac{y}{\delta} = U\cdot\eta$

Hence $\quad \tau_w = \mu\cdot\frac{du}{dy} = \mu\cdot\frac{U}{\delta} \qquad (1) \qquad$ but we need $\delta(x)$

We also have $\quad \tau_w = \rho\cdot U^2\cdot\frac{d\delta}{dx}\cdot\int_0^1 \frac{u}{U}\cdot\left(1-\frac{u}{U}\right) d\eta = \rho\cdot U^2\cdot\frac{d\delta}{dx}\cdot\int_0^1 \eta\cdot(1-\eta)\, d\eta$

The integral is $\quad \int_0^1 \left(\eta-\eta^2\right) dx = \frac{1}{6} \qquad$ so $\qquad \tau_w = \rho\cdot U^2\cdot\frac{d\delta}{dx} = \frac{1}{6}\cdot\rho\cdot U^2\cdot\frac{d\delta}{dx} \qquad (2)$

Comparing Eqs 1 and 2 $\quad \tau_w = \mu\cdot\frac{U}{\delta} = \frac{1}{6}\cdot\rho\cdot U^2\cdot\frac{d\delta}{dx}$

Separating variables $\quad \delta\cdot d\delta = \frac{6\cdot\mu}{\rho\cdot U}\cdot dx \qquad$ or $\qquad \frac{\delta^2}{2} = \frac{6\cdot\mu}{\rho\cdot U}\cdot x + c \qquad$ but $\delta(0) = 0$ so $c = 0$

Hence $\quad \delta = \sqrt{\frac{12\cdot\mu}{\rho\cdot U}\cdot x} \qquad$ or $\qquad \frac{\delta}{x} = \sqrt{\frac{12}{Re_x}} = \frac{3.46}{Re_x}$

Then $\quad F_D = 2\cdot\int \tau_w\, dA = 2\cdot W\cdot\int_0^L \mu\cdot\frac{U}{\delta}\, dx = 2\cdot W\cdot\int_0^L \mu\cdot U\cdot\sqrt{\frac{\rho\cdot U}{12\cdot\mu}}\cdot x^{-\frac{1}{2}}\, dx = \frac{\mu\cdot W\cdot U}{\sqrt{3}}\cdot\sqrt{\frac{U}{\nu}}\cdot\int_0^L x^{-\frac{1}{2}}\, dx$

The integral is $\quad \int_0^L x^{-\frac{1}{2}}\, dx = 2\cdot\sqrt{L} \qquad$ so $\qquad F_D = \frac{2\cdot\mu\cdot W\cdot U}{\sqrt{3}}\cdot\sqrt{\frac{U\cdot L}{\nu}}$

$F_D = \frac{2}{\sqrt{3}}\cdot\rho\cdot W\cdot\sqrt{\nu\cdot L\cdot U^3} \qquad F_D = 0.557N$

Problem 9.50

[Difficulty: 2]

9.50 A horizontal surface, with length $L = 0.8$ m and width $b = 1.9$ m, is immersed in a stream of standard air flowing at $U = 5.3$ m/s. Assume a laminar boundary layer forms and approximate the velocity profile as linear. Plot δ, δ^*, and τ_w versus x/L for the plate.

Given: Horizontal surface immersed in a stream of standard air. Laminar BL with linear profile forms.

$$L = 0.8 \cdot m \quad b = 1.9 \cdot m \quad U = 5.3 \cdot \frac{m}{s} \quad \nu = 1.46 \times 10^{-5} \cdot \frac{m^2}{s} \quad \frac{u}{U} = \frac{y}{\delta}$$

Find: Plot δ, δ^*, and τ_w versus x/L for the plate

Solution: We will determine the drag force from the shear stress at the wall

Governing Equations:

$$\tau_w = \rho \cdot U^2 \cdot \left(\frac{d}{dx}\theta\right) = \mu \cdot \left(\frac{\partial}{\partial y}u\right) \text{ at } y = 0 \qquad \text{(Wall shear stress)}$$

$$\frac{\delta_{disp}}{\delta} = \int_0^1 \left(1 - \frac{u}{U}\right) d\eta \qquad \text{(Displacement thickness)}$$

$$\frac{\theta}{\delta} = \int_0^1 \frac{u}{U} \cdot \left(1 - \frac{u}{U}\right) d\eta \qquad \text{(Momentum thickness)}$$

For the linear velocity profile: $\frac{\theta}{\delta} = \int_0^1 \eta \cdot (1 - \eta)\, d\eta = \int_0^1 \left(\eta - \eta^2\right) d\eta$ Evaluating this integral: $\frac{\theta}{\delta} = \frac{1}{6} = 0.1667$

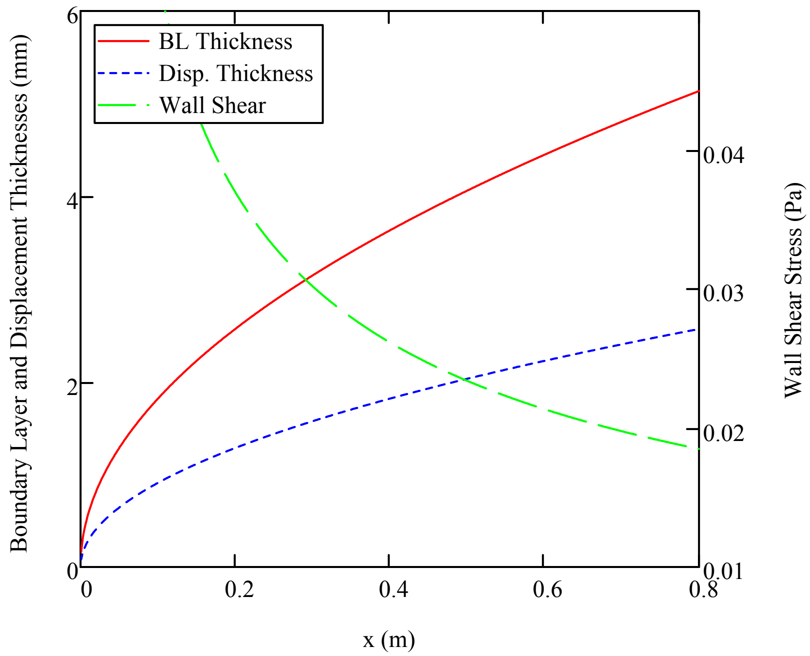
Therefore it follows that $\frac{d}{dx}\theta = \frac{d}{d\delta}\theta \cdot \left(\frac{\partial}{\partial x}\delta\right) = \frac{1}{6} \cdot \left(\frac{\partial}{\partial x}\delta\right)$ To determine the wall shear stress: $\tau_w = \frac{\mu \cdot U}{\delta} = \frac{\rho \cdot U^2}{6} \cdot \left(\frac{\partial}{\partial x}\delta\right)$

Separating variables yields: $\frac{6 \cdot \mu}{\rho \cdot U} \cdot dx = \delta \cdot d\delta$ Integrating yields: $\frac{\delta^2}{2} = \frac{6 \cdot \mu}{\rho \cdot U} \cdot x$ Solving this expression for δ/x: $\frac{\delta}{x} = \frac{3.46}{\sqrt{Re_x}}$

Also, $\frac{\delta_{disp}}{\delta} = \int_0^1 (1 - \eta)\, d\eta$ Evaluating this integral: $\frac{\delta_{disp}}{\delta} = \frac{1}{2}$ $\qquad \frac{\delta_{disp}}{\delta} = \frac{1}{2}$

The Reynolds number is related to x through: $Re_x = 3.63 \times 10^5 \cdot x$ where x is measured in meters.

Plots of δ, δ_{disp} and τ_w as functions of x are shown on the next page.



BL Thickness
Disp. Thickness
Wall Shear
Boundary Layer and Displacement Thicknesses (mm)
Wall Shear Stress (Pa)
x (m)
0
2
4
6
0.01
0.02
0.03
0.04
0
0.2
0.4
0.6
0.8

Problem 9.51

[Difficulty: 2]

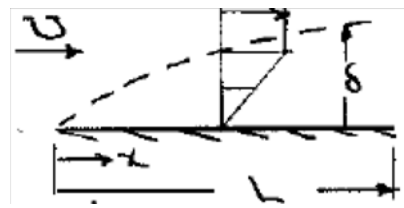
9.51 For the flow conditions of Problem 9.50, develop an algebraic expression for the variation of wall shear stress with distance along the surface. Integrate to obtain an algebraic expression for the total skin-friction drag on the surface. Evaluate the drag for the given conditions.

Given: Horizontal surface immersed in a stream of standard air. Laminar BL with linear profile forms.

$$L = 0.8 \cdot m \quad b = 1.9 \cdot m \quad U = 5.3 \cdot \frac{m}{s} \quad \nu = 1.46 \times 10^{-5} \cdot \frac{m^2}{s} \quad \frac{u}{U} = \frac{y}{\delta}$$

Find: Algebraic expressions for wall shear stress and drag; evaluate at given conditions

Solution: We will determine the drag force from the shear stress at the wall



Governing Equations:

$$\tau_w = \rho \cdot U^2 \cdot \left(\frac{d}{dx}\theta\right) = \mu \cdot \left(\frac{\partial}{\partial y}u\right) \text{ at } y = 0 \qquad \text{(Wall shear stress)}$$

$$\frac{\theta}{\delta} = \int_0^1 \frac{u}{U} \cdot \left(1 - \frac{u}{U}\right) d\eta \qquad \text{(Momentum thickness)}$$

For the linear velocity profile: $\frac{\theta}{\delta} = \int_0^1 \eta \cdot (1 - \eta)\, d\eta = \int_0^1 \left(\eta - \eta^2\right) d\eta$ Evaluating this integral: $\frac{\theta}{\delta} = \frac{1}{6} = 0.1667$

Therefore it follows that $\frac{d}{dx}\theta = \frac{d}{d\delta}\theta \cdot \left(\frac{\partial}{\partial x}\delta\right) = \frac{1}{6} \cdot \left(\frac{\partial}{\partial x}\delta\right)$ To determine the wall shear stress: $\tau_w = \frac{\mu \cdot U}{\delta} = \frac{\rho \cdot U^2}{6} \cdot \left(\frac{\partial}{\partial x}\delta\right)$

Separating variables yields: $\frac{6 \cdot \mu}{\rho \cdot U} \cdot dx = \delta \cdot d\delta$ Integrating yields: $\frac{\delta^2}{2} = \frac{6 \cdot \mu}{\rho \cdot U} \cdot x$ Solving this expression for δ/x: $\frac{\delta}{x} = \sqrt{\frac{12}{Re_x}}$

Substituting this back into the expression for wall shear stress: $\tau_w = \frac{\mu \cdot U}{\delta} = \frac{\mu \cdot U}{x \cdot \sqrt{\frac{12}{Re_x}}} = \frac{1}{\sqrt{12}} \cdot \frac{\mu \cdot U}{x} \cdot \sqrt{Re_x}$ $\quad \tau_w = 0.289 \cdot \frac{\mu \cdot U}{x} \cdot \sqrt{Re_x}$

The drag force is given by: $F_D = \int \tau_w\, dA = \int_0^L \tau_w \cdot b\, dx = \int_0^L \rho \cdot U^2 \cdot \frac{d\theta}{dx} \cdot b\, dx = b \cdot \int_0^{\theta_L} \rho \cdot U^2\, d\theta$ $\qquad F_D = \rho \cdot U^2 \cdot b \cdot \theta_L$

For the given conditions: $Re_L = \frac{U \cdot L}{\nu} = 2.90 \times 10^5$

$$\delta_L = L \cdot \sqrt{\frac{12}{Re_L}} = 5.14 \cdot mm$$

$$\theta_L = \frac{\delta_L}{6} = 0.857 \cdot mm$$

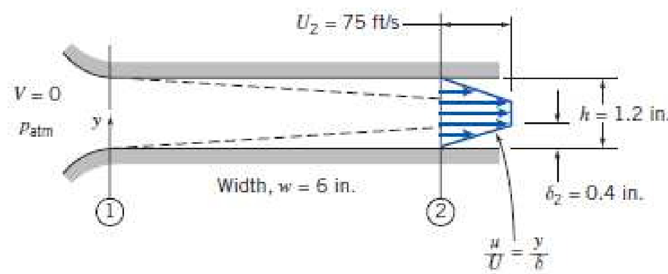
$$F_D = \rho \cdot U^2 \cdot b \cdot \theta_L = 0.0563\, N$$

$$F_D = 0.0563\, N$$

Problem 9.52

[Difficulty: 3]

9.52 Standard air flows from the atmosphere into the wide, flat channel shown. Laminar boundary layers form on the top and bottom walls of the channel (ignore boundary-layer effects on the side walls). Assume the boundary layers behave as on a flat plate, with linear velocity profiles. At any axial distance from the inlet, the static pressure is uniform across the channel. Assume uniform flow at section ①. Indicate where the Bernoulli equation can be applied in this flow field. Find the static pressure (gage) and the displacement thickness at section ②. Plot the stagnation pressure (gage) across the channel at section ②, and explain the result. Find the static pressure (gage) at section ①, and compare to the static pressure (gage) at section ②.



Given: Data on flow in a channel

Find: Static pressures; plot of stagnation pressure

Solution:

The given data is $h = 1.2\cdot\text{in}$ $\quad \delta_2 = 0.4\cdot\text{in}$ $\quad U_2 = 75\cdot\frac{\text{ft}}{\text{s}}$ $\quad w = 6\cdot\text{in}$

Appendix A $\quad \rho = 0.00239\cdot\frac{\text{slug}}{\text{ft}^3}$

Governing equations: Mass

$$\frac{\partial}{\partial t}\int_{CV}\rho\, d\forall + \int_{CS}\rho\vec{V}\cdot d\vec{A} = 0 \qquad (4.12)$$

Before entering the duct, and in the the inviscid core, the Bernoulli equation holds

$$\frac{p}{\rho} + \frac{V^2}{2} + g\cdot z = \text{constant} \qquad (4.24)$$

Assumptions: (1) Steady flow
(2) No body force in *x* direction

For a linear velocity profile, from Table 9.2 the displacement thickness at location 2 is

$$\delta_{disp2} = \frac{\delta_2}{2} \qquad \delta_{disp2} = 0.2\cdot\text{in}$$

From the definition of the displacement thickness, to compute the flow rate, the uniform flow at location 2 is assumed to take place in the entire duct, minus the displacement thicknesses at top and bottom

$$A_2 = w\cdot\left(h - 2\cdot\delta_{disp2}\right) \qquad A_2 = 4.80\cdot\text{in}^2$$

Then $\quad Q = A_2\cdot U_2 \qquad Q = 2.50\cdot\frac{\text{ft}^3}{\text{s}}$

Mass conservation (Eq. 4.12) leads to U_2

$$U_1\cdot A_1 = U_2\cdot A_2 \quad \text{where} \quad A_1 = w\cdot h \qquad A_1 = 7.2\cdot\text{in}^2$$

$$U_1 = \frac{A_2}{A_1}\cdot U_2 \qquad U_1 = 50\cdot\frac{\text{ft}}{\text{s}}$$

The Bernoulli equation applied between atmosphere and location 1 is

$$\frac{p_{atm}}{\rho} = \frac{p_1}{\rho} + \frac{U_1^2}{2}$$

or, working in gage pressures

$$p_1 = -\frac{1}{2}\cdot\rho\cdot U_1^2 \qquad p_1 = -0.0207\cdot psi$$

(Static pressure)

Similarly, between atmosphere and location 2 (gage pressures)

$$p_2 = -\frac{1}{2}\cdot\rho\cdot U_2^2 \qquad p_2 = -0.0467\cdot psi$$

(Static pressure)

The static pressure falls continuously in the entrance region as the fluid in the central core accelerates into a decreasing core. The stagnation pressure at location 2 (measured, e.g., with a Pitot tube as in Eq. 6.12), is indicated by an application of the Bernoulli equation at a point

$$\frac{p_t}{\rho} = \frac{p}{\rho} + \frac{u^2}{2}$$

where p_t is the total or stagnation pressure, $p = p_2$ is the static pressure, and u is the local velocity, given by

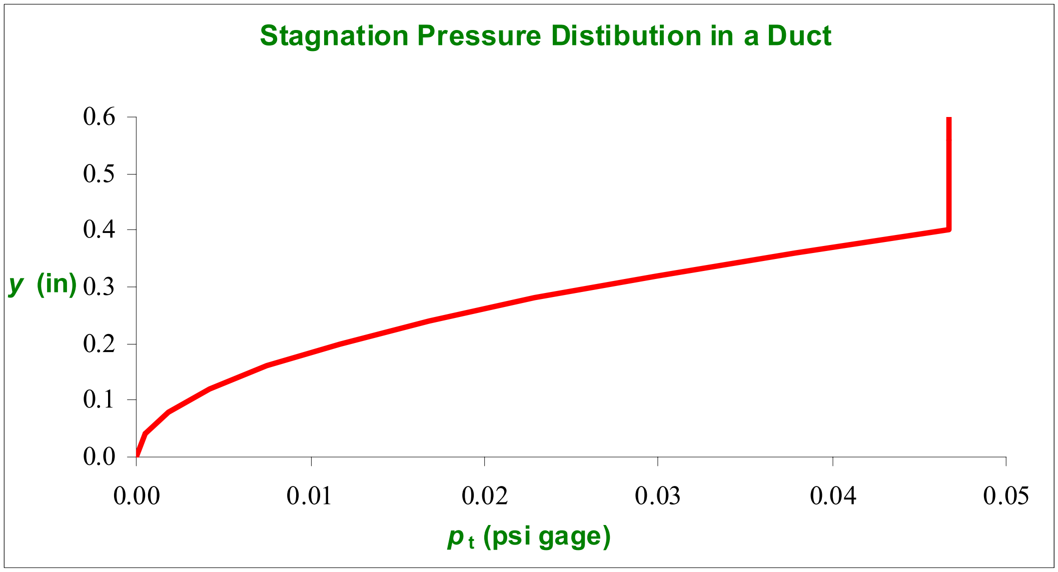
$$\frac{u}{U_2} = \frac{y}{\delta_2} \qquad y \le \delta_2$$

$$u = U_2 \qquad \delta_2 < y \le \frac{h}{2}$$

(Flow and pressure distibutions are symmetric about centerline)

Hence $\quad p_t = p_2 + \frac{1}{2}\cdot\rho\cdot u^2 \quad$ The plot of stagnation pressure is shown in the *Excel* sheet below

y **(in)**	***u*** **(ft/s)**	p_t **(psi)**
0.00	0.00	0.000
0.04	7.50	0.000
0.08	15.00	0.002
0.12	22.50	0.004
0.16	30.00	0.007
0.20	37.50	0.012
0.24	45.00	0.017
0.28	52.50	0.023
0.32	60.00	0.030
0.36	67.50	0.038
0.40	75.00	0.047
0.44	75.00	0.047
0.48	75.00	0.047
0.52	75.00	0.047
0.56	75.00	0.047
0.60	75.00	0.047

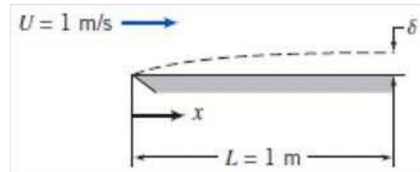


The stagnation pressure indicates total mechanical energy - the curve indicates significant loss close to the walls and no loss of energy in the central core.

Problem 9.53

[Difficulty: 3]

9.53 For the flow conditions of Example 9.4, develop an algebraic expression for the variation of wall shear stress with distance along the surface. Integrate to obtain an algebraic expression for the total skin friction drag on the surface. Evaluate the drag for the given conditions.



Given: Turbulent boundary layer flow of water, 1/7-power profile

The given or available data (Table A.9) is

$$U = 1\cdot\frac{m}{s} \qquad L = 1\cdot m \qquad \nu = 1.00\times 10^{-6}\cdot\frac{m^2}{s} \qquad \rho = 999\cdot\frac{kg}{m^3}$$

Find:
(a) Expression for wall shear stress
(b) Integrate to obtain expression for skin friction drag
(c) Evaluate for conditions shown

Solution:

Basic Equation: $C_f = \dfrac{\tau_w}{\frac{1}{2}\cdot\rho\cdot U^2} = \dfrac{0.0594}{Re_x^{\frac{1}{5}}}$ (Skin friction factor)

Assumptions:
1) Steady flow
2) No pressure force
3) No body force in *x* direction
4) Uniform flow at *ab*

Solving the above expression for the wall shear stress: $\tau_w = 0.0594\cdot\left(\frac{1}{2}\cdot\rho\cdot U^2\right)\cdot Re_x^{-\frac{1}{5}}$ $\qquad \tau_w = 0.0594\cdot\left(\frac{1}{2}\cdot\rho\cdot U^2\right)\cdot\left(\frac{U}{\nu}\right)^{-\frac{1}{5}}\cdot x^{-\frac{1}{5}}$

Integrating to find the drag: $F_D = \int_0^L \tau_w\cdot b\,dx = \int_0^L 0.0594\cdot\left(\frac{1}{2}\cdot\rho\cdot U^2\right)\cdot\left(\frac{U}{\nu}\right)^{-\frac{1}{5}}\cdot x^{-\frac{1}{5}}\cdot b\,dx = c\cdot b\cdot\int_0^L x^{-\frac{1}{5}}\,dx$ where c is defined:

$c = 0.0594\cdot\left(\frac{1}{2}\cdot\rho\cdot U^2\right)\cdot\left(\frac{U}{\nu}\right)^{-\frac{1}{5}}$ Therefore the drag is: $F_D = \frac{5}{4}\cdot c\cdot b\cdot L^{\frac{4}{5}} = \frac{5}{4}\cdot 0.0594\cdot\frac{1}{2}\cdot\rho\cdot U^2\cdot b\cdot L\cdot\left(\frac{U\cdot L}{\nu}\right)^{-\frac{1}{5}}$

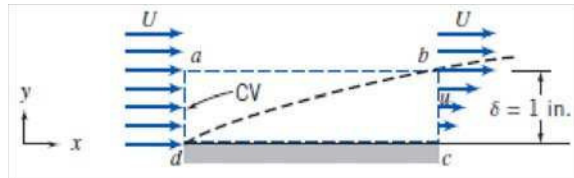
Upon simplification: $F_D = \frac{1}{2}\cdot\rho\cdot U^2\cdot b\cdot L\cdot\dfrac{0.0721}{Re_L^{\frac{1}{5}}}$

Evaluating, with $b = 1\cdot m$ $\qquad Re_L = \dfrac{U\cdot L}{\nu} = 1\times 10^6$ $\qquad F_D = \frac{1}{2}\cdot\rho\cdot U^2\cdot b\cdot L\cdot\dfrac{0.0721}{Re_L^{\frac{1}{5}}}$ $\qquad F_D = 2.27\,N$

Problem 9.54

[Difficulty: 3]

*9.54 A developing boundary layer of standard air on a flat plate is shown in Fig. P9.18. The freestream flow outside the boundary layer is undisturbed with $U = 50$ m/s. The plate is 3 m wide perpendicular to the diagram. Assume flow in the boundary layer is turbulent, with a $\frac{1}{7}$-power velocity profile, and that $\delta = 19$ mm at surface *bc*. Calculate the mass flow rate across surface *ad* and the mass flux across surface *ab*. Evaluate the *x* momentum flux across surface *ab*. Determine the drag force exerted on the flat plate between *d* and *c*. Estimate the distance from the leading edge at which transition from laminar to turbulent flow may be expected.

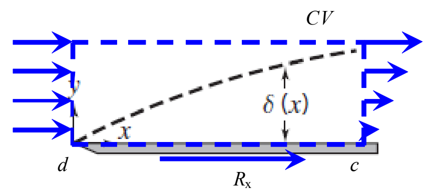


Note: Figure data applies to problem 9.18 only

Given: Data on fluid and turbulent boundary layer

Find: Mass flow rate across *ab*; Momentum flux across *bc*; Distance at which turbulence occurs

Solution:



Basic equations:

Mass $$\frac{\partial}{\partial t}\int_{CV}\rho\, dV + \int_{CS}\rho \vec{V}\cdot d\vec{A} = 0$$

Momentum $$F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\, dV + \int_{CS} u\,\rho\vec{V}\cdot d\vec{A}$$

Assumptions: 1) Steady flow 2) No pressure force 3) No body force in *x* direction 4) Uniform flow at *ab*

The given or available data (Table A.10) is

$$U = 50\cdot\frac{m}{s} \qquad \delta = 19\cdot mm \qquad b = 3\cdot m \qquad \rho = 1.23\cdot\frac{kg}{m^3} \qquad \nu = 1.50\times 10^{-5}\cdot\frac{m^2}{s}$$

Consider CV *abcd* $\quad m_{ad} = -\rho\cdot U\cdot b\cdot\delta \qquad m_{ad} = -3.51\cdot\frac{kg}{s}$ (Note: Software cannot render a dot)

Mass $\quad m_{ad} + \int_0^{\delta}\rho\cdot u\cdot b\,dy + m_{ab} = 0$ and in the boundary layer $\quad \frac{u}{U} = \left(\frac{y}{\delta}\right)^{\frac{1}{7}} = \eta^{\frac{1}{7}} \qquad dy = d\eta\cdot\delta$

Hence $$m_{ab} = \rho\cdot U\cdot b\cdot\delta - \int_0^1 \rho\cdot U\cdot\eta^{\frac{1}{7}}\cdot\delta\,d\eta = \rho\cdot U\cdot b\cdot\delta - \frac{7}{8}\cdot\rho\cdot U\cdot b\cdot\delta \qquad m_{ab} = \frac{1}{8}\cdot\rho\cdot U\cdot b\cdot\delta \qquad m_{ab} = 0.438\cdot\frac{kg}{s}$$

The momentum flux across *bc* is $$mf_{bc} = \int_0^{\delta} u\cdot\rho\cdot\vec{V}\,d\vec{A} = \int_0^{\delta} u\cdot\rho\cdot u\cdot b\,dy = \int_0^1 \rho\cdot U^2\cdot b\cdot\delta\cdot\eta^{\frac{2}{7}}\,d\eta = \rho\cdot U^2\cdot b\cdot\delta\cdot\frac{7}{9}$$

$$mf_{bc} = \frac{7}{9}\cdot\rho\cdot U^2\cdot b\cdot\delta \qquad mf_{bc} = 136.3\cdot\frac{kg\cdot m}{s^2}$$

From momentum $\quad -R_x = U\cdot(-\rho\cdot U\cdot\delta) + m_{ab}\cdot u_{ab} + mf_{bc} \qquad R_x = \rho\cdot U^2\cdot b\cdot\delta - m_{ab}\cdot U - mf_{bc} \qquad R_x = 17.04\cdot N$

Transition occurs at $\quad Re_x = 5\times 10^5$ and $\quad Re_x = \frac{U\cdot x}{\nu} \qquad x_{trans} = \frac{Re_x\cdot\nu}{U} \qquad x_{trans} = 0.1500\cdot m$

Problem 9.55

[Difficulty: 3]

9.55 Consider flow of air over a flat plate of length 5 m. On one graph, plot the boundary-layer thickness as a function of distance along the plate for freestream speed $U = 10$ m/s assuming (a) a completely laminar boundary layer, (b) a completely turbulent boundary layer, and (c) a laminar boundary layer that becomes turbulent at $Re_x = 5 \times 10^5$. Use *Excel's Goal Seek* or *Solver* to find the speeds U for which transition occurs at the trailing edge, and at $x = 4$ m, 3 m, 2 m, and 1 m.

Given: Data on flow over a flat plate $\quad U = 10 \cdot \frac{m}{s} \quad L = 5 \cdot m \quad \nu = 1.45 \times 10^{-5} \cdot \frac{m^2}{s}$ (from Table A.10)

Find: Plot of laminar and turbulent boundary layer; Speeds for transition at trailing edge

Solution:

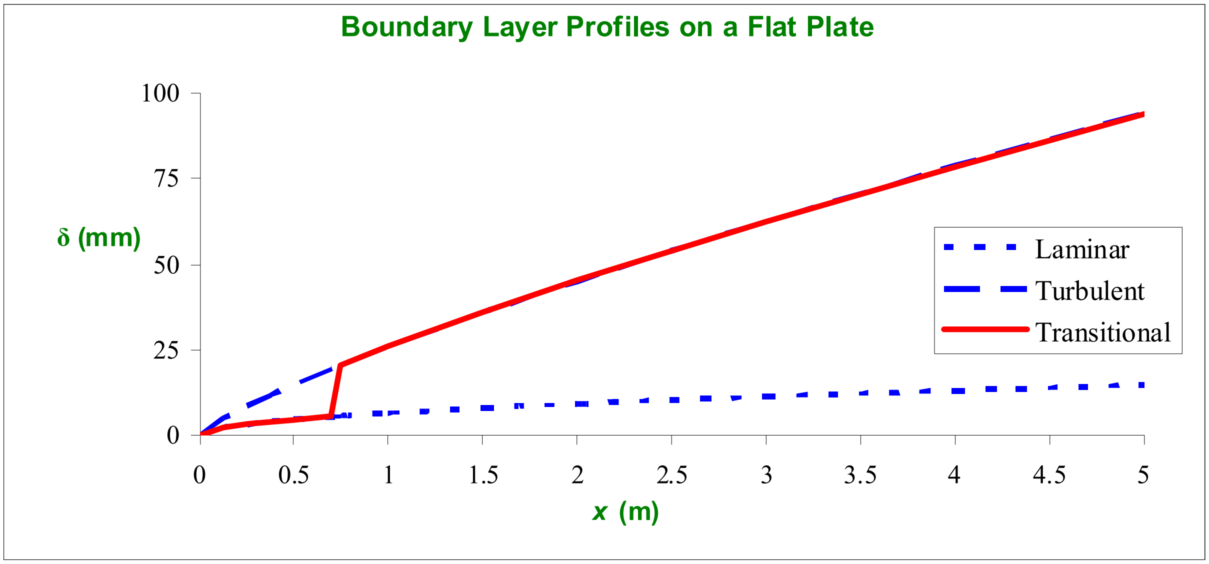
Governing Equations: For laminar flow $\quad \frac{\delta}{x} = \frac{5.48}{\sqrt{Re_x}} \quad$ (9.21) $\quad$ and $\quad Re_x = \frac{U \cdot x}{\nu} \quad$ so $\quad \delta = 5.48 \cdot \sqrt{\frac{\nu \cdot x}{U}} \quad$ (1)

The critical Reynolds number is $\quad Re_{crit} = 500000 \quad$ Hence, for velocity U the critical length x_{crit} is $\quad x_{crit} = 500000 \cdot \frac{\nu}{U} \quad$ (2)

For turbulent flow $\quad \frac{\delta}{x} = \frac{0.382}{Re_x^{\frac{1}{5}}} \quad$ (9.26) $\quad$ so $\quad \delta = 0.382 \cdot \left(\frac{\nu}{U}\right)^{\frac{1}{5}} \cdot x^{\frac{4}{5}} \quad$ (3)

For (a) completely laminar flow Eq. 1 holds; for (b) completely turbulent flow Eq. 3 holds; for (c) transitional flow Eq.1 or 3 holds depending on x_{crit} in Eq. 2. Results are shown below from *Excel*.

***x* (m)**	***Re* $_x$**	**(a) Laminar δ (mm)**	**(b) Turbulent δ (mm)**	**(c) Transition δ (mm)**
0.00	0.00E+00	0.00	0.00	0.00
0.125	8.62E+04	2.33	4.92	2.33
0.250	1.72E+05	3.30	8.56	3.30
0.375	2.59E+05	4.04	11.8	4.04
0.500	3.45E+05	4.67	14.9	4.67
0.700	4.83E+05	5.52	19.5	5.5
0.75	5.17E+05	5.71	20.6	20.6
1.00	6.90E+05	6.60	26.0	26.0
1.50	1.03E+06	8.08	35.9	35.9
2.00	1.38E+06	9.3	45.2	45.2
3.00	2.07E+06	11.4	62.5	62.5
4.00	2.76E+06	13.2	78.7	78.7
5.00	3.45E+06	14.8	94.1	94.1



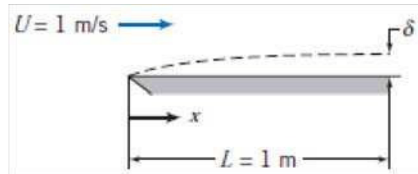
The speeds U at which transition occurs at specific points are shown below

x_{trans} (m)	U (m/s)
5	1.45
4	1.81
3	2.42
2	3.63
1	7.25

Problem 9.56

[Difficulty: 2]

9.56 Assume the flow conditions given in Example 9.4. Plot δ, δ^*, and τ_w versus x/L for the plate.



Given: Turbulent boundary layer flow of water

$$L = 1 \cdot m \quad U = 1 \cdot \frac{m}{s} \quad \nu = 1.00 \times 10^{-6} \cdot \frac{m^2}{s} \quad \frac{u}{U} = \left(\frac{y}{\delta}\right)^{\frac{1}{7}}$$

Find: Plot δ, δ^*, and τ_w versus x/L for the plate

Solution: We will determine the drag force from the shear stress at the wall

Governing Equations:

$$\frac{\delta}{x} = \frac{0.382}{Re_x^{\frac{1}{5}}} \qquad \text{(Boundary layer thickness)}$$

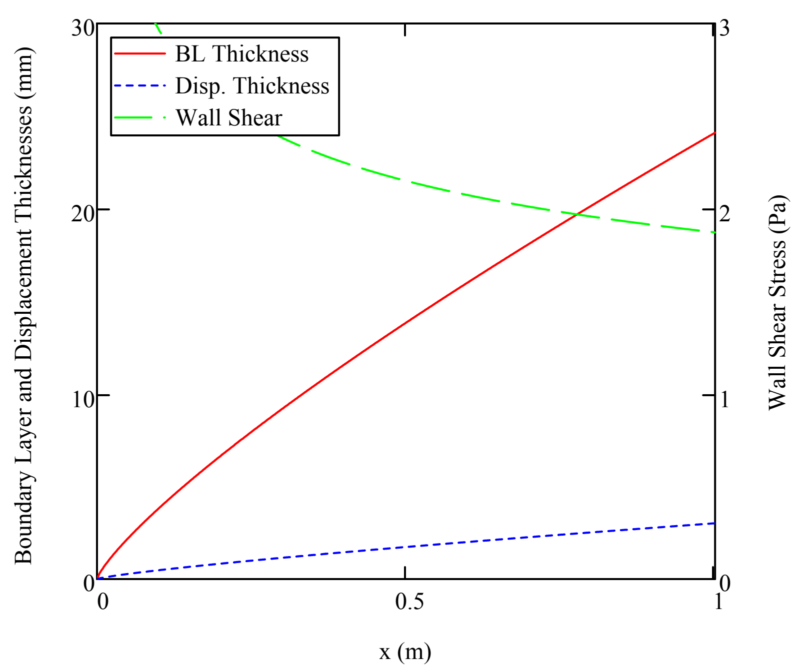
$$\frac{\delta_{disp}}{\delta} = \frac{1}{8} \qquad \text{(Displacement thickness)}$$

$$C_f = \frac{\tau_w}{\frac{1}{2} \cdot \rho \cdot U^2} = \frac{0.0594}{Re_x^{\frac{1}{5}}} \qquad \text{(Skin friction factor)}$$

Assumption: Boundary layer is turbulent from x = 0

For the conditions given: $Re_L = \frac{U \cdot L}{\nu} = 1.0 \times 10^6$ $\quad q = \frac{1}{2} \cdot \rho \cdot U^2 = 500\ Pa$ $\quad \tau_w = \frac{0.0594}{Re_x^{\frac{1}{5}}} \cdot q = 29.7 \cdot Pa \cdot Re_x^{-\frac{1}{5}}$

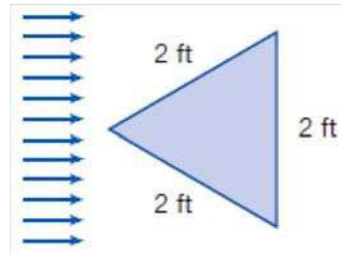
Here is the plot of boundary layer thickness and wall shear stress:



Problem 9.57

[Difficulty: 3]

9.57 Repeat Problem 9.42, except that the air flow is now at 80 ft/s (assume turbulent boundary-layer flow).



Given: Triangular plate

Find: Drag

Solution:

Basic equations: $c_f = \dfrac{\tau_w}{\frac{1}{2}\cdot\rho\cdot U^2}$ $\quad c_f = \dfrac{0.0594}{Re_x^{\frac{1}{5}}}$

$L = 2\cdot ft\cdot\dfrac{\sqrt{3}}{2}$ $\quad L = 1.732\cdot ft$ $\quad W = 2\cdot ft$ $\quad U = 80\cdot\dfrac{ft}{s}$

From Table A.10 at 70ºF $\quad \nu = 1.62\times 10^{-4}\cdot\dfrac{ft^2}{s}$ $\quad \rho = 0.00234\cdot\dfrac{slug}{ft^3}$

First determine the nature of the boundary layer $\quad Re_L = \dfrac{U\cdot L}{\nu}$ $\quad Re_L = 9\times 10^5$ $\quad$ so definitely still laminar over a significant portion of the plate, but we are told to assume turbulent!

The drag (one side) is $\quad F_D = \displaystyle\int \tau_w\, dA$ $\quad F_D = \displaystyle\int_0^L \tau_w\cdot w(x)\, dx$ $\quad w(x) = W\cdot\dfrac{x}{L}$

We also have $\quad \tau_w = c_f\cdot\dfrac{1}{2}\cdot\rho\cdot U^2 = \dfrac{1}{2}\cdot\rho\cdot U^2\cdot\dfrac{0.0594}{Re_x^{\frac{1}{5}}}$

Hence
$$F_D = \frac{1}{2}\cdot\rho\cdot U^2\cdot\frac{W}{L}\cdot\int_0^L \frac{0.0594\cdot x}{\left(\frac{U\cdot x}{\nu}\right)^{\frac{1}{5}}}\, dx = \frac{0.0594}{2}\cdot\rho\cdot U^{\frac{9}{5}}\cdot\frac{W}{L}\cdot\nu^{\frac{1}{5}}\cdot\int_0^L x^{\frac{4}{5}}\, dx$$

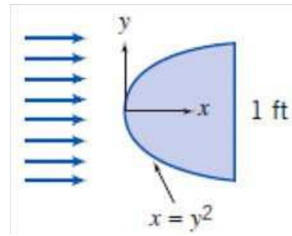
The integral is $\quad \displaystyle\int_0^L x^{\frac{4}{5}}\, dx = \frac{5}{9}\cdot L^{\frac{9}{5}}$ $\quad$ so $\quad F_D = 0.0165\cdot\rho\cdot W\cdot\left(L^4\cdot\nu\cdot U^9\right)^{\frac{1}{5}}$ $\quad F_D = 0.0557\cdot lbf$

Note: For two-sided solution $\quad 2\cdot F_D = 0.1114\cdot lbf$

Problem 9.58

[Difficulty: 3]

9.58 Repeat Problem 9.44, except that the air flow is now at 80 ft/s (assume turbulent boundary-layer flow).



Given: Parabolic plate

Find: Drag

Solution:

Basic equations:

$$c_f = \frac{\tau_w}{\frac{1}{2}\cdot\rho\cdot U^2} \qquad c_f = \frac{0.0594}{Re_x^{\frac{1}{5}}}$$

$$W = 1\cdot ft \qquad L = \frac{\left(\frac{W}{2}\right)^2}{1\cdot ft} \qquad L = 3\cdot in \qquad U = 80\cdot\frac{ft}{s}$$

Note: "y" is the equation of the upper and lower surfaces, so y = W/2 at x = L

From Table A.9 at 70°F $\quad \nu = 1.63\times 10^{-4}\cdot\frac{ft^2}{s} \qquad \rho = 0.00233\cdot\frac{slug}{ft^3}$

First determine the nature of the boundary layer $\quad Re_L = \frac{U\cdot L}{\nu} \qquad Re_L = 1.23\times 10^5 \quad$ so still laminar, but we are told to assume turbulent!

The drag (one side) is $\quad F_D = \int \tau_w\, dA \qquad F_D = \int_0^L \tau_w\cdot w(x)\, dx \qquad w(x) = W\cdot\sqrt{\frac{x}{L}}$

We also have $\quad \tau_w = c_f\cdot\frac{1}{2}\cdot\rho\cdot U^2 = \frac{1}{2}\cdot\rho\cdot U^2\cdot\frac{0.0594}{Re_x^{\frac{1}{5}}}$

Hence

$$F_D = \frac{1}{2}\cdot\rho\cdot U^2\cdot W\cdot\int_0^L \frac{0.0594\cdot\sqrt{\frac{x}{L}}}{\left(\frac{U\cdot x}{\nu}\right)^{\frac{1}{5}}}\, dx = \frac{0.0594}{2}\cdot\rho\cdot U^{\frac{9}{5}}\cdot W\cdot L^{-\frac{1}{2}}\cdot\nu^{\frac{1}{5}}\cdot\int_0^L x^{\frac{3}{10}}\, dx$$

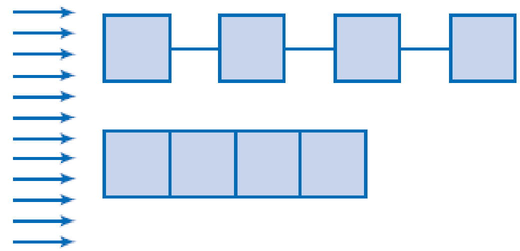
$$F_D = 0.0228\cdot\rho\cdot W\cdot\left(\nu\cdot L^4\cdot U^9\right)^{\frac{1}{5}} \qquad F_D = 0.00816\cdot lbf$$

Note: For two-sided solution $\quad 2\cdot F_D = 0.01632\cdot lbf$

Problem 9.59

[Difficulty: 3]

9.59 Repeat Problem 9.46, except that the air flow is now at 80 ft/s (assume turbulent boundary-layer flow).



Given: Pattern of flat plates

Find: Drag on separate and composite plates

Solution:

Basic equations: $c_f = \dfrac{\tau_w}{\frac{1}{2}\cdot\rho\cdot U^2}$ $\qquad c_f = \dfrac{0.0594}{Re_x^{\frac{1}{5}}}$

For separate plates $\quad L = 3\cdot in \quad W = 3\cdot in \quad U = 80\cdot\dfrac{ft}{s}$

From Table A.7 at 70°F $\quad \nu = 1.06\times 10^{-5}\cdot\dfrac{ft^2}{s} \quad \rho = 1.93\cdot\dfrac{slug}{ft^3}$

First determine the nature of the boundary layer $\quad Re_L = \dfrac{U\cdot L}{\nu} \quad Re_L = 1.89\times 10^6$ so turbulent

The drag (one side) is $\quad F_D = \int \tau_w\,dA \qquad F_D = \int_0^L \tau_w\cdot W\,dx$

We also have $\quad \tau_w = c_f\cdot\frac{1}{2}\cdot\rho\cdot U^2 = \frac{1}{2}\cdot\rho\cdot U^2\cdot\dfrac{0.0594}{Re_x^{\frac{1}{5}}}$

Hence
$$F_D = \frac{1}{2}\cdot\rho\cdot U^2\cdot W\cdot\int_0^L \frac{0.0594}{\left(\frac{U\cdot x}{\nu}\right)^{\frac{1}{5}}}\,dx = \frac{0.0594}{2}\cdot\rho\cdot U^{\frac{9}{5}}\cdot W\cdot\nu^{\frac{1}{5}}\cdot\int_0^L x^{-\frac{1}{5}}\,dx$$

The integral is $\quad \int_0^L x^{-\frac{1}{5}}\,dx = \frac{5}{4}\cdot L^{\frac{4}{5}}$ so $\quad F_D = 0.0371\cdot\rho\cdot W\cdot\left(\nu\cdot L^4\cdot U^9\right)^{\frac{1}{5}} \qquad F_D = 1.59\cdot lbf$

This is the drag on one plate. The total drag is then $\quad F_{Total} = 4\cdot F_D \qquad F_{Total} = 6.37\cdot lbf$

For both sides: $\quad 2\cdot F_{Total} = 12.73\cdot lbf$

For the composite plate $L = 4 \times 3 \cdot in$ $L = 12.00 \cdot in$ and since the Reynolds number for the single plate was turbulent, we know that the flow around the composite plate will be turbulent as well.

$$F_{Composite} = 0.0371 \cdot \rho \cdot W \cdot \left(\nu \cdot L^4 \cdot U^9\right)^{\frac{1}{5}} \qquad F_{Composite} = 4.82 \cdot lbf$$

For both sides: $2 \cdot F_{Composite} = 9.65 \cdot lbf$

The drag is much lower on the composite compared to the separate plates. This is because τ_w is largest near the leading edges and falls off rapidly; in this problem the separate plates experience leading edges four times!

Problem 9.60

[Difficulty: 3]

9.60 The velocity profile in a turbulent boundary-layer flow at zero pressure gradient is approximated by the $\frac{1}{6}$-power profile expression,

$$\frac{u}{U} = \eta^{1/6}, \quad \text{where} \quad \eta = \frac{y}{\delta}$$

Use the momentum integral equation with this profile to obtain expressions for δ/x and C_f. Compare with results obtained in Section 9.5 for the $\frac{1}{7}$-power profile.

Given: Turbulent boundary layer flow with 1/6 power velocity profile: $\frac{u}{U} = \left(\frac{y}{\delta}\right)^{\frac{1}{6}} = \eta^{\frac{1}{6}}$

Find: Expressions for δ/x and C_f using the momentum integral equation; compare to 1/7-power rule results.

Solution: We will apply the momentum integral equation

Governing Equations:

$$\frac{\tau_w}{\rho} = \frac{d}{dx}\left(U^2\cdot\theta\right) + \delta_{disp}\cdot U\cdot\left(\frac{d}{dx}U\right) \qquad \text{(Momentum integral equation)}$$

$$C_f = \frac{\tau_w}{\frac{1}{2}\cdot\rho\cdot U^2} \qquad \text{(Skin friction coefficient)}$$

Assumptions:
(1) Zero pressure gradient, so U is constant and dp/dx = 0
(2) δ is a function of x only, and $\delta = 0$ at x = 0
(3) Incompressible flow
(4) Wall shear stress is: $\tau_w = 0.0233\cdot\rho\cdot U^2\cdot\left(\frac{\nu}{U\cdot\delta}\right)^{0.25}$

Applying the assumptions to the momentum integral equation yields: $\tau_w = \rho\cdot U^2\cdot\left(\frac{d}{dx}\theta\right) = \rho\cdot U^2\cdot\left[\frac{d}{dx}\left[\delta\cdot\int_0^1 \frac{u}{U}\cdot\left(1 - \frac{u}{U}\right)d\eta\right]\right]$

Substituting for the velocity profile: $\tau_w = \rho\cdot U^2\cdot\left[\frac{d}{dx}\left[\delta\cdot\int_0^1 \left(\eta^{\frac{1}{6}} - \eta^{\frac{2}{6}}\right)d\eta\right]\right] = \rho\cdot U^2\cdot\frac{6}{56}\cdot\left(\frac{d}{dx}\delta\right)$ Setting our two τ_w's equal:

$0.0233\cdot\rho\cdot U^2\cdot\left(\frac{\nu}{U\cdot\delta}\right)^{0.25} = \rho\cdot U^2\cdot\frac{6}{56}\cdot\left(\frac{d}{dx}\delta\right)$ Simplifying and separating variables: $\delta^{\frac{1}{4}}\cdot d\delta = 0.0233\cdot\frac{56}{6}\cdot\left(\frac{\nu}{U}\right)^{\frac{1}{4}}\cdot dx$

Integrating both sides: $\frac{4}{5}\cdot\delta^{\frac{5}{4}} = 0.0233\cdot\frac{56}{6}\cdot\left(\frac{\nu}{U}\right)^{\frac{1}{4}}\cdot x + C$ but C = 0 since $\delta = 0$ at x = 0. Therefore: $\delta = \left[\frac{5}{4}\cdot 0.0233\cdot\frac{56}{6}\cdot\left(\frac{\nu}{U}\right)^{\frac{1}{4}}\cdot x\right]^{\frac{4}{5}}$

In terms of the Reynolds number: $\frac{\delta}{x} = \frac{0.353}{Re_x^{\frac{1}{5}}}$

For the skin friction factor:

$$C_f = \frac{\tau_w}{\frac{1}{2}\cdot\rho\cdot U^2} = \frac{0.0233\cdot\rho\cdot U^2\cdot\left(\frac{\nu}{U\cdot\delta}\right)^{\frac{1}{4}}}{\frac{1}{2}\cdot\rho\cdot U^2} = 0.0466\cdot\left(\frac{\nu}{U\cdot x}\right)^{\frac{1}{4}}\cdot\left(\frac{x}{\delta}\right)^{\frac{1}{4}} = 0.0466\cdot Re_x^{-\frac{1}{4}}\cdot\left(\frac{Re_x^{\frac{1}{5}}}{0.353}\right)^{\frac{1}{4}}$$

Upon simplification:

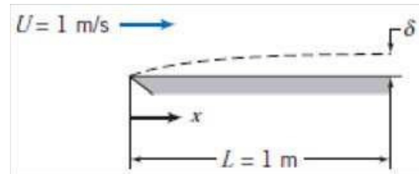
$$C_f = \frac{0.0605}{Re_x^{\frac{1}{5}}}$$

These results compare to $\frac{\delta}{x} = \frac{0.353}{Re_x^{\frac{1}{5}}}$ and $C_f = \frac{0.0605}{Re_x^{\frac{1}{5}}}$ for the 1/7-power profile.

Problem 9.61

[Difficulty: 3]

9.61 For the flow conditions of Example 9.4, but using the $\frac{1}{6}$ power velocity profile of Problem 9.60, develop an algebraic expression for the variation of wall shear stress with distance along the surface. Integrate to obtain an algebraic expression for the total skin friction drag on the surface. Evaluate the drag for the given conditions.



Given: Turbulent boundary layer flow with 1/6 power velocity profile: $\frac{u}{U} = \left(\frac{y}{\delta}\right)^{\frac{1}{6}} = \eta^{\frac{1}{6}}$

The given or available data (Table A.9) is $U = 1\cdot\frac{m}{s}$ $\quad L = 1\cdot m$ $\quad \nu = 1.00\times 10^{-6}\cdot\frac{m^2}{s}$ $\quad \rho = 999\cdot\frac{kg}{m^3}$

Find: Expressions for δ/x and C_f using the momentum integral equation; evaluate drag for the conditions given

Solution: We will apply the momentum integral equation

Governing Equations:

$$\frac{\tau_w}{\rho} = \frac{d}{dx}\left(U^2\cdot\theta\right) + \delta_{disp}\cdot U\cdot\left(\frac{d}{dx}U\right) \qquad \text{(Momentum integral equation)}$$

$$C_f = \frac{\tau_w}{\frac{1}{2}\cdot\rho\cdot U^2} \qquad \text{(Skin friction coefficient)}$$

Assumptions:
(1) Zero pressure gradient, so U is constant and dp/dx = 0
(2) δ is a function of x only, and δ = 0 at x = 0
(3) Incompressible flow
(4) Wall shear stress is: $\tau_w = 0.0233\cdot\rho\cdot U^2\cdot\left(\frac{\nu}{U\cdot\delta}\right)^{0.25}$

Applying the assumptions to the momentum integral equation yields: $\tau_w = \rho\cdot U^2\cdot\left(\frac{d}{dx}\theta\right) = \rho\cdot U^2\cdot\left[\frac{d}{dx}\left[\delta\cdot\int_0^1 \frac{u}{U}\cdot\left(1-\frac{u}{U}\right)d\eta\right]\right]$

Substituting for the velocity profile: $\tau_w = \rho\cdot U^2\cdot\left[\frac{d}{dx}\left[\delta\cdot\int_0^1 \left(\eta^{\frac{1}{6}} - \eta^{\frac{2}{6}}\right)d\eta\right]\right] = \rho\cdot U^2\cdot\frac{6}{56}\cdot\left(\frac{d}{dx}\delta\right)$ Setting our two τ_w's equal:

$0.0233\cdot\rho\cdot U^2\cdot\left(\frac{\nu}{U\cdot\delta}\right)^{0.25} = \rho\cdot U^2\cdot\frac{6}{56}\cdot\left(\frac{d}{dx}\delta\right)$ Simplifying and separating variables: $\delta^{\frac{1}{4}}\cdot d\delta = 0.0233\cdot\frac{56}{6}\cdot\left(\frac{\nu}{U}\right)^{\frac{1}{4}}\cdot dx$

Integrating both sides: $\frac{4}{5}\cdot\delta^{\frac{5}{4}} = 0.0233\cdot\frac{56}{6}\cdot\left(\frac{\nu}{U}\right)^{\frac{1}{4}}\cdot x + C$ but C = 0 since δ = 0 at x = 0. Therefore: $\delta = \left[\frac{5}{4}\cdot 0.0233\cdot\frac{56}{6}\cdot\left(\frac{\nu}{U}\right)^{\frac{1}{4}}\cdot x\right]^{\frac{4}{5}}$

In terms of the Reynolds number: $\dfrac{\delta}{x} = \dfrac{0.353}{Re_x^{\frac{1}{5}}}$

For the skin friction factor:

$$C_f = \frac{\tau_w}{\frac{1}{2}\cdot\rho\cdot U^2} = \frac{0.0233\cdot\rho\cdot U^2\cdot\left(\frac{\nu}{U\cdot\delta}\right)^{\frac{1}{4}}}{\frac{1}{2}\cdot\rho\cdot U^2} = 0.0466\cdot\left(\frac{\nu}{U\cdot x}\right)^{\frac{1}{4}}\cdot\left(\frac{x}{\delta}\right)^{\frac{1}{4}} = 0.0466\cdot Re_x^{-\frac{1}{4}}\cdot\left(\frac{Re_x^{\frac{1}{5}}}{0.353}\right)^{\frac{1}{4}}$$

Upon simplification:

$$C_f = \frac{0.0605}{Re_x^{\frac{1}{5}}}$$

The drag force is:

$$F_D = \int_0^L \tau_w\cdot b\,dx = \int_0^L 0.0605\cdot\frac{1}{2}\cdot\rho\cdot U^2\cdot\left(\frac{\nu}{U}\right)^{\frac{1}{5}}\cdot x^{-\frac{1}{5}}\cdot b\,dx = \frac{0.0605}{2}\cdot\rho\cdot U^2\cdot\left(\frac{\nu}{U}\right)^{\frac{1}{5}}\cdot b\cdot\int_0^L x^{-\frac{1}{5}}\,dx$$

Evaluating the integral: $F_D = \dfrac{0.0605}{2}\cdot\rho\cdot U^2\cdot\left(\dfrac{\nu}{U}\right)^{\frac{1}{5}}\cdot b\cdot\dfrac{5}{4}\cdot L^{\frac{4}{5}}$ In terms of the Reynolds number: $F_D = \dfrac{0.0378\cdot\rho\cdot U^2\cdot b\cdot L}{Re_L^{\frac{1}{5}}}$

For the given conditions and assuming that b = 1 m: $Re_L = 1.0\times 10^6$ and therefore:

$$F_D = 2.38\,\text{N}$$

Problem 9.62

[Difficulty: 3]

9.62 Repeat Problem 9.60, using the $\frac{1}{8}$-power profile expression.

Given: Turbulent boundary layer flow with 1/8 power velocity profile: $\frac{u}{U} = \left(\frac{y}{\delta}\right)^{\frac{1}{8}} = \eta^{\frac{1}{8}}$

Find: Expressions for δ/x and C_f using the momentum integral equation; compare to 1/7-power rule results.

Solution: We will apply the momentum integral equation

Governing Equations:

$$\frac{\tau_w}{\rho} = \frac{d}{dx}\left(U^2\cdot\theta\right) + \delta_{disp}\cdot U\cdot\left(\frac{d}{dx}U\right) \qquad \text{(Momentum integral equation)}$$

$$C_f = \frac{\tau_w}{\frac{1}{2}\cdot\rho\cdot U^2} \qquad \text{(Skin friction coefficient)}$$

Assumptions:
(1) Zero pressure gradient, so U is constant and dp/dx = 0
(2) δ is a function of x only, and $\delta = 0$ at $x = 0$
(3) Incompressible flow
(4) Wall shear stress is: $\tau_w = 0.0233\cdot\rho\cdot U^2\cdot\left(\frac{\nu}{U\cdot\delta}\right)^{0.25}$

Applying the assumptions to the momentum integral equation yields: $\tau_w = \rho\cdot U^2\cdot\left(\frac{d}{dx}\theta\right) = \rho\cdot U^2\cdot\left[\frac{d}{dx}\left[\delta\cdot\int_0^1 \frac{u}{U}\cdot\left(1-\frac{u}{U}\right)d\eta\right]\right]$

Substituting for the velocity profile: $\tau_w = \rho\cdot U^2\cdot\left[\frac{d}{dx}\left[\delta\cdot\int_0^1\left(\eta^{\frac{1}{8}} - \eta^{\frac{2}{8}}\right)d\eta\right]\right] = \rho\cdot U^2\cdot\frac{8}{90}\cdot\left(\frac{d}{dx}\delta\right)$ Setting our two τ_w's equal:

$0.0233\cdot\rho\cdot U^2\cdot\left(\frac{\nu}{U\cdot\delta}\right)^{0.25} = \rho\cdot U^2\cdot\frac{6}{56}\cdot\left(\frac{d}{dx}\delta\right)$ Simplifying and separating variables: $\delta^{\frac{1}{4}}\cdot d\delta = 0.262\cdot\left(\frac{\nu}{U}\right)^{\frac{1}{4}}\cdot dx$

Integrating both sides: $\frac{4}{5}\cdot\delta^{\frac{5}{4}} = 0.262\cdot\left(\frac{\nu}{U}\right)^{\frac{1}{4}}\cdot x + C$ but C = 0 since $\delta = 0$ at $x = 0$. Therefore: $\delta = \left[\frac{5}{4}\cdot 0.262\cdot\left(\frac{\nu}{U}\right)^{\frac{1}{4}}\cdot x\right]^{\frac{4}{5}}$

In terms of the Reynolds number: $\frac{\delta}{x} = \frac{0.410}{Re_x^{\frac{1}{5}}}$

For the skin friction factor:

$$C_f = \frac{\tau_w}{\frac{1}{2}\cdot\rho\cdot U^2} = \frac{0.0233\cdot\rho\cdot U^2\cdot\left(\frac{\nu}{U\cdot\delta}\right)^{\frac{1}{4}}}{\frac{1}{2}\cdot\rho\cdot U^2} = 0.0466\cdot\left(\frac{\nu}{U\cdot x}\right)^{\frac{1}{4}}\cdot\left(\frac{x}{\delta}\right)^{\frac{1}{4}} = 0.0466\cdot Re_x^{-\frac{1}{4}}\cdot\left(\frac{Re_x^{\frac{1}{5}}}{0.410}\right)^{\frac{1}{4}}$$

Upon simplification:

$$C_f = \frac{0.0582}{Re_x^{\frac{1}{5}}}$$

These results compare to $\frac{\delta}{x} = \frac{0.353}{Re_x^{\frac{1}{5}}}$ and $C_f = \frac{0.0605}{Re_x^{\frac{1}{5}}}$ for the 1/7-power profile.

Problem 9.63

[Difficulty: 3]

9.63 Standard air flows over a horizontal smooth flat plate at freestream speed $U = 20$ m/s. The plate length is $L = 1.5$ m and its width is $b = 0.8$ m. The pressure gradient is zero. The boundary layer is tripped so that it is turbulent from the leading edge; the velocity profile is well represented by the $\frac{1}{7}$-power expression. Evaluate the boundary-layer thickness, δ, at the trailing edge of the plate. Calculate the wall shear stress at the trailing edge of the plate. Estimate the skin friction drag on the portion of the plate between $x = 0.5$ m and the trailing edge.

Given: Turbulent boundary layer flow of water, 1/7-power profile

The given or available data (Table A.9) is

$$U = 20 \cdot \frac{m}{s} \qquad L = 1.5 \cdot m \qquad b = 0.8 \cdot m \quad \nu = 1.46 \times 10^{-5} \cdot \frac{m^2}{s} \qquad \rho = 1.23 \cdot \frac{kg}{m^3} \qquad x_1 = 0.5 \cdot m$$

Find:
(a) δ at $x = L$
(b) τ_w at $x = L$
(c) Drag force on the portion $0.5\ m < x < L$

Solution:

Basic equations:

$$\frac{\delta}{x} = \frac{0.382}{Re_x^{\frac{1}{5}}} \qquad \text{(Boundary Layer Thickness)}$$

$$C_f = \frac{\tau_w}{\frac{1}{2} \cdot \rho \cdot U^2} = \frac{0.0594}{Re_x^{\frac{1}{5}}} \qquad \text{(Skin friction factor)}$$

Assumptions:
1) Steady flow
2) No pressure force
3) No body force in *x* direction

At the trailing edge of the plate: $Re_L = \frac{U \cdot L}{\nu} = 2.05 \times 10^6$ Therefore $\delta_L = L \cdot \frac{0.382}{Re_L^{\frac{1}{5}}}$ $\qquad \delta_L = 31.3 \cdot mm$

Similarly, the wall shear stress is: $\tau_{wL} = \frac{1}{2} \cdot \rho \cdot U^2 \cdot \frac{0.0594}{Re_L^{\frac{1}{5}}}$ $\qquad \tau_{wL} = 0.798 \cdot Pa$

To find the drag: $F_D = \int_{x_1}^{L} \tau_w \cdot b\, dx = \int_{x_1}^{L} 0.0594 \cdot \left(\frac{1}{2} \cdot \rho \cdot U^2\right) \cdot \left(\frac{U}{\nu}\right)^{-\frac{1}{5}} \cdot x^{-\frac{1}{5}} \cdot b\, dx = c \cdot b \cdot \int_0^L x^{-\frac{1}{5}}\, dx$ where c is defined:

$c = 0.0594 \cdot \left(\frac{1}{2} \cdot \rho \cdot U^2\right) \cdot \left(\frac{U}{\nu}\right)^{-\frac{1}{5}}$ Therefore the drag is: $F_D = \frac{5}{4} \cdot c \cdot b \cdot L^{\frac{4}{5}} = \frac{5}{4} \cdot \frac{1}{2} \cdot \rho \cdot U^2 \cdot b \cdot \left(L \cdot C_{fL} - x_1 \cdot C_{fx1}\right)$

At $x = x_1$: $Re_{x1} = \frac{U \cdot x_1}{\nu} = 6.849 \times 10^5$ $C_{fx1} = \frac{0.0594}{Re_{x1}^{\frac{1}{5}}} = 4.043 \times 10^{-3}$ and at $x = L$ $C_{fL} = \frac{0.0594}{Re_L^{\frac{1}{5}}} = 3.245 \times 10^{-3}$

Therefore the drag is: $F_D = 0.700\,N$

Alternately, we could solve for the drag using the momentum thickness: $F_D = \rho \cdot U^2 \cdot b \cdot \left(\theta_L - \theta_{x1}\right)$ where $\theta = \frac{7}{72} \cdot \delta$

At $x = L$ $\delta_L = 31.304 \cdot mm$ $\theta_L = \frac{7}{72} \cdot \delta_L = 3.043 \cdot mm$ At $x = x_1$: $\delta_{x1} = x_1 \cdot \frac{0.382}{Re_{x1}^{\frac{1}{5}}} = 12.999 \cdot mm$ $\theta_{x1} = \frac{7}{72} \cdot \delta_{x1} = 1.264 \cdot mm$

Therefore the drag is: $F_D = 0.700\,N$

Problem 9.64

[Difficulty: 3]

9.64 Air at standard conditions flows over a flat plate. The freestream speed is 30 ft/s. Find δ and τ_w at $x = 3$ ft from the leading edge assuming (a) completely laminar flow (assume a parabolic velocity profile) and (b) completely turbulent flow (assume a $\frac{1}{7}$-power velocity profile).

Given: Air at standard conditions flowing over a flat plate

The given or available data (Table A.10) is

$$U = 30 \cdot \frac{ft}{s} \qquad x = 3 \cdot ft \qquad \nu = 1.57 \times 10^{-4} \cdot \frac{ft^2}{s} \qquad \rho = 0.00238 \frac{slug}{ft^3}$$

Find: δ and τ_w at x assuming:
(a) completely laminar flow (parabolic velocity profile)
(b) completely turbulent flow (1/7-power velocity profile)

Solution:

(Laminar Flow) (Turbulent Flow)

Basic equations:

$$\frac{\delta}{x} = \frac{5.48}{\sqrt{Re_x}} \qquad \frac{\delta}{x} = \frac{0.382}{Re_x^{\frac{1}{5}}} \qquad \text{(Boundary Layer Thickness)}$$

$$C_f = \frac{\tau_w}{\frac{1}{2} \cdot \rho \cdot U^2} = \frac{0.730}{\sqrt{Re_x}} \qquad C_f = \frac{\tau_w}{\frac{1}{2} \cdot \rho \cdot U^2} = \frac{0.0594}{Re_x^{\frac{1}{5}}} \qquad \text{(Skin friction factor)}$$

The Reynolds number is: $$Re_x = \frac{U \cdot x}{\nu} = 5.73 \times 10^5$$

For laminar flow: $$\delta_{lam} = x \cdot \frac{5.48}{\sqrt{Re_x}} \qquad \delta_{lam} = 0.261 \cdot in$$

$$\tau_{wlam} = \frac{1}{2} \cdot \rho \cdot U^2 \cdot \frac{0.730}{\sqrt{Re_x}} \qquad \tau_{wlam} = 7.17 \times 10^{-6} \cdot psi$$

For turbulent flow: $$\delta_{turb} = x \cdot \frac{0.382}{Re_x^{\frac{1}{5}}} \qquad \delta_{turb} = 0.970 \cdot in$$

$$\tau_{wturb} = \frac{1}{2} \cdot \rho \cdot U^2 \cdot \frac{0.0594}{Re_x^{\frac{1}{5}}} \qquad \tau_{wturb} = 3.12 \times 10^{-5} \cdot psi$$

Comparing results: $$\frac{\delta_{turb}}{\delta_{lam}} = 3.72$$

The turbulent boundary layer has a much larger skin friction, which causes it to grow more rapidly than the laminar boundary layer.

$$\frac{\tau_{wturb}}{\tau_{wlam}} = 4.34$$

Problem 9.65

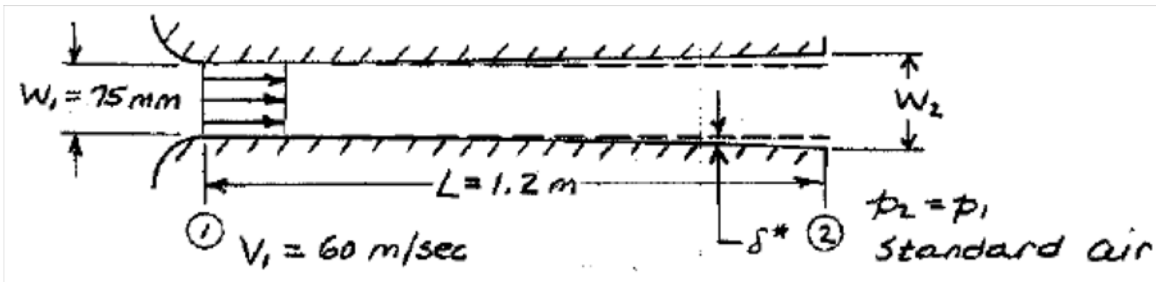
[Difficulty: 3]

9.65 A uniform flow of standard air at 60 m/s enters a plane-wall diffuser with negligible boundary-layer thickness. The inlet width is 75 mm. The diffuser walls diverge slightly to accommodate the boundary-layer growth so that the pressure gradient is negligible. Assume flat-plate boundary-layer behavior. Explain why the Bernoulli equation is applicable to this flow. Estimate the diffuser width 1.2 m downstream from the entrance.

Given: Air at standard conditions flowing through a plane-wall diffuser with negligible BL thickness. Walls diverge slightly to accomodate BL growth, so p = constant.

The given or available data (Table A.9) is

$$U = 60\cdot\frac{m}{s} \qquad L = 1.2\cdot m \qquad W_1 = 75\cdot mm \qquad \nu = 1.46\times 10^{-5}\cdot\frac{m^2}{s} \qquad \rho = 1.23\cdot\frac{kg}{m^3}$$



Find: (a) why Bernoulli is applicable to this flow.
(b) diffuser width W_2 at x = L

Solution:

Basic equations:

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + g\cdot z_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + g\cdot z_2 \qquad \text{(Bernoulli Equation)}$$

$$\frac{\partial}{\partial t}\int \rho\, dV + \int \rho\cdot\vec{V}\, d\vec{A} = 0 \qquad \text{(Continuity)}$$

Assumptions:
(1) Steady flow
(2) Turbulent, 1/7-power velocity profile in boundary layer
(3) z = constant
(4) p = constant

The Bernoulli equation may be applied along a streamline in any steady, incompressible flow in the absence of friction. The given flow is steady and incompressible. Frictional effects are confined to the thin wall boundary layers. Therefore, the Bernoulli equation may be applied along any streamline in the core flow outside the boundary layers. (In addition, since there is no streamline curvature, the pressure is uniform across sections 1 and 2.

From the assumptions, Bernoulli reduces to: $V_1 = V_2$ and from continuity: $-\rho\cdot V_1\cdot A_1 + \rho\cdot V_2\cdot A_{2eff} = 0$

or $A_{2eff} = \left(W_2 - 2\cdot\delta_{disp2}\right)\cdot b = W_1\cdot b$ Therefore: $W_2 = W_1 + 2\cdot\delta_{disp2}$

The Reynolds number is: $Re_L = \frac{U\cdot L}{\nu} = 4.932\times 10^6$ From turbulent BL theory: $\delta_2 = L\cdot\frac{0.382}{Re_L^{\frac{1}{5}}} = 21.02\cdot mm$

The displacement thickness is determined from: $\delta_{disp2} = \delta_2 \cdot \int_0^1 \left(1 - \frac{u}{U}\right) d\eta$ where $\frac{u}{U} = \eta^{\frac{1}{7}}$ $\eta = \frac{y}{\delta}$

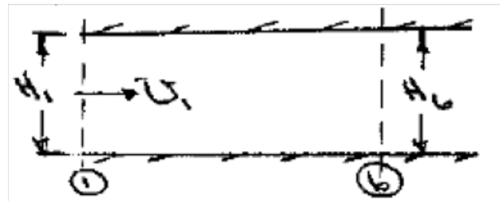
Substituting the velocity profile and valuating the integral: $\delta_{disp2} = \delta_2 \cdot \int_0^1 \left(1 - \eta^{\frac{1}{7}}\right) d\eta = \frac{\delta_2}{8}$ $\delta_{disp2} = 2.628 \cdot mm$

Therefore: $W_2 = W_1 + 2 \cdot \delta_{disp2}$ $W_2 = 80.3 \cdot mm$

Problem 9.66

[Difficulty: 3]

9.66 A laboratory wind tunnel has a flexible upper wall that can be adjusted to compensate for boundary-layer growth, giving zero pressure gradient along the test section. The wall boundary layers are well represented by the $\frac{1}{7}$-power-velocity profile. At the inlet the tunnel cross section is square, with height H_1 and width W_1, each equal to 1 ft. With freestream speed $U_1 = 90$ ft/s, measurements show that $\delta_1 = 0.5$ in. and downstream $\delta_6 = 0.65$ in. Calculate the height of the tunnel walls at ⑥. Determine the equivalent length of a flat plate that would produce the inlet boundary layer thickness. Estimate the streamwise distance between sections ① and ⑥ in the tunnel. Assume standard air.



Given: Laboratory wind tunnel has flexible wall to accomodate BL growth. BL's are well represented by 1/7-power profile. Information at two stations are known:

The given or available data (Table A.9) is

$$U = 90\cdot\frac{ft}{s} \quad H_1 = 1\cdot ft \quad W_1 = 1\cdot ft \quad \delta_1 = 0.5\cdot in \quad \delta_6 = 0.65\cdot in \quad \nu = 1.57\times10^{-4}\cdot\frac{ft^2}{s} \quad \rho = 0.00238\cdot\frac{slug}{ft^3}$$

Find: (a) Height of tunnel walls at section 6.
(b) Equivalent length of flat plate that would produce the inlet BL
(c) Estimate length of tunnel between stations 1 and 6.

Solution:

Basic equations: $\frac{\partial}{\partial t}\int \rho\, dV + \int \rho\cdot\vec{V}\, d\vec{A} = 0$ (Continuity)

Assumptions: (1) Steady flow
(2) Turbulent, 1/7-power velocity profile in boundary layer
(3) z = constant
(4) p = constant

Applying continuity between 1 and 6: $A_1\cdot U_1 = A_6\cdot U_6$ where A is the effective flow area. The velocities at 1 and 6 must be equal since pressure is constant. In terms of the duct dimensions:

$$\left(W_1 - 2\cdot\delta_{disp1}\right)\left(H_1 - 2\cdot\delta_{disp1}\right) = \left(W_1 - 2\cdot\delta_{disp6}\right)\cdot\left(H_6 - 2\cdot\delta_{disp6}\right)$$

solving for the height at 6: $H_6 = \frac{\left(W_1 - 2\cdot\delta_{disp1}\right)\left(H_1 - 2\cdot\delta_{disp1}\right)}{\left(W_1 - 2\cdot\delta_{disp6}\right)} + 2\cdot\delta_{disp6}$

The displacement thickness is determined from: $\delta_{disp} = \delta\cdot\int_0^1 \left(1 - \frac{u}{U}\right) d\eta$ where $\frac{u}{U} = \eta^{\frac{1}{7}}$ $\eta = \frac{y}{\delta}$

Substituting the velocity profile and valuating the integral: $\delta_{disp} = \delta\cdot\int_0^1 \left(1 - \eta^{\frac{1}{7}}\right) d\eta = \frac{\delta}{8}$ Therefore: $\delta_{disp1} = 0.0625\cdot in$ $\delta_{disp6} = 0.0813\cdot in$

We may now determine the height at 6: $H_6 = 1.006\cdot ft$

For a flat plate turbulent boundary layer with 1/7-power law profile: $\delta_1 = L_1 \cdot \dfrac{0.382}{Re_1^{\frac{1}{5}}} = 0.382 \cdot \left(\dfrac{\nu}{U}\right)^{\frac{1}{5}} \cdot L_1^{\frac{4}{5}}$ Solving for L_1:

$$L_1 = \left(\frac{\delta_1}{0.382}\right)^{\frac{5}{4}} \cdot \left(\frac{U}{\nu}\right)^{\frac{1}{4}}$$

$$L_1 = 1.725 \cdot ft$$

To estimate the length between 1 and 6, we determine length necessary to build the BL at section 6:

$$L_6 = \left(\frac{\delta_6}{0.382}\right)^{\frac{5}{4}} \cdot \left(\frac{U}{\nu}\right)^{\frac{1}{4}} = 2.394 \cdot ft$$

Therefore, the distance between 1 and 6 is: $L = L_6 - L_1$

$$L = 0.669 \cdot ft$$

Problem 9.67

[Difficulty: 3]

9.67 Small wind tunnels in an undergraduate laboratory have 305-mm square test sections. Measurements show the boundary layers on the tunnel walls are fully turbulent and well represented by $\frac{1}{7}$-power profiles. At cross section ① with freestream speed $U_1 = 26.1$ m/s, data show that $\delta_1 = 12.2$ mm; at section ②, located downstream, $\delta_2 = 16.6$ mm. Evaluate the change in static pressure between sections ① and ②. Estimate the distance between the two sections.

Given: Laboratory wind tunnel has fixed walls. BL's are well represented by 1/7-power profile. Information at two stations are known:

The given or available data (Table A.9) is

$$U_1 = 26.1 \cdot \frac{m}{s} \quad H = 305 \cdot mm \quad W = 305 \cdot mm \quad \delta_1 = 12.2 \cdot mm \quad \delta_2 = 16.6 \cdot mm \quad \nu = 1.46 \times 10^{-5} \cdot \frac{m^2}{s} \quad \rho = 1.23 \cdot \frac{kg}{m^3}$$

Find: (a) Change in static pressure between 1 and 2
(b) Estimate length of tunnel between stations 1 and 2.

Solution:

Basic equations:

$$\frac{\partial}{\partial t}\int \rho\, dV + \int \rho \cdot \vec{V}\, d\vec{A} = 0 \qquad \text{(Continuity)}$$

Assumptions:
(1) Steady flow
(2) Turbulent, 1/7-power velocity profile in boundary layer
(3) z = constant

Applying continuity between 1 and 6: $A_1 \cdot U_1 = A_2 \cdot U_2$ where A is the effective flow area. In terms of the duct dimensions:

$$\left(W - 2\cdot\delta_{disp1}\right)\left(H - 2\cdot\delta_{disp1}\right)\cdot U_1 = \left(W - 2\cdot\delta_{disp2}\right)\cdot\left(H - 2\cdot\delta_{disp2}\right)\cdot U_2$$

solving for the speed at 2:

$$U_2 = U_1 \cdot \frac{\left(W - 2\cdot\delta_{disp1}\right)\left(H - 2\cdot\delta_{disp1}\right)}{\left(W - 2\cdot\delta_{disp2}\right)\cdot\left(H - 2\cdot\delta_{disp2}\right)}$$

The displacement thickness is determined from: $\delta_{disp} = \delta \cdot \int_0^1 \left(1 - \frac{u}{U}\right) d\eta$ where $\frac{u}{U} = \eta^{\frac{1}{7}}$ $\quad \eta = \frac{y}{\delta}$

Substituting the velocity profile and valuating the integral: $\delta_{disp} = \delta \cdot \int_0^1 \left(1 - \eta^{\frac{1}{7}}\right) d\eta = \frac{\delta}{8}$ Therefore: $\delta_{disp1} = 1.525 \cdot mm$ $\quad \delta_{disp2} = 2.075 \cdot mm$

We may now determine the speed at 2: $U_2 = 26.3\,\frac{m}{s}$

Applying Bernoulli between 1 and 2: $\frac{p_1}{\rho} + \frac{U_1^2}{2} = \frac{p_2}{\rho} + \frac{U_2^2}{2}$ Solving for the pressure change: $\Delta p = \frac{1}{2}\cdot\rho\cdot\left(U_1^2 - U_2^2\right)$

Substituting given values: $\Delta p = -6.16\text{Pa}$

For a flat plate turbulent boundary layer with 1/7-power law profile: $\delta = x \cdot \frac{0.382}{Re_x^{\frac{1}{5}}} = 0.382 \cdot \left(\frac{\nu}{U}\right)^{\frac{1}{5}} \cdot x^{\frac{4}{5}}$ Solving for location at 1:

$$x_1 = \left(\frac{\delta_1}{0.382}\right)^{\frac{5}{4}} \cdot \left(\frac{U_1}{\nu}\right)^{\frac{1}{4}} = 0.494\ \text{m}$$

To estimate the length between 1 and 6, we determine length necessary to build the BL at section 2:

$$x_2 = \left(\frac{\delta_2}{0.382}\right)^{\frac{5}{4}} \cdot \left(\frac{U_2}{\nu}\right)^{\frac{1}{4}} = 0.727\ \text{m}$$

Therefore, the distance between 1 and 2 is: $L = x_2 - x_1$

$$L = 0.233\ \text{m}$$

Problem 9.68

[Difficulty: 3]

9.68 Air flows in a cylindrical duct of diameter $D = 6$ in. At section ①, the turbulent boundary layer is of thickness $\delta_1 = 0.4$ in. and the velocity in the inviscid central core is $U_1 = 80$ ft/s. Further downstream, at section ②, the boundary layer is of thickness $\delta_2 = 1.2$ in. The velocity profile in the boundary layer is approximated well by the $\frac{1}{7}$-power expression. Find the velocity, U_2, in the inviscid central core at the second section, and the pressure drop between the two sections. Does the magnitude of the pressure drop indicate that we are justified in approximating the flow between sections ① and ② as one with zero pressure gradient? Estimate the length of duct between sections ① and ②. Estimate the distance downstream from section ① at which the boundary layer thickness is $\delta = 0.6$ in. Assume standard air.

Given: Data on flow in a duct

Find: Velocity at location 2; pressure drop; length of duct; position at which boundary layer is 20 mm

Solution:

The given data is $\quad D = 6\cdot in \quad \delta_1 = 0.4\cdot in \quad \delta_2 = 1.2\cdot in \quad U_1 = 80\cdot\frac{ft}{s}$

Table A.9 $\quad \rho = 0.00234\cdot\frac{slug}{ft^3} \quad \nu = 1.56\times 10^{-4}\cdot\frac{ft^2}{s}$

Governing equations

Mass
$$\frac{\partial}{\partial t}\int_{CV}\rho\, dV + \int_{CS}\rho\vec{V}\cdot d\vec{A} = 0 \qquad (4.12)$$

In the boundary layer
$$\frac{\delta}{x} = \frac{0.382}{Re_x^{\frac{1}{5}}} \qquad (9.26)$$

In the the inviscid core, the Bernoulli equation holds
$$\frac{p}{\rho} + \frac{V^2}{2} + g\cdot z = \text{constant} \qquad (4.24)$$

Assumptions: (1) Steady flow
(2) No body force (gravity) in *x* direction

For a 1/7-power law profile, from Example 9.4 the displacement thickness is $\delta_{disp} = \frac{\delta}{8}$

Hence $\quad \delta_{disp1} = \frac{\delta_1}{8} \quad \delta_{disp1} = 0.0500\cdot in$

$\delta_{disp2} = \frac{\delta_2}{8} \quad \delta_{disp2} = 0.1500\cdot in$

From the definition of the displacement thickness, to compute the flow rate, the uniform flow at locations 1 and 2 is assumed to take place in the entire duct, minus the displacement thicknesses

$$A_1 = \frac{\pi}{4}\cdot\left(D - 2\cdot\delta_{disp1}\right)^2 \qquad A_1 = 0.1899\cdot ft^2$$

$$A_2 = \frac{\pi}{4}\cdot\left(D - 2\cdot\delta_{disp2}\right)^2 \qquad A_2 = 0.1772\cdot ft^2$$

Mass conservation (Eq. 4.12) leads to U_2

$$\left(-\rho\cdot U_1\cdot A_1\right) + \left(\rho\cdot U_2\cdot A_2\right) = 0 \qquad \text{or} \qquad U_2 = U_1\cdot\frac{A_1}{A_2} \qquad U_2 = 85.7\cdot\frac{ft}{s}$$

The Bernoulli equation applied between locations 1 and 2 is

$$\frac{p_1}{\rho} + \frac{U_1^{\,2}}{2} = \frac{p_2}{\rho} + \frac{U_2^{\,2}}{2}$$

or the pressure drop is $p_1 - p_2 = \Delta p = \frac{\rho}{2}\cdot\left(U_2^{\,2} - U_1^{\,2}\right)$ $\qquad \Delta p = 7.69 \times 10^{-3}\cdot psi$ (Depends on ρ value selected)

The static pressure falls continuously in the entrance region as the fluid in the central core accelerates into a decreasing core.

If we assume the stagnation pressure is atmospheric, a change in pressure of about 0.008 psi is not significant; in addition, the velocity changes by about 5%, again not a large change to within engineering accuracy

To compute distances corresponding to boundary layer thicknesses, rearrange Eq.9.26

$$\frac{\delta}{x} = \frac{0.382}{Re_x^{\frac{1}{5}}} = 0.382\cdot\left(\frac{\nu}{U\cdot x}\right)^{\frac{1}{5}} \qquad \text{so} \qquad x = \left(\frac{\delta}{0.382}\right)^{\frac{5}{4}}\cdot\left(\frac{U}{\nu}\right)^{\frac{1}{4}}$$

Applying this equation to locations 1 and 2 (using $U = U_1$ or U_2 as approximations)

$$x_1 = \left(\frac{\delta_1}{0.382}\right)^{\frac{5}{4}}\cdot\left(\frac{U_1}{\nu}\right)^{\frac{1}{4}} \qquad x_1 = 1.269\cdot ft$$

$$x_2 = \left(\frac{\delta_2}{0.382}\right)^{\frac{5}{4}}\cdot\left(\frac{U_2}{\nu}\right)^{\frac{1}{4}} \qquad x_2 = 5.098\cdot ft$$

$x_2 - x_1 = 3.83\cdot ft$ (Depends on ν value selected)

For location 3 $\qquad \delta_3 = 0.6\cdot in \qquad \delta_{disp3} = \frac{\delta_3}{8} \qquad \delta_{disp3} = 0.075\cdot in$

$$A_3 = \frac{\pi}{4}\cdot\left(D - 2\cdot\delta_{disp3}\right)^2 \qquad A_3 = 0.187\cdot ft^2$$

$$U_3 = U_1\cdot\frac{A_1}{A_3} \qquad U_3 = 81.4\cdot\frac{ft}{s}$$

$$x_3 = \left(\frac{\delta_3}{0.382}\right)^{\frac{5}{4}}\cdot\left(\frac{U_2}{\nu}\right)^{\frac{1}{4}} \qquad x_3 = 2.143\cdot ft$$

$x_3 - x_1 = 0.874\cdot ft$ (Depends on ν value selected)

Problem 9.69

[Difficulty: 3]

9.69 Consider the linear, sinusoidal, and parabolic laminar boundary-layer approximations of Problem 9.10. Compare the momentum fluxes of these profiles. Which is most likely to separate first when encountering an adverse pressure gradient?

Given: Linear, sinusoidal and parabolic velocity profiles

Find: Momentum fluxes

Solution:

The momentum flux is given by $$mf = \int_0^\delta \rho \cdot u^2 \cdot w \, dy$$

where *w* is the width of the boundary layer

For a linear velocity profile $$\frac{u}{U} = \frac{y}{\delta} = \eta \qquad (1)$$

For a sinusoidal velocity profile $$\frac{u}{U} = \sin\left(\frac{\pi}{2} \cdot \frac{y}{\delta}\right) = \sin\left(\frac{\pi}{2} \cdot \eta\right) \qquad (2)$$

For a parabolic velocity profile $$\frac{u}{U} = 2 \cdot \left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2 = 2 \cdot \eta - (\eta)^2 \qquad (3)$$

For each of these $$u = U \cdot f(\eta) \qquad y = \delta \cdot \eta$$

Using these in the momentum flux equation $$mf = \rho \cdot U^2 \cdot \delta \cdot w \cdot \int_0^1 f(\eta)^2 \, d\eta \qquad (4)$$

For the linear profile Eqs. 1 and 4 give $$mf = \rho \cdot U^2 \cdot \delta \cdot w \cdot \int_0^1 \eta^2 \, d\eta \qquad mf = \frac{1}{3} \cdot \rho \cdot U^2 \cdot \delta \cdot w$$

For the sinusoidal profile Eqs. 2 and 4 give $$mf = \rho \cdot U^2 \cdot \delta \cdot w \cdot \int_0^1 \sin\left(\frac{\pi}{2} \cdot \eta\right)^2 d\eta \qquad mf = \frac{1}{2} \cdot \rho \cdot U^2 \cdot \delta \cdot w$$

For the parabolic profile Eqs. 3 and 4 give $$mf = \rho \cdot U^2 \cdot \delta \cdot w \cdot \int_0^1 \left[2 \cdot \eta - (\eta)^2\right]^2 d\eta \qquad mf = \frac{8}{15} \cdot \rho \cdot U^2 \cdot \delta \cdot w$$

The linear profile has the smallest momentum, so would be most likely to separate

Problem 9.70

[Difficulty: 3]

9.70 Perform a cost-effectiveness analysis on a typical large tanker used for transporting petroleum. Determine, as a percentage of the petroleum cargo, the amount of petroleum that is consumed in traveling a distance of 2000 miles. Use data from Example 9.5, and the following: Assume the petroleum cargo constitutes 75% of the total weight, the propeller efficiency is 70%, the wave drag and power to run auxiliary equipment constitute losses equivalent to an additional 20%, the engines have a thermal efficiency of 40%, and the petroleum energy is 20,000 Btu/lbm. Also compare the performance of this tanker to that of the Alaskan Pipeline, which requires about 120 Btu of energy for each ton-mile of petroleum delivery.

Given: Data on a large tanker

Find: Cost effectiveness of tanker; compare to Alaska pipeline

Solution:

The given data is $L = 360\cdot m$ $\quad B = 70\cdot m$ $\quad D = 25\cdot m$ $\quad \rho = 1020\cdot\frac{kg}{m^3}$ $\quad U = 6.69\cdot\frac{m}{s}$ $\quad x = 2000\cdot mi$

$P = 9.7\cdot MW$ $\quad P = 1.30\times 10^4\cdot hp$ (Power consumed by drag)

The power to the propeller is $P_{prop} = \frac{P}{70\cdot\%}$ $\quad P_{prop} = 1.86\times 10^4\cdot hp$

The shaft power is $P_s = 120\%\cdot P_{prop}$ $\quad P_s = 2.23\times 10^4\cdot hp$

The efficiency of the engines is $\eta = 40\cdot\%$

Hence the heat supplied to the engines is $Q = \frac{P_s}{\eta}$ $\quad Q = 1.42\times 10^8\cdot\frac{BTU}{hr}$

The journey time is $t = \frac{x}{U}$ $\quad t = 134\cdot hr$

The total energy consumed is $Q_{total} = Q\cdot t$ $\quad Q_{total} = 1.9\times 10^{10}\cdot BTU$

From buoyancy the total ship weight equals the displaced seawater volume

$M_{ship}\cdot g = \rho\cdot g\cdot L\cdot B\cdot D$ $\quad M_{ship} = \rho\cdot L\cdot B\cdot D$ $\quad M_{ship} = 1.42\times 10^9\cdot lb$

Hence the mass of oil is $M_{oil} = 75\%\cdot M_{ship}$ $\quad M_{oil} = 1.06\times 10^9\cdot lb$

The chemical energy stored in the petroleum is $q = 20000\cdot\frac{BTU}{lb}$

The total chemical energy is $E = q\cdot M_{oil}$ $\quad E = 2.13\times 10^{13}\cdot BTU$

The equivalent percentage of petroleum cargo used is then $\frac{Q_{total}}{E} = 0.089\cdot\%$

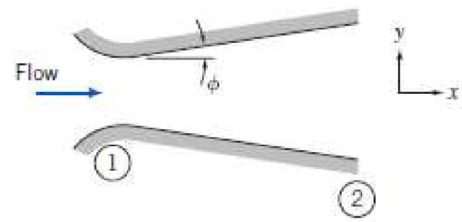
The Alaska pipeline uses $e_{pipeline} = 120\cdot\frac{BTU}{ton\cdot mi}$ but for the ship $\quad e_{ship} = \frac{Q_{total}}{M_{oil}\cdot x}$ $\quad e_{ship} = 17.8\cdot\frac{BTU}{ton\cdot mi}$

The ship uses only about 15% of the energy of the pipeline!

Problem 9.71

[Difficulty: 2]

9.71 Consider the plane-wall diffuser shown in Fig. P9.71. First, assume the fluid is inviscid. Describe the flow pattern, including the pressure distribution, as the diffuser angle ϕ is increased from zero degrees (parallel walls). Second, modify your description to allow for boundary layer effects. Which fluid (inviscid or viscous) will generally have the highest exit pressure?



Given: Plane-wall diffuser

Find: (a) For inviscid flow, describe flow pattern and pressure distribution as φ is increased from zero
(b) Redo part (a) for a viscous fluid
(c) Which fluid will have the higher exit pressure?

Solution:

For the inviscid fluid:
With $\varphi = 0$ (straight channel) there will be no change in the velocity, and hence no pressure gradient.
With $\varphi > 0$ (diverging channel) the velocity will decrease, and hence the pressure will increase.

For the viscous fluid:
With $\varphi = 0$ (straight channel) the boundary layer will grow, decreasing the effective flow area. As a result, velocity will increase, and the pressure will drop.
With $\varphi > 0$ (diverging channel) the pressure increase due to the flow divergence will cause in increase in the rate of boundary layer growth. If φ is too large, the flow will separate from one or both walls.

The inviscid fluid will have the higher exit pressure. (The pressure gradient with the real fluid is reduced by the boundary layer development for all values of φ.)

Problem 9.72

[Difficulty: 4]

*9.72 Table 9.1 shows the numerical results obtained from Blasius exact solution of the laminar boundary-layer equations. Plot the velocity distribution (note that from Eq. 9.13 we see that $\eta \approx 5.0\frac{y}{\delta}$). On the same graph, plot the turbulent velocity distribution given by the $\frac{1}{7}$-power expression of Eq. 9.24. Which is most likely to separate first when encountering an adverse pressure gradient? To justify your answer, compare the momentum fluxes of these profiles (the laminar data can be integrated using a numerical method such as Simpson's rule).

Given: Laminar (Blasius) and turbulent (1/7-power) velocity distributions

Find: Plot of distributions; momentum fluxes

Solution:

The momentum flux is given by $$mf = \int_0^\delta \rho \cdot u^2 \, dy$$ per unit width of the boundary layer

Using the substitutions $$\frac{u}{U} = f(\eta) \qquad \frac{y}{\delta} = \eta$$

the momentum flux becomes $$mf = \rho \cdot U^2 \cdot \delta \cdot \int_0^1 f(\eta)^2 \, d\eta$$

For the Blasius solution a numerical evaluation (a Simpson's rule) of the integral is needed

$$mf_{lam} = \rho \cdot U^2 \cdot \delta \cdot \frac{\Delta\eta}{3} \cdot \left(f(\eta_0)^2 + 4 \cdot f(\eta_1)^2 + 2 \cdot f(\eta_2)^2 + f(\eta_N)^2 \right)$$

where $\Delta\eta$ is the step size and N the number of steps

The result for the Blasius profile is $$mf_{lam} = 0.525 \cdot \rho \cdot U^2 \cdot \delta$$

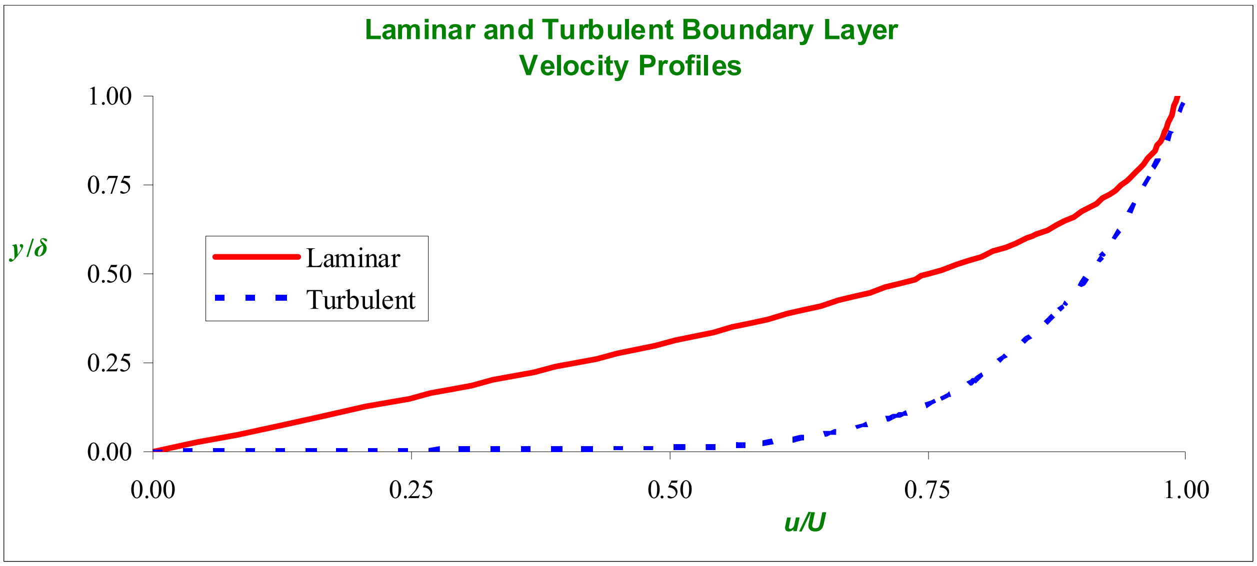
For a 1/7 power velocity profile $$mf_{turb} = \rho \cdot U^2 \cdot \delta \cdot \int_0^1 \eta^{\frac{2}{7}} \, d\eta \qquad mf_{turb} = \frac{7}{9} \cdot \rho \cdot U^2 \cdot \delta$$

The laminar boundary has less momentum, so will separate first when encountering an adverse pressure gradient. The computed results were generated in *Excel* and are shown below:

(Table 9.1) (Simpsons Rule)

η	Laminar u/U	Weight w	Weight x $(u/U)^2$
0.0	0.000	1	0.00
0.5	0.166	4	0.11
1.0	0.330	2	0.22
1.5	0.487	4	0.95
2.0	0.630	2	0.79
2.5	0.751	4	2.26
3.0	0.846	2	1.43
3.5	0.913	4	3.33
4.0	0.956	2	1.83
4.5	0.980	4	3.84
5.0	0.992	1	0.98
		Simpsons':	**0.525**

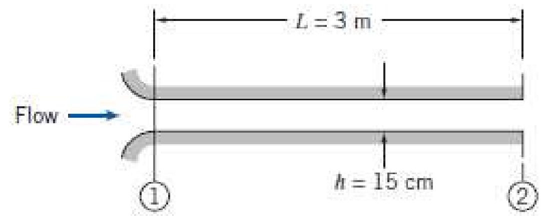
$y/\delta = \eta$	t u/U
0.0	0.00
0.0125	0.53
0.025	0.59
0.050	0.65
0.10	0.72
0.15	0.76
0.2	0.79
0.4	0.88
0.6	0.93
0.8	0.97
1.0	1.00



Problem 9.73

[Difficulty: 5]

9.73 Cooling air is supplied through the wide, flat channel shown. For minimum noise and disturbance of the outlet flow, laminar boundary layers must be maintained on the channel walls. Estimate the maximum inlet flow speed at which the outlet flow will be laminar. Assuming parabolic velocity profiles in the laminar boundary layers, evaluate the pressure drop, $p_1 - p_2$. Express your answer in inches of water.



Given: Channel flow with laminar boundary layers

Find: Maximum inlet speed for laminar exit; Pressure drop for parabolic velocity in boundary layers

Solution:

Basic equations: $Re_{trans} = 5\times 10^5$ $\quad \dfrac{\delta}{x} = \dfrac{5.48}{\sqrt{Re_x}}$ $\quad \dfrac{p}{\rho} + \dfrac{V^2}{2} + g\cdot z = \text{const}$

Assumptions: 1) Steady flow 2) Incompressible 3) z = constant

From Table A.10 at 20ºC $\quad \nu = 1.50\times 10^{-5}\cdot\dfrac{m^2}{s}$ $\quad \rho = 1.21\cdot\dfrac{kg}{m^3}$ $\quad L = 3\cdot m$ $\quad h = 15\cdot cm$

Then $\quad Re_{trans} = \dfrac{U_{max}\cdot L}{\nu}$ $\quad U_{max} = \dfrac{Re_{trans}\cdot\nu}{L}$ $\quad U_{max} = 2.50\,\dfrac{m}{s}$ $\quad U_1 = U_{max}$ $\quad U_1 = 2.50\,\dfrac{m}{s}$

For $\quad Re_{trans} = 5\times 10^5$ $\quad \delta_2 = L\cdot\dfrac{5.48}{\sqrt{Re_{trans}}}$ $\quad \delta_2 = 0.0232\,m$

For a parabolic profile $\quad \dfrac{\delta_{disp}}{\delta} = \displaystyle\int_0^1 \left(1 - \frac{u}{U}\right)d\lambda = \int_0^1 \left(1 - 2\cdot\lambda + \lambda^2\right)d\lambda = \frac{1}{3}$ $\quad$ where δ_{trans} is the displacement thickness

$\delta_{disp2} = \dfrac{1}{3}\cdot\delta_2$ $\quad \delta_{disp2} = 0.00775\,m$

From continuity $\quad U_1\cdot w\cdot h = U_2\cdot w\cdot\left(h - 2\cdot\delta_{disp2}\right)$ $\quad U_2 = U_1\cdot\dfrac{h}{h - 2\cdot\delta_{disp2}}$ $\quad U_2 = 2.79\,\dfrac{m}{s}$

Since the boundary layers do not meet Bernoulli applies in the core

$$\frac{p_1}{\rho} + \frac{U_1^2}{2} = \frac{p_2}{\rho} + \frac{U_2^2}{2} \qquad \Delta p = p_1 - p_2 = \frac{\rho}{2}\cdot\left(U_2^2 - U_1^2\right)$$

$$\Delta p = \frac{\rho}{2}\cdot\left(U_2^2 - U_1^2\right) \qquad \Delta p = 0.922\,Pa$$

From hydrostatics $\quad \Delta p = \rho_{H2O}\cdot g\cdot\Delta h$ $\quad$ with $\quad \rho_{H2O} = 1000\cdot\dfrac{kg}{m^3}$

$$\Delta h = \frac{\Delta p}{\rho_{H2O}\cdot g} \qquad \Delta h = 0.0940\cdot mm \qquad \Delta h = 0.00370\cdot in$$

Problem 9.74

[Difficulty: 3]

9.74 Boundary-layer separation occurs when the shear stress at the surface becomes zero. Assume a polynomial representation for the laminar boundary layer of the form, $u/U = a + b\lambda + c\lambda^2 + d\lambda^3$, where $\lambda = y/\delta$. Specify boundary conditions on the velocity profile at separation. Find appropriate constants, a, b, c, and d, for the separation profile. Calculate the shape factor H at separation. Plot the profile and compare with the parabolic approximate profile.

Given: Laminar boundary layer with velocity profile $\frac{u}{U} = a + b\cdot\lambda + c\cdot\lambda^2 + d\cdot\lambda^3$ $\qquad \lambda = \frac{y}{\delta}$

Separation occurs when shear stress at the surface becomes zero.

Find:
(a) Boundary conditions on the velocity profile at separation
(b) Appropriate constants a, b, c, d for the profile
(c) Shape factor H at separation
(d) Plot the profile and compare with the parabolic approximate profile

Solution:

Basic equations: $\frac{u}{U} = 2\cdot\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$ $\qquad$ (Parabolic profile)

The boundary conditions for the separation profile are: $\quad$ at $y = 0 \quad u = 0 \quad \tau = \mu\cdot\frac{du}{dy} = 0$

at $y = \delta \quad u = U \quad \tau = \mu\cdot\frac{du}{dy} = 0$

Four boundary conditions for four coefficients a, b, c, d

The velocity gradient is defined as: $\frac{du}{dy} = \frac{U}{\delta}\cdot\left[\frac{d}{d\lambda}\left(\frac{u}{U}\right)\right] = \frac{U}{\delta}\cdot\left(b + 2\cdot c\cdot\lambda + 3\cdot d\cdot\lambda^2\right)$

Applying the boundary conditions: $y = 0 \quad \lambda = 0 \quad \frac{u}{U} = a + b\cdot 0 + c\cdot 0^2 + d\cdot 0^3 = 0$ $\quad$ Therefore: $\quad a = 0$

$\frac{du}{dy} = \frac{U}{\delta}\cdot\left(b + 2\cdot c\cdot 0 + 3\cdot d\cdot 0^2\right) = 0$ $\quad$ Therefore: $\quad b = 0$

The velocity profile and gradient may now be written as: $\frac{u}{U} = c\cdot\lambda^2 + d\cdot\lambda^3 \qquad \frac{du}{dy} = \frac{U}{\delta}\cdot\left(2\cdot c\cdot\lambda + 3\cdot d\cdot\lambda^2\right)$

Applying the other boundary conditions:

$y = \delta \quad \lambda = 1 \quad \frac{u}{U} = c\cdot 1^2 + d\cdot 1^3 = 1 \qquad c + d = 1$

$\frac{du}{dy} = \frac{U}{\delta}\cdot\left(2\cdot c\cdot 1 + 3\cdot d\cdot 1^2\right) = 0 \qquad 2\cdot c + 3\cdot d = 0$

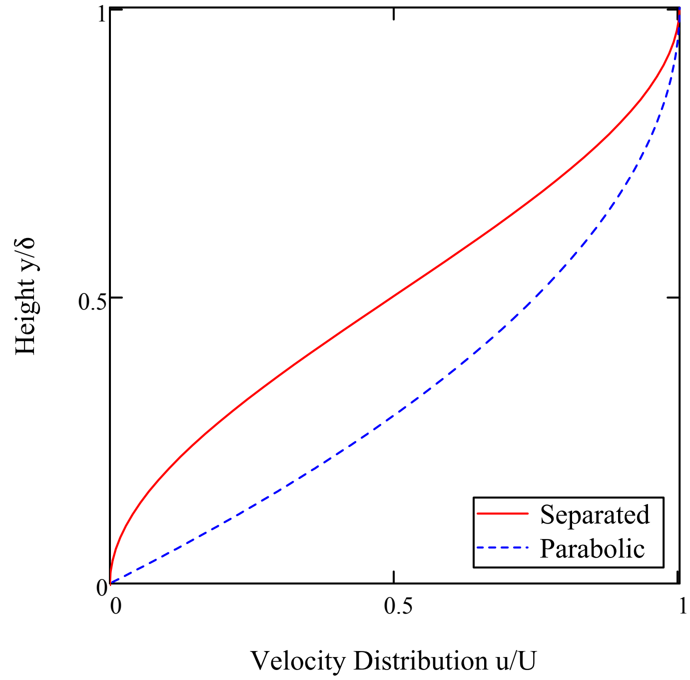
Solving this system of equations yields: $\quad c = 3 \quad d = -2$

The velocity profile is: $\frac{u}{U} = 3\cdot\lambda^2 - 2\cdot\lambda^3$ $\quad$ The shape parameter is defined as: $\quad H = \frac{\delta_{disp}}{\theta} = \frac{\delta_{disp}}{\delta}\cdot\frac{\delta}{\theta}$

$\frac{\delta_{disp}}{\delta} = \int_0^1 \left(1 - 3\cdot\lambda^2 + 2\cdot\lambda^3\right) d\lambda = 1 - 1 + \frac{1}{2} = \frac{1}{2} \qquad \frac{\theta}{\delta} = \int_0^1 \left(3\cdot\lambda^2 - 2\cdot\lambda^3\right)\cdot\left(1 - 3\cdot\lambda^2 + 2\cdot\lambda^3\right) d\lambda$ $\quad$ Expanding out the integrand yields:

$\frac{\theta}{\delta} = \int_0^1 \left(3\cdot\lambda^2 - 2\cdot\lambda^3 - 9\cdot\lambda^4 + 12\cdot\lambda^5 - 4\cdot\lambda^6\right) d\lambda = 1 - \frac{1}{2} - \frac{9}{5} + 2 - \frac{4}{7} = \frac{9}{70}$ $\quad$ Thus $\quad H = \frac{1}{2}\times\frac{70}{9} \qquad H = 3.89$

The two velocity profiles are plotted here:



Problem 9.75

[Difficulty: 4]

9.75 For flow over a flat plate with zero pressure gradient, will the shear stress increase, decrease, or remain constant along the plate? Justify your answer. Does the momentum flux increase, decrease, or remain constant as the flow proceeds along the plate? Justify your answer. Compare the behavior of laminar flow and turbulent flow (both from the leading edge) over a flat plate. At a given distance from the leading edge, which flow will have the larger boundary-layer thickness? Does your answer depend on the distance along the plate? How would you justify your answer?

Discussion: Shear stress decreases along the plate because the freestream flow speed remains constant while the boundary-layer thickness increases.

The momentum flux decreases as the flow proceeds along the plate. Momentum thickness θ (actually proportional to the defect in momentum within the boundary layer) increases, showing that momentum flux decreases. The forct that must be applied to hold the plate stationary reduces the momentum flux of the stream and boundary layer.

The laminar boundary layer has less shear stress than the turbulent boundary layer. Therefore laminar boundary layer flow from the leading edge produces a thinner boundary layer and less shear stress everywhere along the plate than a turbulent boundary layer from the leading edge.

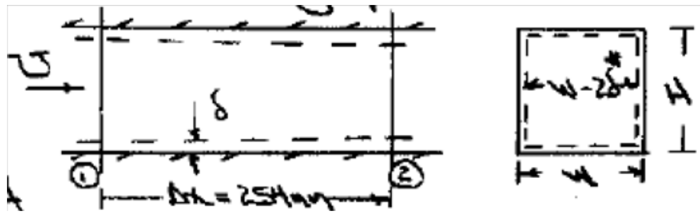
Since both boundary layers continue to grow with increasing distance from the leading edge, and the turbulent boundary layer continues to grow more rapidly because of its higher shear stress, this comparison will be the same no matter the distance from the leading edge.

Problem 9.76

[Difficulty: 5]

9.76 A laboratory wind tunnel has a test section that is square in cross section, with inlet width W_1 and height H_1, each equal to 1 ft. At freestream speed $U_1 = 80$ ft/s, measurements show the boundary-layer thickness is $\delta_1 = 0.4$ in. with a $\frac{1}{7}$-power turbulent velocity profile. The pressure gradient in this region is given approximately by $dp/dx = -0.035$ in. H_2O/in. Evaluate the reduction in effective flow area caused by the boundary layers on the tunnel bottom, top, and walls at section ①. Calculate the rate of change of boundary-layer momentum thickness, $d\theta/dx$, at section ①. Estimate the momentum thickness at the end of the test section, located at $L = 10$ in downstream.

Given: Laboratory wind tunnel has fixed walls. BL's are well represented by 1/7-power profile. Information at two stations are known:



The given or available data (Table A.9) is

$$U_1 = 80\cdot\frac{ft}{s} \qquad H_1 = 1\cdot ft \qquad W_1 = 1\cdot ft \qquad \delta_1 = 0.4\cdot in$$

$$\frac{dp}{dx} = -0.035\cdot\frac{in\cdot H_2O}{in} \qquad L = 10\cdot in \qquad \nu = 1.62\times 10^{-4}\cdot\frac{ft^2}{s} \qquad \rho = 0.00234\cdot\frac{slug}{ft^3}$$

Find:
(a) Reduction in effective flow area at section 1
(b) $d\theta/dx$ at section 1
(c) θ at section 2

Solution:

Basic equations:

$$\frac{\partial}{\partial t}\int \rho\, dV + \int \rho\cdot\vec{V}\, d\vec{A} = 0 \qquad \text{(Continuity)}$$

$$\frac{\tau_w}{\rho} = \frac{d}{dx}\left(U^2\cdot\theta\right) + \delta_{disp}\cdot U\cdot\left(\frac{d}{dx}U\right) \qquad \text{(Momentum integral equation)}$$

Assumptions:
(1) Steady flow
(2) Turbulent, 1/7-power velocity profile in boundary layer
(3) z = constant

The percent reduction in flow area at 1 is given as:
$$\frac{A_{eff} - A}{A} = \frac{\left(W_1 - 2\cdot\delta_{disp}\right)\cdot\left(H_1 - 2\cdot\delta_{disp}\right) - W_1\cdot H_1}{W_1\cdot H_1}$$

The displacement thickness is determined from:
$$\delta_{disp} = \delta\cdot\int_0^1\left(1 - \frac{u}{U}\right)d\eta \qquad \text{where} \qquad \frac{u}{U} = \eta^{\frac{1}{7}} \qquad \eta = \frac{y}{\delta}$$

Substituting the velocity profile and valuating the integral:
$$\delta_{disp} = \delta\cdot\int_0^1\left(1 - \eta^{\frac{1}{7}}\right)d\eta = \frac{\delta}{8} \qquad \text{Therefore:} \qquad \delta_{disp1} = 0.0500\cdot in$$

$$\text{Thus:} \qquad \frac{A_{eff} - A}{A} = -1.66\cdot\%$$

Solving the momentum integral equation for the momentum thickness gradient: $\dfrac{d\theta}{dx} = \dfrac{\tau_w}{\rho \cdot U^2} - (H+2)\cdot\dfrac{\theta}{U}\cdot\dfrac{dU}{dx}$

At station 1: $\dfrac{\tau_{w1}}{\rho \cdot {U_1}^2} = 0.0233\cdot\left(\dfrac{\nu}{U_1\cdot\delta_1}\right)^{\frac{1}{4}}$ $\quad 0.0233\cdot\left(\dfrac{\nu}{U_1\cdot\delta_1}\right)^{\frac{1}{4}} = 2.057\times 10^{-3}$

$\dfrac{\theta}{\delta} = \displaystyle\int_0^1 \dfrac{u}{U}\cdot\left(1-\dfrac{u}{U}\right)d\eta = \int_0^L \left(\eta^{\frac{1}{7}} - \eta^{\frac{2}{7}}\right)d\eta = \dfrac{7}{8} - \dfrac{7}{9} = \dfrac{7}{72}$ Thus: $\theta_1 = \dfrac{7}{72}\cdot\delta_1 = 0.0389\cdot\text{in}$ $\quad H = \dfrac{\delta_{disp1}}{\theta_1} = 1.286$

Now outside the boundary layer $p + \dfrac{1}{2}\cdot\rho\cdot U^2 = \text{constant}$ from the Bernoulli equation. Then: $\dfrac{dp}{dx} = -\rho\cdot U\cdot\dfrac{dU}{dx}$

Solving for the velocity gradient: $\dfrac{1}{U}\cdot\dfrac{dU}{dx} = -\dfrac{1}{\rho\cdot U^2}\cdot\dfrac{dp}{dx} = 0.1458\cdot\dfrac{1}{\text{ft}}$ Substituting all of this information into the above expression:

$$\frac{d\theta}{dx} = 4.89\times 10^{-4} = 0.00587\cdot\frac{\text{in}}{\text{ft}}$$

We approximate the momentum thickness at 2 from: $\theta_2 = \theta_1 + \dfrac{d\theta}{dx}\cdot L$ $\qquad \theta_2 = 0.0438\cdot\text{in}$

Problem 9.77

[Difficulty: 5]

9.77 The variable-wall concept is proposed to maintain constant boundary-layer thickness in the wind tunnel of Problem 9.76. Beginning with the initial conditions of Problem 9.76, evaluate the freestream velocity distribution needed to maintain constant boundary-layer thickness. Assume constant width, W_1. Estimate and plot the top-height settings along the test section from $x = 0$ at section (1) to $x = 10$ in. at section (2) downstream.

Given: Laboratory wind tunnel of Problem 9.76 with a movable top wall:

The given or available data (Table A.9) is

$$U_1 = 80 \cdot \frac{ft}{s} \qquad H_1 = 1 \cdot ft \qquad W = 1 \cdot ft \qquad \delta = 0.4 \cdot in \qquad L = 10 \cdot in \qquad \nu = 1.57 \times 10^{-4} \cdot \frac{ft^2}{s} \qquad \rho = 0.00234 \cdot \frac{slug}{ft^3}$$

Find: (a) Velocity distribution needed for constant boundary layer thickness
(b) Tunnel height distribution h(x) from 0 to L

Solution:

Basic equations:

$$\frac{\partial}{\partial t}\int \rho\, dV + \int \rho \cdot \vec{V}\, d\vec{A} = 0 \qquad \text{(Continuity)}$$

$$\frac{\tau_w}{\rho} = \frac{d}{dx}\left(U^2 \cdot \theta\right) + \delta_{disp} \cdot U \cdot \left(\frac{d}{dx}U\right) \qquad \text{(Momentum integral equation)}$$

Assumptions:
(1) Steady flow
(2) Incompressible flow
(3) Turbulent, 1/7-power velocity profile in boundary layer
(4) δ = constant

From the 1/7-power profile: $\delta_{disp} = \frac{\delta}{8} \qquad \theta = \frac{7}{72} \cdot \delta \qquad H = \frac{72}{56}$

After applying assumptions, the momentum integral equation is: $\frac{\tau_w}{\rho \cdot U^2} = (H + 2) \cdot \frac{\theta}{U} \cdot \left(\frac{d}{dx}U\right)$

To integrate, we need to make an assumption about the wall shear stress:

Case 1: assume constant τ_w: $U \cdot dU = \frac{\tau_w}{\rho \cdot \theta \cdot (H + 2)} \cdot dx$ Integrating: $\frac{U^2 - U_1^{\,2}}{2} = \frac{\tau_w}{\rho \cdot \theta \cdot (H + 2)} \cdot x$

$$\frac{U}{U_1} = \sqrt{1 + \frac{2 \cdot \tau_w}{\rho \cdot U_1^{\,2}} \cdot \frac{x}{\theta \cdot (H + 2)}}$$ which may be rewritten as: $$\frac{U}{U_1} = \sqrt{1 + \frac{C_f}{\theta \cdot (H + 2)} \cdot x}$$

Case 2: assume τ_w has the form: $\tau_w = 0.0233 \cdot \rho \cdot U^2 \cdot \left(\frac{\nu}{U \cdot \delta}\right)^{\frac{1}{4}}$ Substituting and rearranging yields the following expression:

$$\frac{\tau_w}{\rho \cdot U^2} = 0.0233 \cdot \left(\frac{\nu}{U \cdot \delta}\right)^{\frac{1}{4}} = (H+2) \cdot \frac{\theta}{U} \cdot \left(\frac{d}{dx} U\right) \quad \text{or} \quad \frac{dU}{U^{0.75}} = 0.0233 \cdot \left(\frac{\nu}{\delta}\right)^{0.25} \cdot \frac{dx}{(H+2) \cdot \theta}$$

Integrating this yields:

$$4 \cdot \left(U^{0.25} - U_1^{\ 0.25}\right) = 0.0233 \cdot \left(\frac{\nu}{\delta}\right)^{0.25} \cdot \frac{x}{(H+2) \cdot \theta}$$

or:

$$\frac{U}{U_1} = \left[1 + 0.00583 \cdot \left(\frac{\nu}{U_1 \cdot \delta}\right)^{0.25} \cdot \frac{x}{(H+2) \cdot \theta}\right]^4$$

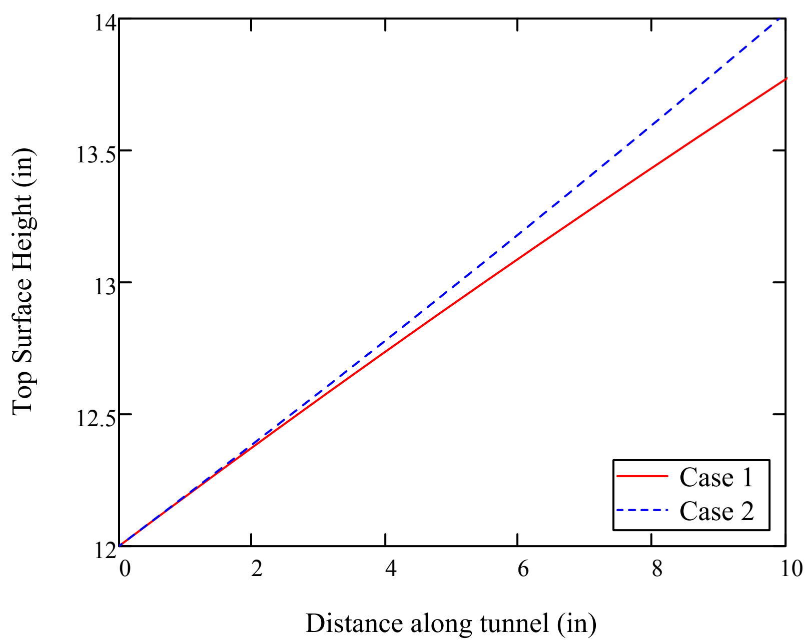
From continuity: $U_1 \cdot A_1 = U \cdot A$ which may be rewritten as: $U_1 \cdot \left(W - 2 \cdot \delta_{disp}\right)\left(H_1 - 2 \cdot \delta_{disp}\right) = U \cdot \left(W - 2.\delta_{disp}\right) \cdot \left(h - 2 \cdot \delta_{disp}\right)$

Thus: $\frac{A}{A_1} = \left(\frac{U}{U_1}\right)^{-1}$ and $\frac{h - 2 \cdot \delta_{disp}}{H_1 - 2 \cdot \delta_{disp}} = \frac{U_1}{U}$ solving for h: $\frac{h}{W} = \left(1 - 2 \cdot \frac{\delta_{disp}}{H_1}\right) \cdot \frac{U_1}{U} + 2 \cdot \frac{\delta_{disp}}{H_1}$

Evaluating using the given data: $\delta_{disp} = \frac{\delta}{8} = 0.0500 \cdot in$ $\quad \theta = \frac{7}{72} \cdot \delta = 0.0389 \cdot in$ $\quad Re_{\delta 1} = \frac{U_1 \cdot \delta}{\nu} = 1.699 \times 10^4$

$C_f = 0.0466 \cdot Re_{\delta 1}^{\ -0.25} = 4.082 \times 10^{-3}$

The results for both wall profiles are shown in the plot here:



Problem 9.78

[Difficulty: 3]

9.78 A flat-bottomed barge, 80 ft long and 35 ft wide, submerged to a depth of 5 ft, is to be pushed up a river (the river water is at 60°F). Estimate and plot the power required to overcome skin friction for speeds ranging up to 15 mph.

Given: Barge pushed upriver

$L = 80 \cdot ft \quad B = 35 \cdot ft \quad D = 5 \cdot ft$ From Table A.7: $\nu = 1.321 \times 10^{-5} \cdot \frac{ft^2}{s} \quad \rho = 1.94 \cdot \frac{slug}{ft^3}$

Find: Power required to overcome friction; Plot power versus speed

Solution:

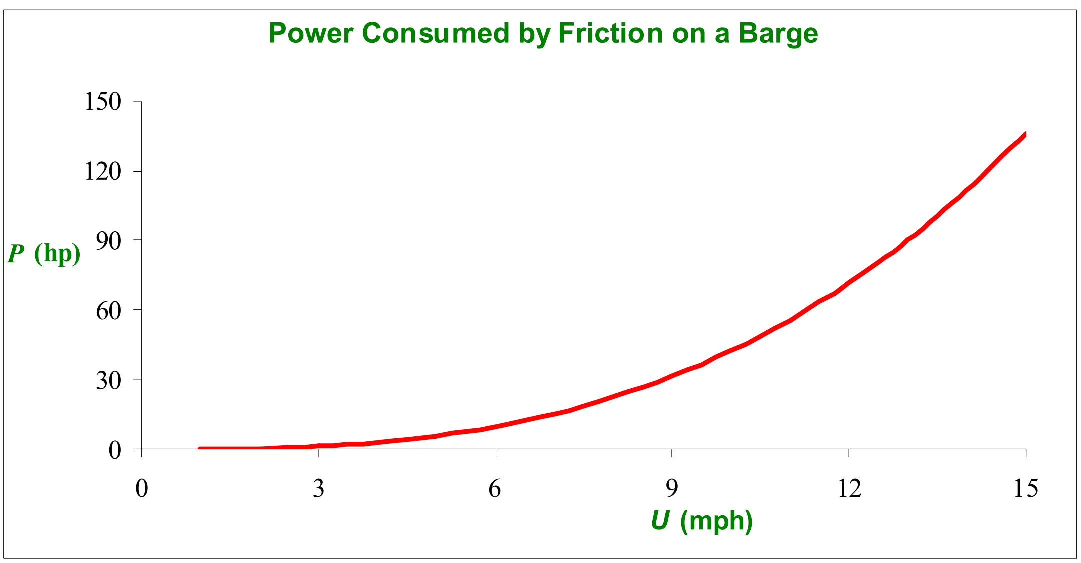
Basic equations:

$$C_D = \frac{F_D}{\frac{1}{2} \cdot \rho \cdot U^2 \cdot A} \quad (9.32) \qquad C_D = \frac{0.455}{\left(\log\left(Re_L\right)\right)^{2.58}} - \frac{1610}{Re_L} \quad (9.37b) \qquad Re_L = \frac{U \cdot L}{\nu}$$

From Eq. 9.32 $\quad F_D = C_D \cdot A \cdot \frac{1}{2} \cdot \rho \cdot U^2 \quad$ and $\quad A = L \cdot (B + 2 \cdot D) \quad A = 3600 \, ft^2$

The power consumed is $\quad P = F_D \cdot U \quad P = C_D \cdot A \cdot \frac{1}{2} \cdot \rho \cdot U^3 \quad$ The calculated results and the plot were generated in *Excel*:

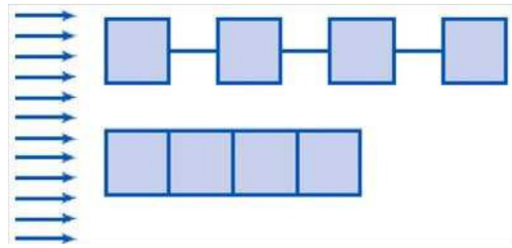
U (mph)	*Re* $_L$	*C* $_D$	*P* (hp)
1	9.70E+06	0.00285	0.0571
2	1.94E+07	0.00262	0.421
3	2.91E+07	0.00249	1.35
4	3.88E+07	0.00240	3.1
5	4.85E+07	0.00233	5.8
6	5.82E+07	0.00227	9.8
7	6.79E+07	0.00222	15
8	7.76E+07	0.00219	22
9	8.73E+07	0.00215	31
10	9.70E+07	0.00212	42
11	1.07E+08	0.00209	56
12	1.16E+08	0.00207	72
13	1.26E+08	0.00205	90
14	1.36E+08	0.00203	111
15	1.45E+08	0.00201	136



Problem 9.79

[Difficulty: 3]

9.79 Repeat Problem 9.46, except that the water flow is now at 30 ft/s. (Use formulas for C_D from Section 9.7.)



Given: Pattern of flat plates

Find: Drag on separate and composite plates

Solution:

Basic equations: $C_D = \dfrac{F_D}{\frac{1}{2}\cdot\rho\cdot V^2\cdot A}$

For separate plates $L = 3\cdot in$ $W = 3\cdot in$ $A = W\cdot L$ $A = 9.000\cdot in^2$ $V = 30\cdot\dfrac{ft}{s}$

From Table A.8 at 70ºF $\nu = 1.06\times 10^{-5}\cdot\dfrac{ft^2}{s}$ $\rho = 1.93\cdot\dfrac{slug}{ft^3}$

First determine the Reynolds number $Re_L = \dfrac{V\cdot L}{\nu}$ $Re_L = 7.08\times 10^5$ so use Eq. 9.34

$C_D = \dfrac{0.0742}{Re_L^{\frac{1}{5}}}$ $C_D = 0.00502$

The drag (one side) is then $F_D = C_D\cdot\dfrac{1}{2}\cdot\rho\cdot V^2\cdot A$ $F_D = 0.272\cdot lbf$

This is the drag on one plate. The total drag is then $F_{Total} = 4\cdot F_D$ $F_{Total} = 1.09\cdot lbf$

For both sides: $2\cdot F_{Total} = 2.18\cdot lbf$

For the composite plate $L = 4\times 3\cdot in$ $L = 12.000\cdot in$ $A = W\cdot L$ $A = 36\cdot in^2$

First determine the Reylolds number $Re_L = \dfrac{V\cdot L}{\nu}$ $Re_L = 2.83\times 10^6$ so use Eq. 9.34

$C_D = \dfrac{0.0742}{Re_L^{\frac{1}{5}}}$ $C_D = 0.00380$

The drag (one side) is then $F_D = C_D\cdot\dfrac{1}{2}\cdot\rho\cdot V^2\cdot A$ $F_D = 0.826\cdot lbf$ For both sides: $2\cdot F_D = 1.651\cdot lbf$

The drag is much lower on the composite compared to the separate plates. This is because τ_w is largest near the leading edges and falls off rapidly; in this problem the separate plates experience leading edges four times!

Problem 9.80

[Difficulty: 3]

9.80 A towboat for river barges is tested in a towing tank. The towboat model is built at a scale ratio of 1:13.5. Dimensions of the model are overall length 3.5 m, beam 1 m, and draft 0.2 m. (The model displacement in fresh water is 5500 N.) Estimate the average length of wetted surface on the hull. Calculate the skin friction drag force of the prototype at a speed of 7 knots relative to the water.

Given: Towboat model at 1:13.5 scale to be tested in towing tank.

$L_m = 3.5 \cdot m \quad B_m = 1 \cdot m \quad d_m = 0.2 \cdot m \quad U_p = 7 \cdot knot = 3.601 \frac{m}{s} \quad Disp_m = 5500 \cdot N$

Find: (a) Estimate average length of wetted surface on the hull
(b) Calculate skin friction drag force on the prototype

Solution:

Basic equations:

$$F_D = C_D \cdot \frac{1}{2} \cdot \rho \cdot U^2 \cdot A \qquad \text{(Drag)}$$

$$C_D = \frac{0.455}{\left(\log\left(Re_L\right)\right)^{2.58}} - \frac{1610}{Re_L} \qquad \text{(Drag Coefficient)}$$

We will represent the towboat as a rectangular solid of length L_{av}, with the displacement of the boat. From buoyancy:

$$W = \rho \cdot g \cdot V = \rho \cdot g \cdot L_{av} \cdot B_m \cdot d_m \quad \text{thus:} \quad L_{av} = \frac{W}{\rho \cdot g \cdot B_m \cdot d_m} \qquad L_{av} = 2.80 \cdot m$$

For the prototype: $L_p = 13.5 \cdot L_{av}$ $\qquad L_p = 37.9 \cdot m$

The Reynolds number is: $Re_L = \frac{U_p \cdot L_p}{\nu}$ $\qquad Re_L = 1.36 \times 10^8$

This flow is predominantly turbulent, so we will use a turbulent analysis.

The drag coefficient is: $C_D = \frac{0.455}{\left(\log\left(Re_L\right)\right)^{2.58}} - \frac{1610}{Re_L} = 0.00203$

The area is: $A = 13.5^2 \cdot L_{av} \cdot \left(B_m + 2 \cdot d_m\right) = 716\, m^2$

The drag force would then be: $F_D = C_D \cdot \frac{1}{2} \cdot \rho \cdot U_p^{\,2} \cdot A$ $\qquad F_D = 9.41 \cdot kN$

This is skin friction only.

Problem 9.81

[Difficulty: 3]

9.81 A jet transport aircraft cruises at 12 km in steady level flight at 800 km/h. Model the aircraft fuselage as a circular cylinder with diameter $D = 4$ m and length $L = 38$ m. Neglecting compressibility effects, estimate the skin friction drag force on the fuselage. Evaluate the power needed to overcome this force.

Given: Aircraft cruising at 12 km

Find: Skin friction drag force; Power required

Solution:

Basic equations: $C_D = \dfrac{F_D}{\frac{1}{2}\cdot\rho\cdot V^2\cdot A}$

We "unwrap" the cylinder to obtain an equivalent flat plate

$L = 38\cdot m$ $\quad D = 4\cdot m$ $\quad A = L\cdot\pi\cdot D$ $\quad A = 478\,m^2$ $\quad V = 800\cdot\dfrac{km}{hr}$

From Table A.3, with $z = 12000\cdot m$ $\quad \dfrac{\rho}{\rho_{SL}} = 0.2546$ $\quad \rho_{SL} = 1.225\cdot\dfrac{kg}{m^3}$

$\rho = 0.2546\cdot\rho_{SL}$ $\quad \rho = 0.3119\,\dfrac{kg}{m^3}$ $\quad$ and also $\quad T = 216.7\cdot K$

From Appendix A-3 $\quad \mu = \dfrac{b\cdot T^{\frac{1}{2}}}{1+\dfrac{S}{T}}$ $\quad$ with $\quad b = 1.458\times 10^{-6}\cdot\dfrac{kg}{m\cdot s\cdot K^{\frac{1}{2}}}$ $\quad S = 110.4\cdot K$

Hence $\quad \mu = \dfrac{b\cdot T^{\frac{1}{2}}}{1+\dfrac{S}{T}}$ $\quad \mu = 1.42\times 10^{-5}\cdot\dfrac{N\cdot s}{m^2}$

Next we need the Reynolds number $\quad Re_L = \dfrac{\rho\cdot V\cdot L}{\mu}$ $\quad Re_L = 1.85\times 10^{8}$ $\quad$ so use Eq. 9.35

$C_D = \dfrac{0.455}{\log\left(Re_L\right)^{2.58}}$ $\quad C_D = 0.00196$

The drag is then $\quad F_D = C_D\cdot\frac{1}{2}\cdot\rho\cdot V^2\cdot A$ $\quad F_D = 7189\,N$

The power consumed is $\quad P = F_D\cdot V$ $\quad P = 1.598\cdot MW$

Problem 9.82

[Difficulty: 3]

9.82 Resistance of a barge is to be determined from model test data. The model is constructed to a scale ratio of 1:13.5 and has length, beam, and draft of 7.00 m, 1.4 m, and 0.2 m, respectively. The test is to simulate performance of the prototype at 10 knots. What must the model speed be for the model and prototype to exhibit similar wave drag behavior? Is the boundary layer on the prototype predominantly laminar or turbulent? Does the model boundary layer become turbulent at the comparable point? If not, the model boundary layer could be artificially triggered to turbulent by placing a tripwire across the hull. Where could this be placed? Estimate the skin-friction drag on model and prototype.

Given: Towboat model at 1:13.5 scale to be tested in towing tank.

$L_m = 7.00\cdot m \quad B_m = 1.4\cdot m \quad d_m = 0.2\cdot m \quad V_p = 10\cdot knot$

Find:
(a) Model speed in order to exhibit similar wave drag behavior
(b) Type of boundary layer on the prototype
(c) Where to place BL trips on the model
(d) Estimate skin friction drag on prototype

Solution:

Basic equations: $F_D = C_D\cdot\frac{1}{2}\cdot\rho\cdot V^2\cdot A$ (Drag)

The test should be conducted to match Froude numbers: $\frac{V_m}{\sqrt{g\cdot L_m}} = \frac{V_p}{\sqrt{g\cdot L_p}} \quad V_m = V_p\cdot\sqrt{\frac{L_m}{L_p}}$ $\quad V_m = 2.72\cdot knot$

The Reynolds number is: $Re_p = \frac{V_p\cdot L_p}{\nu}$ $\quad Re_p = 4.85\times 10^8$

Therefore the boundary layer is turbulent. Transition occurs at $Re_t = 5\times 10^5$ so $\frac{x_t}{L} = \frac{Re_t}{Re_p} = 0.00155$

Thus the location of transition would be: $x_t = 0.00155\cdot L_m$ $\quad x_t = 0.0109\,m$

The wetted area is: $A = L\cdot(B + 2\cdot d)$ We calculate the drag coefficient from turbulent BL theory:

$C_D = 1.25\cdot C_f = 1.25\times\frac{0.0594}{Re_L^{0.2}} = \frac{0.0743}{Re_L^{0.2}}$ For the model: $L_m = 7\,m \quad Re_m = \frac{V_m\cdot L_m}{\nu} = 9.77\times 10^6 \quad A_m = 12.6\,m^2$

Therefore $C_{Dm} = \frac{0.0743}{Re_m^{0.2}} = 2.97\times 10^{-3}$ and the drag force is: $F_{Dm} = C_{Dm}\cdot\frac{1}{2}\cdot\rho\cdot V_m^2\cdot A_m$ $\quad F_{Dm} = 36.70\,N$

For the prototype: $C_{Dp} = \frac{0.455}{\left(\log\left(Re_p\right)\right)^{2.56}} - \frac{1610}{Re_p}$ $\quad C_{Dp} = 1.7944\times 10^{-3} \quad A_p = 2.30\times 10^3\cdot m^2$

$F_{Dp} = C_{Dp}\cdot\frac{1}{2}\cdot\rho\cdot V_p^2\cdot A_p$ $\quad F_{Dp} = 54.5\cdot kN$

Problem 9.83

[Difficulty: 2]

9.83 A vertical stabilizing fin on a land-speed-record car is $L = 1.65$ m long and $H = 0.785$ m tall. The automobile is to be driven at the Bonneville Salt Flats in Utah, where the elevation is 1340 m and the summer temperature reaches 50°C. The car speed is 560 km/hr. Evaluate the length Reynolds number of the fin. Estimate the location of transition from laminar to turbulent flow in the boundary layers. Calculate the power required to overcome skin friction drag on the fin.

Given: Stabilizing fin on Bonneville land speed record auto

$z = 1340 \cdot m \quad V = 560 \cdot \frac{km}{hr} \quad H = 0.785 \cdot m \quad L = 1.65 \cdot m$

Find:
(a) Evaluate Reynolds number of fin
(b) Estimate of location for transition in the boundary layer
(c) Power required to overcome skin friction drag

Solution:

Basic equations: $F_D = C_D \cdot \frac{1}{2} \cdot \rho \cdot V^2 \cdot A$ (Drag)

Assumptions: (1) Standard atmosphere (use table A.3)

At this elevation: $T = 279 \cdot K \quad \rho = 0.877 \times 1.23 \cdot \frac{kg}{m^3} = 1.079 \frac{kg}{m^3} \quad \mu = 1.79 \times 10^{-5} \cdot \frac{kg}{m \cdot s}$

The Reynolds number on the fin is: $Re_L = \frac{\rho \cdot V \cdot L}{\mu}$ $\qquad Re_L = 1.547 \times 10^7$

Assume transition occurs at: $Re_t = 5 \times 10^5$ The location for transition would then be: $x_t = \frac{Re_t \cdot \mu}{\rho \cdot V}$ $\qquad x_t = 53.3 \cdot mm$

From Figure 9.8, the drag coefficient is: $C_D = 0.0029$ The area is: $A = 2 \cdot L \cdot H = 2.591\ m^2$ (both sides of the fin)

The drag force would then be: $F_D = C_D \cdot \frac{1}{2} \cdot \rho \cdot V^2 \cdot A$ $\qquad F_D = 98.0\,N$

The power required would then be: $P = F_D \cdot V$ $\qquad P = 15.3 \cdot kW$

If we check the drag coefficient using Eq. 9.37b: $C_D = \frac{0.455}{\left(\log\left(Re_L\right)\right)^{2.58}} - \frac{1610}{Re_L} = 0.0027$

This is slightly less than the graph, but still reasonable agreement.

Problem 9.84

[Difficulty: 4]

9.84 A nuclear submarine cruises fully submerged at 27 knots. The hull is approximately a circular cylinder with diameter $D = 11.0$ m and length $L = 107$ m. Estimate the percentage of the hull length for which the boundary layer is laminar. Calculate the skin friction drag on the hull and the power consumed.

Given: Nuclear submarine cruising submerged. Hull approximated by circular cylinder

$L = 107 \cdot m \qquad D = 11.0 \cdot m \qquad V = 27 \cdot knot$

Find:
(a) Percentage of hull length for which BL is laminar
(b) Skin friction drag on hull
(c) Power consumed

Solution:

Basic equations: $F_D = C_D \cdot \frac{1}{2} \cdot \rho \cdot V^2 \cdot A$ (Drag)

Transition occurs at $Re_t = 5 \times 10^5$ so the location of transition would be: $\frac{x_t}{L} = \frac{Re_t \cdot \nu}{V \cdot L}$ $\qquad \frac{x_t}{L} = 0.0353\%$

We will therefore assume that the BL is completely turbulent.

The Reynolds number at x = L is: $Re_L = \frac{V \cdot L}{\nu} = 1.42 \times 10^9$ For this Reynolds number: $C_D = \frac{0.455}{\left(\log\left(Re_L\right)\right)^{2.58}} = 1.50 \times 10^{-3}$

The wetted area of the hull is: $A = \pi \cdot D \cdot L = 3698 \cdot m^2$

So the drag force is: $F_D = C_D \cdot \frac{1}{2} \cdot \rho \cdot V^2 \cdot A$ $\qquad F_D = 5.36 \times 10^5 N$

The power consumed is: $P = F_D \cdot V$ $\qquad P = 7.45 \cdot MW$

Problem 9.85

[Difficulty: 3]

9.85 You are asked by your college crew to estimate the skin friction drag on their eight-seat racing shell. The hull of the shell may be approximated as half a circular cylinder with 457 mm diameter and 7.32 m length. The speed of the shell through the water is 6.71 m/s. Estimate the location of transition from laminar to turbulent flow in the boundary layer on the hull of the shell. Calculate the thickness of the turbulent boundary layer at the rear of the hull. Determine the total skin friction drag on the hull under the given conditions.

Given: Racing shell for crew approximated by half-cylinder:

$$L = 7.32\cdot m \qquad D = 457\cdot mm \qquad V = 6.71\cdot\frac{m}{s}$$

Find:
(a) Location of transition on hull
(b) Thickness of turbulent BL at the rear of the hull
(c) Skin friction drag on hull

Solution:

Basic equations: $F_D = C_D\cdot\frac{1}{2}\cdot\rho\cdot V^2\cdot A$ (Drag)

Transition occurs at $Re_t = 5\times 10^5$ so the location of transition would be: $x_t = \frac{Re_t\cdot\nu}{V}$ $\qquad x_t = 0.0745\,m$

For the turbulent boundary layer $\frac{\delta}{x} = \frac{0.382}{Re_x^{0.2}}$ Therefore $\delta = \frac{0.382\cdot L}{Re_L^{0.2}}$

The Reynolds number at x = L is: $Re_L = \frac{V\cdot L}{\nu} = 4.91\times 10^7$ so the BL thickness is: $\delta = \frac{0.382\cdot L}{Re_L^{0.2}}$ $\qquad \delta = 0.0810\,m$

The wetted area of the hull is: $A = \frac{\pi\cdot D}{2}\cdot L = 5.2547\cdot m^2$ For this Reynolds number: $C_D = \frac{0.455}{\left(\log\left(Re_L\right)\right)^{2.58}} = 2.36\times 10^{-3}$

So the drag force is: $F_D = C_D\cdot\frac{1}{2}\cdot\rho\cdot V^2\cdot A$ $\qquad F_D = 278\,N$

Note that the rowers must produce an average power of $P = F_D\cdot V = 1.868\cdot kW$ to move the shell at this speed.

Problem 9.86

[Difficulty: 3]

9.86 A sheet of plastic material 0.5 in. thick, with specific gravity SG = 1.7, is dropped into a large tank containing water. The sheet is 2 ft × 4 ft. Estimate the terminal speed of the sheet as it falls with (a) the short side vertical and (b) the long side vertical. Assume that the drag is due only to skin friction, and that the boundary layers are turbulent from the leading edge.

Given: Plastic sheet falling in water

Find: Terminal speed both ways

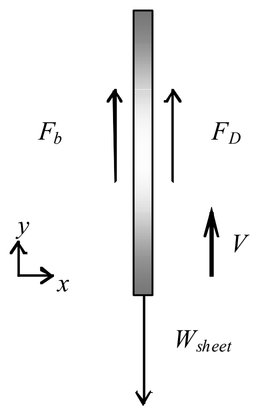
Solution:

Basic equations: $\Sigma F_y = 0$ for terminal speed $\qquad C_D = \dfrac{F_D}{\frac{1}{2}\cdot\rho\cdot V^2\cdot A} \qquad C_D = \dfrac{0.0742}{Re_L^{\frac{1}{5}}}$ (9.34) (assuming $5 \times 10^5 < Re_L < 10^7$)

$h = 0.5\cdot in \qquad W = 4\cdot ft \qquad L = 2\cdot ft \qquad SG = 1.7$ From Table A.8 at 70ºF $\nu = 1.06\times 10^{-5}\cdot\dfrac{ft^2}{s} \qquad \rho = 1.94\cdot\dfrac{slug}{ft^3}$

$A = W\cdot L$

A free body diagram of the sheet is shown here. Summing the forces in the vertical (y) direction:



$$F_D + F_b - W_{sheet} = 0 \qquad F_D = W_{sheet} - F_b = \rho\cdot g\cdot h\cdot A\cdot(SG - 1)$$

Also, we can generate an expression for the drag coefficient in terms of the geometry of the sheet and the water properties:

$$F_D = 2\cdot C_D\cdot A\cdot\frac{1}{2}\cdot\rho\cdot V^2 = 2\cdot\frac{0.0742}{Re_L^{\frac{1}{5}}}\cdot A\cdot\frac{1}{2}\cdot\rho\cdot V^2 = \frac{0.0742}{\left(\frac{V\cdot L}{\nu}\right)^{\frac{1}{5}}}\cdot W\cdot L\cdot\rho\cdot V^2 = 0.0742\cdot W\cdot L^{\frac{4}{5}}\cdot\nu^{\frac{1}{5}}\cdot\rho\cdot V^{\frac{9}{5}}$$

(Note that we double F_D because drag acts on both sides of the sheet.)

Hence $\rho_{H2O}\cdot g\cdot h\cdot W\cdot(SG - 1) = 0.0742\cdot W\cdot L^{-\frac{1}{5}}\cdot\nu^{\frac{1}{5}}\cdot\rho\cdot V^{\frac{9}{5}}$ Solving for V $V = \left[\dfrac{g\cdot h\cdot(SG-1)}{0.0742}\cdot\left(\dfrac{L}{\nu}\right)^{\frac{1}{5}}\right]^{\frac{5}{9}}$ $\qquad V = 15.79\cdot\dfrac{ft}{s}$

Check the Reynolds number $Re_L = \dfrac{V\cdot L}{\nu}$ $\qquad Re_L = 2.98\times 10^6$ Hence Eq. 9.34 is reasonable

Repeating for $L = 4\cdot ft$ $\qquad V = \left[\dfrac{g\cdot h\cdot(SG-1)}{0.0742}\cdot\left(\dfrac{L}{\nu}\right)^{\frac{1}{5}}\right]^{\frac{5}{9}}$ $\qquad V = 17.06\cdot\dfrac{ft}{s}$

Check the Reynolds number $Re_L = \dfrac{V\cdot L}{\nu}$ $\qquad Re_L = 6.44\times 10^6$ Eq. 9.34 is still reasonable

The short side vertical orientation falls more slowly because the largest friction is at the region of the leading edge (τ tails off as the boundary layer progresses); its leading edge area is larger. Note that neither orientation is likely - the plate will flip around in a chaotic manner.

Problem 9.87

[Difficulty: 4]

9.87 The 600-seat jet transport aircraft proposed by Airbus Industrie has a fuselage that is 240 ft long and 25 ft in diameter. The aircraft is to operate 14 hr per day, 6 days per week; it will cruise at 575 mph ($M = 0.87$) at 12-km altitude. The engines consume fuel at the rate of 0.6 lbm/hr for each pound force of thrust produced. Estimate the skin friction drag force on the aircraft fuselage at cruise. Calculate the annual fuel savings for the aircraft if friction drag on the fuselage could be reduced by 1 percent by modifying the surface coating.

Given: 600-seat jet transport to operate 14 hr/day, 6 day/wk

$L = 240 \cdot ft \quad D = 25 \cdot ft \qquad V = 575 \cdot mph \quad z = 12 \cdot km \qquad TSFC = 0.6 \cdot \frac{lbm}{hr \cdot lbf}$

Find:
(a) Skin friction drag on fuselage at cruise
(b) Annual fuel savings if drag is reduced by 1%

Solution:

Basic equations: $F_D = C_D \cdot \frac{1}{2} \cdot \rho \cdot V^2 \cdot A$ (Drag)

From the atmosphere model: $T = 216.7 \cdot K = 390.1 \cdot R \quad \rho = 0.2546 \times 0.002377 \cdot \frac{slug}{ft^3} = 6.0518 \times 10^{-4} \cdot \frac{slug}{ft^3}$

From the Sutherland model for viscosity: $\mu = \dfrac{b \cdot \sqrt{T}}{1 + \dfrac{S}{T}} = 1.422 \times 10^{-5} \frac{kg}{m \cdot s}$ So the Reynolds number is

$Re_L = \dfrac{\rho \cdot V \cdot L}{\mu} = 4.1247 \times 10^8$ For this Reynolds number: $C_D = \dfrac{0.455}{\left(\log\left(Re_L\right)\right)^{2.58}} = 1.76 \times 10^{-3}$

The wetted area of the fuselage is: $A = \pi \cdot D \cdot L = 1.885 \times 10^4 \cdot ft^2$

So the drag force is: $F_D = C_D \cdot \frac{1}{2} \cdot \rho \cdot V^2 \cdot A \quad F_D = 7.13 \times 10^3 \cdot lbf$

If there were a 1% savings in drag, the drop in drag force would be: $\Delta F_D = 1 \cdot \% \cdot F_D = 71.31 \cdot lbf$

The savings in fuel would be: $\Delta m_{fuel} = TSFC \cdot \Delta F_D \times 14 \cdot \frac{hr}{day} \times \left(\frac{6}{7} \times 52\right) \cdot \frac{day}{yr} \qquad \Delta m_{fuel} = 2.670 \times 10^4 \cdot \frac{lbm}{yr}$

If jet fuel costs $1 per gallon, this would mean a savings of over $4,400 per aircraft per year.

Problem 9.88

[Difficulty: 4]

9.88 A supertanker displacement is approximately 600,000 tons. The ship has length $L = 1000$ ft, beam (width) $b = 270$ ft, and draft (depth) $D = 80$ ft. The ship steams at 15 knots through seawater at 40°F. For these conditions, estimate (a) the thickness of the boundary layer at the stern of the ship, (b) the total skin friction drag acting on the ship, and (c) the power required to overcome the drag force.

Given: Supertanker in seawater at 40°F

$L = 1000 \cdot ft \quad B = 270 \cdot ft \quad D = 80 \cdot ft \qquad V = 15 \cdot knot = 25.32 \cdot \frac{ft}{s} \qquad SG = 1.025$

$\nu = 1.05 \times 1.65 \times 10^{-5} \cdot \frac{ft^2}{s} = 1.73 \times 10^{-5} \cdot \frac{ft^2}{s} \qquad \rho = 1.9888 \cdot \frac{slug}{ft^3}$

Find: (a) Thickness of the boundary layer at the stern of the ship
(b) Skin friction drag on the ship
(b) Power required to overcome the drag force

Solution:

Basic equations: $F_D = C_D \cdot \frac{1}{2} \cdot \rho \cdot V^2 \cdot A$ (Drag)

The Reynolds number is $Re_L = \frac{V \cdot L}{\nu} = 1.4613 \times 10^9$ So the BL is turbulent. The BL thickness is calculated using:

$\frac{\delta}{x} = \frac{0.382}{Re_x^{0.20}}$ At the stern of the ship: $\delta = L \cdot \frac{0.382}{Re_L^{0.20}}$ $\qquad \delta = 5.61 \cdot ft$

The wetted area of the hull is: $A = L \cdot (B + 2 \cdot D) = 4.30 \times 10^5 \cdot ft^2$ For this Reynolds number: $C_D = \frac{0.455}{\left(\log\left(Re_L\right)\right)^{2.58}} = 1.50 \times 10^{-3}$

So the drag force is: $F_D = C_D \cdot \frac{1}{2} \cdot \rho \cdot V^2 \cdot A$ $\qquad F_D = 4.11 \times 10^5 \cdot lbf$

The power consumed to overcome the skin friction drag is: $P = F_D \cdot V$ $\qquad P = 1.891 \times 10^4 \cdot hp$

$P = 1.040 \times 10^7 \cdot \frac{ft \cdot lbf}{s}$

Problem 9.89

[Difficulty: 4]

9.89 In Section 7.6 the wave resistance and viscous resistance on a model and prototype ship were discussed. For the prototype, $L = 130$ m and $A = 1800$ m^2. From the data of Figs 7.2 and 7.3, plot on one graph the wave, viscous, and total resistance (N) experienced by the prototype, as a function of speed. Plot a similar graph for the model. Discuss your results. Finally, plot the power (kW) required for the prototype ship to overcome the total resistance.

Given: "Resistance" data on a ship

$L_p = 130 \cdot m$ $\quad A_p = 1800 \cdot m^2$ $\quad \rho = 1023 \cdot \frac{kg}{m^3}$ $\quad \mu = 1.08 \times 10^{-3} \cdot \frac{N \cdot s}{m^2}$

$L_m = \frac{L_p}{80} = 1.625\ m$ $\quad A_m = \frac{A_p}{80^2} = 0.281\ m^2$

Find: Plot of wave, viscous and total drag (prototype and model); power required by prototype

Solution:

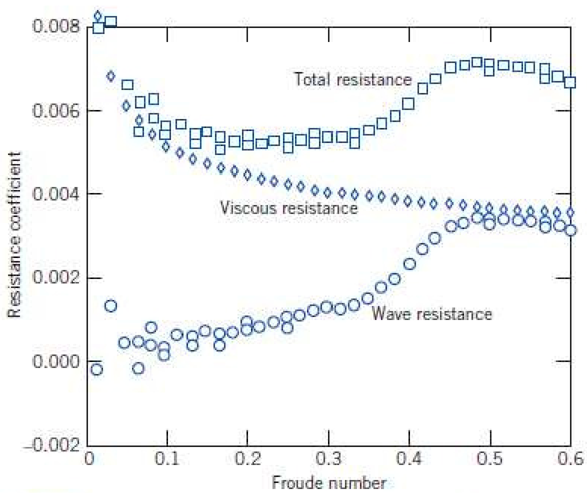


Fig. 7.2 Data from test of 1:80 scale model of U.S. Navy guided missile frigate *Oliver Hazard Perry* (FFG-7). (Data from U.S. Naval Academy Hydromechanics Laboratory, courtesy of Professor Bruce Johnson.)

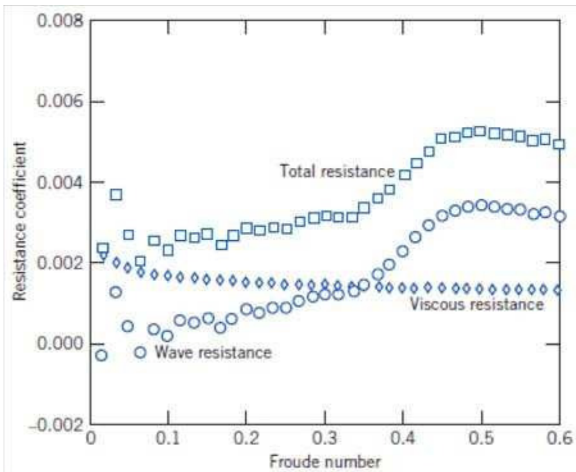


Fig. 7.3 Resistance of full-scale ship predicted from model test results. (Data from U.S. Naval Academy Hydromechanics Laboratory, courtesy of Professor Bruce Johnson.)

Basic equations:

$$C_D = \frac{F_D}{\frac{1}{2} \cdot \rho \cdot U^2 \cdot A} \qquad (9.32) \qquad Fr = \frac{U}{\sqrt{gL}}$$

From Eq. 9.32 $\quad F_D = C_D \cdot A \cdot \frac{1}{2} \cdot \rho \cdot U^2$

This applies to each component of the drag (wave and viscous) as well as to the total

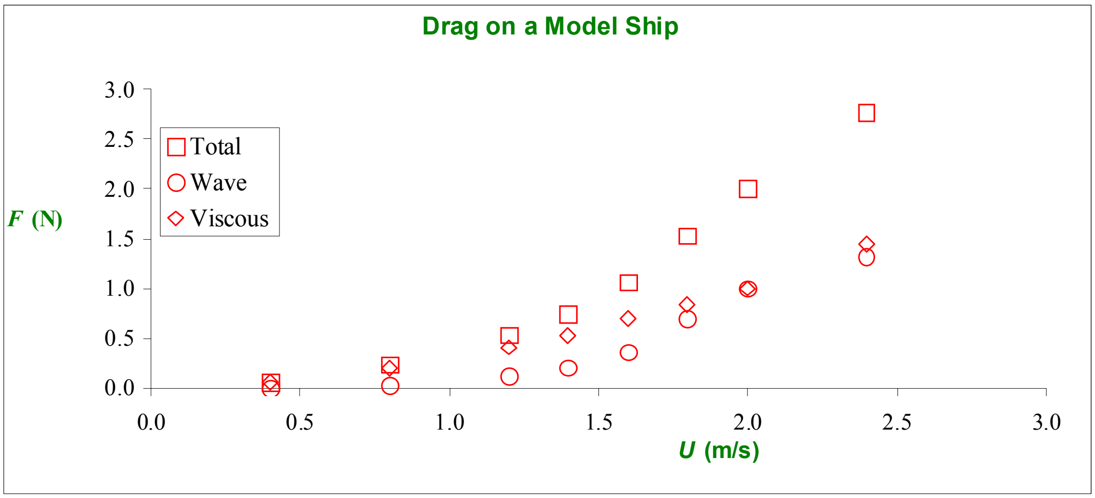
The power consumed is $\quad P = F_D \cdot U \qquad P = C_D \cdot A \cdot \frac{1}{2} \cdot \rho \cdot U^3$

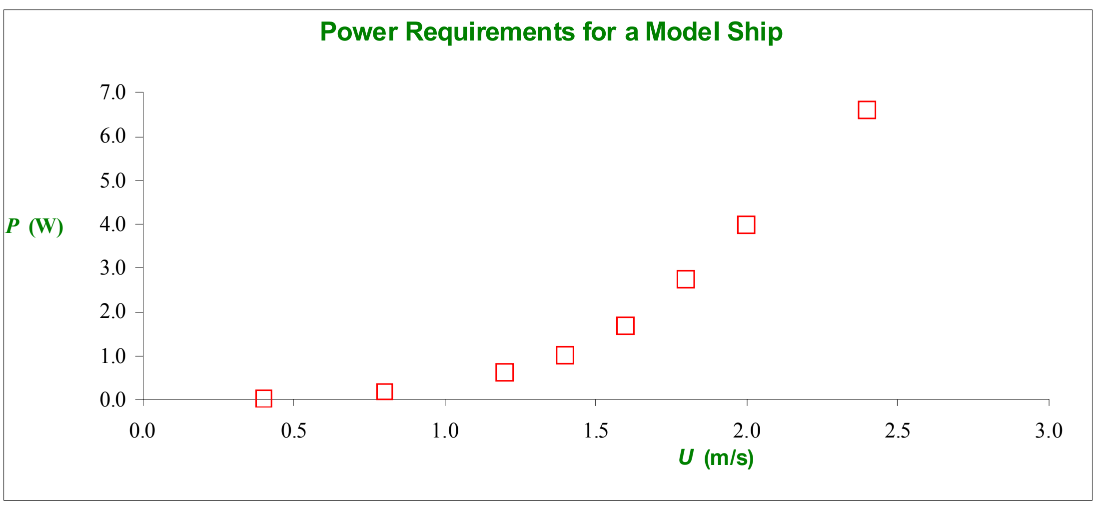
From the Froude number $\quad U = Fr \cdot \sqrt{gL}$

The solution technique is: For each speed *Fr* value from the graph, compute *U*; compute the drag from the corresponding "resistance" value from the graph. The results were generated in *Excel* and are shown below:

Model

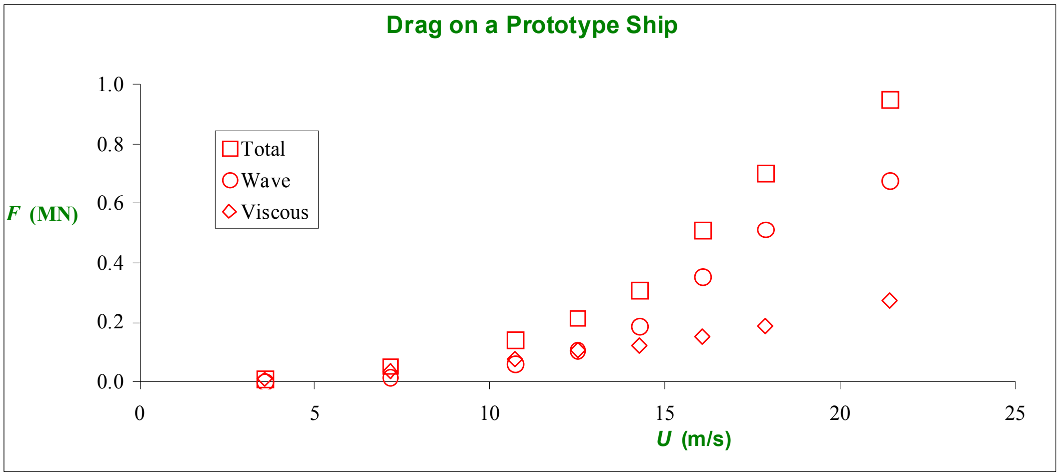
Fr	Wave "Resistance"	Viscous "Resistance"	Total "Resistance"	*U* (m/s)	Wave Drag (N)	Viscous Drag (N)	Total Drag (N)	Power (W)
0.10	0.00050	0.0052	0.0057	0.40	0.0057	0.0596	0.0654	0.0261
0.20	0.00075	0.0045	0.0053	0.80	0.0344	0.2064	0.2408	0.1923
0.30	0.00120	0.0040	0.0052	1.20	0.1238	0.4128	0.5366	0.6427
0.35	0.00150	0.0038	0.0053	1.40	0.2107	0.5337	0.7444	1.0403
0.40	0.00200	0.0038	0.0058	1.60	0.3669	0.6971	1.0640	1.6993
0.45	0.00300	0.0036	0.0066	1.80	0.6966	0.8359	1.5324	2.7533
0.50	0.00350	0.0035	0.0070	2.00	1.0033	1.0033	2.0065	4.0057
0.60	0.00320	0.0035	0.0067	2.40	1.3209	1.4447	2.7656	6.6252

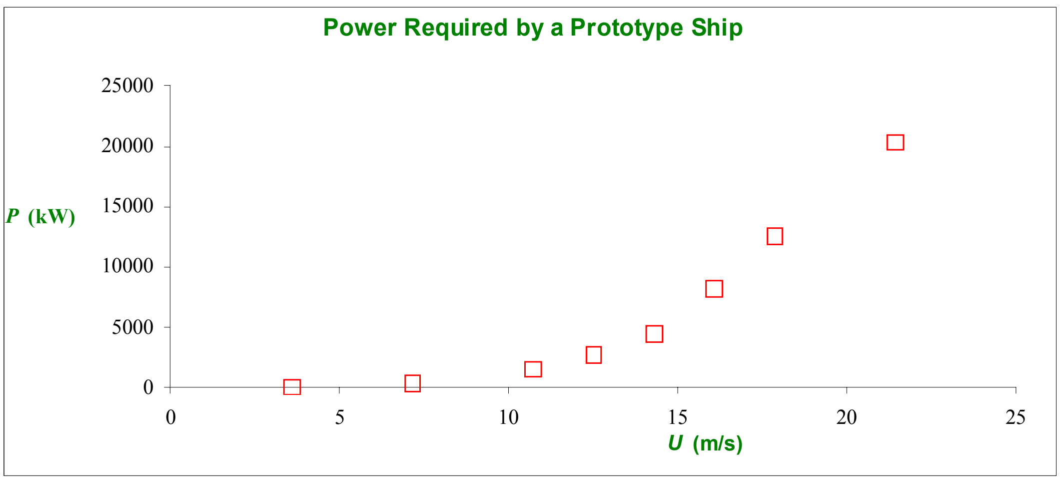




Prototype

Fr	Wave "Resistance"	Viscous "Resistance"	Total "Resistance"	*U* (m/s)	Wave Drag (MN)	Viscous Drag (MN)	Total Drag (MN)	Power (kW)	Power (hp)
0.10	0.00050	0.0017	0.0022	3.6	0.0029	0.0100	0.0129	46.1	61.8
0.20	0.00075	0.0016	0.0024	7.1	0.0176	0.0376	0.0552	394.1	528.5
0.30	0.00120	0.0015	0.0027	10.7	0.0634	0.0793	0.1427	1528.3	2049.5
0.35	0.00150	0.0015	0.0030	12.5	0.1079	0.1079	0.2157	2696.6	3616.1
0.40	0.00200	0.0013	0.0033	14.3	0.1879	0.1221	0.3100	4427.7	5937.6
0.45	0.00300	0.0013	0.0043	16.1	0.3566	0.1545	0.5112	8214.7	11015.9
0.50	0.00350	0.0013	0.0048	17.9	0.5137	0.1908	0.7045	12578.7	16868.1
0.60	0.00320	0.0013	0.0045	21.4	0.6763	0.2747	0.9510	20377.5	27326.3





For the prototype wave resistance is a much more significant factor at high speeds! However, note that for both scales, the primary source of drag changes with speed. At low speeds, viscous effects dominate, and so the primary source of drag is viscous drag. At higher speeds, inertial effects dominate, and so the wave drag is the primary source of drag.

Problem 9.90

[Difficulty: 2]

9.90 As part of the 1976 bicentennial celebration, an enterprising group hung a giant American flag (194 ft high and 367 ft wide) from the suspension cables of the Verrazano Narrows Bridge. They apparently were reluctant to make holes in the flag to alleviate the wind force, and hence they effectively had a flat plate normal to the flow. The flag tore loose from its mountings when the wind speed reached 10 mph. Estimate the wind force acting on the flag at this wind speed. Should they have been surprised that the flag blew down?

Given: Flag mounted vertically

$$H = 194\cdot ft \quad W = 367\cdot ft \qquad V = 10\cdot mph = 14.67\cdot\frac{ft}{s} \quad \rho = 0.00234\cdot\frac{slug}{ft^3} \quad \nu = 1.62\times10^{-4}\cdot\frac{ft^2}{s}$$

Find: Force acting on the flag. Was failure a surprise?

Solution:

Basic equations: $F_D = C_D\cdot\frac{1}{2}\cdot\rho\cdot V^2\cdot A$ (Drag)

We should check the Reynolds number to be sure that the data of Fig. 9.10 are applicable: $Re_W = \frac{V\cdot W}{\nu} = 3.32\times10^7$

(We used W as our length scale here since it is the lesser of the two dimensions of the flag.) Since the Reynolds number is less than 1000, we may use Figure 9.10 to find the drag coefficient.

The area of the flag is: $A = H\cdot W = 7.12\times10^4\cdot ft^2$ $\quad AR = \frac{W}{H} = 1.89$ $\qquad$ From Fig. 9.10: $C_D = 1.15$

So the drag force is: $F_D = C_D\cdot\frac{1}{2}\cdot\rho\cdot V^2\cdot A$ $\qquad F_D = 2.06\times10^4\cdot lbf$

This is a large force. Failure should have been expected.

Problem 9.91

[Difficulty: 3]

9.91 A fishing net is made of 0.75-mm diameter nylon thread assembled in a rectangular pattern. The horizontal and vertical distances between adjacent thread centerlines are 1 cm. Estimate the drag on a 2 m × 12 m section of this net when it is dragged (perpendicular to the flow) through 15°C water at 6 knots. What is the power required to maintain this motion?

Given: Fishing net

Find: Drag; Power to maintain motion

$\frac{3}{8}\cdot in = 9.525\cdot mm$

Solution:

Basic equations: $C_D = \dfrac{F_D}{\frac{1}{2}\cdot\rho\cdot V^2\cdot A}$

We convert the net into an equivalent cylinder (we assume each segment does not interfere with its neighbors)

$L = 12\cdot m$ $\quad W = 2\cdot m$ $\quad d = 0.75\cdot mm$ Spacing: $D = 1\cdot cm$ $\quad V = 6\cdot knot$ $\quad V = 3.09\cdot\frac{m}{s}$

Total number of threads of length L is $n_1 = \frac{W}{D}$ $\quad n_1 = 200$ $\quad$ Total length $L_1 = n_1\cdot L$ $\quad L_1 = 2400\cdot m$

Total number of threads of length W is $n_2 = \frac{L}{D}$ $\quad n_2 = 1200$ $\quad$ Total length $L_2 = n_2\cdot W$ $\quad L_2 = 2400\cdot m$

Total length of thread $L_T = L_1 + L_2$ $\quad L_T = 4800\cdot m$ $\quad L_T = 2.98\cdot mile$ A lot!

The frontal area is then $A = L_T\cdot d$ $\quad A = 3.60\cdot m^2$ $\quad$ Note that $L\cdot W = 24.00\cdot m^2$

From Table A.8 $\quad \rho = 999\cdot\frac{kg}{m^3}$ $\quad \nu = 1.01\times 10^{-6}\cdot\frac{m^2}{s}$

The Reynolds number is $Re_d = \frac{V\cdot d}{\nu}$ $\quad Re_d = 2292$

For a cylinder in a crossflow at this Reynolds number, from Fig. 9.13, approximately $C_D = 0.8$

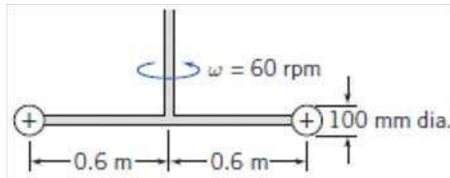
Hence $\quad F_D = C_D\cdot\frac{1}{2}\cdot\rho\cdot V^2\cdot A$ $\quad F_D = 13.71\cdot kN$

The power required is $P = F_D\cdot V$ $\quad P = 42.3\cdot kW$

Problem 9.92

[Difficulty: 2]

9.92 A rotary mixer is constructed from two circular disks as shown. The mixer is rotated at 60 rpm in a large vessel containing a brine solution (SG = 1.1). Neglect the drag on the rods and the motion induced in the liquid. Estimate the minimum torque and power required to drive the mixer.



Given: Rotary mixer rotated in a brine solution

$R = 0.6 \cdot m \quad \omega = 60 \cdot rpm \quad d = 100 \cdot mm \quad SG = 1.1 \qquad \rho = \rho_w \cdot SG \qquad \rho = 1100 \cdot \frac{kg}{m^3}$

$\nu = 1.05 \times 1.55 \times 10^{-6} \cdot \frac{m^2}{s} = 1.63 \times 10^{-6} \cdot \frac{m^2}{s}$

Find: (a) Torque on mixer
(b) Horsepower required to drive mixer

Solution:

Basic equations:

$F_D = C_D \cdot \frac{1}{2} \cdot \rho \cdot V^2 \cdot A$ (Drag)

$T = 2 \cdot R \cdot F_D$ (Torque)

$P = T \cdot \omega$ (Power)

Assumptions: Drag on rods and motion induced in the brine can be neglected.

The speed of the disks through the brine is: $V = R \cdot \omega = 3.77 \frac{m}{s}$ From Table 9.2: $C_D = 1.17$ for a disk.

The area of one disk is: $A = \frac{\pi}{4} \cdot d^2 = 0.00785 \cdot m^2$

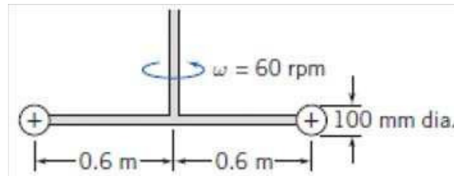
So the drag force is: $F_D = C_D \cdot \frac{1}{2} \cdot \rho \cdot V^2 \cdot A = 71.8\,N$ and the torque is: $T = 2 \cdot R \cdot F_D \qquad T = 86.2 \cdot N \cdot m$

The power consumed to run the mixer is: $P = T \cdot \omega = 542\,W \qquad P = 0.726 \cdot hp$

Problem 9.93

[Difficulty: 3]

9.93 As a young design engineer you decide to make the rotary mixer look more "cool" by replacing the disks with rings. The rings may have the added benefit of making the mixer mix more effectively. If the mixer absorbs 350 W at 60 rpm, redesign the device. There is a design constraint that the outer diameter of the rings not exceed 125 mm.



Given: Data on a rotary mixer

Find: New design dimensions

Solution:

The given data or available data is

$$R = 0.6 \cdot m \qquad P = 350 \cdot W \qquad \omega = 60 \cdot rpm \qquad \rho = 1099 \cdot \frac{kg}{m^3}$$

For a ring, from Table 9.3 $\quad C_D = 1.2$

The torque at the specified power and speed is

$$T = \frac{P}{\omega} \qquad T = 55.7 \cdot N \cdot m$$

The drag on *each* ring is then $\quad F_D = \frac{1}{2} \cdot \frac{T}{R} \qquad F_D = 46.4\,N$

The linear velocity of each ring is $\quad V = R \cdot \omega \qquad V = 3.77\,\frac{m}{s}$

The drag and velocity of each ring are related using the definition of drag coefficient

$$C_D = \frac{F_D}{\frac{1}{2} \cdot \rho \cdot A \cdot V^2}$$

Solving for the ring area $\quad A = \frac{F_D}{\frac{1}{2} \cdot \rho \cdot V^2 \cdot C_D} \qquad A = 4.95 \times 10^{-3}\,m^2$

But $\quad A = \frac{\pi}{4} \cdot \left(d_o^{\,2} - d_i^{\,2} \right)$

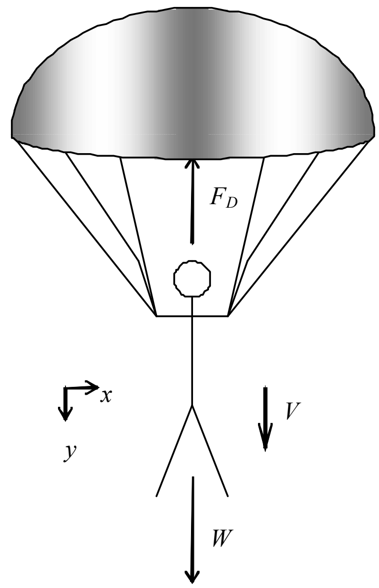
The outer diameter is $\quad d_o = 125 \cdot mm$

Hence the inner diameter is $\quad d_i = \sqrt{d_o^{\,2} - \frac{4 \cdot A}{\pi}} \qquad d_i = 96.5 \cdot mm$

Problem 9.94

[Difficulty: 2]

9.94 The vertical component of the landing speed of a parachute is to be less than 20 ft/s. The total weight of the jumper and the chute is 250 lb. Determine the minimum diameter of the open parachute.



Given: Man with parachute

$$W = 250 \cdot \text{lbf} \quad V = 20 \cdot \frac{\text{ft}}{\text{s}} \qquad \rho = 0.00234 \cdot \frac{\text{slug}}{\text{ft}^3}$$

Find: Minimum diameter of parachute

Solution:

Basic equations: $F_D = C_D \cdot \frac{1}{2} \cdot \rho \cdot V^2 \cdot A$ (Drag)

Assumptions: (1) Standard air
(2) Parachute behaves as open hemisphere
(3) Vertical speed is constant

For constant speed: $\Sigma F_y = M \cdot g - F_D = 0$ Therefore: $F_D = W$

In terms of the drag coefficient: $C_D \cdot \frac{1}{2} \cdot \rho \cdot V^2 \cdot A = W$

Solving for the area: $A = \frac{2 \cdot W}{C_D \cdot \rho \cdot V^2}$ From Table 9.2: $C_D = 1.42$ for an open hemisphere. The area is: $A = \frac{\pi}{4} \cdot D^2$

Setting both areas equal: $\frac{\pi}{4} \cdot D^2 = \frac{2 \cdot W}{C_D \cdot \rho \cdot V^2}$ Solving for the diameter of the parachute: $D = \sqrt{\frac{8}{\pi} \cdot \frac{W}{C_D \cdot \rho \cdot V^2}}$

Therefore the diameter is: $D = 21.9 \cdot \text{ft}$

Problem 9.95

[Difficulty: 3]

9.95 As a young design engineer you are asked to design an emergency braking parachute system for use with a military aircraft of mass 9500 kg. The plane lands at 350 km/hr, and the parachute system alone must slow the airplane to 100 km/hr in less than 1200 m. Find the minimum diameter required for a single parachute, and for three non-interfering parachutes. Plot the airplane speed versus distance and versus time. What is the maximum "g-force" experienced?

Given: Data on airplane landing

$$M = 9500\cdot kg \quad V_i = 350\cdot\frac{km}{hr} \quad V_f = 100\cdot\frac{km}{hr} \quad x_f = 1200\cdot m \quad C_D = 1.43 \text{ (Table 9.3)} \quad \rho = 1.23\cdot\frac{kg}{m^3}$$

Find: Single and three-parachute sizes; plot speed against distance and time; maximum "g"s

Solution:

Basic equations:

$$F_D = C_D\cdot\frac{1}{2}\cdot\rho\cdot V^2\cdot A \qquad \text{(Drag)}$$

Assumptions:
(1) Standard air
(2) Parachute behaves as open hemisphere
(3) Vertical speed is constant

Newton's second law for the aircraft is $\quad M\cdot\frac{dV}{dt} = -C_D\cdot\frac{1}{2}\cdot\rho\cdot A\cdot V^2$

where A and CD are the single parachute area and drag coefficient

Separating variables $\quad \frac{dV}{V^2} = -\frac{C_D\cdot\rho\cdot A}{2\cdot M}\cdot dt$

Integrating, with IC $V = V_i$

$$V(t) = \frac{V_i}{1 + \frac{C_D\cdot\rho\cdot A}{2\cdot M}\cdot V_i\cdot t} \qquad (1)$$

Integrating again with respect to t

$$x(t) = \frac{2\cdot M}{C_D\cdot\rho\cdot A}\cdot\ln\left(1 + \frac{C_D\cdot\rho\cdot A}{2\cdot M}\cdot V_i\cdot t\right) \qquad (2)$$

Eliminating t from Eqs. 1 and 2

$$x = \frac{2\cdot M}{C_D\cdot\rho\cdot A}\cdot\ln\left(\frac{V_i}{V}\right) \qquad (3)$$

To find the minimum parachute area we must solve Eq 3 for A with $x = x_f$ when $V = V_f$

$$A = \frac{2\cdot M}{C_D\cdot\rho\cdot x_f}\cdot\ln\left(\frac{V_i}{V_f}\right) \qquad (4)$$

For three parachutes, the analysis is the same except A is replaced with $3A$, leading to

$$A = \frac{2\cdot M}{3\cdot C_D\cdot\rho\cdot x_f}\cdot\ln\left(\frac{V_i}{V_f}\right) \qquad (5)$$

The "g"'s are given by $\dfrac{\frac{dV}{dt}}{g} = \dfrac{-C_D \cdot \rho \cdot A \cdot V^2}{2 \cdot M \cdot g}$ which has a maximum at the initial instant ($V = V_i$)

The results generated in *Excel* are shown below:

Single:

A = 11.4 m^2

D = 3.80 m

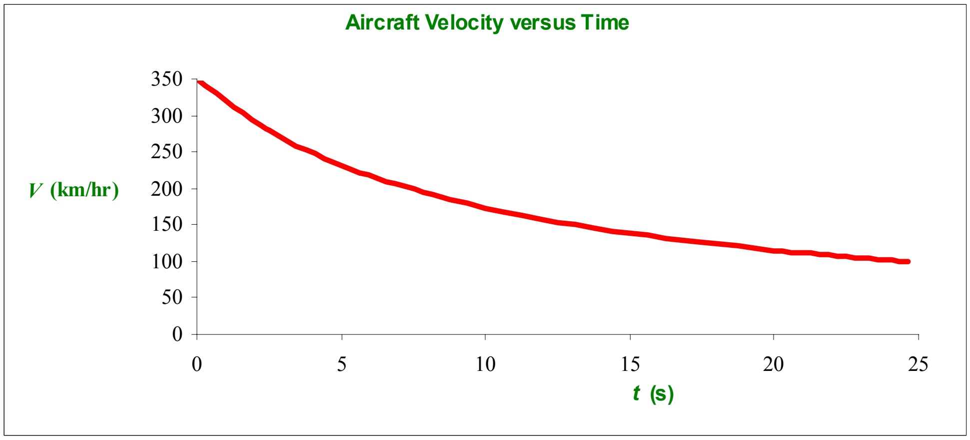
Triple:

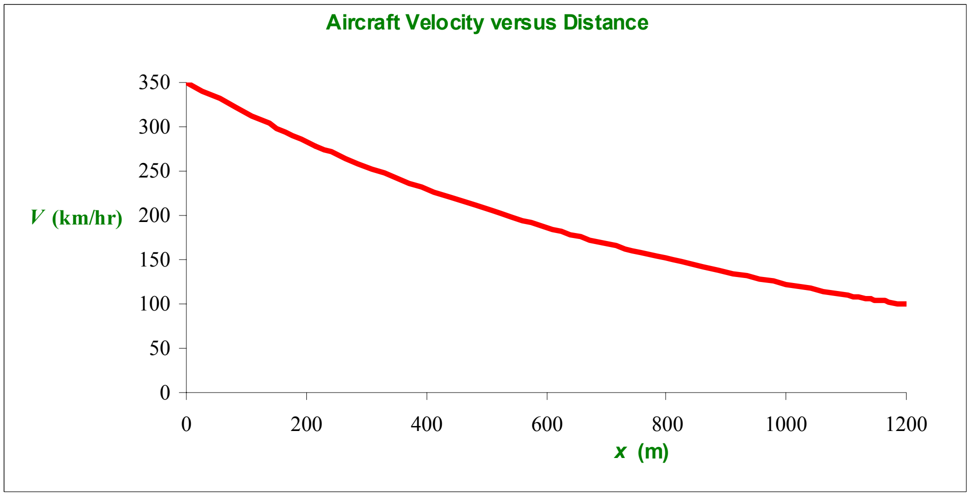
A = 3.8 m^2

D = 2.20 m

"g"'s = -1.01 Max

t (s)	x (m)	V (km/hr)
0.00	0.0	350
2.50	216.6	279
5.00	393.2	232
7.50	542.2	199
10.0	671.1	174
12.5	784.7	154
15.0	886.3	139
17.5	978.1	126
20.0	1061.9	116
22.5	1138.9	107
24.6	1200.0	100





Problem 9.96

[Difficulty: 3]

9.96 An emergency braking parachute system on a military aircraft consists of a large parachute of diameter 6 m. If the airplane mass is 8500 kg, and it lands at 400 km/hr, find the time and distance at which the airplane is slowed to 100 km/hr by the parachute alone. Plot the aircraft speed versus distance and versus time. What is the maximum "g-force" experienced? An engineer proposes that less space would be taken up by replacing the large parachute with three non-interfering parachutes each of diameter 3.75 m. What effect would this have on the time and distance to slow to 100 km/hr?

Given: Data on airplane and parachute

Find: Time and distance to slow down; plot speed against distance and time; maximum "g"'s

Solution:

The given data or available data is

$$M = 8500 \cdot kg \quad V_i = 400 \cdot \frac{km}{hr} \quad V_f = 100 \cdot \frac{km}{hr} \quad C_D = 1.42 \quad \rho = 1.23 \cdot \frac{kg}{m^3} \quad D_{single} = 6 \cdot m \quad D_{triple} = 3.75 \cdot m$$

$$A_{single} = \frac{\pi}{4} \cdot D_{single}^2 = 28.274\,m^2 \qquad A_{triple} = \frac{\pi}{4} \cdot D_{triple}^2 = 11.045\,m^2$$

Newton's second law for the aircraft is $$M \cdot \frac{dV}{dt} = -C_D \cdot \frac{1}{2} \cdot \rho \cdot A \cdot V^2$$

where A and C_D are the single parachute area and drag coefficient

Separating variables $$\frac{dV}{V^2} = -\frac{C_D \cdot \rho \cdot A}{2 \cdot M} \cdot dt$$

Integrating, with IC $V = V_i$ $$V(t) = \frac{V_i}{1 + \frac{C_D \cdot \rho \cdot A}{2 \cdot M} \cdot V_i \cdot t} \qquad (1)$$

Integrating again with respect to t $$x(t) = \frac{2 \cdot M}{C_D \cdot \rho \cdot A} \cdot \ln\left(1 + \frac{C_D \cdot \rho \cdot A}{2 \cdot M} \cdot V_i \cdot t\right) \qquad (2)$$

Eliminating t from Eqs. 1 and 2 $$x = \frac{2 \cdot M}{C_D \cdot \rho \cdot A} \cdot \ln\left(\frac{V_i}{V}\right) \qquad (3)$$

To find the time and distance to slow down to 100 km/hr, Eqs. 1 and 3 are solved with $V = 100$ km/hr (or use *Goal Seek*)

The "g"'s are given by $$\frac{\frac{dV}{dt}}{g} = \frac{-C_D \cdot \rho \cdot A \cdot V^2}{2 \cdot M \cdot g}$$ which has a maximum at the initial instant ($V = V_i$)

For three parachutes, the analysis is the same except A is replaced with $3A$. leading to

$$V(t) = \frac{V_i}{1 + \frac{3 \cdot C_D \cdot \rho \cdot A}{2 \cdot M} \cdot V_i \cdot t}$$

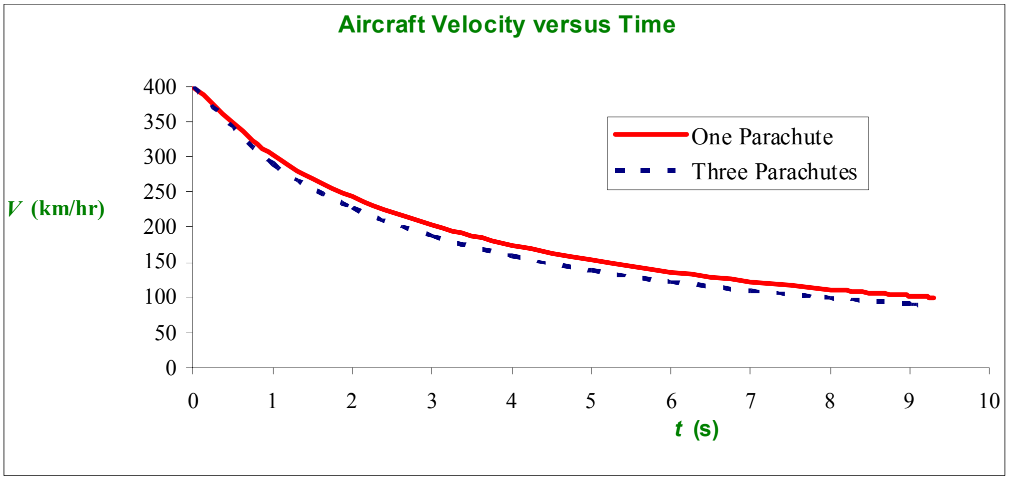
$$x(t) = \frac{2 \cdot M}{3 \cdot C_D \cdot \rho \cdot A} \cdot \ln\left(1 + \frac{3 \cdot C_D \cdot \rho \cdot A}{2 \cdot M} \cdot V_i \cdot t\right)$$

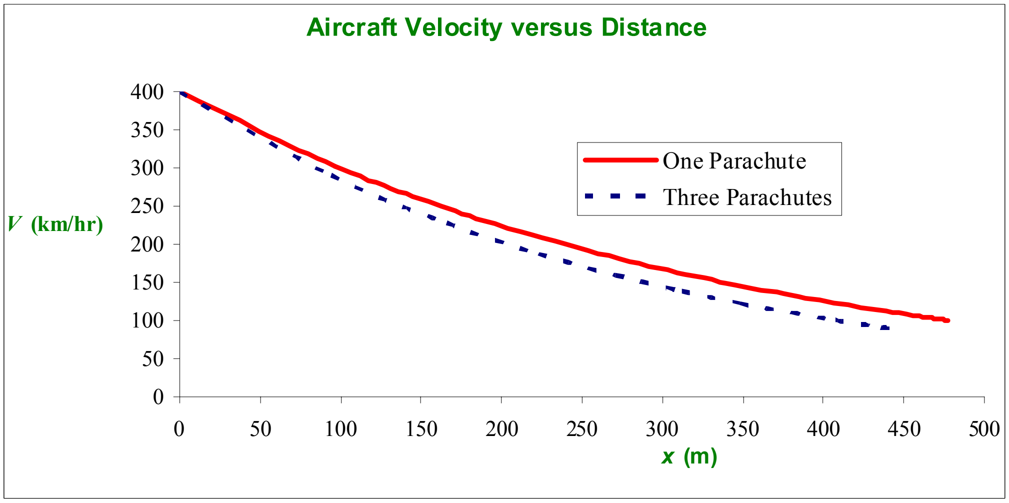
The results generated in *Excel* are shown here:

t (s)	*x* (m)	*V* (km/hr)
0.0	0.0	400
1.0	96.3	302
2.0	171	243
3.0	233	203
4.0	285	175
5.0	331	153
6.0	371	136
7.0	407	123
8.0	439	112
9.0	469	102
9.29	477	100

t (s)	*x* (m)	*V* (km/hr)
0.0	0.0	400
1.0	94.2	290
2.0	165	228
3.0	223	187
4.0	271	159
5.0	312	138
6.0	348	122
7.0	380	110
7.93	407	100
9.0	436	91
9.3	443	89

"g"'s = -3.66 Max

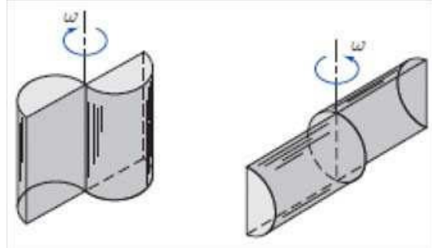




Problem 9.97

[Difficulty: 2]

9.97 It has been proposed to use surplus 55 gal oil drums to make simple windmills for underdeveloped countries. (It is a simple Savonius turbine.) Two possible configurations are shown. Estimate which would be better, why, and by how much. The diameter and length of a 55 gal drum are $D = 24$ in. and $H = 29$ in.



Given: Windmills are to be made from surplus 55 gallon oil drums

$$D = 24\cdot\text{in} \qquad H = 29\cdot\text{in}$$

Find: Which configuration would be better, why, and by how much

Solution:

Basic equations:

$$F_D = C_D\cdot\frac{1}{2}\cdot\rho\cdot V^2\cdot A \qquad \text{(Drag)}$$

Assumptions:
(1) Standard air
(2) Neglect friction in the pivot
(3) Neglect interference between the flow over the two halves

For the first configuration:

$$\Sigma M = \frac{D}{2}\cdot F_u - \frac{D}{2}\cdot F_d = \frac{D}{2}\cdot\left(F_u - F_d\right)$$

Where F_u is the force on the half "catching" the wind and F_d is the force on the half "spilling" the wind.

$$\Sigma M = \frac{D}{2}\cdot\left(C_{Du}\cdot\frac{1}{2}\cdot\rho\cdot V^2\cdot A - C_{Dd}\cdot\frac{1}{2}\cdot\rho\cdot V^2\cdot A\right) = \frac{D}{2}\cdot\left(C_{Du} - C_{Dd}\right)\cdot\left(\frac{1}{2}\cdot\rho\cdot V^2\cdot A\right)$$

For the second configuration:

$$\Sigma M = \frac{H}{2}\cdot F_u - \frac{H}{2}\cdot F_d = \frac{H}{2}\cdot\left(F_u - F_d\right)$$

$$\Sigma M = \frac{H}{2}\cdot\left(C_{Du}\cdot\frac{1}{2}\cdot\rho\cdot V^2\cdot A - C_{Dd}\cdot\frac{1}{2}\cdot\rho\cdot V^2\cdot A\right) = \frac{H}{2}\cdot\left(C_{Du} - C_{Dd}\right)\cdot\left(\frac{1}{2}\cdot\rho\cdot V^2\cdot A\right)$$

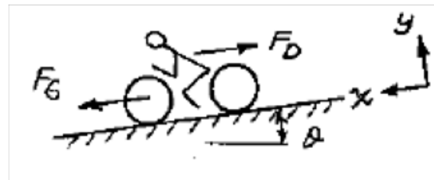
Since H > D, the second configuration will be superior.

The improvement will be: $\dfrac{H - D}{D} = 20.8\cdot\%$

Problem 9.98

[Difficulty: 2]

9.98 The resistance to motion of a good bicycle on smooth pavement is nearly all due to aerodynamic drag. Assume that the total weight of rider and bike is $W = 210$ lbf. The frontal area measured from a photograph is $A = 5\ \text{ft}^2$. Experiments on a hill, where the road grade is 9 percent, show that terminal speed is $V_t = 50$ ft/s. From these data, the drag coefficient is estimated as $C_D = 1.25$. Verify this calculation of drag coefficient. Estimate the distance needed for the bike and rider to decelerate from 50 ft/s to 30 ft/s while coasting after reaching level road.



Given: Bike and rider at terminal speed on hill with 8% grade.

$$W = 210 \cdot \text{lbf} \quad A = 5 \cdot \text{ft}^2 \quad V_t = 50 \cdot \frac{\text{ft}}{\text{s}} \quad C_D = 1.25$$

Find: (a) Verify drag coefficient
(b) Estimate distance needed for bike and rider to decelerate to 10 m/s after reaching level road

Solution:

Basic equations: $F_D = C_D \cdot \frac{1}{2} \cdot \rho \cdot V^2 \cdot A$ (Drag)

Assumptions: (1) Standard air
(2) Neglect all losses other than aerodynamic drag

The angle of incline is: $\theta = \text{atan}(9 \cdot \%) = 5.143 \cdot \text{deg}$ Summing forces in the x-direction: $\Sigma F_x = F_G - F_D = 0$

Expanding out both force terms: $M \cdot g \cdot \sin(\theta) = C_D \cdot \frac{1}{2} \cdot \rho \cdot V_t^2 \cdot A$ Solving this expression for the drag coefficient:

$$C_D = \frac{2 \cdot W \cdot \sin(\theta)}{\rho \cdot V_t^2 \cdot A} \qquad C_D = 1.26$$

The original estimate for the drag coefficient was good.

Once on the flat surface: $\Sigma F_x = -F_D = \frac{W}{g} \cdot \left(\frac{d}{dt}V\right) = \frac{W}{g} \cdot V \cdot \left(\frac{d}{ds}V\right)$ Therefore: $\frac{W}{g} \cdot V \cdot \left(\frac{d}{ds}V\right) = -C_D \cdot \frac{1}{2} \cdot \rho \cdot V^2 \cdot A$

Separating variables: $ds = -\frac{2 \cdot W}{C_D \cdot \rho \cdot g \cdot A} \cdot \frac{dV}{V}$ Integrating both sides yields: $\Delta s = -\frac{2 \cdot W}{C_D \cdot \rho \cdot g \cdot A} \cdot \ln\left(\frac{V_2}{V_1}\right)$ $\Delta s = 447 \cdot \text{ft}$

Problem 9.99

[Difficulty: 2]

9.99 A cyclist is able to attain a maximum speed of 30 km/hr on a calm day. The total mass of rider and bike is 65 kg. The rolling resistance of the tires is $F_R = 7.5$ N, and the drag coefficient and frontal area are $C_D = 1.2$ and $A = 0.25\ \text{m}^2$. The cyclist bets that today, even though there is a headwind of 10 km/hr, she can maintain a speed of 24 km/hr. She also bets that, cycling with wind support, she can attain a top speed of 40 km/hr. Which, if any, bets does she win?

Given:

Data on cyclist performance on a calm day

Find:

Performance hindered and aided by wind

Solution:

The given data or available data is

$$F_R = 7.5 \cdot N \qquad M = 65 \cdot kg \qquad A = 0.25 \cdot m^2$$

$$C_D = 1.2 \qquad \rho = 1.23 \cdot \frac{kg}{m^3} \qquad V = 30 \cdot \frac{km}{hr}$$

The governing equation is

$$F_D = \frac{1}{2} \cdot \rho \cdot A \cdot V^2 \cdot C_D \qquad F_D = 12.8\,N$$

The power steady power generated by the cyclist is

$$P = (F_D + F_R) \cdot V \qquad P = 169\,W \qquad P = 0.227 \cdot hp$$

Now, with a headwind we have

$$V_w = 10 \cdot \frac{km}{hr} \qquad V = 24 \cdot \frac{km}{hr}$$

The aerodynamic drag is greater because of the greater effective wind speed

$$F_D = \frac{1}{2} \cdot \rho \cdot A \cdot (V + V_w)^2 \cdot C_D \qquad F_D = 16.5\,N$$

The power required is that needed to overcome the total force $F_D + F_R$, moving at the cyclist's speed

$$P = V \cdot (F_D + F_R) \qquad P = 160\,W$$

This is less than the power she can generate

She wins the bet!

With the wind supporting her the effective wind speed is substantially lower

$$V_W = 10 \cdot \frac{km}{hr} \qquad V = 40 \cdot \frac{km}{hr}$$

$$F_D = \frac{1}{2} \cdot \rho \cdot A \cdot (V - V_W)^2 \cdot C_D \qquad F_D = 12.8\,N$$

The power required is that needed to overcome the total force $F_D + F_R$, moving at the cyclist's speed

$$P = V \cdot (F_D + F_R) \qquad P = 226\,W$$

This is more than the power she can generate

She loses the bet

Problem 9.100

[Difficulty: 2]

9.100 Ballistic data obtained on a firing range show that aerodynamic drag reduces the speed of a .44 magnum revolver bullet from 250 m/s to 210 m/s as it travels over a horizontal distance of 150 m. The diameter and mass of the bullet are 11.2 mm and 15.6 g, respectively. Evaluate the average drag coefficient for the bullet.

Given: Ballistic data for .44 magnum revolver bullet

$$V_i = 250 \cdot \frac{m}{s} \quad V_f = 210 \cdot \frac{m}{s} \quad \Delta x = 150 \cdot m \quad M = 15.6 \cdot gm \quad D = 11.2 \cdot mm$$

Find: Average drag coefficient

Solution:

Basic equations:

$$F_D = C_D \cdot \frac{1}{2} \cdot \rho \cdot V^2 \cdot A \qquad \text{(Drag)}$$

Assumptions: (1) Standard air
(2) Neglect all losses other than aerodynamic drag

Newton's 2nd law: $\Sigma F_x = -F_D = M \cdot \left(\frac{d}{dt} V\right) = M \cdot V \cdot \left(\frac{d}{ds} V\right)$ Therefore: $M \cdot V \cdot \left(\frac{d}{dx} V\right) = -C_D \cdot \frac{1}{2} \cdot \rho \cdot V^2 \cdot A$

Separating variables: $dx = -\frac{2 \cdot M}{C_D \cdot \rho \cdot A} \cdot \frac{dV}{V}$ Integrating both sides yields: $\Delta x = -\frac{2 \cdot M}{C_D \cdot \rho \cdot A} \cdot \ln\left(\frac{V_f}{V_i}\right)$

Solving this expression for the drag coefficient: $C_D = -\frac{2 \cdot M}{\Delta x \cdot \rho \cdot A} \cdot \ln\left(\frac{V_f}{V_i}\right)$ The area is: $A = \frac{\pi}{4} \cdot D^2 = 98.52 \cdot mm^2$

Therefore the drag coefficent is: $C_D = 0.299$

Problem 9.101

[Difficulty: 3]

9.101 Consider the cyclist in Problem 9.99. She is having a bad day, because she has to climb a hill with a 5° slope. What is the speed she is able to attain? What is the maximum speed if there is also a headwind of 10 km/hr? She reaches the top of the hill, and turns around and heads down the hill. If she still pedals as hard as possible, what will be her top speed (when it is calm, and when the wind is present)? What will be her maximum speed if she decides to coast down the hill (with and without the aid of the wind)?

Given: Data on cyclist performance on a calm day

Find: Performance on a hill with and without wind

Solution:

The given data or available data is

$F_R = 7.5 \cdot N$ $\quad M = 65 \cdot kg$ $\quad A = 0.25 \cdot m^2$

$C_D = 1.2$ $\quad \rho = 1.23 \cdot \frac{kg}{m^3}$ $\quad V = 30 \cdot \frac{km}{hr}$

The governing equation is $\quad F_D = \frac{1}{2} \cdot \rho \cdot A \cdot V^2 \cdot C_D \quad F_D = 12.8\,N$

Power steady power generated by the cyclist is $\quad P = \left(F_D + F_R\right) \cdot V \quad P = 169\,W \quad P = 0.227 \cdot hp$

Riding up the hill (no wind) $\quad \theta = 5 \cdot deg$

For steady speed the cyclist's power is consumed by working against the net force (rolling resistance, drag, and gravity)

Cycling up the hill: $\quad P = \left(F_R + \frac{1}{2} \cdot \rho \cdot A \cdot V^2 \cdot C_D + M \cdot g \cdot \sin(\theta)\right) \cdot V$

This is a cubic equation for the speed which can be solved analytically, or by iteration, or using *Excel*'s *Goal Seek* or *Solver*. The solution is obtained from the associated *Excel* workbook

From *Solver* $\quad V = 9.47 \cdot \frac{km}{hr}$

Now, with a headwind we have $\quad V_w = 10 \cdot \frac{km}{hr}$

The aerodynamic drag is greater because of the greater effective wind speed

$$F_D = \frac{1}{2} \cdot \rho \cdot A \cdot \left(V + V_w\right)^2 \cdot C_D$$

The power required is that needed to overcome the total force (rolling resistance, drag, and gravity) moving at the cyclist's speed is

Uphill against the wind: $\quad P = \left[F_R + \frac{1}{2} \cdot \rho \cdot A \cdot \left(V + V_w\right)^2 \cdot C_D + M \cdot g \cdot \sin(\theta)\right] \cdot V$

This is again a cubic equation for *V*

From *Solver* $\quad V = 8.94 \cdot \frac{km}{hr}$

Pedalling downhill (no wind) gravity helps increase the speed; the maximum speed is obtained from

Cycling down the hill: $$P = \left(F_R + \frac{1}{2}\cdot\rho\cdot A\cdot V^2\cdot C_D - M\cdot g\cdot\sin(\theta)\right)\cdot V$$

This cubic equation for *V* is solved in the associated *Excel* workbook

From *Solver* $$V = 63.6\cdot\frac{km}{hr}$$

Pedalling downhill (wind assisted) gravity helps increase the speed; the maximum speed is obtained from

Wind-assisted downhill: $$P = \left[F_R + \frac{1}{2}\cdot\rho\cdot A\cdot\left(V - V_w\right)^2\cdot C_D - M\cdot g\cdot\sin(\theta)\right]\cdot V$$

This cubic equation for *V* is solved in the associated *Excel* workbook

From *Solver* $$V = 73.0\cdot\frac{km}{hr}$$

Freewheeling downhill, the maximum speed is obtained from the fact that the net force is zero

Freewheeling downhill: $$F_R + \frac{1}{2}\cdot\rho\cdot A\cdot V^2\cdot C_D - M\cdot g\cdot\sin(\theta) = 0$$

$$V = \sqrt{\frac{M\cdot g\cdot\sin(\theta) - F_R}{\frac{1}{2}\cdot\rho\cdot A\cdot C_D}} \qquad V = 58.1\cdot\frac{km}{hr}$$

Wind assisted: $$F_R + \frac{1}{2}\cdot\rho\cdot A\cdot\left(V - V_w\right)^2\cdot C_D - M\cdot g\cdot\sin(\theta) = 0$$

$$V = V_w + \sqrt{\frac{M\cdot g\cdot\sin(\theta) - F_R}{\frac{1}{2}\cdot\rho\cdot A\cdot C_D}} \qquad V = 68.1\cdot\frac{km}{hr}$$

Problem 9.102

[Difficulty: 3]

9.102 Consider the cyclist in Problem 9.99. Determine the maximum speeds she is actually able to attain today (with the 10 km/hr wind) cycling into the wind, and cycling with the wind. If she were to replace the tires with high-tech ones that had a rolling resistance of only 3.5 N, determine her maximum speed on a calm day, cycling into the wind, and cycling with the wind. If she in addition attaches an aerodynamic fairing that reduces the drag coefficient to $C_D = 0.9$, what will be her new maximum speeds?

Given: Data on cyclist performance on a calm day

Find: Performance hindered and aided by wind; repeat with high-tech tires; with fairing

Solution:

The given data or available data is

$F_R = 7.5 \cdot N$ $\quad M = 65 \cdot kg$ $\quad A = 0.25 \cdot m^2$

$C_D = 1.2$ $\quad \rho = 1.23 \cdot \frac{kg}{m^3}$ $\quad V = 30 \cdot \frac{km}{hr}$

The governing equation is $\quad F_D = \frac{1}{2} \cdot \rho \cdot A \cdot V^2 \cdot C_D \qquad F_D = 12.8\,N$

Power steady power generated by the cyclist is $\quad P = \left(F_D + F_R\right) \cdot V \qquad P = 169\ W \qquad P = 0.227 \cdot hp$

Now, with a headwind we have $\quad V_w = 10 \cdot \frac{km}{hr}$

The aerodynamic drag is greater because of the greater effective wind speed

$$F_D = \frac{1}{2} \cdot \rho \cdot A \cdot \left(V + V_w\right)^2 \cdot C_D \qquad (1)$$

The power required is that needed to overcome the total force $F_D + F_R$, moving at the cyclist's speed is

$$P = V \cdot \left(F_D + F_R\right) \qquad (2)$$

Combining Eqs 1 and 2 we obtain an expression for the cyclist's maximum speed V cycling into a headwind (where $P = 169$ W is the cyclist's power)

Cycling into the wind: $$P = \left[F_R + \frac{1}{2} \cdot \rho \cdot A \cdot \left(V + V_w\right)^2 \cdot C_D\right] \cdot V \qquad (3)$$

This is a cubic equation for V; it can be solved analytically, or by iterating. It is convenient to use *Excel*'s *Goal Seek* (or *Solver*). From the associated *Excel* workbook

From *Solver* $\quad V = 24.7 \cdot \frac{km}{hr}$

By a similar reasoning:

Cycling with the wind: $$P = \left[F_R + \frac{1}{2} \cdot \rho \cdot A \cdot \left(V - V_w\right)^2 \cdot C_D\right] \cdot V \qquad (4)$$

From *Solver* $V = 35.8 \cdot \frac{km}{hr}$

With improved tires $F_R = 3.5 \cdot N$

Maximum speed on a calm day is obtained from $P = \left(F_R + \frac{1}{2} \cdot \rho \cdot A \cdot V^2 \cdot C_D\right) \cdot V$

This is a again a cubic equation for V; it can be solved analytically, or by iterating. It is convenient to use *Excel*'s *Goal Seek* (or *Solver*). From the associated *Excel* workbook

From *Solver* $V = 32.6 \cdot \frac{km}{hr}$

Equations 3 and 4 are repeated for the case of improved tires

From *Solver* Against the wind $V = 26.8 \cdot \frac{km}{hr}$ With the wind $V = 39.1 \cdot \frac{km}{hr}$

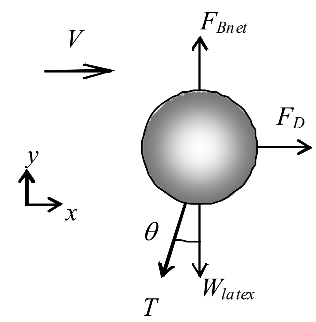
For improved tires and fairing, from *Solver*

$V = 35.7 \cdot \frac{km}{hr}$ Against the wind $V = 29.8 \cdot \frac{km}{hr}$ With the wind $V = 42.1 \cdot \frac{km}{hr}$

Problem 9.103

[Difficulty: 3]

9.103 At a surprise party for a friend you've tied a series of 20-cm-diameter helium balloons to a flagpole, each tied with a short string. The first one is tied 1 m above the ground, and the other eight are tied at 1 m spacings, so that the last is tied at a height of 9 m. Being quite a nerdy engineer, you notice that in the steady wind, each balloon is blown by the wind so it looks like the angles that the strings make with the vertical are about 10°, 20°, 30°, 35°, 40°, 45°, 50°, 60°, and 65°. Estimate and plot the wind velocity profile for the 9-m range. Assume the helium is at 20°C and 10 kPa gage and that each balloon is made of 3 g of latex.



Given: Series of party balloons

Find: Wind velocity profile; Plot

Solution:

Basic equations: $C_D = \dfrac{F_D}{\frac{1}{2}\cdot\rho\cdot V^2\cdot A}$ $\qquad F_B = \rho_{air}\cdot g\cdot Vol$ $\qquad \Sigma\vec{F} = 0$

This problem is ideal for computing and plotting in *Excel*, but we will go through the details here.

The above figure applies to each balloon

For the horizontal forces $\quad F_D - T\cdot\sin(\theta) = 0 \qquad (1)$

For the vertical forces $\quad -T\cdot\cos(\theta) + F_{Bnet} - W_{latex} = 0 \qquad (2)$

Here $\quad F_{Bnet} = F_B - W = \left(\rho_{air} - \rho_{He}\right)\cdot g\cdot\dfrac{\pi\cdot D^3}{6}$

$D = 20\cdot cm \qquad M_{latex} = 3\cdot gm \qquad W_{latex} = M_{latex}\cdot g \qquad W_{latex} = 0.02942\cdot N$

We have (Table A.6) $\quad R_{He} = 2077\cdot\dfrac{N\cdot m}{kg\cdot K} \qquad p_{He} = 111\cdot kPa \qquad T_{He} = 293\cdot K \qquad \rho_{He} = \dfrac{p_{He}}{R_{He}\cdot T_{He}} \qquad \rho_{He} = 0.1824\cdot\dfrac{kg}{m^3}$

$R_{air} = 287\cdot\dfrac{N\cdot m}{kg\cdot K} \qquad p_{air} = 101\cdot kPa \qquad T_{air} = 293\cdot K \qquad \rho_{air} = \dfrac{p_{air}}{R_{air}\cdot T_{air}} \qquad \rho_{air} = 1.201\cdot\dfrac{kg}{m^3}$

$F_{Bnet} = \left(\rho_{air} - \rho_{He}\right)\cdot g\cdot\dfrac{\pi\cdot D^3}{6} \qquad F_{Bnet} = 0.0418\cdot N$

Applying Eqs 1 and 2 to the top balloon, for which $\quad \theta = 65\cdot deg$

$$F_D = T\cdot\sin(\theta) = \frac{F_{Bnet} - W_{latex}}{\cos(\theta)}\cdot\sin(\theta)$$

Hence $\quad F_D = \left(F_{Bnet} - W_{latex}\right)\cdot\tan(\theta) \qquad F_D = 0.0266\,N$

But we have $\quad F_D = C_D\cdot\dfrac{1}{2}\cdot\rho_{air}\cdot V^2\cdot A = C_D\cdot\dfrac{1}{2}\cdot\rho_{air}\cdot V^2\cdot\dfrac{\pi\cdot D^2}{4}$ with $\quad C_D = 0.4$ from Fig. 9.11 (we will check Re later)

$V = \sqrt{\dfrac{8\cdot F_D}{C_D\cdot\rho_{air}\cdot\pi\cdot D^2}} \qquad V = 1.88\cdot\dfrac{m}{s}$

From Table A.9 $\quad \nu = 1.50\times 10^{-5}\cdot\dfrac{m^2}{s}$ The Reynolds number is $\quad Re_d = \dfrac{V\cdot D}{\nu} \qquad Re_d = 2.51\times 10^4$ We are okay!

For the next balloon $\theta = 60\cdot deg$ $F_D = (F_{Bnet} - W_{latex})\cdot \tan(\theta)$ $F_D = 0.0215\cdot N$ with $C_D = 0.4$

$$V = \sqrt{\frac{8\cdot F_D}{C_D\cdot \rho_{air}\cdot \pi\cdot D^2}} \qquad V = 1.69\,\frac{m}{s}$$

The Reynolds number is $Re_d = \frac{V\cdot D}{\nu}$ $Re_d = 2.25\times 10^4$ We are okay!

For the next balloon $\theta = 50\cdot deg$ $F_D = (F_{Bnet} - W_{latex})\cdot \tan(\theta)$ $F_D = 0.01481\,N$ with $C_D = 0.4$

$$V = \sqrt{\frac{8\cdot F_D}{C_D\cdot \rho_{air}\cdot \pi\cdot D^2}} \qquad V = 1.40\,\frac{m}{s}$$

The Reynolds number is $Re_d = \frac{V\cdot D}{\nu}$ $Re_d = 1.87\times 10^4$ We are okay!

For the next balloon $\theta = 45\cdot deg$ $F_D = (F_{Bnet} - W_{latex})\cdot \tan(\theta)$ $F_D = 0.01243\,N$ with $C_D = 0.4$

$$V = \sqrt{\frac{8\cdot F_D}{C_D\cdot \rho_{air}\cdot \pi\cdot D^2}} \qquad V = 1.28\,\frac{m}{s}$$

The Reynolds number is $Re_d = \frac{V\cdot D}{\nu}$ $Re_d = 1.71\times 10^4$ We are okay!

For the next balloon $\theta = 40\cdot deg$ $F_D = (F_{Bnet} - W_{latex})\cdot \tan(\theta)$ $F_D = 0.01043\,N$ with $C_D = 0.4$

$$V = \sqrt{\frac{8\cdot F_D}{C_D\cdot \rho_{air}\cdot \pi\cdot D^2}} \qquad V = 1.18\,\frac{m}{s}$$

The Reynolds number is $Re_d = \frac{V\cdot D}{\nu}$ $Re_d = 1.57\times 10^4$ We are okay!

For the next balloon $\theta = 35\cdot deg$ $F_D = (F_{Bnet} - W_{latex})\cdot \tan(\theta)$ $F_D = 0.00870\,N$ with $C_D = 0.4$

$$V = \sqrt{\frac{8\cdot F_D}{C_D\cdot \rho_{air}\cdot \pi\cdot D^2}} \qquad V = 1.07\,\frac{m}{s}$$

The Reynolds number is $Re_d = \frac{V\cdot D}{\nu}$ $Re_d = 1.43\times 10^4$ We are okay!

For the next balloon $\theta = 30\cdot deg$ $F_D = (F_{Bnet} - W_{latex})\cdot \tan(\theta)$ $F_D = 0.00717\,N$ with $C_D = 0.4$

$$V = \sqrt{\frac{8\cdot F_D}{C_D\cdot \rho_{air}\cdot \pi\cdot D^2}} \qquad V = 0.97\,\frac{m}{s}$$

The Reynolds number is $Re_d = \frac{V\cdot D}{\nu}$ $Re_d = 1.30\times 10^4$ We are okay!

For the next balloon $\theta = 20\cdot deg$ $F_D = (F_{Bnet} - W_{latex})\cdot \tan(\theta)$ $F_D = 0.00452\,N$ with $C_D = 0.4$

$$V = \sqrt{\frac{8\cdot F_D}{C_D\cdot \rho_{air}\cdot \pi\cdot D^2}} \qquad V = 0.77\,\frac{m}{s}$$

The Reynolds number is $Re_d = \frac{V\cdot D}{\nu}$ $Re_d = 1.03\times 10^4$ We are okay!

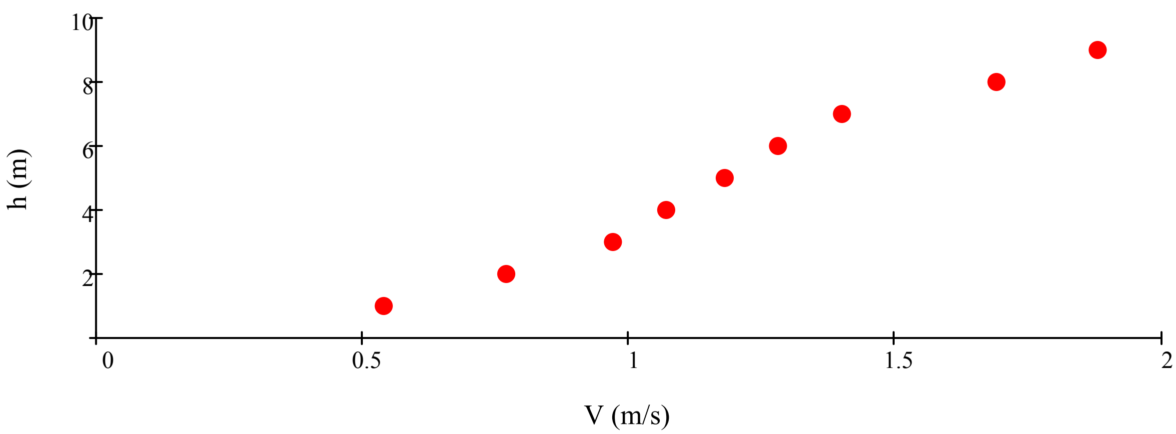
For the next balloon $\theta = 10\cdot deg$ $F_D = (F_{Bnet} - W_{latex})\cdot \tan(\theta)$ $F_D = 0.002191\,N$ with $C_D = 0.4$

$$V = \sqrt{\frac{8 \cdot F_D}{C_D \cdot \rho_{air} \cdot \pi \cdot D^2}} \qquad V = 0.54\,\frac{m}{s}$$

The Reynolds number is $Re_d = \dfrac{V \cdot D}{\nu}$ $\qquad Re_d = 7184.21$ $\qquad$ We are okay!

In summary we have $V = (0.54 \;\; 0.77 \;\; 0.97 \;\; 1.07 \;\; 1.18 \;\; 1.28 \;\; 1.40 \;\; 1.69 \;\; 1.88) \cdot \dfrac{m}{s}$

$h = (1 \;\; 2 \;\; 3 \;\; 4 \;\; 5 \;\; 6 \;\; 7 \;\; 8 \;\; 9) \cdot m$

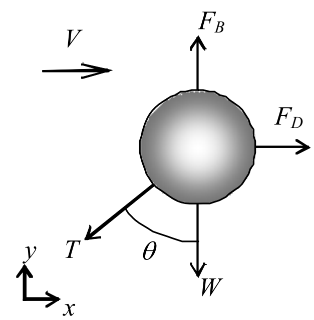


This does not seem like an unreasonable profile for the lowest portion of an atmospheric boundary layer - over cities or rough terrain the atmospheric boundary layer is typically 300-400 meters, so a near-linear profile over a small fraction of that distance is not out of the question.

Problem 9.104

[Difficulty: 2]

9.104 A 0.5-m-diameter hollow plastic sphere containing pollution test equipment is being dragged through the Hudson River in New York by a diver riding an underwater jet device. The sphere (with an effective specific gravity of SG = 0.30) is fully submerged, and it is tethered to the diver by a thin 1.5-m-long wire. What is the angle the wire makes with the horizontal if the velocity of the diver and sphere relative to the water is 5 m/s? The water is at 10°C.



Given: Sphere dragged through river

Find: Relative velocity of sphere

Solution:

Basic equations: $C_D = \dfrac{F_D}{\frac{1}{2}\cdot\rho\cdot V^2\cdot A}$ $\quad F_B = \rho\cdot g\cdot Vol \quad \Sigma\vec{F} = 0$

The above figure applies to the sphere

For the horizontal forces $\quad F_D - T\cdot\sin(\theta) = 0 \qquad (1)$

For the vertical forces $\quad -T\cdot\cos(\theta) + F_B - W = 0 \qquad (2)$

Here $\quad V = 5\cdot\dfrac{m}{s} \quad D = 0.5\cdot m \quad SG = 0.30$ and from Table A.8 $\nu = 1.30\times10^{-6}\cdot\dfrac{m^2}{s} \quad \rho = 1000\cdot\dfrac{kg}{m^3}$

The Reynolds number is $Re_d = \dfrac{V\cdot D}{\nu} \quad Re_d = 1.92\times10^6$ Therefore we estimate the drag coefficient: $C_D = 0.15$ (Fig 9.11)

$$F_D = T\cdot\sin(\theta) = \frac{F_B - W}{\cos(\theta)}\cdot\sin(\theta) = \rho\cdot g\cdot Vol\cdot(1-SG)\cdot\tan(\theta)$$

Hence $\quad F_D = \rho\cdot g\cdot\dfrac{\pi\cdot D^3}{6}\cdot(1-SG)\cdot\tan(\theta)$ $\quad$ But we have $\quad F_D = C_D\cdot\dfrac{1}{2}\cdot\rho\cdot V^2\cdot A = C_D\cdot\dfrac{1}{2}\cdot\rho\cdot V^2\cdot\dfrac{\pi\cdot D^2}{4}$

Therefore $\quad C_D\cdot\dfrac{1}{2}\cdot\rho\cdot V^2\cdot\dfrac{\pi\cdot D^2}{4} = \rho\cdot g\cdot\dfrac{\pi\cdot D^3}{6}\cdot(1-SG)\cdot\tan(\theta)$

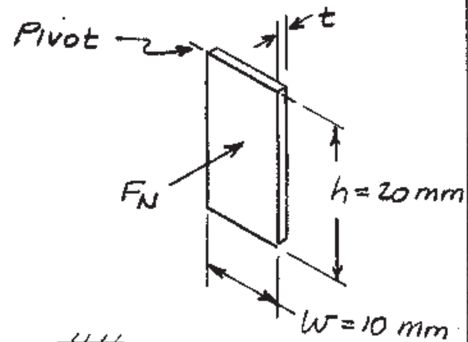
Solving for θ: $\quad \tan(\theta) = \dfrac{3}{4}\cdot\dfrac{C_D\cdot V^2}{g\cdot D\cdot(1-SG)}$

$\theta = \operatorname{atan}\left[\dfrac{3}{4}\cdot\dfrac{C_D\cdot V^2}{g\cdot D\cdot(1-SG)}\right]$ $\quad$ The angle with the horizontal is: $\quad \alpha = 90\cdot deg - \theta \qquad \alpha = 50.7\cdot deg$

Problem 9.105

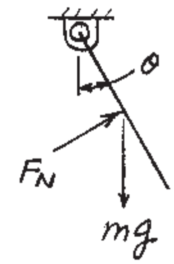
[Difficulty: 2]

9.105 A simple but effective anemometer to measure wind speed can be made from a thin plate hinged to deflect in the wind. Consider a thin plate made from brass that is 20 mm high and 10 mm wide. Derive a relationship for wind speed as a function of deflection angle, θ. What thickness of brass should be used to give $\theta = 30°$ at 10 m/s?



Solution: Sum moments about pivot.

$$\Sigma M = F_N \frac{h}{2} - mg\frac{h}{2}\sin\theta = 0$$



$$F_N = C_D A \tfrac{1}{2}\rho V_n^2 = mg\sin\theta$$

$$C_D A \tfrac{1}{2}\rho V^2\cos^2\theta = mg\sin\theta \qquad (1)$$

$$V = \left[\frac{2mg\sin\theta}{C_D A\rho\cos^2\theta}\right]^{1/2}$$ V

From plate geometry, $m = \rho wht = SG\rho_{H_2O}wht$. From Eq. 1,

$$SG\rho_{H_2O}gwht\sin\theta = C_D A\tfrac{1}{2}\rho V^2\cos^2\theta \qquad \{\text{From Table A.1, } SG = 8.55 \text{ for brass.}\}$$

$$t = \frac{C_D A\rho V^2\cos^2\theta}{2SG\rho_{H_2O}wh\sin\theta g} = \frac{C_D\rho V^2\cos^2\theta}{2SG\rho_{H_2O}\sin\theta g} \qquad \text{since } A = wh$$

From Fig. 9.10, $C_D = 1.2$ at $b/h = 2.0$, so

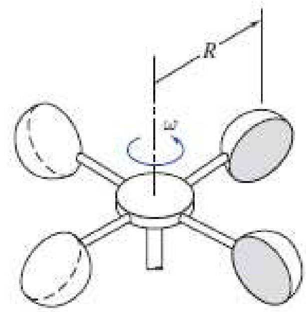
$$t = \frac{1.2}{8.55}\times\frac{1.23}{2}\frac{kg}{m^3}\times(10)^2\frac{m^2}{s^2}\times\cos^2(30°)\times\frac{m^3}{999\,kg}\times\frac{1}{\sin(30°)}\times\frac{s^2}{9.81\,m}\times 1000\frac{mm}{m}$$

$$t = 1.30\text{ mm}$$ t

Problem 9.106

[Difficulty: 3]

9.106 An anemometer to measure wind speed is made from four hemispherical cups of 2-in. diameter, as shown. The center of each cup is placed at $R = 3$ in. from the pivot. Find the theoretical calibration constant, k, in the calibration equation $V = k\omega$, where V (mph) is the wind speed and ω (rpm) is the rotation speed. In your analysis, base the torque calculations on the drag generated at the instant when two of the cups are orthogonal and the other two cups are parallel, and ignore friction in the bearings. Explain why, in the absence of friction, at any given wind speed, the anemometer runs at constant speed rather than accelerating without limit. If the actual anemometer bearing has (constant) friction such that the anemometer needs a minimum wind speed of 0.5 mph to begin rotating, compare the rotation speeds with and without friction for $V = 20$ mph.



Given: Data on dimensions of anemometer

Find: Calibration constant; compare to actual with friction

Solution:

The given data or available data is $D = 2\cdot\text{in}$ $\quad R = 3\cdot\text{in}$ $\quad \rho = 0.00234\cdot\dfrac{\text{slug}}{\text{ft}^3}$

The drag coefficients for a cup with open end facing the airflow and a cup with open end facing downstream are, respectively, from Table 9.3

$$C_{Dopen} = 1.42 \qquad C_{Dnotopen} = 0.38$$

The equation for computing drag is $F_D = \dfrac{1}{2}\cdot\rho\cdot A\cdot V^2\cdot C_D$ (1)

where $A = \dfrac{\pi\cdot D^2}{4}$ $\qquad A = 0.0218\cdot\text{ft}^2$

Assuming steady speed ω at steady wind speed *V* the sum of moments will be zero. The two cups that are momentarily parallel to the flow will exert no moment; the two cups with open end facing and not facing the flow will exert a moment beacuse of their drag forces. For each, the drag is based on Eq. 1 (with the *relative* velocity used!). In addition, friction of the anemometer is neglected

$$\Sigma M = 0 = \left[\frac{1}{2}\cdot\rho\cdot A\cdot(V - R\cdot\omega)^2\cdot C_{Dopen}\right]\cdot R - \left[\frac{1}{2}\cdot\rho\cdot A\cdot(V + R\cdot\omega)^2\cdot C_{Dnotopen}\right]\cdot R$$

or $$(V - R\cdot\omega)^2\cdot C_{Dopen} = (V + R\cdot\omega)^2\cdot C_{Dnotopen}$$

This indicates that the anemometer reaches a steady speed even in the abscence of friction because it is the *relative* velocity on each cup that matters: the cup that has a higher drag coefficient has a lower relative velocity

Rearranging for $k = \dfrac{V}{\omega}$ $\qquad \left(\dfrac{V}{\omega} - R\right)^2\cdot C_{Dopen} = \left(\dfrac{V}{\omega} + R\right)^2\cdot C_{Dnotopen}$

Hence $$k = \frac{\left(1 + \sqrt{\frac{C_{Dnotopen}}{C_{Dopen}}}\right)}{\left(1 - \sqrt{\frac{C_{Dnotopen}}{C_{Dopen}}}\right)} \cdot R \qquad k = 9.43 \cdot in \qquad k = 0.0561 \cdot \frac{mph}{rpm}$$

For the actual anemometer (with friction), we first need to determine the torque produced when the anemometer is stationary but about to rotate

Minimum wind for rotation is $V_{min} = 0.5 \cdot mph$

The torque produced at this wind speed is

$$T_f = \left(\frac{1}{2} \cdot \rho \cdot A \cdot V_{min}^2 \cdot C_{Dopen}\right) \cdot R - \left(\frac{1}{2} \cdot \rho \cdot A \cdot V_{min}^2 \cdot C_{Dnotopen}\right) \cdot R$$

$$T_f = 3.57 \times 10^{-6} \cdot ft \cdot lbf$$

A moment balance at wind speed *V*, including this friction, is

$$\Sigma M = 0 = \left[\frac{1}{2} \cdot \rho \cdot A \cdot (V - R \cdot \omega)^2 \cdot C_{Dopen}\right] \cdot R - \left[\frac{1}{2} \cdot \rho \cdot A \cdot (V + R \cdot \omega)^2 \cdot C_{Dnotopen}\right] \cdot R - T_f$$

or $$(V - R \cdot \omega)^2 \cdot C_{Dopen} - (V + R \cdot \omega)^2 \cdot C_{Dnotopen} = \frac{2 \cdot T_f}{R \cdot \rho \cdot A}$$

This quadratic equation is to be solved for ω when $V = 20 \cdot mph$

After considerable calculations $\omega = 356.20 \cdot rpm$

This must be compared to the rotation for a frictionless model, given by

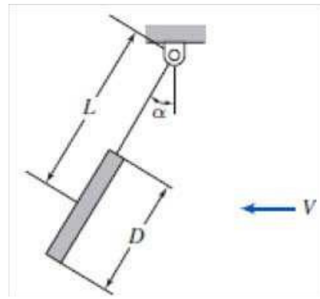
$$\omega_{frictionless} = \frac{V}{k} \qquad \omega_{frictionless} = 356.44 \cdot rpm$$

The error in neglecting friction is $\left|\frac{\omega - \omega_{frictionless}}{\omega}\right| = 0.07 \cdot \%$

Problem 9.107

[Difficulty: 2]

9.107 A circular disk is hung in an air stream from a pivoted strut as shown. In a wind-tunnel experiment, performed in air at 15 m/s with a 25-mm diameter disk, α was measured at 10°. For these conditions determine the mass of the disk. Assume the drag coefficient for the disk applies when the component of wind speed normal to the disk is used. Assume drag on the strut and friction in the pivot are negligible. Plot a theoretical curve of α as a function of air speed.



Given: Circular disk in wind

Find: Mass of disk; Plot α versus V

Solution:

Basic equations: $C_D = \dfrac{F_D}{\frac{1}{2}\cdot\rho\cdot V^2\cdot A}$ $\qquad \Sigma\vec{M} = 0$

Summing moments at the pivo $W\cdot L\cdot\sin(\alpha) - F_n\cdot L = 0$ and $F_n = \frac{1}{2}\cdot\rho\cdot V_n^{\ 2}\cdot A\cdot C_D$

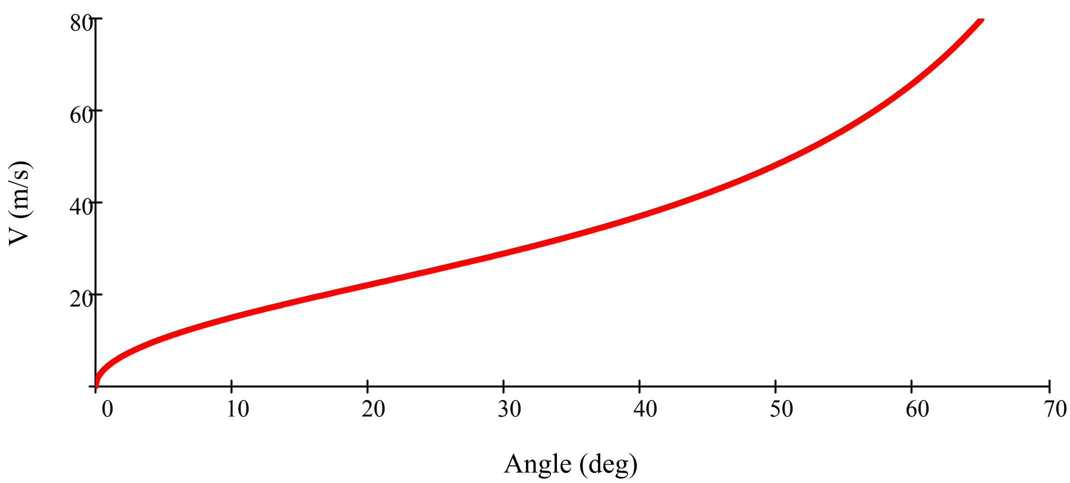
Hence $M\cdot g\cdot\sin(\alpha) = \frac{1}{2}\cdot\rho\cdot(V\cdot\cos(\alpha))^2\cdot\frac{\pi\cdot D^2}{4}\cdot C_D$

The data is $\rho = 1.225\cdot\frac{kg}{m^3}$ $\quad V = 15\cdot\frac{m}{s}$ $\quad D = 25\cdot mm$ $\quad \alpha = 10\cdot deg$ $\quad C_D = 1.17$ (Table 9.3)

$$M = \frac{\pi\cdot\rho\cdot V^2\cdot\cos(\alpha)^2\cdot D^2\cdot C_D}{8\cdot g\cdot\sin(\alpha)} \qquad M = 0.0451\,kg$$

Rearranging $V = \sqrt{\dfrac{8\cdot M\cdot g}{\pi\cdot\rho\cdot D^2\cdot C_D}}\cdot\sqrt{\dfrac{\tan(\alpha)}{\cos(\alpha)}}$ $\qquad V = 35.5\cdot\frac{m}{s}\cdot\sqrt{\dfrac{\tan(\alpha)}{\cos(\alpha)}}$

We can plot this by choosing α and computing V



This graph can be easily plotted in *Excel*

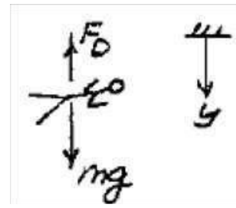
Problem 9.108

[Difficulty: 2]

9.108 Experimental data [16] suggest that the maximum and minimum drag area ($C_D A$) for a skydiver varies from about 9.1 ft^2 for a prone, spread-eagle position to about 1.2 ft^2 for vertical fall. Estimate the terminal speeds for a 170-lb skydiver in each position. Calculate the time and distance needed for the skydiver to reach 90 percent of terminal speed at an altitude of 9800 ft on a standard day.

Given: Mass, maximum and minimum drag areas for a skydiver

Find: (a) Terminal speeds for skydiver in each position
(b) Time and distance needed to reach percentage of terminal speed from given altitude



Solution:

Basic equation $\quad C_D = \dfrac{F_D}{\frac{1}{2}\cdot\rho\cdot U^2\cdot A}$

The given or available data are: $\quad W = 170\cdot lbf \quad AC_{Dmin} = 1.2\cdot ft^2 \quad AC_{Dmax} = 9.1\cdot ft^2 \quad H = 9800\cdot ft = 2987\,m$

From Table A.3 we can find the density: $\quad \dfrac{\rho}{\rho_{SL}} = 0.7433 \qquad \rho = 0.002377\cdot\dfrac{slug}{ft^3} \times 0.7433 = 1.767\times 10^{-3}\cdot\dfrac{slug}{ft^3}$

To find terminal speed, we take FBD of the skydiver: $\quad \Sigma F_y = 0 \quad M\cdot g - F_D = 0 \quad F_D = \frac{1}{2}\cdot\rho\cdot U^2\cdot A\cdot C_D = M\cdot g = W$

Solving for the speed: $\quad U_t = \sqrt{\dfrac{2\cdot W}{\rho\cdot A\cdot C_D}}$ $\quad$ For the minimum drag area: $\quad U_{tmax} = \sqrt{\dfrac{2\cdot W}{\rho\cdot AC_{Dmin}}} \qquad U_{tmax} = 400\cdot\dfrac{ft}{s}$

For the maximum drag area: $\quad U_{tmin} = \sqrt{\dfrac{2\cdot W}{\rho\cdot AC_{Dmax}}} \qquad U_{tmin} = 145.4\cdot\dfrac{ft}{s}$

To find the time needed to reach a fraction of the terminal velocity, we re-write the force balance:

$$\Sigma F_y = M\cdot a_y \qquad M\cdot g - F_D = M\cdot a_y \qquad M\cdot g - \frac{1}{2}\cdot\rho\cdot U^2\cdot A\cdot C_D = M\cdot\frac{dU}{dt} = M\cdot U\cdot\frac{dU}{dy}$$

In terms of the weight: $\quad W - \frac{1}{2}\cdot\rho\cdot U^2\cdot A\cdot C_D = \dfrac{W}{g}\cdot\dfrac{dU}{dt} = \dfrac{W}{g}\cdot U\cdot\dfrac{dU}{dy}$ $\quad$ To normalize the derivatives by the terminal speed:

$$\frac{W}{g}\cdot\frac{dU}{dt} = \frac{W\cdot U_t}{g}\cdot\frac{d}{dt}\left(\frac{U}{U_t}\right) \quad \text{and} \quad \frac{W}{g}\cdot U\cdot\frac{dU}{dy} = \frac{W\cdot U_t^2}{g}\cdot\left(\frac{U}{U_t}\right)\cdot\frac{d}{dy}\left(\frac{U}{U_t}\right)$$

We may now re-write the above equation as:

$$W - \frac{1}{2}\cdot\rho\cdot\left(\frac{U}{U_t}\right)^2\cdot A\cdot C_D\cdot\left(\frac{2\cdot W}{\rho\cdot A\cdot C_D}\right) = \frac{W\cdot U_t}{g}\cdot\frac{d}{dt}\left(\frac{U}{U_t}\right) = \frac{W\cdot U_t^2}{g}\cdot\left(\frac{U}{U_t}\right)\cdot\frac{d}{dy}\left(\frac{U}{U_t}\right)$$

where we have substituted for the terminal speed.

Simplifying this expression: $\quad 1 - \left(\dfrac{U}{U_t}\right)^2 = \dfrac{U_t}{g}\cdot\dfrac{d}{dt}\left(\dfrac{U}{U_t}\right) = \dfrac{U_t^2}{g}\cdot\left(\dfrac{U}{U_t}\right)\cdot\dfrac{d}{dy}\left(\dfrac{U}{U_t}\right)$

Now we can integrate with respect to time and distance:

If we let $U_n = \dfrac{U}{U_t}$ we can rewrite the equations:

$1 - U_n^{\,2} = \dfrac{U_t}{g}\cdot\dfrac{dU_n}{dt}$ Separating variables: $\displaystyle\int_0^t \frac{g}{U_t}\,dt = \int_0^{0.90} \frac{1}{1 - U_n^{\,2}}\,dU_n$ Integrating we get: $\dfrac{g\cdot t}{U_t} = \text{atanh}(0.9) - \text{atanh}(0)$

Evaluating the inverse hyperbolic tangents: $t = \dfrac{1.472\cdot U_t}{g}$ so: $t_{min} = \dfrac{1.472\cdot U_{tmin}}{g} = 6.65\,\text{s}$ $t_{max} = \dfrac{1.472\cdot U_{tmax}}{g} = 18.32\,\text{s}$

Now to find the distance: $1 - U_n^{\,2} = \dfrac{U_t^{\,2}}{g}\cdot U_n\cdot\dfrac{dU_n}{dy}$ Separating variables: $\displaystyle\int_0^y \frac{g}{U_t^{\,2}}\,dy = \int_0^{0.9} \frac{U_n}{1 - U_n^{\,2}}\,dU_n$

Integrating we get: $\dfrac{g\cdot y}{U_t^{\,2}} = -\dfrac{1}{2}\cdot\ln\!\left(\dfrac{1 - 0.9^2}{1 - 0}\right) = 0.8304$ Solving for the distance: $y = \dfrac{0.8304\cdot U_t^{\,2}}{g}$

so: $y_{min} = \dfrac{0.8304\cdot U_{tmin}^{\,2}}{g} = 166.4\,\text{m}$ $y_{max} = \dfrac{0.8304\cdot U_{tmax}^{\,2}}{g} = 1262\,\text{m}$

Problem 9.109

[Difficulty: 3]

9.109 A vehicle is built to try for the land-speed record at the Bonneville Salt Flats, elevation 4400 ft. The engine delivers 500 hp to the rear wheels, and careful streamlining has resulted in a drag coefficient of 0.15, based on a 15 ft^2 frontal area. Compute the theoretical maximum ground speed of the car (a) in still air and (b) with a 20 mph headwind.

Solution: Apply definitions of power, drag coefficient.

Computing equations: $\mathbb{P} = F_D V$, $C_D = \dfrac{F_D}{\frac{1}{2}\rho (V+V_w)^2 A}$

Assumptions: (1) Neglect rolling drag

(2) $\rho \simeq 0.878\,\rho_0$ (Table A.3)

For no wind case, $V_w = 0$, and

$$\mathbb{P} = F_D V = C_D \tfrac{1}{2}\rho V^2 A V = C_D \tfrac{1}{2}\rho V^3 A$$

$$V = \left[\frac{2\mathbb{P}}{\rho C_D A}\right]^{1/3} = \left[2 \times 500\,hp \times \frac{ft^3}{(0.878)0.00238\,slug} \times \frac{1}{0.15} \times \frac{1}{15\,ft^2} \times 550\,\frac{ft\cdot lbf}{hp\cdot s} \times \frac{slug\cdot ft}{lbf\cdot s^2}\right]^{1/3}$$

$V = 489$ ft/s (333 mph) $\qquad V$

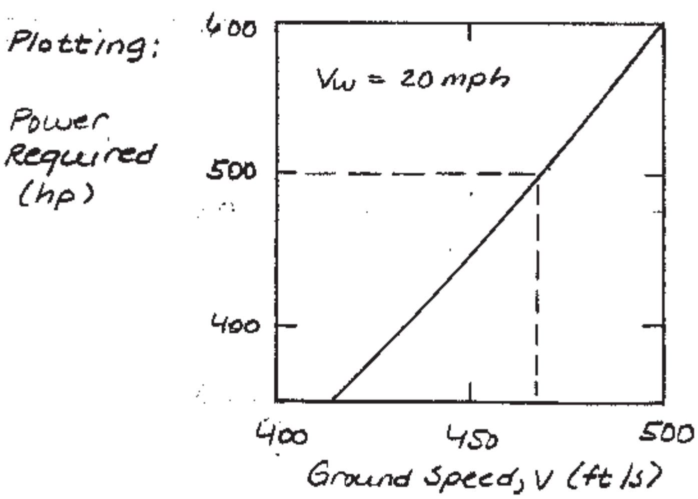
With a head wind,

$$\mathbb{P} = F_D V = C_D \tfrac{1}{2}\rho (V+V_w)^2 A V \quad \text{or} \quad \mathbb{P}(hp) = 4.27\times 10^{-6}(V+V_w)^2 V \;(ft^3/s^3)$$

This can be solved by iteration. Using $V_w = 20$ mph or 29.3 ft/s,

V (ft/s)	$\mathbb{P}$ (hp)
400	315
470	501
500	599

Plotting:



From the plot, $V \simeq 468$ ft/s (319 mph) $\qquad V$

{Note that the maximum speed is not reduced by 20 mph when wind is present, because drag is nonlinear.}

Problem 9.110

[Difficulty: 3]

9.110 An F-4 aircraft is slowed after landing by dual parachutes deployed from the rear. Each parachute is 12 ft in diameter. The F-4 weighs 32,000 lbf and lands at 160 knots. Estimate the time and distance required to decelerate the aircraft to 100 knots, assuming that the brakes are not used and the drag of the aircraft is negligible.

Solution: Apply Newton's second law of motion, definition of C_D.

Basic equations: $\Sigma F_x = m a_x$

$$C_D = \frac{F_D}{\frac{1}{2}\rho V^2 A}$$

Then

$$\Sigma F_x = -2F_D = -C_D \rho V^2 A = m a_x = \frac{W}{g}\frac{dV}{dt} \qquad (1)$$

or $\dfrac{dV}{V^2} = -\dfrac{C_D \rho g A}{W}\, dt$

Integrating,

$$\int_{V_i}^{V_f} \frac{dV}{V^2} = -\frac{1}{V}\Big]_{V_i}^{V_f} = \frac{1}{V_i} - \frac{1}{V_f} = \int_0^t -\frac{C_D \rho g A}{W}\, dt = -\frac{C_D \rho g A}{W} t$$

or

$$t = \frac{W}{C_D \rho g A}\left[\frac{1}{V_f} - \frac{1}{V_i}\right]$$

Since two chutes (assume hemispheres), $A = 2\left(\frac{\pi D^2}{4}\right) = \frac{\pi D^2}{2}$

From Table 9.3, $C_D = 1.42$ for hemisphere facing stream. For standard air, $\rho g = \gamma \simeq 0.075 \text{ lbf/ft}^3$, and

$$t = 32{,}000 \text{ lbf} \times \frac{1}{1.42} \times \frac{\text{ft}^3}{0.075 \text{ lbf}} \times \frac{2}{\pi} \times \frac{1}{(12)^2 \text{ft}^2}\left[\frac{1}{100} - \frac{1}{160}\right]\frac{\text{hr}}{\text{nm}} \times 3600\frac{\text{s}}{\text{hr}} \times \frac{\text{nm}}{6080 \text{ ft}}$$

or

$$t = 2.95 \text{ s}$$ t

To find distance, set $a_x = \frac{dV}{dt} = V\frac{dV}{dx}$. Then, from Eq. 1,

$$-2C_D \frac{\rho V^2}{2} A = \frac{W}{g} V \frac{dV}{dx}$$

and $\dfrac{dV}{V} = -\dfrac{C_D A \rho g}{W}\, dx$

Integrating,

$$\int_{V_i}^{V_f} \frac{dV}{V} = \ln V\Big]_{V_i}^{V_f} = \ln\frac{V_f}{V_i} = -\frac{C_D A \rho g}{W} x \quad \text{or} \quad x = -\frac{W}{C_D A \rho g}\ln\frac{V_f}{V_i}$$

Thus

$$x = -\frac{1}{1.42} \times 32{,}000 \text{ lbf} \times \frac{2}{\pi (12)^2 \text{ft}^2} \times \frac{\text{ft}^3}{0.075 \text{ lbf}} \times \ln\left(\frac{100}{160}\right) = 624 \text{ ft}$$ x

Problem 9.111

[Difficulty: 3]

9.111 A tractor-trailer rig has frontal area $A = 102\ \text{ft}^2$ and drag coefficient $C_D = 0.9$. Rolling resistance is 6 lbf per 1000 lbf of vehicle weight. The specific fuel consumption of the diesel engine is 0.34 lbm of fuel per horsepower hour, and drivetrain efficiency is 92 percent. The density of diesel fuel is 6.9 lbm/gal. Estimate the fuel economy of the rig at 55 mph if its gross weight is 72,000 lbf. An air fairing system reduces aerodynamic drag 15 percent. The truck travels 120,000 miles per year. Calculate the fuel saved per year by the roof fairing.

Solution: Tractive force is $F_T = F_R + F_D$ ← aerodynamic force

↖ rolling resistance force

Engine power is $P_e = \dfrac{P_T}{\eta_d} = \dfrac{F_T V}{\eta_d}$

Thus

$$F_R = C_R W = 0.006 \times 72{,}000\,\text{lbf} = 432\ \text{lbf}$$

$$F_D = C_D A \tfrac{1}{2}\rho V^2 \qquad V = 55\,\frac{\text{mi}}{\text{hr}} \times 5280\,\frac{\text{ft}}{\text{mi}} \times \frac{\text{hr}}{3600\,\text{s}} = 80.7\ \text{ft/s}$$

$$F_D = 0.9 \times 102\,\text{ft}^2 \times \frac{1}{2} \times 0.00238\,\frac{\text{slug}}{\text{ft}^3} \times (80.7)^2\,\frac{\text{ft}^2}{\text{s}^2} \times \frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}} = 711\ \text{lbf}$$

so

$$F_T = F_R + F_D = 432 + 711 = 1140\ \text{lbf}$$

$$P_e = 1140\,\text{lbf} \times 80.7\,\frac{\text{ft}}{\text{s}} \times \frac{1}{0.92} \times \frac{\text{hp}\cdot\text{s}}{550\,\text{ft}\cdot\text{lbf}} = 182\ \text{hp}$$

Finally

$$FE = \frac{\rho V}{P_e\, BSFC} = 6.9\,\frac{\text{lbm}}{\text{gal}} \times 55\,\frac{\text{mi}}{\text{hr}} \times \frac{1}{182\,\text{hp}} \times \frac{\text{hp}\cdot\text{hr}}{0.34\,\text{lbm}} = 6.13\ \text{mi/gal}$$

FE

With the air deflector,

$$F_D = (1 - 0.15)\,711\ \text{lbf} = 604\ \text{lbf}$$

$$F_T = F_R + F_D = 432 + 604 = 1040\ \text{lbf}$$

$$P_e = 1040\,\text{lbf} \times 80.7\,\frac{\text{ft}}{\text{s}} \times \frac{1}{0.92} \times \frac{\text{hp}\cdot\text{s}}{550\,\text{ft}\cdot\text{lbf}} = 166\ \text{hp}$$

and $$FE = 6.9\,\frac{\text{lbm}}{\text{gal}} \times 55\,\frac{\text{mi}}{\text{hr}} \times \frac{1}{166\,\text{hp}} \times \frac{\text{hp}\cdot\text{hr}}{0.34\,\text{lbm}} = 6.72\ \text{mi/gal}$$

The fuel saving would be

$$\Delta Q = \left(\frac{1}{FE}\Big)_{\text{without}} - \frac{1}{FE_{\text{with}}}\right)\text{mileage} = \left(\frac{1}{6.13} - \frac{1}{6.72}\right)\frac{\text{gal}}{\text{mi}} \times 120{,}000\,\frac{\text{mi}}{\text{yr}} = 1720\ \text{gal/yr}$$

Q

The percentage saving would be

$$\frac{\Delta Q}{Q} = \frac{\left(\dfrac{1}{FE_{\text{without}}} - \dfrac{1}{FE_{\text{with}}}\right)}{\dfrac{1}{FE}\Big)_{\text{without}}} = 0.0878 \text{ or } 8.78 \text{ percent savings}$$

Problem 9.112

[Difficulty: 3]

9.112 A bus travels at 80 km/h in standard air. The frontal area of the vehicle is 7.5 m^2, and the drag coefficient is 0.92. How much power is required to overcome aerodynamic drag? Estimate the maximum speed of the bus if the engine is rated at 465 hp. A young engineer proposes adding fairings on the front and rear of the bus to reduce the drag coefficient. Tests indicate that this would reduce the drag coefficient to 0.86 without changing the frontal area. What would the required power be at 80 km/h and the new top speed? If the fuel cost for the bus is currently \$300/day, how long would the modification take to pay for itself if it costs \$4800 to install?

Given: Data on a bus

Find: Power to overcome drag; Maximum speed; Recompute with new fairing; Time for fairing to pay for itself

Solution:

Basic equation: $F_D = \frac{1}{2}\cdot\rho\cdot A\cdot V^2\cdot C_D \qquad P = F_D\cdot V$

The given data or available data is $\quad V = 80\cdot\frac{km}{hr} \qquad V = 22.2\cdot\frac{m}{s} \qquad A = 7.5\cdot m^2 \qquad C_D = 0.92 \qquad \rho = 1.23\cdot\frac{kg}{m^3}$

$F_D = \frac{1}{2}\cdot\rho\cdot A\cdot V^2\cdot C_D \qquad F_D = 2096\cdot N \qquad P = F_D\cdot V \qquad P = 46.57\cdot kW$ The power available is $P_{max} = 465\cdot hp = 346.75\cdot kW$

The maximum speed corresponding to this maximum power is obtained from

$$P_{max} = \left(\frac{1}{2}\cdot\rho\cdot A\cdot V_{max}{}^2\cdot C_D\right)\cdot V_{max} \quad \text{or} \quad V_{max} = \left(\frac{P_{max}}{\frac{1}{2}\cdot\rho\cdot A\cdot C_D}\right)^{\frac{1}{3}} \qquad V_{max} = 43.4\cdot\frac{m}{s} \quad V_{max} = 156.2\cdot\frac{km}{hr}$$

We repeat these calculations with the new fairing, for which $C_D = 0.86$

$$F_D = \frac{1}{2}\cdot\rho\cdot A\cdot V^2\cdot C_D \qquad F_D = 1959\cdot N \qquad P_{new} = F_D\cdot V \qquad P_{new} = 43.53\cdot kW$$

The maximum speed is now
$$V_{max} = \left(\frac{P_{max}}{\frac{1}{2}\cdot\rho\cdot A\cdot C_D}\right)^{\frac{1}{3}} \qquad V_{max} = 44.4\cdot\frac{m}{s} \quad V_{max} = 159.8\cdot\frac{km}{hr}$$

The initial cost of the fairing is $\quad Cost = 4500\cdot dollars \qquad$ The fuel cost is $\quad Cost_{day} = 300\cdot\frac{dollars}{day}$

The cost per day is reduced by improvement in the bus performance at 80 km/h $\quad Gain = \frac{P_{new}}{P} \qquad Gain = 93.5\cdot\%$

The new cost per day is then $\quad Cost_{daynew} = Gain\cdot Cost_{day} \qquad Cost_{daynew} = 280\cdot\frac{dollars}{day}$

Hence the savings per day is $\quad Saving = Cost_{day} - Cost_{daynew} \qquad Saving = 19.6\cdot\frac{dollars}{day}$

The initial cost will be paid for in $\quad \tau = \frac{Cost}{Saving} \qquad \tau = 7.56\cdot month$

Problem 9.113

[Difficulty: 2]

9.113 Compare and plot the power (hp) required by a typical large American sedan of the 1970s and a current midsize sedan to overcome aerodynamic drag versus road speed in standard air, for a speed range of 20 mph to 100 mph. Use the following as representative values:

	Weight (lbf)	Drag Coefficient	Frontal Area (ft^2)
1970s Sedan	4500	0.5	24
Current Sedan	3500	0.3	20

If rolling resistance is 1.5 percent of curb weight, determine for each vehicle the speed at which the aerodynamic force exceeds frictional resistance.

Given: Data on 1970's and current sedans

Find: Plot of power versus speed; Speeds at which aerodynamic drag exceeds rolling drag

Solution:

Basic equation: $$C_D = \frac{F_D}{\frac{1}{2}\cdot\rho\cdot V^2\cdot A}$$

The aerodynamic drag is $F_D = C_D\cdot\frac{1}{2}\cdot\rho\cdot V^2\cdot A$ The rolling resistance is $F_R = 0.015\cdot W$

Total resistance $F_T = F_D + F_R$ The results generated in *Excel* are shown below:

ρ = 0.00234 slug/ft^3 (Table A.9)

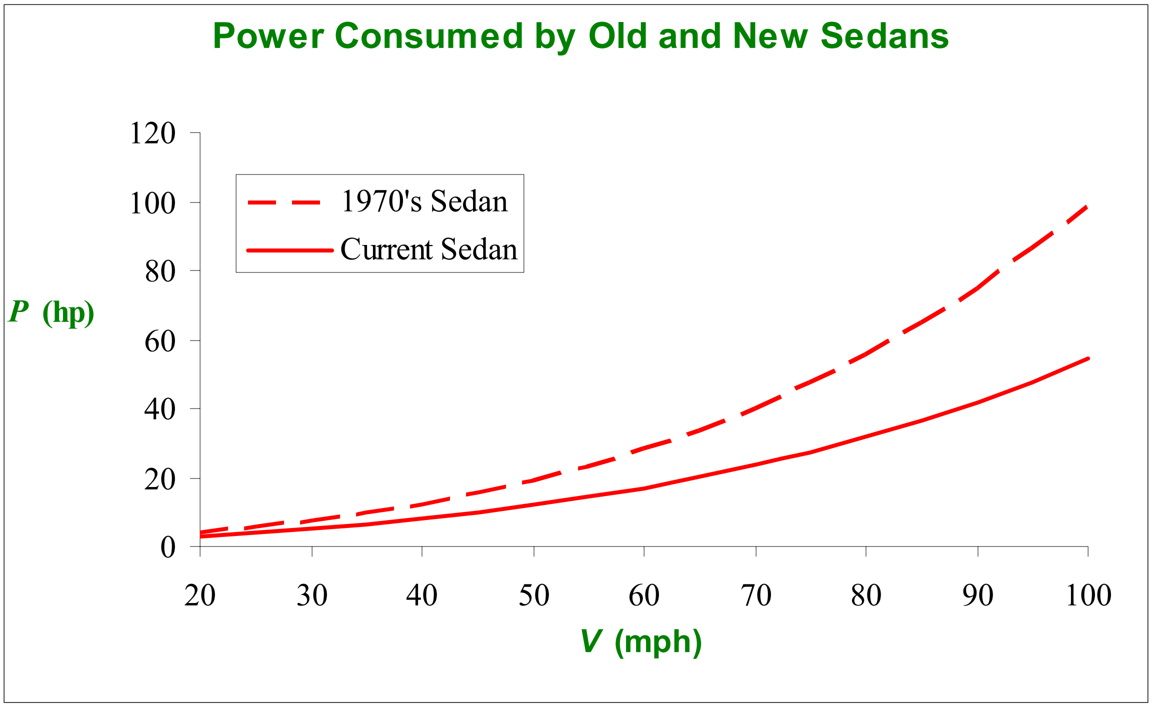
Computed results:

	1970's Sedan			Current Sedan		
V **(mph)**	F_D **(lbf)**	F_T **(lbf)**	***P*** **(hp)**	F_D **(lbf)**	F_T **(lbf)**	***P*** **(hp)**
20	12.1	79.6	4.24	6.04	58.5	3.12
25	18.9	86.4	5.76	9.44	61.9	4.13
30	27.2	94.7	7.57	13.6	66.1	5.29
35	37.0	104	9.75	18.5	71.0	6.63
40	48.3	116	12.4	24.2	76.7	8.18
45	61.2	129	15.4	30.6	83.1	10.0
50	75.5	143	19.1	37.8	90.3	12.0
55	91.4	159	23.3	45.7	98.2	14.4
60	109	176	28.2	54.4	107	17.1
65	128	195	33.8	63.8	116	20.2
70	148	215	40.2	74.0	126	23.6
75	170	237	47.5	84.9	137	27.5
80	193	261	55.6	96.6	149	31.8
85	218	286	64.8	109	162	36.6
90	245	312	74.9	122	175	42.0
95	273	340	86.2	136	189	47.8
100	302	370	98.5	151	204	54.3

V **(mph)**	F_D **(lbf)**	F_R **(lbf)**
47.3	67.5	67.5

V **(mph)**	F_D **(lbf)**	F_R **(lbf)**
59.0	52.5	52.5

The two speeds above were obtained using *Solver*



Power Consumed by Old and New Sedans
120
100
80
60
40
20
0
P (hp)
1970's Sedan
Current Sedan
20
30
40
50
60
70
80
90
100
V (mph)

Problem 9.114

[Difficulty: 4]

9.114 A 180-hp sports car of frontal area 1.72 m^2, with a drag coefficient of 0.31, requires 17 hp to cruise at 100 km/h. At what speed does aerodynamic drag first exceed rolling resistance? (The rolling resistance is 1.2 percent of the car weight, and the car mass is 1250 kg.) Find the drivetrain efficiency. What is the maximum acceleration at 100 km/h? What is the maximum speed? Which redesign will lead to a higher maximum speed: improving the drive train efficiency by 6 percent from its current value, reducing the drag coefficient to 0.29, or reducing the rolling resistance to 0.91 percent of the car weight?

Given: Data on a sports car

Find: Speed for aerodynamic drag to exceed rolling resistance; maximum speed & acceleration at 100 km/h; Redesign change that has greatest effect

Solution:

Basic equation: $F_D = \frac{1}{2}\cdot\rho\cdot A\cdot V^2\cdot C_D$ $\qquad P = F_D\cdot V$

The given data or available data is $\quad M = 1250\cdot kg \quad A = 1.72\cdot m^2 \quad C_D = 0.31$

$P_{engine} = 180\cdot hp = 134.23\cdot kW \quad F_R = 0.012 \times M\cdot g \quad \rho = 1.23\cdot\frac{kg}{m^3}$

The rolling resistance is then $\quad F_R = 147.1\cdot N$

To find the speed at which aerodynamic drag first equals rolling resistance, set the two forces equal $\quad \frac{1}{2}\cdot\rho\cdot V^2\cdot A\cdot C_D = F_R$

Hence $V = \sqrt{\frac{2\cdot F_R}{\rho\cdot A\cdot C_D}} \qquad V = 21.2\cdot\frac{m}{s} \qquad V = 76.2\cdot\frac{km}{hr}$

To find the drive train efficiency we use the data at a speed of $\quad V = 100\cdot\frac{km}{hr} \quad V = 27.8\cdot\frac{m}{s} \quad P_{engine} = 17\cdot hp = 12.677\cdot kW$

The aerodynamic drag at this speed is $\quad F_D = \frac{1}{2}\cdot\rho\cdot V^2\cdot A\cdot C_D \qquad F_D = 253\cdot N$

The power consumed by drag and rolling resistance at this speed is $\quad P_{used} = \left(F_D + F_R\right)\cdot V \qquad P_{used} = 11.1\cdot kW$

Hence the drive train efficiency is $\quad \eta = \frac{P_{used}}{P_{engine}} \qquad \eta = 87.7\cdot\%$

The acceleration is obtained from Newton's second law $\quad M\cdot a = \Sigma F = T - F_R - F_D$

where *T* is the thrust produced by the engine, given by $\quad T = \frac{P}{V}$

The maximum acceleration at 100 km/h is when full engine power is used. $\quad P_{engine} = 180\cdot hp = 134.2\cdot kW$

Because of drive train inefficiencies the maximum power at the wheels is $\quad P_{max} = \eta\cdot P_{engine} \qquad P_{max} = 118\cdot kW$

Hence the maximum thrust is $\quad T_{max} = \frac{P_{max}}{V} \qquad T_{max} = 4237\cdot N$

The maximum acceleration is then $\quad a_{max} = \frac{T_{max} - F_D - F_R}{M} \qquad a_{max} = 3.07\cdot\frac{m}{s^2}$

The maximum speed is obtained when the maximum engine power is just balanced by power consumed by drag and rolling resistance

For maximum speed: $$P_{max} = \left(\frac{1}{2}\cdot\rho\cdot V_{max}^{2}\cdot A\cdot C_D + F_R\right)\cdot V_{max}$$

This is a cubic equation that can be solved by iteration or by using *Excel*'s *Goal Seek* or *Solver* $\quad V_{max} = 248\cdot\frac{km}{hr}$

We are to evaluate several possible improvements:

For improved drive train $\quad \eta = \eta + 6\cdot\% \quad \eta = 93.7\cdot\% \quad P_{max} = \eta\cdot P_{engine} \quad P_{max} = 126\cdot kW$

$$P_{max} = \left(\frac{1}{2}\cdot\rho\cdot V_{max}^{2}\cdot A\cdot C_D + F_R\right)\cdot V_{max}$$

Solving the cubic (using *Solver*) $\quad V_{max} = 254\cdot\frac{km}{hr}$

Improved drag coefficient: $\quad C_{Dnew} = 0.29 \quad P_{max} = 118\cdot kW$

$$P_{max} = \left(\frac{1}{2}\cdot\rho\cdot V_{max}^{2}\cdot A\cdot C_{Dnew} + F_R\right)\cdot V_{max}$$

Solving the cubic (using *Solver*) $\quad V_{max} = 254\cdot\frac{km}{hr}$ This is the best option!

Reduced rolling resistance: $\quad F_{Rnew} = 0.91\cdot\%\cdot M\cdot g \quad F_{Rnew} = 111.6\,N$

$$P_{max} = \left(\frac{1}{2}\cdot\rho\cdot V_{max}^{2}\cdot A\cdot C_D + F_{Rnew}\right)\cdot V_{max}$$

Solving the cubic (using *Solver*) $\quad V_{max} = 250\cdot\frac{km}{hr}$

The improved drag coefficient is the best option.

Problem 9.115

[Difficulty: 3]

9.115 Consider a negatively charged spherical particle of radius a bearing a charge, Q_s, suspended in a pure dielectric fluid (containing no ions). When subject to a uniform electric field, $\vec{E}_\infty$, the particle will translate under the influence of the electric force acting on it. The induced particle motion refers to electrophoresis, which has been widely used to characterize and purify molecules and colloidal particles. The net electrical force on the charged particle will simply be $\vec{F}_E = Q_S\vec{E}_\infty$. As soon as the particle starts to move under the influence of this electric force, it encounters an oppositely directed fluid drag force.

(a) Under the Stokes flow regime and neglecting the gravitational force and the buoyancy force acting on the microparticle, derive an expression to calculate the particle's steady-state translational velocity.

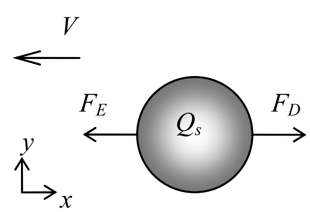
(b) Based on the above results, explain why electrophoresis can be used to separate biological samples.

(c) Calculate the translational velocities of two particles of radius $a = 1\ \mu$m and $10\ \mu$m using $Q_s = -10^{-12}$ C, $E_\infty = 1000$ V/m, and $\mu = 10^{-3}$ Pa·s.

Given: zero net force acting on the particle; drag force and electrostatic force

Find:

Solution:



(a) Under steady-state, the net force acting on the particle is zero. The forces acting on the particle contain the electrostatic force $\mathbf{F}_E$ and the drag force $\mathbf{F}_D$ (Page 418, the first equation right after Fig.8.11).

$$\mathbf{F_E} + \mathbf{F_d} = 0 \rightarrow Q_s\mathbf{E}_\infty + 6\pi\mu\mathbf{U}a = 0 \qquad (1)$$

where **U** is the particle velocity relative to the stationary liquid.

Then one obtains $$\mathbf{U} = \frac{Q_s\mathbf{E}_\infty}{6\pi\mu a} \qquad (2)$$

(b) From the solution, we can know that the particle velocity depends on its size. Smaller particles run faster than larger ones, and thus they can be separated.

(c) Substituting the values of a, Q_s, $\mathbf{E}_\infty$, and μ into equation (2), we obtains the velocity for a=1 μm

$$\mathbf{U} = \frac{-10^{-12}}{6\pi\times10^{-3}}\frac{\text{C}}{\text{Pa}\cdot\text{s}}\times\frac{1000}{1\times10^{-6}}\frac{\text{V/m}}{\text{m}} = -0.053\frac{\text{N}}{\text{Pa}\cdot\text{s}\cdot\text{m}} = -0.053\text{m/s}$$

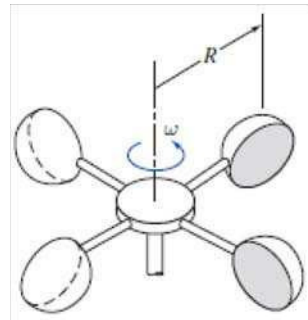
and **U** = 0.0053 m/s for a = 10 μm.

The negatively charged particle moves in the direction opposite to that of the electric field applied.

Problem 9.116

[Difficulty: 5]

9.116 Repeat the analysis for the frictionless anemometer of Problem 9.106, except this time base the torque calculations on the more realistic model that the average torque is obtained by integrating, over one revolution, the instantaneous torque generated by each cup (i.e., as the cup's orientation to the wind varies).



Given: Data on dimensions of anemometer

Find: Calibration constant

Solution:

The given data or available data is $D = 2 \cdot in$ $\quad R = 3 \cdot in$ $\quad \rho = 0.00234 \cdot \frac{slug}{ft^3}$

The drag coefficients for a cup with open end facing the airflow and a cup with open end facing downstream are, respectively, from Table 9.3

$$C_{Dopen} = 1.42 \qquad\qquad C_{Dnotopen} = 0.38$$

Assume the anemometer achieves steady speed ω due to steady wind speed *V*

The goal is to find the calibration constant *k*, defined by $\quad k = \frac{V}{\omega}$

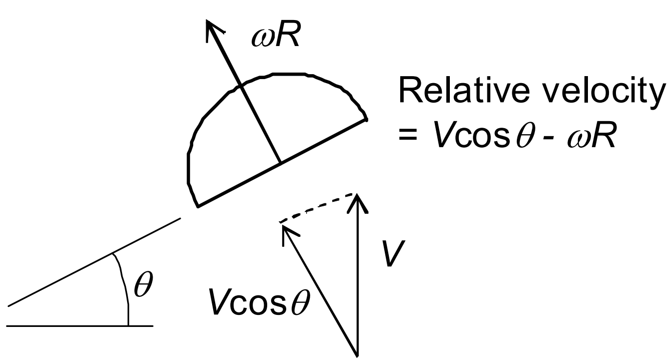
We will analyze each cup separately, with the following assumptions

1) Drag is based on the instantaneous normal component of velocity (we ignore possible effects on drag coefficient of velocity component parallel to the cup)

2) Each cup is assumed unaffected by the others - as if it were the only object present

3) Swirl is neglected

4) Effects of struts is neglected



In this more sophisticated analysis we need to compute the instantaneous normal relative velocity. From the sketch, when a cup is at angle θ, the normal component of relative velocity is

$$V_n = V\cdot\cos(\theta) - \omega\cdot R \qquad (1)$$

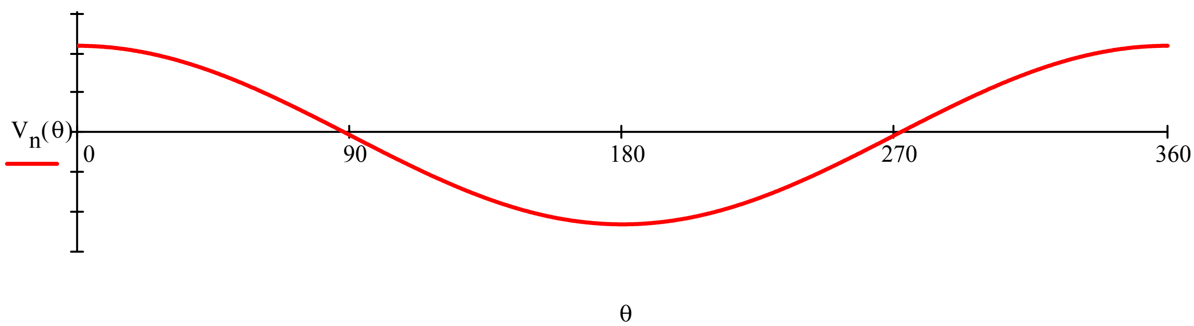
The relative velocity is sometimes positive, and sometimes negatiive. From Eq. 1, this is determined by

$$\theta_c = \operatorname{acos}\left(\frac{\omega\cdot R}{V}\right) \qquad (2)$$

For $0 < \theta < \theta_c$ $\quad V_n > 0$

$\theta_c < \theta < 2\cdot\pi - \theta_c$ $\quad V_n < 0$

$\theta_c < \theta < 2\cdot\pi$ $\quad V_n > 0$



The equation for computing drag is $\quad F_D = \frac{1}{2}\cdot\rho\cdot A\cdot V_n^{\,2}\cdot C_D \qquad (3)$

where $\quad A = \frac{\pi\cdot D^2}{4} \qquad A = 3.14\cdot in^2$

In Eq. 3, the drag coefficient, and whether the drag is postive or negative, depend on the sign of the relative velocity

For $0 < \theta < \theta_c$ $\quad C_D = C_{Dopen}$ $\quad F_D > 0$

$\theta_c < \theta < 2\cdot\pi - \theta_c$ $\quad C_D = C_{Dnotopen}$ $\quad F_D < 0$

$\theta_c < \theta < 2\cdot\pi$ $\quad C_D = C_{Dopen}$ $\quad F_D > 0$

The torque is $\quad T = F_D\cdot R = \frac{1}{2}\cdot\rho\cdot A\cdot V_n^{\,2}\cdot C_D\cdot R$

The average torque is $\quad T_{av} = \frac{1}{2\cdot\pi}\cdot\int_0^{2\cdot\pi} T\,d\theta = \frac{1}{\pi}\cdot\int_0^{\pi} T\,d\theta$

where we have taken advantage of symmetry

Evaluating this, allowing for changes when $\theta = \theta_c$ $\quad T_{av} = \frac{1}{\pi}\cdot\int_0^{\theta_c} \frac{1}{2}\cdot\rho\cdot A\cdot V_n^{\,2}\cdot C_{Dopen}\cdot R\,d\theta - \frac{1}{\pi}\cdot\int_{\theta_c}^{\pi} \frac{1}{2}\cdot\rho\cdot A\cdot V_n^{\,2}\cdot C_{Dnotopen}\cdot R\,d\theta$

Using Eq. 1

$$T_{av} = \frac{\rho \cdot A \cdot R}{2 \cdot \pi} \cdot \left[C_{Dopen} \cdot \int_0^{\theta_c} (V \cdot \cos(\theta) - \omega \cdot R)^2 \, d\theta - C_{Dnotopen} \cdot \int_{\theta_c}^{\pi} (V \cdot \cos(\theta) - \omega \cdot R)^2 \, d\theta \right]$$

$$T_{av} = \frac{\rho \cdot A \cdot R \cdot \omega^2}{2 \cdot \pi} \cdot \left[C_{Dopen} \cdot \int_0^{\theta_c} \left(\frac{V}{\omega} \cdot \cos(\theta) - R \right)^2 d\theta - C_{Dnotopen} \cdot \int_{\theta_c}^{\pi} \left(\frac{V}{\omega} \cdot \cos(\theta) - R \right)^2 d\theta \right]$$

and note that $\frac{V}{\omega} = k$

The integral is $\int (k \cdot \cos(\theta) - R)^2 \, d\theta = k^2 \cdot \left(\frac{1}{2} \cdot \cos(\theta) \cdot \sin(\theta) + \frac{1}{2} \cdot \theta \right) - 2 \cdot k \cdot R \cdot \sin(\theta) + R^2 \cdot \theta$

For convenience define $f(\theta) = k^2 \cdot \left(\frac{1}{2} \cdot \cos(\theta) \cdot \sin(\theta) + \frac{1}{2} \cdot \theta \right) - 2 \cdot k \cdot R \cdot \sin(\theta) + R^2 \cdot \theta$

Hence $T_{av} = \frac{\rho \cdot A \cdot R}{2 \cdot \pi} \cdot \left[C_{Dopen} \cdot f(\theta_c) - C_{Dnotopen} \cdot \left(f(\pi) - f(\theta_c) \right) \right]$

For steady state conditions the torque (of each cup, and of all the cups) is zero. Hence

$$C_{Dopen} \cdot f(\theta_c) - C_{Dnotopen} \cdot \left(f(\pi) - f(\theta_c) \right) = 0$$

or $f(\theta_c) = \frac{C_{Dnotopen}}{C_{Dopen} + C_{Dnotopen}} \cdot f(\pi)$

Hence $k^2 \cdot \left(\frac{1}{2} \cdot \cos(\theta_c) \cdot \sin(\theta_c) + \frac{1}{2} \cdot \theta_c \right) - 2 \cdot k \cdot R \cdot \sin(\theta_c) + R^2 \cdot \theta_c = \frac{C_{Dnotopen}}{C_{Dopen} + C_{Dnotopen}} \cdot \left(k^2 \cdot \frac{\pi}{2} + R^2 \cdot \pi \right)$

Recall from Eq 2 that $\theta_c = \text{acos}\left(\frac{\omega \cdot R}{V} \right)$ or $\theta_c = \text{acos}\left(\frac{R}{k} \right)$

Hence $k^2 \cdot \left(\frac{1}{2} \cdot \frac{R}{k} \cdot \sin\left(\text{acos}\left(\frac{R}{k} \right) \right) + \frac{1}{2} \cdot \text{acos}\left(\frac{R}{k} \right) \right) - 2 \cdot k \cdot R \cdot \sin\left(\text{acos}\left(\frac{R}{k} \right) \right) + R^2 \cdot \text{acos}\left(\frac{R}{k} \right) = \frac{C_{Dnotopen}}{C_{Dopen} + C_{Dnotopen}} \cdot \left(k^2 \cdot \frac{\pi}{2} + R^2 \cdot \pi \right)$

This equation is to be solved for the coefficient *k*. The equation is highly nonlinear; it can be solved by iteration or using *Excel*'s *Goal Seek* or *Solver*

From the associated *Excel* workbook

$k = 0.990 \cdot ft$ $\qquad k = 0.0707 \cdot \frac{mph}{rpm}$

The result from Problem 9.106 was $k = 0.0561 \cdot \frac{mph}{rpm}$ This represents a difference of 20.6%. The difference can be attributed to the fact that we had originally averaged the flow velocity, rather than integrated over a complete revolution.

Problem 9.117

[Difficulty: 4]

9.117 A round thin disk of radius R is oriented perpendicular to a fluid stream. The pressure distributions on the front and back surfaces are measured and presented in the form of pressure coefficients. The data are modeled with the following expressions for the front and back surfaces, respectively:

Front Surface $\quad C_p = 1 - \left(\frac{r}{R}\right)^6$

Rear Surface $\quad C_p = -0.42$

Calculate the drag coefficient for the disk.

Solution: Computing equations are

$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho V^2} \qquad C_D = \frac{F_D}{\frac{1}{2}\rho V^2 A} \qquad A = \pi R^2$$

Assumptions: (1) Steady, incompressible flow
(2) Neglect skin friction drag (disk thin, edge area small)

Then $$C_D = \frac{F_D}{\frac{1}{2}\rho V^2 A} = \frac{\int_A (p_f - p_r)\,dA}{\frac{1}{2}\rho V^2 \pi R^2} = \frac{\int_0^R (p_f - p_r)\,2\pi r\,dr}{\frac{1}{2}\rho V^2 \pi R^2}$$

From defn of C_p, $p_f = p_\infty + C_{p_f}\frac{1}{2}\rho V^2$, $p_r = p_\infty + C_{p_r}\frac{1}{2}\rho V^2$

and $$p_f - p_r = (C_{p_f} - C_{p_r})\frac{1}{2}\rho V^2$$

Substituting,

$$C_D = \frac{\frac{1}{2}\rho V^2 \int_0^R (C_{p_f} - C_{p_r})\,2\pi r\,dr}{\frac{1}{2}\rho V^2 \pi R^2} = \frac{2}{R^2}\int_0^R \left[1 - \left(\frac{r}{R}\right)^6 + 0.42\right] r\,dr$$

$$= 2\int_0^1 \left[1.42 - \left(\frac{r}{R}\right)^7\right]\left(\frac{r}{R}\right) d\left(\frac{r}{R}\right) = 2\left[\frac{1.42}{2}\left(\frac{r}{R}\right)^2 - \frac{1}{8}\left(\frac{r}{R}\right)^8\right]_0^1$$

$$= 2\,(0.710 - 0.125)$$

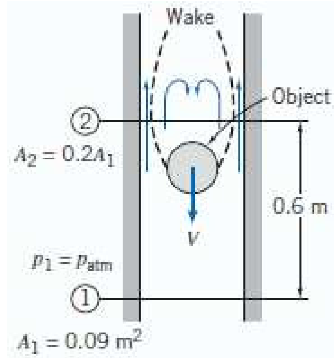
$$C_D = 1.17$$

C_D

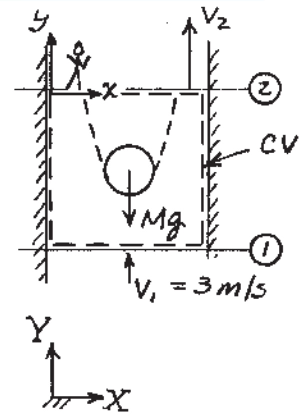
Problem 9.118

[Difficulty: 4]

9.118 An object falls in air down a long vertical chute. The speed of the object is constant at 3 m/s. The flow pattern around the object is shown. The static pressure is uniform across sections (1) and (2); pressure is atmospheric at section (1). The effective flow area at section (2) is 20 percent of the chute area. Frictional effects between sections (1) and (2) are negligible. Evaluate the flow speed relative to the object at section (2). Calculate the static pressure at section (2). Determine the mass of the object.



Solution: Choose a CV moving with object. Apply continuity, Bernoulli, and y momentum.



Basic equations:

$$0 = \underbrace{\frac{\partial}{\partial t}\int_{CV}\rho\, d\forall}_{=0(1)} + \int_{CS}\rho \vec{V}\cdot d\vec{A}$$

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + g z_2$$

$$F_{S_y} + F_{B_y} = \underbrace{\frac{\partial}{\partial t}\int_{CV} v\rho\, d\forall}_{=0(1)} + \int_{CS} v\rho \vec{V}\cdot d\vec{A}$$

Assumptions:
(1) Steady flow relative to CV
(2) Incompressible flow
(3) Neglect friction
(4) Flow along a streamline
(5) Neglect Δz
(6) Uniform flow and pressures at (1) and (2).
(7) No net flow in wake

Then from continuity,

$$0 = \{-\rho V_1 A_1\} + \{+\rho V_2 A_2\} \quad \text{so} \quad V_2 = V_1\frac{A_1}{A_2} = 3\frac{m}{s}\times\frac{1}{0.2} = 15\ m/s$$

V_2

From Bernoulli,

$$p_2 = p_1 + \frac{1}{2}\rho V_1^2 - \frac{1}{2}\rho V_2^2 = p_1 + \frac{1}{2}\rho V_1^2\left[1-\left(\frac{V_2}{V_1}\right)^2\right]$$

$$p_2(\text{gage}) = \frac{1}{2}\times 1.23\frac{kg}{m^3}\times(3)^2\frac{m^2}{s^2}\left[1-\left(\frac{1}{0.2}\right)^2\right]\frac{N\cdot s^2}{kg\cdot m} = -133\ N/m^2\ (\text{gage})$$

p_2

From momentum

$$p_1A_1 - p_2A_2 - Mg = v_1\{-\rho V_1 A_1\} + v_2\{+\rho V_2 A_2\} = \rho V_1 A_1(V_2 - V_1) = \rho V_1^2 A_1\left(\frac{V_2}{V_1}-1\right)$$

$$v_1 = V_1 \qquad v_2 = V_2$$

Thus

$$M = \frac{A_1}{g}\left[p_1 - p_2 - \rho V_1^2\left(\frac{V_2}{V_1}-1\right)\right]$$

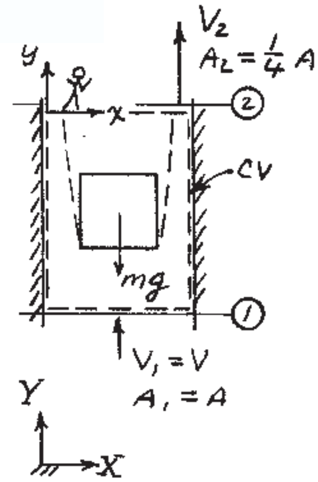
$$M = 0.09\ m^2\times\frac{s^2}{9.81\ m}\left[133\frac{N}{m^2}\times\frac{kg\cdot m}{N\cdot s^2} - 1.23\frac{kg}{m^3}\times(3)^2\frac{m^2}{s^2}\left(\frac{1}{0.2}-1\right)\right] = 0.814\ kg$$

M

Problem 9.119

[Difficulty: 4]

9.119 An object of mass m, with cross-sectional area equal to half the size of the chute, falls down a mail chute. The motion is steady. The wake area is $\frac{3}{4}$ the size of the chute at its maximum area. Use the assumption of constant pressure in the wake. Apply the continuity, Bernoulli, and momentum equations to develop an expression for terminal speed of the object in terms of its mass and other quantities.



<u>Solution</u>: Choose a CV moving <u>with</u> the object.
Apply continuity, Bernoulli, and y momentum.

Basic equations:

$$0 = \underbrace{\frac{\partial}{\partial t}\int_{CV} \rho\, d\forall}_{=0(1)} + \int_{CS} \rho \vec{V}\cdot d\vec{A}$$

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + g z_2$$

$$F_{S_y} + F_{B_y} = \underbrace{\frac{\partial}{\partial t}\int_{CV} v \rho\, d\forall}_{=0(1)} + \int_{CS} v \rho \vec{V}\cdot d\vec{A}$$

Assumptions: (1) Steady flow relative to CV
(2) Incompressible flow
(3) Neglect friction
(4) Flow along a streamline
(5) Uniform flow and pressures at ① and ②.
(6) Neglect Δz
(7) No net flow in wake

From continuity,

$$0 = \{-\rho V_1 A_1\} + \{\rho V_2 A_2\} \quad \text{so} \quad V_2 = V_1 \frac{A_1}{A_2} = V\frac{A}{A_2}$$

From Bernoulli

$$p_1 - p_2 = \frac{1}{2}\rho V_2^2 - \frac{1}{2}\rho V_1^2 = \frac{1}{2}\rho V^2\left[\left(\frac{V_2}{V}\right)^2 - 1\right] = \frac{1}{2}\rho V^2\left[\left(\frac{A}{A_2}\right)^2 - 1\right]$$

From momentum

$$p_1 A - p_2 A - mg = v_1\{-\rho V_1 A_1\} + v_2\{+\rho V_2 A_2\} = \rho V A (V_2 - V) = \rho V^2 A\left(\frac{V_2}{V} - 1\right)$$

$$v_1 = V \qquad v_2 = V_2$$

or

$$(p_1 - p_2)A - mg = \rho V^2 A\left(\frac{A}{A_2} - 1\right)$$

Substituting for $p_1 - p_2$

$$\frac{1}{2}\rho V^2 A\left[\left(\frac{A}{A_2}\right)^2 - 1\right] - mg = \rho V^2 A\left(\frac{A}{A_2} - 1\right) \quad \text{or} \quad mg = \frac{1}{2}\rho V^2 A\left[\left(\frac{A}{A_2}\right)^2 - 2\left(\frac{A}{A_2}\right) + 1\right]$$

Thus

$$V = \left[\frac{2mg}{\rho A}\,\frac{1}{\left(\frac{A}{A_2}\right)^2 - 2\left(\frac{A}{A_2}\right) + 1}\right]^{1/2}$$

V

where A_2 is the net flow area at Section ②.

Problem 9.120

[Difficulty: 2]

9.120 A light plane tows an advertising banner over a football stadium on a Saturday afternoon. The banner is 4 ft tall and 45 ft long. According to Hoerner [16], the drag coefficient based on area (Lh) for such a banner is approximated by $C_D = 0.05\ L/h$, where L is the banner length and h is the banner height. Estimate the power required to tow the banner at $V = 55$ mph. Compare with the drag of a rigid flat plate. Why is the drag larger for the banner?

Given: Data on advertising banner

Find: Power to tow banner; Compare to flat plate; Explain discrepancy

Solution:

Basic equation: $F_D = \frac{1}{2}\cdot\rho\cdot A\cdot V^2\cdot C_D$ $\quad P = F_D\cdot V$

The given data or available data is $\quad V = 55\cdot mph \quad V = 80.7\cdot\frac{ft}{s} \quad L = 45\cdot ft \quad h = 4\cdot ft \quad \rho = 0.00234\frac{slug}{ft^3}$

$A = L\cdot h \quad A = 180\cdot ft^2 \quad C_D = 0.05\cdot\frac{L}{h} \quad C_D = 0.563$

$F_D = \frac{1}{2}\cdot\rho\cdot A\cdot V^2\cdot C_D \quad F_D = 771\cdot lbf \quad P = F_D\cdot V \quad P = 6.22\times 10^4\cdot\frac{ft\cdot lbf}{s} \quad P = 113\cdot hp$

For a flate plate, check Re $\quad \nu = 1.62\times 10^{-4}\cdot\frac{ft^2}{s}$ (Table A.9, 69ºF)

$Re_L = \frac{V\cdot L}{\nu} \quad Re_L = 2.241\times 10^7$ so flow is fully turbulent. Hence use Eq 9.37b

$$C_D = \frac{0.455}{\log(Re_L)^{2.58}} - \frac{1610}{Re_L} \qquad C_D = 0.00258$$

$F_D = \frac{1}{2}\cdot\rho\cdot A\cdot V^2\cdot C_D \quad F_D = 3.53\cdot lbf$

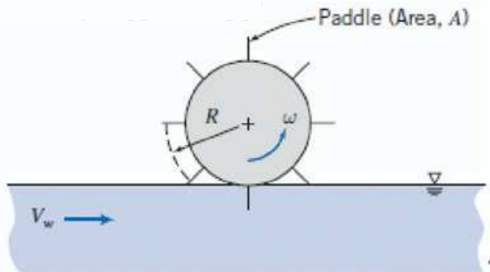
This is the drag on one side. The total drag is then $2\cdot F_D = 7.06\cdot lbf$. This is VERY much less than the banner drag.

The banner drag allows for banner flutter and other secondary motion which induces significant form drag.

Problem 9.121

[Difficulty: 5]

9.121 A large paddle wheel is immersed in the current of a river to generate power. Each paddle has area A and drag coefficient C_D; the center of each paddle is located at radius R from the centerline of the paddle wheel. Assume the equivalent of one paddle is submerged continuously in the flowing stream. Obtain an expression for the drag force on a single paddle in terms of geometric variables, current speed, V, and linear speed of the paddle center, $U = R\omega$. Develop expressions for the torque and power produced by the paddle wheel. Find the speed at which the paddle wheel should rotate to obtain maximum power output from the wheel in a given current.



Solution: Computing equations $F_D = C_D A \frac{1}{2}\rho V_{rel}^2$

$T = F_D R \;,\; P = T\omega$

Assumptions: (1) Neglect air resistance, since $\rho_{air} \ll \rho_{water}$
(2) Use velocity relative to the paddle

Thus $V_{rel} = V - U = V - R\omega$

$$F_D = C_D A \frac{1}{2}\rho V_{rel}^2 = C_D A \frac{1}{2}\rho (V-U)^2$$ F_D

The torque is

$$T = F_D R = C_D A \frac{1}{2}\rho (V-U)^2 R$$ T

The power is

$$P = T\omega = C_D A \frac{1}{2}\rho (V-U)^2 R\omega = C_D A \frac{1}{2}\rho (V-U)^2 U$$ P

To optimize power, set $\frac{dP}{dU} = 0$

$$\frac{dP}{dU} = C_D A \frac{1}{2}\rho \left[2(V-U)U(-1) + (V-U)^2 \right] = 0$$

Cancelling a factor $(V-U)$ gives $-2U + V - U = 0$ or $V - 3U = 0$

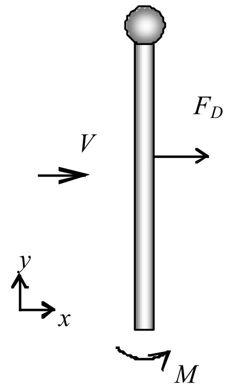
Thus $U = R\omega = \frac{V}{3}$ so

$$\omega_{opt} = \frac{V}{3R}$$ ω_{opt}

Problem 9.122

[Difficulty: 1]

9.122 The antenna on a car is 10 mm in diameter and 1.8 m long. Estimate the bending moment that tends to snap it off if the car is driven at 120 km/hr on a standard day.



Given: Data on car antenna

Find: Bending moment

Solution:

Basic equation: $F_D = \frac{1}{2}\cdot\rho\cdot A\cdot V^2\cdot C_D$

The given or available data is $V = 120\cdot\frac{km}{hr} = 33.333\frac{m}{s}$ $\quad L = 1.8\cdot m$ $\quad D = 10\cdot mm$

$A = L\cdot D$ $\quad A = 0.018\,m^2$

$\rho = 1.225\cdot\frac{kg}{m^3}$ $\quad \nu = 1.50\times 10^{-5}\cdot\frac{m^2}{s}$ $\quad$ (Table A.10, 20°C)

For a cylinder, check Re $\quad Re = \frac{V\cdot D}{\nu}$ $\quad Re = 2.22\times 10^4$

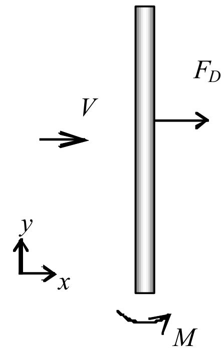
From Fig. 9.13 $\quad C_D = 1.0$ $\quad F_D = \frac{1}{2}\cdot\rho\cdot A\cdot V^2\cdot C_D$ $\quad F_D = 12.3\,N$

The bending moment is then $\quad M = F_D\cdot\frac{L}{2}$ $\quad M = 11.0\cdot N\cdot m$

Problem 9.123

[Difficulty: 1]

9.123 A large three-blade horizontal axis wind turbine (HAWT) can be damaged if the wind speed is too high. To avoid this, the blades of the turbine can be oriented so that they are parallel to the flow. Find the bending moment at the base of each blade when the wind speed is 85 knots. Model each blade as a flat plate 115 ft wide × 1.5 ft long.



Given: Data on wind turbine blade

Find: Bending moment

Solution:

Basic equation: $F_D = \frac{1}{2}\cdot\rho\cdot A\cdot V^2\cdot C_D$

The given or available data is $V = 85\cdot\text{knot} = 143.464\cdot\frac{\text{ft}}{\text{s}}$ $\quad L = 1.5\cdot\text{ft} \quad W = 115\cdot\text{ft}$

$A = L\cdot W \qquad A = 172.5\cdot\text{ft}^2$

$\rho = 0.00233\cdot\frac{\text{slug}}{\text{ft}^3} \qquad \nu = 1.63\times 10^{-4}\cdot\frac{\text{ft}^2}{\text{s}}$ (Table A.9, 70°F)

For a flat plate, check Re $\quad Re_L = \frac{V\cdot L}{\nu} \qquad Re_L = 1.32\times 10^6$ so use Eq. 9.37a

$$C_D = \frac{0.0742}{Re_L^{\frac{1}{5}}} - \frac{1740}{Re_L} \qquad C_D = 0.00311$$

$$F_D = 2\cdot\frac{1}{2}\cdot\rho\cdot A\cdot V^2\cdot C_D \qquad F_D = 25.7\cdot\text{lbf}$$

The bending moment is then $\quad M = F_D\cdot\frac{W}{2} \qquad M = 1480\cdot\text{ft}\cdot\text{lbf}$

Problem 9.124

[Difficulty: 4]

9.124 The HAWT of Problem 9.123 is not self-starting. The generator is used as an electric motor to get the turbine up to the operating speed of 25 rpm. To make this easier, the blades are aligned so that they lie in the plane of rotation. Assuming an overall efficiency of motor and drive train of 60 percent, find the power required to maintain the turbine at the operating speed. As an approximation, model each blade as a series of flat plates (the outer region of each blade moves at a significantly higher speed than the inner region).

Given: Data on wind turbine blade

Find: Power required to maintain operating speed

Solution:

Basic equation: $$F_D = \frac{1}{2}\cdot\rho\cdot A\cdot V^2\cdot C_D$$

The given or available data is $\omega = 25\cdot rpm$ $\quad L = 1.5\cdot ft$ $\quad W = 115\cdot ft$

$$\rho = 0.00233\cdot\frac{slug}{ft^3} \qquad \nu = 1.63\times 10^{-4}\cdot\frac{ft^2}{s} \qquad \text{(Table A.9, 70°F)}$$

The velocity is a function of radial position, $V(r) = r\cdot\omega$, so Re varies from 0 to $Re_{max} = \frac{V(W)\cdot L}{\nu}$ $\qquad Re_{max} = 2.77\times 10^6$

The transition Reynolds number is 500,000 which therefore occurs at about 1/4 of the maximum radial distance; the boundary layer is laminar for the first quarter of the blade. We approximate the entire blade as turbulent - the first 1/4 of the blade will not exert much moment in any event

Hence $$Re(r) = \frac{L}{\nu}\cdot V(r) = \frac{L\cdot\omega}{\nu}\cdot r$$

Using Eq. 9.37a $$C_D = \frac{0.0742}{Re_L^{\frac{1}{5}}} - \frac{1740}{Re_L} = \frac{0.0742}{\left(\frac{L\cdot\omega}{\nu}\cdot r\right)^{\frac{1}{5}}} - \frac{1740}{\frac{L\cdot\omega}{\nu}\cdot r} = 0.0742\cdot\left(\frac{\nu}{L\cdot\omega}\right)^{\frac{1}{5}}\cdot r^{-\frac{1}{5}} - 1740\cdot\left(\frac{\nu}{L\cdot\omega}\right)\cdot r^{-1}$$

The drag on a differential area is $dF_D = \frac{1}{2}\cdot\rho\cdot dA\cdot V^2\cdot C_D = \frac{1}{2}\cdot\rho\cdot L\cdot V^2\cdot C_D\cdot dr$ $\quad$ The bending moment is then $dM = dF_D\cdot r$

Hence $$M = \int 1\,dM = \int_0^W \frac{1}{2}\cdot\rho\cdot L\cdot V^2\cdot C_D\cdot r\,dr$$

$$M = \int_0^W \frac{1}{2}\cdot\rho\cdot L\cdot\omega^2\cdot r^3\cdot\left[0.0742\cdot\left(\frac{\nu}{L\cdot\omega}\right)^{\frac{1}{5}}\cdot r^{-\frac{1}{5}} - 1740\cdot\left(\frac{\nu}{L\cdot\omega}\right)\cdot r^{-1}\right]dr$$

$$M = \frac{1}{2}\cdot\rho\cdot L\cdot\omega^2\cdot\int_0^W \left[0.0742\cdot\left(\frac{\nu}{L\cdot\omega}\right)^{\frac{1}{5}}\cdot r^{\frac{14}{5}} - 1740\cdot\left(\frac{\nu}{L\cdot\omega}\right)\cdot r^2\right]dr \qquad M = \frac{1}{2}\cdot\rho\cdot L\cdot\omega^2\cdot\left[\frac{5\cdot 0.0742}{19}\cdot\left(\frac{\nu}{L\cdot\omega}\right)^{\frac{1}{5}}\cdot W^{\frac{19}{5}} - \frac{1740}{3}\cdot\left(\frac{\nu}{L\cdot\omega}\right)\cdot W^3\right]$$

$M = 1666\cdot ft\cdot lbf$ $\qquad$ Hence the power is $\quad P = M\cdot\omega \qquad P = 7.93\cdot hp$

Problem 9.125

[Difficulty: 2]

9.125 A runner maintains a speed of 7.5 mph during a 4-mi run. The runner's route consists of running straight down a road for 2 mi, then turning around and returning the 2 mi straight home. The C_DA for the runner is 9 ft^2. On a windless day, how many calories (kcal) will the runner burn overcoming drag? On a day in which the wind is blowing 5 mph directly along the runner's route how many calories (kcal) will the runner burn overcoming drag?

Given: A runner running during different wind conditions.

Find: Calories burned for the two different cases

Solution:

Governing equation:

$$C_D = \frac{F_D}{\frac{1}{2}\rho V^2 A} \qquad F_D = \frac{1}{2} C_D \rho V^2 A$$

Assumption: 1) $C_DA = 9\ \text{ft}^2$ 2) Runner maintains speed of 7.5 mph regardless of wind conditions

No wind:

$$V = 7.5\ \text{mph} = 11\ \text{ft/s} \qquad \rho = 0.00238\ \text{slug/ft}^3$$

The drag force on the runner is: $$F_D = \frac{1}{2} \times 9\ \text{ft}^2 \times 0.00238 \frac{\text{slug}}{\text{ft}^3} \times (11)^2 \frac{\text{ft}^2}{\text{s}^2} = 1.296\ \text{lbf}$$

Energy burned: $$E = Power \times time = F_D \times V_{runner} \times time$$

Where $$time = 4\ \text{mi} \times \frac{\text{hr}}{7.5\ \text{mi}} \times \frac{3600\ \text{s}}{\text{hr}} = 1920\ \text{s}$$

Hence $$E = 1.296\ \text{lbf} \times \frac{11\text{f}}{\text{s}} \times 1920\ \text{s} \times \frac{0.0003238\ \text{kcal}}{\text{ft} \cdot \text{lbf}} \qquad E = 8.86\ \text{kcal}$$

With 5 mph wind:

Going upwind: $$V_{rel} = 12.5\ \text{mph} = 18.33 \frac{\text{ft}}{\text{s}}$$

The drag force on the runner is: $$F_D = \frac{1}{2} \times 9\ \text{ft}^2 \times 0.00238 \frac{\text{slug}}{\text{ft}^3} \times (18.33)^2 \frac{\text{ft}^2}{\text{s}^2} = 3.598\ \text{lbf}$$

$$time = 2\ \text{mi} \times \frac{\text{hr}}{7.5\ \text{mi}} \times \frac{3600\ \text{s}}{\text{hr}} = 960\ \text{s}$$

$$E_{upwind} = 3.598\ \text{lbf} \times \frac{11\text{f}}{\text{s}} \times 960\ \text{s} \times \frac{0.0003238\ \text{kcal}}{\text{ft} \cdot \text{lbf}} = 12.30\ \text{kcal}$$

Going downwind: $V_{rel} = 2.5\ \text{mph} = 3.67\ \frac{\text{ft}}{\text{s}}$

The drag force on the runner is: $F_D = \frac{1}{2} \times 9\ \text{ft}^2 \times 0.00238 \frac{\text{slug}}{\text{ft}^3} \times (3.67)^2\ \frac{\text{ft}^2}{\text{s}^2} = 0.144\ \text{lbf}$

$$time = 2\ \ \text{mi} \times \frac{\text{hr}}{7.5\ \text{mi}} \times \frac{3600\ \text{s}}{\text{hr}} = 960\ \text{s}$$

$$E_{downwind} = 0.144\ \text{lbf} \times \frac{11\text{f}}{\text{s}} \times 960\ \text{s} \times \frac{0.0003238\ \text{kcal}}{\text{ft} \cdot \text{lbf}} = 0.49\ \text{kcal}$$

Hence the total energy burned to overcome drag when the wind is 5 mph is:

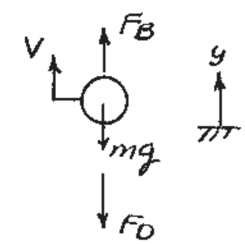
$E = 12.30\ \text{kcal} + 0.49\ \text{kcal} = 12.79\ \text{kcal}$ this is 44% higher

Problem 9.126

[Difficulty: 2]

9.126 Consider small oil droplets (SG = 0.85) rising in water. Develop a relation for calculating terminal speed of a droplet (in m/s) as a function of droplet diameter (in mm) assuming Stokes flow. For what range of droplet diameter is Stokes flow a reasonable assumption?

<u>Solution</u>: Draw free-body diagram of droplet, apply Newton's second law.



Basic equation: $\Sigma F_y = -mg + F_B - F_D = ma_y$

Assume: Stokes' drag law, $F_D = 3\pi\mu V D$, for $Re < 10$

Then $-\rho \forall g + \rho_{H_2O} \forall g - 3\pi\mu V_t D = 0$ at terminal speed, V_t.

Solving, $V_t = \dfrac{(\rho_{H_2O} - \rho_o)\forall g}{3\pi\mu D} = \rho_{H_2O}(1 - SG_o)\dfrac{\pi D^3}{6}\dfrac{g}{3\pi\mu D} = \dfrac{(1-SG_o)D^2 g}{18\,\nu}$

Evaluating,

$$V_t(m/s) = \frac{(1-0.85)}{18} \times D^2\,mm^2 \times 9.81\frac{m}{s} \times \frac{s}{1.00\,10^{-6}\,m^2} \times \frac{m^2}{10^6\,mm^2} \qquad (T = 20°C)$$

$$V_t(m/s) = 0.0818[D(mm)]^2$$

V_t

For Stokes flow, $Re < 1$, so

$$Re = \frac{\rho V_t D}{\mu} = \frac{V_t D}{\nu} = \frac{(1-SG_o)D^3 g}{18\,\nu^2} \le 1$$

Thus

$$D^3 \le \frac{18\,\nu^2}{(1-SG_o)g} \quad \text{or} \quad D \le \left[\frac{18\,\nu^2}{(1-SG_o)g}\right]^{1/3}$$

Evaluating,

$$D \le \left[\frac{18}{(1-0.85)}(1.00\times10^{-6})^2\frac{m^4}{s^2} \times \frac{s^2}{9.81\,m}\right]^{1/3} = 2.31\times10^{-4}\,m \;(0.231\,mm)$$

D

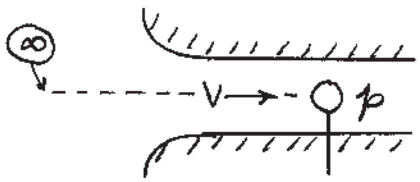
Thus Stokes' flow will be a valid assumption for $D < 0.231$ mm.

Problem 9.127

[Difficulty: 2]

9.127 Standard air is drawn into a low-speed wind tunnel. A 30-mm diameter sphere is mounted on a force balance to measure lift and drag. An oil-filled manometer is used to measure static pressure inside the tunnel; the reading is −40 mm of oil (SG = 0.85). Calculate the freestream air speed in the tunnel, the Reynolds number of flow over the sphere, and the drag force on the sphere. Are the boundary layers on the sphere laminar or turbulent? Explain.

Solution: Apply Bernoulli



$$p_\infty + \frac{1}{2}\rho \underbrace{V_\infty^2}_{\approx 0\,(6)} + \rho g \underbrace{z_\infty}_{(5)} = p + \frac{1}{2}\rho V^2 + \rho g \underbrace{z}_{(5)}$$

Assume: (1) Steady flow
(2) Incompressible flow
(3) Flow along a streamline
(4) No friction (neglect honeycomb and/or screens
(5) Neglect z
(6) $V_\infty \approx 0$

Then $p_\infty = p + \frac{1}{2}\rho V^2$ or $V = \sqrt{\frac{2(p_\infty - p)}{\rho}}$

But $p_\infty - p = -\rho_{oil} g \Delta h = -SG\,\rho_{H_2O}\, g \Delta h$

$$V = \left[\frac{-2\,SG\,\rho_{H_2O}\, g\,\Delta h}{\rho}\right]^{\frac{1}{2}}$$

$$V = \left[-2(0.85) \times 1000\,\frac{kg}{m^3} \times 9.81\,\frac{m}{s^2} \times (-0.04\,m) \times \frac{m^3}{1.23\,kg}\right]^{\frac{1}{2}} = 23.3\,m/s$$

V

and

$$Re = \frac{VD}{\nu} = 23.3\,\frac{m}{s} \times 0.03\,m \times \frac{s}{1.45\times10^{-5}\,m^2} = 48{,}200$$

Re

Re is subcritical; BLs are laminar, and $C_D = 0.47$

$$F_D = C_D A \frac{1}{2}\rho V^2 \qquad A = \frac{\pi D^2}{4} = 7.07\times10^{-4}\,m^2$$

$$= 0.47 \times 7.07\times10^{-4}\,m^2 \times \frac{1}{2} \times 1.23\,\frac{kg}{m^3} \times (23.3)^2\,\frac{m^2}{s^2} \times \frac{N\cdot s^2}{kg\cdot m}$$

$$F_D = 0.111\,N$$

F_D

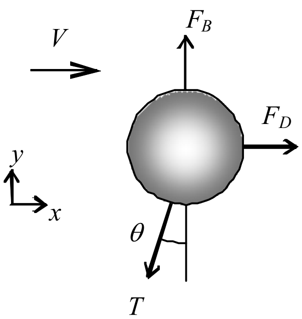
Problem 9.128

[Difficulty: 2]

9.128 A spherical helium-filled balloon, 20 in. in diameter, exerts an upward force of 0.3 lbf on a restraining string when held stationary in standard air with no wind. With a wind speed of 10 ft/s, the string holding the balloon makes an angle of 55° with the horizontal. Calculate the drag coefficient of the balloon under these conditions, neglecting the weight of the string.

Given: Data on helium-filled balloon, angle balloon string makes when subjected to wind

Find: Drag coefficient for the balloon



Solution:

Basic equations: $F_D = \frac{1}{2}\cdot\rho\cdot A\cdot V^2\cdot C_D \qquad \Sigma F_y = 0$

The given or available data is $D = 20\cdot in \qquad F_B = 0.3\cdot lbf \qquad V = 10\cdot\frac{ft}{s} \qquad \theta = 55\cdot deg$

$\rho = 0.00233\cdot\frac{slug}{ft^3} \qquad \nu = 1.63\times 10^{-4}\cdot\frac{ft^2}{s}$ (Table A.9, 70ºF)

Based on a free body diagram of the balloon, $F_D = F_B\cdot\tan(90\cdot deg - \theta) = 0.2101\cdot lbf$

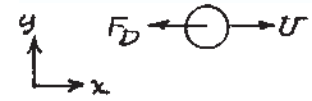
The reference area for the balloon is: $A = \frac{\pi}{4}\cdot D^2 = 2.182\cdot ft^2$ so the drag coefficient is: $C_D = \frac{F_D}{\frac{1}{2}\cdot\rho\cdot V^2\cdot A} = 0.826$

Problem 9.129

[Difficulty: 2]

9.129 A field hockey ball has diameter $D = 73$ mm and mass $m = 160$ g. When struck well, it leaves the stick with initial speed $U_0 = 50$ m/s. The ball is essentially smooth. Estimate the distance traveled in horizontal flight before the speed of the ball is reduced 10 percent by aerodynamic drag.

<u>Solution</u>: Apply Newton's second law of motion:



Basic equation: $\Sigma F_x = ma_x = m\dfrac{dU}{dt} = mU\dfrac{dU}{dx}$

Thus $-F_D = -C_D A \frac{1}{2}\rho U^2 = mU\dfrac{dU}{dx}$ or $dx = -\dfrac{2m}{C_D A \rho}\dfrac{dU}{U}$

Check Re to find C_D (use ν at $T = 15°C$ from Table A.10):

$$Re \le \frac{U_0 D}{\nu} = 50\,\frac{m}{s} \times 0.075\,m \times \frac{s}{1.46\times10^{-5}\,m^2} = 2.57\times10^5 \qquad \text{(standard air)}$$

From Fig. 9.11, flow is <u>subcritical</u> and $C_D \simeq 0.47 = \text{constant}$.

Thus
$$x = \int_0^x dx = \int_{U_0}^{U} -\frac{2m}{C_D A\rho}\frac{dU}{U} = -\frac{2m}{C_D A\rho}\ln U\Big]_{U_0}^{U} = -\frac{2m}{C_D A\rho}\ln\left(\frac{U}{U_0}\right)$$

or
$$x = -2\times 0.160\,kg \times \frac{1}{0.47}\times\frac{4}{\pi(0.073)^2 m^2}\times\frac{m^3}{1.23\,kg}\ln(0.9) = 13.9\,m$$

x

Problem 9.130

[Difficulty: 2]

9.130 Compute the terminal speed of a 3-mm-diameter raindrop (assume spherical) in standard air.

Given: 3 mm raindrop

Find: Terminal speed

Solution:

Basic equation: $F_D = \frac{1}{2}\cdot\rho\cdot A\cdot V^2\cdot C_D$ $\qquad \Sigma F = 0$

Given or available data is $D = 3\cdot mm$ $\quad \rho_{H2O} = 1000\cdot\frac{kg}{m^3}$ $\quad \rho_{air} = 1.225\cdot\frac{kg}{m^3}$ $\quad \nu = 1.50\times 10^{-5}\cdot\frac{m^2}{s}$ (Table A.10, 20°C)

Summing vertical forces $\quad M\cdot g - F_D = M\cdot g - \frac{1}{2}\cdot\rho_{air}\cdot A\cdot V^2\cdot C_D = 0$ $\qquad$ Buoyancy is negligible

$$M = \rho_{H2O}\cdot\frac{\pi\cdot D^3}{6} \qquad M = 1.41\times 10^{-5}\,kg \qquad A = \frac{\pi\cdot D^2}{4} \qquad A = 7.07\times 10^{-6}\,m^2$$

Assume the drag coefficient is in the flat region of Fig. 9.11 and verify Re later $\qquad C_D = 0.4$

$$V = \sqrt{\frac{2\cdot M\cdot g}{C_D\cdot\rho_{air}\cdot A}} \qquad V = 8.95\,\frac{m}{s}$$

Check Re $\quad Re = \frac{V\cdot D}{\nu}$ $\qquad Re = 1.79\times 10^3$ which does place us in the flat region of the curve

Actual raindrops are not quite spherical, so their speed will only be approximated by this result

Problem 9.131

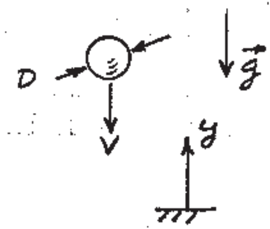
[Difficulty: 3]

9.131 A small sphere ($D = 6$ mm) is observed to fall through castor oil at a terminal speed of 60 mm/s. The temperature is 20°C. Compute the drag coefficient for the sphere. Determine the density of the sphere. If dropped in water, would the sphere fall slower or faster? Why?

Solution: Apply Newton's second law of motion, definition of C_D.

Basic equations: $\Sigma F_y = ma_y$

$$C_D = \frac{F_D}{\frac{1}{2}\rho V^2 A}$$



Assume $Re < 1$, so Stokes flow, $F_D = 3\pi\mu DV$.

From the definition, noting A is the frontal area, $A = \frac{\pi D^2}{4}$,

$$C_D = \frac{F_D}{\frac{1}{2}\rho V^2 A} = \frac{3\pi\mu DV}{\frac{1}{2}\rho V^2 \frac{\pi D^2}{4}} = \frac{24\mu}{\rho VD} = \frac{24}{Re}$$

For castor oil at 20 C, $\mu = 0.9\ N \cdot s/m^2$ (Fig. A.2) and $SG \approx 0.97$ (Table A.2).

$$Re = \frac{SG\,\rho_{H_2O} VD}{\mu} = (0.97)\,999\,\frac{kg}{m^3} \times 0.06\,\frac{m}{s} \times 0.006\,m \times \frac{m^2}{0.9\,N\cdot s} \times \frac{N \cdot s^2}{kg \cdot m} = 0.388 < 1 \quad ✓✓$$

$$C_D = \frac{24}{Re} = \frac{24}{0.388} = 61.9 \qquad C_D$$

From Newton's second law, $F_D + F_{buoyancy} - mg = 0$ or $F_D = mg - F_b$. Thus

$$F_D = C_D \tfrac{1}{2}\rho V^2 A = mg - F_b = (SG_s - SG_o)\rho_{H_2O} g \forall \text{ or } SG_s = SG_o + \frac{C_D \frac{1}{2}\rho V^2 A}{\rho_{H_2O} g \forall} = SG_o\left(1 + \frac{C_D V^2 A}{2g\forall}\right)$$

But $\frac{A}{\forall} = \frac{\pi r^2}{\frac{4}{3}\pi r^3} = \frac{3}{4r} = \frac{3}{2D}$, so $SG_s = SG_o\left(1 + \frac{3}{4}\frac{C_D V^2}{gD}\right)$

$$SG_s = 0.97\left(1 + \frac{3}{4} \times 61.9 \times (0.06)^2 \frac{m^2}{s^2} \cdot \frac{s^2}{9.81\,m} \times \frac{1}{0.006\,m}\right) = 3.72, \text{ so } \rho_s = 3720\,\frac{kg}{m^3} \qquad \rho_s$$

In water, the net weight and drag must balance, or

$$F_D = \tfrac{1}{2}\rho V^2 C_D A = W - F_b = \gamma_{H_2O}(SG_s - 1)\forall \quad \text{or} \quad V = \left[\frac{4(SG-1)gD}{3C_D}\right]^{\frac{1}{2}}$$

However, C_D is a function of Re, so iteration is needed. From Fig. 9.11, $C_D \approx 0.4$ over a range of Re. Using $C_D = 0.4$,

$$V = \left[\frac{4}{3}\left(\frac{2.72}{0.4}\right) 9.81\,\frac{m}{s^2} \times 0.006\,m\right]^{\frac{1}{2}} = 0.731\ m/s \qquad V$$

Check Re:

$$Re = \frac{VD}{\nu} = 0.731\,\frac{m}{s} \times 0.006\,m \times \frac{s}{1.1 \times 10^{-6}\,m^2} = 3990 \quad \text{ok!}$$

Problem 9.132

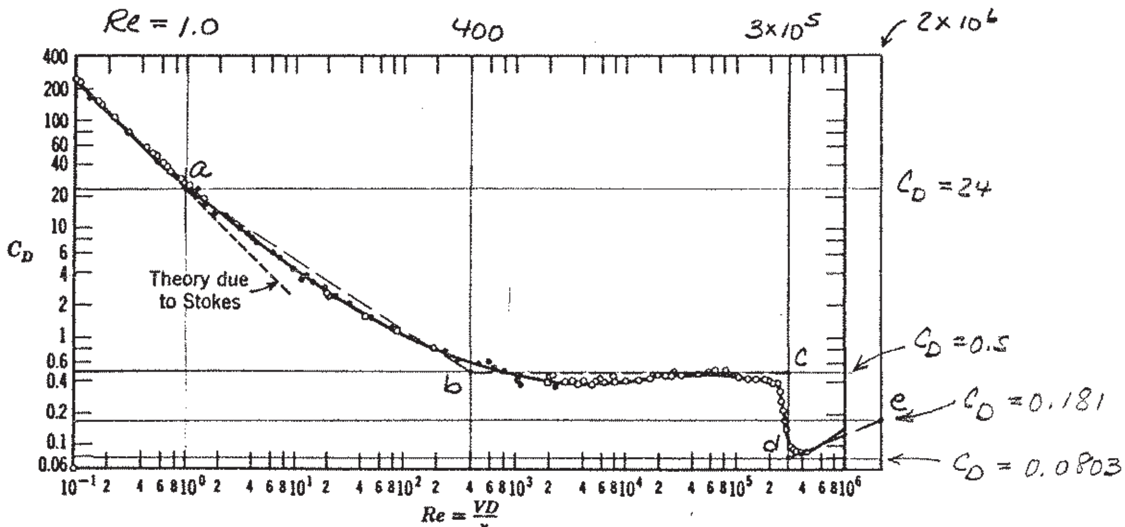
[Difficulty: 3]

9.132 The following curve-fit for the drag coefficient of a smooth sphere as a function of Reynolds number has been proposed by Chow [36]:

$C_D = 24/Re$	$Re \leq 1$
$C_D = 24/Re^{0.646}$	$1 < Re \leq 400$
$C_D = 0.5$	$400 < Re \leq 3 \times 10^5$
$C_D = 0.000366\, Re^{0.4275}$	$3 \times 10^5 < Re \leq 2 \times 10^6$
$C_D = 0.18$	$Re > 2 \times 10^6$

Use data from Fig. 9.11 to estimate the magnitude and location of the maximum error between the curve fit and data.

Solution: The curve-fit segments are plotted on Fig. 9.11 below:

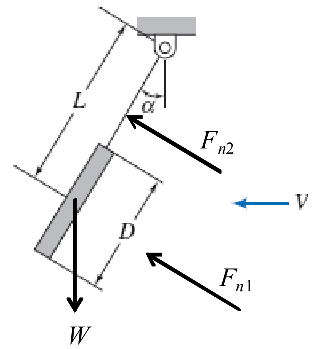


The maximum significant error occurs in the region where C_D is modeled as equal to the constant value, $C_D = 0.5$. The curve fit appears to be about 10 percent high in the region from $Re \simeq 10^3$ to $Re \simeq 10^4$.

Problem 9.133

[Difficulty: 3]

9.133 Problem 9.107 showed a circular disk hung in an air stream from a cylindrical strut. Assume the strut is L = 40 mm long and d = 3 mm in. diameter. Solve Problem 9.107 including the effect of drag on the support.



Given: Circular disk in wind

Find: Mass of disk; Plot α versus V

Solution:

Basic equations: $$C_D = \frac{F_D}{\frac{1}{2}\cdot\rho\cdot V^2\cdot A} \qquad \Sigma\vec{M} = 0$$

Summing moments at the pivot $W\cdot L\cdot\sin(\alpha) - F_{n1}\cdot L - \frac{1}{2}\cdot\left(L - \frac{D}{2}\right)\cdot F_{n2} = 0$ (1) and for each normal drag $F_n = \frac{1}{2}\cdot\rho\cdot V_n^{\,2}\cdot A\cdot C_D$

Assume 1) No pivot friction 2) C_D is valid for $V_n = V\cos(\alpha)$

The data is $\rho = 1.225\cdot\frac{kg}{m^3}$ $\mu = 1.8\times 10^{-5}\cdot\frac{N\cdot s}{m^2}$ $V = 15\cdot\frac{m}{s}$

$D = 25\cdot mm$ $d = 3\cdot mm$ $L = 40\cdot mm$ $\alpha = 10\cdot deg$

$C_{D1} = 1.17$ (Table 9.3) $Re_d = \frac{\rho\cdot V\cdot d}{\mu}$ $Re_d = 3063$ so from Fig. 9.13 $C_{D2} = 0.9$

Hence $$F_{n1} = \frac{1}{2}\cdot\rho\cdot(V\cdot\cos(\alpha))^2\cdot\frac{\pi\cdot D^2}{4}\cdot C_{D1} \qquad F_{n1} = 0.077\,N$$

$$F_{n2} = \frac{1}{2}\cdot\rho\cdot(V\cdot\cos(\alpha))^2\cdot\left(L - \frac{D}{2}\right)\cdot d\cdot C_{D2} \qquad F_{n2} = 0.00992\,N$$

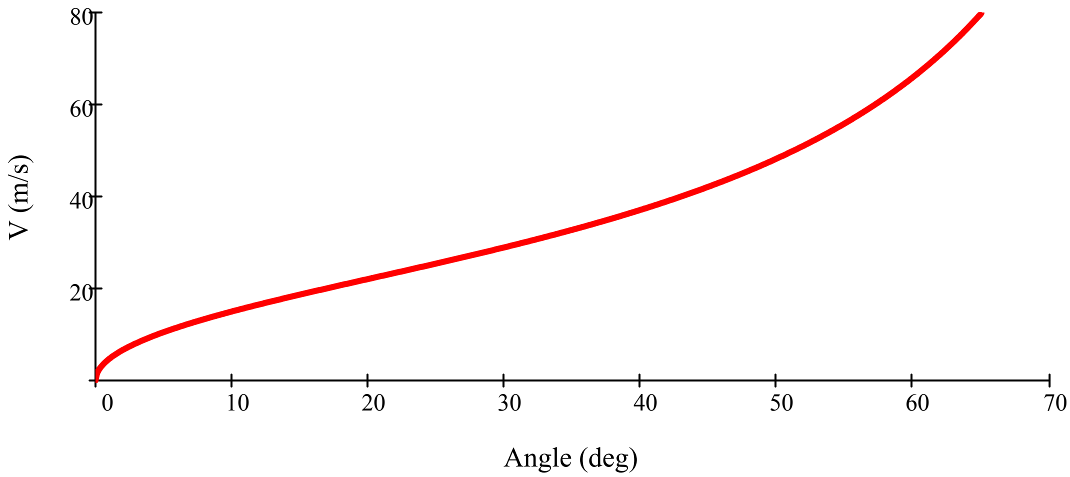
The drag on the support is much less than on the disk (and moment even less), so results will not be much different from those of Problem 9.105

Hence Eq. 1 becomes $$M\cdot L\cdot g\cdot\sin(\alpha) = L\cdot\frac{1}{2}\cdot\rho\cdot(V\cdot\cos(\alpha))^2\cdot\frac{\pi\cdot D^2}{4}\cdot C_{D1} + \frac{1}{2}\cdot\left(L - \frac{D}{2}\right)\cdot\left[\frac{1}{2}\cdot\rho\cdot(V\cdot\cos(\alpha))^2\cdot\left(L - \frac{D}{2}\right)\cdot d\cdot C_{D2}\right]$$

$$M = \frac{\rho\cdot V^2\cdot\cos(\alpha)^2}{4\cdot g\cdot\sin(\alpha)}\cdot\left[\frac{1}{2}\cdot\pi\cdot D^2\cdot C_{D1} + \left(1 - \frac{D}{2\cdot L}\right)\cdot\left(L - \frac{D}{2}\right)\cdot d\cdot C_{D2}\right] \qquad M = 0.0471\,kg$$

Rearranging $$V = \sqrt{\frac{4\cdot M\cdot g}{\rho}}\cdot\sqrt{\frac{\tan(\alpha)}{\cos(\alpha)}}\cdot\frac{1}{\sqrt{\left[\frac{1}{2}\cdot\pi\cdot D^2\cdot C_{D1} + \left(1 - \frac{D}{2\cdot L}\right)\cdot\left(L - \frac{D}{2}\right)\cdot d\cdot C_{D2}\right]}}$$ $$V = 35.5\cdot\frac{m}{s}\cdot\sqrt{\frac{\tan(\alpha)}{\cos(\alpha)}}$$

We can plot this by choosing α and computing V



This graph can be easily plotted in *Excel*

Problem 9.134

[Difficulty: 3]

9.134 A tennis ball with a mass of 57 g and diameter of 64 mm is dropped in standard sea level air. Calculate the terminal velocity of the ball. Assuming as an approximation that the drag coefficient remains constant at its terminal-velocity value, estimate the time and distance required for the ball to reach 95% of its terminal speed.

Given: Data on a tennis ball

Find: Terminal speed time and distance to reach 95% of terminal speed

Solution:

The given data or available data is $M = 57\cdot gm$ $\quad D = 64\cdot mm$ $\quad \nu = 1.45\cdot 10^{-5}\cdot\frac{m^2}{s}$ $\quad \rho = 1.23\cdot\frac{kg}{m^3}$

Then $A = \frac{\pi\cdot D^2}{4}$ $\qquad A = 3.22\times 10^{-3}\,m^2$

Assuming high Reynolds number $C_D = 0.5$ (from Fig. 9.11)

At terminal speed drag equals weight $F_D = M\cdot g$

The drag at speed V is given by $F_D = \frac{1}{2}\cdot\rho\cdot A\cdot V^2\cdot C_D$

Hence the terminal speed is $V_t = \sqrt{\frac{M\cdot g}{\frac{1}{2}\cdot\rho\cdot A\cdot C_D}}$ $\qquad V_t = 23.8\,\frac{m}{s}$

Check the Reynolds number $Re = \frac{V_t\cdot D}{\nu}$ $\qquad Re = 1.05\times 10^5$ $\qquad$ Check!

For motion before terminal speed Newton's second law applies

$$M\cdot a = M\cdot\frac{dV}{dt} = M\cdot g - \frac{1}{2}\cdot\rho\cdot V^2\cdot A\cdot C_D \quad \text{or} \quad \frac{d}{dt}V = g - k\cdot V^2 \quad \text{where} \quad k = \frac{\rho\cdot A\cdot C_D}{2\cdot M} \qquad k = 0.0174\,\frac{1}{m}$$

Separating variables $\displaystyle\int_0^V \frac{1}{g - k\cdot V^2}\,dV = t$ $\qquad \displaystyle\int \frac{1}{g - k\cdot V^2}\,dV = \frac{1}{\sqrt{g\cdot k}}\cdot \operatorname{atanh}\left(\sqrt{\frac{k}{g}}\cdot V\right)$

Hence $V(t) = \sqrt{\frac{g}{k}}\cdot\tanh\left(\sqrt{g\cdot k}\cdot t\right)$

Evaluating at $V = 0.95V_t$ $\quad 0.95\cdot V_t = \sqrt{\frac{g}{k}}\cdot\tanh\left(\sqrt{g\cdot k}\cdot t\right)$ $\qquad t = \frac{1}{\sqrt{g\cdot k}}\cdot\operatorname{atanh}\left(0.95\cdot V_t\cdot\sqrt{\frac{k}{g}}\right)$ $\qquad t = 4.44\,s$

For distance x versus time, integrate $\quad \frac{dx}{dt} = \sqrt{\frac{g}{k}}\cdot\tanh\left(\sqrt{g\cdot k}\cdot t\right)$ $\qquad x = \displaystyle\int_0^t \sqrt{\frac{g}{k}}\cdot\tanh\left(\sqrt{g\cdot k}\cdot t\right)dt$

Note that $\int \tanh(a \cdot t)\, dt = \frac{1}{a} \cdot \ln(\cosh(a \cdot t))$

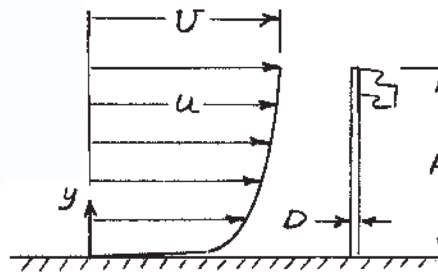
Hence $x(t) = \frac{1}{k} \cdot \ln\left(\cosh\left(\sqrt{g \cdot k} \cdot t\right)\right)$

Evaluating at $V = 0.95 V_t$ $t = 4.44\ \text{s}$ so $x(t) = 67.1\ \text{m}$

Problem 9.135

[Difficulty: 4]

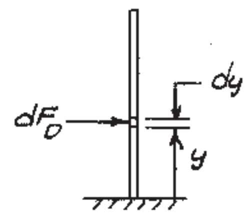
9.135 Consider a cylindrical flag pole of height H. For constant drag coefficient, evaluate the drag force and bending moment on the pole if wind speed varies as $u/U = (y/H)^{1/7}$, where y is distance measured from the ground. Compare with drag and moment for a uniform wind profile with constant speed U.



Solution: Apply definition of drag coefficient, $C_D = \dfrac{F_D}{\frac{1}{2}\rho V^2 A}$

Assume: (1) C_D = constant
(2) C_D same as circular cylinder

On an element of the pole,



$$dF_D = C_D \tfrac{1}{2}\rho u^2 dA = C_D \tfrac{1}{2}\rho U^2 \left(\frac{y}{H}\right)^{2/7} D\,dy$$

Thus

$$F_D = \int_0^H dF_D = \int_0^H C_D \tfrac{1}{2}\rho U^2 \left(\frac{y}{H}\right)^{2/7} D\,dy$$

$$F_D = C_D \tfrac{1}{2}\rho U^2 DH \int_0^1 \left(\frac{y}{H}\right)^{2/7} d\left(\frac{y}{H}\right) = C_D \tfrac{1}{2}\rho U^2 DH \left[\frac{7}{9}\left(\frac{y}{H}\right)^{9/7}\right]_0^1 = \frac{7}{9} C_D \tfrac{1}{2}\rho U^2 DH \qquad F_D$$

On an element of the pole,

$$dM = y\,dF_D = y\,C_D \tfrac{1}{2}\rho u^2 dA = y\,C_D \tfrac{1}{2}\rho u^2 D\,dy$$

Thus

$$M = \int_0^H dM = \int_0^H y\,C_D \tfrac{1}{2}\rho U^2 \left(\frac{y}{H}\right)^{2/7} D\,dy = C_D \tfrac{1}{2}\rho U^2 DH^2 \int_0^1 \left(\frac{y}{H}\right)\left(\frac{y}{H}\right)^{2/7} d\left(\frac{y}{H}\right)$$

$$M = C_D \tfrac{1}{2}\rho U^2 DH^2 \int_0^1 \left(\frac{y}{H}\right)^{9/7} d\left(\frac{y}{H}\right) = C_D \tfrac{1}{2}\rho U^2 DH^2 \left[\frac{7}{16}\left(\frac{y}{H}\right)^{16/7}\right]_0^1 = \frac{7}{16} C_D \tfrac{1}{2}\rho U^2 DH^2 \qquad M$$

Comparing,

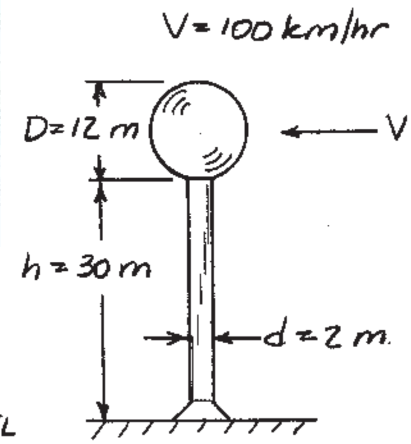
$$\frac{F_D(\text{1/7-profile})}{F_D(\text{uniform})} = \frac{\frac{7}{9} C_D \frac{1}{2}\rho U^2 DH}{C_D \frac{1}{2}\rho U^2 DH} = \frac{7}{9}$$

$$\frac{M(\text{1/7-profile})}{M(\text{uniform})} = \frac{\frac{7}{16} C_D \frac{1}{2}\rho U^2 DH^2}{C_D \frac{1}{2}\rho U^2 DH \frac{H}{2}} = \frac{7/16}{1/2} = \frac{7}{8} \qquad \frac{M}{M_U}$$

Problem 9.136

[Difficulty: 3]

9.136 A water tower consists of a 12-m-diameter sphere on top of a vertical tower 30 m tall and 2 m in diameter. Estimate the bending moment exerted on the base of the tower due to the aerodynamic force imposed by a 100 km/hr wind on a standard day. Neglect interference at the joint between the sphere and tower.



Solution: Apply definition of C_D, sum moments about base.

Computing equations: $C_D \equiv \dfrac{F_D}{\frac{1}{2}\rho V^2 A}$; $\sum \overset{\curvearrowleft +}{M} = \sum FL$

Assumptions: (1) F_{Ds} acts at center of sphere; F_{Dc} at center of cylinder
(2) Neglect interference between sphere and cylinder.

Then $M = F_{Ds}\left(h + \frac{D}{2}\right) + F_{Dc}\left(\frac{h}{2}\right)$ $\quad A_s = \frac{\pi D^2}{4} = \frac{\pi}{4}(12)^2\,m^2 = 113\ m^2$

$A_c = hd = 30\ m \times 2\ m = 60\ m^2$

$$V = 100\ \frac{km}{hr} \times 1000\ \frac{m}{km} \times \frac{hr}{3600\ s} = 27.8\ m/s$$

$$q = \frac{1}{2}\rho V^2 = \frac{1}{2} \times 1.23\ \frac{kg}{m^3} \times (27.8)^2 \frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m} = 475\ N/m^2$$

$C_D = C_D(Re)$. For standard air (Table A.10), $\nu = 1.46 \times 10^{-5}\ m^2/s$, so

$$Re_s = \frac{VD}{\nu} = 27.8\ \frac{m}{s} \times 12\ m \times \frac{s}{1.46\times10^{-5}\ m^2} = 2.28 \times 10^7$$

This Re is too large for Fig. 9.11. Thus guess $C_{Ds} = 0.18$ (Problem 9.125).

$$F_{Ds} = C_{Ds}\, q\, A_s = 0.18 \times 475\ \frac{N}{m^2} \times 113\ m^2 = 9.66\ kN$$

$$Re_c = \frac{Vd}{\nu} = 27.8\ \frac{m}{s} \times 2.0\ m \times \frac{s}{1.46\times10^{-5}\ m^2} = 3.81 \times 10^6$$

This Re is too large for Fig. 9.13. Thus guess $C_{Dc} \simeq 0.4$.

$$F_{Dc} = C_{Dc}\, q\, A_c = 0.4 \times 475\ \frac{N}{m^2} \times 60\ m^2 = 11.4\ kN$$

The moment is

$$M = 9.66\ kN\left(30\ m + \frac{12\ m}{2}\right) + 11.4\ kN\left(\frac{30\ m}{2}\right) = 519\ kN\cdot m$$

M

Problem 9.137

[Difficulty: 3]

9.137 A model airfoil of chord 15 cm and span 60 cm is placed in a wind tunnel with an air flow of 30 m/s (the air is at 20°C). It is mounted on a cylindrical support rod 2 cm in diameter and 25 cm tall. Instruments at the base of the rod indicate a vertical force of 50 N and a horizontal force of 6 N. Calculate the lift and drag coefficients of the airfoil.

Given: Data on model airfoil

Find: Lift and drag coefficients

Solution:

Basic equation: $C_D = \dfrac{F_D}{\frac{1}{2}\cdot\rho\cdot A\cdot V^2}$ $\quad C_L = \dfrac{F_L}{\frac{1}{2}\cdot\rho\cdot A\cdot V^2}$ where A is plan area for airfoil, frontal area for rod

Given or available data is $D = 2\cdot cm$ $\quad L = 25\cdot cm$ (Rod) $\quad b = 60\cdot cm$ $\quad c = 15\cdot cm$ (Airfoil)

$V = 30\cdot\dfrac{m}{s}$ $\quad F_L = 50\cdot N$ $\quad F_H = 6\cdot N$

Note that the horizontal force F_H is due to drag on the airfoil AND on the rod

$\rho = 1.225\cdot\dfrac{kg}{m^3}$ $\quad \nu = 1.50\times 10^{-5}\cdot\dfrac{m^2}{s}$ (Table A.10, 20oC)

For the rod $\quad Re_{rod} = \dfrac{V\cdot D}{\nu}$ $\quad Re_{rod} = 4\times 10^4$ so from Fig. 9.13 $\quad C_{Drod} = 1.0$

$A_{rod} = L\cdot D$ $\quad A_{rod} = 5\times 10^{-3}\, m^2$

$F_{Drod} = C_{Drod}\cdot\dfrac{1}{2}\cdot\rho\cdot A_{rod}\cdot V^2$ $\quad F_{Drod} = 2.76\, N$

Hence for the airfoil $\quad A = b\cdot c$ $\quad F_D = F_H - F_{Drod}$ $\quad F_D = 3.24\, N$

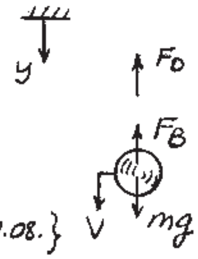
$C_D = \dfrac{F_D}{\frac{1}{2}\cdot\rho\cdot A\cdot V^2}$ $\quad C_D = 0.0654$ $\quad C_L = \dfrac{F_L}{\frac{1}{2}\cdot\rho\cdot A\cdot V^2}$ $\quad C_L = 1.01$ $\quad \dfrac{C_L}{C_D} = 15.4$

Problem 9.138

9.138 A cast-iron "12-pounder" cannonball rolls off the deck of a ship and falls into the ocean at a location where the depth is 1000 m. Estimate the time that elapses before the cannonball hits the sea bottom.

Solution: Apply Newton's second law of motion, definition of C_D.

Computing equations: $\qquad \sum F_y = ma_y \qquad C_D = \dfrac{F_D}{\frac{1}{2}\rho V^2 A}$



First find diameter of ball. In air,

$$W = mg = \rho \forall g = SG\rho_{H_2O}\frac{\pi D^3}{6}g = 12\text{ lbf} \quad \{\text{From Table A.1, } SG = 7.08.\}$$

Thus

$$D = \left[\frac{6W}{\pi SG\rho_{H_2O} g}\right]^{1/3} = \left[\frac{6}{\pi}\times 12\text{ lbf}\times\frac{1}{7.08}\times\frac{ft^3}{1.94\text{ slug}}\times\frac{s^2}{32.2\text{ ft}}\times\frac{\text{slug}\cdot\text{ft}}{\text{lbf}\cdot s^2}\right]^{1/3} = 0.373\text{ ft}$$

or

$$D = 0.373\text{ ft}\times 0.3048\frac{m}{ft} = 0.114\text{ m} \qquad D$$

At terminal speed, $V = V_t$, and $a_y = 0$. Summing forces,

$$mg - F_B - F_D = SG\rho_{H_2O}\forall g - SG_{sw}\rho_{H_2O}\forall g - C_D A\frac{1}{2}\rho V_t^2 = 0$$

or

$$V_t = \left[\frac{2(SG_{ci} - SG_{sw})\rho_{H_2O}\forall g}{C_D\, SG_{sw}\rho_{H_2O} A}\right]^{1/2}$$

Introducing $\forall = \frac{\pi D^3}{6}$ and $A = \frac{\pi D^2}{4}$, then

$$V_t = \left[\frac{4}{3}\frac{(SG_{ci}/SG_{sw} - 1)Dg}{C_D}\right]^{1/2} = \left[\frac{4}{3}(7.08/1.025 - 1)\,0.114\text{ m}\times 9.81\frac{m}{s^2}\times\frac{1}{C_D}\right]^{1/2} = \frac{2.97}{\sqrt{C_D}}\text{ m/s}$$

Choose $C_D = 0.47$ from flat range of curve:

$$V_t = \frac{2.97}{\sqrt{0.47}}\text{ m/sec} = 4.33\text{ m/s} \qquad \{\text{At } T = 20°C,\ \nu_{sw} = 1.05\,\nu_{H_2O} \text{ (Table A.2).}\}$$

Then $Re = \dfrac{\rho V_t D}{\mu} = \dfrac{V_t D}{\nu_{sw}} = 4.33\dfrac{m}{s}\times 0.114\text{ m}\times\dfrac{s}{1.05\times 10^{-6}\text{ m}^2} = 4.70\times 10^5$

This is a supercritical Re, so choose $C_D \approx 0.09$ (Fig. 9.11). Then

$$V_t = \frac{2.97}{\sqrt{0.09}}\text{ m/s} = 9.90\text{ m/s} \qquad V_t$$

Then $Re = 9.90\dfrac{m}{s}\times 0.114\text{ m}\times\dfrac{s}{1.05\times 10^{-6}\text{ m}^2} = 1.07\times 10^6$

From Fig. 9.11, $C_D \approx 0.14$. Therefore $V_t = \dfrac{2.97}{\sqrt{0.14}} = 7.94$ m/s

$$t = \frac{d}{V_t} = 1000\text{ m}\times\frac{s}{7.94\text{ m}} = 126\text{ s} \qquad t$$

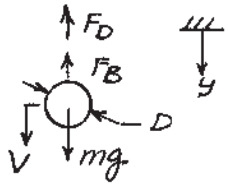
Problem 9.139

[Difficulty: 4]

9.139 The Stokes drag law for smooth spheres is to be verified experimentally by dropping steel ball bearings in glycerin. Evaluate the largest diameter steel ball for which $Re < 1$ at terminal speed. Calculate the height of glycerin column needed for a bearing to reach 95 percent of terminal speed.

Solution: Draw free-body diagram of ball, apply Newton's second law.

Basic equation: $\Sigma F_y = mg - F_B - F_D = m\frac{dV}{dt} = mV\frac{dV}{dy}$ (1)



$$\forall = \frac{\pi D^3}{6} \qquad m = \rho_s \forall \qquad F_B = \rho_g \forall g \qquad F_D = 3\pi\mu V D$$

At terminal speed, V_t, acceleration is zero. Thus

$$\rho_s\forall g - \rho_g \forall g - 3\pi\mu V_t D = 0 \quad \text{or} \quad V_t = (\rho_s - \rho_g)\frac{\pi D^3}{6} g \frac{1}{3\pi\mu D} = \frac{(\rho_s - \rho_g)D^2 g}{18\mu}$$

$$\text{or} \quad V_t = \frac{(\rho_s/\rho_g - 1)\rho_g D^2 g}{18\mu} = \frac{(SG_s/SG_g - 1)D^2 g}{18\nu} = \frac{(7.8/1.26 - 1)D^2 g}{18\nu} = 0.288\, D^2 g/\nu$$

(from Table A.2, $SG_g = 1.26$). Stokes' drag law holds for $Re < 1$. Thus

$$Re = \frac{\rho V_t D}{\mu} = \frac{V_t D}{\nu} = \frac{0.288 D^3 g}{\nu^2} \leq 1 \quad \text{or} \quad D^3 \leq \frac{1}{0.288}\frac{\nu^2}{g} \quad \text{or} \quad D \leq \left[\frac{3.47\nu^2}{g}\right]^{1/3}$$

Assuming $T = 20°C$, then from Fig. A.3, $\nu = 0.0012\ m^2/s$, so

$$D \leq \left[3.47 \times (0.0012)^2 \frac{m^4}{s^2} \times \frac{s^2}{9.81\ m}\right]^{1/3} = 0.00799\ m \quad (7.99\ mm)$$

D

From Eq. 1,

$$\rho_s \forall g - \rho_g \forall g - 3\pi\mu V D = \rho_s \forall V \frac{dV}{dy}$$

Dividing by $(\rho_s - \rho_g)\forall g$ gives

$$1 - \frac{3\pi\mu D}{(\rho_s - \rho_g)\frac{\pi D^3}{6} g} V = 1 - \frac{V}{V_t} = \frac{\rho_s \forall}{(\rho_s - \rho_g)\forall g} V \frac{dV}{dy} = \left(\frac{\rho_s}{\rho_s - \rho_g}\right)\frac{V}{g}\frac{dV}{dy} = \left(\frac{\rho_s}{\rho_s - \rho_g}\right)\frac{V_t^2}{g}\frac{V}{V_t}\frac{d(V/V_t)}{dy}$$

Separating variables, $dy = \left(\frac{\rho_s}{\rho_s - \rho_g}\right)\frac{V_t^2}{g}\frac{(V/V_t)\,d(V/V_t)}{1 - V/V_t} = \left(\frac{\rho_s}{\rho_s - \rho_g}\right)\frac{V_t^2}{g}\frac{r\,dr}{1 - r}$

Integrating,

$$\int_0^{0.95} \frac{r\,dr}{1-r} = \int_1^{0.05} \frac{(1-x)(-dx)}{x} = \int_1^{0.05} dx - \int \frac{dx}{x} = x - \ln x \Big]_1^{0.05} = -0.95 - \ln(0.05)$$

Thus

$$y = 2.05\left(\frac{\rho_s}{\rho_s - \rho_g}\right)\frac{V_t^2}{g} = 2.05\left(\frac{SG_s}{SG_s - SG_g}\right)\frac{V_t^2}{g}$$

But $V_t = 0.288\frac{D^2 g}{\nu} = 0.288 \times (0.0172)^2\, m^2 \times 9.81\frac{m}{s^2} \times \frac{s}{0.0012\ m^2} = 0.697\ m/s$

so

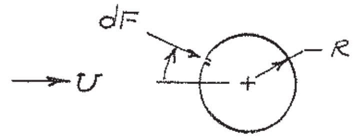
$$y = 2.05 \times \frac{7.8}{(7.8 - 1.26)} \times (0.697)^2 \frac{m^2}{s^2} \times \frac{s^2}{9.81\ m} = 0.121\ m \quad (121\ mm)$$

y

Problem 9.140

9.140 The plot shows pressure difference versus angle, measured for air flow around a circular cylinder at $Re = 80{,}000$. Use these data to estimate C_D for this flow. Compare with data from Fig. 9.13. How can you explain the difference?

Solution: Consider the geometry sketched.
Apply the definition of drag coefficient.



Computing equation: $C_D = \dfrac{F_D}{\frac{1}{2}\rho U^2 A}$

Assumption: Neglect viscous force; $dF_D = dF\cos\theta = p\,dA\cos\theta = p\,w\,R\,d\theta\cos\theta$

Then $F_D = \int_A dF_D = \int_0^{2\pi} p\,w\,R\,d\theta\cos\theta = 2\int_0^{\pi} p\,w\,R\,d\theta\cos\theta = \int_0^{\pi} p\cos\theta\,(w\,2R)\,d\theta$

Since $\int_0^{\pi} p_\infty\cos\theta\,d\theta = 0$, then $F_D = \int_0^{\pi}(p-p_\infty)\cos\theta\,(w\,2R)\,d\theta$

The stagnation pressure is $p(0) - p_\infty = \frac{1}{2}\rho U^2$; so

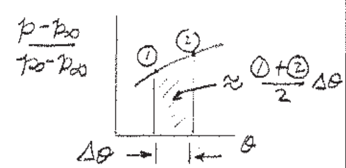
$$C_D = \frac{F_D}{\frac{1}{2}\rho U^2 A} = \frac{\int_0^{\pi}(p-p_\infty)\cos\theta\,(w\,2R)\,d\theta}{(p_0-p_\infty)(w\,2R)} = \int_0^{\pi}\left(\frac{p-p_\infty}{p_0-p_\infty}\right)\cos\theta\,d\theta$$

Tabulating:

θ (deg)	$p-p_\infty$ (in. H_2O)	$\dfrac{p-p_\infty}{p_0-p_\infty}$	$\left(\dfrac{p-p_\infty}{p_0-p_\infty}\right)\cos\theta$
0	0.52	1.00	1.00
10	0.46	0.885	0.871
20	0.33	0.635	0.596
30	0.13	0.250	0.217
40	−0.13	−0.250	−0.192
50	−0.40	−0.769	−0.494
60	−0.57	−1.10	−0.548
70	−0.60	−1.15	−0.395
80	−0.55	−1.06	−0.184
90	−0.52	−1.00	0.00
100	−0.51	−0.981	0.170
110	−0.52	−1.00	0.342
120	−0.52	−1.00	0.500
130	−0.52	−1.00	0.643
140	−0.52	−1.00	0.766
150	−0.52	−1.00	0.866
160	−0.54	−1.04	0.976
170	−0.54	−1.04	1.02
180	−0.54	−1.04	1.04

$\Sigma = 7.194$

Trapezoidal rule:



Half of the end points are subtracted to avoid double counting.

$$C_D \approx \left\{\Sigma\left(\frac{p-p_\infty}{p_0-p_\infty}\right)\cos\theta - \frac{1}{2}(\,)_0 - \frac{1}{2}(\,)_{180}\right\}\Delta\theta = \left\{7.194 - \frac{1}{2}(1.00 + 1.04)\right\} 10\ \text{deg} \times \frac{\pi\ \text{rad}}{180\ \text{deg}} = 1.08 \qquad C_D$$

From Fig. 9.13, $C_D \approx 1.2$. The difference is due to skin friction effects.

Problem 9.141

[Difficulty: 4]

9.141 Consider the tennis ball of Problem 9.134. Use the equations for drag coefficient given in Problem 9.132, and a numerical integration scheme (e.g., Simpson's rule) to compute the time and distance required for the ball to reach 95% of its terminal speed.

Given: Data on a tennis ball

Find: Terminal speed time and distance to reach 95% of terminal speed

Solution:

The given data or available data is $M = 57\cdot gm$ $\quad D = 64\cdot mm$ $\quad \nu = 1.45\cdot 10^{-5}\cdot\frac{m^2}{s}$ $\quad \rho = 1.23\cdot\frac{kg}{m^3}$

Then $A = \frac{\pi\cdot D^2}{4}$ $\quad A = 3.22\times 10^{-3}\,m^2$

From Problem 9.132

$$C_D = \frac{24}{Re} \qquad Re \le 1$$

$$C_D = \frac{24}{Re^{0.646}} \qquad 1 < Re \le 400$$

$$C_D = 0.5 \qquad 400 < Re \le 3\times 10^5$$

$$C_D = 0.000366\cdot Re^{0.4275} \qquad 3\times 10^5 < Re \le 2\times 10^6$$

$$C_D = 0.18 \qquad Re > 2\times 10^6$$

At terminal speed drag equals weight $F_D = M\cdot g$

The drag at speed V is given by $F_D = \frac{1}{2}\cdot\rho\cdot A\cdot V^2\cdot C_D$

Assume $C_D = 0.5$

Hence the terminal speed is $V_t = \sqrt{\frac{2\cdot M\cdot g}{\rho\cdot A\cdot C_D}}$ $\quad V_t = 23.8\,\frac{m}{s}$

Check the Reynolds number $Re = \frac{V_t\cdot D}{\nu}$ $\quad Re = 1.05\times 10^5$

This is consistent with the tabulated C_D values!

For motion before terminal speed, Newton's second law is $M \cdot a = M \cdot \frac{dV}{dt} = M \cdot g \cdot - \frac{1}{2} \cdot \rho \cdot V^2 \cdot A \cdot C_D$

Hence the time to reach 95% of terminal speed is obtained by separating variables and integrating

$$t = \int_0^{0.95 \cdot V_t} \frac{1}{g - \frac{\rho \cdot A \cdot C_D}{2 \cdot M} \cdot V^2} \, dV$$

For the distance to reach terminal speed Newton's second law is written in the form

$$M \cdot a = M \cdot V \cdot \frac{dV}{dx} = M \cdot g \cdot - \frac{1}{2} \cdot \rho \cdot V^2 \cdot A \cdot C_D$$

Hence the distance to reach 95% of terminal speed is obtained by separating variables and integrating

$$x = \int_0^{0.95 \cdot V_t} \frac{V}{g - \frac{\rho \cdot A \cdot C_D}{2 \cdot M} \cdot V^2} \, dV$$

These integrals are quite difficult because the drag coefficient varies with Reynolds number, which varies with speed. They are best evaluated numerically. A form of Simpson's Rule is

$$\int f(V)\, dV = \frac{\Delta V}{3} \cdot \left(f\left(V_0\right) + 4 \cdot f\left(V_1\right) + 2 \cdot f\left(V_2\right) + 4 \cdot f\left(V_3\right) + f\left(V_N\right)\right)$$

where ΔV is the step size, and V_0, V_1 etc., are the velocities at points 0, 1, ... N.

Here $V_0 = 0$ $\quad V_N = 0.95 \cdot V_t$ $\quad \Delta V = \frac{0.95 \cdot V_t}{N}$

From the associated *Excel* calculations (shown below): $t = 4.69 \cdot s$ $\quad x = 70.9 \cdot m$

These results compare to 4.44 s and 67.1 m from Problem 9.132, which assumed the drag coefficient was constant and analytically integrated. Note that the drag coefficient IS essentially constant, so numerical integration was not really necessary!

For the time:

V (m/s)	*Re*	C_D	W	$f(V)$	Wx$f(V)$
0	0	5438	1	0.102	0.102
1.13	4985	0.500	4	0.102	0.409
2.26	9969	0.500	2	0.103	0.206
3.39	14954	0.500	4	0.104	0.416
4.52	19938	0.500	2	0.106	0.212
5.65	24923	0.500	4	0.108	0.432
6.78	29908	0.500	2	0.111	0.222
7.91	34892	0.500	4	0.115	0.458
9.03	39877	0.500	2	0.119	0.238
10.2	44861	0.500	4	0.125	0.499
11.3	49846	0.500	2	0.132	0.263
12.4	54831	0.500	4	0.140	0.561
13.6	59815	0.500	2	0.151	0.302
14.7	64800	0.500	4	0.165	0.659
15.8	69784	0.500	2	0.183	0.366
16.9	74769	0.500	4	0.207	0.828
18.1	79754	0.500	2	0.241	0.483
19.2	84738	0.500	4	0.293	1.17
20.3	89723	0.500	2	0.379	0.758
21.5	94707	0.500	4	0.550	2.20
22.6	99692	0.500	1	1.05	1.05

Total time: 4.69 s

For the distance:

$f(V)$	Wx$f(V)$
0.00	0.000
0.115	0.462
0.232	0.465
0.353	1.41
0.478	0.955
0.610	2.44
0.752	1.50
0.906	3.62
1.08	2.15
1.27	5.07
1.49	2.97
1.74	6.97
2.05	4.09
2.42	9.68
2.89	5.78
3.51	14.03
4.36	8.72
5.62	22.5
7.70	15.4
11.8	47.2
23.6	23.6

Total distance: 70.9 m

Problem 9.142

[Difficulty: 4]

9.142 The air bubble of Problem 3.10 expands as it rises in water. Find the time it takes for the bubble to reach the surface. Repeat for bubbles of diameter 5 mm and 15 mm. Compute and plot the depth of the bubbles as a function of time.

Given:

Data on an air bubble

Find:

Time to reach surface

Solution:

The given data or available data is

$$h = 100 \cdot ft = 30.48\,m \qquad \rho = 1025 \cdot \frac{kg}{m^3} \quad \text{(Table A.2)} \qquad C_D = 0.5 \quad \text{(Fig. 9.11)} \qquad p_{atm} = 101 \cdot kPa$$

To find the location we have to integrate the velocity over time: $dx = V \cdot dt$ where $V = \sqrt{\frac{4 \cdot g \cdot d_0}{3 \cdot C_D} \cdot \left[\frac{p_{atm} + \rho \cdot g \cdot h}{p_{atm} + \rho \cdot g \cdot (h - x)}\right]^{\frac{1}{6}}}$

The results (generated in Excel) for each bubble diameter are shown below:

$d_0 =$ 0.3 in
$d_0 =$ 7.62 mm

t (s)	*x* (m)	*V* (m/s)
0	0	0.446
5	2.23	0.451
10	4.49	0.455
15	6.76	0.460
20	9.1	0.466
25	11.4	0.472
30	13.8	0.478
35	16.1	0.486
40	18.6	0.494
45	21.0	0.504
50	23.6	0.516
63.4	30.5	0.563

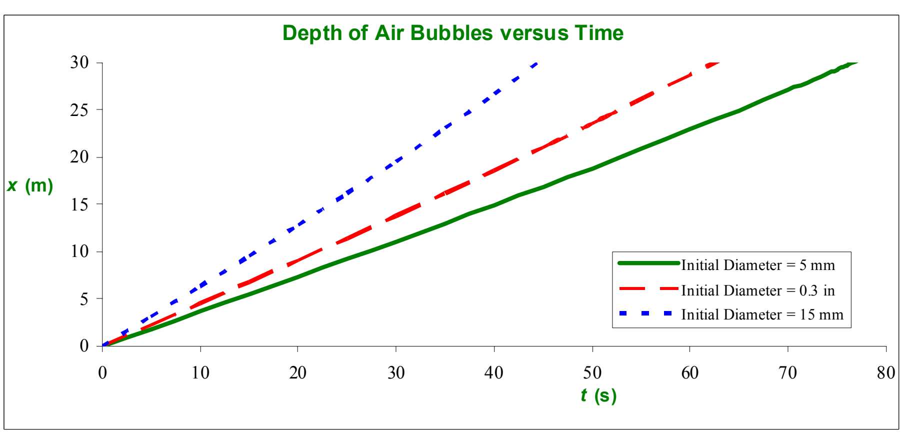
$d_0 =$ 5 mm

t (s)	*x* (m)	*V* (m/s)
0	0	0.362
5	1.81	0.364
10	3.63	0.367
15	5.47	0.371
20	7.32	0.374
25	9.19	0.377
30	11.1	0.381
35	13.0	0.386
40	14.9	0.390
45	16.9	0.396
50	18.8	0.401
55	20.8	0.408
60	22.9	0.415
65	25.0	0.424
70	27.1	0.435
75	29.3	0.448
77.8	30.5	0.456

$d_0 =$ 15 mm

t (s)	*x* (m)	*V* (m/s)
0.0	0	0.626
5.0	3.13	0.635
10.0	6.31	0.644
15.0	9.53	0.655
20.0	12.8	0.667
25.0	16.1	0.682
30.0	19.5	0.699
35.0	23.0	0.721
40.0	26.6	0.749
45.1	30.5	0.790

Use *Goal Seek* for the last time step to make *x* = *h* !



Problem 9.143

[Difficulty: 4]

9.143 Consider the tennis ball of Problem 9.134. Suppose it is hit so that it has an initial upward speed of 50 m/s. Estimate the maximum height of the ball, assuming (a) a constant drag coefficient and (b) using the equations for drag coefficient given in Problem 9.132, and a numerical integration scheme (e.g., a Simpson's rule).

Given: Data on a tennis ball

Find: Maximum height

Solution:

The given data or available data is $M = 57\cdot gm$ $\quad D = 64\cdot mm$ $\quad V_i = 50\cdot\frac{m}{s}$ $\quad \nu = 1.45\cdot 10^{-5}\cdot\frac{m^2}{s}$ $\quad \rho = 1.23\cdot\frac{kg}{m^3}$

Then $A = \frac{\pi\cdot D^2}{4}$ $\quad A = 3.22\times 10^{-3}\,m^2$

From Problem 9.132

$$C_D = \frac{24}{Re} \qquad Re \le 1$$

$$C_D = \frac{24}{Re^{0.646}} \qquad 1 < Re \le 400$$

$$C_D = 0.5 \qquad 400 < Re \le 3\times 10^5$$

$$C_D = 0.000366\cdot Re^{0.4275} \qquad 3\times 10^5 < Re \le 2\times 10^6$$

$$C_D = 0.18 \qquad Re > 2\times 10^6$$

The drag at speed V is given by $F_D = \frac{1}{2}\cdot\rho\cdot A\cdot V^2\cdot C_D$

For motion before terminal speed, Newton's second law (x upwards) is $M\cdot a = M\cdot\frac{dV}{dt} = -\frac{1}{2}\cdot\rho\cdot V^2\cdot A\cdot C_D - M\cdot g$

For the maximum height Newton's second law is written in the form $M\cdot a = M\cdot V\cdot\frac{dV}{dx} = -\frac{1}{2}\cdot\rho\cdot V^2\cdot A\cdot C_D - M\cdot g$

Hence the maximum height is

$$x_{max} = \int_{V_i}^{0}\frac{V}{-\frac{\rho\cdot A\cdot C_D}{2\cdot M}\cdot V^2 - g}\,dV = \int_{0}^{V_i}\frac{V}{\frac{\rho\cdot A\cdot C_D}{2\cdot M}\cdot V^2 + g}\,dV$$

This integral is quite difficult because the drag coefficient varies with Reynolds number, which varies with speed. It is best evaluated numerically. A form of Simpson's Rule is

$$\int f(V)\,dV = \frac{\Delta V}{3}\cdot\left(f(V_0) + 4\cdot f(V_1) + 2\cdot f(V_2) + 4\cdot f(V_3) + f(V_N)\right)$$

where ΔV is the step size, and V_0, V_1 etc., are the velocities at points 0, 1, ... N.

Here $V_0 = 0$ $V_N = V_i$ $\Delta V = \frac{V_i}{N}$

From the associated *Excel* workbook (shown here) $x_{max} = 48.7 \cdot m$

V (m/s)	*Re*	C_D	*W*	*f*(*V*)	*W*x*f*(*V*)
0.0	0	0.000	1	0.000	0.000
2.5	11034	0.500	4	0.252	1.01
5.0	22069	0.500	2	0.488	0.976
7.5	33103	0.500	4	0.695	2.78
10.0	44138	0.500	2	0.866	1.73
12.5	55172	0.500	4	1.00	3.99
15.0	66207	0.500	2	1.09	2.19
17.5	77241	0.500	4	1.16	4.63
20.0	88276	0.500	2	1.19	2.39
22.5	99310	0.500	4	1.21	4.84
25.0	110345	0.500	2	1.21	2.42
27.5	121379	0.500	4	1.20	4.80
30.0	132414	0.500	2	1.18	2.36
32.5	143448	0.500	4	1.15	4.62
35.0	154483	0.500	2	1.13	2.25
37.5	165517	0.500	4	1.10	4.38
40.0	176552	0.500	2	1.06	2.13
42.5	187586	0.500	4	1.03	4.13
45.0	198621	0.500	2	1.00	2.00
47.5	209655	0.500	4	0.970	3.88
50.0	220690	0.500	1	0.940	0.940

If we assume $C_D = 0.5$

the integral
$$x_{max} = \int_0^{V_i} \frac{V}{\frac{\rho \cdot A \cdot C_D}{2 \cdot M} \cdot V^2 + g} \, dV$$

becomes
$$x_{max} = \frac{M}{\rho \cdot A \cdot C_D} \cdot \ln\left(\frac{\rho \cdot A \cdot C_D}{2 \cdot M \cdot g} \cdot V_i^2 + 1\right)$$ $x_{max} = 48.7\,m$

The two results agree very closely! This is because the integrand does not vary much after the first few steps so the numerical integral is accurate, and the analytic solution assumes $C_D = 0.5$, which it essentially does!

Problem 9.144

[Difficulty: 4]

9.144 Why is it possible to kick a football farther in a spiral motion than in an end-over-end tumbling motion?

Discussion: A football has a prolate spheroid shape. It is almost circular when viewed from the front (parallel to the major axis), and longer and more elliptical when viewed from the side (along a minor axis). The football has more frontal area when traveling with the major axis perpendicular to the motion that when it is "spiraling" with the major axis parallel to the direction of travel.

The drag coefficient of the ball when parallel to the flow in spiral motion undoubtedly is less than when perpendicular to the flow. As a rough approximation, the perpendicular drag coefficient might be similar to that of a cylinder ($C_D = 1.2$), whereas the spiral drag coefficient probably is less (perhaps $C_D \approx 0.2 - 0.3$) than that of a sphere ($C_D = 0.5$). Thus the drag coefficient when traveling with the long axis perpendicular to the flow may be 4 to 6 times as large as when traveling in spiral motion with the long axis parallel to the flow. The difference in the drag-area product $C_D A$ will be even larger.

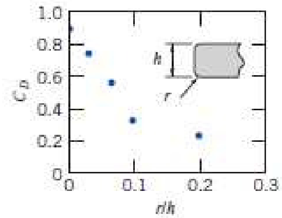
In tumbling motion the drag-area product varies cyclically between the two extremes we have discussed. On average the drag-area product for the tumbling ball is considerably larger, perhaps 2 to 3 times as large, as when the ball is in spiral motion. Therefore the maximum range (travel distance) that can be achieved with tumbling motion is much less than that for spiral motion.

Also, a well kicked or thrown spiral is a thing of beauty. Perhaps function follows form here!

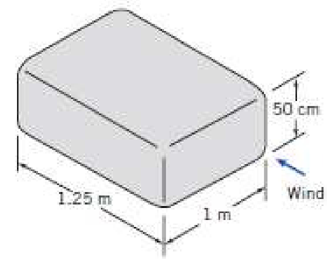
Problem 9.145

[Difficulty: 3]

9.145 Approximate dimensions of a rented rooftop carrier are shown. Estimate the drag force on the carrier ($r = 10$ cm) at 100 km/hr. If the drivetrain efficiency of the vehicle is 0.85 and the brake specific fuel consumption of its engine is 0.3 kg/(kW · hr), estimate the additional rate of fuel consumption due to the carrier. Compute the effect on fuel economy if the auto achieves 12.75 km/L without the carrier. The rental company offers you a cheaper, square-edged carrier at a price \$5 less than the current carrier. Estimate the extra cost of using this carrier instead of the round-edged one for a 750 km trip, assuming fuel is \$3.50 per gallon. Is the cheaper carrier really cheaper?



Drag coefficient v. radius ratio [37]



Given: Data on rooftop carrier

Find: Drag on carrier; Additional fuel used; Effect on economy; Effect of "cheaper" carrier

Solution:

Basic equation: $$C_D = \frac{F_D}{\frac{1}{2}\cdot\rho\cdot A\cdot V^2}$$

Given or available data is $w = 1\cdot m$ $\quad h = 50\cdot cm$ $\quad r = 10\cdot cm$ $\quad \eta_d = 85\cdot\%$

$V = 100\cdot\frac{km}{hr}$ $\quad V = 27.8\,\frac{m}{s}$ $\quad FE = 12.75\cdot\frac{km}{L}$ $\quad FE = 30.0\cdot\frac{mi}{gal}$

$\rho_{H2O} = 1000\cdot\frac{kg}{m^3}$ $\quad A = w\cdot h$ $\quad A = 0.5\,m^2$ $\quad BSFC = 0.3\cdot\frac{kg}{kW\cdot hr}$

$\rho = 1.225\cdot\frac{kg}{m^3}$ $\quad \nu = 1.50\times 10^{-5}\cdot\frac{m^2}{s}$ $\quad$ (Table A.10, 20ºF)

From the diagram $\frac{r}{h} = 0.2$ so $\quad C_D = 0.25$ $\quad F_D = C_D\cdot\frac{1}{2}\cdot\rho\cdot A\cdot V^2$ $\quad F_D = 59.1\,N$

Additional power is $\Delta P = \frac{F_D\cdot V}{\eta_d}$ $\quad \Delta P = 1.93\cdot kW$

Additional fuel is $\Delta FC = BSFC\cdot\Delta P$ $\quad \Delta FC = 1.61\times 10^{-4}\,\frac{kg}{s}$ $\quad \Delta FC = 0.00965\cdot\frac{kg}{min}$

Fuel consumption of the car only is (with $SG_{gas} = 0.72$ from Table A.2)

$FC = \frac{V}{FE}\cdot SG_{gas}\cdot\rho_{H2O}$ $\quad FC = 1.57\times 10^{-3}\,\frac{kg}{s}$ $\quad FC = 0.0941\cdot\frac{kg}{min}$

The total fuel consumption is then $FC_T = FC + \Delta FC$ $\quad FC_T = 1.73\times 10^{-3}\,\frac{kg}{s}$ $\quad FC_T = 0.104\cdot\frac{kg}{min}$

Fuel economy with the carrier is $FE = \frac{V}{FC_T}\cdot SG_{gas}\cdot\rho_{H2O}$ $\quad FE = 11.6\cdot\frac{km}{L}$ $\quad FE = 27.2\cdot\frac{mi}{gal}$

For the square-edged: $\frac{r}{h} = 0$ so $\quad C_D = 0.9$ $\quad F_D = C_D\cdot\frac{1}{2}\cdot\rho\cdot A\cdot V^2$ $\quad F_D = 213\,N$

Additional power is $\Delta P = \frac{F_D\cdot V}{\eta_d}$ $\quad \Delta P = 6.95\cdot kW$

Additional fuel is $\Delta FC = BSFC \cdot \Delta P$ $\Delta FC = 5.79 \times 10^{-4} \frac{kg}{s}$ $\Delta FC = 0.0348 \cdot \frac{kg}{min}$

The total fuel consumption is then $FC_T = FC + \Delta FC$ $FC_T = 2.148 \times 10^{-3} \frac{kg}{s}$ $FC_T = 0.129 \cdot \frac{kg}{min}$

Fuel economy withy the carrier is now $FE = \frac{V}{FC_T} \cdot SG_{gas} \cdot \rho_{H2O}$ $FE = 9.3 \cdot \frac{km}{L}$ $FE = 21.9 \cdot \frac{mi}{gal}$

The cost of the trip of distance $d = 750 \cdot km$ for fuel costing $p = \frac{\$ \cdot 3.50}{gal}$ with a rental $discount = \$ \cdot 5$ less than the rounded carrier is then

$$Cost = \frac{d}{FE} \cdot p - discount \qquad Cost = 69.47$$ plus the rental fee

The cost of the trip of with the rounded carrier ($FE = 11.6 \cdot \frac{km}{L}$) is then

$$Cost = \frac{d}{FE} \cdot p \qquad Cost = 59.78$$ plus the rental fee

Hence the "cheaper" carrier is more expensive (AND the environment is significantly more damaged!)

Problem 9.146

[Difficulty: 4]

9.146 A barge weighing 8820 kN that is 10 m wide, 30 m long, and 7 m tall has come free from its tug boat in the Mississippi River. It is in a section of river which has a current of 1 m/s, and there is a wind blowing straight upriver at 10 m/s. Assume that the drag coefficient is 1.3 for both the part of the barge in the wind as well as the part below water. Determine the speed at which the barge will be steadily moving. Is it moving upriver or downriver?

Given: Data on barge and river current

Find: Speed and direction of barge

Solution:

Basic equation:
$$C_D = \frac{F_D}{\frac{1}{2}\cdot\rho\cdot A\cdot V^2}$$

Given or available data is $W = 8820\cdot kN$ $\quad w = 10\cdot m$ $\quad L = 30\cdot m$ $\quad h = 7\cdot m$ $\quad V_{river} = 1\cdot\frac{m}{s}$ $\quad V_{wind} = 10\cdot\frac{m}{s}$ $\quad C_{Dw} = 1.3$

$C_{Da} = 1.3$ $\quad \rho_w = 998\cdot\frac{kg}{m^3}$ $\quad \nu_w = 1.01\times 10^{-6}\cdot\frac{m^2}{s}$ $\quad \rho_a = 1.21\cdot\frac{kg}{m^3}$ $\quad \nu_a = 1.50\times 10^{-5}\cdot\frac{m^2}{s}$ (Water data from Table A.8, air data from Table A.10, 20ºC)

First we need to calculate the amount of the barge submerged in the water. From Archimedes' Principle: $W = \rho_w\cdot g\cdot V_{sub}$

The submerged volume can be expressed as: $V_{sub} = w\cdot L\cdot h_{sub}$ Combining these expressions and solving for the depth:

$$h_{sub} = \frac{W}{\rho_w\cdot g\cdot w\cdot L} = 3.00\,m$$

Therefore the height of barge exposed to the wind is: $h_{air} = h - h_{sub} = 4.00\,m$

Assuming the barge is floating downstream, the velocities of the water and air relative to the barge is:

$$V_w = V_{river} - V_{barge} \qquad V_a = V_{wind} + V_{barge}$$

Assuming that the barge is rectangular, the areas exposed to the air and water are: $A_a = L\cdot w + 2\cdot(L + w)\cdot h_{air} = 620\,m^2$

$$A_w = L\cdot w + 2\cdot(L + w)\cdot h_{sub} = 540\,m^2$$

In order for the barge to be traveling at a constant speed, the drag forces due to the air and water must match:

$\frac{1}{2}\cdot C_{Dw}\cdot\rho_w\cdot V_w^2\cdot A_w = \frac{1}{2}\cdot C_{Da}\cdot\rho_a\cdot V_a^2\cdot A_a$ Since the drag coefficients are equal, we can simplify: $\rho_w\cdot V_w^2\cdot A_w = \rho_a\cdot V_a^2\cdot A_a$

Solving for the speed relative to the water: $V_w^2 = V_a^2\cdot\frac{\rho_a}{\rho_w}\cdot\frac{A_a}{A_w}$ Since the speeds must be in opposite directions:

$V_w = -V_a\cdot\sqrt{\frac{\rho_a}{\rho_w}\cdot\frac{A_a}{A_w}}$ In terms of the barge speed: $V_{river} - V_{barge} = -\left(V_{wind} + V_{barge}\right)\cdot\sqrt{\frac{\rho_a}{\rho_w}\cdot\frac{A_a}{A_w}}$

So solving for the barge speed:
$$V_{barge} = \frac{V_{river} + V_{wind}\cdot\sqrt{\frac{\rho_a}{\rho_w}\cdot\frac{A_a}{A_w}}}{1 - \sqrt{\frac{\rho_a}{\rho_w}\cdot\frac{A_a}{A_w}}}$$

$V_{barge} = 1.426\,\frac{m}{s}$ downstream

Problem 9.147

[Difficulty: 4]

9.147 Coastdown tests, performed on a level road on a calm day, can be used to measure aerodynamic drag and rolling resistance coefficients for a full-scale vehicle. Rolling resistance is estimated from dV/dt measured at low speed, where aerodynamic drag is small. Rolling resistance then is deducted from dV/dt measured at high speed to determine the aerodynamic drag. The following data were obtained during a test with a vehicle, of weight $W = 25{,}000$ lbf and frontal area $A = 79\ \text{ft}^2$:

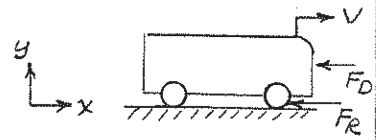
V(mph)	5	55
$\dfrac{dV}{dt}\left(\dfrac{\text{mph}}{\text{s}}\right)$	−0.150	−0.475

Estimate the aerodynamic drag coefficient for this vehicle. At what speed does the aerodynamic drag first exceed rolling resistance?

Solution: Apply Newton's second law of motion, definition of C_D.

Computing equations:

$$\Sigma F_x = ma_x \qquad C_D = \frac{F_D}{\frac{1}{2}\rho V^2 A}$$



Summing forces, $-F_R - F_D = ma_x$

At 55 mph $\quad -F_R - F_{D55} = ma_{x55}$

At 5 mph $\quad -F_R - \cancel{F_{D5}}^{\approx 0} = ma_{x5}$

Subtracting, obtain $\quad -F_{D55} = m(a_{x55} - a_{x5}) = m\left[\frac{dV}{dt}\Big)_{55} - \frac{dV}{dt}\Big)_5\right] = -C_D A \rho \frac{V^2}{2}$

Thus

$$C_D = \frac{m\left[\frac{dV}{dt}\Big)_5 - \frac{dV}{dt}\Big)_{55}\right]}{\frac{1}{2}\rho V^2 A} = \frac{2W\left[\frac{dV}{dt}\Big)_5 - \frac{dV}{dt}\Big)_{55}\right]}{\rho g V^2 A}$$

Evaluating, assuming standard air, with $\rho g = 0.0765\ \text{lbf/ft}^3$

$$C_D = 2 \times 25000\ \text{lbf} \times \frac{\text{ft}^3}{0.0765\ \text{lbf}} \times \frac{\text{hr}^2}{(55)^2\ \text{mi}^2} \times \frac{1}{79\ \text{ft}^2} \times [-0.150 - (-0.475)]\frac{\text{mi}}{\text{hr}\cdot\text{s}} \times \frac{\text{mi}}{5280\ \text{ft}} \times 3600\ \frac{\text{s}}{\text{hr}}$$

$$C_D = 0.606$$

C_D

At $V = 5$ mph, $F_D \approx 0 \quad \therefore \quad -F_R = ma_{x5}$

$$F_R = -ma_{x5} = -\frac{25000\ \text{lbf}}{32.2\ \text{ft}}\,\text{s}^2 \times -0.150\ \frac{\text{mi}}{\text{hr}\cdot\text{s}} \times \frac{\text{hr}}{3600\ \text{s}} \times \frac{5280\ \text{ft}}{\text{mi}} = 171\ \text{lbf}$$

For $F_R = F_D = C_D \frac{1}{2}\rho V^2 A$, then $V = \left[\frac{2F_R}{\rho A C_D}\right]^{1/2}$

$$V = \left[2 \times 171\ \text{lbf} \times \frac{\text{ft}^3}{0.00238\ \text{slug}} \times \frac{1}{79\ \text{ft}^2} \times \frac{1}{0.606} \times \frac{\text{slug}\cdot\text{ft}}{\text{lbf}\cdot\text{s}^2}\right]^{1/2}$$

$V = 54.8\ \frac{\text{ft}}{\text{s}} = 37.4$ mph.

$V_{F_R = F_D}$

Problem 9.148

[Difficulty: 4]

9.148 A spherical sonar transducer with 15 in. diameter is to be towed in seawater. The transducer must be fully submerged at 55 ft/s. To avoid cavitation, the minimum pressure on the surface of the transducer must be greater than 5 psia. Calculate the hydrodynamic drag force acting on the transducer at the required towing speed. Estimate the minimum depth to which the transducer must be submerged to avoid cavitation.

Given: Data on sonar transducer

Find: Drag force at required towing speed; minimum depth necessary to avoid cavitation

Solution:

Basic equation: $C_D = \dfrac{F_D}{\frac{1}{2}\cdot\rho\cdot A\cdot V^2}$ $\qquad C_P = \dfrac{p - p_{inf}}{\frac{1}{2}\cdot\rho\cdot V^2}$ $\qquad p = p_{atm} + \rho\cdot g\cdot h$

Given or available data is $D = 15\cdot in$ $\quad V = 55\cdot\dfrac{ft}{s}$ $\quad p_{min} = 5\cdot psi$ $\quad \rho = 1.93\cdot\dfrac{slug}{ft^3}$ $\quad \nu = 1.06\times10^{-5}\cdot\dfrac{ft^2}{s}$ (Table A.7, 70ºF)

The Reynolds number of the flow is: $Re = \dfrac{V\cdot D}{\nu} = 6.486\times10^6$ From Fig. 9.11, we estimate the drag coefficient: $C_D = 0.18$

The area is: $A = \dfrac{\pi}{4}\cdot D^2 = 1.227\cdot ft^2$ Therefore the drag force is: $F_D = \dfrac{1}{2}\cdot C_D\cdot\rho\cdot V^2\cdot A$ $\qquad F_D = 645\cdot lbf$

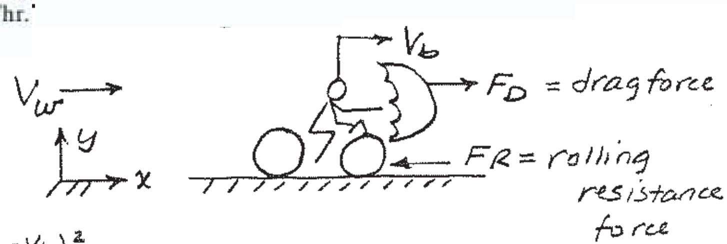
From Fig. 9.12 the minimum pressure occurs where $C_P = -1.2$ Therefore: $p_{inf} = p_{min} - \dfrac{1}{2}\cdot C_P\cdot\rho\cdot V^2 = 29.326\cdot psi$

Solving for the required depth: $h = \dfrac{p_{inf} - p_{atm}}{\rho\cdot g}$ $\qquad h = 33.9\cdot ft$

Problem 9.149

[Difficulty: 4]

9.149 While walking across campus one windy day, Floyd Fluids speculates about using an umbrella as a "sail" to propel a bicycle along the sidewalk. Develop an algebraic expression for the speed a bike could reach on level ground with the umbrella "propulsion system." The frontal area of bike and rider is estimated as 0.3 m^2, and the drag coefficient is about 1.2. Assume the rolling resistance is 0.75 percent of the bike and rider weight; the combined mass is 75 kg. Evaluate the bike speed that could be achieved with an umbrella 1.22 m in diameter in a wind that blows at 24 km/hr. Discuss the practicality of this propulsion system.



Sum forces in x direction:

$$\Sigma F_x = F_D - F_R = 0$$

But $F_D = (C_{Du} A_u + C_{Db} A_b) \frac{1}{2} \rho (V_w - V_b)^2$

$$A_u = \frac{\pi D_u^2}{4} = \frac{\pi}{4}(1.22)^2 m^2 = 1.17\ m^2$$

$$F_R = C_R mg$$

Choose $C_{Du} = 1.42$ (Table 9.3).

$C_R = 0.75\%$, $m = 75$ kg, so $F_R = 0.0075 \times 75\ kg \times 9.81 \frac{m}{s^2} \times \frac{N \cdot s^2}{kg \cdot m} = 5.52\ N$

Then $V_b = V_w - \left[\frac{2F_R}{\rho(C_{Du}A_u + C_{Db}A_b)}\right]^{\frac{1}{2}}$ $\quad V_b$

But

$$V_w = 24 \frac{km}{hr} \times 1000 \frac{m}{km} \times \frac{hr}{3600\ s} = 6.67\ m/s$$

$$V_b = 6.67 \frac{m}{s} - \left[2 \times 5.52\ N \times \frac{m^3}{1.23\ kg} \frac{1}{(1.42)1.17 m^2 + (1.20)0.3 m^2} \times \frac{kg \cdot m}{N \cdot s^2}\right]^{\frac{1}{2}}$$

$$V_b = 6.67 \frac{m}{s} - 2.11 \frac{m}{s} = 4.56 \frac{m}{s} \text{ or } 16.4 \frac{km}{hr}$$ $\quad V_b$

Thus Floyd's bicycle (with the umbrella propelling it) travels at 68.3% wind speed.

{Without the umbrella, $V_b = 1.68 \frac{m}{s}$ or $6.04 \frac{km}{hr}$, by setting $C_{Du} = 0$ above.}

Discussion: Floyd is confused about his fluid mechanics principles if he thinks he can exceed the wind speed. It is impossible to obtain a propulsive force from aerodynamic drag unless the bicycle is moving more slowly than the wind. The drag force must be sufficient to overcome the rolling resistance of the bike and rider. At equilibrium speed the drag force and rolling resistance force must be equal and opposite.

The only benefit could be achieved by adding drag force more rapidly than rolling resistance. An umbrella, with its relatively high drag and low weight, is ideal for this purpose.

However, one would somehow have to hold the umbrella perpendicular to the wind while riding the bike. This would be dangerous at best, especially if the bike had hand-activated brakes.

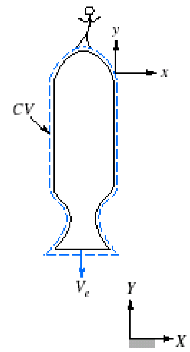
Since the umbrella must be held perpendicular to the wind, it would be very effective at blocking the rider's view of the road ahead!

In summary, this "system" of propulsion appears quite impractical.

Problem 9.150

[Difficulty: 4]

9.150 Motion of a small rocket was analyzed in Example 4.12 assuming negligible aerodynamic drag. This was not realistic at the final calculated speed of 369 m/s. Use the Euler method of Section 5.5 for approximating the first derivatives, in an *Excel* workbook, to solve the equation of motion for the rocket. Plot the rocket speed as a function of time, assuming $C_D = 0.3$ and a rocket diameter of 700 mm. Compare with the results for $C_D = 0$.



Given: Data on a rocket

Find: Plot of rocket speed with and without drag

Solution:

From Example 4.12, with the addition of drag the momentum equation becomes

$$F_{B_y} + F_{S_y} - \int_{\text{CV}} a_{rf_y}\,\rho\, d\forall = \frac{\partial}{\partial t}\int_{\text{CV}} v_{xyz}\,\rho\, d\forall + \int_{\text{CV}} v_{xyz}\,\rho \vec{V}_{xyz}\cdot d\vec{A}$$

where the surface force is

$$F_{S_y} = -\frac{1}{2}\rho A V^2 C_{\text{D}}$$

Following the analysis of the example problem, we end up with

$$\frac{dV_{\text{CV}}}{dt} = \frac{V_e \dot{m}_e - \frac{1}{2}\rho A V_{\text{CV}}{}^2 C_{\text{D}}}{M_0 - \dot{m}_e t} - g$$

This can be written (dropping the subscript for convenience)

$$\frac{dV}{dt} = f(V,t) \qquad (1)$$

where

$$f(V,t) = \frac{V_e \dot{m}_e - \frac{1}{2}\rho A V^2 C_{\text{D}}}{M_0 - \dot{m}_e t} - g \qquad (2)$$

Equation 1 is a differential equation for speed V.

It can be solved using Euler's numerical method

$$V_{\text{n+1}} \approx V_{\text{n}} + \Delta t\, f_{\text{n}}$$

where $V_{\text{n+1}}$ and V_{n} are the n + 1$^{\text{th}}$ and n$^{\text{th}}$ values of V, f_{n} is the function given by Eq. 2 evaluated at the n$^{\text{th}}$ step, and Δt is the time step.

The initial condition is $\qquad V_0 = 0 \quad \text{at} \quad t = 0$

Given or available data:

M_0 = 400 kg
m_e = 5 kg/s
V_e = 3500 m/s
ρ = 1.23 kg/m^3
D = 700 mm
C_D = 0.3

Computed results:

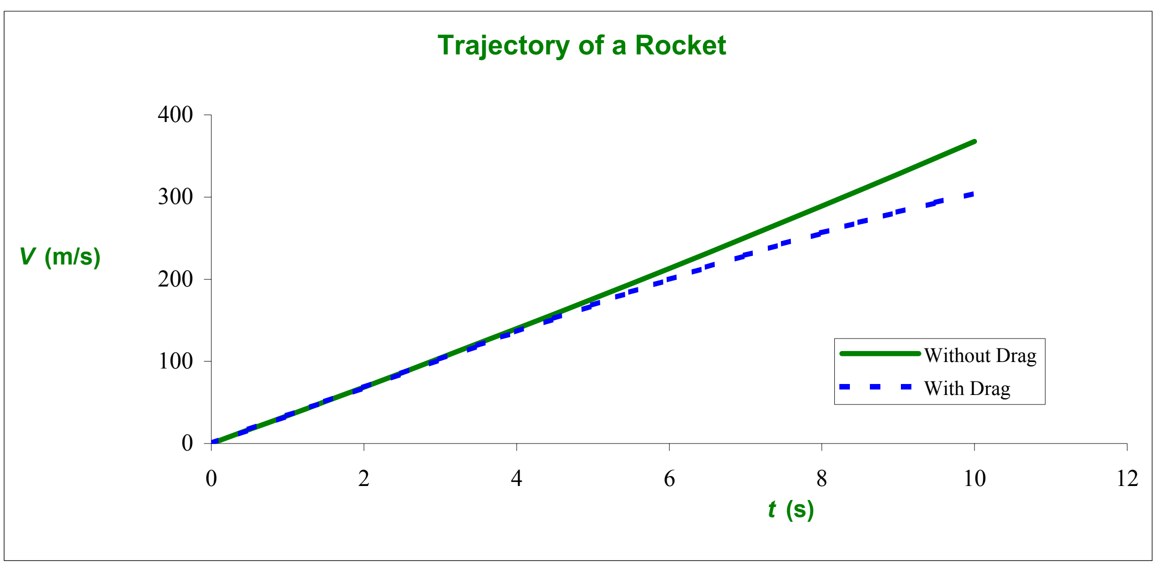
A = 0.38 m^2
N = 20
Δt = 0.50 s

With drag:

n	t_n (s)	V_n (m/s)	f_n	V_{n+1} (m/s)
0	0.0	0.0	33.9	17.0
1	0.5	17.0	34.2	34.1
2	1.0	34.1	34.3	51.2
3	1.5	51.2	34.3	68.3
4	2.0	68.3	34.2	85.5
5	2.5	85.5	34.0	102
6	3.0	102	33.7	119
7	3.5	119	33.3	136
8	4.0	136	32.8	152
9	4.5	152	32.2	168
10	5.0	168	31.5	184
11	5.5	184	30.7	200
12	6.0	200	29.8	214
13	6.5	214	28.9	229
14	7.0	229	27.9	243
15	7.5	243	26.9	256
16	8.0	256	25.8	269
17	8.5	269	24.7	282
18	9.0	282	23.6	293
19	9.5	293	22.5	305
20	10.0	305	21.4	315

Without drag:

V_n (m/s)	f_n	V_{n+1} (m/s)
0.0	33.9	17.0
17.0	34.2	34.1
34.1	34.5	51.3
51.3	34.8	68.7
68.7	35.1	86.2
86.2	35.4	104
104	35.6	122
122	35.9	140
140	36.2	158
158	36.5	176
176	36.9	195
195	37.2	213
213	37.5	232
232	37.8	251
251	38.1	270
270	38.5	289
289	38.8	308
308	39.1	328
328	39.5	348
348	39.8	368
368	40.2	388



Problem 9.151

[Difficulty: 4]

9.151 A baseball is popped straight up with an initial velocity of 25 m/s. The baseball has a diameter of 0.073 m and a mass of 0.143 kg. The drag coefficient for the baseball can be estimated as 0.47 for $Re < 10^4$ and 0.10 for $Re > 10^4$. Determine how long the ball will be in the air and how high it will go.

Given: Baseball popped up, drag estimates based on Reynolds number

Find: Time of flight and maximum height

Solution:

Basic equation: $C_D = \dfrac{F_D}{\frac{1}{2}\cdot\rho\cdot A\cdot V^2}$ $\qquad \Sigma F_y = M\cdot a_y \qquad a_y = \dfrac{dV_y}{dt}$

Given or available data is $M = 0.143\,kg \qquad V_{0y} = 25\cdot\dfrac{m}{s} \qquad D = 0.073\,m$

We solve this problem by discretizing the flight of the ball: $\Delta V_y = a_y\cdot\Delta t = \dfrac{\Sigma F_y}{M}\cdot\Delta t \qquad \Delta y = V_y\cdot\Delta t$

Here are the calculations performed in *Excel*:

Given or available data:

$M =$	0.143	kg
$V_{0y} =$	25	m/s
$D =$	0.073	m
$\rho =$	1.21	kg/m^3
$\nu =$	1.50E-05	m^2/s

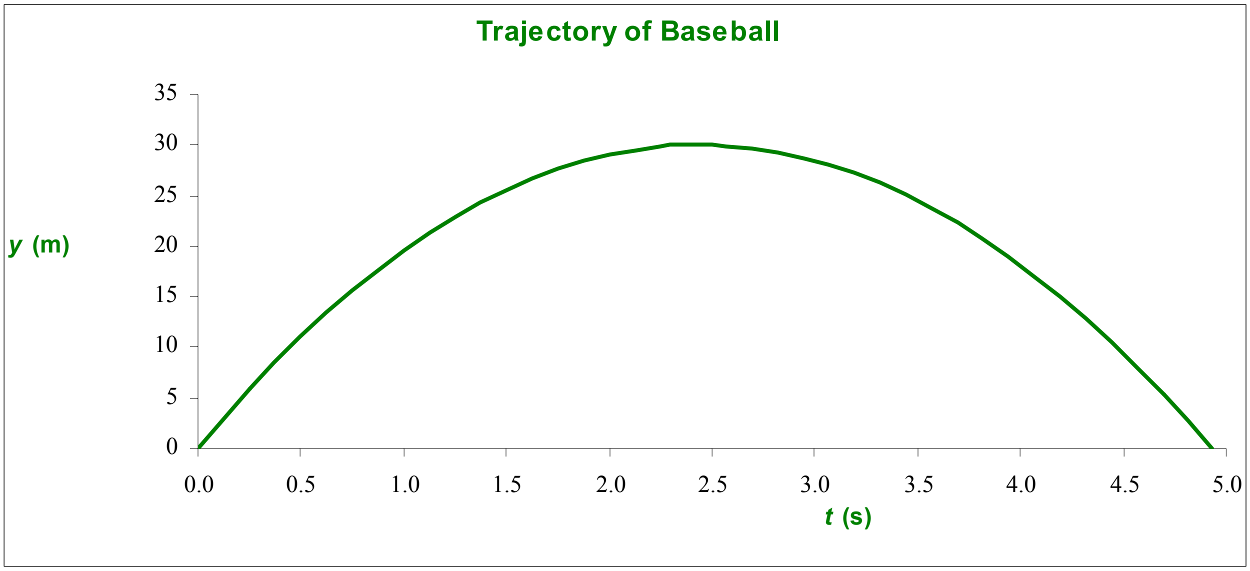
Computed results:

$A =$	0.00419	m^2
$\Delta t =$	0.25	s

t_n (s)	y (m)	V_y (m/s)	Re	C_D	a_y (m/s^2)	V_{ynew} (m/s)
0.00	0.0	25.0	1.22E+05	0.10	-10.917	22.3
0.25	5.9	22.3	1.08E+05	0.10	-10.688	19.6
0.50	11.1	19.6	9.54E+04	0.10	-10.490	17.0
0.75	15.7	17.0	8.26E+04	0.10	-10.320	14.4
1.00	19.6	14.4	7.01E+04	0.10	-10.177	11.9
1.25	22.9	11.9	5.77E+04	0.10	-10.059	9.3
1.50	25.6	9.3	4.54E+04	0.10	-9.964	6.8
1.75	27.6	6.8	3.33E+04	0.10	-9.893	4.4
2.00	29.0	4.4	2.13E+04	0.10	-9.844	1.9
2.25	29.8	1.9	9.30E+03	0.47	-9.840	0.0
2.44	30.0	0.0	0.00E+00	0.47	-9.810	-2.5
2.69	29.7	-2.5	1.19E+04	0.10	-9.799	-4.9
2.94	28.7	-4.9	2.39E+04	0.10	-9.767	-7.3
3.19	27.2	-7.3	3.57E+04	0.10	-9.714	-9.8
3.44	25.1	-9.8	4.76E+04	0.10	-9.641	-12.2
3.69	22.3	-12.2	5.93E+04	0.10	-9.547	-14.6
3.94	19.0	-14.6	7.09E+04	0.10	-9.434	-16.9
4.19	15.0	-16.9	8.24E+04	0.10	-9.303	-19.3
4.44	10.5	-19.3	9.37E+04	0.10	-9.154	-21.5
4.69	5.4	-21.5	1.05E+05	0.10	-8.988	-23.7
4.93	0.0	-23.7	1.15E+05	0.10	-8.816	

The results are plotted below.
The answers are:

height =	30.0	m
time =	4.93	s


Trajectory of Baseball
y (m)
35
30
25
20
15
10
5
0
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
4.5
5.0
t (s)

9.152 Wiffle™ balls made from light plastic with numerous holes are used to practice baseball and golf. Explain the purpose of the holes and why they work. Explain how you could test your hypothesis experimentally.

Discussion: The basic concept of the Wiffle ball is a low-mass, high-drag configuration that can be hit or struck with full force, but will not fly fast or far. Thus the Wiffle ball can be used for practice in a limited space.

The low mass is achieved by making the ball of relatively thin plastic material. This gives it low mass for its size, and is a step toward making the drag force relatively high compared to the weight of the ball.

Even higher drag force is achieved by perforating the surface of the Wiffle ball with numerous large holes. These holes further reduce the mass of the Wiffle ball.

In the sub-critical flow regime (below $Re_D \approx 2 \times 10^5$) skin friction drag accounts for less than 5 percent of the total drag of a sphere. The holes increase the skin friction drag of the ball by allowing boundary-layer fluid to escape into the interior of the ball. Each new bit of surface then sees essentially a new boundary layer developing, with attendant high shear stress.

Pressure drag accounts for the majority of the drag of a sphere at any Reynolds number above about 1000. The holes disrupt the flow pattern around the ball and probably trigger early separation. This ensures that the ball remains in the high-drag sub-critical flow regime no matter what its actual Reynolds number.

This hypothesis could be tested experimentally by comparing the performance of two balls, one with holes and one without. (The balls should have nearly the same mass and diameter.) With the help of an assistant, drop the balls from some height (for example, down a stairwell). After each ball has reached terminal speed, measure the time required for it to fall through a fixed distance. Then calculate and compare the drag coefficients for the two balls. If the drag coefficient for the ball with holes is significantly larger than for the ball without holes, the hypothesis is confirmed.

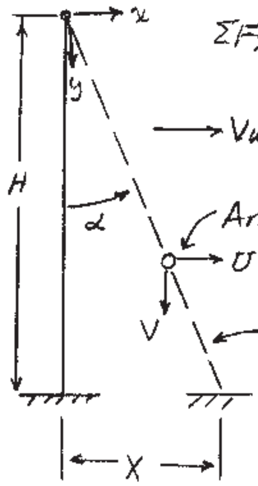
Several balls of each type might be evaluated experimentally to obtain an idea of the Reynolds number dependence of the results.

Problem 9.153

[Difficulty: 5]

9.153 Towers for television transmitters may be up to 500 m in height. In the winter, ice forms on structural members. When the ice thaws, chunks break off and fall to the ground. How far from the base of a tower would you recommend placing a fence to limit danger to pedestrians from falling ice chunks?

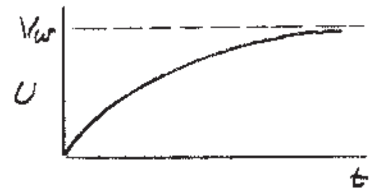
Analysis: An ice chunk detaching from a tower starts at rest, falls by gravity, and simultaneously is blown sideways by wind. Because drag is proportional to relative speed squared, it may be treated in separate x and y components.



$$\Sigma F_x = F_D = m\frac{dU}{dt} = C_D A \tfrac{1}{2}\rho (V_w - U)^2 = m\frac{dU}{dt} = -m\frac{d(V_w - U)}{dt} \qquad (1)$$

This equation is solved in Example Problem 4.11. The result is

$$\frac{U}{V_w} = \frac{bt}{1+bt}$$



$$b = \frac{\rho C_D A}{2m} V_w$$

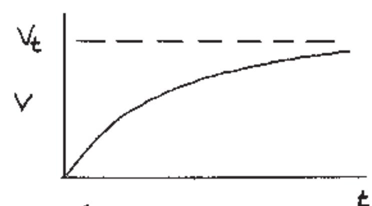
U quickly approaches V_w

$$\Sigma F_y = mg - F_D = m\frac{dV}{dt} = mV\frac{dV}{dt} = mg - C_D A \tfrac{1}{2}\rho V^2 = mg - kV^2 \qquad (2)$$

This equation is solved in Example Problem 1.2. The result is

$$V = \left\{\frac{mg}{k}\left(1 - e^{-\frac{2k}{m}y}\right)\right\}^{1/2} \quad \text{or} \quad \frac{V}{V_t} = 1 - e^{-2\frac{k}{m}y}$$

$$V_t = \left[\frac{mg}{k}\right]^{\frac{1}{2}}$$



Model the chunk as a geometric sphere, but with larger C_D because of jagged edges. Both b and V_t depend on diameter; $V_t = [8.11\, D(mm)]^{1/2}$; or 11.0 m/s for D = 15 mm.

Thus a reasonable approximation is falling at V_t and moving sideways at V_w. For V_w = 10 mph (4.47 m/s), and V_t = 11.0 m/s, then $\alpha = \tan^{-1}(4.47/11.0) = 22.1°$, and $X = H \tan 22.1° = 203$ m. ← X

To more precisely compute distance, obtain x(t) and y(t) by solving Eqs. 1 and 2 numerically, and plot the particle path.

Discussion: Because towers may be very tall, ice chunks can travel long distances from the base even in moderate winds. Considerable area around the base of a tower must be fenced to keep personnel on the ground safe from falling ice.

The analysis in this problem would be accurate if the drag-area product C_DA for an ice chunk were known precisely. However, the size of the structural members and the thickness of the ice coating are both unknown. Therefore it is difficult to choose the most probable drag-area product. We recommend you bracket the sizes of known ice chunks, pick a reasonable range of drag coefficients, and then use the analysis to develop guidelines for the safety of personnel.

9.154 The "shot tower," used to produce spherical lead shot, has been recognized as a mechanical engineering landmark. In a shot tower, molten lead is dropped from a high tower; as the lead solidifies, surface tension pulls each shot into a spherical shape. Discuss the possibility of increasing the "hang time," or of using a shorter tower, by dropping molten lead into an air stream that is moving upward. Support your discussion with appropriate calculations.

Analysis: This problem may be analyzed parametrically, in terms of shot diameter. Consider the range from "bird shot" of about 1 mm to musket balls of about 15 mm diameter to illustrate the results.

Analysis with still air: Terminal speed is reached when aerodynamic drag force exactly equals the weight of the shot. The first plot shows terminal speed versus diameter of lead shot.

The solution for shot speed versus distance traveled, with no upward air movement, parallels the solution of Example Problem 1.2, which gives the fraction of terminal speed reached in a tower of specified height. For any tower height (choose 50 m to illustrate the results) the fraction of terminal speed reached decreases with increasing shot diameter (see the second plot).

The solution for hang time versus diameter is shown in the third plot.

Analysis with upward flow of air: The solution for shot speed versus distance traveled is more complex when air in the shot tower flows upward. Introducing upward air flow in the tower increases drag force compared to shot weight. Therefore the shot accelerate more slowly in the upward flow. It is possible to obtain an analytical solution, but the result is so complex that it is difficult to interpret. Results can be obtained for specific cases by integrating the differential equations numerically.

The solution for shot speed versus time also is more complex when air flows upward. Again numerical integration can be used to obtain results for specific cases.

Outline of Procedure: Derive a differential equation for shot acceleration from a free-body diagram. Integrate once to obtain shot velocity as a function of time. Integrate again to obtain shot position as a function of time.

From the results of the second integration, identify the "hang time" when the shot reaches the bottom of the tower. Plot hang time versus diameter and compare with results for the case with no upward air flow.

Set the upward air flow velocity to zero and compare numerical results with the analytical results for the case without flow to validate your model.

Discussion: The terminal speed reached by small shot in still air is quite low. Therefore, the "hang time" of small shot can be increased significantly by providing upward flow of air at reasonable speed in the tower.

Larger shot have higher terminal speeds. However, the higher terminal speed does not reduce hang time much because the large shot reach only a smaller fraction of their terminal speed in the 50 m tower height.

Introducing upward flow of air in the shot tower increases the drag force and results in slower acceleration of the shot. Therefore the hang time is increased. The increase in hang time allows more time for cooling, and should result in the production of more nearly spherical shot.

Problem 9.154

Input data:

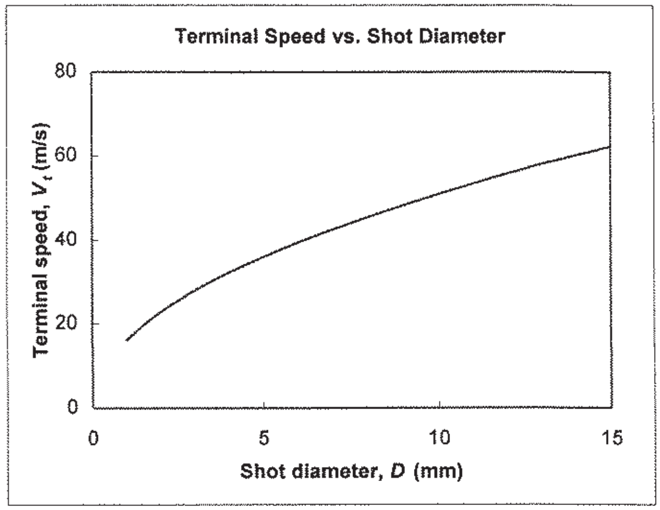
$C_D =$	0.47	(---)	Drag coefficient of sphere
$SG_s =$	11.4	(---)	Specific gravity of (lead) shot
$\Delta z =$	50	m	Height of shot tower
$\rho_{air} =$	1.23	kg/m^3	Density of air

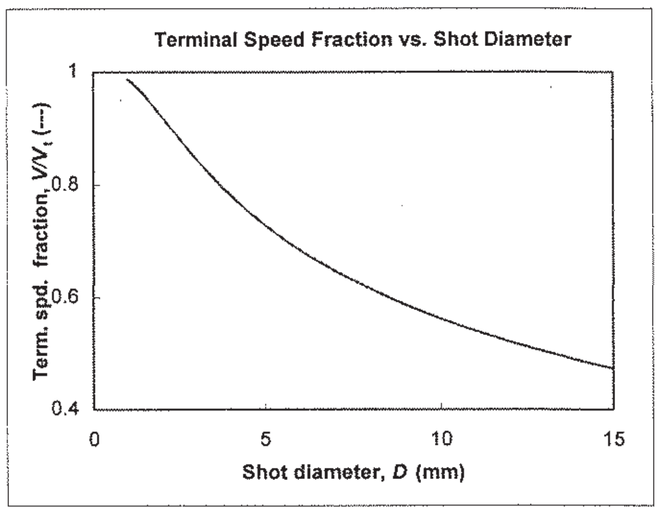
Calculated parameters:

$k/D^2 =$	2.27E-07	kg/m-mm^2	Drag factor, $F_D = kV^2$
$m/D^3 =$	5.969E-06	kg/mm^3	Mass of shot

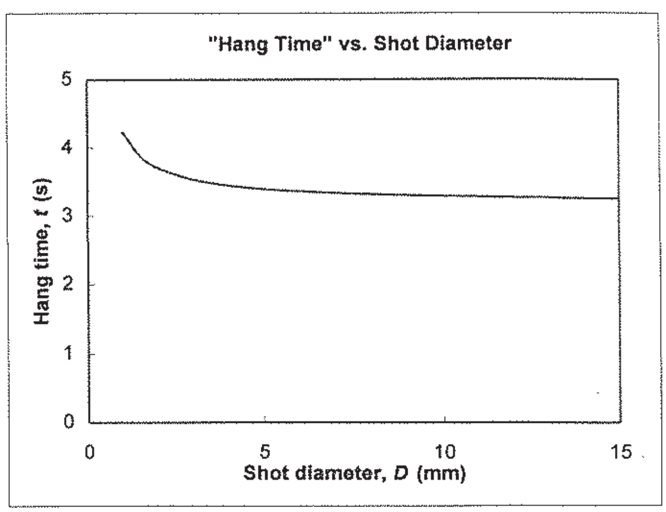
(1) Shot falling in still air:

D (mm)	V_t (m/s)	V/V_t (---)
1	16.1	0.989
1.5	19.7	0.960
2	22.7	0.922
3	27.8	0.848
4	32.1	0.783
5	35.9	0.730
6	39.3	0.685
7	42.5	0.647
8	45.4	0.615
9	48.2	0.587
10	50.8	0.562
11	53.3	0.541
12	55.6	0.521
13	57.9	0.504
14	60.1	0.488
15	62.2	0.473





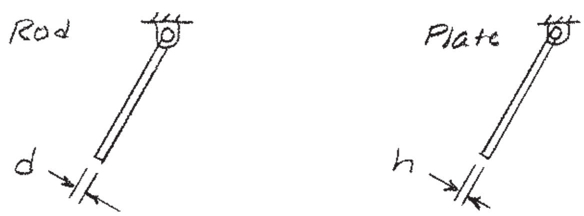
D (mm)	V_t (m/s)	V/V_t (---)	t (s)
1	16.1	0.989	4.24
1.5	19.7	0.960	3.89
2	22.7	0.922	3.71
3	27.8	0.848	3.54
4	32.1	0.783	3.45
5	35.9	0.730	3.40
6	39.3	0.685	3.36
7	42.5	0.647	3.34
8	45.4	0.615	3.32
9	48.2	0.587	3.31
10	50.8	0.562	3.29
11	53.3	0.541	3.29
12	55.6	0.521	3.28
13	57.9	0.504	3.27
14	60.1	0.488	3.27
15	62.2	0.473	3.26

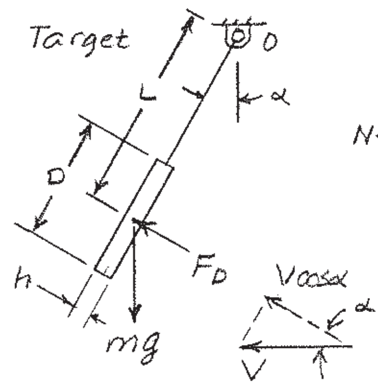


Problem 9.155

9.155 Design a wind anemometer that uses aerodynamic drag to move or deflect a member or linkage, producing an output that can be related to wind speed, for the range from 1 to 10 m/s in standard air. Consider three alternative design concepts. Select the best concept and prepare a detailed design. Specify the shape, size, and material for each component. Quantify the relation between wind speed and anemometer output. Present results as a theoretical "calibration curve" of anemometer output versus wind speed. Discuss reasons why you rejected the alternative designs and chose your final design concept.

Analysis: The "target" concept was chosen for analysis. The drag force acting on the target is calculated, then moments are summed about the pivot (see Problem 9.105). The results are:





Neglect drag on strut.

$$\sum M_O = mg\cancel{L}\sin\alpha - F_D\cancel{L} = 0$$

$$F_D = C_D A \tfrac{1}{2}\rho (V\cos\alpha)^2 = C_D A \tfrac{1}{2}\rho V^2 \cos^2\alpha$$

So $$C_D A \tfrac{1}{2}\rho V^2 \cos^2\alpha = mg\sin\alpha = \rho_m \forall_m g \sin\alpha = SG\rho_{H_2O} A h g \sin\alpha \qquad (1)$$

Assume $\alpha = 60°$ at highest wind speed (when $V = 10$ m/s; $q = \tfrac{1}{2}\rho V^2 = 61.5\ N/m^2$).

Then $$\frac{\sin\alpha}{\cos^2\alpha} = \frac{C_D A q}{SG\rho_{H_2O} A g h} = \frac{C_D q}{SG\rho_{H_2O} g h} = f(\alpha) \qquad \text{At } \alpha = 60°,\ f(\alpha) = 3.46$$

$$h = \frac{C_D q}{SG\rho_{H_2O} g f(\alpha)} = 1.2 \times 61.5\frac{N}{m^2} \times \frac{m^3}{(SG)1000\,kg} \times \frac{s^2}{9.81\,m} \times \frac{1}{3.46} \times \frac{kg \cdot m}{N \cdot s^2} = \frac{0.00217\,m}{SG} = \frac{2.17\ mm}{SG}$$

Choose aluminum, with SG = 2.64 (Table A.1). Then

$$h = \frac{2.17\ mm}{2.64} = 0.822\ mm$$ h

Solve Eq. 1 for velocity

$$V = \left[\frac{2SG\rho_{H_2O} h g}{\rho C_D} \times f(\alpha)\right]^{\frac{1}{2}} \qquad \text{(see plot)}$$ V

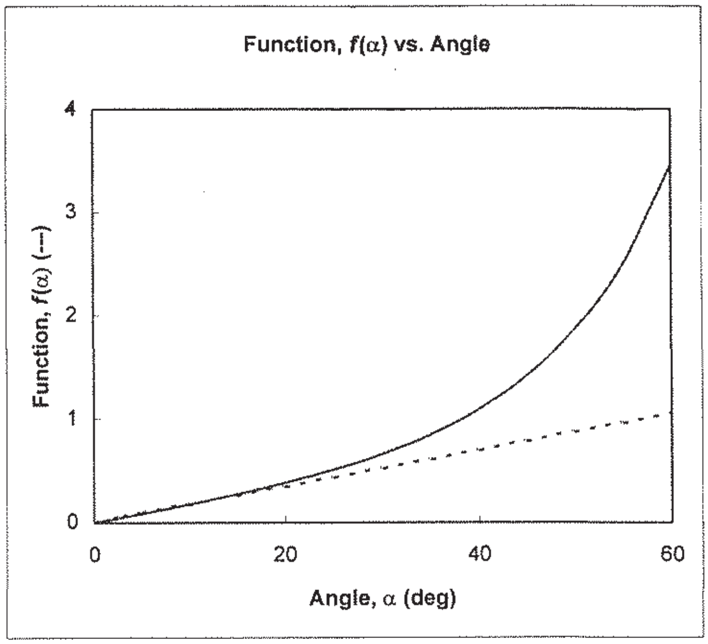
Discussion: Concepts considered included a manometer that sensed stagnation pressure, a parallelogram linkage supporting a vertical target, and bending of a thin member in the air stream. The three final concepts chosen were variations on the theme of a single hanging member supported from a single pivot, and were chosen for their simplicity.

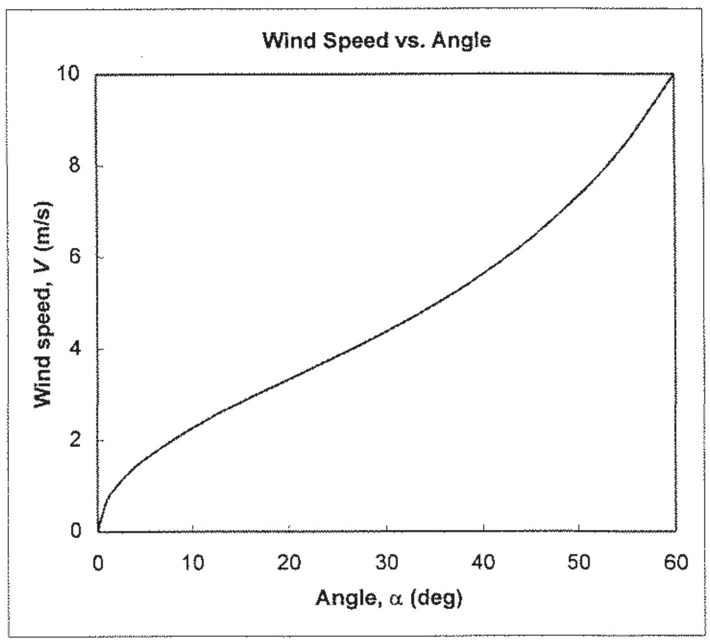
The major advantage of the target concept is that different materials can be used for the rod and target; this concept can be tailored to give the largest deflection angle for a given wind speed. Therefore this device should be capable of the most accurate indication at low wind speeds.

Drag force on the target is assumed to depend on the component of wind velocity acting normal to the target. This model could be improved by using actual experimental data for the drag coefficient of a disk at angle of attack.

Problem 9.155

α (deg)	α (rad)	$f(\alpha)$ (---)	V (m/s)
0	0	0	0
1	0.0175	0.0175	0.71
2	0.0349	0.0349	1.00
3	0.0524	0.0525	1.23
5	0.0873	0.0878	1.59
10	0.175	0.179	2.27
15	0.262	0.277	2.83
20	0.349	0.387	3.34
25	0.436	0.515	3.85
30	0.524	0.667	4.39
35	0.611	0.855	4.97
40	0.698	1.10	5.62
45	0.785	1.41	6.39
50	0.873	1.85	7.31
55	0.960	2.49	8.47
60	1.05	3.46	10.0





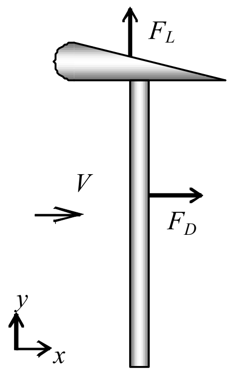
Problem 9.156

[Difficulty: 3]

9.156 A model airfoil of chord 6 in. and span 30 in. is placed in a wind tunnel with an air flow of 100 ft/s (the air is at 70°F). It is mounted on a cylindrical support rod 1 in. in diameter and 10 in. tall. Instruments at the base of the rod indicate a vertical force of 10 lbf and a horizontal force of 1.5 lbf. Calculate the lift and drag coefficients of the airfoil.

Given: Data on airfoil and support in wind tunnel, lift and drag measurements

Find: Lift and drag coefficients of airfoil



Solution:

Basic equations: $C_D = \dfrac{F_D}{\frac{1}{2}\cdot\rho\cdot A\cdot V^2}$ $\quad C_L = \dfrac{F_L}{\frac{1}{2}\cdot\rho\cdot A\cdot V^2}$

The given or available data is $L = 6\cdot in$ $\quad W = 30\cdot in$ $\quad V = 100\cdot\frac{ft}{s}$ $\quad D_{cyl} = 1\cdot in$ $\quad L_{cyl} = 10\cdot in$

$F_L = 10\cdot lbf$ $\quad F_D = 1.5\cdot lbf$ $\quad \rho = 0.00233\cdot\frac{slug}{ft^3}$ $\quad \nu = 1.63\times 10^{-4}\cdot\frac{ft^2}{s}$

We need to determine the cylindrical support's contribution to the total drag force: $F_D = F_{Dcyl} + F_{Dairfoil}$

Compute the Reynolds number $\quad Re = \frac{V\cdot D_{cyl}}{\nu}$ $\quad Re = 5.112\times 10^4$ Therefore: $C_{Dcyl} = 1$

So the drag force on the support is: $F_{Dcyl} = \frac{1}{2}\cdot C_{Dcyl}\cdot\rho\cdot V^2\cdot L_{cyl}\cdot D_{cyl} = 0.809\cdot lbf$

So the airfoil drag is: $F_{Dairfoil} = F_D - F_{Dcyl} = 0.691\cdot lbf$ The reference area for the airfoil is: $A = L\cdot W = 1.25\cdot ft^2$

The lift and drag coefficients are:

$$C_L = \frac{F_L}{\frac{1}{2}\cdot\rho\cdot V^2\cdot A} \qquad C_L = 0.687$$

$$C_D = \frac{F_{Dairfoil}}{\frac{1}{2}\cdot\rho\cdot V^2\cdot A} \qquad C_D = 0.0474$$

Problem 9.157

[Difficulty: 2]

9.157 An antique airplane carries 50 m of external guy wires stretched normal to the direction of motion. The wire diameter is 5 mm. Estimate the maximum power saving that results from an optimum streamlining of the wires at a plane speed of 175 km/hr in standard air at sea level.

Given: Antique airplane guy wires

Find: Maximum power saving using optimum streamlining

Solution:

Basic equation: $C_D = \dfrac{F_D}{\frac{1}{2}\cdot\rho\cdot A\cdot V^2}$ $\quad P = F_D\cdot V$

Given or available data is $L = 50\cdot m$ $\quad D = 5\cdot mm$ $\quad V = 175\cdot\dfrac{km}{hr}$ $\quad V = 48.6\,\dfrac{m}{s}$

$A = L\cdot D$ $\quad A = 0.25\,m^2$

$\rho = 1.21\cdot\dfrac{kg}{m^3}$ $\quad \nu = 1.50\times 10^{-5}\cdot\dfrac{m^2}{s}$ $\quad$ (Table A.10, 20°C)

The Reynolds number is $Re = \dfrac{V\cdot D}{\nu}$ $\quad Re = 1.62\times 10^4$ $\quad$ so from Fig. 9.13 $\quad C_D = 1.0$

Hence $P = \left(C_D\cdot\frac{1}{2}\cdot\rho\cdot A\cdot V^2\right)\cdot V$ $\quad P = 17.4\cdot kW$ $\quad$ with standard wires

Figure 9.19 suggests we could reduce the drag coefficient to $C_D = 0.06$

Hence $P_{faired} = \left(C_D\cdot\frac{1}{2}\cdot\rho\cdot A\cdot V^2\right)\cdot V$ $\quad P_{faired} = 1.04\cdot kW$

The maximum power saving is then $\Delta P = P - P_{faired}$ $\quad \Delta P = 16.3\cdot kW$

Thus $\dfrac{\Delta P}{P} = 94\cdot\%$ $\quad$ which is a HUGE savings! It's amazing the antique planes flew!

Problem 9.158

[Difficulty: 4]

9.158 Why do modern guns have rifled barrels?

Discussion: Almost all projectiles fired by modern guns have smoothly rounded noses and abruptly tapered ("boat-tailed") or square rear ends. The minimum drag for these shapes is obtained when the projectile travels with its axis parallel to the direction of motion and its nose pointed forward.

Rifling in a gun barrel imparts spin about the longitudinal axis of the projectile. This rotation about the longitudinal axis causes the projectile to act as a gyroscope and stabilizes it during flight to keep its nose pointed in the direction of motion.

Early smoothbore guns primarily used ball projectiles. The balls were spherical and molded from lead. Since the ball shape was spherical and had no preferred orientation, no benefit would have been achieved from rifling that caused spin. Therefore the gun barrels were bored smooth, i.e., without rifling grooves, hence these guns were called "smoothbore" guns.

9.159 How do cab-mounted wind deflectors for tractor-trailer trucks work? Explain using diagrams of the flow pattern around the truck and pressure distribution on the surface of the truck.

Discussion: Consider both the cab and the trailer flow patterns and pressure distributions in no-wind and crosswind situations.

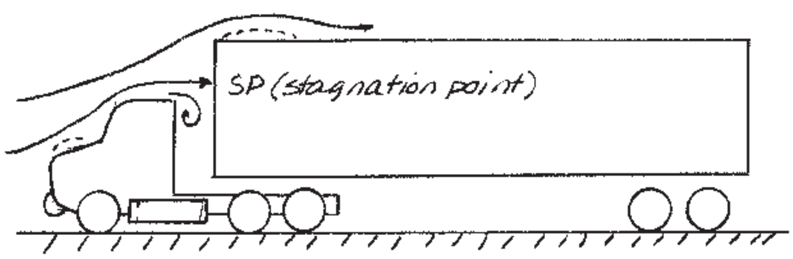
No-wind situation: Without the deflector, flow separation from the roof and sides of the tractor creates a low-pressure wake and high drag force on the tractor. (Flow patterns and pressure distributions on the tractor and the front of the trailer are sketched below.) The cab-mounted deflector reduces the pressures on the front of the tractor, thus reducing the aerodynamic drag force on the tractor.

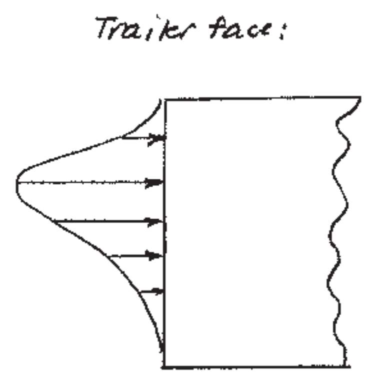
Without the deflector, high-speed air separates from the roof of the tractor and impinges on the vertical front face of the trailer. The cab-mounted deflector reduces the amount of high-speed air hitting the front of the trailer, reducing the net aerodynamic drag force on the trailer. (Ideally air from the deflector flows smoothly along the top and sides of the trailer.)

Crosswind situation: Without the deflector, flow separates from the lee side of the tractor, altering the pressure field and increasing the drag on the tractor. The cab-mounted deflector, especially in combination with side seals, minimizes the increase in drag by reducing the amount of separation around the tractor.

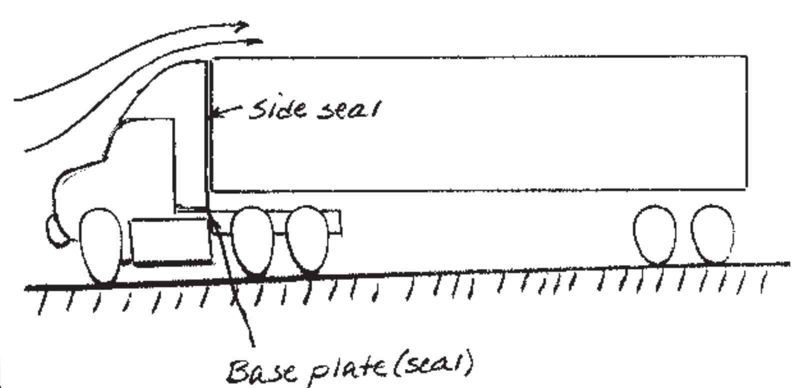
Without the deflector, the front face of the trailer is impacted by the high-speed air from the freestream flow. Massive separation occurs on the lee side of the trailer, thus altering the pressure field and increasing the drag on the trailer. With the cab-mounted deflector, the amount of high-speed air impacting the trailer is markedly reduced. This alters the flow pattern and minimizes the increase in drag caused by the crosswind.

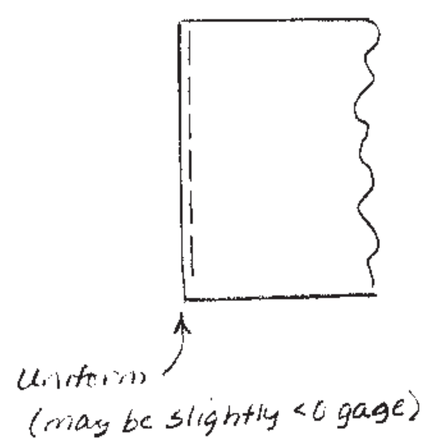
Without cab-mounted wind deflector:





With cab-mounted wind deflector:





Problem 9.160

9.160 An airplane with an effective lift area of 25 m^2 is fitted with airfoils of NACA 23012 section (Fig. 9.23). The maximum flap setting that can be used at takeoff corresponds to configuration (2) in Fig. 9.23. Determine the maximum gross mass possible for the airplane if its takeoff speed is 150 km/hr at sea level (neglect added lift due to ground effect). Find the minimum takeoff speed required for this gross mass if the airplane is instead taking off from Denver (elevation approximately 1.6 km).

Solution: Apply definition of lift coefficient.

Basic equation: $C_L = \dfrac{F_L}{\frac{1}{2}\rho V^2 A_p}$

Assumption: Lift force must equal gravity force at takeoff.

$$F_L = mg = C_D A_p \tfrac{1}{2}\rho V^2$$

For maximum mass, need maximum lift, so use $C_{L,max}$:

$$m_{max} = \frac{C_{L,max} A_p \rho V^2}{2g}$$

From Fig. 9.23, $C_{L,max} = 2.67$ for condition (2). Then for std. air,

$$m_{max} = \frac{2.67}{2} \times 25\, m^2 \times 1.23 \frac{kg}{m^3}\left(150 \times 10^3 \frac{m}{hr} \times \frac{hr}{3600\, s}\right)^2 \frac{s^2}{9.81\, m}$$

$$m_{max} = 7260\ kg$$

m_{max}

{This represents the maximum mass theoretically possible when the aircraft is on the verge of stalling. To attempt takeoff at such a large mass would be ill-advised.}

(b) In Denver, $z = 1.61$ km. From Table A.3, at $z = 1.61$ km, $\rho/\rho_0 = 0.855$.

At the same gross mass, the lift force remains the same. Thus

$$F_{L_0} = C_D A \tfrac{1}{2}\rho_0 V_0^2 = F_{L_D} = C_D A \tfrac{1}{2}\rho_D V_D^2 \quad \text{or} \quad \rho_0 V_0^2 = \rho_D V_D^2$$

and

$$V_D = V_0\left(\frac{\rho_0}{\rho_D}\right)^{1/2} = 150\ kph\left(\frac{1}{0.855}\right)^{1/2} = 162\ kph$$

V_D

{The takeoff speed must increase about 8 percent.}

Problem 9.161

[Difficulty: 1]

9.161 An aircraft is in level flight at 225 km/hr through air at standard conditions. The lift coefficient at this speed is 0.45 and the drag coefficient is 0.065. The mass of the aircraft is 900 kg. Calculate the effective lift area for the craft, and the required engine thrust and power.

Given: Aircraft in level flight

Find: Effective lift area; Engine thrust and power

Solution:

Basic equation: $C_D = \dfrac{F_D}{\frac{1}{2}\cdot\rho\cdot A\cdot V^2}$ $\quad C_L = \dfrac{F_L}{\frac{1}{2}\cdot\rho\cdot A\cdot V^2}$ $\quad P = T\cdot V$

For level, constant speed $\quad F_D = T \quad F_L = W$

Given or available data is $\quad V = 225\cdot\dfrac{km}{hr} \quad V = 62.5\,\dfrac{m}{s} \quad C_L = 0.45 \quad C_D = 0.065 \quad M = 900\cdot kg$

$\rho = 1.21\cdot\dfrac{kg}{m^3}$ (Table A.10, 20°C)

Hence $\quad F_L = C_L\cdot\dfrac{1}{2}\cdot\rho\cdot A\cdot V^2 = M\cdot g \quad A = \dfrac{2\cdot M\cdot g}{C_L\cdot\rho\cdot V^2} \quad A = 8.30\cdot m^2$

Also $\quad \dfrac{F_L}{F_D} = \dfrac{C_L}{C_D} \quad F_L = M\cdot g \quad F_L = 8826\,N \quad F_D = F_L\cdot\dfrac{C_D}{C_L} \quad F_D = 1275\,N$

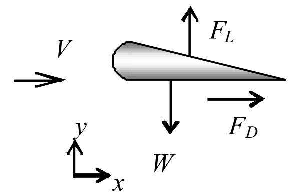
$T = F_D \quad T = 1275\,N$

The power required is then $\quad P = T\cdot V \quad P = 79.7\cdot kW$

Problem 9.162

[Difficulty: 2]

9.162 The foils of a surface-piercing hydrofoil watercraft have a total effective area of 7.5 ft^2. Their coefficients of lift and drag are 1.5 and 0.63, respectively. The total weight of the craft in running trim is 4000 lb. Determine the minimum speed at which the craft is supported by the hydrofoils. At this speed, find the power required to overcome water resistance. If the craft is fitted with a 150-hp engine, estimate its top speed.



Given: Data on a hydrofoil

Find: Minimum speed, power required, top speed

Solution:

Assumption: The drag on the hydrofoil is much greater than any other drag force on the craft once the foil supports the craft.

The given data or available data is $\rho = 1.94 \cdot \frac{slug}{ft^3}$ $\quad A = 7.5 \cdot ft^2$ $\quad C_L = 1.5$ $\quad C_D = 0.63$ $\quad W = 4000 \cdot lbf$ $\quad P_{max} = 150 \cdot hp$

To support the hydrofoil, the lift force must equal the weight: $F_L = W = 4000 \cdot lbf$

Based on the required lift force, the speed must be: $V_{min} = \sqrt{\frac{2 \cdot F_L}{\rho \cdot A \cdot C_L}}$ $\qquad V_{min} = 19.1 \cdot \frac{ft}{s}$

The drag force at this speed is $F_D = \frac{1}{2} \cdot \rho \cdot A \cdot V_{min}{}^2 \cdot C_D$ $\qquad F_D = 1680 \cdot lbf$

Engine thrust required $T = F_D$ $\qquad T = 1680 \cdot lbf$

The power required is $P = T \cdot V_{min}$ $\qquad P = 58.5 \cdot hp$

As the speed increases, the lift will increase such that the lift and weight are still balanced. Therefore:

$P_{max} = \frac{C_D}{C_L} \cdot W \cdot V_{max}$ Solving for the maximum speed: $V_{max} = \frac{P_{max}}{W} \cdot \frac{C_L}{C_D}$ $\qquad V_{max} = 49.1 \cdot \frac{ft}{s}$

Problem 9.163

[Difficulty: 2]

9.163 A high school project involves building a model ultralight airplane. Some of the students propose making an airfoil from a sheet of plastic 5 ft long × 7 ft wide at an angle of attack of 10°. At this airfoil's aspect ratio and angle of attack the lift and drag coefficients are $C_L = 0.75$ and $C_D = 0.19$. If the airplane is designed to fly at 40 ft/s, what is the maximum total payload? What will be the required power to maintain flight? Does this proposal seem feasible?

Given: Data on an airfoil

Find: Maximum payload; power required

Solution:

The given data or available data is $\rho = 0.00234 \cdot \frac{slug}{ft^3}$ $\quad L = 5 \cdot ft$ $\quad w = 7 \cdot ft$ $\quad V = 40 \cdot \frac{ft}{s}$ $\quad C_L = 0.75$ $\quad C_D = 0.19$

Then $\quad A = w \cdot L \quad A = 35 \cdot ft^2$

The governing equations for steady flight are $\quad W = F_L \quad$ and $\quad T = F_D$

where *W* is the model total weight and *T* is the thrust

The lift is given by $\quad F_L = \frac{1}{2} \cdot \rho \cdot A \cdot V^2 \cdot C_L \quad F_L = 49.1 \cdot lbf$

The payload is then given by $\quad W = M \cdot g = F_L$

or $\quad M = \frac{F_L}{g} \quad M = 49.1 \cdot lb$

The drag is given by $\quad F_D = \frac{1}{2} \cdot \rho \cdot A \cdot V^2 \cdot C_D \quad F_D = 12.4\, lbf$

Engine thrust required $\quad T = F_D \quad T = 12.4\, lbf$

The power required is $\quad P = T \cdot V \quad P = 498 \cdot \frac{ft \cdot lbf}{s} \quad P = 0.905\, hp$

The ultralight model is just feasible: it is possible to find an engine that can produce about 1 hp that weighs less than about 50 lb.

Problem 9.164

[Difficulty: 3]

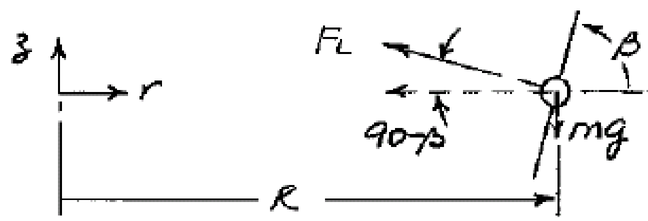
9.164 The U.S. Air Force F-16 fighter aircraft has wing planform area $A = 300\ \text{ft}^2$; it can achieve a maximum lift coefficient of $C_L = 1.6$. When fully loaded, its weight is 26,000 lb. The airframe is capable of maneuvers that produce $9g$ vertical accelerations. However, student pilots are restricted to $5g$ maneuvers during training. Consider a turn flown in level flight with the aircraft banked. Find the minimum speed in standard air at which the pilot can produce a $5g$ total acceleration. Calculate the corresponding flight radius. Discuss the effect of altitude on these results.

Given: Data on F-16 fighter

Find: Minimum speed at which pilot can produce 5g acceleration; flight radius, effect of altitude on results

Solution:

The given data or available data is $\rho = 0.00234 \cdot \frac{\text{slug}}{\text{ft}^3}$ $\quad A = 300 \cdot \text{ft}^2$ $\quad C_L = 1.6$ $\quad W = 26000 \cdot \text{lbf}$



At 5g acceleration, the corresponding force is:

$$F_L = 5 \cdot W = 130000 \cdot \text{lbf}$$

The minimum velocity corresponds to the maximum lift coefficient:

$$V_{min} = \sqrt{\frac{2 \cdot F_L}{\rho \cdot A \cdot C_L}} = 481 \cdot \frac{\text{ft}}{\text{s}} \qquad V_{min} = 481 \cdot \frac{\text{ft}}{\text{s}}$$

To find the flight radius, we perform a vertical force balance: $F_L \cdot \sin(90 \cdot \text{deg} - \beta) - W = 0$ $\quad \beta = 90 \cdot \text{deg} - \text{asin}\left(\frac{W}{F_L}\right) = 78.5 \cdot \text{deg}$

Now set the horizontal force equal to the centripetal acceleration: $F_L \cdot \cos(90 \cdot \text{deg} - \beta) = \frac{W}{g} \cdot a_c$

$$a_c = g \cdot \frac{F_L}{W} \cdot \cos(90 \cdot \text{deg} - \beta) \qquad a_c = 157.6 \frac{\text{ft}}{\text{s}^2}$$

The flight radius corresponding to this acceleration is: $R = \frac{V_{min}^2}{a_c}$ $\qquad R = 1469 \cdot \text{ft}$

As altitude increases, the density decreases, and both the velocity and radius will increase.

Problem 9.165

[Difficulty: 3]

9.165 The teacher of the students designing the airplane of Problem 9.163 is not happy with the idea of using a sheet of plastic for the airfoil. He asks the students to evaluate the expected maximum total payload, and required power to maintain flight, if the sheet of plastic is replaced with a conventional section (NACA 23015) airfoil with the same aspect ratio and angle of attack. What are the results of the analysis?

Given: Data on an airfoil

Find: Maximum payload; power required

Solution:

The given data or available data is $V = 40\cdot\frac{\text{ft}}{\text{s}}$ $\rho = 0.00234\cdot\frac{\text{slug}}{\text{ft}^3}$ $c = 5\cdot\text{ft}$ $b = 7\cdot\text{ft}$

Then the area is $A = b\cdot c$ $A = 35.00\cdot\text{ft}^2$

and the aspect ratio is $ar = \frac{b}{c}$ $ar = 1.4$

The governing equations for steady flight are

$W = F_L$ and $T = F_D$

where W is the model total weight and T is the thrust

At a 10° angle of attack, from Fig. 9.17 $C_L = 1.2$ $C_{Di} = 0.010$

where C_{Di} is the section drag coefficient

The wing drag coefficient is given by Eq. 9.42 $C_D = C_{Di} + \frac{C_L^{\,2}}{\pi\cdot ar}$ $C_D = 0.337$

The lift is given by $F_L = \frac{1}{2}\cdot\rho\cdot A\cdot V^2\cdot C_L$ $F_L = 78.6\cdot\text{lbf}$

The payload is then given by $W = M\cdot g = F_L$

or $M = \frac{F_L}{g}$ $M = 78.6\cdot\text{lb}$

The drag is given by $F_D = \frac{1}{2}\cdot\rho\cdot A\cdot V^2\cdot C_D$ $F_D = 22.1\cdot\text{lbf}$

Engine thrust required $T = F_D$ $T = 22.1\cdot\text{lbf}$

The power required is $P = T\cdot V$ $P = 1.61\cdot\text{hp}$

NOTE: Strictly speaking we have TWO extremely stubby wings, so a recalculation of drag effects (lift is unaffected) gives

$b = 3.5\cdot\text{ft}$ $c = 5.00\cdot\text{ft}$

and $A = b\cdot c$ $A = 1.63\,\text{m}^{2.00}$ $ar = \frac{b}{c}$ $ar = 0.70$

so the wing drag coefficient is $C_D = C_{Di} + \dfrac{C_L^{\ 2}}{\pi \cdot ar}$ $C_D = 0.665$

The drag is $F_D = 2 \cdot \dfrac{1}{2} \cdot \rho \cdot A \cdot V^2 \cdot C_D$ $F_D = 43.6 \cdot lbf$

Engine thrust is $T = F_D$ $T = 43.6 \cdot lbf$

The power required is $P = T \cdot V$ $P = 3.17 \cdot hp$

In this particular case it would seem that the ultralight model makes more sense - we need a smaller engine and smaller lift requirements. However, on a per unit weight basis, the motor required for this aircraft is actually *smaller*. In other words, it should probably be easier to find a 3.5 hp engine that weighs less than 80 lb (22.9 lb/hp) than a 1 hp engine that weighs less than 50 lb (50 lb/hp).

Problem 9.166

[Difficulty: 3]

9.166 A light airplane, with mass $M = 1000$ kg, has a conventional-section (NACA 23015) wing of planform area $A = 10$ m^2. Find the angle of attack of the wing for a cruising speed of $V = 63$ m/s. What is the required power? Find the maximum instantaneous vertical "g force" experienced at cruising speed if the angle of attack is suddenly increased.

Given: Data on a light airplane

Find: Angle of attack of wing; power required; maximum "g" force

Solution:

The given data or available data is $\rho = 1.23 \cdot \frac{kg}{m^3}$ $\quad M = 1000 \cdot kg$ $\quad A = 10 \cdot m^2$

$V = 63 \cdot \frac{m}{s}$ $\quad C_L = 0.72$ $\quad C_D = 0.17$

The governing equations for steady flight are $\quad W = M \cdot g = F_L \quad T = F_D$

where W is the weight T is the engine thrust

The lift coeffcient is given by $\quad F_L = \frac{1}{2} \cdot \rho \cdot A \cdot V^2 \cdot C_d$

Hence the required lift coefficient is $\quad C_L = \frac{M \cdot g}{\frac{1}{2} \cdot \rho \cdot A \cdot V^2} \quad C_L = 0.402$

From Fig 9.17, for at this lift coefficient $\quad \alpha = 3 \cdot deg$

and the drag coefficient at this angle of attack is $\quad C_D = 0.0065$

(Note that this does NOT allow for aspect ratio effects on lift and drag!)

Hence the drag is $\quad F_D = \frac{1}{2} \cdot \rho \cdot A \cdot V^2 \cdot C_D \quad F_D = 159\,N$

and $\quad T = F_D \quad T = 159\,N$

The power required is then $\quad P = T \cdot V \quad P = 10 \cdot kW$

The maximum "g"'s occur when the angle of attack is suddenly increased to produce the maximum lift

From Fig. 9.17 $\quad C_{L.max} = 1.72$

$F_{Lmax} = \frac{1}{2} \cdot \rho \cdot A \cdot V^2 \cdot C_{L.max} \quad F_{Lmax} = 42 \cdot kN$

The maximum "g"s are given by application of Newton's second law

$M \cdot a_{perp} = F_{Lmax}$

where a_{perp} is the acceleration perpendicular to the flight direction

Hence $$a_{perp} = \frac{F_{Lmax}}{M} \qquad a_{perp} = 42\frac{m}{s^2}$$

In terms of "g"s $$\frac{a_{perp}}{g} = 4.28$$

Note that this result occurs when the airplane is banking at 90°, i.e, when the airplane is flying momentarily in a circular flight path in the horizontal plane. For a straight horizontal flight path Newton's second law is

$$M \cdot a_{perp} = F_{Lmax} - M \cdot g$$

Hence $$a_{perp} = \frac{F_{Lmax}}{M} - g \qquad a_{perp} = 32.2\frac{m}{s^2}$$

In terms of "g"s $$\frac{a_{perp}}{g} = 3.28$$

Problem 9.167

[Difficulty: 3]

9.167 A light airplane has 35-ft effective wingspan and 5.5-ft chord. It was originally designed to use a conventional (NACA 23015) airfoil section. With this airfoil, its cruising speed on a standard day near sea level is 150 mph. A conversion to a laminar-flow (NACA 66_2–215) section airfoil is proposed. Determine the cruising speed that could be achieved with the new airfoil section for the same power.

Given: Data on a light airplane

Find: Cruising speed achieved using a new airfoil design

Solution:

The given data or available data is $V = 150 \cdot mph = 220.00 \cdot \frac{ft}{s}$ $\quad \rho = 0.00234 \cdot \frac{slug}{ft^3}$ $\quad c = 5.5 \cdot ft$ $\quad b = 35 \cdot ft$

Then the area is $A = b \cdot c$ $\quad A = 192.50 \cdot ft^2$

and the aspect ratio is $ar = \frac{b}{c}$ $\quad ar = 6.36$

The governing equations for steady flight are

$$W = F_L \quad \text{and} \quad T = F_D$$

where W is the total weight and T is the thrust

For the NACA 23015 airfoil: $C_L = 0.3$ $\quad C_{Di} = 0.0062$

where C_{Di} is the section drag coefficient

The wing drag coefficient is given by Eq. 9.42 $C_D = C_{Di} + \frac{C_L^{\,2}}{\pi \cdot ar}$ $\quad C_D = 0.011$

The drag is given by $F_D = \frac{1}{2} \cdot \rho \cdot A \cdot V^2 \cdot C_D$ $\quad F_D = 116.7 \cdot lbf$

Engine thrust required $T = F_D$ $\quad T = 116.7 \cdot lbf$

The power required is $P = T \cdot V$ $\quad P = 46.66 \cdot hp$

For the NACA 66_2-215 airfoil: $C_L = 0.2$ $\quad C_{Di} = 0.0031$

The wing drag coefficient is given by Eq. 9.42 $C_D = C_{Di} + \frac{C_L^{\,2}}{\pi \cdot ar}$ $\quad C_D = 5.101 \times 10^{-3}$

The power is: $P = \frac{1}{2} \cdot \rho \cdot A \cdot V^3 \cdot C_D$ so the new speed is: $V_{new} = \sqrt[3]{\frac{2 \cdot P}{\rho \cdot A \cdot C_D}}$ $\quad V_{new} = 282 \cdot \frac{ft}{s}$ $\quad V_{new} = 192.0 \cdot mph$

Problem 9.168

[Difficulty: 3]

9.168 Instead of a new laminar-flow airfoil, a redesign of the light airplane of Problem 9.167 is proposed in which the current conventional airfoil section is replaced with another conventional airfoil section of the same area, but with aspect ratio $AR = 8$. Determine the cruising speed that could be achieved with this new airfoil for the same power.

Given: Data on an airfoil

Find: Maximum payload; power required

Solution:

The given data or available data is $V_{old} = 150 \cdot mph$ $\quad \rho = 0.00234 \cdot \frac{slug}{ft^3}$ $\quad A = 192.5 \cdot ft^2$ $\quad ar_{old} = \frac{35}{5.5}$ $\quad ar_{old} = 6.36$

Assuming the old airfoil operates at close to design lift, from Fig. 9.19 $C_L = 0.3$ $\quad C_{Di} = 0.0062$ $\quad$ (C_{Di} is the old airfoil's section drag coefficient)

Then

$$C_{Dold} = C_{Di} + \frac{C_L^{\;2}}{\pi \cdot ar_{old}} \qquad C_{Dold} = 0.0107$$

The new wing aspect ratio is $ar_{new} = 8$

Hence

$$C_{Dnew} = C_{Di} + \frac{C_L^{\;2}}{\pi \cdot ar_{new}} \qquad C_{Dnew} = 0.00978$$

The power required is

$$P = T \cdot V = F_D \cdot V = \frac{1}{2} \cdot \rho \cdot A \cdot V^2 \cdot C_D \cdot V$$

If the old and new designs have the same available power, then

$$\frac{1}{2} \cdot \rho \cdot A \cdot V_{new}^{\;2} \cdot C_{Dnew} \cdot V_{new} = \frac{1}{2} \cdot \rho \cdot A \cdot V_{old}^{\;2} \cdot C_{Dold} \cdot V_{old}$$

or

$$V_{new} = V_{old} \cdot \sqrt[3]{\frac{C_{Dold}}{C_{Dnew}}} \qquad V_{new} = 227 \cdot \frac{ft}{s}$$

Problem 9.169

[Difficulty: 3]

9.169 Assume the Boeing 727 aircraft has wings with NACA 23012 section, planform area of 1600 ft^2, double-slotted flaps, and effective aspect ratio of 6.5. If the aircraft flies at 150 knots in standard air at 175,000 lb gross weight, estimate the thrust required to maintain level flight.

Solution: For steady, level flight, thrust equals drag and lift equals weight.

Computing equations: $F_L = W = C_L \frac{1}{2}\rho V^2 A$ (1)

$F_D = T = C_D \frac{1}{2}\rho V^2 A$ (2)

$$C_D = C_{D,\infty} + C_{D,i} = C_{D,0} + \frac{C_L^2}{\pi ar}$$

Assumptions: (1) Standard air
(2) Data from Fig. 9.23 apply

$$V = 150 \frac{nm}{hr} \times 6076 \frac{ft}{nm} \times \frac{hr}{3600\,sec} = 253\ ft/sec$$

$$q = \frac{1}{2}\rho V^2 = \frac{1}{2} \times 0.00238 \frac{slug}{ft^3} \times (253)^2 \frac{ft^2}{s^2} \times \frac{lbf \cdot s^2}{slug \cdot ft} = 76.2\ lbf/ft^2$$

From Eq. 1, $C_L = \frac{W}{qA} = 175000\ lbf \times \frac{ft^2}{76.2\ lbf} \times \frac{1}{1600\ ft^2} = 1.44$

From Fig. 9.23, this corresponds to operation with a single slot open, and $C_{D,0} \cong 0.04$. Thus

$$C_D = C_{D,0} + \frac{C_L^2}{\pi ar} = 0.04 + \frac{(1.44)^2}{\pi(6.5)} = 0.142$$

To find thrust, note

$$\frac{T}{F_L} = \frac{C_D}{C_L}\frac{qA}{qA} = \frac{C_D}{C_L} = \frac{0.142}{1.44} = 0.0986$$

Thus

$$T = F_L \frac{C_D}{C_L} = W \frac{C_D}{C_L} = 175{,}000\ lbf \times 0.0986 = 17{,}300\ lbf$$

T

Problem 9.170

[Difficulty: 3]

9.170 An airplane with mass of 10,000 lb is flown at constant elevation and speed on a circular path at 150 mph. The flight circle has a radius of 3250 ft. The plane has lifting area of 225 ft^2 and is fitted with NACA 23015 section airfoils with effective aspect ratio of 7. Estimate the drag on the aircraft and the power required.

Given: Aircraft in circular flight

Find: Drag and power

Solution:

Basic equations: $C_D = \frac{F_D}{\frac{1}{2}\cdot\rho\cdot A\cdot V^2}$ $\quad C_L = \frac{F_L}{\frac{1}{2}\cdot\rho\cdot A\cdot V^2}$ $\quad P = F_D\cdot V$ $\quad \Sigma\vec{F} = M\cdot\vec{a}$

The given data or available data are

$\rho = 0.002377\cdot\frac{\text{slug}}{\text{ft}^3}$ $\quad R = 3250\cdot\text{ft}$ $\quad M = 10000\cdot\text{lbm}$ $\quad M = 311\cdot\text{slug}$

$V = 150\cdot\text{mph}$ $\quad V = 220\cdot\frac{\text{ft}}{\text{s}}$ $\quad A = 225\cdot\text{ft}^2$ $\quad ar = 7$

Assuming the aircraft is flying banked at angle β, the vertical force balance is

$$F_L\cdot\cos(\beta) - M\cdot g = 0 \quad \text{or} \quad \frac{1}{2}\cdot\rho\cdot A\cdot V^2\cdot C_L\cdot\cos(\beta) = M\cdot g \qquad (1)$$

The horizontal force balance is

$$-F_L\cdot\sin(\beta) = M\cdot a_r = -\frac{M\cdot V^2}{R} \quad \text{or} \quad \frac{1}{2}\cdot\rho\cdot A\cdot V^2\cdot C_L\cdot\sin(\beta) = \frac{M\cdot V^2}{R} \qquad (2)$$

Equations 1 and 2 enable the bank angle β to be found $\quad \tan(\beta) = \frac{V^2}{R\cdot g}$ $\quad \beta = \operatorname{atan}\left(\frac{V^2}{R\cdot g}\right)$ $\quad \beta = 24.8\cdot\text{deg}$

Then from Eq 1 $\quad F_L = \frac{M\cdot g}{\cos(\beta)}$ $\quad F_L = 1.10\times 10^4\cdot\text{lbf}$

Hence $\quad C_L = \frac{F_L}{\frac{1}{2}\cdot\rho\cdot A\cdot V^2}$ $\quad C_L = 0.851$

For the section, $C_{Dinf} = 0.0075$ at $C_L = 0.851$ (from Fig. 9.19), so $\quad C_D = C_{Dinf} + \frac{C_L^{\,2}}{\pi\cdot ar}$ $\quad C_D = 0.040$

Hence $\quad F_D = F_L\cdot\frac{C_D}{C_L}$ $\quad F_D = 524\cdot\text{lbf}$

The power is $\quad P = F_D\cdot V$ $\quad P = 1.15\times 10^5\cdot\frac{\text{ft}\cdot\text{lbf}}{\text{s}}$ $\quad P = 209\cdot\text{hp}$

Problem 9.171

[Difficulty: 4]

9.171 Find the minimum and maximum speeds at which the airplane of Problem 9.170 can fly on a 3250 ft radius circular flight path, and estimate the drag on the aircraft and power required at these extremes.

Given: Aircraft in circular flight

Find: Maximum and minimum speeds; Drag and power at these extremes

Solution:

Basic equations: $C_D = \dfrac{F_D}{\frac{1}{2}\cdot\rho\cdot A\cdot V^2}$ $\quad C_L = \dfrac{F_L}{\frac{1}{2}\cdot\rho\cdot A\cdot V^2}$ $\quad P = F_D\cdot V$ $\quad \Sigma\cdot\vec{F} = M\cdot\vec{a}$

The given data or available data are

$\rho = 0.002377\cdot\dfrac{slug}{ft^3}$ $\quad R = 3250\cdot ft$ $\quad M = 10000\cdot lbm$ $\quad M = 311\cdot slug$

$A = 225\cdot ft^2$ $\quad ar = 7$

The minimum velocity will be when the wing is at its maximum lift condition. From Fig . 9. 17 or Fig. 9.19

$C_L = 1.72$ $\quad C_{Dinf} = 0.02$

where C_{Dinf} is the section drag coefficient

The wing drag coefficient is then $\quad C_D = C_{Dinf} + \dfrac{{C_L}^2}{\pi\cdot ar}$ $\quad C_D = 0.155$

Assuming the aircraft is flying banked at angle β, the vertical force balance is

$F_L\cdot\cos(\beta) - M\cdot g = 0$ $\quad$ or $\quad \frac{1}{2}\cdot\rho\cdot A\cdot V^2\cdot C_L\cdot\cos(\beta) = M\cdot g$ $\quad$ (1)

The horizontal force balance is

$-F_L\cdot\sin(\beta) = M\cdot a_r = -\dfrac{M\cdot V^2}{R}$ $\quad$ or $\quad \frac{1}{2}\cdot\rho\cdot A\cdot V^2\cdot C_L\cdot\sin(\beta) = \dfrac{M\cdot V^2}{R}$ $\quad$ (2)

Equations 1 and 2 enable the bank angle β and the velocity V to be determined

$$\sin(\beta)^2 + \cos(\beta)^2 = \left(\frac{\frac{M\cdot V^2}{R}}{\frac{1}{2}\cdot\rho\cdot A\cdot V^2\cdot C_L}\right)^2 + \left(\frac{M\cdot g}{\frac{1}{2}\cdot\rho\cdot A\cdot V^2\cdot C_L}\right)^2 = 1$$

or

$$\frac{M^2\cdot V^4}{R^2} + M^2\cdot g^2 = \frac{\rho^2\cdot A^2\cdot V^4\cdot {C_L}^2}{4}$$

$$V = \sqrt[4]{\frac{M^2\cdot g^2}{\frac{\rho^2\cdot A^2\cdot {C_L}^2}{4} - \frac{M^2}{R^2}}}$$ $\quad V = 149\cdot\dfrac{ft}{s}$ $\quad V = 102\cdot mph$

$\tan(\beta) = \dfrac{V^2}{R\cdot g}$ $\quad \beta = \operatorname{atan}\left(\dfrac{V^2}{R\cdot g}\right)$ $\quad \beta = 12.0\cdot deg$

The drag is then $F_D = \frac{1}{2}\cdot\rho\cdot A\cdot V^2\cdot C_D$ $F_D = 918\cdot\text{lbf}$

The power required to overcome drag is $P = F_D\cdot V$ $P = 1.37\times 10^5\cdot\frac{\text{ft}\cdot\text{lbf}}{\text{s}}$ $P = 249\cdot\text{hp}$

The analysis is repeated for the maximum speed case, when the lift/drag coefficient is at its minimum value. From Fig. 9.19, reasonable values are

$C_L = 0.3$ $C_{Dinf} = \frac{C_L}{47.6}$ corresponding to $\alpha = 2^o$ (Fig. 9.17)

The wing drag coefficient is then $C_D = C_{Dinf} + \frac{C_L^{\ 2}}{\pi\cdot ar}$ $C_D = 0.0104$

From Eqs. 1 and 2 $V = \sqrt[4]{\frac{M^2\cdot g^2}{\frac{\rho^2\cdot A^2\cdot C_L^{\ 2}}{4} - \frac{M^2}{R^2}}}$ $V = (309.9 + 309.9i)\cdot\frac{\text{ft}}{\text{s}}$ Obviously unrealistic (lift is just too low, and angle of attack is too low to generate sufficient lift)

We try instead a larger, more reasonable, angle of attack

$C_L = 0.55$ $C_{Dinf} = 0.0065$ corresponding to $\alpha = 4^o$ (Fig. 9.17)

The wing drag coefficient is then $C_D = C_{Dinf} + \frac{C_L^{\ 2}}{\pi\cdot ar}$ $C_D = 0.0203$

From Eqs. 1 and 2 $V = \sqrt[4]{\frac{M^2\cdot g^2}{\frac{\rho^2\cdot A^2\cdot C_L^{\ 2}}{4} - \frac{M^2}{R^2}}}$ $V = 91.2\,\frac{\text{m}}{\text{s}}$ $V = 204\cdot\text{mph}$

$\tan(\beta) = \frac{V^2}{R\cdot g}$ $\beta = \text{atan}\left(\frac{V^2}{R\cdot g}\right)$ $\beta = 40.6\cdot\text{deg}$

The drag is then $F_D = \frac{1}{2}\cdot\rho\cdot A\cdot V^2\cdot C_D$ $F_D = 485\cdot\text{lbf}$

The power required to overcome drag is $P = F_D\cdot V$ $P = 1.45\times 10^5\cdot\frac{\text{ft}\cdot\text{lbf}}{\text{s}}$ $P = 264\cdot\text{hp}$

Problem 9.172

[Difficulty: 3]

9.172 Jim Hall's Chaparral 2F sports-racing cars in the 1960s pioneered use of airfoils mounted above the rear suspension to enhance stability and improve braking performance. The airfoil was effectively 6 ft wide (span) and had a 1-ft chord. Its angle of attack was variable between 0 and minus 12 degrees. Assume lift and drag coefficient data are given by curves (for conventional section) in Fig. 9.17. Consider a car speed of 120 mph on a calm day. For an airfoil deflection of 12° down, calculate (a) the maximum downward force and (b) the maximum increase in deceleration force produced by the airfoil.

Solution: Apply definitions of C_L and C_D.

Computing equations: $C_L = \frac{F_L}{\frac{1}{2}\rho V^2 A}$ $\quad C_D = \frac{F_D}{\frac{1}{2}\rho V^2 A} = C_{D,\infty} + \frac{C_L^2}{\pi ar}$ $\quad ar = \frac{s}{c}$

From Fig. 9.17, at $\alpha = 12°$, $C_L = 1.4$ and $C_{D,\infty} = 0.013$. Thus, since $\alpha = -12°$,

$$F_L = -C_L A \frac{1}{2}\rho V^2 \qquad A = sc = 6\,ft^2$$

$$= -1.4 \times 6\,ft^2 \times \frac{1}{2} \times 0.00238\,\frac{slug}{ft^3} \times \left(120\,\frac{mi}{hr} \times 5280\,\frac{ft}{mi} \times \frac{hr}{3600\,s}\right)^2 \frac{lbf \cdot s^2}{slug \cdot ft}$$

$$F_L = -310\ lbf \text{ (downward force)}$$

F_L

Then $F_D = F_L \frac{C_D}{C_L} = \frac{0.013 + \frac{(1.4)^2}{\pi(6)}}{1.4} \times 310\ lbf = 25.9\ lbf$

Braking thrust increases as drag increases and as normal force increases tire adhesion (friction). Thus

$$\Delta F_B = \mu_k F_L + F_D$$

For $\mu_k = 1.0$ (probably conservative for racing tires),

$$\Delta F_B = 1.0 \times 310\ lbf + 25.9\ lbf = 336\ lbf$$

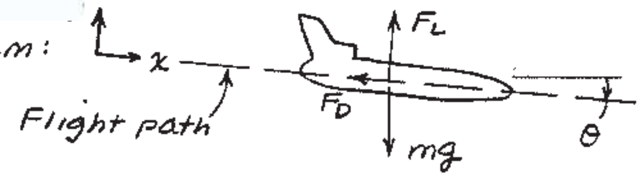
ΔF_B

Problem 9.173

[Difficulty: 4]

9.173 The glide angle for unpowered flight is such that lift, drag, and weight are in equilibrium. Show that the glide slope angle, θ, is such that $\tan\theta = C_D/C_L$. The minimum glide slope occurs at the speed where C_L/C_D is a maximum. For the conditions of Example 9.8, evaluate the minimum glide slope angle for a Boeing 727-200. How far could this aircraft glide from an initial altitude of 10 km on a standard day?

Solution: Consider free-body diagram:



Sum forces along (x) and normal to (y) flight path:

$$\Sigma F_x = -F_D + mg\sin\theta = 0 \qquad mg\sin\theta = F_D$$

$$\Sigma F_y = F_L - mg\cos\theta = 0 \qquad mg\cos\theta = F_L$$

$$\tan\theta = \frac{F_D}{F_L} = \frac{C_D}{C_L}$$ tanθ

Use relationships from Section 9-8:

Computing equation: $C_D = C_{D,0} + \dfrac{C_L^2}{\pi ar} = C_{D,0} + C_{D,i}$

Thus $$\frac{C_D}{C_L} = \frac{C_{D,0}}{C_L} + \frac{C_L}{\pi ar} \qquad (1)$$

To minimize, set $d(C_D/C_L)/dC_L = 0$

$$\frac{d}{dC_L}\left(\frac{C_D}{C_L}\right) = (-1)\frac{C_{D,0}}{C_L^2} + \frac{1}{\pi ar} = 0 \quad \text{when} \quad C_{D,0} = \frac{C_L^2}{\pi ar} = C_{D,i}$$

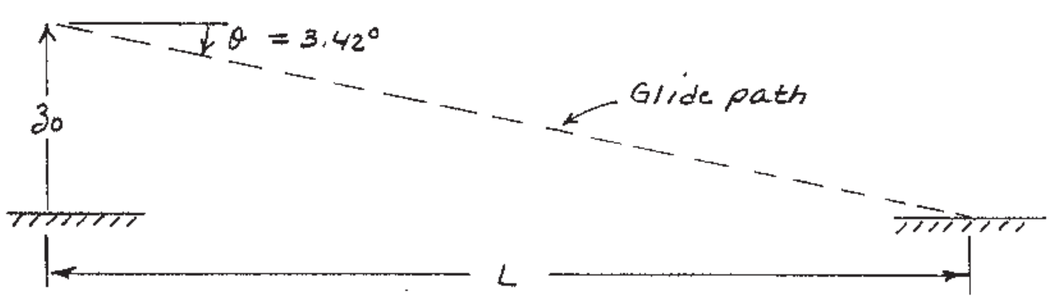
From Example Problem 9.8, $C_{D,0} = 0.0182$ and $ar = 6.5$. Thus optimum is

$$C_L = (\pi ar\, C_{D,0})^{1/2} = [\pi(6.5)0.0182]^{1/2} = 0.610$$

and from Eq. 1,

$$\frac{C_D}{C_L} = \frac{0.0182}{0.61} + \frac{0.61}{\pi(6.5)} = 0.0597 = \tan\theta\,;\ \theta = \tan^{-1}(0.0597) = 3.42^\circ$$ θ

Note θ is independent of atmospheric conditions. Thus θ = constant



$$\frac{z_0}{L} = \tan\theta\,;\quad L = \frac{z_0}{\tan\theta} = \frac{10\text{ km}}{0.0597} = 168\text{ km}$$ L

9.174 The wing loading of the Gossamer Condor is 0.4 lbf/ft^2 of wing area. Crude measurements showed drag was approximately 6 lbf at 12 mph. The total weight of the Condor was 200 lbf. The effective aspect ratio of the Condor is 17. Estimate the minimum power required to fly the aircraft. Compare to the 0.39 hp that pilot Brian Allen could sustain for 2 hr.

<u>Solution</u>: Apply relationships from Section 9-8:

Computing equations: $W = F_L = C_L A \frac{1}{2}\rho V^2 \qquad P = V F_D$

$$T = F_D = C_D A \tfrac{1}{2}\rho V^2 \; ; \; C_D = C_{D,0} + \frac{C_L^2}{\pi a r}$$

The task is to find V to minimize P:

$$P = V F_D = V\left(C_D A \tfrac{1}{2}\rho V^2\right) = \left(C_{D,0} + \frac{C_L^2}{\pi a r}\right) A \tfrac{1}{2}\rho V^3 \qquad (1)$$

But C_L varies with aircraft speed:

$$F_L = W = C_L A \tfrac{1}{2}\rho V^2 \; ; \quad C_L = \frac{2W}{\rho V^2 A} \; ; \quad C_L^2 = \left(\frac{2W}{\rho A}\right)^2 \frac{1}{V^4}$$

Substituting into Eq. 1,

$$P = \left[C_{D,0} + \frac{1}{\pi a r}\left(\frac{2W}{\rho A}\right)^2 \frac{1}{V^4}\right] A \tfrac{1}{2}\rho V^3$$

To minimize power, set $dP/dV = 0$. Then

$$\frac{dP}{dV} = C_{D,0} A \tfrac{1}{2}\rho V^2 (3) + (-1)\frac{1}{\pi a r}\left(\frac{2W}{\rho A}\right)^2 \frac{1}{V^2} A \tfrac{1}{2}\rho = 0$$

Thus at minimum power,

$$3C_{D,0} = \frac{1}{\pi a r}\left(\frac{2W}{\rho A}\right)^2 \frac{1}{V^4} = \frac{C_L^2}{\pi a r} = C_{D,i} \quad \text{or} \quad C_{D,i} = 3C_{D,0}$$ minimum power

From given data, at 12 mph (17.6 ft/s):

$$\tfrac{1}{2}\rho V^2 = \tfrac{1}{2} \times 0.00238 \frac{\text{slug}}{\text{ft}^3} \times (17.6)^2 \frac{\text{ft}^2}{\text{s}^2} \times \frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}} = 0.369 \text{ lbf/ft}^2$$

$$C_L = \frac{F_L}{\frac{1}{2}\rho V^2 A} = \frac{W/A}{\frac{1}{2}\rho V^2} = 0.4 \frac{\text{lbf}}{\text{ft}^2} \times \frac{\text{ft}^2}{0.369 \text{ lbf}} = 1.08 \; ; \; A = \frac{W}{W/A} = 200 \text{ lbf} \times \frac{\text{ft}^2}{0.4 \text{ lbf}} = 500 \text{ ft}^2$$

$$C_D = \frac{F_D}{\frac{1}{2}\rho V^2 A} = 6 \text{ lbf} \times \frac{\text{ft}^2}{0.369 \text{ lbf}} \times \frac{1}{500 \text{ ft}^2} = 0.0325$$

$$C_{D,0} = C_D - C_{D,i} = C_D - \frac{C_L^2}{\pi a r} = 0.0325 - \frac{(1.08)^2}{\pi (17)} = 0.0107 \qquad (C_{D,0} = \text{constant})$$

At flight speed for minimum power, $C_{D,i} = 3C_{D,0}$

Thus at minimum power

$$C_{D,i} = \frac{C_L^2}{\pi ar} = 3C_{D,0} = 3(0.0107) = 0.0321$$

so

$$C_L = (0.0321\,\pi\, ar)^{1/2} = 1.31 \quad \text{(minimum power)}$$

Since

$$F_L = W = C_L A \tfrac{1}{2}\rho V^2$$

then
$$V = \sqrt{\frac{2W}{C_L \rho A}} \quad \text{and} \quad \frac{V_{min}}{V} = \sqrt{\frac{C_L}{C_{L_{min}}}}$$

Thus at minimum power

$$V_{min} = 12\text{ mph}\sqrt{\frac{1.08}{1.31}} = 10.9\text{ mph } (16.0\text{ ft/s}) \quad \text{(minimum power)}$$ V_{min}

The power requirement would be

$$P_{pilot} = \frac{P_{flight}}{\eta_{drive}\eta_{prop}}$$

$$P_{flight} = VF_D = V C_D A \tfrac{1}{2}\rho V^2 = (C_{D,0} + 3C_{D,0}) A \tfrac{1}{2}\rho V^3 = 2C_{D,0} A \rho V^3$$

$$= 2(0.0107)\, 500\text{ ft}^2 \times 0.00238\frac{\text{slug}}{\text{ft}^3} \times (16.0)^3 \frac{\text{ft}^3}{\text{s}^3} \times \frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}} \times \frac{\text{hp}\cdot\text{s}}{550\text{ ft}\cdot\text{lbf}}$$

$$P_{flight} = 0.190\text{ hp} \quad \text{(power for flight)}$$

If $\eta_{drive} = 0.9$ and $\eta_{prop} = 0.7$,

$$P_{pilot} \approx \frac{0.190}{(0.9)(0.7)} = 0.302\text{ hp} \quad \text{(minimum power)}$$ P_{pilot}

Thus $P_{pilot} < 0.39$ hp!

Problem 9.175

[Difficulty: 4]

9.175 Some cars come with a "spoiler," a wing section mounted on the rear of the vehicle that salespeople sometimes claim significantly increases traction of the tires at highway speeds. Investigate the validity of this claim. Are these devices really just cosmetic?

Given: Car spoiler

Find: Whether they are effective

Solution:

To perform the investigation, consider some typical data

For the spoiler, assume $b = 4 \cdot ft$ $\quad c = 6 \cdot in$ $\quad \rho = 1.23 \cdot \frac{kg}{m^3}$ $\quad A = b \cdot c$ $\quad A = 2 \cdot ft^2$

From Fig. 9.17 a reasonable lift coefficient for a conventional airfoil section is $C_L = 1.4$

Assume the car speed is $V = 55 \cdot mph$

Hence the "negative lift" is $F_L = \frac{1}{2} \cdot \rho \cdot A \cdot V^2 \cdot C_L$ $\quad F_L = 21.7 \cdot lbf$

This is a relatively minor negative lift force (about four bags of sugar); it is not likely to produce a noticeable difference in car traction

The picture gets worse at 30 mph: $F_L = 6.5 \cdot lbf$

For a race car, such as that shown on the cover of the text, typical data might be

$b = 5 \cdot ft$ $\quad c = 18 \cdot in$ $\quad A = b \cdot c$ $\quad A = 7.5 \cdot ft^2$ $\quad V = 200 \cdot mph$

In this case: $F_L = 1078 \cdot lbf$

Hence, for a race car, a spoiler can generate very significant negative lift!

Problem 9.176

[Difficulty: 5]

9.176 Roadside signs tend to oscillate in a twisting motion when a strong wind blows. Discuss the phenomena that must occur to cause this behavior.

Discussion: Many roadside signs are mounted on a single post formed from stamped steel. The post has an open "C" cross-section, which provides little torsional rigidity. Wind gusts can excite oscillations in a sign, which acts as a flat plate at an angle of attack relative to the oncoming wind. When at an angle of attack, a plate develops both a lift force and a moment that tends to twist the sign farther from its equilibrium position. While the sign twists, the post provides a resisting torque as a result of being twisted from its equilibrium position.

As the sign twists, the angle of attack relative to the oncoming air increases. An overshoot phenomenon called dynamic stall allows the flow to remain attached and the angle of attack to grow larger before stall occurs than if the change in angle of attack had been slow and gradual. Once stall occurs, the lift force and moment decrease, and the motion is no longer forced. Then the sign tends to return to its undisturbed position.

The moment of inertia of the sign causes it to overshoot the equilibrium position. The sign continues beyond equilibrium and develops a lift force and a moment tending to move it farther past equilibrium. The process repeats, with growing amplitude, until a more-or-less steady-state oscillation is reached.

The sign and post form a spring-mass-damper mechanical system. The sign is the mass, the post is the spring, and hysteresis and aerodynamic resistance to oscillation provide the damping. The "steady" oscillation occurs near the natural frequency of the system.

At steady state, the rate at which energy is added to the sign by the gusting wind exactly balances the rate at which energy is dissipated by hysteresis in the sign motion and its supporting post. The oscillations can continue almost indefinitely, and with considerable amplitude, as can be observed on a windy day. In some cases the oscillations lead to fatigue failure of the sign post.

Problem 9.177

[Difficulty: 5]

9.177 How does a Frisbee™ fly? What causes it to curve left or right? What is the effect of spin on its flight?

Discussion: When viewed from the side, the Frisbee shape has a rounded upper surface and a flat bottom surface. Such a shape is capable of generating lift as it travels through air.

When a Frisbee is not spinning, the lift vector probably acts slightly forward of the maximum thickness on the profile. When spinning, the motion of the surface likely affects the development and separation of the boundary layers. This may displace the center of lift slightly to the right or left of center, depending on the direction of spin.

A Frisbee is not stable when thrown without spin: it will tend to tumble as it moves through the air. Spin is used to stabilize the motion (just as a spinning gyroscope tends to remain upright). The combination of spin and the off-center lift vector cause the Frisbee to precess as a gyroscope. Therefore its spin axis can change from vertical while in flight, causing the flight path to curve right or left.

The Frisbee also can be thrown intentionally to curve right or left. This is done by inclining the spin axis so that it is not vertical at launch. When the spin axis is inclined to the left (as seen by the thrower), the Frisbee drifts to the left along a more-or-less constant radius path. Inclining the spin axis to the right causes the opposite effect.

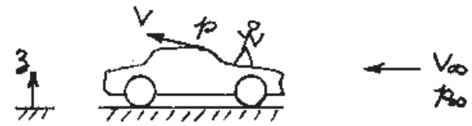
Problem 9.178

[Difficulty: 2]

9.178 Air moving over an automobile is accelerated to speeds higher than the travel speed, as shown in Fig. 9.25. This causes changes in interior pressure when windows are opened or closed. Use the data of Fig. 9.25 to estimate the pressure reduction when a window is opened slightly at a speed of 100 km/hr. What is the air speed in the freestream near the window opening?

Solution: Apply the Bernoulli equation and pressure coefficient definition.

Basic equations:
$$\frac{p_\infty}{\rho} + \frac{V_\infty^2}{2} + g\cancel{z_\infty} = \frac{p}{\rho} + \frac{V^2}{2} + g\cancel{z} \qquad C_p = \frac{p - p_\infty}{\frac{1}{2}\rho V_\infty^2}$$



Assumptions: (1) Steady flow seen from auto
(2) Incompressible flow
(3) No friction
(4) Flow along a streamline
(5) Neglect changes in elevation

From Fig. 9.25, C_p near driver's window ranges between -1.23 and -0.40.

At $V_\infty = 100$ km/hr,

$$q = \frac{1}{2}\rho V_\infty^2 = \frac{1}{2} \times 1.23\,\frac{kg}{m^3} \times \left[100\,\frac{km}{hr} \times 1000\,\frac{m}{km} \times \frac{hr}{3600\,s}\right]^2 \frac{N \cdot s^2}{kg \cdot m} = 475\ N/m^2$$

Thus the pressures outside may be between

$$p - p_\infty = C_p \frac{1}{2}\rho V_\infty^2 = -1.23 \times 475\,\frac{N}{m^2} = -584\ N/m^2\ \text{(gage)}$$

and $$p - p_\infty = -0.40 \times 475\,\frac{N}{m^2} = -190\ N/m^2\ \text{(gage)}$$ p

From the Bernoulli equation,

$$\frac{V^2}{2} = \frac{V_\infty^2}{2} + \frac{p_\infty - p}{\rho} = \frac{V_\infty^2}{2}\left(1 - \frac{p - p_\infty}{\frac{1}{2}\rho V_\infty^2}\right) = \frac{V_\infty^2}{2}(1 - C_p)$$

Thus
$$V = V_\infty \sqrt{1 - C_p}$$

The local flow speeds range from

$$V = V_\infty \sqrt{1 - (-1.23)} = 100\,\frac{km}{hr}\sqrt{2.23} = 149\ km/hr\ \ (41.5\ m/s)$$

$$V = V_\infty \sqrt{1 - (-0.40)} = 100\,\frac{km}{hr}\sqrt{1.40} = 118\ km/hr\ \ (32.9\ m/s)$$ V

Thus local flow speeds are significantly higher than V_∞.

Problem 9.179

[Difficulty: 5]

9.179 An automobile travels down the road with a bicycle attached to a carrier across the rear of the trunk. The bicycle wheels rotate slowly. Explain why and in what direction the rotation occurs.

Discussion: All objects moving in ground effect generate lift (the air flows over the top faster than over the bottom because of the shape of the automobile). Any object that produces lift carries with it a bound vortex that creates circulation about the profile that accompanies lift.

The bound vortex creates two trailing vortices, one on each side of the car, which rotate in opposite directions as they follow in the wake of the automobile. When viewed from the rear of the auto, the left side trailing vortex rotates clockwise and the right side trailing vortex rotates counterclockwise.

The swirl in the trailing vortex motion is responsible for the motion of the bicycle wheels (check it out on your next auto trip during the summer months!). The swirl causes shear stresses that tend to rotate the bicycle wheels in the same senses as the trailing vortices. Again viewing from the rear of the auto, the left wheel rotates clockwise and the right wheel counterclockwise.

(Sometimes the rear wheel of the bicycle cannot freewheel. In this case only the front wheel turns slowly as the car drives down the road.)

Problem 9.180

[Difficulty: 2]

9.180 A class demonstration showed that lift is present when a cylinder rotates in an air stream. A string wrapped around a paper cylinder and pulled causes the cylinder to spin and move forward simultaneously. Assume a cylinder of 5 cm diameter and 30 cm length is given a rotational speed of 240 rpm and a forward speed of 1.5 m/s. Estimate the approximate lift force that acts on the cylinder.

Given: Data on rotating cylinder

Find: Lift force on cylinder

Solution:

Basic equations: $$C_L = \frac{F_L}{\frac{1}{2}\cdot\rho\cdot A\cdot V^2}$$

The given or available data is $\rho = 1.21\cdot\frac{kg}{m^3}$ $\nu = 1.50\cdot10^{-5}\cdot\frac{m^2}{s}$ $L = 30\cdot cm$ $D = 5\cdot cm$ $\omega = 240\cdot rpm$ $V = 1.5\cdot\frac{m}{s}$

The spin ratio is: $\frac{\omega\cdot D}{2\cdot V} = 0.419$ From Fig. 9.29, we can estimate the maximum lift coefficient: $C_L = 1.0$

The area is $A = D\cdot L = 0.015\,m^2$ Therefore, the lift force is: $F_L = \frac{1}{2}\cdot C_L\cdot\rho\cdot A\cdot V^2$ $F_L = 0.0204\,N$

Problem 9.181

[Difficulty: 2]

9.181 A golf ball (diameter $D = 43$ mm) with circular dimples is hit from a sand trap at 20 m/s with backspin of 2000 rpm. The mass of the ball is 48 g. Evaluate the lift and drag forces acting on the ball. Express your results as fractions of the weight of the ball.

Solution: Use data from Fig. 9.28 for lift and drag coefficients.

Computing equations: $F_L = C_L A \frac{1}{2}\rho V^2 \qquad F_D = C_D A \frac{1}{2}\rho V^2$

At $V = 20$ m/sec, then $q = \frac{1}{2}\rho V^2 = \frac{1}{2} \times 1.23 \frac{kg}{m^3} \times (20)^2 \frac{m^2}{s^2} \times \frac{N \cdot s^2}{kg \cdot m} = 246\ N/m^2$

From Fig. 9.28, $C_L = C_L\left(Re, \frac{\omega D}{2V}\right)$

$$Re = \frac{VD}{\nu} = 20\frac{m}{s} \times 0.043\,m \times \frac{s}{1.45 \times 10^{-5}\,m^2} = 5.93 \times 10^4 \text{ (assume close to } 1.26 \times 10^5\text{)}$$

$$\frac{\omega D}{2V} = \frac{1}{2} \times 2000\frac{rev}{min} \times 0.043\,m \times \frac{s}{20\,m} \times 2\pi\frac{rad}{rev} \times \frac{min}{60\,s} = 0.225$$

Then $C_L \simeq 0.23$ and $F_L = C_L A \frac{1}{2}\rho V^2 = C_L A q$

$$F_L = 0.23 \times \frac{\pi(0.043)^2}{4} m^2 \times 246 \frac{N}{m^2} = 0.0822\ N$$

F_L

From Fig. 9.28 $C_D = 0.31$, so

$$F_D = \frac{C_D}{C_L} F_L = \frac{0.31}{0.23} \times 0.0822\ N = 0.111\ N$$

F_D

For the ball, $mg = 0.048\ kg \times 9.81 \frac{m}{sec^2} \times \frac{N \cdot sec^2}{kg \cdot m} = 0.471\ N$

Thus $$\frac{F_L}{mg} = \frac{0.0822\ N}{0.471\ N} = 0.175$$

$$\frac{F_D}{mg} = \frac{0.111\ N}{0.471\ N} = 0.236$$

Ratios

Problem 9.182

[Difficulty: 2]

9.182 Rotating cylinders were proposed as a means of ship propulsion in 1924 by the German engineer, Flettner. The original Flettner rotor ship had two rotors, each about 10 ft in diameter and 50 ft high, rotating at up to 800 rpm. Calculate the maximum lift and drag forces that act on each rotor in a 30-mph wind. Compare the total force to that produced at the optimum L/D at the same wind speed. Estimate the power needed to spin the rotor at 800 rpm.

Given: Data on original Flettner rotor ship

Find: Maximum lift and drag forces, optimal force at same wind speed, power requirement

Solution:

Basic equations: $$C_L = \frac{F_L}{\frac{1}{2}\cdot\rho\cdot A\cdot V^2}$$

The given or available data is $\rho = 0.00234\cdot\frac{\text{slug}}{\text{ft}^3}$ $\quad L = 50\cdot\text{ft}$ $\quad D = 10\cdot\text{ft}$ $\quad \omega = 800\cdot\text{rpm}$ $\quad V = 30\cdot\text{mph} = 44\cdot\frac{\text{ft}}{\text{s}}$

$\nu = 1.62\times 10^{-4}\cdot\frac{\text{ft}^2}{\text{s}}$

The spin ratio is: $\frac{\omega\cdot D}{2\cdot V} = 9.52$ From Fig. 9.29, we can estimate the lift and drag coefficients: $C_L = 9.5$ $\quad C_D = 3.5$

The area is $A = D\cdot L = 500\cdot\text{ft}^2$ Therefore, the lift force is: $F_L = \frac{1}{2}\cdot C_L\cdot\rho\cdot A\cdot V^2$ $\qquad F_L = 1.076\times 10^4\cdot\text{lbf}$

The drag force is: $F_D = \frac{1}{2}\cdot C_D\cdot\rho\cdot A\cdot V^2$ $\qquad F_D = 3.964\times 10^3\cdot\text{lbf}$

This appears to be close to the optimum L/D ratio. The total force is: $F = \sqrt{{F_L}^2 + {F_D}^2}$ $\qquad F = 1.147\times 10^4\cdot\text{lbf}$

To determine the power requirement, we need to estimate the torque on the cylinder. $T = \tau\cdot A\cdot R = \tau\cdot\pi\cdot L\cdot D\cdot\frac{D}{2} = \frac{\pi\cdot\tau\cdot D^2\cdot L}{2}$

In this expression τ is the average wall shear stress. We can estimate this stress using the flat plate approximation:

$Re = \frac{\left(V + \omega\cdot\frac{D}{2}\right)\cdot D}{\nu} = 2.857\times 10^7$ For a cylinder at this Reynolds number: $C_D = 0.003$ Therefore, the shear stress is:

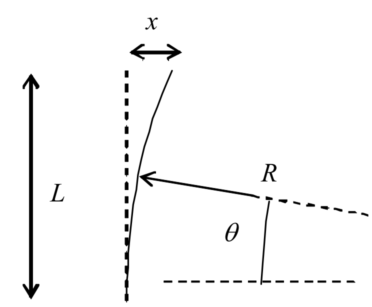
$\tau = \frac{F_D}{A}$ $\qquad \tau = \frac{1}{2}\cdot\rho\cdot V^2\cdot C_D = 6.795\times 10^{-3}\cdot\frac{\text{lbf}}{\text{ft}^2}$ So the torque is: $T = \frac{\pi\cdot\tau\cdot D^2\cdot L}{2} = 53.371\cdot\text{ft}\cdot\text{lbf}$

The power is: $P = T\cdot\omega = 4471\cdot\frac{\text{ft}\cdot\text{lbf}}{\text{s}}$ $\qquad P = 8.13\cdot\text{hp}$

Problem 9.183

[Difficulty: 4]

9.183 A baseball pitcher throws a ball at 80 mph. Home plate is 60 ft away from the pitcher's mound. What spin should be placed on the ball for maximum horizontal deviation from a straight path? (A baseball has a mass of 5 oz and a circumference of 9 in.) How far will the ball deviate from a straight line?



Given: Baseball pitch

Find: Spin on the ball

Solution:

Basic equations: $C_L = \dfrac{F_L}{\frac{1}{2}\cdot\rho\cdot A\cdot V^2}$ $\quad \Sigma\cdot\vec{F} = M\cdot\vec{a}$

The given or available data is $\rho = 0.00234\cdot\dfrac{slug}{ft^3}$ $\quad \nu = 1.62\times 10^{-4}\cdot\dfrac{ft^2}{s}$ $\quad L = 60\cdot ft$

$M = 5\cdot oz$ $\quad C = 9\cdot in$ $\quad D = \dfrac{C}{\pi}$ $\quad D = 2.86\cdot in$ $\quad A = \dfrac{\pi\cdot D^2}{4}$ $\quad A = 6.45\cdot in^2$ $\quad V = 80\cdot mph$

Compute the Reynolds number $Re = \dfrac{V\cdot D}{\nu}$ $\quad Re = 1.73\times 10^5$

This Reynolds number is slightly beyond the range of Fig. 9.27; we use Fig. 9.27 as a rough estimate

The ball follows a trajectory defined by Newton's second law. In the horizontal plane (*x* coordinate)

$$F_L = M\cdot a_R = M\cdot a_x = M\cdot\frac{V^2}{R} \quad \text{and} \quad F_L = \frac{1}{2}\cdot\rho\cdot A\cdot V^2\cdot C_L$$

where R is the instantaneous radius of curvature of the trajectory

From Eq 1 we see the ball trajectory has the smallest radius (i.e. it curves the most) when C_L is as large as possible. From Fig. 9.27 we see this is when $C_L = 0.4$

Solving for R $\quad R = \dfrac{2\cdot M}{C_L\cdot A\cdot\rho}$ $\quad (1)$ $\quad R = 463.6\cdot ft$

Also, from Fig. 9.27 $\quad \dfrac{\omega\cdot D}{2\cdot V} = 1.5$ $\quad$ to $\quad \dfrac{\omega\cdot D}{2\cdot V} = 1.8$ $\quad$ defines the best range

Hence $\quad \omega = 1.5\cdot\dfrac{2\cdot V}{D}$ $\quad \omega = 14080\cdot rpm$ $\quad \omega = 1.8\cdot\dfrac{2\cdot V}{D}$ $\quad \omega = 16896\cdot rpm$

From the trajectory geometry $\quad x + R\cdot\cos(\theta) = R$ $\quad$ where $\quad \sin(\theta) = \dfrac{L}{R}$

Hence $\quad x + R\cdot\sqrt{1 - \left(\dfrac{L}{R}\right)^2} = R$

Solving for x $\quad x = R - R\cdot\sqrt{1 - \left(\dfrac{L}{R}\right)^2}$ $\quad x = 3.90\cdot ft$

Problem 9.184

[Difficulty: 3]

9.184 American and British golf balls have slightly different diameters but the same mass (see Problems 1.39 and 1.42). Assume a professional golfer hits each type of ball from a tee at 85 m/s with backspin of 9000 rpm. Evaluate the lift and drag forces on each ball. Express your answers as fractions of the weight of each ball. Estimate the radius of curvature of the trajectory of each ball. Which ball should have the longer range for these conditions?

Solution: Apply definitions of lift and drag coefficients, data from Fig. 9.28.

Computing equations: $C_L = \dfrac{F_L}{\frac{1}{2}\rho V^2 A} = \dfrac{F_L}{qA}$; $C_D = \dfrac{F_D}{qA}$; $q = \frac{1}{2}\rho V^2$; $A = \dfrac{\pi D^2}{4}$

The parameters are Re_D and $\omega D/2V$; tabulate results:

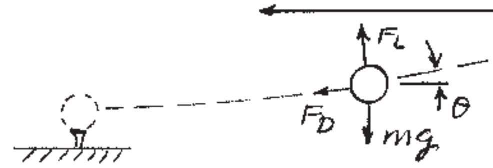
Ball	m (oz)	D (in.)	$\frac{\omega D}{2V}$ (---)	Re (---)	C_L (---)	C_D (---)	F_L (N)	F_D (N)
American	1.62	1.68	0.236	2.50×10^5	0.27	0.32	1.71	2.03
British	1.62	1.62	0.228	2.41×10^5	0.26	0.31	1.53	1.83

Taking ratios for the American ball: British ball:

$F_L/mg = 1.71\text{ N} \times \dfrac{1}{1.62\text{ oz}} \times \dfrac{16\text{ oz}}{\text{lbf}} \times \dfrac{\text{lbf}}{4.448\text{ N}} = 3.80$ 3.40 F_L/mg

F_D/mg $= 4.51$ 4.07 F_D/mg

Draw FBD to compute trajectory:



$\Sigma F_\perp$ to path $= F_L - mg\cos\theta = m\dfrac{V^2}{R}$

Assume θ small, so $\cos\theta \approx 1$. Then

$$R \approx \frac{mV^2}{F_L - mg} = \frac{V^2/g}{F_L/mg - 1} = \frac{1}{3.80-1} \times (85)^2\frac{m^2}{s^2} \times \frac{s^2}{9.81\,m} = 263\text{ m (American)}$$

$$= \frac{1}{3.40-1} \times (85)^2\frac{m^2}{s^2} \times \frac{s^2}{9.81\,m} = 307\text{ m (British)}$$ R

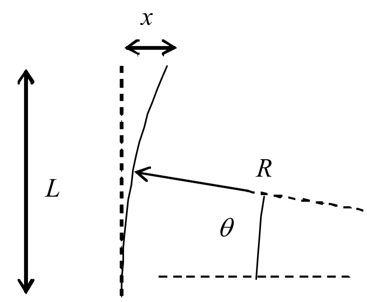
(Note because $F_L/mg > 1$, the balls actually rise!)

Drag probably is more important than lift in affecting range of a drive. Therefore one probably would expect the British ball to carry farther. Range

Problem 9.185

[Difficulty: 4]

9.185 A soccer player takes a free kick. Over a distance of 10 m, the ball veers to the right by about 1 m. Estimate the spin the player's kick put on the ball if its speed is 30 m/s. The ball has a mass of 420 gm and has a circumference of 70 cm.



Given: Soccer free kick

Find: Spin on the ball

Solution:

Basic equations: $C_L = \dfrac{F_L}{\frac{1}{2}\cdot\rho\cdot A\cdot V^2}$ $\quad \Sigma\cdot\vec{F} = M\cdot\vec{a}$

The given or available data is $\rho = 1.21\cdot\dfrac{kg}{m^3}$ $\quad \nu = 1.50\cdot 10^{-5}\cdot\dfrac{m^2}{s}$ $\quad L = 10\cdot m$ $\quad x = 1\cdot m$

$M = 420\cdot gm$ $\quad C = 70\cdot cm$ $\quad D = \dfrac{C}{\pi}$ $\quad D = 22.3\cdot cm$ $\quad A = \dfrac{\pi\cdot D^2}{4}$ $\quad A = 0.0390\cdot m^2$ $\quad V = 30\cdot\dfrac{m}{s}$

Compute the Reynolds number $Re = \dfrac{V\cdot D}{\nu}$ $\quad Re = 4.46\times 10^5$

This Reynolds number is beyond the range of Fig. 9.27; however, we use Fig. 9.27 as a rough estimate

The ball follows a trajectory defined by Newton's second law. In the horizontal plane (*x* coordinate)

$$F_L = M\cdot a_R = M\cdot a_x = M\cdot\frac{V^2}{R} \quad \text{and} \quad F_L = \frac{1}{2}\cdot\rho\cdot A\cdot V^2\cdot C_L$$

where R is the instantaneous radius of curvature of the trajectory

Hence, solving for R $\quad R = \dfrac{2\cdot M}{C_L\cdot A\cdot\rho}$ $\quad$ (1)

From the trajectory geometry $\quad x + R\cdot\cos(\theta) = R$ $\quad$ where $\quad \sin(\theta) = \dfrac{L}{R}$

Hence $\quad x + R\cdot\sqrt{1 - \left(\dfrac{L}{R}\right)^2} = R$

Solving for *R* $\quad R = \dfrac{\left(L^2 + x^2\right)}{2\cdot x}$ $\quad R = 50.5\cdot m$

Hence, from Eq 1 $\quad C_L = \dfrac{2\cdot M}{R\cdot A\cdot\rho}$ $\quad C_L = 0.353$

For this lift coefficient, from Fig. 9.27 $\quad \dfrac{\omega\cdot D}{2\cdot V} = 1.2$

Hence $\quad \omega = 1.2\cdot\dfrac{2\cdot V}{D}$ $\quad \omega = 3086\cdot rpm$

(And of course, Beckham still kind of rules!)

Problem 10.1

[Difficulty: 2]

10.1 Dimensions of a centrifugal pump impeller are

Parameter	Inlet, Section (1)	Outlet, Section (2)
Radius, r (mm)	175	500
Blade width, b (mm)	50	30
Blade angle, β (deg)	65	70

The pump handles water and is driven at 750 rpm. Calculate the theoretical head and mechanical power input if the flow rate is 0.75 m^3/s.

Solution:

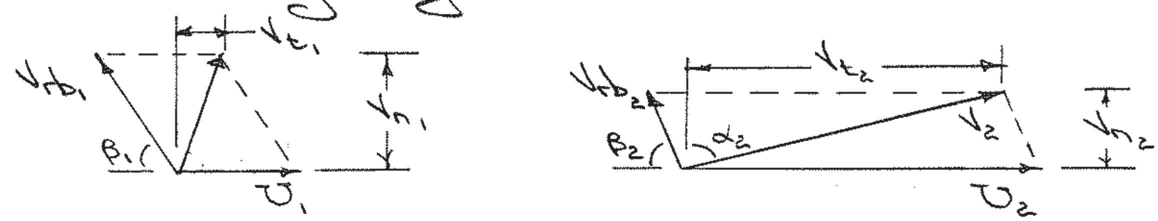
Computing equations: $\dot{W}_m = (U_2 V_{t_2} - U_1 V_{t_1})\dot{m}$ (10.2b)

$H = \frac{1}{g}(U_2 V_{t_2} - U_1 V_{t_1})$ (10.2c)

Assume: (1) uniform flow at blade inlet and outlet

(2) flow enters and leaves tangent to the blade

Draw velocity diagrams:



From continuity, $V_n = \frac{Q}{2\pi r b} = V_{rb}\sin\beta \quad \therefore V_{rb} = \frac{V_n}{\sin\beta}$

From geometry, $V_t = U - V_{rb}\cos\beta = U - \frac{V_n}{\sin\beta}\cos\beta = U - \frac{Q}{2\pi r b}\cot\beta$

Substituting numerical values,

$\omega = 750\,\frac{\text{rev}}{\text{min}} \times 2\pi\,\frac{\text{rad}}{\text{rev}} \times \frac{\text{min}}{60\,\text{s}} = 78.5\ \text{rad/s}$

$U_1 = \omega r_1 = 78.5\,\frac{\text{rad}}{\text{s}} \times 0.175\,\text{m} = 13.7\ \text{m/s}$; $U_2 = 39.3\ \text{m/s}$

$V_{t_1} = U_1 - \frac{Q}{2\pi r_1 b_1}\cot\beta_1 = 13.7\,\frac{\text{m}}{\text{s}} - \frac{0.75}{2\pi}\frac{\text{m}^3}{\text{s}} \times \frac{1}{0.175\,\text{m}} \times \frac{\cot 65^\circ}{0.05\,\text{m}} = 7.34\ \text{m/s}$

$V_{t_2} = 39.3\,\frac{\text{m}}{\text{s}} - \frac{0.75}{2\pi}\frac{\text{m}^3}{\text{s}} \times \frac{1}{0.50\,\text{m}} \times \frac{\cot 70^\circ}{0.03\,\text{m}} = 36.4\ \text{m/s}$

$H = \frac{1}{g}(U_2 V_{t_2} - U_1 V_{t_1}) = \frac{\text{s}^2}{9.81\,\text{m}}\left(39.3\,\frac{\text{m}}{\text{s}} \times 36.4\,\frac{\text{m}}{\text{s}} - 13.7\,\frac{\text{m}}{\text{s}} \times 7.34\,\frac{\text{m}}{\text{s}}\right) = 135\ \text{m}$ $\quad H$

$\dot{W}_m = (U_2 V_{t_2} - U_1 V_{t_1})\dot{m} = gH\rho Q = 9.81\,\frac{\text{m}}{\text{s}^2} \times 135\,\text{m} \times 999\,\frac{\text{kg}}{\text{m}^3} \times 0.75\,\frac{\text{m}^3}{\text{s}} \times \frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}} \times \frac{\text{W}\cdot\text{s}}{\text{N}\cdot\text{m}}$

$\dot{W}_m = 994\ \text{kW}$ $\quad \dot{W}_m$

Problem 10.2

[Difficulty: 2]

10.2 The geometry of a centrifugal water pump is $r_1 = 10$ cm, $r_2 = 20$ cm, $b_1 = b_2 = 4$ cm, $\beta_1 = 30°$, $\beta_2 = 15°$, and it runs at speed 1600 rpm. Estimate the discharge required for axial entry, the power generated in the water in watts, and the head produced.

Given: Geometry of centrifugal pump

Find: Estimate discharge for axial entry; Head

Solution:

Basic equations: $$\dot{W}_m = (U_2 V_{t_2} - U_1 V_{t_1})\dot{m} \qquad \text{(Eq. 10.2b)}$$

$$H = \frac{\dot{W}_m}{\dot{m}g} = \frac{1}{g}(U_2 V_{t_2} - U_1 V_{t_1}) \qquad \text{(Eq. 10.2c)}$$

The given or available data is

$$\rho = 999\cdot\frac{kg}{m^3} \qquad r_1 = 10\cdot cm \qquad r_2 = 20\cdot cm \qquad b_1 = 4\cdot cm \qquad b_2 = 4\cdot cm$$

$$\omega = 1600\cdot rpm \qquad \beta_1 = 30\cdot deg \qquad \beta_2 = 15\cdot deg$$

From continuity $$V_n = \frac{Q}{2\cdot\pi\cdot r\cdot b} = w\cdot\sin(\beta) \qquad w = \frac{V_n}{\sin(\beta)}$$

From geometry $$V_t = U - w\cdot\cos(\beta) = U - \frac{V_n}{\sin(\beta)}\cdot\cos(\beta) = U - \frac{Q}{2\cdot\pi\cdot r\cdot b}\cdot\cot(\beta)$$

For an axial entry $$V_{t1} = 0 \qquad \text{so} \qquad U_1 - \frac{Q}{2\cdot\pi\cdot r_1\cdot b_1}\cdot\cot(\beta_1) = 0$$

Using given data $$U_1 = \omega\cdot r_1 \qquad U_1 = 16.755\frac{m}{s}$$

Hence $$Q = 2\cdot\pi\cdot r_1\cdot b_1\cdot U_1\cdot\tan(\beta_1) \qquad Q = 0.2431\cdot\frac{m^3}{s}$$

To find the power we need U_2, V_{t2}, and m_{rate}

The mass flow rate is $$m_{rate} = \rho\cdot Q \qquad m_{rate} = 242.9\cdot\frac{kg}{s}$$

$$U_2 = \omega\cdot r_2 \qquad U_2 = 33.5\cdot\frac{m}{s}$$

$$V_{t2} = U_2 - \frac{Q}{2\cdot\pi\cdot r_2\cdot b_2}\cdot\cot(\beta_2) \qquad V_{t2} = 15.5\cdot\frac{m}{s}$$

Hence $$W_m = (U_2\cdot V_{t2} - U_1\cdot V_{t1})\cdot m_{rate} \qquad W_m = 1.258\times 10^5\cdot\frac{J}{s} \qquad W_m = 126\cdot kW$$

The head is $$H = \frac{W_m}{m_{rate}\cdot g} \qquad H = 52.8\cdot m$$

Problem 10.3

[Difficulty: 2]

10.3 A centrifugal pump running at 3000 rpm pumps water at a rate of 0.6 m^3/min. The water enters axially and leaves the impeller at 5.4 m/s relative to the blades, which are radial at the exit. If the pump requires 5 kW and is 72 percent efficient, estimate the basic dimensions (impeller exit diameter and width), using the Euler turbomachine equation.

Given: Data on centrifugal pump

Find: Estimate basic dimensions

Solution:

Basic equations: $\dot{W}_m = (U_2 V_{t_2} - U_1 V_{t_1})\dot{m}$ (Eq. 10.2b, directly derived from the Euler turbomachine equation)

The given or available data is

$\rho = 999 \cdot \frac{kg}{m^3}$ $\quad Q = 0.6 \cdot \frac{m^3}{min}$ $\quad Q = 0.0100 \frac{m^3}{s}$ $\quad W_{in} = 5 \cdot kW$ $\quad \eta = 72 \cdot \%$

$\omega = 3000 \cdot rpm$ $\quad w_2 = 5.4 \cdot \frac{m}{s}$ $\quad \beta_2 = 90 \cdot deg$

For an axial inlet $\quad V_{t1} = 0$

From the outlet geometry $\quad V_{t2} = U_2 - V_{rb2} \cdot \cos(\beta_2) = U_2$ $\quad$ and $\quad U_2 = r_2 \cdot \omega$

Hence, in Eq. 10.2b $\quad W_m = U_2^2 \cdot m_{rate} = r_2^2 \cdot \omega^2 \cdot m_{rate}$

with $\quad W_m = \eta \cdot W_{in}$ $\quad W_m = 3.6 \cdot kW$

and $\quad m_{rate} = \rho \cdot Q$ $\quad m_{rate} = 9.99 \cdot \frac{kg}{s}$

Hence $\quad r_2 = \sqrt{\frac{W_m}{m_{rate} \cdot \omega^2}}$ $\quad r_2 = 0.06043 \cdot m$ $\quad r_2 = 6.04 \cdot cm$

Also $\quad V_{n2} = w_2 \cdot \sin(\beta_2)$ $\quad V_{n2} = 5.40 \cdot \frac{m}{s}$

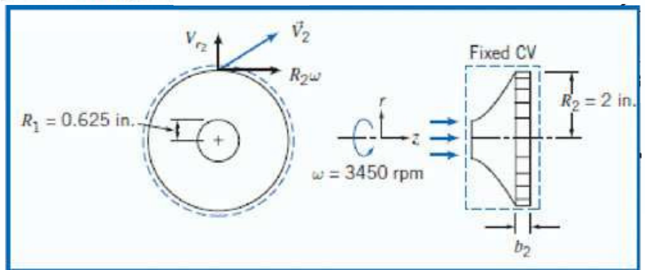
From continuity $\quad V_{n2} = \frac{Q}{2 \cdot \pi \cdot r_2 \cdot b_2}$

Hence $\quad b_2 = \frac{Q}{2 \cdot \pi \cdot r_2 \cdot V_{n2}}$ $\quad b_2 = 4.8776 \times 10^{-3} \cdot m$ $\quad b_2 = 0.488 \cdot cm$

Problem 10.4

[Difficulty: 2]

10.4 Consider the centrifugal pump impeller dimensions given in Example 10.1. Estimate the ideal head rise and mechanical power input if the outlet blade angle is changed to 60°, 70°, 80°, or 85°.



Solution:

Computing equations: $\dot{W}_M = (U_2 V_{t_2} - \cancel{U_1 V_{t_1}})\dot{m}$ (10.2b)

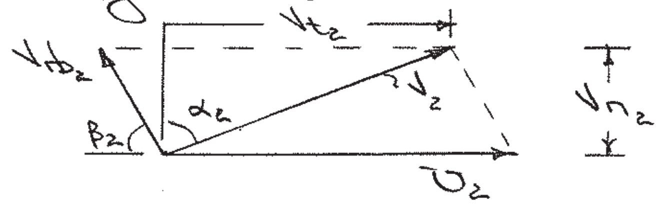
$H = \frac{1}{g}(U_2 V_{t_2} - \cancel{U_1 V_{t_1}})$ (10.2c)

$U_1 V_{t_1} = 0(1)$

Assumptions: (1) axial inlet flow (given) so $V_{t_1} = 0$

(2) at blade outlet, flow is uniform and leaves tangent to blade

Exit velocity diagram:



From continuity, $V_{n_2} = \frac{Q}{2\pi r_2 b_2} = V_{rb_2} \sin\beta_2$ $\therefore V_{rb_2} = \frac{V_{n_2}}{\sin\beta_2}$

From geometry, $V_{t_2} = U_2 - V_{rb_2}\cos\beta_2 = U_2 - \frac{V_{n_2}}{\sin\beta_2}\cos\beta_2 = U_2 - V_{n_2}\cot\beta_2$

Substituting numerical values, for $\beta = 60°$

$$U_2 = \omega r_2 = 3450\,\frac{\text{rev}}{\text{min}} \times 2\pi\,\frac{\text{rad}}{\text{rev}} \times \frac{\text{min}}{60\,\text{s}} \times 2.0\,\text{in} \times \frac{\text{ft}}{12\,\text{in}} = 60.2\,\text{ft/s}$$

$$V_{n_2} = \frac{Q}{2\pi r_2 b_2} = \frac{1}{2\pi} \times 150\,\frac{\text{gal}}{\text{min}} \times \frac{\text{min}}{60\,\text{s}} \times \frac{\text{ft}^3}{7.48\,\text{gal}} \times \frac{1}{2.0\,\text{in}} \times \frac{1}{0.383\,\text{in}} \times \frac{144\,\text{in}^2}{\text{ft}^2} = 10.0\,\text{ft/s}$$

$$V_{t_2} = U_2 - V_{n_2}\cot\beta_2 = 60.2\,\frac{\text{ft}}{\text{s}} - 10.0\,\frac{\text{ft}}{\text{s}}\cot 60° = 54.4\,\text{ft/s}$$

$$\dot{m} = \rho Q = 1.94\,\frac{\text{slug}}{\text{ft}^3} \times 150\,\frac{\text{gal}}{\text{min}} \times \frac{\text{min}}{60\,\text{s}} \times \frac{\text{ft}^3}{7.48\,\text{gal}} = 0.648\,\text{slug/s}$$

$$H = \frac{1}{g}U_2 V_{t_2} = \frac{\text{s}^2}{32.2\,\text{ft}} \times 60.2\,\frac{\text{ft}}{\text{s}} \times 54.4\,\frac{\text{ft}}{\text{s}} = 102\,\text{ft}$$ ← H $\beta = 60°$

$$\dot{W}_M = \dot{m}U_2 V_{t_2} = \dot{m}gH = 0.648\,\frac{\text{slug}}{\text{s}} \times 32.2\,\frac{\text{ft}}{\text{s}^2} \times 102\,\text{ft} \times \frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}} \times \frac{\text{hp}\cdot\text{s}}{550\,\text{ft}\cdot\text{lbf}}$$

$\dot{W}_M = 3.87$ hp ← $\dot{W}_M$ $\beta = 60°$

For $\beta = 70°$, $V_{t_2} = 56.6$ ft/s, $H = 106$ ft, $\dot{W}_M = 4.02$ hp ←

$\beta = 80°$, $V_{t_2} = 58.4$ ft/s, $H = 109$ ft, $\dot{W}_M = 4.15$ hp ←

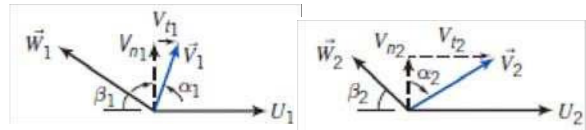
$\beta = 85°$, $V_{t_2} = 59.3$ ft/s, $H = 111$ ft, $\dot{W}_M = 4.21$ hp ←

Problem 10.5

[Difficulty: 2]

10.5 Dimensions of a centrifugal pump impeller are

Parameter	Inlet, Section (1)	Outlet, Section (2)
Radius, r (in.)	15	45
Blade width, b (in.)	4.75	3.25
Blade angle, β (deg)	40	60



The pump is driven at 575 rpm and the fluid is water. Calculate the theoretical head and mechanical power if the flow rate is 80,000 gpm.

Given: Geometry of centrifugal pump

Find: Theoretical head; Power input for given flow rate

Solution:

Basic equations:

$$\dot{W}_m = (U_2 V_{t_2} - U_1 V_{t_1})\dot{m} \qquad \text{(Eq. 10.2b)}$$

$$H = \frac{\dot{W}_m}{\dot{m}g} = \frac{1}{g}(U_2 V_{t_2} - U_1 V_{t_1}) \qquad \text{(Eq. 10.2c)}$$

The given or available data is

$$\rho = 1.94 \cdot \frac{\text{slug}}{\text{ft}^3} \qquad r_1 = 15 \cdot \text{in} \qquad r_2 = 45 \cdot \text{in} \qquad b_1 = 4.75 \cdot \text{in} \qquad b_2 = 3.25 \cdot \text{in}$$

$$\omega = 575 \cdot \text{rpm} \qquad \beta_1 = 40 \cdot \text{deg} \qquad \beta_2 = 60 \cdot \text{deg} \qquad Q = 80000 \cdot \text{gpm} \qquad Q = 178 \cdot \frac{\text{ft}^3}{\text{s}}$$

From continuity

$$V_n = \frac{Q}{2 \cdot \pi \cdot r \cdot b} = V_{rb} \cdot \sin(\beta) \qquad V_{rb} = \frac{V_n}{\sin(\beta)}$$

From geometry

$$V_t = U - V_{rb} \cdot \cos(\beta) = U - \frac{V_n}{\sin(\beta)} \cdot \cos(\beta) = U - \frac{Q}{2 \cdot \pi \cdot r \cdot b} \cdot \cot(\beta)$$

Using given data

$$U_1 = \omega \cdot r_1 \qquad U_1 = 75.3 \cdot \frac{\text{ft}}{\text{s}} \qquad U_2 = \omega \cdot r_2 \qquad U_2 = 226 \cdot \frac{\text{ft}}{\text{s}}$$

$$V_{t1} = U_1 - \frac{Q}{2 \cdot \pi \cdot r_1 \cdot b_1} \cdot \cot(\beta_1) \qquad V_{t1} = 6.94 \cdot \frac{\text{ft}}{\text{s}}$$

$$V_{t2} = U_2 - \frac{Q}{2 \cdot \pi \cdot r_2 \cdot b_2} \cdot \cot(\beta_2) \qquad V_{t2} = 210 \cdot \frac{\text{ft}}{\text{s}}$$

The mass flow rate is

$$m_{rate} = \rho \cdot Q \qquad m_{rate} = 346 \cdot \frac{\text{slug}}{\text{s}}$$

Hence

$$W_m = (U_2 \cdot V_{t2} - U_1 \cdot V_{t1}) \cdot m_{rate} \qquad W_m = 1.62 \times 10^7 \cdot \frac{\text{ft} \cdot \text{lbf}}{\text{s}} \qquad W_m = 2.94 \times 10^4 \cdot \text{hp}$$

The head is

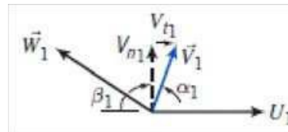
$$H = \frac{W_m}{m_{rate} \cdot g} \qquad H = 1455 \cdot \text{ft}$$

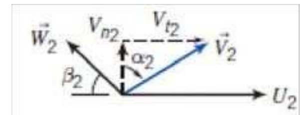
Problem 10.6

[Difficulty: 2]

10.6 Dimensions of a centrifugal pump impeller are

Parameter	Inlet, Section (1)	Outlet, Section (2)
Radius, r (in.)	3	9.75
Blade width, b (in.)	1.5	1.125
Blade angle, β (deg)	60	70





The pump is driven at 1250 rpm while pumping water. Calculate the theoretical head and mechanical power input if the flow rate is 1500 gpm.

Given: Geometry of centrifugal pump

Find: Theoretical head; Power input for given flow rate

Solution:

Basic equations:

$$\dot{W}_m = (U_2 V_{t_2} - U_1 V_{t_1})\dot{m} \qquad \text{(Eq. 10.2b)}$$

$$H = \frac{\dot{W}_m}{\dot{m} g} = \frac{1}{g}(U_2 V_{t_2} - U_1 V_{t_1}) \qquad \text{(Eq. 10.2c)}$$

The given or available data is

$\rho = 1.94 \cdot \frac{\text{slug}}{\text{ft}^3}$ $\quad r_1 = 3 \cdot \text{in}$ $\quad r_2 = 9.75 \cdot \text{in}$ $\quad b_1 = 1.5 \cdot \text{in}$ $\quad b_2 = 1.125 \cdot \text{in}$

$\omega = 1250 \cdot \text{rpm}$ $\quad \beta_1 = 60 \cdot \text{deg}$ $\quad \beta_2 = 70 \cdot \text{deg}$ $\quad Q = 1500 \cdot \text{gpm}$ $\quad Q = 3.34 \cdot \frac{\text{ft}^3}{\text{s}}$

From continuity $\quad V_n = \frac{Q}{2 \cdot \pi \cdot r \cdot b} = V_{rb} \cdot \sin(\beta)$ $\quad V_{rb} = \frac{V_n}{\sin(\beta)}$

From geometry $\quad V_t = U - V_{rb} \cdot \cos(\beta) = U - \frac{V_n}{\sin(\beta)} \cdot \cos(\beta) = U - \frac{Q}{2 \cdot \pi \cdot r \cdot b} \cdot \cot(\beta)$

Using given data $\quad U_1 = \omega \cdot r_1$ $\quad U_1 = 32.7 \cdot \frac{\text{ft}}{\text{s}}$ $\quad U_2 = \omega \cdot r_2$ $\quad U_2 = 106.4 \cdot \frac{\text{ft}}{\text{s}}$

$V_{t1} = U_1 - \frac{Q}{2 \cdot \pi \cdot r_1 \cdot b_1} \cdot \cot(\beta_1)$ $\quad V_{t1} = 22.9 \cdot \frac{\text{ft}}{\text{s}}$

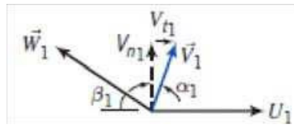
$V_{t2} = U_2 - \frac{Q}{2 \cdot \pi \cdot r_2 \cdot b_2} \cdot \cot(\beta_2)$ $\quad V_{t2} = 104 \cdot \frac{\text{ft}}{\text{s}}$

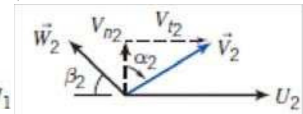
The mass flow rate is $\quad m_{rate} = \rho \cdot Q$ $\quad m_{rate} = 6.48 \cdot \frac{\text{slug}}{\text{s}}$

Hence $\quad W_m = (U_2 \cdot V_{t2} - U_1 \cdot V_{t1}) \cdot m_{rate}$ $\quad W_m = 66728 \cdot \frac{\text{ft} \cdot \text{lbf}}{\text{s}}$ $\quad W_m = 121 \cdot \text{hp}$

The head is $\quad H = \frac{W_m}{m_{rate} \cdot g}$ $\quad H = 320 \cdot \text{ft}$

10.7 For the impeller of Problem 10.6, determine the rotational speed for which the tangential component of the inlet velocity is zero if the volume flow rate is 4000 gpm. Calculate the theoretical head and mechanical power input.





Given: Geometry of centrifugal pump

Find: Rotational speed for zero inlet velocity; Theoretical head; Power input

Solution:

Basic equations:

$$\dot{W}_m = (U_2 V_{t_2} - U_1 V_{t_1})\dot{m} \qquad \text{(Eq. 10.2b)}$$

$$H = \frac{\dot{W}_m}{\dot{m}g} = \frac{1}{g}(U_2 V_{t_2} - U_1 V_{t_1}) \qquad \text{(Eq. 10.2c)}$$

The given or available data is

$$\rho = 1.94 \cdot \frac{\text{slug}}{\text{ft}^3} \qquad r_1 = 3 \cdot \text{in} \qquad r_2 = 9.75 \cdot \text{in} \qquad b_1 = 1.5 \cdot \text{in} \qquad b_2 = 1.125 \cdot \text{in}$$

$$\beta_1 = 60 \cdot \text{deg} \qquad \beta_2 = 70 \cdot \text{deg} \qquad Q = 4000 \cdot \text{gpm} \qquad Q = 8.91 \cdot \frac{\text{ft}^3}{\text{s}}$$

From continuity $\qquad V_n = \dfrac{Q}{2 \cdot \pi \cdot r \cdot b} = V_{rb} \cdot \sin(\beta) \qquad V_{rb} = \dfrac{V_n}{\sin(\beta)}$

From geometry $\qquad V_t = U - V_{rb} \cdot \cos(\beta) = U - \dfrac{V_n}{\sin(\beta)} \cdot \cos(\beta) = U - \dfrac{Q}{2 \cdot \pi \cdot r \cdot b} \cdot \cot(\beta)$

For $V_{t1} = 0$ we get $\qquad U_1 - \dfrac{Q}{2 \cdot \pi \cdot r_1 \cdot b_1} \cdot \cot(\beta_1) = 0 \qquad$ or $\qquad \omega \cdot r_1 - \dfrac{Q}{2 \cdot \pi \cdot r_1 \cdot b_1} \cdot \cot(\beta_1) = 0$

Hence, solving for ω $\qquad \omega = \dfrac{Q}{2 \cdot \pi \cdot r_1^{\ 2} \cdot b_1} \cdot \cot(\beta_1) \qquad \omega = 105 \cdot \dfrac{\text{rad}}{\text{s}} \qquad \omega = 1001 \cdot \text{rpm}$

We can now find U_2 $\qquad U_2 = \omega \cdot r_2 \qquad U_2 = 85.2 \cdot \dfrac{\text{ft}}{\text{s}}$

$$V_{t2} = U_2 - \frac{Q}{2 \cdot \pi \cdot r_2 \cdot b_2} \cdot \cot(\beta_2) \qquad V_{t2} = 78.4 \cdot \frac{\text{ft}}{\text{s}}$$

The mass flow rate is $\qquad m_{rate} = \rho \cdot Q \qquad m_{rate} = 17.3 \cdot \dfrac{\text{slug}}{\text{s}}$

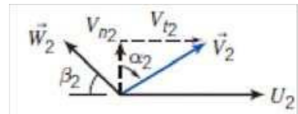
Hence Eq 10.2b becomes $\qquad W_m = U_2 \cdot V_{t2} \cdot m_{rate} \qquad W_m = 1.15 \times 10^5 \cdot \dfrac{\text{ft} \cdot \text{lbf}}{\text{s}} \qquad W_m = 210 \cdot \text{hp}$

The head is $\qquad H = \dfrac{W_m}{m_{rate} \cdot g} \qquad H = 208 \cdot \text{ft}$

Problem 10.8

[Difficulty: 2]

10.8 A centrifugal water pump, with 15 cm diameter impeller and axial inlet flow, is driven at 1750 rpm. The impeller vanes are backward-curved ($\beta_2 = 65°$) and have axial width $b_2 = 2$ cm. For a volume flow rate of 225 m³/hr determine the theoretical head rise and power input to the pump.



Given: Geometry of centrifugal pump

Find: Theoretical head; Power input for given flow rate

Solution:

Basic equations: $$\dot{W}_m = (U_2 V_{t_2} - U_1 V_{t_1})\dot{m} \qquad \text{(Eq. 10.2b)}$$

$$H = \frac{\dot{W}_m}{\dot{m}g} = \frac{1}{g}(U_2 V_{t_2} - U_1 V_{t_1}) \qquad \text{(Eq. 10.2c)}$$

The given or available data is

$\rho = 1000 \cdot \frac{kg}{m^3}$ $\quad r_2 = 7.5 \cdot cm$ $\quad b_2 = 2 \cdot cm$ $\quad \beta_2 = 65 \cdot deg$

$\omega = 1750 \cdot rpm$ $\quad Q = 225 \cdot \frac{m^3}{hr}$ $\quad Q = 0.0625 \frac{m^3}{s}$

From continuity $\quad V_{n2} = \frac{Q}{2 \cdot \pi \cdot r_2 \cdot b_2}$ $\quad V_{n2} = 6.63 \frac{m}{s}$

From geometry $\quad V_{t2} = U_2 - V_{rb2} \cdot \cos(\beta_2) = U_2 - \frac{V_{n2}}{\sin(\beta_2)} \cdot \cos(\beta_2)$

Using given data $\quad U_2 = \omega \cdot r_2$ $\quad U_2 = 13.7 \frac{m}{s}$

Hence $\quad V_{t2} = U_2 - \frac{Q}{2 \cdot \pi \cdot r_2 \cdot b_2} \cdot \cot(\beta_2)$ $\quad V_{t2} = 10.7 \frac{m}{s}$ $\quad V_{t1} = 0$ (axial inlet)

The mass flow rate is $\quad m_{rate} = \rho \cdot Q$ $\quad m_{rate} = 62.5 \frac{kg}{s}$

Hence $\quad W_m = U_2 \cdot V_{t2} \cdot m_{rate}$ $\quad W_m = 9.15 \cdot kW$

The head is $\quad H = \frac{W_m}{m_{rate} \cdot g}$ $\quad H = 14.9 \cdot m$

10.9 For the impeller of Problem 10.1, operating at 750 rpm, determine the volume flow rate for which the tangential component of the inlet velocity is zero. Calculate the theoretical head and mechanical power input.

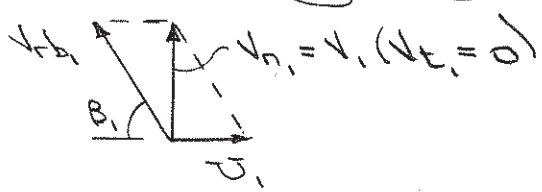
Solution:

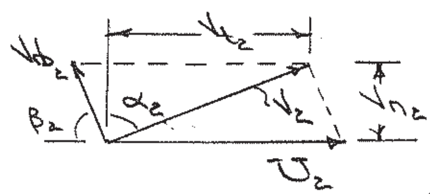
Computing equations: $\dot{W}_m = (U_2 V_{t_2} - \cancel{U_1 V_{t_1}}^{0(3)})\dot{m}$ (10.2b)

$H = \frac{1}{g}(U_2 V_{t_2} - \cancel{U_1 V_{t_1}}^{0(3)})$ (10.2c)

Assume: (1) uniform flow at blade inlet and outlet
(2) flow enters and leaves tangent to the blade
(3) $V_{t_1} = 0$ (given)

Draw velocity diagrams:





From continuity, $V_n = \frac{Q}{2\pi r b} = V_{rb}\sin\beta \quad \therefore V_{rb} = \frac{V_n}{\sin\beta}$

From geometry, $V_t = U - V_{rb}\cos\beta = U - \frac{V_n}{\sin\beta}\cos\beta = U - \frac{Q}{2\pi r b}\cot\beta$

For $V_{t_1} = 0$, then $U_1 - \frac{Q}{2\pi r_1 b_1}\cot\beta_1 = 0$ and $Q = \frac{2\pi r_1 b_1 U_1}{\cot\beta_1}$

Substituting numerical values

$U_1 = \omega_1 r_1 = 750 \frac{\text{rev}}{\text{min}} \times 2\pi \frac{\text{rad}}{\text{rev}} \times \frac{\text{min}}{60\text{s}} \times 0.175\text{ m} = 13.7 \text{ m/s}; \; U_2 = 39.3 \text{ m/s}$

$Q = \frac{2\pi}{\cot 65^\circ} \times 0.175\text{m} \times 0.050\text{m} \times 13.7 \frac{\text{m}}{\text{s}} = 1.62 \text{ m}^3/\text{s}$ ← Q

$V_{t_2} = U_2 - \frac{Q}{2\pi r_2 b_2}\cot\beta_2 = 39.3 \frac{\text{m}}{\text{s}} - 1.62 \frac{\text{m}^3}{\text{s}} \times \frac{1}{2\pi} \times \frac{1}{0.50\text{m}} \times \frac{\cot 70^\circ}{0.03\text{m}} = 33.0 \text{ m/s}$

$H = \frac{1}{g} U_2 V_{t_2} = \frac{\text{s}^2}{9.81\text{m}} \times 39.3 \frac{\text{m}}{\text{s}} \times 33.0 \frac{\text{m}}{\text{s}} = 132 \text{ m}$ ← H

$\dot{W}_m = \dot{m} U_2 V_{t_2} = \rho Q g H = 999 \frac{\text{kg}}{\text{m}^3} \times 1.62 \frac{\text{m}^3}{\text{s}} \times 9.81 \frac{\text{m}}{\text{s}^2} \times 132\text{m} \times \frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}} \times \frac{\text{W}\cdot\text{s}}{\text{N}\cdot\text{m}}$

$\dot{W}_m = 2{,}100 \text{ kW}$ ←

Problem 10.10

[Difficulty: 2]

10.10 Consider the geometry of the idealized centrifugal pump described in Problem 10.11. Draw inlet and outlet velocity diagrams assuming b = constant. Calculate the inlet blade angles required for "shockless" entry flow at the design flow rate. Evaluate the theoretical power input to the pump at the design flow rate.

Given: Geometry of centrifugal pump

Find: Draw inlet and exit velocity diagrams; Inlet blade angle; Power

Solution:

Basic equations: $\dot{W}_m = (U_2 V_{t_2} - U_1 V_{t_1})\dot{m}$ $\qquad V_n = \dfrac{Q}{2\cdot\pi\cdot r\cdot b}$

The given or available data is

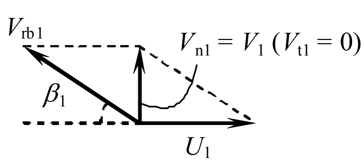
$R_1 = 1\cdot in$ $\qquad R_2 = 7.5\cdot in$ $\qquad b_2 = 0.375\cdot in$ $\qquad \omega = 2000\,rpm$

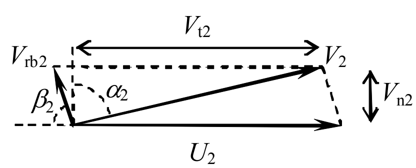
$\rho = 1.94\cdot\dfrac{slug}{ft^3}$ $\qquad Q = 800\cdot gpm$ $\qquad Q = 1.8\cdot\dfrac{ft^3}{s}$ $\qquad \beta_2 = 75\cdot deg$

$U_1 = \omega\cdot R_1$ $\qquad U_1 = 17.5\cdot\dfrac{ft}{s}$ $\qquad U_2 = \omega\cdot R_2$ $\qquad U_2 = 131\cdot\dfrac{ft}{s}$

$V_{n2} = \dfrac{Q}{2\cdot\pi\cdot R_2\cdot b_2}$ $\qquad V_{n2} = 14.5\cdot\dfrac{ft}{s}$ $\qquad V_{n1} = \dfrac{R_2}{R_1}\cdot V_{n2}$ $\qquad V_{n1} = 109\cdot\dfrac{ft}{s}$

Velocity diagrams:





Then $\qquad \beta_1 = \operatorname{atan}\left(\dfrac{V_{n1}}{U_1}\right)$ $\qquad \beta_1 = 80.9\cdot deg$ $\qquad$ (Essentially radial entry)

From geometry $\qquad V_{t1} = U_1 - V_{n1}\cdot\cos(\beta_1)$ $\qquad V_{t1} = 0.2198\cdot\dfrac{ft}{s}$ $\qquad V_{t2} = U_2 - V_{n2}\cdot\cos(\beta_2)$ $\qquad V_{t2} = 127.1\cdot\dfrac{ft}{s}$

Then $\qquad W_m = (U_2\cdot V_{t2} - U_1\cdot V_{t1})\cdot\rho\cdot Q$ $\qquad W_m = 5.75\times 10^4\cdot\dfrac{ft\cdot lbf}{s}$ $\qquad W_m = 105\cdot hp$

Problem 10.11

[Difficulty: 3]

10.11 Consider a centrifugal water pump whose geometry and flow conditions are as follows:

Impeller inlet radius, R_1	2.5 cm
Impeller outlet radius, R_2	18 cm
Impeller outlet width, b_2	1 cm
Design speed, N	1800 rpm
Design flow rate, Q	30 m^3/min
Backward-curved vanes (outlet blade angle), β_2	75°
Required flow rate range	50–150% of design

Assume ideal pump behavior with 100 percent efficiency. Find the shutoff head. Calculate the absolute and relative discharge velocities, the total head, and the theoretical power required at the design flow rate.

Given: Geometry of centrifugal pump

Find: Shutoff head; Absolute and relative exit velocitiesTheoretical head; Power input

Solution:

Basic equations:

$$\dot{W}_m = (U_2 V_{t_2} - U_1 V_{t_1})\dot{m} \qquad \text{(Eq. 10.2b)}$$

$$H = \frac{\dot{W}_m}{\dot{m}g} = \frac{1}{g}(U_2 V_{t_2} - U_1 V_{t_1}) \qquad \text{(Eq. 10.2c)}$$

The given or available data is

$\rho = 999 \cdot \frac{kg}{m^3}$ $\quad R_1 = 2.5 \cdot cm$ $\quad R_2 = 18 \cdot cm$ $\quad b_2 = 1 \cdot cm$

$\omega = 1800 \cdot rpm$ $\quad \beta_2 = 75 \cdot deg$ $\quad Q = 30 \cdot \frac{m^3}{min}$ $\quad Q = 0.500 \cdot \frac{m^3}{s}$

At the exit $\quad U_2 = \omega \cdot R_2 \quad U_2 = 33.9 \cdot \frac{m}{s}$

At shutoff $\quad V_{t2} = U_2 \quad V_{t2} = 33.9 \cdot \frac{m}{s} \quad H_0 = \frac{1}{g} \cdot (U_2 \cdot V_{t2}) \quad H_0 = 117 \cdot m$

At design. from continuity $\quad V_{n2} = \frac{Q}{2 \cdot \pi \cdot R_2 \cdot b_2} \quad V_{n2} = 44.2 \cdot \frac{m}{s}$

From the velocity diagram $\quad V_{n2} = w_2 \cdot \sin(\beta_2) \quad w_2 = \frac{V_{n2}}{\sin(\beta_2)} \quad w_2 = 45.8 \frac{m}{s}$

$V_{t2} = U_2 - V_{n2} \cdot \cot(\beta_2) \quad V_{t2} = 22.1 \frac{m}{s}$

Hence we obtain $\quad V_2 = \sqrt{V_{n2}^2 + V_{t2}^2} \quad V_2 = 49.4 \frac{m}{s}$

with $\quad \alpha_2 = atan\left(\frac{V_{t2}}{V_{n2}}\right) \quad \alpha_2 = 26.5 \cdot deg$

For $V_{t1} = 0$ we get $\quad W_m = U_2 \cdot V_{t2} \cdot \rho \cdot Q = 374\,kW \quad H = \frac{W_m}{\rho \cdot Q \cdot g} = 76.4\,m$

Problem 10.12

[Difficulty: 3]

10.12 Consider the centrifugal pump impeller dimensions given in Example 10.1. Construct the velocity diagram for shockless flow at the impeller inlet, if b = constant. Calculate the effective flow angle with respect to the radial impeller blades for the case of no inlet swirl. Investigate the effects on flow angle of (a) variations in impeller width and (b) inlet swirl velocities.

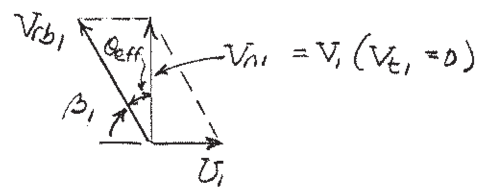
Solution: $Q = 150 \frac{gal}{min} \times \frac{ft^3}{7.48\,gal} \times \frac{min}{60\,s} = 0.334\ ft^3/sec$; $r_1 = 0.0521\ ft$

$b = 0.0319\ ft$; $\omega = 3450 \frac{rev}{min} \times 2\pi \frac{rad}{rev} \times \frac{min}{60\,s} = 361\ rad/s$

From continuity, $V_{n1} = \frac{Q}{2\pi r_1 b_1} = \frac{1}{2\pi} \times 0.334 \frac{ft^3}{s} \times \frac{1}{0.0521\,ft} \times \frac{1}{0.0319\,ft} = 32.0\ ft/s$

$U_1 = \omega r_1 = 361 \frac{rad}{s} \times 0.0521\ ft = 18.8\ ft/s$

$\beta_1 = \tan^{-1} \frac{V_{n1}}{U_1} = \tan^{-1}\left(\frac{32.0}{18.8}\right) = 59.6°$



Thus for radial vanes,

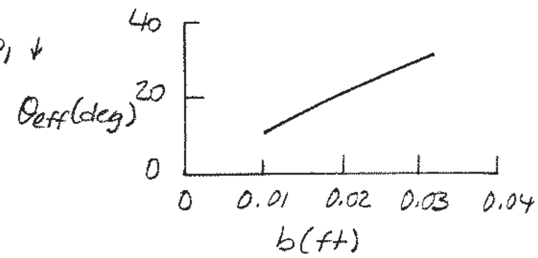
$\theta_{eff} = \frac{\pi}{2} - \beta_1 = 90° - 59.6° = 30.4°$

θ_{eff}

To change θ_{eff} : (a) vary b with no inlet swirl: $V_1 = V_{n1} = \frac{Q}{2\pi r_1 b_1}$

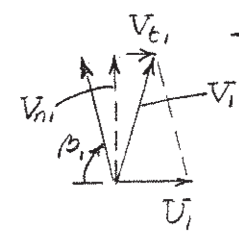
$\beta_1 = \tan^{-1} \frac{Q}{2\pi r_1 b_1 U_1}$ so $\beta_1 \uparrow$ as $b_1 \downarrow$

$\theta_{eff} = 90° - \beta_1$



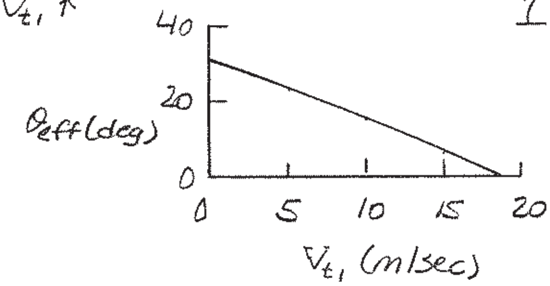
b

(b) Vary inlet swirl (V_{t1}) with $b = 0.0319$ ft:



$\beta_1 = \tan^{-1} \frac{V_{n1}}{U_1 - V_{t1}}$ so $\beta_1 \uparrow$ as $V_{t1} \uparrow$

$\theta_{eff} = 90° - \beta_1$



V_{t1}

Problem 10.13

[Difficulty: 2]

10.13 For the impeller of Problem 10.5, determine the inlet blade angle for which the tangential component of the inlet velocity is zero if the volume flow rate is 125,000 gpm. Calculate the theoretical head and mechanical power input.

Given: Geometry of centrifugal pump

Find: Inlet blade angle for no tangential inlet velocity at 125,000 gpm; Head; Power

Solution:

Basic equations: $\dot{W}_m = (U_2 V_{t_2} - U_1 V_{t_1})\dot{m}$ (Eq. 10.2b)

$$H = \frac{\dot{W}_m}{\dot{m}g} = \frac{1}{g}(U_2 V_{t_2} - U_1 V_{t_1})$$ (Eq. 10.2c)

The given or available data is

$\rho = 1.94 \cdot \frac{\text{slug}}{\text{ft}^3}$ $\quad r_1 = 15 \cdot \text{in}$ $\quad r_2 = 45 \cdot \text{in}$ $\quad b_1 = 4.75 \cdot \text{in}$ $\quad b_2 = 3.25 \cdot \text{in}$

$\omega = 575 \cdot \text{rpm}$ $\quad \beta_2 = 60 \cdot \text{deg}$ $\quad Q = 125000 \cdot \text{gpm}$ $\quad Q = 279 \cdot \frac{\text{ft}^3}{\text{s}}$

From continuity $\quad V_n = \frac{Q}{2 \cdot \pi \cdot r \cdot b} = V_{rb} \cdot \sin(\beta)$ $\quad V_{rb} = \frac{V_n}{\sin(\beta)}$

From geometry $\quad V_t = U - V_{rb} \cdot \cos(\beta) = U - \frac{V_n}{\sin(\beta)} \cdot \cos(\beta) = U - \frac{Q}{2 \cdot \pi \cdot r \cdot b} \cdot \cot(\beta)$

For $V_{t1} = 0$ we obtain $\quad U_1 - \frac{Q}{2 \cdot \pi \cdot r_1 \cdot b_1} \cdot \cot(\beta_1) = 0$ $\quad$ or $\quad \cot(\beta_1) = \frac{2 \cdot \pi \cdot r_1 \cdot b_1 \cdot U_1}{Q}$

Using given data $\quad U_1 = \omega \cdot r_1$ $\quad U_1 = 75.3 \cdot \frac{\text{ft}}{\text{s}}$

Hence $\quad \beta_1 = \text{acot}\left(\frac{2 \cdot \pi \cdot r_1 \cdot b_1 \cdot U_1}{Q}\right)$ $\quad \beta_1 = 50 \cdot \text{deg}$

Also $\quad U_2 = \omega \cdot r_2$ $\quad U_2 = 226 \cdot \frac{\text{ft}}{\text{s}}$

$V_{t2} = U_2 - \frac{Q}{2 \cdot \pi \cdot r_2 \cdot b_2} \cdot \cot(\beta_2)$ $\quad V_{t2} = 201 \cdot \frac{\text{ft}}{\text{s}}$

The mass flow rate is $\quad m_{rate} = \rho \cdot Q$ $\quad m_{rate} = 540 \cdot \frac{\text{slug}}{\text{s}}$

Hence $\quad W_m = (U_2 \cdot V_{t2} - U_1 \cdot V_{t1}) \cdot m_{rate}$ $\quad W_m = 2.45 \times 10^7 \cdot \frac{\text{ft} \cdot \text{lbf}}{\text{s}}$ $\quad W_m = 44497 \cdot \text{hp}$

The head is $\quad H = \frac{W_m}{m_{rate} \cdot g}$ $\quad H = 1408 \cdot \text{ft}$

10.14 A centrifugal water pump designed to operate at 1300 rpm has dimensions

Parameter	Inlet	Outlet
Radius, r (mm)	100	175
Blade width, b (mm)	10	7.5
Blade angle, β (deg)		40

Draw the inlet velocity diagram for a volume flow rate of 35 L/s. Determine the inlet blade angle for which the entering velocity has no tangential component. Draw the outlet velocity diagram. Determine the outlet absolute flow angle (measured relative to the normal direction). Evaluate the hydraulic power delivered by the pump, if its efficiency is 75 percent. Determine the head developed by the pump.

Solution: Apply continuity and the Euler turbomachine equation.

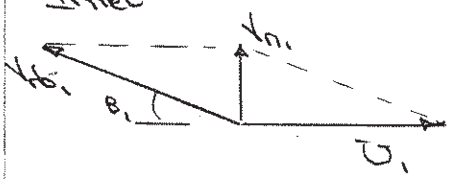
Computing equations: $V_n = \dfrac{Q}{2\pi r b}$ $\quad \dot{W}_m = \rho Q(U_2 V_{t_2} - U_1 V_{t_1})$

$$\omega = 1300 \frac{\text{rev}}{\text{min}} \times 2\pi \frac{\text{rad}}{\text{rev}} \times \frac{\text{min}}{60\,\text{s}} = 136 \text{ rad/s} \;;\; U_1 = 13.6 \text{ m/s} \;,\; U_2 = 23.8 \text{ m/s}$$

$$V_{n_1} = \frac{Q}{2\pi r_1 b_1} = \frac{1}{2\pi} \times 35 \frac{\text{L}}{\text{s}} \times \frac{\text{m}^3}{10^3\,\text{L}} \times \frac{1}{0.1\,\text{m}} \times \frac{1}{0.01\,\text{m}} = 5.57 \text{ m/s}.$$

$$V_{n_2} = \frac{r_1 b_1}{r_2 b_2} V_{n_1} = \frac{100}{175} \times \frac{10}{7.5} \times 5.57 \text{ m/s} = 4.24 \text{ m/s}.$$

Inlet

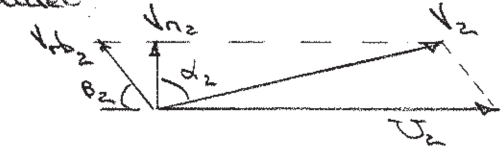


$$\tan\beta_1 = \frac{V_{n_1}}{U_1}$$

$$\beta_1 = \tan^{-1}\left(\frac{5.57}{13.6}\right)$$

$$\beta_1 = 22.3^\circ \qquad \beta_1$$

Outlet



From the outlet diagram, $V_{t_2} = U_2 - V_{n_2}\cot\beta_2 = 23.8 \frac{\text{m}}{\text{s}} - 4.24 \frac{\text{m}}{\text{s}} \times \frac{1}{\tan 40^\circ}$

$$V_{t_2} = 18.8 \text{ m/s}$$

$$\alpha_2 = \tan^{-1} \frac{V_{t_2}}{V_{n_2}} = \tan^{-1}\left(\frac{18.8}{4.24}\right) = 77.3^\circ \qquad \alpha_2$$

$$\dot{W}_m = \rho Q(U_2 V_{t_2} - U_1 V_{t_1}) = 999 \frac{\text{kg}}{\text{m}^3} \times 35 \frac{\text{L}}{\text{s}} \times \frac{\text{m}^3}{10^3\,\text{L}} \left[23.8 \frac{\text{m}}{\text{s}} \times 18.8 \frac{\text{m}}{\text{s}} - 0\right] \frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}} \times 0.75$$

$$\dot{W}_m = 11.7 \text{ kW} \qquad \dot{W}_m$$

$$H = \frac{\dot{W}_m}{\rho g Q} = \frac{U_2 V_{t_2} - U_1 V_{t_1}}{g} = 23.8 \frac{\text{m}}{\text{s}} \times 18.8 \frac{\text{m}}{\text{s}} \times \frac{\text{s}^2}{9.81\,\text{m}} \times 0.75 = 34.2 \text{ m} \qquad H$$

Problem 10.15

[Difficulty: 3]

10.15 A centrifugal pump runs at 1750 rpm while pumping water at a rate of 50 L/s. The water enters axially, and leaves tangential to the impeller blades. The impeller exit diameter and width are 300 mm and 10 mm, respectively. If the pump requires 45 kW, and is 75 percent efficient, estimate the exit angle of the impeller blades.

Given: Data on a centrifugal pump

Find: Estimate exit angle of impeller blades

Solution:

The given or available data is $\rho = 999 \cdot \frac{kg}{m^3}$ $\quad Q = 50 \cdot \frac{L}{s}$ $\quad W_{in} = 45 \cdot kW$ $\quad \eta = 75 \cdot \%$

$\omega = 1750 \cdot rpm$ $\quad b_2 = 10 \cdot mm$ $\quad D = 300 \cdot mm$

The governing equation (derived directly from the Euler turbomachine equation) is

$$\dot{W}_m = (U_2 V_{t_2} - U_1 V_{t_1})\dot{m} \qquad (10.2b)$$

For an axial inlet $\quad V_{t1} = 0 \quad$ hence $\quad V_{t2} = \frac{W_m}{U_2 \cdot \rho \cdot Q}$

We have $\quad U_2 = \frac{D}{2} \cdot \omega \quad U_2 = 27.5 \frac{m}{s} \quad$ and $\quad W_m = \eta \cdot W_{in} \quad W_m = 33.8 \cdot kW$

Hence $\quad V_{t2} = \frac{W_m}{U_2 \cdot \rho \cdot Q} \quad V_{t2} = 24.6 \frac{m}{s}$

From continuity $\quad V_{n2} = \frac{Q}{\pi \cdot D \cdot b_2} \quad V_{n2} = 5.31 \frac{m}{s}$

With the exit velocities determined, β can be determined from exit geometry

$$\tan(\beta) = \frac{V_{n2}}{U_2 - V_{t2}} \quad \text{or} \quad \beta = \operatorname{atan}\left(\frac{V_{n2}}{U_2 - V_{t2}}\right) \quad \beta = 61.3 \cdot deg$$

Problem 10.16

[Difficulty: 3]

10.16 A centrifugal water pump designed to operate at 1200 rpm has dimensions

Parameter	Inlet	Outlet
Radius, *r* (mm)	90	150
Blade width, *b* (mm)	10	7.5
Blade angle, β (deg)	25	45

Determine the flow rate at which the entering velocity has no tangential component. Draw the outlet velocity diagram, and determine the outlet absolute flow angle (measured relative to the normal direction) at this flow rate. Evaluate the hydraulic power delivered by the pump if its efficiency is 70 percent. Determine the head developed by the pump.

Given: Data on a centrifugal pump

Find: Flow rate for zero inlet tangential velocity; outlet flow angle; power; head developed

Solution:

The given or available data is $\rho = 999 \cdot \frac{kg}{m^3}$ $\quad \omega = 1200 \cdot rpm$ $\quad \eta = 70 \cdot \%$

$r_1 = 90 \cdot mm \quad b_1 = 10 \cdot mm \quad \beta_1 = 25 \cdot deg \quad r_2 = 150 \cdot mm \quad b_2 = 7.5 \cdot mm \quad \beta_2 = 45 \cdot deg$

The governing equations (derived directly from the Euler turbomachine equation) are

$$\dot{W}_m = (U_2 V_{t_2} - U_1 V_{t_1})\dot{m} \qquad (10.2b)$$

$$H = \frac{\dot{W}_m}{\dot{m} g} = \frac{1}{g}(U_2 V_{t_2} - U_1 V_{t_1}) \qquad (10.2c)$$

We also have from geometry $\quad \alpha_2 = \operatorname{atan}\left(\frac{V_{t2}}{V_{n2}}\right) \qquad (1)$

From geometry $\quad V_{t1} = 0 = U_1 - V_{rb1} \cdot \cos(\beta_1) = r_1 \cdot \omega \cdot -\frac{V_{n1}}{\sin(\beta_1)} \cdot \cos(\beta_1)$

and from continuity $\quad V_{n1} = \frac{Q}{2 \cdot \pi \cdot r_1 \cdot b_1}$

Hence $\quad r_1 \cdot \omega - \frac{Q}{2 \cdot \pi \cdot r_1 \cdot b_1 \cdot \tan(\beta_1)} = 0 \qquad Q = 2 \cdot \pi \cdot r_1^{\,2} \cdot b_1 \cdot \omega \cdot \tan(\beta_1) \qquad Q = 29.8 \cdot \frac{L}{s} \qquad Q = 0.0298 \frac{m^3}{s}$

The power, head and absolute angle α at the exit are obtained from direct computation using Eqs. 10.2b, 10.2c, and 1 above

$U_1 = r_1 \cdot \omega \qquad U_1 = 11.3 \frac{m}{s} \qquad U_2 = r_2 \cdot \omega \qquad U_2 = 18.8 \frac{m}{s} \qquad V_{t1} = 0 \cdot \frac{m}{s}$

From geometry $\quad V_{t2} = U_2 - V_{rb2} \cdot \cos(\beta_2) = r_2 \cdot \omega \cdot -\frac{V_{n2}}{\sin(\beta_2)} \cdot \cos(\beta_2)$

and from continuity $\quad V_{n2} = \frac{Q}{2 \cdot \pi \cdot r_2 \cdot b_2} \qquad V_{n2} = 4.22 \frac{m}{s}$

Hence $V_{t2} = r_2 \cdot \omega - \dfrac{V_{n2}}{\tan(\beta_2)}$ $\qquad V_{t2} = 14.6 \dfrac{m}{s}$

Using these results in Eq. 1 $\alpha_2 = \mathrm{atan}\left(\dfrac{V_{t2}}{V_{n2}}\right)$ $\qquad \alpha_2 = 73.9 \cdot deg$

Using them in Eq. 10.2b $W_m = (U_2 \cdot V_{t2} - U_1 \cdot V_{t1}) \cdot \rho \cdot Q$ $\qquad W_m = 8.22 \cdot kW$

Using them in Eq. 10.2c $H = \dfrac{1}{g} \cdot (U_2 \cdot V_{t2} - U_1 \cdot V_{t1})$ $\qquad H = 28.1\, m$

This is the power and head assuming no inefficiency; with $\eta = 70\%$, we have (from Eq. 10.4c)

$$W_h = \eta \cdot W_m \qquad W_h = 5.75 \cdot kW$$

$$H_p = \eta \cdot H \qquad H_p = 19.7\, m$$

(This last result can also be obtained from Eq. 10.4a $W_h = \rho \cdot Q \cdot g \cdot H_p$)

Problem 10.17

[Difficulty: 1]

10.17 Repeat the analysis for determining the optimum speed for an impulse turbine of Example 10.13, using the Euler turbomachine equation.

Given: Impulse turbibe

Find: Optimum speed using the Euler turbomachine equation

Solution:

The governing equation is the Euler turbomachine equation

$$T_{shaft} = (r_2 V_{t_2} - r_1 V_{t_1})\dot{m} \qquad (10.1c)$$

In terms of the notation of Example 10.13, for a stationary CV

$r_1 = r_2 = R$ $\quad U_1 = U_2 = U$ $\quad V_{t1} = V - U$ $\quad V_{t2} = (V - U)\cdot\cos(\theta)$ and $\quad m_{flow} = \rho\cdot Q$

Hence $\quad T_{shaft} = [R\cdot(V - U)\cdot\cos(\theta) - R\cdot(V - U)]\cdot\rho\cdot Q \qquad T_{out} = T_{shaft} = \rho\cdot Q\cdot R\cdot(V - U)\cdot(1 - \cos(\theta))$

The power is $\quad W_{out} = \omega\cdot T_{out} = \rho\cdot Q\cdot R\cdot\omega\cdot(V - U)\cdot(1 - \cos(\theta)) \qquad W_{out} = \rho\cdot Q\cdot U\cdot(V - U)\cdot(1 - \cos(\theta))$

These results are identical to those of Example 10.13. The proof that maximum power is when $U = V/2$ is hence also the same and will not be repeated here.

Problem 10.18

[Difficulty: 1]

10.18 Kerosene is pumped by a centrifugal pump. When the flow rate is 350 gpm, the pump requires 18 hp input, and its efficiency is 82 percent. Calculate the pressure rise produced by the pump. Express this result as (a) feet of water and (b) feet of kerosene.

Given: Data on centrifugal pump

Find: Pressure rise; Express as ft of water and kerosene

Solution:

Basic equations: $\eta = \dfrac{\rho \cdot Q \cdot g \cdot H}{W_m}$

The given or available data is $\rho_w = 1.94 \cdot \dfrac{slug}{ft^3}$ $\quad Q = 350 \cdot gpm \quad Q = 0.780 \cdot \dfrac{ft^3}{s}$

$W_m = 18 \cdot hp \quad \eta = 82 \cdot \%$

Solving for H $\quad H = \dfrac{\eta \cdot W_m}{\rho_w \cdot Q \cdot g}$ $\quad H = 166.8 \cdot ft$

For kerosene, from Table A.2 $\quad SG = 0.82 \quad H_k = \dfrac{\eta \cdot W_m}{SG \cdot \rho_w \cdot Q \cdot g}$ $\quad H_k = 203 \cdot ft$

Problem 10.19

[Difficulty: 3]

10.19 A centrifugal pump designed to deliver water at 70 cfm has dimensions

Parameter	Inlet	Outlet
Radius, r (in.)	14	7
Blade width, b (in.)	0.4	0.3
Blade angle, β (°)	20	45

Draw the inlet velocity diagram. Determine the design speed if the entering velocity has no tangential component. Draw the outlet velocity diagram. Determine the outlet absolute flow angle (measured relative to the normal direction). Evaluate the theoretical head developed by the pump. Estimate the minimum mechanical power delivered to the pump.

Given: Geometry of centrifugal pump

Find: Draw inlet velocity diagram; Design speed for no inlet tangential velocity; Outlet angle; Head; Power

Solution:

Basic equations: $\dot{W}_m = (U_2 V_{t_2} - U_1 V_{t_1})\dot{m}$ (Eq. 10.2b)

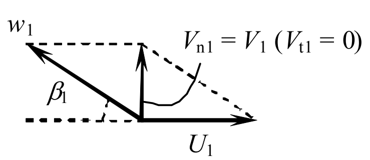
$$H = \frac{\dot{W}_m}{\dot{m}g} = \frac{1}{g}(U_2 V_{t_2} - U_1 V_{t_1}) \quad \text{(Eq. 10.2c)}$$

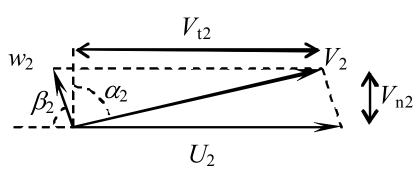
The given or available data is

$r_1 = 4 \cdot in$ $\quad r_2 = 7 \cdot in$ $\quad b_1 = 0.4 \cdot in$ $\quad b_2 = 0.3 \cdot in$ $\quad \beta_1 = 20 \cdot deg$ $\quad \beta_2 = 45 \cdot deg$

$\rho = 1.94 \cdot \frac{slug}{ft^3}$ $\quad Q = 70 \cdot cfm$ $\quad Q = 1.167 \cdot \frac{ft^3}{s}$

Velocity diagrams:





From continuity $\quad V_n = \frac{Q}{2 \cdot \pi \cdot r \cdot b} = w \cdot \sin(\beta)$ $\quad w = \frac{V_n}{\sin(\beta)}$ $\quad \frac{V_{n1}}{V_{n2}} = \frac{A_2}{A_1} = \frac{r_2 \cdot b_2}{r_1 \cdot b_1}$

From geometry $\quad V_t = U - V_{rb} \cdot \cos(\beta) = U - \frac{V_n}{\sin(\beta)} \cdot \cos(\beta) = U - \frac{Q}{2 \cdot \pi \cdot r \cdot b} \cdot \cot(\beta)$

For $V_{t1} = 0$ we obtain $\quad U_1 - \frac{Q}{2 \cdot \pi \cdot r_1 \cdot b_1} \cdot \cot(\beta_1) = 0$ $\quad$ or $\quad \omega \cdot r_1 - \frac{Q}{2 \cdot \pi \cdot r_1 \cdot b_1} \cdot \cot(\beta_1) = 0$

Solving for ω $\quad \omega = \frac{Q}{2 \cdot \pi \cdot r_1^{\,2} \cdot b_1} \cdot \cot(\beta_1)$ $\quad \omega = 138 \cdot \frac{rad}{s}$ $\quad \omega = 1315 \cdot rpm$

Hence $\quad U_1 = \omega \cdot r_1$ $\quad U_1 = 45.9 \cdot \frac{ft}{s}$ $\quad U_2 = \omega \cdot r_2$ $\quad U_2 = 80.3 \cdot \frac{ft}{s}$

$$V_{n2} = \frac{Q}{2\cdot\pi\cdot r_2\cdot b_2} \qquad V_{n2} = 12.73\cdot\frac{ft}{s} \qquad V_{t2} = U_2 - \frac{Q}{2\cdot\pi\cdot r_2\cdot b_2}\cdot\cot(\beta_2) \qquad V_{t2} = 67.6\cdot\frac{ft}{s}$$

From the sketch $\alpha_2 = \operatorname{atan}\left(\frac{V_{t2}}{V_{n2}}\right)$ $\qquad \alpha_2 = 79.3\cdot deg$

Hence $W_m = U_2\cdot V_{t2}\cdot\rho\cdot Q \qquad W_m = 1.230\times 10^4\cdot\frac{ft\cdot lbf}{s} \qquad W_m = 22.4\cdot hp$

The head is $H = \frac{W_m}{\rho\cdot Q\cdot g} \qquad H = 169\cdot ft$

Problem 10.20

[Difficulty: 4]

10.20 In the water pump of Problem 10.8, the pump casing acts as a diffuser, which converts 60 percent of the absolute velocity head at the impeller outlet to static pressure rise. The head loss through the pump suction and discharge channels is 0.75 times the radial component of velocity head leaving the impeller. Estimate the volume flow rate, head rise, power input, and pump efficiency at the maximum efficiency point. Assume the torque to overcome bearing, seal, and spin losses is 10 percent of the ideal torque at $Q = 0.065\ m^3/s$.

Given: Geometry of centrifugal pump with diffuser casing

Find: Flow rate; Theoretical head; Power; Pump efficiency at maximum efficiency point

Solution:

Basic equations:

$$\dot{W}_m = (U_2 V_{t_2} - U_1 V_{t_1})\dot{m} \qquad \text{(Eq. 10.2b)}$$

$$H = \frac{\dot{W}_m}{\dot{m}g} = \frac{1}{g}(U_2 V_{t_2} - U_1 V_{t_1}) \qquad \text{(Eq. 10.2c)}$$

The given or available data is

$$\rho = 1000\frac{kg}{m^3} \qquad r_2 = 7.5\cdot cm \qquad b_2 = 2\cdot cm \qquad \beta_2 = 65\cdot deg$$

$$\omega = 1750\cdot rpm \qquad \omega = 183\cdot\frac{rad}{s}$$

Using given data $\quad U_2 = \omega\cdot r_2 \qquad U_2 = 13.7\frac{m}{s}$

Illustrate the procedure with $\quad Q = 0.065\cdot\frac{m^3}{s}$

From continuity $\quad V_{n2} = \dfrac{Q}{2\cdot\pi\cdot r_2\cdot b_2} \qquad V_{n2} = 6.9\frac{m}{s}$

From geometry $\quad V_{t2} = U_2 - V_{rb2}\cdot\cos(\beta_2) = U_2 - \dfrac{V_{n2}}{\sin(\beta_2)}\cdot\cos(\beta_2)$

Hence $\quad V_{t2} = U_2 - \dfrac{Q}{2\cdot\pi\cdot r_2\cdot b_2}\cdot\cot(\beta_2) \qquad V_{t2} = 10.5\frac{m}{s} \qquad V_{t1} = 0 \quad$ (axial inlet)

$$V_2 = \sqrt{V_{n2}^2 + V_{t2}^2} \qquad V_2 = 12.6\frac{m}{s}$$

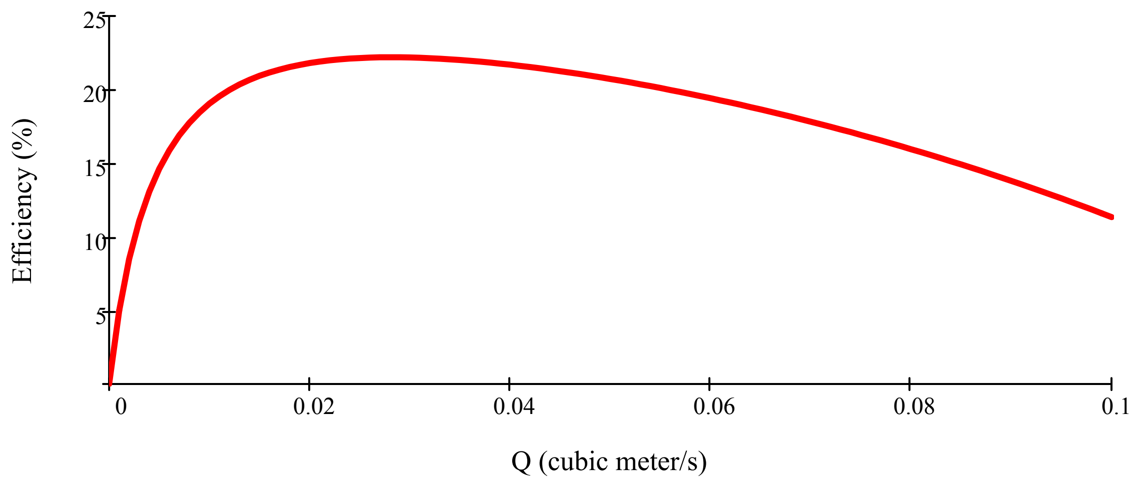
$$H_{ideal} = \frac{U_2\cdot V_{t2}}{g} \qquad H_{ideal} = 14.8\cdot m$$

$$T_{friction} = 10\cdot\%\cdot\frac{W_{mideal}}{\omega} = 10\cdot\%\cdot\frac{\rho\cdot Q\cdot H_{ideal}}{\omega}$$

$$T_{friction} = 10\cdot\%\cdot\frac{Q\cdot\rho\cdot g\cdot H_{ideal}}{\omega} \qquad T_{friction} = 5.13\cdot N\cdot m$$

$$H_{actual} = 60\cdot\%\cdot\frac{V_2^{\,2}}{2\cdot g} - 0.75\cdot\frac{V_{n2}^{\,2}}{2\cdot g} \qquad H_{actual} = 3.03\,m$$

$$\eta = \frac{Q\cdot\rho\cdot g\cdot H_{actual}}{Q\cdot\rho\cdot g\cdot H_{ideal} + \omega\cdot T_{friction}} \qquad \eta = 18.7\cdot\%$$



The above graph can be plotted in Excel. In addition, Solver can be used to vary Q to maximize η. The results are

$$Q = 0.0282\cdot\frac{m^3}{s} \qquad \eta = 22.2\cdot\% \qquad H_{ideal} = 17.3\,m \qquad H_{actual} = 4.60\,m$$

$$W_m = Q\cdot\rho\cdot g\cdot H_{ideal} + \omega\cdot T_{friction} \qquad W_m = 5.72\cdot kW$$

Problem 10.21

[Difficulty: 4]

10.21 The theoretical head delivered by a centrifugal pump at shutoff depends on the discharge radius and angular speed of the impeller. For preliminary design, it is useful to have a plot showing the theoretical shutoff characteristics and approximating the actual performance. Prepare a log-log plot of impeller radius versus theoretical head rise at shutoff with standard motor speeds as parameters. Assume the fluid is water and the actual head at the design flow rate is 70 percent of the theoretical shutoff head. (Show these as dashed lines on the plot.) Explain how this plot might be used for preliminary design.

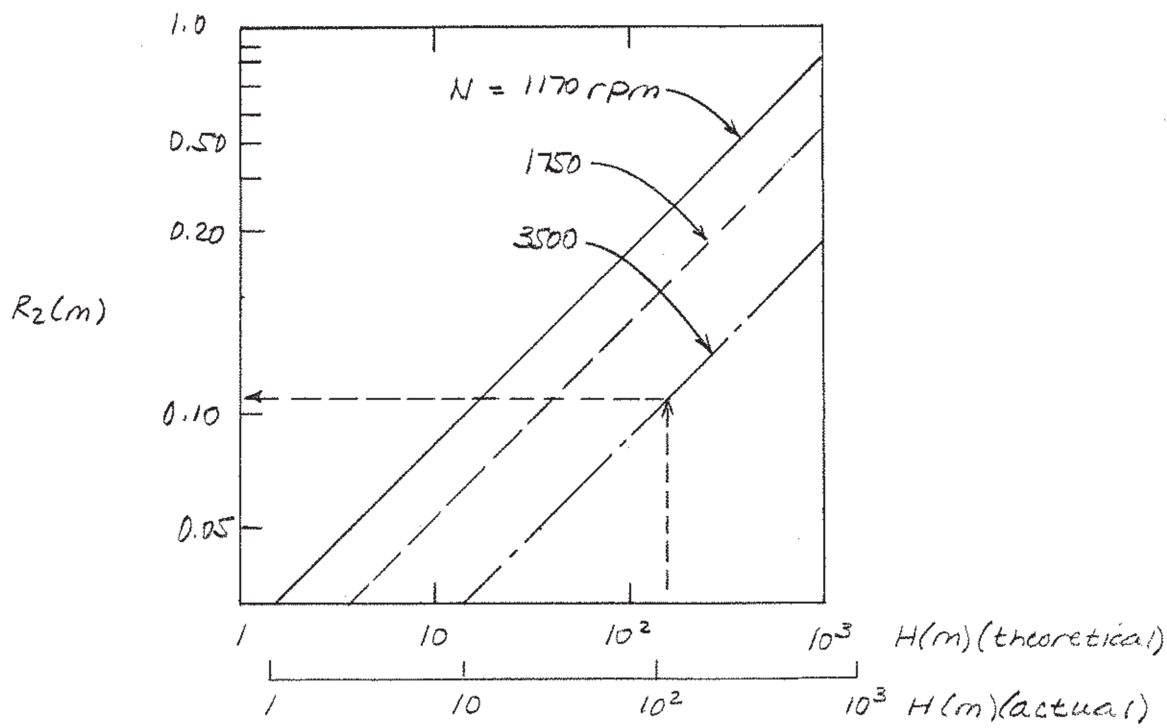
Solution: Apply the Euler turbomachine equation.

Computing equation: $H = \frac{1}{g}(U_2 V_{t2} - U_1 \cancel{V_{t1}})$ $\approx 0 (2)$

Assumptions: (1) No through flow, (2) Neglect V_{t1}

Then $H = \frac{1}{g}(\omega R_2 \, \omega R_2) = \frac{\omega^2 R_2^2}{g}$ or $\log H = 2\log\omega + 2\log R_2 - \log g$

These will be straight lines on a plot of $\log R_2$ vs. $\log H$ (at constant ω):



For a given application, enter the abscissa with the desired head, move up to the desired driver speed, then move left to the ordinate and read the required impeller radius. The example (--- line) illustrates.

Problem 10.22

[Difficulty: 2]

10.22 Use data from Appendix D to choose points from the performance curves for a Peerless horizontal split case Type 16A18B pump at 705 and 880 nominal rpm. Obtain and plot curve-fits of total head versus delivery for this pump, with an 18.0-in.-diameter impeller.

Solution: Tabulate data from Figs. D.9 (705 rpm) and D.10 (880 rpm):

705 rpm:

Q (gpm)	0	2000	4000	6000	8000
H (ft)	59	56	50	43	32

Curve-fit: $\hat{H}(ft) = 57.8 - 4.09\times10^{-7}[Q(gpm)]^2$; $r^2 = 0.994$

$\hat{H}$ (ft)	57.8	56.2	51.3	43.1	31.6

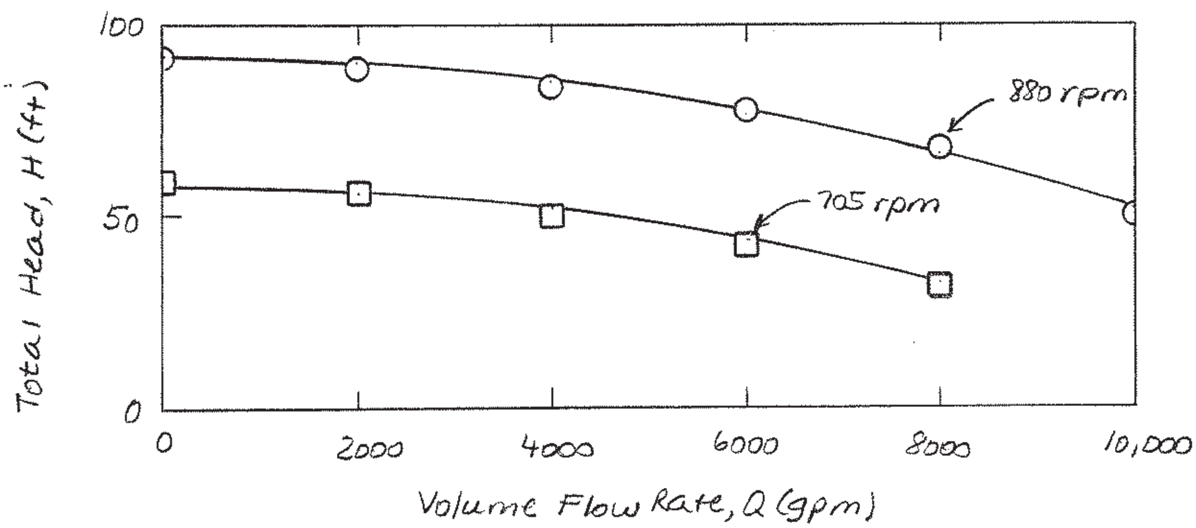
880 rpm:

Q (gpm)	0	2000	4000	6000	8000	10,000
H (ft)	92	89	84	78	68	50

Curve-fit: $\hat{H}(ft) = 91.5 - 4.01\times10^{-7}[Q(gpm)]^2$; $r^2 = 0.992$

$\hat{H}$ (ft)	91.5	89.9	85.1	77.1	65.9	51.5

Plot:



Problem 10.23

[Difficulty: 2]

10.23 Use data from Appendix D to choose points from the performance curves for a Peerless horizontal split case Type 4AE12 pump at 1750 and 3550 nominal rpm. Obtain and plot curve-fits for total head versus delivery at each speed for this pump, with a 12-in.-diameter impeller.

Solution: Tabulate data from Figs. D.4 (1750 rpm) and D.5 (3550 rpm):

1750 rpm:

Q (gpm)	0	200	400	600
H (ft)	155	150	137	106

Curve-fit: $\hat{H}(ft) = 156 - 1.36 \times 10^{-4}[Q(gpm)]^2$; $r^2 = 0.994$

$\hat{H}$ (ft)	156	151	134	107

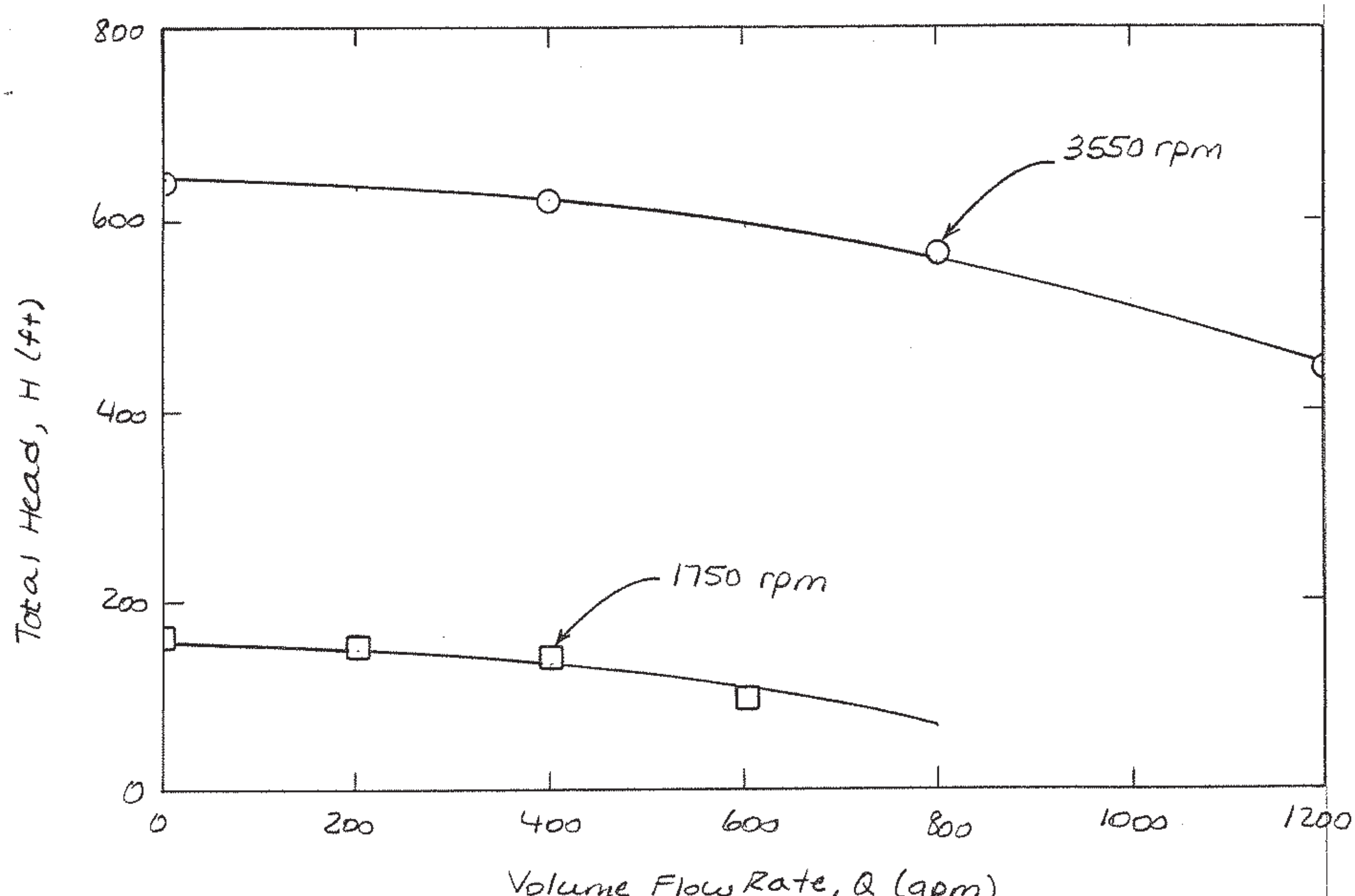
3550 rpm:

Q (gpm)	0	400	800	1200
H (ft)	635	620	565	445

Curve-fit: $\hat{H}(ft) = 641 - 1.33 \times 10^{-4}[Q(gpm)]^2$; $r^2 = 0.994$

$\hat{H}$ (ft)	641	619	556	449

Plot:



Problem 10.24

[Difficulty: 3]

10.24 Data from tests of a water suction pump operated at 2000 rpm with a 12-in. diameter impeller are

Flow rate, Q (cfm)	36	50	74	88	125
Total head, H (ft)	190	195	176	162	120
Power input, $\mathscr{P}$ (hp)	25	30	35	40	46

Plot the performance curves for this pump; include a curve of efficiency versus volume flow rate. Locate the best efficiency point and specify the pump rating at this point.

Given: Data on suction pump

Find: Plot of performance curves; Best effiiciency point

Solution:

Basic equations: $\eta_p = \frac{P_h}{P_m}$ $P_h = \rho \cdot Q \cdot g \cdot H$ (Note: Software cannot render a dot!)

$\rho =$ 1.94 slug/ft^3

Q (cfm)	H (ft)	$\mathscr{P}_m$ (hp)	$\mathscr{P}_h$ (hp)	η (%)
36	190	25	12.9	51.7%
50	195	30	18.4	61.5%
74	176	35	24.6	70.4%
88	162	40	27.0	67.4%
125	120	46	28.4	61.7%

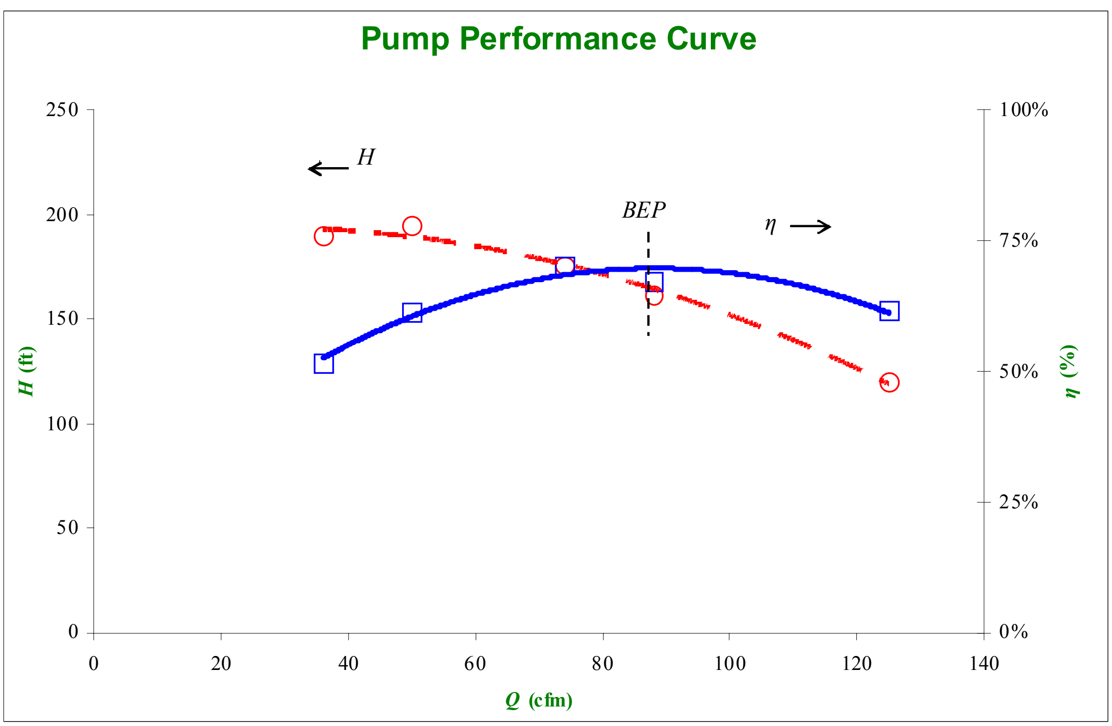
Fitting a 2nd order polynomial to each set of data we find

$H = -0.00759Q^2 + 0.390Q + 189.1$

$\eta = -6.31\text{x}10^{-5}Q^2 + 0.01113Q + 0.207$

Finally, we use Solver to maximize η by varying Q:

Q (cfm)	H (ft)	η (%)
88.2	164.5	69.8%



Problem 10.25

[Difficulty: 3]

10.25 A 9-in.-diameter centrifugal pump, running at 900 rpm with water at 68°F generates the following performance data:

Flow rate, Q (cfm)	0	200	400	600	800	1000
Total head, H (ft)	23.0	22.3	21.0	19.5	17.0	12.5
Power input, $\mathcal{P}$ (hp)	3.13	3.50	4.06	4.47	4.88	5.09

Plot the performance curves for this pump; include a curve of efficiency versus volume flow rate. Locate the best efficiency point. What is the specific speed for this pump?

Given: Data on suction pump

Find: Plot of performance curves; Best effiiciency point

Solution:

Basic equations: $\eta_p = \dfrac{P_h}{P_m}$ $\quad P_h = \rho \cdot Q \cdot g \cdot H$ $\quad N_s = \dfrac{N \cdot \sqrt{Q}}{(g \cdot H)^{0.75}}$ (Note: Software cannot render a dot!)

$\rho =$ 1.94 slug/ft^3

Fitting a 2nd order polynomial to each set of data we find

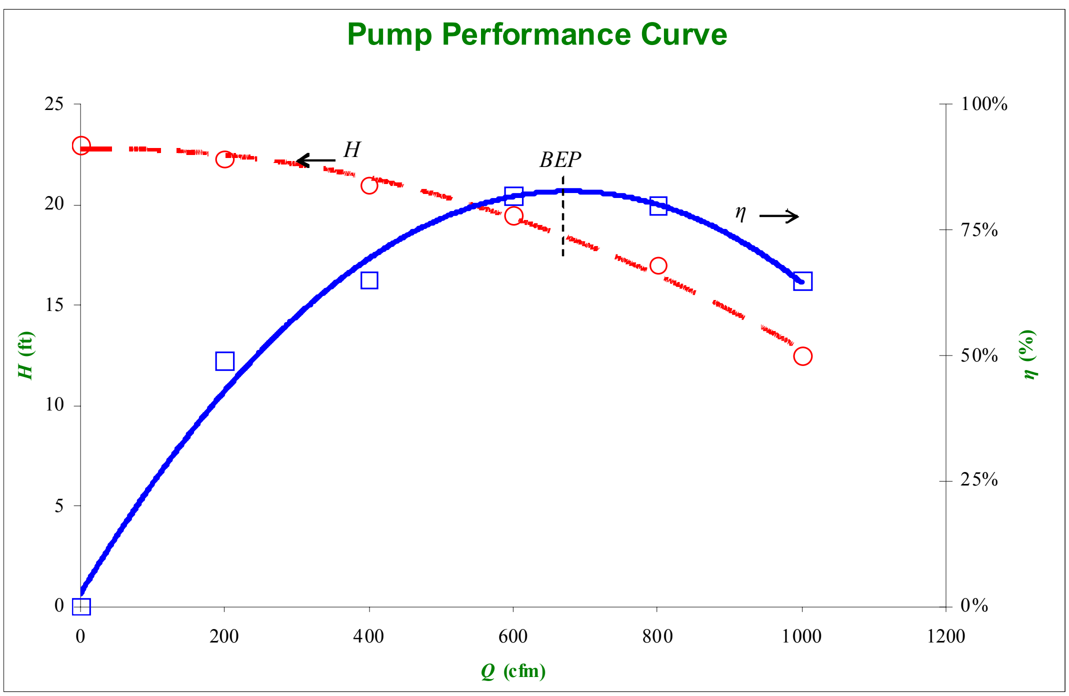
$H = -1.062 \times 10^{-5} Q^2 + 6.39 \times 10^{-4} Q + 22.8$

$\eta = -1.752 \times 10^{-6} Q^2 + 0.00237 Q + 0.0246$

Q (cfm)	H (ft)	$\mathcal{P}_m$ (hp)	$\mathcal{P}_h$ (hp)	η (%)
0	23.0	15.2	0.0	0.0%
200	22.3	17.2	8.4	49.0%
400	21.0	24.4	15.9	65.1%
600	19.5	27.0	22.1	82.0%
800	17.0	32.2	25.7	79.9%
1000	12.5	36.4	23.6	65.0%

Finally, we use Solver to maximize η by varying Q:

Q (cfm)	H (ft)	η (%)
676	18.4	82.6%



The Specific Speed for this pump is: 2.639

Problem 10.26

[Difficulty: 3]

10.26 An axial-flow fan operates in seal-level air at 1350 rpm and has a blade tip diameter of 3 ft and a root diameter of 2.5 ft. The inlet angles are $\alpha_1 = 55°$, $\beta_1 = 30°$, and at the exit $\beta_2 = 60°$. Estimate the flow volumetric flow rate, horsepower, and the outlet angle, α_2.

Given: Data on axial flow fan

Find: Volumetric flow rate, horsepower, flow exit angle

Solution:

Basic equations: $\dot{W}_m = (U_2 V_{t_2} - U_1 V_{t_1})\dot{m}$ (Eq. 10.2b)

$$H = \frac{\dot{W}_m}{\dot{m}g} = \frac{1}{g}(U_2 V_{t_2} - U_1 V_{t_1}) \quad \text{(Eq. 10.2c)}$$

The given or available data is

$$\rho = 0.002377 \cdot \frac{\text{slug}}{\text{ft}^3} \quad \omega = 1350 \cdot \text{rpm} \quad d_{tip} = 3 \cdot \text{ft} \quad d_{root} = 2.5 \cdot \text{ft} \quad \alpha_1 = 55 \cdot \text{deg} \quad \beta_1 = 30 \cdot \text{deg} \quad \beta_2 = 60 \cdot \text{deg}$$

The mean radius would be half the mean diameter: $r = \frac{1}{2} \cdot \frac{d_{tip} + d_{root}}{2}$ $\quad r = 1.375 \cdot \text{ft}$

Therefore, the blade speed is: $U = r \cdot \omega$ $\quad U = 194.39 \cdot \frac{\text{ft}}{\text{s}}$

From velocity triangles we can generate the following two equations: $V_1 \cdot \cos(\alpha_1) = w_1 \cdot \sin(\beta_1)$ (axial component)

$V_1 \cdot \sin(\alpha_1) + w_1 \cdot \cos(\beta_1) = U$ (tangential component)

Combining the two equations: $V_1 = \dfrac{U}{\sin(\alpha_1) + \dfrac{\cos(\alpha_1)}{\tan(\beta_1)}}$ $\quad V_1 = 107.241 \cdot \frac{\text{ft}}{\text{s}}$ $\quad w_1 = V_1 \cdot \dfrac{\cos(\alpha_1)}{\sin(\beta_1)}$ $\quad w_1 = 123.021 \cdot \frac{\text{ft}}{\text{s}}$

So the entrance velocity components are: $V_{n1} = V_1 \cdot \cos(\alpha_1)$ $\quad V_{n1} = 61.511 \cdot \frac{\text{ft}}{\text{s}}$ $\quad V_{t1} = V_1 \cdot \sin(\alpha_1)$ $\quad V_{t1} = 87.846 \cdot \frac{\text{ft}}{\text{s}}$

The volumetric flow rate would then be: $Q = V_{n1} \cdot \frac{\pi}{4} \cdot \left(d_{tip}^2 - d_{root}^2\right)$ $\quad Q = 132.9 \cdot \frac{\text{ft}^3}{\text{s}}$

Since axial velocity does not change: $V_{n2} = V_{n1}$

The exit speed relative to the blade is: $w_2 = \dfrac{V_{n2}}{\sin(\beta_2)}$ $\quad w_2 = 71.026 \cdot \frac{\text{ft}}{\text{s}}$ so the tangential component of absolute velocity is:

$V_{t2} = U - w_2 \cdot \cos(\beta_2)$ $\quad V_{t2} = 158.873 \cdot \frac{\text{ft}}{\text{s}}$ Into the expression for power: $W_m = U \cdot (V_{t2} - V_{t1}) \cdot \rho \cdot Q$ $\quad W_m = 7.93 \cdot \text{hp}$

The flow exit angle is: $\alpha_2 = \text{atan}\left(\dfrac{V_{t2}}{V_{n2}}\right)$ $\quad \alpha_2 = 68.8 \cdot \text{deg}$

Problem 10.27

[Difficulty: 2]

10.27 Write the turbine specific speed in terms of the flow coefficient and the head coefficient.

Solution:

$$N_s = \omega \mathcal{P}^{1/2} / \rho^{1/2} h^{5/4} \quad \text{------ } 10.18a$$

Power coefficient $\quad \Pi_3 = \dfrac{\mathcal{P}}{\rho \omega^3 D^5}$

Head coefficient $\quad \Pi_2 = \dfrac{h}{\omega^2 D^2}$

Then

$$N_s = \left[\frac{\mathcal{P}}{\rho \omega^3 D^5}\right]^{1/2} \left[\frac{\omega^2 D^2}{h}\right]^{5/4} = \frac{\mathcal{P}^{1/2}}{\rho^{1/2} \omega^{3/2} D^{5/2}} \times \frac{\omega^{5/2} D^{5/2}}{h^{5/4}} = \frac{\mathcal{P}^{1/2} \omega}{\rho^{1/2} h^{5/4}}$$

$$N_s = \Pi_3^{1/2} / \Pi_2^{5/4} \longleftarrow N_s$$

Problem 10.28

[Difficulty: 3]

10.28 Data measured during tests of a centrifugal pump driven at 3000 rpm are

Parameter	Inlet, Section ①	Outlet, Section ②
Gage pressure, p (psi)	12.5	
Elevation above datum, z (ft)	6.5	32.5
Average speed of flow, $\overline{V}$ (ft/s)	6.5	15

The flow rate is 65 gpm and the torque applied to the pump shaft is 4.75 lbf·ft. The pump efficiency is 75 percent, and the electric motor efficiency is 85 percent. Find the electric power required, and the gage pressure at section ②.

Given: Data on centrifugal pump

Find: Electric power required; gage pressure at exit

Solution:

Basic equations: $\dot{W}_h = \rho Q g H_p$ (Eq. 10.8a)

$$H_p = \left(\frac{p}{\rho g} + \frac{\bar{V}^2}{2g} + z\right)_{\text{discharge}} - \left(\frac{p}{\rho g} + \frac{\bar{V}^2}{2g} + z\right)_{\text{suction}} \quad \text{(Eq. 10.8b)} \qquad \eta_p = \frac{\dot{W}_h}{\dot{W}_m} = \frac{\rho Q g H_p}{\omega T} \quad \text{(Eq. 10.8c)}$$

The given or available data is

$\rho = 1.94\cdot\frac{\text{slug}}{\text{ft}^3}$ $\quad \omega = 3000\cdot\text{rpm}$ $\quad \eta_p = 75\cdot\%$ $\quad \eta_e = 85\cdot\%$ $\quad Q = 65\cdot\text{gpm}$ $\quad Q = 0.145\cdot\frac{\text{ft}^3}{\text{s}}$

$T = 4.75\cdot\text{lbf}\cdot\text{ft}$ $\quad p_1 = 12.5\cdot\text{psi}$ $\quad z_1 = 6.5\cdot\text{ft}$ $\quad V_1 = 6.5\cdot\frac{\text{ft}}{\text{s}}$ $\quad z_2 = 32.5\cdot\text{ft}$ $\quad V_2 = 15\cdot\frac{\text{ft}}{\text{s}}$

From Eq. 10.8c $\quad H_p = \frac{\omega\cdot T\cdot\eta_p}{\rho\cdot Q\cdot g}$ $\quad H_p = 124\cdot\text{ft}$

Hence, from Eq. 10.8b $\quad p_2 = p_1 + \frac{\rho}{2}\cdot\left(V_1{}^2 - V_2{}^2\right) + \rho\cdot g\cdot\left(z_1 - z_2\right) + \rho\cdot g\cdot H_p$ $\quad p_2 = 53.7\cdot\text{psi}$

Also $\quad W_h = \rho\cdot g\cdot Q\cdot H_p$ $\quad W_h = 1119\cdot\frac{\text{ft}\cdot\text{lbf}}{\text{s}}$ $\quad W_h = 2.03\cdot\text{hp}$

The shaft work is then $\quad W_m = \frac{W_h}{\eta_p}$ $\quad W_m = 1492\cdot\frac{\text{ft}\cdot\text{lbf}}{\text{s}}$ $\quad W_m = 2.71\cdot\text{hp}$

Hence, electrical input is $\quad W_e = \frac{W_m}{\eta_e}$ $\quad W_e = 1756\cdot\frac{\text{ft}\cdot\text{lbf}}{\text{s}}$ $\quad W_e = 2.38\cdot\text{kW}$

Problem 10.29

[Difficulty: 2]

10.29 The *kilogram force* (kgf), defined as the force exerted by a kilogram mass in standard gravity, is commonly used in European practice. The *metric horsepower* (hpm) is defined as 1 hpm ≡ 75 m·kgf/s. Develop a conversion relating metric horsepower to U.S. horsepower. Relate the specific speed for a hydraulic turbine—calculated in units of rpm, metric horsepower, and meters—to the specific speed calculated in U.S. customary units.

Solution:

$$1\ hp\,(U.S.) = 550\,\frac{ft\cdot lbf}{s} \times 0.305\,\frac{m}{ft} \times 0.4536\,\frac{kgf}{lbf} \times \frac{hpm\cdot s}{75\ m\cdot kgf} = 1.01\ hpm$$

hpm

$$N_{s\,cu} = \frac{N(\mathcal{P})^{1/2}}{(h)^{5/4}} = \frac{N(rpm)[\mathcal{P}(hpUS)]^{1/2}}{[h(ft)]^{5/4}}$$

$$= N(rpm)\frac{[\mathcal{P}(hpUS)]^{1/2}}{[\mathcal{P}(hpm)]^{1/2}} \times [\mathcal{P}(hpm)]^{\frac{1}{2}} \times \frac{[h(m)]^{5/4}}{[h(ft)]^{5/4}} \times \frac{1}{[h(m)]^{5/4}}$$

$$= N(rpm)\frac{[\mathcal{P}(hpm)]^{1/2}}{[h(m)]^{5/4}} \times \left[\frac{\mathcal{P}(hpUS)}{\mathcal{P}(hpm)}\right]^{1/2} \times \left[\frac{h(m)}{h(ft)}\right]^{5/4}$$

$$= N_s(rpm, hpm, m) \times (1.01)^{1/2} \times (0.305)^{5/4}$$

$$N_{s\,cu} = 0.228\ N_s(rpm, hpm, m)$$

N_s

Check: $N = 1\,rpm,\ \mathcal{P} = 1\,hp,\ h = 1\,ft\,;\ N_s\,(USCS) = 1$

$$N = 1\,rpm,\ \mathcal{P} = 1\,hpm,\ h = 1\,m\,;\ N_s = \frac{(1)\left(\frac{1}{1.01}\right)^{1/2}}{(0.305)^{5/4}} = 4.39$$

$$\frac{N_s(USCS)}{N_s(rpm, hpm, m)} = \frac{1}{4.39} = 0.228$$ ✓✓

Problem 10.30

[Difficulty: 2]

10.30 Write the pump specific speed in terms of the flow coefficient and the head coefficient.

Solution:

$$N_s = \frac{\omega Q^{1/2}}{h^{3/4}} \quad \text{------- (7.6a)}$$

Flow coefficient $\pi_1 = \frac{Q}{\omega D^3}$

Head coefficient $\pi_2 = \frac{h}{\omega^2 D^2}$

Then

$$N_s = \left[\frac{Q}{\omega D^3}\right]^{1/2} \cdot \left[\frac{\omega^2 D^2}{h}\right]^{3/4} = \frac{Q^{1/2}}{\omega^{1/2} \cancel{D^{3/2}}} \cdot \frac{\omega^{3/2} \cancel{D^{3/2}}}{h^{3/4}} = \frac{\omega Q^{1/2}}{h^{3/4}}$$

$$N_s = \pi_1^{1/2} / \pi_2^{3/4} \longleftarrow N_s$$

Problem 10.31

[Difficulty: 2]

10.31 A small centrifugal pump, when tested at $N = 2875$ rpm with water, delivered $Q = 0.016\ \text{m}^3/\text{s}$ and $H = 40$ m at its best efficiency point ($\eta = 0.70$). Determine the specific speed of the pump at this test condition. Sketch the impeller shape you expect. Compute the required power input to the pump.

Given: Data on small centrifugal pump

Find: Specific speed; Sketch impeller shape; Required power input

Solution:

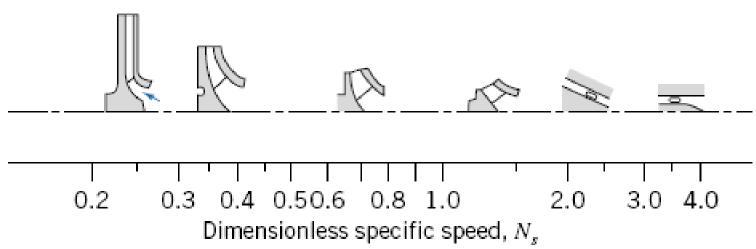
Basic equation: $$N_S = \frac{\omega Q^{1/2}}{h^{3/4}} \quad \text{(Eq. 7.22a)} \qquad \eta_p = \frac{\dot{W}_h}{\dot{W}_m} = \frac{\rho Q g H_p}{\omega T} \quad \text{(Eq. 10.3c)}$$

The given or available data is

$$\rho = 1000 \cdot \frac{\text{kg}}{\text{m}^3} \qquad \omega = 2875 \cdot \text{rpm} \qquad \eta_p = 70 \cdot \% \qquad Q = 0.016 \cdot \frac{\text{m}^3}{\text{s}} \qquad H = 40 \cdot \text{m}$$

Hence $h = g \cdot H$ $\qquad h = 392 \frac{\text{m}^2}{\text{s}^2}$ $\qquad$ (H is energy/weight. h is energy/mass)

Then $N_S = \dfrac{\omega \cdot Q^{\frac{1}{2}}}{h^{\frac{3}{4}}}$ $\qquad N_S = 0.432$



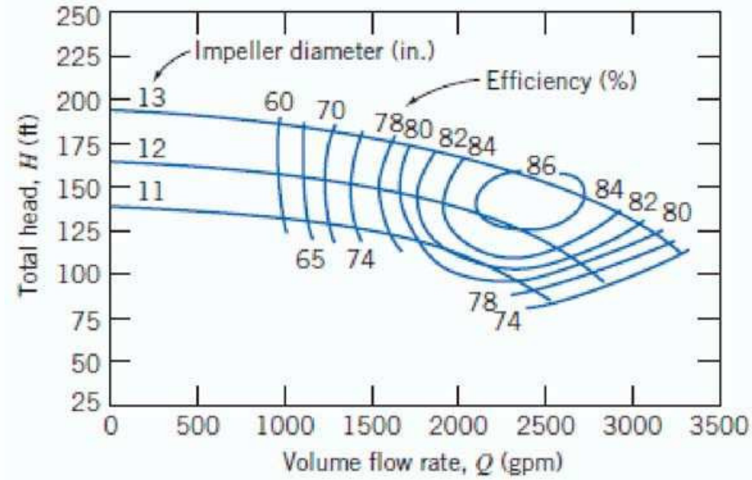
From the figure we see the impeller will be centrifugal

The power input is (from Eq. 10.3c) $\quad W_m = \dfrac{W_h}{\eta_p} \qquad W_m = \dfrac{\rho \cdot Q \cdot g \cdot H}{\eta_p} \qquad W_m = 8.97 \cdot \text{kW}$

Problem 10.32

[Difficulty: 2]

10.32 Typical performance curves for a centrifugal pump, tested with three different impeller diameters in a single casing, are shown. Specify the flow rate and head produced by the pump at its best efficiency point with a 12-in. diameter impeller. Scale these data to predict the performance of this pump when tested with 11 in. and 13 in. impellers. Comment on the accuracy of the scaling procedure.



Solution: From the graph, BEP occurs for the 12 in. impeller at

$Q \simeq 2200$ gpm and $H \simeq 130$ ft — BEP_{12}

From Section 10-4.3, scaling rules are

$$Q_2 = Q_1\left(\frac{D_2}{D_1}\right)^3 ; \quad Q_{11} = 2200\text{ gpm}\left(\frac{11}{12}\right)^3 = 1690\text{ gpm}$$

$$Q_{13} = 2200\text{ gpm}\left(\frac{13}{12}\right)^3 = 2800\text{ gpm}$$

$$H_2 = H_1\left(\frac{D_2}{D_1}\right)^2 ; \quad H_{11} = 130\text{ ft}\left(\frac{11}{12}\right)^2 = 109\text{ ft}$$

$$H_{13} = 130\text{ ft}\left(\frac{13}{12}\right)^2 = 153\text{ ft}$$

Thus BEP_{11} is at $Q = 1690$ gpm, $H = 109$ ft — BEP_{11}

BEP_{13} is at $Q = 2800$ gpm, $H = 153$ ft — BEP_{13}

The complete scaling rules tend to move the volume flow rate too far. Accuracy would be improved using $Q_2 = Q_1(D_2/D_1)^2$, since the impeller width does not change, and $H_2 = H_1(D_2/D_1)^2$, since $H \simeq V^2$. With these modified rules

$(Q_{11}, H_{11}) = 1850$ gpm, 109 ft and $(Q_{13}, H_{13}) = 2580$ gpm, 153 ft

These modified scaling points are closer to the measured BEPs.

Problem 10.33

[Difficulty: 3]

10.33 A pump with $D = 500$ mm delivers $Q = 0.725\ \mathrm{m^3/s}$ of water at $H = 10$ m at its best efficiency point. If the specific speed of the pump is 1.74, and the required input power is 90 kW, determine the shutoff head, H_0, and best efficiency, η. What type of pump is this? If the pump is now run at 900 rpm, by scaling the performance curve, estimate the new flow rate, head, shutoff head, and required power.

Given: Data on a pump

Find: Shutoff head; best efficiency; type of pump; flow rate, head, shutoff head and power at 900 rpm

Solution:

The given or available data is

$$\rho = 999 \cdot \frac{\mathrm{kg}}{\mathrm{m}^3} \qquad N_s = 1.74 \qquad D = 500 \cdot \mathrm{mm} \qquad Q = 0.725 \cdot \frac{\mathrm{m}^3}{\mathrm{s}} \qquad H = 10 \cdot \mathrm{m} \qquad W_m = 90 \cdot \mathrm{kW} \qquad \omega' = 900 \cdot \mathrm{rpm}$$

The governing equations are

$$W_h = \rho \cdot Q \cdot g \cdot H \qquad N_s = \frac{\omega \cdot Q^{\frac{1}{2}}}{h^{\frac{3}{4}}} \qquad H_0 = C_1 = \frac{U_2^{\,2}}{g}$$

Similarity rules:

$$\frac{Q_1}{\omega_1 \cdot D_1^{\,3}} = \frac{Q_2}{\omega_2 \cdot D_2^{\,3}} \qquad \frac{h_1}{\omega_1^{\,2} \cdot D_1^{\,2}} = \frac{h_2}{\omega_2^{\,2} \cdot D_2^{\,2}} \qquad \frac{P_1}{\rho_1 \cdot \omega_1^{\,3} \cdot D_1^{\,5}} = \frac{P_2}{\rho_2 \cdot \omega_2^{\,3} \cdot D_2^{\,5}}$$

$$h = g \cdot H = 98.1 \frac{\mathrm{J}}{\mathrm{kg}} \quad \text{Hence} \quad \omega = \frac{N_s \cdot h^{\frac{3}{4}}}{Q^{\frac{1}{2}}} \qquad \omega = 63.7 \cdot \frac{\mathrm{rad}}{\mathrm{s}} \qquad W_h = \rho \cdot Q \cdot g \cdot H = 71.0\,\mathrm{kW} \qquad \eta_p = \frac{W_h}{W_m} = 78.9\,\%$$

The shutoff head is given by

$$H_0 = \frac{U_2^{\,2}}{g} \qquad U_2 = \frac{D}{2} \cdot \omega \qquad U_2 = 15.9 \frac{\mathrm{m}}{\mathrm{s}} \quad \text{Hence} \quad H_0 = \frac{U_2^{\,2}}{g} \qquad H_0 = 25.8\,\mathrm{m}$$

with $D_1 = D_2$:

$$\frac{Q_1}{\omega_1} = \frac{Q_2}{\omega_2} \quad \text{or} \quad \frac{Q}{\omega} = \frac{Q'}{\omega'} \qquad Q' = Q \cdot \frac{\omega'}{\omega} = 1.073 \frac{\mathrm{m}^3}{\mathrm{s}} \qquad \frac{h_1}{\omega_1^{\,2}} = \frac{h_2}{\omega_2^{\,2}} \quad \text{or} \quad \frac{H}{\omega^2} = \frac{H'}{\omega'^2} \qquad H' = H \cdot \left(\frac{\omega'}{\omega}\right)^2 = 21.9\,\mathrm{m}$$

Also

$$\frac{H_0}{\omega^2} = \frac{H'_0}{\omega'^2} \qquad H'_0 = H_0 \cdot \left(\frac{\omega'}{\omega}\right)^2 \qquad H'_0 = 56.6\,\mathrm{m}$$

$$\frac{P_1}{\rho \cdot \omega_1^{\,3}} = \frac{P_2}{\rho \cdot \omega_2^{\,3}} \quad \text{or} \quad \frac{W_m}{\omega^3} = \frac{W'_m}{\omega'^3} \qquad W'_m = W_m \cdot \left(\frac{\omega'}{\omega}\right)^3 \qquad W'_m = 292 \cdot \mathrm{kW}$$

Problem 10.34

[Difficulty: 3]

10.34 At its best efficiency point ($\eta = 0.87$), a mixed-flow pump, with $D = 16$ in., delivers $Q = 2500$ cfm of water at $H = 140$ ft when operating at $N = 1350$ rpm. Calculate the specific speed of this pump. Estimate the required power input. Determine the curve-fit parameters of the pump performance curve based on the shutoff point and the best efficiency point. Scale the performance curve to estimate the flow, head, efficiency, and power input required to run the same pump at 820 rpm.

Given: Data on a pump at BEP

Find:
(a) Specific Speed
(b) Required power input
(c) Curve fit parameters for the pump performance curve.
(d) Performance of pump at 820 rpm

Solution:

The given or available data is

$\rho = 1.94 \cdot \frac{slug}{ft^3}$ $\quad \eta = 87\%$ $\quad D = 16 \cdot in$ $\quad Q = 2500 \cdot cfm$ $\quad H = 140 \cdot ft$ $\quad \omega = 1350 \cdot rpm$ $\quad \omega' = 820 \cdot rpm$

The governing equations are $\quad N_s = \frac{\omega \cdot \sqrt{Q}}{(g \cdot H)^{0.75}}$ $\quad W_h = \rho \cdot Q \cdot g \cdot H$ $\quad W = \frac{W_h}{\eta}$ $\quad H_0 = \frac{U_2^{\;2}}{g}$

The specific speed is: $\quad N_s = 1.66$

The power is: $\quad W = 761 \cdot hp$

At shutoff $\quad U_2 = \frac{D}{2} \cdot \omega$ $\quad U_2 = 94.248 \cdot \frac{ft}{s}$ $\quad$ Therefore: $\quad H_0 = \frac{U_2^{\;2}}{g}$ $\quad H_0 = 276.1 \cdot ft$

Since $\quad H = H_0 - A \cdot Q^2$ $\quad$ it follows that $\quad A = \frac{H_0 - H}{Q^2}$ $\quad A = 2.18 \times 10^{-5} \cdot \frac{min^2}{ft^5}$

Another way to write this is: $\quad H(ft) = 276.1 - 2.18 \times 10^{-5} \cdot Q(cfm)^2$

$\omega' = 820 \cdot rpm$ $\quad H'_0 = H_0 \cdot \left(\frac{\omega'}{\omega}\right)^2$ $\quad$ and $\quad A' = A$ $\quad$ Thus: $\quad H'_0 = 101.9 \cdot ft$ $\quad A' = 2.18 \times 10^{-5} \cdot \frac{min^2}{ft^5}$

At BEP: $\quad Q' = Q \cdot \left(\frac{\omega'}{\omega}\right)$ $\quad Q' = 1519 \cdot cfm$ $\quad H' = H \cdot \left(\frac{\omega'}{\omega}\right)^2$ $\quad H' = 51.7 \cdot ft$ $\quad \eta' = \eta = 87 \cdot \%$

$W_m = W \cdot \left(\frac{\omega'}{\omega}\right)^3$ $\quad W_m = 170.5 \cdot hp$

At

Problem 10.35

[Difficulty: 3]

10.35 A pumping system must be specified for a lift station at a wastewater treatment facility. The average flow rate is 110 million liters per day and the required lift is 10 m. Non-clogging impellers must be used; about 65 percent efficiency is expected. For convenient installation, electric motors of 37.5 kW or less are desired. Determine the number of motor/pump units needed and recommend an appropriate operating speed.

Given: Data on pumping system

Find: Number of pumps needed; Operating speed

Solution:

Basic equations: $W_h = \rho \cdot Q \cdot g \cdot H \qquad \eta_p = \dfrac{W_h}{W_m}$

The given or available data is

$$\rho = 1000 \cdot \frac{kg}{m^3} \qquad Q_{total} = 110 \times 10^6 \cdot \frac{L}{day} \qquad Q_{total} = 1.273 \frac{m^3}{s} \qquad H = 10 \cdot m \qquad \eta = 65 \cdot \%$$

Then for the system $W_h = \rho \cdot Q_{total} \cdot g \cdot H \qquad W_h = 125 \cdot kW$

The required total power is $W_m = \dfrac{W_h}{\eta} \qquad W_m = 192 \cdot kW$

Hence the total number of pumps must be $\dfrac{192}{37.5} = 5.12$, or at least six pumps

The flow rate per pump will then be $Q = \dfrac{Q_{total}}{6} \qquad Q = 0.212 \dfrac{m^3}{s} \qquad Q = 212 \cdot \dfrac{L}{s}$

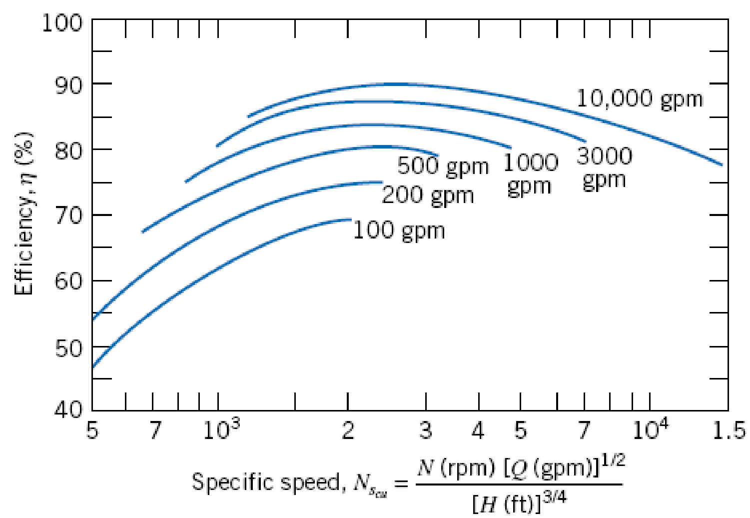
From Fig. 10.15 the peak effiiciency is at a specific speed of about

$$N_{Scu} = 2000$$

We also need $H = 32.8 \cdot ft \qquad Q = 3363 \cdot gpm$

Hence $N = N_{Scu} \cdot \dfrac{H^{\frac{3}{4}}}{Q^{\frac{1}{2}}} \qquad N = 473$

The nearest standard speed to $N = 473$ rpm should be used



Problem 10.36

[Difficulty: 2]

10.36 A centrifugal water pump operates at 1750 rpm; the impeller has backward-curved vanes with $\beta_2 = 60°$ and $b_2 = 1.25$ cm. At a flow rate of 0.025 m³/s, the radial outlet velocity is $V_{n_2} = 3.5$ m/s. Estimate the head this pump could deliver at 1150 rpm.

Given: Data on centrifugal pump

Find: Head at 1150 rpm

Solution:

Basic equation:

$$H = \frac{\dot{W}_m}{\dot{m}g} = \frac{1}{g}\left(U_2 V_{t_2} - U_1 V_{t_1}\right) \qquad \text{(Eq. 10.2c)}$$

The given or available data is

$$\rho = 1000 \cdot \frac{kg}{m^3} \qquad Q = 0.025 \cdot \frac{m^3}{s} \qquad \beta_2 = 60 \cdot deg \qquad b_2 = 1.25 \cdot cm$$

$$\omega = 1750 \cdot rpm \qquad \omega' = 1150 \cdot rpm \qquad V_{n2} = 3.5 \cdot \frac{m}{s}$$

From continuity $\quad V_{n2} = \dfrac{Q}{2 \cdot \pi \cdot r_2 \cdot b_2}$

Hence $\quad r_2 = \dfrac{Q}{2 \cdot \pi \cdot b_2 \cdot V_{n2}} \qquad r_2 = 0.0909\,m \qquad r_2 = 9.09 \cdot cm$

Then $\quad V'_{n2} = \dfrac{\omega'}{\omega} \cdot V_{n2} \qquad V'_{n2} = 2.30\,\dfrac{m}{s}$

Also $\quad U'_2 = \omega' \cdot r_2 \qquad U'_2 = 11.0\,\dfrac{m}{s}$

From the outlet geometry $\quad V'_{t2} = U'_2 - V'_{n2} \cdot \cos(\beta_2) \qquad V'_{t2} = 9.80\,\dfrac{m}{s}$

Finally $\quad H' = \dfrac{U'_2 \cdot V'_{t2}}{g} \qquad H' = 10.9\,m$

Problem 10.37

[Difficulty: 3]

10.37 A set of eight 30-kW motor-pump units is used to deliver water through an elevation of 30 m. The efficiency of the pumps is specified to be 65 percent. Estimate the delivery (liters per day) and select an appropriate operating speed.

Given: Data on pumping system

Find: Total delivery; Operating speed

Solution:

Basic equations: $W_h = \rho \cdot Q \cdot g \cdot H \qquad \eta_p = \dfrac{W_h}{W_m}$

The given or available data is

$$\rho = 1000 \cdot \frac{kg}{m^3} \qquad W_m = 30 \cdot kW \qquad H = 30 \cdot m \qquad H = 98.425 \cdot ft \quad \eta = 65 \cdot \%$$

Then for the system $\quad W_{mTotal} = 8 \cdot W_m \qquad W_{mTotal} = 240 \cdot kW$

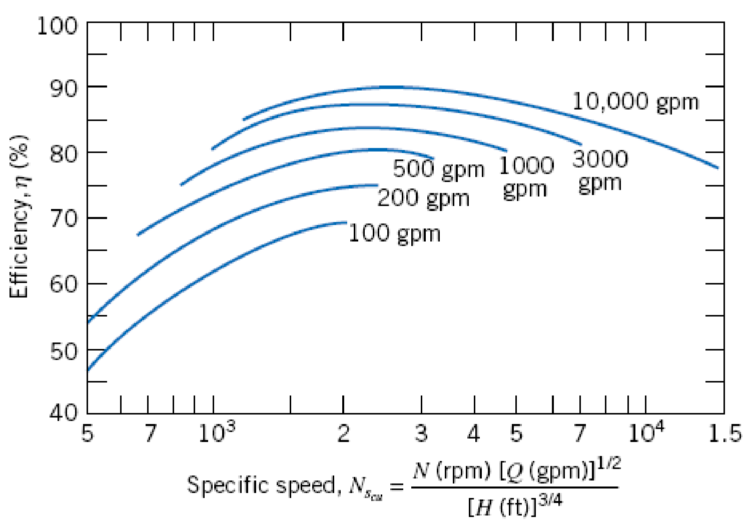
The hydraulic total power is $\quad W_{hTotal} = W_{mTotal} \cdot \eta \qquad W_{hTotal} = 156 \cdot kW$

The total flow rate will then be $\quad Q_{Total} = \dfrac{W_{hTotal}}{\rho \cdot g \cdot H} \qquad Q_{Total} = 0.53 \cdot \dfrac{m^3}{s} \qquad Q_{Total} = 4.58 \times 10^7 \cdot \dfrac{L}{day}$

The flow rate per pump is $\quad Q = \dfrac{Q_{Total}}{8} \qquad Q = 0.066 \cdot \dfrac{m^3}{s} \qquad Q = 1051 \cdot gpm$

From Fig. 10.15 the peak effiiciency is at a specific speed of about

$$N_{Scu} = 2500$$



Hence $\quad N = N_{Scu} \cdot \dfrac{H^{\frac{3}{4}}}{Q^{\frac{1}{2}}} \qquad N = 2410$

The nearest standard speed to $N = 2410$ rpm should be used

Problem 10.38

[Difficulty: 2]

10.38 Appendix D contains area bound curves for pump model selection and performance curves for individual pump models. Use these data to verify the similarity rules for a Peerless Type 4AE12 pump, with impeller diameter $D = 11.0$ in., operated at 1750 and 3550 nominal rpm.

Solution: From Figs. D.4 and D.5, at the best efficiency point (BEP):

N (rpm)	Q (gpm)	H (ft)	$\dot{W}_m$ (hp)	η (%)
1750	470	104	17	73^+
3550	970	430	135	74^+

The similarity rules are

$$\frac{Q_1}{\omega_1 D_1^3} = \frac{Q_2}{\omega_2 D_2^3}, \quad \frac{H_1}{\omega_1^2 D_1^2} = \frac{H_2}{\omega_2^2 D_2^2}, \quad \frac{\mathcal{P}_1}{\omega_1^3 D_1^5} = \frac{\mathcal{P}_2}{\omega_2^3 D_2^5}, \text{ and } \eta_1 = \eta_2$$

Evaluating, with $D_1 = D_2$,

$$Q_1 = Q_2 \frac{\omega_1}{\omega_2} = 970 \text{ gpm} \frac{1750 \text{ rpm}}{3550 \text{ rpm}} = 478 \text{ gpm}$$

$$H_1 = H_2 \left(\frac{\omega_1}{\omega_2}\right)^2 = 430 \text{ ft} \left(\frac{1750 \text{ rpm}}{3550 \text{ rpm}}\right)^2 = 104 \text{ ft}$$

$$\mathcal{P}_1 = \mathcal{P}_2 \left(\frac{\omega_1}{\omega_2}\right)^3 = 135 \text{ hp} \left(\frac{1750 \text{ rpm}}{3550 \text{ rpm}}\right)^3 = 16.2 \text{ hp}$$

$$\eta_1 = \eta_2 = 0.74^+$$

Comparing shows excellent agreement.

Problem 10.39

[Difficulty: 3]

10.39 Appendix D contains area bound curves for pump model selection and performance curves for individual pump models. Use these data and the similarity rules to predict and plot the curves of head H (ft) versus Q (gpm) of a Peerless Type 10AE12 pump, with impeller diameter $D=12$ in., for nominal speeds of 1000, 1200, 1400, and 1600 rpm.

Given: Data on Peerless Type 10AE12 pump at 1720 rpm

Find: Data at speeds of 1000, 1200, 1400, and 1600 rpm

Solution:

The governing equations are the similarity rules: $$\frac{Q_1}{\omega_1 \cdot D_1^{\ 3}} = \frac{Q_2}{\omega_2 \cdot D_2^{\ 3}} \qquad \frac{h_1}{\omega_1^{\ 2} \cdot D_1^{\ 2}} = \frac{h2}{\omega_2^{\ 2} \cdot D_2^{\ 2}} \qquad \text{where} \qquad h = g \cdot H$$

For scaling from speed ω_1 to speed ω_2: $$Q_2 = Q_1 \cdot \frac{\omega_2}{\omega_1} \qquad H_2 = H_1 \cdot \left(\frac{\omega_2}{\omega_1}\right)^2$$ Here are the results generated in *Excel*:

Speed (rpm) = 1760

Q (gal/min)	Q^2	H (ft)	H (fit)
0	0	170	161
500	250000	160	160
1000	1000000	155	157
1500	2250000	148	152
2000	4000000	140	144
2500	6250000	135	135
3000	9000000	123	123
3500	12250000	110	109
4000	16000000	95	93

Speed (rpm) = 1000

Q (gal/min)	H (ft)
0	52.0
284	51.7
568	50.7
852	49.0
1136	46.6
1420	43.5
1705	39.7
1989	35.3
2273	30.2

Speed (rpm) = 1200

Q (gal/min)	H (ft)
0	74.9
341	74.5
682	73.0
1023	70.5
1364	67.1
1705	62.6
2045	57.2
2386	50.8
2727	43.5

Speed (rpm) = 1400

Q (gal/min)	H (ft)
0	102.0
398	101.3
795	99.3
1193	96.0
1591	91.3
1989	85.3
2386	77.9
2784	69.2
3182	59.1

Speed (rpm) = 1600

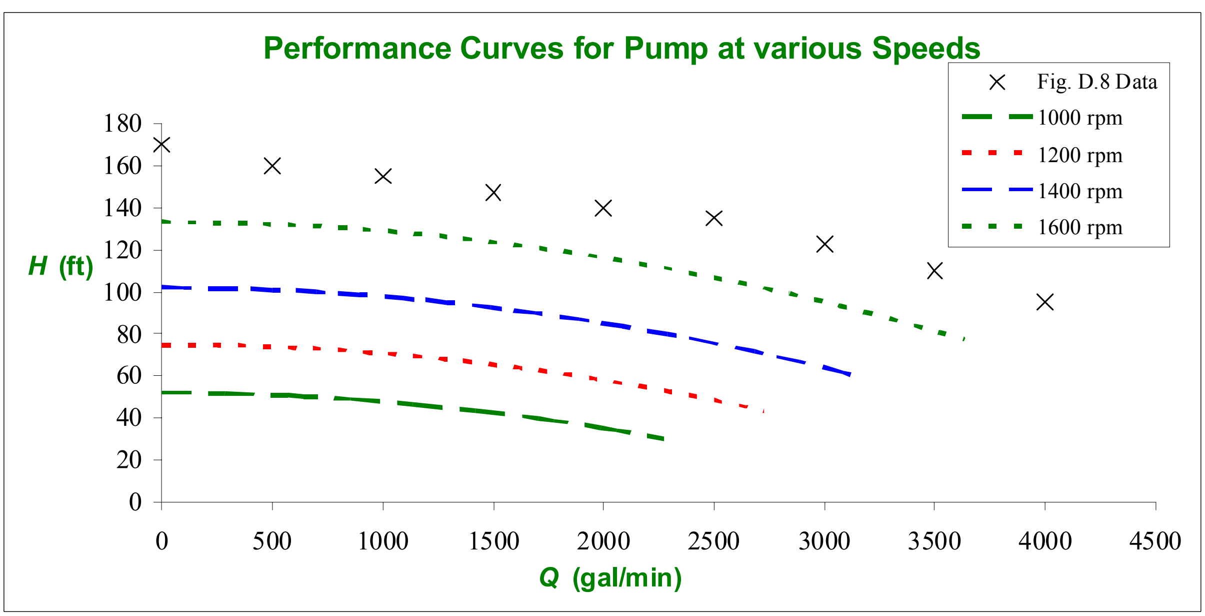
Q (gal/min)	H (ft)
0	133.2
455	132.4
909	129.7
1364	125.4
1818	119.2
2273	111.4
2727	101.7
3182	90.4
3636	77.2

Data from Fig. D.8 is "eyeballed"

The fit to data is obtained from a least squares fit to $H = H_0 - AQ^2$

$H_0 =$ 161 ft

$A =$ 4.23E-06 ft/(gal/min)



Problem 10.40

[Difficulty: 3]

10.40 Consider the Peerless Type 16A18B horizontal split case centrifugal pump (Appendix D). Use these performance data to verify the similarity rules for (a) impeller diameter change and (b) operating speeds of 705 and 880 rpm (note the scale change between speeds).

Solution: From Figs. D.9 and D.10 at the best efficiency point (BEP):

N (rpm)	D (in.)	Q (gpm)	H (ft)	$\dot{W}_m$ (hp)	η (%)
705	18.0	6250	42	76	86+
	17.0	5850	37	63	86+
	16.0	5600	32	54	86
880	18.0	7900	69	155	87+
	17.0	7400	59	125	87
	16.0	7100	50	105	85

The similarity rules are:

$$\frac{Q_1}{\omega_1 D_1^3} = \frac{Q_2}{\omega_2 D_2^3}, \quad \frac{H_1}{\omega_1^2 D_1^2} = \frac{H_2}{\omega_2^2 D_2^2}, \quad \frac{\mathcal{P}_1}{\omega_1^3 D_1^5} = \frac{\mathcal{P}_2}{\omega_2^3 D_2^5}, \text{ and } \eta_1 = \eta_2$$

Evaluating with $\omega_1 = \omega_2 = 705$ rpm,

$Q_{17} = Q_{18}\left(\frac{17}{18}\right)^3 = 5270$ gpm, $Q_{16} = 4390$ gpm; $H_{17} = H_{18}\left(\frac{17}{18}\right)^2 = 37.5$ ft,

$H_{16} = 33.2$ ft; $\mathcal{P}_{17} = \mathcal{P}_{18}\left(\frac{17}{18}\right)^5 = 57.1$ hp, $\mathcal{P}_{16} = 42.2$ hp; $\eta =$ constant

At 880 rpm, $Q_{17} = 6660$ gpm, $Q_{16} = 5550$ gpm; $H_{17} = 61.5$ ft, $H_{16} = 54.5$ ft;

$\mathcal{P}_{17} = 116$ hp, $\mathcal{P}_{16} = 86.0$ hp; $\eta =$ constant

Evaluating with $D_1 = D_2 = 18.0$ in.,

$Q_{705} = Q_{880}\left(\frac{705}{880}\right) = 6330$ gpm; $H_{705} = H_{880}\left(\frac{705}{880}\right)^2 = 44.3$ ft;

$\mathcal{P}_{705} = \mathcal{P}_{880}\left(\frac{705}{880}\right)^3 = 79.7$ hp; $\eta =$ constant

Comparing results with data shows at <u>constant speed</u>:

(1) flow rate scales poorly, (2) head scales well, (3) power scales poorly with changes in diameter.

Comparing results with data shows at <u>constant diameter</u>:

all quantities scale well with changes in speed.

Flow rate scaling may be improved using the modified procedure discussed on page 526, in which $Q \sim D^2$ and $\mathcal{P} \sim D^4$.

Problem 10.41

[Difficulty: 3]

10.41 Use data from Appendix D to verify the similarity rules for the effect of changing the impeller diameter of a Peerless Type 4AE12 pump operated at 1750 and 3550 nominal rpm.

Solution: From Figs. D.4 and D.5 at the best efficiency point (BEP):

N (rpm)	D (in.)	Q (gpm)	H (ft)	$\dot{W}_m$ (hp)	η (%)
1750	10.00	455	87	13	73+
	11.00	470	104	17	73+
	12.12	500	123	22	73+
3550	10.00	930	360	115	74+
	11.00	970	430	135	74+
	12.12	1030	500	180	74+

The similarity rules are:

$$\frac{Q_1}{\omega_1 D_1^3} = \frac{Q_2}{\omega_2 D_2^3}; \quad \frac{H_1}{\omega_1^2 D_1^2} = \frac{H_2}{\omega_2^2 D_2^2}, \quad \frac{\mathcal{P}_1}{\omega_1^3 D_1^5} = \frac{\mathcal{P}_2}{\omega_2^3 D_2^5}, \text{ and } \eta_1 = \eta_2 .$$

Evaluating, with $\omega_1 = \omega_2 = 1750$ rpm,

$Q_{10} = Q_{11}\left(\frac{10}{11}\right)^3 = 353$; $Q_{12} = Q_{11}\left(\frac{12.12}{11}\right)^3 = 629$; $H_{10} = 86.0$ ft, $H_{12} = 126$ ft

$\mathcal{P}_{10} = 10.6$ hp, $\mathcal{P}_{12} = 27.6$ hp; $\eta =$ constant

Evaluating, with $\omega_1 = \omega_2 = 3550$ rpm,

$Q_{10} = 729$ gpm, $Q_{12} = 1300$ gpm; $H_{10} = 355$ ft, $H_{12} = 522$ ft; $\mathcal{P}_{10} = 83.8$ hp,

$\mathcal{P}_{12} = 219$ hp; $\eta =$ constant

Comparing results with data shows:

(1) flow rate is scaled poorly

(2) head is scaled well

(3) power is scaled poorly (because flow rate is scaled poorly)

Better results are obtained using the modified scaling rules (see p. 526); then

$Q \sim D^2$ so $Q_{10} = 388$ gpm and $\mathcal{P} \sim D^4$ so $\mathcal{P}_{10} = 11.6$ hp at 1750 rpm

and $Q_{10} = 802$ gpm and $\mathcal{P}_{10} = 92.2$ hp at 3550 rpm.

Problem 10.42

[Difficulty: 3]

10.42 Performance curves for Peerless horizontal split case pumps are presented in Appendix D. Develop and plot a curve-fit for a Type 10AE12 pump driven at 1150 nominal rpm using the procedure described in Example 10.6.

Solution: Tabulate performance data and curve-fit, for $D = 12$ in. diameter impeller at 1760 nominal rpm:

Q(gpm)	1500	2000	2500	3000	3500	4000
H(ft)	148	141	133	123	110	95
$\hat{H}$(ft)	148	141	133	122	110	95.5

Curve-fit: $\hat{H}(ft) = 157 - 3.83\times10^{-6}[Q(gpm)]^2$; $r^2 = 0.999$

or $\hat{H} = H_0 - AQ^2$

The similarity rules are

$$\frac{Q_1}{\omega_1 D_1^3} = \frac{Q_2}{\omega_2 D_2^3} \; ; \; \frac{H_1}{\omega_1^2 D_1^2} = \frac{H_2}{\omega_2^2 D_2^2}$$

The pump diameter stays constant, so

$$Q_2 = Q_1\left(\frac{\omega_2}{\omega_1}\right) \text{ and } H_2 = H_1\left(\frac{\omega_2}{\omega_1}\right)^2 = H_1\left(\frac{1150}{1760}\right)^2 = 0.427\, H_1$$

Following the procedure of Example Problem 10.6, then at 1150 rpm,

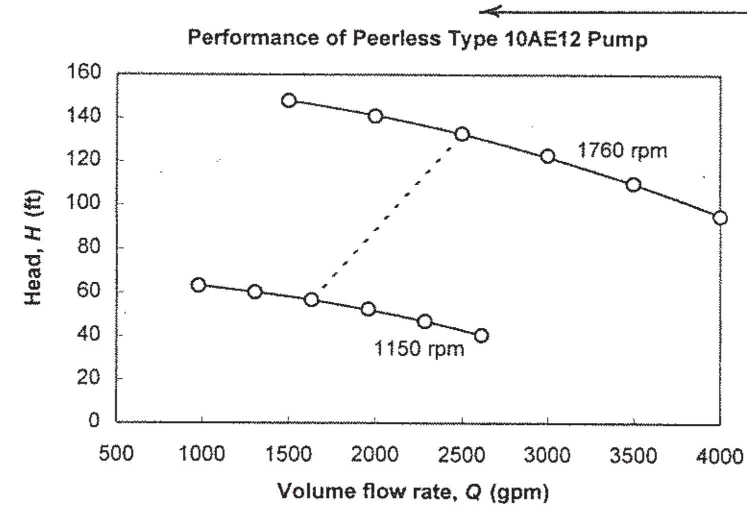
$$\hat{H}(ft) = 0.427\, H_0 - AQ^2$$

$$= (0.427)\, 157\text{ ft} - 3.83\times10^{-6}[Q(gpm)]^2$$

$$\hat{H}(ft) = 67.0\text{ ft} - 3.83\times10^{-6}[Q(gpm)]^2$$

$\hat{H}$(ft) (1150 rpm)

The plot is:



Problem 10.43

[Difficulty: 3]

10.43 Performance curves for Peerless horizontal split case pumps are presented in Appendix D. Develop and plot curve-fits for a Type 16A18B pump, with impeller diameter $D = 18.0$ in., driven at 705 and 880 nominal rpm. Verify the effects of pump speed on scaling pump curves using the procedure described in Example 10.6.

Solution: Tabulate performance data and curve-fits:

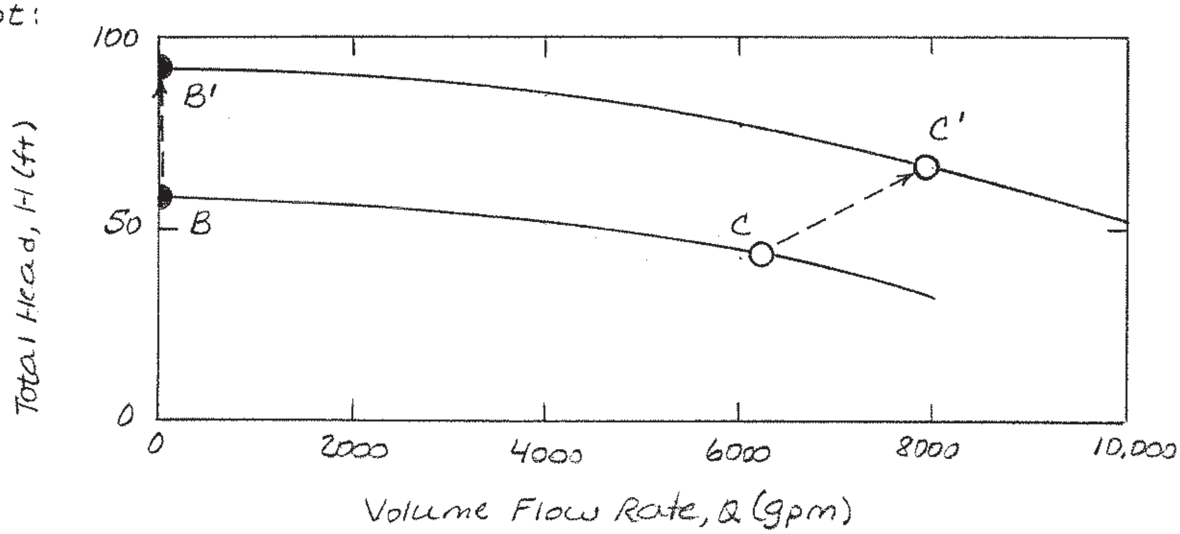
705 rpm:	Q (gpm)	0	2000	4000	6000	8000	BEP:	6250
	H (ft)	59	56	50	43	32		42

Curve-fit: $\hat{H}(ft) = 57.8 - 4.09 \times 10^{-7}[Q(gpm)]^2$; $r^2 = 0.994$

880 rpm:	Q (gpm)	0	2000	4000	6000	8000	BEP:	7900
	H (ft)	92	89	84	78	68		69

Curve-fit: $\hat{H}(ft) = 91.5 - 4.01 \times 10^{-7}[Q(gpm)]^2$; $r^2 = 0.992$

Plot:



Using the procedure of Example Problem 10.6:

$$Q_{B'} = Q_B = 0; \; H_{B'} = H_B\left(\frac{\omega'}{\omega}\right)^2 = 59\,ft\left(\frac{880}{705}\right)^2 = 91.9\,ft$$

$$Q_{C'} = Q_C\left(\frac{\omega'}{\omega}\right) = 6250\,gpm\left(\frac{880}{705}\right) = 7800\,gpm$$

$$H_{C'} = H_C\left(\frac{\omega'}{\omega}\right)^2 = 42\,ft\left(\frac{880}{705}\right)^2 = 65.4\,ft$$

Comparing the curve-fit parameters shows good agreement:

$\hat{H} = H_0 - AQ^2$ $\quad H_0 = 91.5$ ft compared to $H_{B'} = 91.9$ ft

$$\left.\begin{aligned} A' &= 4.01 \times 10^{-7}\,ft/(gpm)^2 \\ A' &= 4.09 \times 10^{-7}\,ft/(gpm)^2 \end{aligned}\right\} \text{within } 2.0\%$$

Problem 10.44

[Difficulty: 4]

10.44 Catalog data for a centrifugal water pump at design conditions are $Q = 250$ gpm and $\Delta p = 18.6$ psi at 1750 rpm. A laboratory flume requires 200 gpm at 32 ft of head. The only motor available develops 3 hp at 1750 rpm. Is this motor suitable for the laboratory flume? How might the pump/motor match be improved?

Solution: To obtain efficiency and pump power requirement, find specific speed.

$$H = \frac{\Delta p}{\rho g} = 18.6\,\frac{\text{lbf}}{\text{in.}^2} \times \frac{\text{ft}^3}{62.4\ \text{lbf}} \times 144\,\frac{\text{in.}^2}{\text{ft}^2} = 42.9\ \text{ft} \quad ; \quad Q = 250\,\frac{\text{gal}}{\text{min}} = 0.557\ \text{cfs}$$

$$N_{s_{cu}} = \frac{N Q^{1/2}}{H^{3/4}} = \frac{1750\ \text{rpm}\,(250\ \text{gpm})^{1/2}}{(42.9\ \text{ft})^{3/4}} = 1650$$

From Fig. 10.15, $\eta \approx 0.73$. Thus

$$\dot{W}_m = \frac{\dot{W}_h}{\eta} = \frac{\rho Q g H}{\eta} = \frac{1}{0.73} \times 62.4\,\frac{\text{lbf}}{\text{ft}^3} \times 0.557\,\frac{\text{ft}^3}{\text{s}} \times 42.9\ \text{ft} \times \frac{\text{hp}\cdot\text{s}}{550\ \text{ft}\cdot\text{lbf}} = 3.71\ \text{hp}$$

The motor is not suitable to drive the pump directly.

The pump at 1750 rpm produces more head and flow than necessary. It may be run at reduced speed, e.g., by using a belt drive.

To produce $Q_f = 200$ gpm, solve $\dfrac{Q_p}{\omega_p D_p^3} = \dfrac{Q_f}{\omega_f D_f^3}$; $\omega_f = \dfrac{200}{250} \times 1750 = 1400$ rpm

To produce $H_f = 32$ ft, solve $\dfrac{H_p}{\omega_p^2 D_p^2} = \dfrac{H_f}{\omega_f^2 D_f^2}$; $\omega_f = \sqrt{\dfrac{H_f}{H_p}}\,\omega_p = \sqrt{\dfrac{32}{42.9}} \times 1750 = 1510$ rpm

At 1510 rpm the power requirement will be given by $\dfrac{\mathcal{P}_p}{\omega_p^3 D_p^5} = \dfrac{\mathcal{P}_f}{\omega_f^3 D_f^5}$, so

$$\mathcal{P}_f = \mathcal{P}_p\left(\frac{\omega_f}{\omega_p}\right)^3 = 3.71\ \text{hp}\left(\frac{1510}{1750}\right)^3 = 2.38\ \text{hp}$$

This is well within the capability of the 3 hp motor. Therefore run pump at 1510 rpm.

Problem 10.45

10.45 Problem 10.21 suggests that pump head at best efficiency is typically about 70 percent of shutoff head. Use pump data from Appendix D to evaluate this suggestion. A further suggestion in Section 10.4 is that the appropriate scaling for tests of a pump casing with different impeller diameters is $Q \propto D^2$. Use pump data to evaluate this suggestion.

Discussion: Data selected from pump performance curves in Appendix D is tabulated and plotted on the next page. Data were selected at the maximum efficiency point for the largest (D_{max}) and smallest (D_{min}) diameter impellers with which each pump was tested.

The head at the best efficiency point with the largest impeller was selected to compare with the shutoff head for the same impeller. These data are shown in the first graph, where they are compared to the average ratio, $H_{BEP} = 0.766\ H_0$. There is some scatter, but the trend of agreement is fairly clear. The actual values suggest a higher ratio than the 0.7 mentioned in Problem 10.20.

The flow rate ratio Q_{max}/Q_{min} was compared with the square of the impeller diameter ratio $(D_{max}/D_{min})^2$. These data ratios are shown in the second graph, where they are compared to the correlation line. Agreement is not perfect, but the trend supports a positive correlation of 0.751. The predicted relationship between diameter and flow rate is $Q_{max}/Q_{min} = 0.751\ (D_{max}/D_{min})^2$.

(Use of three significant figures probably is not justified in this problem. The data are read from small graphs in the Appendix that have already been smoothed by the manufacturer. Also there is some uncertainty in selecting the best efficiency point on each curve.)

Problem 10.45

Sample	Fig.	Model	Speed (rpm)	H_0 (ft)	H_{BEP} (ft)	H_{BEP}/H_0 (---)
1	D.3	4AE11	1750	113	95	0.84
2	D.5	4AE12	3550	636	500	0.79
3	D.6	6AE14	1750	209	160	0.77
4	D.7	8AE20G	1770	430	365	0.85
5	D.8	10AE12	1760	170	112	0.66
6	D.9	16A18B	705	59	42	0.71
7	D.10	16A18B	880	92	69	0.75
					Average:	0.766

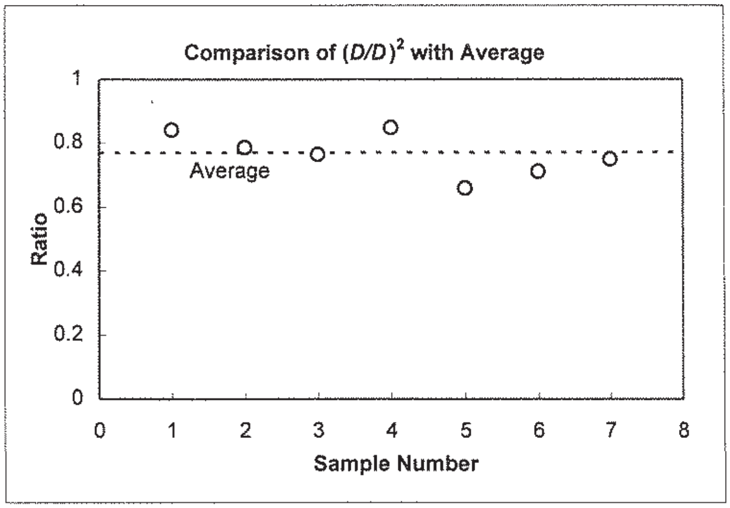
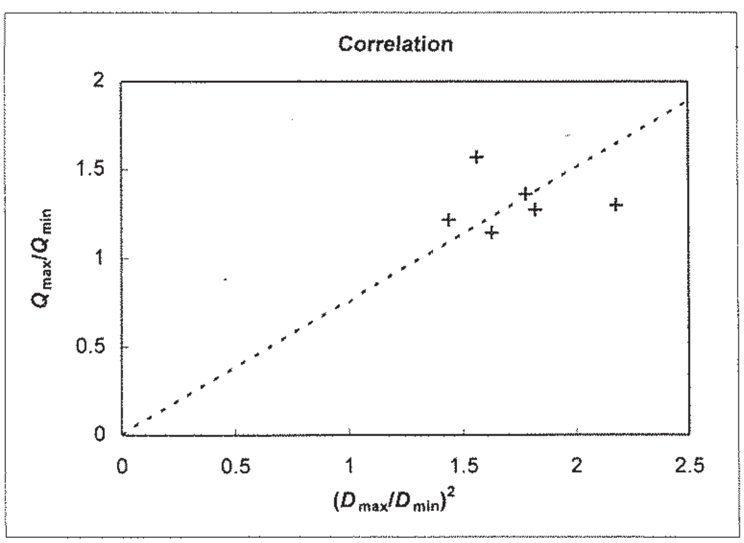


Fig.	Model	Speed (rpm)	D_{min} (in.)	Q_{BEP} (gpm)	D_{max} (in.)	Q_{BEP} (gpm)	$(D_{max}/D_{min})^2$ (---)	Q_{max}/Q_{min} (---)
D.3	4AE11	1750	7.62	740	11.25	960	2.2	1.3
D.5	4AE12	3550	9.5	910	12.12	1040	1.6	1.1
D.6	6AE14	1750	10.38	1375	14.0	1750	1.8	1.3
D.7	8AE20G	1770	16.0	2200	20.0	3450	1.6	1.6
D.8	10AE12	1760	9.0	2500	12.0	3400	1.8	1.4
D.9	16A18B	705	15.0	5100	18.0	6200	1.4	1.2
D.10	16A18B	880	15.0	6500	18.0	7900	1.4	1.2
						Correlation:		0.751



Problem 10.46

[Difficulty: 5]

10.46 White [53] suggests modeling the efficiency for a centrifugal pump using the curve-fit, $\eta = aQ - bQ^3$, where a and b are constants. Describe a procedure to evaluate a and b from experimental data. Evaluate a and b using data for the Peerless Type 10AE12 pump, with impeller diameter $D = 12.0$ in., at 1760 rpm (Appendix D). Plot and illustrate the accuracy of the curve-fit by comparing measured and predicted efficiencies for this pump.

Solution: From Fig. D.8, data are:

η (%)	70	75	80	84	86	86	84
Q (gpm)	1850	2100	2400	2780	3100	3700	4075

Two equations are needed to solve for constants a and b directly. A second equation may be obtained by differentiating. At peak efficiency,

$$\frac{d\eta}{dQ} = a - 3bQ^2 = 0$$

Assume peak efficiency is 87 percent at 3400 gpm. Then

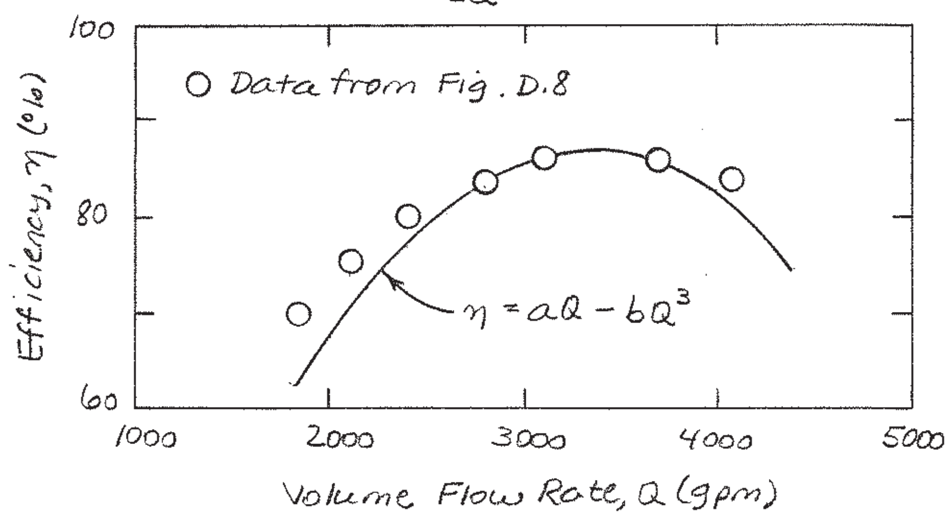
$$\eta_{max} = aQ - bQ^3$$
$$0 = a - 3bQ^2$$

Substituting from the second equation into the first gives

$$\eta_{max} = 3bQ^3 - bQ^3 = 2bQ^3\,;\quad b = \frac{\eta_{max}}{2Q^3} = 1.11 \times 10^{-9}\,;\quad a = 3bQ^2 = 0.0384$$

a, b

Plotting:



The curve-fit does a good job near peak efficiency, but tends to underestimate the measured data elsewhere.

{ An alternative curve-fit procedure is to plot η/Q versus $a - bQ^2$, then do a least-squares fit (using all the data) to obtain a and b. Then $a = 0.0426\,(gpm)^{-1}$, $b = -1.56 \times 10^{-9}\,(gpm)^{-2}$; $r^2 = 0.996$. This underestimates η at $Q > 3500$ gpm. }

Problem 10.47

[Difficulty: 3]

10.47 A fan operates at $Q = 6.3\ \text{m}^3/\text{s}$, $H = 0.15$ m, and $N = 1440$ rpm. A smaller, geometrically similar fan is planned in a facility that will deliver the same head at the same efficiency as the larger fan, but at a speed of 1800 rpm. Determine the volumetric flow rate of the smaller fan.

Given: Data on a model fan, smaller scale similar fan

Find: Scale factor and volumetric flow rate of similar fan

Solution:

Basic equations:

$$\frac{Q_1}{\omega_1 \cdot D_1^{\ 3}} = \frac{Q_2}{\omega_2 \cdot D_2^{\ 3}} \qquad \frac{H_1}{\omega_1^{\ 2} \cdot D_1^{\ 2}} = \frac{H_2}{\omega_2^{\ 2} \cdot D_2^{\ 2}}$$

The given or available data is $\omega_1 = 1440 \cdot \text{rpm}$ $\qquad \omega_2 = 1800 \cdot \text{rpm}$

$$Q_1 = 6.3 \cdot \frac{\text{m}^3}{\text{s}} \qquad H_1 = 0.15 \cdot \text{m} \qquad H_2 = H_1 = 0.15\,\text{m}$$

Solving the head equation for the scale D_2/D_1: $\quad \dfrac{D_2}{D_1} = \dfrac{\omega_1}{\omega_2} \cdot \sqrt{\dfrac{H_2}{H_1}} = 0.8 \qquad \dfrac{D_2}{D_1} = 0.8$

We can use this to find the new flowrate: $\quad Q_2 = Q_1 \cdot \dfrac{\omega_2}{\omega_1} \cdot \left(\dfrac{D_2}{D_1}\right)^3 \qquad Q_2 = 4.03\,\dfrac{\text{m}^3}{\text{s}}$

Problem 10.48

[Difficulty: 3]

10.48 A 1/3 scale model of a centrifugal water pump, when running at $N_m = 5100$ rpm, produces a flow rate of $Q_m = 1\ \mathrm{m^3/s}$ with a head of $H_m = 5.4$ m. Assuming the model and prototype efficiencies are comparable, estimate the flow rate, head, and power requirement if the design speed is 125 rpm.

Given: Data on a model pump

Find: Prototype flow rate, head, and power at 125 rpm

Solution:

Basic equation: $W_h = \rho \cdot Q \cdot g \cdot H$ and similarity rules

$$\frac{Q_1}{\omega_1 \cdot D_1^{\ 3}} = \frac{Q_2}{\omega_2 \cdot D_2^{\ 3}} \quad (10.19a) \qquad \frac{h_1}{\omega_1^{\ 2} \cdot D_1^{\ 2}} = \frac{h2}{\omega_2^{\ 2} \cdot D_2^{\ 2}} \quad (10.19b) \qquad \frac{P_1}{\rho_1 \cdot \omega_1^{\ 3} \cdot D_1^{\ 5}} = \frac{P_2}{\rho_2 \cdot \omega_2^{\ 3} \cdot D_2^{\ 5}} \quad (10.19a)$$

The given or available data is $N_m = 100 \cdot \mathrm{rpm}$ $\quad N_p = 125 \cdot \mathrm{rpm}$ $\quad \rho = 1000 \cdot \frac{\mathrm{kg}}{\mathrm{m^3}}$

$Q_m = 1 \cdot \frac{\mathrm{m^3}}{\mathrm{s}}$ $\quad H_m = 4.5 \cdot \mathrm{m}$

From Eq. 10.8a $\quad W_{hm} = \rho \cdot Q_m \cdot g \cdot H_m \quad W_{hm} = 44.1 \cdot \mathrm{kW}$

From Eq. 10.19a (with $D_m/D_p = 1/3$)

$$\frac{Q_p}{\omega_p \cdot D_p^{\ 3}} = \frac{Q_m}{\omega_m \cdot D_m^{\ 3}} \quad \text{or} \quad Q_p = Q_m \cdot \frac{\omega_p}{\omega_m} \cdot \left(\frac{D_p}{D_m}\right)^3 = 3^3 \cdot Q_m \cdot \frac{\omega_p}{\omega_m}$$

$$Q_p = 27 \cdot Q_m \cdot \frac{N_p}{N_m} \qquad Q_p = 33.8 \frac{\mathrm{m^3}}{\mathrm{s}}$$

From Eq. 10.19b (with $D_m/D_p = 1/3$)

$$\frac{h_p}{\omega_p^{\ 2} \cdot D_p^{\ 2}} = \frac{h_m}{\omega_m^{\ 2} \cdot D_m^{\ 2}} \quad \text{or} \quad \frac{g \cdot H_p}{\omega_p^{\ 2} \cdot D_p^{\ 2}} = \frac{g \cdot H_m}{\omega_m^{\ 2} \cdot D_{pm}^{\ 2}}$$

$$H_p = H_m \cdot \left(\frac{\omega_p}{\omega_m}\right)^2 \cdot \left(\frac{D_p}{D_m}\right)^2 = 3^2 \cdot H_m \cdot \left(\frac{\omega_p}{\omega_m}\right)^2 \qquad H_p = 9 \cdot H_m \cdot \left(\frac{N_p}{N_m}\right)^2 \qquad H_p = 63.3\ \mathrm{m}$$

From Eq. 10.19c (with $D_m/D_p = 1/3$)

$$\frac{P_p}{\rho \cdot \omega_p^{\ 3} \cdot D_p^{\ 5}} = \frac{P_m}{\rho \cdot \omega_m^{\ 3} \cdot D_m^{\ 5}} \quad \text{or} \quad W_{hp} = W_{hm} \cdot \left(\frac{\omega_p}{\omega_m}\right)^3 \cdot \left(\frac{D_p}{D_m}\right)^5 = 3^5 \cdot W_{hm} \cdot \left(\frac{\omega_p}{\omega_m}\right)^3$$

$$W_{hp} = 243 \cdot W_{hm} \cdot \left(\frac{N_p}{N_m}\right)^3 \qquad W_{hp} = 20.9 \cdot \mathrm{MW}$$

Problem 10.49

[Difficulty: 2]

10.49 Sometimes the variation of water viscosity with temperature can be used to achieve dynamic similarity. A model pump delivers 0.10 m^3/s of water at 15°C against a head of 27 m, when operating at 3600 rpm. Determine the water temperature that must be used to obtain dynamically similar operation at 1800 rpm. Estimate the volume flow rate and head produced by the pump at the lower-speed test condition. Comment on the *NPSH* requirements for the two tests.

Given: Data on a model pump

Find: Temperature for dynamically similar operation at 1800 rpm; Flow rate and head; Comment on NPSH

Solution:

Basic equation: $Re_1 = Re_2$ and similarity rules $\dfrac{Q_1}{\omega_1 \cdot D_1^{\,3}} = \dfrac{Q_2}{\omega_2 \cdot D_2^{\,3}}$ $\quad \dfrac{H_1}{\omega_1^{\,2} \cdot D_1^{\,2}} = \dfrac{H_2}{\omega_2^{\,2} \cdot D_2^{\,2}}$

The given or available data is $\omega_1 = 3600 \cdot rpm$ $\quad \omega_2 = 1800 \cdot rpm$ $\quad Q_1 = 0.1 \cdot \dfrac{m^3}{s}$ $\quad H_1 = 27 \cdot m$

From Table A.8 at 15°C $\quad \nu_1 = 1.14 \times 10^{-6} \cdot \dfrac{m^2}{s}$

For D = constant $\quad Re_1 = \dfrac{V_1 \cdot D}{\nu_1} = \dfrac{\omega_1 \cdot D \cdot D}{\nu_1} = Re_2 = \dfrac{\omega_2 \cdot D \cdot D}{\nu_2}$ $\quad$ or $\quad \nu_2 = \nu_1 \cdot \dfrac{\omega_2}{\omega_1}$ $\quad \nu_2 = 5.7 \times 10^{-7} \dfrac{m^2}{s}$

From Table A.8, at $\nu_2 = 5.7 \times 10^{-7} \dfrac{m^2}{s}$, we find, by linear interpolation

$$T_2 = 45 + \frac{(50-45)}{(5.52-6.02)} \cdot (5.70 - 6.02) \qquad T_2 = 48 \quad \text{degrees C}$$

From similar operation $\quad \dfrac{Q_1}{\omega_1 \cdot D^3} = \dfrac{Q_2}{\omega_2 \cdot D^3}$ $\quad$ or $\quad Q_2 = Q_1 \cdot \dfrac{\omega_2}{\omega_1}$ $\quad Q_2 = 0.0500 \cdot \dfrac{m^3}{s}$

and also $\quad \dfrac{H_1}{\omega_1^{\,2} \cdot D^2} = \dfrac{H_2}{\omega_2^{\,2} \cdot D^2}$ $\quad$ or $\quad H_2 = H_1 \cdot \left(\dfrac{\omega_2}{\omega_1}\right)^2$ $\quad H_2 = 6.75\,m$

The water at 48°C is closer to boiling. The inlet pressure would have to be changed to avoid cavitation. The increase between runs 1 and 2 would have to be $\Delta p = p_{v2} - p_{v1}$ where p_{v2} and p_{v1} are the vapor pressures at T_2 and T_1. From the steam tables:

$$p_{v1} = 1.71 \cdot kPa \qquad p_{v2} = 11.276 \cdot kPa \qquad \Delta p = p_{v2} - p_{v1} \qquad \Delta p = 9.57 \cdot kPa$$

Problem 10.50

[Difficulty: 4]

10.50 A large deep fryer at a snack-food plant contains hot oil that is circulated through a heat exchanger by pumps. Solid particles and water droplets coming from the food product are observed in the flowing oil. What special factors must be considered in specifying the operating conditions for the pumps?

Discussion: Any solid particles must be able to pass through the pumps without clogging. If the particles are large, this may require larger than normal clearances within the pumps.

If the water droplets flashed to steam, they would form local pockets of water vapor. The pockets of water vapor would disrupt the flow patterns in the pumps in the same way as cavitation in a homogeneous liquid. To prevent this "cavitation" from occurring, static pressure everywhere in the flow circuit must be maintained above the saturation pressure of the water droplets at the temperature of the flowing oil.

The net positive suction head at the pump inlets must be sufficiently high to prevent any problems from occurring within the pumps themselves.

The solid particles may act as nucleation sites, which would foster the development of vapor pockets in the flow. This might increase the net positive suction head required by the pump above that measured in tests using water. The system must be sized to maintain a large net positive suction head at the design flow rate.

Finally, the viscosity of the oil must be considered. If viscosity is high, pump performance will be degraded compared to pumping water. Then a larger pump must be specified to handle the flow requirement of the hot oil circulation system.

Problem 10.51

[Difficulty: 2]

10.51 Data from tests of a pump operated at 1450 rpm, with a 12.3-in. diameter impeller, are

Flow rate, Q (cfm)	20	40	60	80	100	120	140
Net positive suction head required, $NPSR$ (ft)	7.1	8.0	8.9	10.3	11.8	12.3	16.9

Develop and plot a curve-fit equation for $NPSHR$ versus volume flow rate in the form $NPSHR = a + bQ^2$, where a and b are constants. If the $NPSHA = 20$ ft, estimate the maximum allowable flow rate of this pump.

Given: Data on a NPSHR for a pump

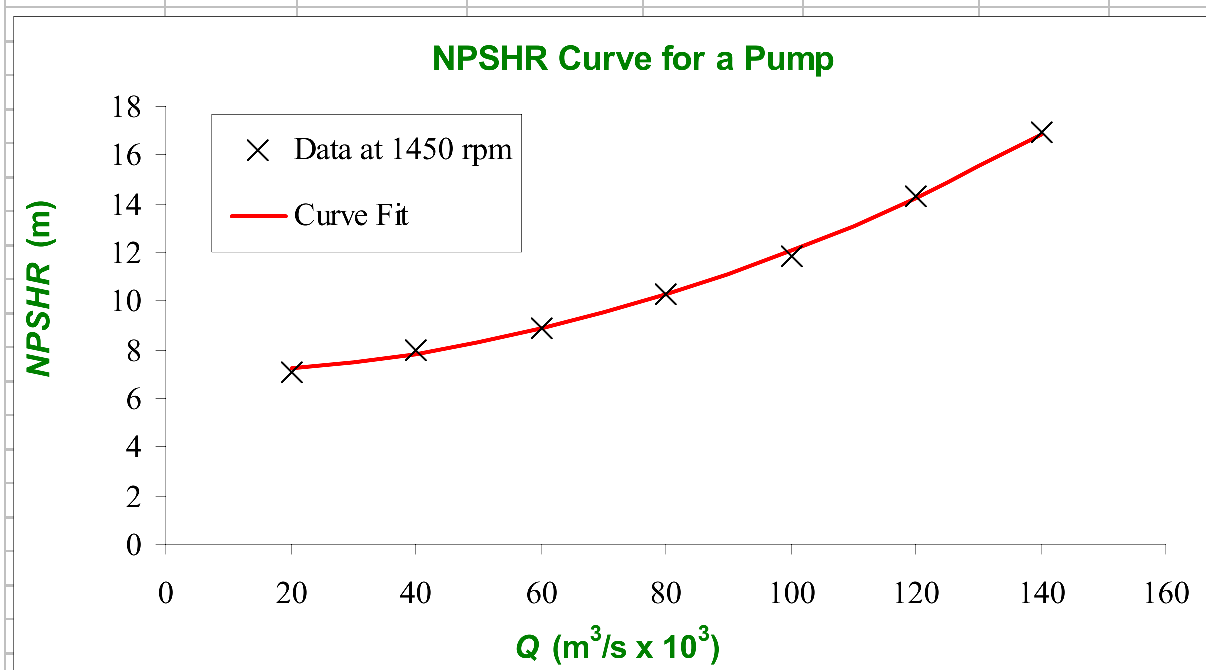
Find: Curve fit; Maximum allowable flow rate

Solution: The results were generated in *Excel*:

Q (cfm)	Q^2	$NPSHR$ (ft)	$NPSHR$ (fit)
20	4.00E+02	7.1	7.2
40	1.60E+03	8.0	7.8
60	3.60E+03	8.9	8.8
80	6.40E+03	10.3	10.2
100	1.00E+04	11.8	12.0
120	1.44E+04	14.3	14.2
140	1.96E+04	16.9	16.9

The fit to data is obtained from a least squares fit to $NPSHR = a + bQ^2$

a =	7.04	ft	Q (cfm)	$NPSHR$ (ft)	
b =	5.01E-04	ft/(cfm)^2	160.9	20.00	Use *Goal Seek* to find Q!



Problem 10.52

[Difficulty: 3]

10.52 A four-stage boiler feed pump has suction and discharge lines of 10 cm and 7.5 cm inside diameter. At 3500 rpm, the pump is rated at 0.025 m^3/s against a head of 125 m while handling water at 115°C. The inlet pressure gage, located 50 cm below the impeller centerline, reads 150 kPa. The pump is to be factory certified by tests at the same flow rate, head rise, and speed, but using water at 27°C. Calculate the *NPSHA* at the pump inlet in the field installation. Evaluate the suction head that must be used in the factory test to duplicate field suction conditions.

Given:
Data on a boiler feed pump

Find:
NPSHA at inlet for field temperature water; Suction head to duplicate field conditions

Solution:

Basic equation: $NPSHA = p_t - p_v = p_g + p_{atm} + \frac{1}{2}\cdot\rho\cdot V^2 - p_v$

Given or available data is $D_s = 10\cdot cm$ $\quad D_d = 7.5\cdot cm$ $\quad H = 125\cdot m$ $\quad Q = 0.025\cdot\frac{m^3}{s}$

$p_{inlet} = 150\cdot kPa$ $\quad p_{atm} = 101\cdot kPa$ $\quad z_{inlet} = -50\cdot cm$ $\quad \rho = 1000\cdot\frac{kg}{m^3}$ $\quad \omega = 3500\cdot rpm$

For field conditions $\quad p_g = p_{inlet} + \rho\cdot g\cdot z_{inlet}$ $\quad p_g = 145\cdot kPa$

From continuity $\quad V_s = \frac{4\cdot Q}{\pi\cdot D_s^2}$ $\quad V_s = 3.18\frac{m}{s}$

From steam tables (try Googling!) at 115°C $\quad p_v = 169\cdot kPa$

Hence $\quad NPSHA = p_g + p_{atm} + \frac{1}{2}\cdot\rho\cdot V_s^2 - p_v$ $\quad NPSHA = 82.2\cdot kPa$

Expressed in meters or feet of water $\quad \frac{NPSHA}{\rho\cdot g} = 8.38m$ $\quad \frac{NPSHA}{\rho\cdot g} = 27.5\cdot ft$

In the laboratory we must have the same NPSHA. From Table A.8 (or steam tables - try Googling!) at 27°C $\quad p_v = 3.57\cdot kPa$

Hence $\quad p_g = NPSHA - p_{atm} - \frac{1}{2}\cdot\rho\cdot V_s^2 + p_v$ $\quad p_g = -20.3\cdot kPa$

The absolute pressure is $\quad p_g + p_{atm} = 80.7\cdot kPa$

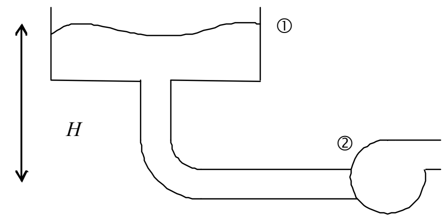
Problem 10.53

[Difficulty: 3]

10.53 The net positive suction head required (*NPSHR*) by a pump may be expressed approximately as a parabolic function of volume flow rate. The *NPSHR* for a particular pump operating at 1800 rpm is given as $H_r = H_0 = AQ^2$, where $H_0 = 10$ ft of water and $A = 7.9$ ft/cfs^2. Assume the pipe system supplying the pump suction consists of a reservoir, whose surface is 22 ft above the pump centerline, a square entrance, 20 ft of 6-in. (nominal) cast-iron pipe, and a 90° elbow. Calculate the maximum volume flow rate at 70°F for which the suction head is sufficient to operate this pump without cavitation.

Given: Pump and supply pipe system

Find: Maximum operational flow rate



Solution:

Basic equations:

$$\left(\frac{p_1}{\rho} + \alpha_1 \cdot \frac{V_1^2}{2} + g \cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \cdot \frac{V_2^2}{2} + g \cdot z_2\right) = h_{lT}$$

$$h_{lT} = f \cdot \frac{L}{D} \cdot \frac{V^2}{2} + f \cdot \frac{L_e}{D} \cdot \frac{V^2}{2} + K \cdot \frac{V^2}{2}$$ L_e for the elbow, and K for the square entrance

$$NPSHA = \frac{p_t - p_v}{\rho \cdot g} \qquad H_r = H_0 + A \cdot Q^2$$

Assumptions: 1) $p_1 = 0$ 2) $V_1 = 0$ 3) $\alpha_2 = 1$ 4) $z_2 = 0$

We must match the NPSHR ($=H_r$) and NPSHA   From the energy equation

$$g \cdot H - \left(\frac{p_2}{\rho} + \frac{V^2}{2}\right) = f \cdot \frac{L}{D} \cdot \frac{V^2}{2} + f \cdot \frac{L_e}{D} \cdot \frac{V^2}{2} + K \cdot \frac{V^2}{2} \qquad \frac{p_2}{\rho \cdot g} = H - \frac{V^2}{2 \cdot g} \cdot \left[1 + f \cdot \left(\frac{L}{D} + \frac{L_e}{D}\right) + K\right]$$

$$NPSHA = \frac{p_t - p_v}{\rho \cdot g} = \frac{p_2}{\rho \cdot g} + \frac{p_{atm}}{\rho \cdot g} + \frac{V_2^2}{2 \cdot g} - \frac{p_v}{\rho \cdot g} \qquad NPSHA = H - \frac{V^2}{2 \cdot g} \cdot \left[f \cdot \left(\frac{L}{D} + \frac{L_e}{D}\right) + K\right] + \frac{(p_{atm} - p_v)}{\rho \cdot g}$$

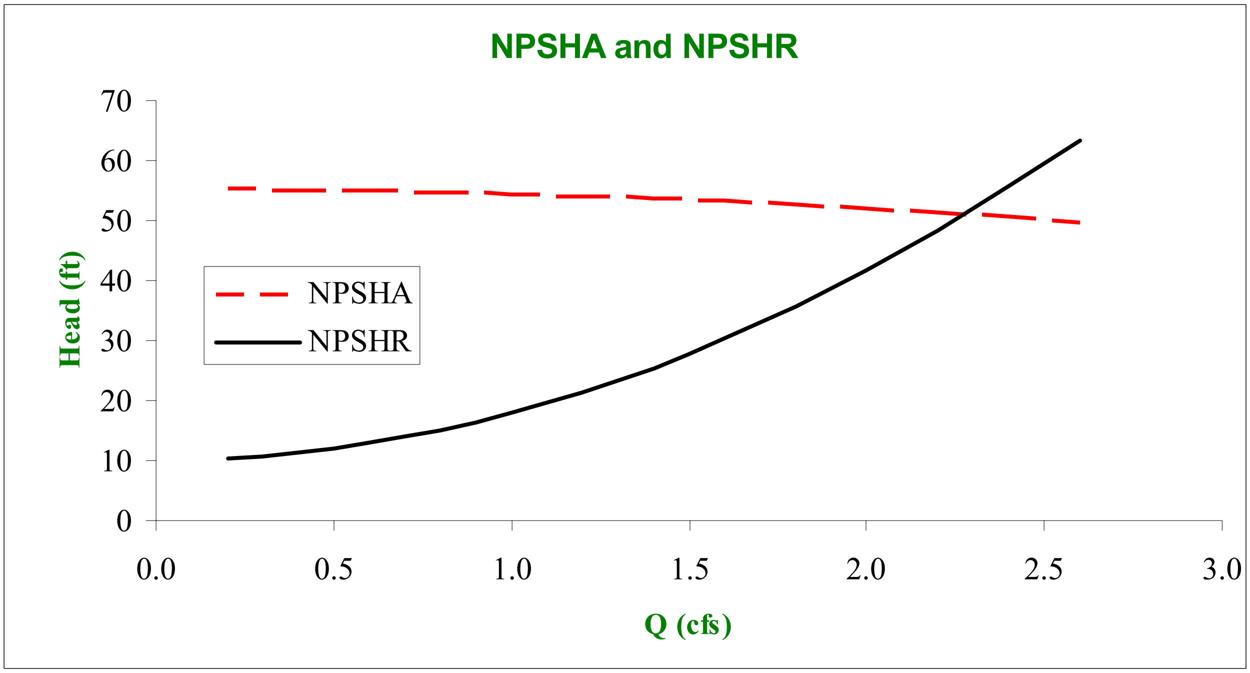
Calculated results and plot were generated using *Excel*:

Given data:

L =	20	ft
e =	0.00085	ft
D =	6.065	in
K_{ent} =	0.5	
L_e/D =	30	
H_0 =	10	ft
A =	7.9	ft/(cfs)^2
H =	22	ft
p_{atm} =	14.7	psia
p_v =	0.363	psia
∪=	1.93	slug/ft^3
∪=	1.06E-05	ft^2/s

Computed results:

Q (cfs)	*V* (ft/s)	*Re*	*f*	*NPSHA* (ft)	*NPSHR* (ft)	Error
0.2	1.00	4.75E+04	0.0259	55.21	10.32	
0.4	1.99	9.51E+04	0.0243	55.11	11.26	
0.6	2.99	1.43E+05	0.0237	54.95	12.84	
0.8	3.99	1.90E+05	0.0234	54.72	15.06	
1.0	4.98	2.38E+05	0.0232	54.43	17.90	
1.2	5.98	2.85E+05	0.0231	54.08	21.38	
1.4	6.98	3.33E+05	0.0230	53.66	25.48	
1.6	7.98	3.80E+05	0.0229	53.18	30.22	
1.8	8.97	4.28E+05	0.0229	52.63	35.60	
2.0	9.97	4.75E+05	0.0228	52.02	41.60	
2.2	10.97	5.23E+05	0.0228	51.35	48.24	
2.4	11.96	5.70E+05	0.0227	50.62	55.50	
2.6	12.96	6.18E+05	0.0227	49.82	63.40	
Crossover point: 2.28	11.36	5.42E+05	0.0228	51.07	51.07	0.00


NPSHA and NPSHR
Head (ft)
Q (cfs)
NPSHA
NPSHR
0
10
20
30
40
50
60
70
0.0
0.5
1.0
1.5
2.0
2.5
3.0

Problem 10.54

[Difficulty: 2]

10.54 A centrifugal pump, operating at $N = 2265$ rpm, lifts water between two reservoirs connected by 300 ft of 6 in. and 100 ft of 3 in. cast-iron pipe in series. The gravity lift is 25 ft. Estimate the head requirement, power needed, and hourly cost of electrical energy to pump water at 200 gpm to the higher reservoir. Assume that electricity costs 12¢/kWhr, and that the electric motor efficiency is 85 percent.

Solution:

Apply the energy equation to the total system for steady, incompressible flow using ③ and ④ at reservoir surfaces.

Computing eq.: $\cancel{\frac{p_3}{\rho g}} + \cancel{\alpha_3 \frac{V_3^2}{2g}} + z_3 + H_a = \cancel{\frac{p_4}{\rho g}} + \cancel{\alpha_4 \frac{V_4^2}{2g}} + z_4 + \frac{h_{l_T}}{g}$ (10.24b)

$$h_{l_T} = f_1 \frac{L_1}{D_1} \frac{V_1^2}{2} + f_2 \frac{L_2}{D_2} \frac{V_2^2}{2}$$

Assumptions: (1) $p_3 = p_4 = p_{atm}$, $V_3 = V_4 \approx 0$
(2) neglect minor losses

then

$$H_a = z_4 - z_3 + f_1 \frac{L_1}{D_1} \frac{V_1^2}{2g} + f_2 \frac{L_2}{D_2} \frac{V_2^2}{2g} \quad \text{------(1)}$$

$$V_1 = \frac{Q}{A_1} = 200 \frac{gal}{min} \times \frac{ft^3}{7.48\,gal} \times \frac{min}{60\,s} \times \frac{4}{\pi (0.50)^2 ft^2} = 2.27 \text{ ft/s}$$

$V_2 = 9.08$ ft/s . For water at 59°F, $\nu = 1.23 \times 10^{-5}$ ft²/s (Table A.7)

$$Re_1 = \frac{V_1 D_1}{\nu} = 2.27 \frac{ft}{s} \times \frac{6}{12} ft \times \frac{s}{1.23\times 10^{-5}\,ft^2} = 9.23 \times 10^4$$

$Re_2 = 1.85 \times 10^5$. From Table 8.1 for cast iron, $e = 0.00085$ ft

$e/D_1 = 0.0017$, $e/D_2 = 0.0034$

From Eq. 8.37 , $f_1 = 0.0244$, $f_2 = 0.0278$. Substituting into (1)

$$H_a = 25\,ft + 0.0244 \times \frac{300\times 12}{6} \times \frac{(2.27)^2}{2} \frac{ft^2}{s^2} \times \frac{s^2}{32.2\,ft} + 0.0278 \times \frac{100\times 12}{3} \times \frac{(9.08)^2}{2} \frac{ft^2}{s^2} \times \frac{s^2}{32.2\,ft}$$

$H_a = 40.4$ ft ← H_a

Specific speed $N_{s_{cu}} = \frac{N Q^{1/2}}{H^{3/4}} = \frac{2265 (200)^{1/2}}{(40.4)^{3/4}} = 2000$

From Fig 10.15, $\eta_p \approx 0.75$. Then $P_m = \frac{\dot{W}_h}{\eta_p} = \frac{\rho Q g H}{\eta_p}$

$$P_m = \frac{1}{0.75} \times 200 \frac{gal}{min} \times \frac{ft^3}{7.48\,gal} \times \frac{min}{60\,s} \times 62.4 \frac{lbf}{ft^3} \times 40.4\,ft \times \frac{hp \cdot s}{550\,ft \cdot lbf} = 2.72 \text{ hp}$$ ← P_m

Cost = $c P_e$. Since $c = \$0.12$ /kw.hr & $\eta_m = 0.85$

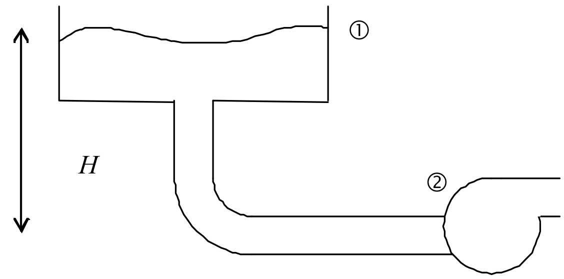
then $P_e = \frac{P_M}{\eta_M}$ and

$$\text{Cost} = c \frac{P_M}{\eta_M} = \frac{\$0.12}{kw \cdot hr} \times \frac{2.72\,hp}{0.85} \times 0.746 \frac{kw}{hp} = 28.7¢ /hr$$ ← C

Problem 10.55

[Difficulty: 5]

10.55 For the pump and flow system of Problem 10.53, calculate the maximum flow rate for hot water at various temperatures and plot versus water temperature. (Be sure to consider the density variation as water temperature is varied.)



Given: Pump and supply pipe system

Find: Maximum operational flow rate as a function of temperature

Solution:

Basic equations:

$$\left(\frac{p_1}{\rho}+\alpha_1\cdot\frac{V_1^{\ 2}}{2}+g\cdot z_1\right)-\left(\frac{p_2}{\rho}+\alpha_2\cdot\frac{V_2^{\ 2}}{2}+g\cdot z_2\right)=h_l\ h_{lT}=f\cdot\frac{L}{D}\cdot\frac{V^2}{2}+f\cdot\frac{L_e}{D}\cdot\frac{V^2}{2}+K\cdot\frac{V^2}{2}$$

L_e for the elbow, and K for the square

$$NPSHA=\frac{p_t-p_v}{\rho\cdot g}\qquad H_r=H_0+A\cdot Q^2$$

Assumptions: 1) $p_1 = 0$ 2) $V_1 = 0$ 3) $\alpha_2 = 0$ 4) $z_2 = 0$

We must match the NPSHR ($=H_r$) and NPSHA

From the energy equation

$$g\cdot H-\left(\frac{p_2}{\rho}+\frac{V^2}{2}\right)=f\cdot\frac{L}{D}\cdot\frac{V^2}{2}+f\cdot\frac{L_e}{D}\cdot\frac{V^2}{2}+K\cdot\frac{V^2}{2}\qquad \frac{p_2}{\rho\cdot g}=H-\frac{V^2}{2\cdot g}\cdot\left[1+f\cdot\left(\frac{L}{D}+\frac{L_e}{D}\right)+K\right]$$

$$NPSHA=\frac{p_t-p_v}{\rho\cdot g}=\frac{p_2}{\rho\cdot g}+\frac{p_{atm}}{\rho\cdot g}+\frac{V_2^{\ 2}}{2\cdot g}-\frac{p_v}{\rho\cdot g}\qquad NPSHA=H-\frac{V^2}{2\cdot g}\cdot\left[f\cdot\left(\frac{L}{D}+\frac{L_e}{D}\right)+K\right]+\frac{\left(p_{atm}-p_v\right)}{\rho\cdot g}$$

The results generated using *Excel* are shown on the next page.

Given data:			Computed results:										

Given data:

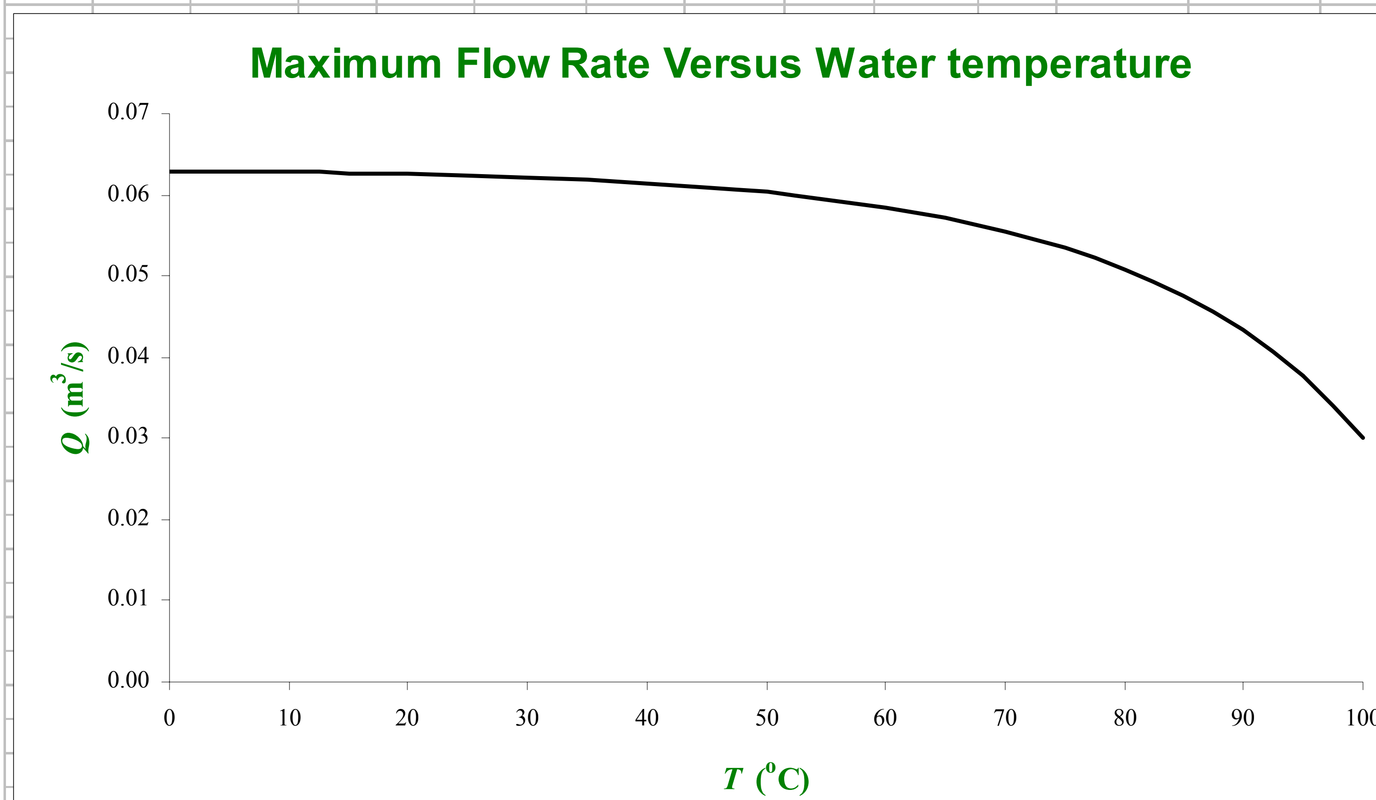
L =	6	m
e =	0.26	mm
D =	15	cm
K_{ent} =	0.5	
L_e/D =	30	
H_0 =	3	m
A =	3000	$m/(m^3/s)^2$
H =	6	m
p_{atm} =	101	kPa
π =	1000	kg/m^3
π =	1.01E-06	m^2/s

Computed results:

T (^oC)	p_v (kPa)	ρ (kg/m^3)	ν (m^3/s)	Q (m^3/s)	V (m/s)	Re	f	$NPSHA$ (m)	$NPSHR$ (m)	$Error$
0	0.661	1000	1.76E-06	0.06290	3.56	3.03E+05	0.0232	14.87	14.87	0.00
5	0.872	1000	1.51E-06	0.06286	3.56	3.53E+05	0.0231	14.85	14.85	0.00
10	1.23	1000	1.30E-06	0.06278	3.55	4.10E+05	0.0230	14.82	14.82	0.00
15	1.71	999	1.14E-06	0.06269	3.55	4.67E+05	0.0230	14.79	14.79	0.00
20	2.34	998	1.01E-06	0.06257	3.54	5.26E+05	0.0229	14.75	14.75	0.00
25	3.17	997	8.96E-07	0.06240	3.53	5.91E+05	0.0229	14.68	14.68	0.00
30	4.25	996	8.03E-07	0.06216	3.52	6.57E+05	0.0229	14.59	14.59	0.00
35	5.63	994	7.25E-07	0.06187	3.50	7.24E+05	0.0228	14.48	14.48	0.00
40	7.38	992	6.59E-07	0.06148	3.48	7.92E+05	0.0228	14.34	14.34	0.00
45	9.59	990	6.02E-07	0.06097	3.45	8.60E+05	0.0228	14.15	14.15	0.00
50	12.4	988	5.52E-07	0.06031	3.41	9.27E+05	0.0228	13.91	13.91	0.00
55	15.8	986	5.09E-07	0.05948	3.37	9.92E+05	0.0228	13.61	13.61	0.00
60	19.9	983	4.72E-07	0.05846	3.31	1.05E+06	0.0228	13.25	13.25	0.00
65	25.0	980	4.40E-07	0.05716	3.23	1.10E+06	0.0227	12.80	12.80	0.00
70	31.2	978	4.10E-07	0.05548	3.14	1.15E+06	0.0227	12.24	12.24	0.00
75	38.6	975	3.85E-07	0.05342	3.02	1.18E+06	0.0227	11.56	11.56	0.00
80	47.4	972	3.62E-07	0.05082	2.88	1.19E+06	0.0227	10.75	10.75	0.00
85	57.8	969	3.41E-07	0.04754	2.69	1.18E+06	0.0227	9.78	9.78	0.00
90	70.1	965	3.23E-07	0.04332	2.45	1.14E+06	0.0227	8.63	8.63	0.00
95	84.6	962	3.06E-07	0.03767	2.13	1.05E+06	0.0228	7.26	7.26	0.00
100	101	958	2.92E-07	0.02998	1.70	8.71E+05	0.0228	5.70	5.70	0.00

Use *Solver* to make the sum of absolute errors between *NPSHA* and *NPSHR* zero by varying the *Q*'s 0.00

NPSHR increases with temperature because the p_v increases; NPHSA decreases because ρ decreases and p_v increases



Problem 10.56

[Difficulty: 3]

10.56 A centrifugal pump is installed in a piping system with $L = 300$ m of $D = 40$ cm cast-iron pipe. The downstream reservoir surface is 15 m lower than the upstream reservoir. Determine and plot the system head curve. Find the volume flow rate (magnitude and direction) through the system when the pump is not operating. Estimate the friction loss, power requirement, and hourly energy cost to pump water at 1 m^3/s through this system.

Given: Pump and reservoir system

Find: System head curve; Flow rate when pump off; Loss, Power required and cost for 1 m^3/s flow rate

Solution:

Basic equations:
$$\left(\frac{p_1}{\rho} + \alpha_1\cdot\frac{V_1^{\,2}}{2} + g\cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2\cdot\frac{V_2^{\,2}}{2} + g\cdot z_2\right) = h_{lT} - h_p \qquad h_{lT} = f\cdot\frac{L}{D}\cdot\frac{V^2}{2} + \Sigma K\cdot\frac{V^2}{2} \text{(K for the exit)}$$

where points 1 and 2 are the reservoir free surfaces, and h_p is the pump head

Note also $\quad H = \dfrac{h}{g} \quad$ Pump efficiency: $\quad \eta_p = \dfrac{W_h}{W_m}$

Assumptions: 1) $p_1 = p_2 = p_{atm}$ 2) $V_1 = V_2 = 0$ 3) $\alpha_2 = 0$ 4) $z_1 = 0$, $z_2 = -15\cdot m$ 4) $K = K_{ent} + K_{ent} = 1.5$

From the energy equation $\quad -g\cdot z_2 = f\cdot\dfrac{L}{D}\cdot\dfrac{V^2}{2} - h_p + K\cdot\dfrac{V^2}{2} \qquad h_p = g\cdot z_2 + f\cdot\dfrac{L}{D}\cdot\dfrac{V^2}{2} + K\cdot\dfrac{V^2}{2} \qquad H_p = z_2 + f\cdot\dfrac{L}{D}\cdot\dfrac{V^2}{2\cdot g} + K\cdot\dfrac{V^2}{2\cdot g}$

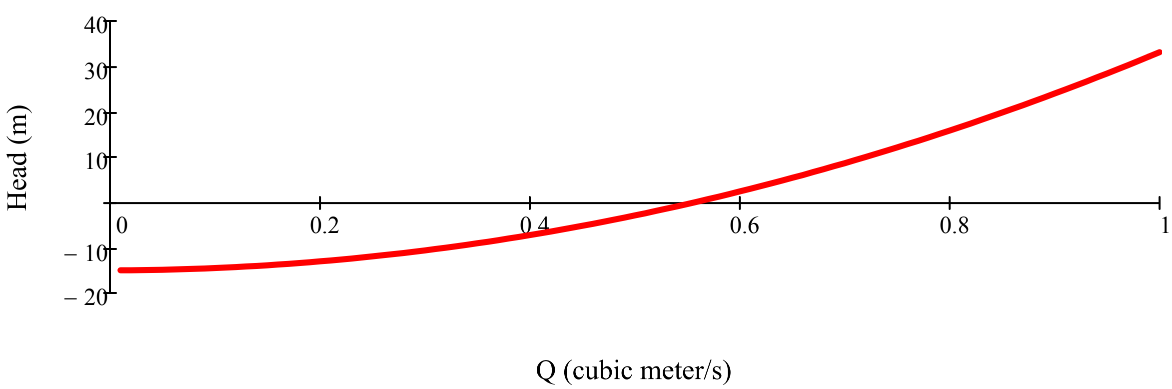
Given or available data $\quad L = 300\cdot m \qquad D = 40\cdot cm \qquad e = 0.26\cdot mm \quad$ (Table 8.1)

$$\rho = 1000\cdot\frac{kg}{m^3} \qquad \nu = 1.01\times 10^{-6}\cdot\frac{m^2}{s} \qquad \text{(Table A.8)}$$

The set of equations to solve for each flow rate Q are

$$V = \frac{4\cdot Q}{\pi\cdot D^2} \qquad Re = \frac{V\cdot D}{\nu} \qquad \frac{1}{\sqrt{f}} = -2.0\cdot\log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re\cdot\sqrt{f}}\right) \qquad H_p = z_2 + f\cdot\frac{L}{D}\cdot\frac{V^2}{2\cdot g} + K\cdot\frac{V^2}{2\cdot g}$$

For example, for $\quad Q = 1\cdot\dfrac{m^3}{s} \qquad V = 7.96\cdot\dfrac{m}{s} \qquad Re = 3.15\times 10^6 \qquad f = 0.0179 \qquad H_p = 33.1\cdot m$



The above graph can be plotted in Excel. In Excel, Solver can be used to find Q for $H_p = 0$ $Q = 0.557\,\frac{m^3}{s}$ (Zero power rate)

At $Q = 1 \cdot \frac{m^3}{s}$ we saw that $H_p = 33.1 \cdot m$

Assuming optimum efficiency at $Q = 1.59 \times 10^4 \cdot gpm$ from Fig. 10.15 $\eta_p = 92 \cdot \%$

Then the hydraulic power is $W_h = \rho \cdot g \cdot H_p \cdot Q$ $W_h = 325 \cdot kW$

The pump power is then $W_m = \frac{W_h}{\eta_p}$ $W_m \cdot 2 = 706 \cdot kW$

If electricity is 10 cents per kW-hr then the hourly cost is about $35

If electricity is 15 cents per kW-hr then the hourly cost is about $53

If electricity is 20 cents per kW-hr then the hourly cost is about $71

Problem 10.57

10.57 Part of the water supply for the South Rim of Grand Canyon National Park is taken from the Colorado River [54]. A flow rate of 600 gpm, taken from the river at elevation 3734 ft, is pumped to a storage tank atop the South Rim at 7022 ft elevation. Part of the pipeline is above ground and part is in a hole directionally drilled at angles up to 70° from the vertical; the total pipe length is approximately 13,200 ft. Under steady flow operating conditions, the frictional head loss is 290 ft of water in addition to the static lift. Estimate the diameter of the commercial steel pipe in the system. Compute the pumping power requirement if the pump efficiency is 61 percent.

Solution:

Apply the energy equation to the total system for steady incompressible flow using ① and ② at inlet and reservoir surface respectively.

Computing eq.: $\frac{p_1}{\rho g} + \alpha_1 \frac{\bar{V}_1^2}{2g} + z_1 + H_a = \frac{p_2}{\rho g} + \alpha_2 \frac{\bar{V}_2^2}{2} + z_2 + \frac{h_{\ell T}}{g}$ (10.24b)

$h_{\ell T} = f \frac{L}{D} \frac{\bar{V}^2}{2}$

Assumptions: (1) $p_1 = p_2 = p_{atm}$, $\bar{V}_1 \approx \bar{V}_2 \approx 0$
(2) neglect minor losses

Then $H_a = z_2 - z_1 + \frac{h_{\ell T}}{g}$ ......(1) and $\frac{h_{\ell T}}{g} = f \frac{L}{D} \frac{\bar{V}^2}{2g} = 290\,ft$

Since $f = f(Re, e/D)$ and D is unknown, we must iterate.

For commercial steel, $e = 0.00015$ ft

The procedure is
- assume D, calculate V, Re; determine f (Eqs 8.37a,b); calculate $h_{\ell T}/g$ and compare to value of 290 ft.

D (in.)	V (ft/s)	Re	f_0	$f^{-0.5}$	f	h_{lT}/g (ft)
12	1.70	1.46E+06	0.0139	8.517	0.0138	8.2
10	2.45	1.75E+06	0.0141	8.433	0.0141	20.8
8	3.83	2.19E+06	0.0146	8.306	0.0145	65.3
6	6.81	2.92E+06	0.0153	8.111	0.0152	289

Thus $D = 6.0$ in ← D

The total pump head is $H_a = 7022 - 3734 + 290 = 3578$ ft.

The pump power is $\mathcal{P}_m = \frac{\dot{W}_h}{\eta_p} = \frac{\rho g Q H}{\eta_p}$

$\mathcal{P}_m = \frac{1}{0.61} \times 62.4 \frac{lbf}{ft^3} \times 600 \frac{gal}{min} \times \frac{ft^3}{7.48\,gal} \times \frac{min}{60\,s} \times 3578\,ft \times \frac{hp \cdot s}{550\,ft \cdot lbf}$

$\mathcal{P}_m = 890$ hp ← $\mathcal{P}_m$

10.58 A Peerless horizontal split-case type 4AE12 pump with 11.0-in.-diameter impeller, operating at 1750 rpm, lifts water between two reservoirs connected by 200 ft of 4 in. and 200 ft of 3 in. cast-iron pipe in series. The gravity lift is 10 ft. Plot the system head curve and determine the pump operating point.

Solution:

Apply the energy equation to the total system for steady, incompressible flow using ③ and ④ at reservoir surfaces.

Computing eq.: $$\cancel{\frac{p_3}{\rho g}} + \cancel{\alpha_3 \frac{\bar{V}_3^2}{2g}} + z_3 + H_a = \cancel{\frac{p_4}{\rho g}} + \cancel{\alpha_4 \frac{\bar{V}_4^2}{2g}} + z_4 + \frac{h_{lT}}{g} \quad (10.24b)$$

$$h_{lT} = f_1 \frac{L_1}{D_1}\frac{\bar{V}_1^2}{2g} + f_2 \frac{L_2}{D_2}\frac{\bar{V}_2^2}{2g}$$

Assumptions: (1) $p_3 = p_4 = p_{atm}$, $V_3 = V_4 \approx 0$

(2) neglect minor losses

then
$$H_a = z_4 - z_3 + f_1 \frac{L_1}{D_1}\frac{\bar{V}_1^2}{2g} + f_2 \frac{L_2}{D_2}\frac{\bar{V}_2^2}{2g} \quad \text{-------} (1)$$

Express V as a function of Q

$$V_1 = \frac{Q}{A_1} = \frac{4Q}{\pi D_1^2} = \frac{4}{\pi} \times \left(\frac{12}{4}\right)^2 \frac{1}{ft^2} Q\left(\frac{gal}{min}\right) \times \frac{ft^3}{7.48\,gal} \times \frac{min}{60s} = 0.0255\,Q(gpm)$$

$V_2 = 0.0454\,Q(gpm)$

The friction factor is determined from the Colebrook eq.

$$\frac{1}{f^{0.5}} = -2.0 \log\left(\frac{e/D}{3.7} + \frac{2.51}{Re\, f^{0.5}}\right) \quad (8.37a)$$

using the equation of Miller for the original estimate

$$f_0 = 0.25\left[\log\left(\frac{e/D}{3.7} + \frac{5.74}{Re^{0.9}}\right)\right]^{-2} \quad (8.37b)$$

Assuming $T = 59°F$, $\nu = 1.23 \times 10^{-5}\ ft^2/s$ (Table A.7)

$Re_1 = \frac{D\bar{V}_1}{\nu} = \frac{4}{12} \times \frac{10^5}{1.23} V_1 = 2.71 \times 10^4\, V_1$, $Re_2 = 2.03 \times 10^4\, V_2$

For cast iron, $e = 0.00085$ ft (Table 8.1)

$e/D_1 = 0.00255$ $\qquad e/D_2 = 0.00340$

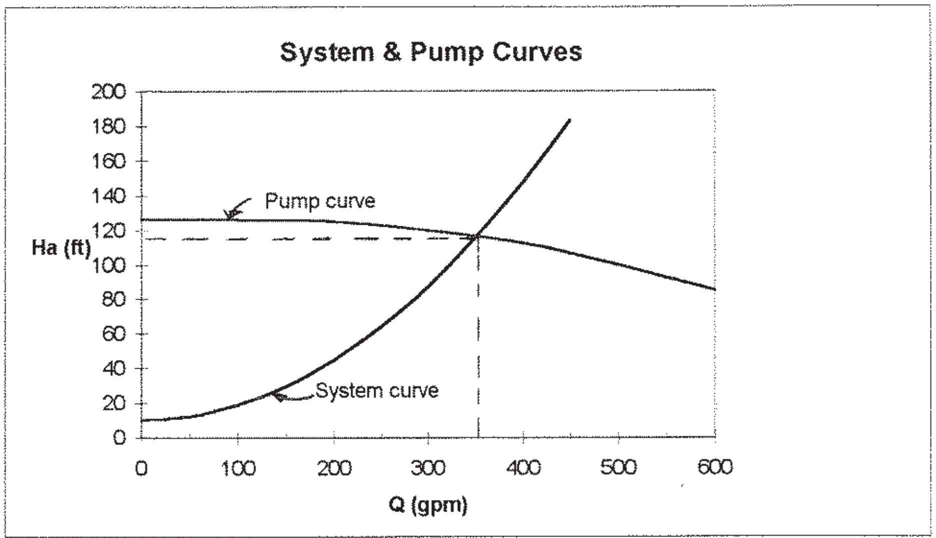
$$H_a = 10\,ft + f_1 \times 9.317\, V_1^2 + f_2 \times 12.42\, V_2^2$$

The pump curve is obtained from Fig. D.4

Problem 10.58

Q (gpm)	V_1 (ft/s)	Re_1	$f_0)_1$	$f^{-0.5})_1$	f_1	V_2 (ft/s)	Re_2	$f_0)_2$	$f^{-0.5})_2$	f_2	H_a (ft)
0	0.0	0.00E+00				0					10.0
50	1.3	3.46E+04	0.0290	5.905	0.0287	2.3	4.61E+04	0.0299	5.817	0.0295	12
100	2.6	6.91E+04	0.0273	6.083	0.0270	4.5	9.22E+04	0.0287	5.933	0.0284	19
150	3.8	1.04E+05	0.0266	6.154	0.0264	6.8	1.38E+05	0.0282	5.977	0.0280	30
200	5.1	1.38E+05	0.0263	6.193	0.0261	9.1	1.84E+05	0.0280	6.000	0.0278	45
250	6.4	1.73E+05	0.0261	6.217	0.0259	11.4	2.30E+05	0.0278	6.014	0.0277	64
300	7.7	2.07E+05	0.0259	6.233	0.0257	13.6	2.76E+05	0.0277	6.023	0.0276	88
350	8.9	2.42E+05	0.0258	6.245	0.0256	15.9	3.23E+05	0.0276	6.030	0.0275	115
400	10.2	2.76E+05	0.0257	6.255	0.0256	18.2	3.69E+05	0.0276	6.036	0.0275	147
450	11.5	3.11E+05	0.0256	6.262	0.0255	20.4	4.15E+05	0.0275	6.040	0.0274	183

Q (gpm)	H_p (ft)
0	126
100	126
200	125
300	120
400	113
500	100
600	85



Pump operates at Q ≈ 350 gpm.
H ≈ 115 ft

Problem 10.59

[Difficulty: 3]

10.59 A pump transfers water from one reservoir to another through two cast-iron pipes in series. The first is 3000 ft of 9 in. pipe and the second is 1000 ft of 6 in. pipe. A constant flow rate of 75 gpm is tapped off at the junction between the two pipes. Obtain and plot the system head versus flow rate curve. Find the delivery if the system is supplied by the pump of Example 10.6, operating at 1750 rpm.

Given: Data on pump and pipe system

Find: Delivery through system

Solution:

Governing Equations:

For the pump and system

$$\left(\frac{p_1}{\rho}+\alpha_1\frac{\bar{V}_1^2}{2}+gz_1\right)-\left(\frac{p_2}{\rho}+\alpha_2\frac{\bar{V}_2^2}{2}+gz_2\right)=h_{l_T}-\Delta h_{\text{pump}} \qquad (8.49)$$

where the total head loss is comprised of major and minor losses

$$h_l = f\frac{L}{D}\frac{\bar{V}^2}{2} \qquad (8.34)$$

$$h_{l_m} = K\frac{\bar{V}^2}{2} \qquad (8.40a)$$

and the pump head (in energy/mass) is given by (from Example 10.6)

$$H_{pump}(ft) = 55.9 - 3.44\times 10^{-5}\cdot Q(gpm)^2$$

Hence, applied between the two reservoir free surfaces ($p_1 = p_2 = 0$, $V_1 = V_2 = 0$, $z_1 = z_2$) we have

$$0 = h_{lT} - \Delta h_{pump}$$

$$h_{lT} = g\cdot H_{system} = \Delta h_{pump} = g\cdot H_{pump}$$

or

$$H_{lT} = H_{pump} \qquad (1)$$

where

$$H_{lT} = \left(f_1\cdot\frac{L_1}{D_1}+K_{ent}\right)\cdot\frac{V_1^2}{2\cdot g}+\left(f_2\cdot\frac{L_2}{D_2}+K_{exit}\right)\cdot\frac{V_2^2}{2}$$

Results generated in *Excel* are shown on the next page.

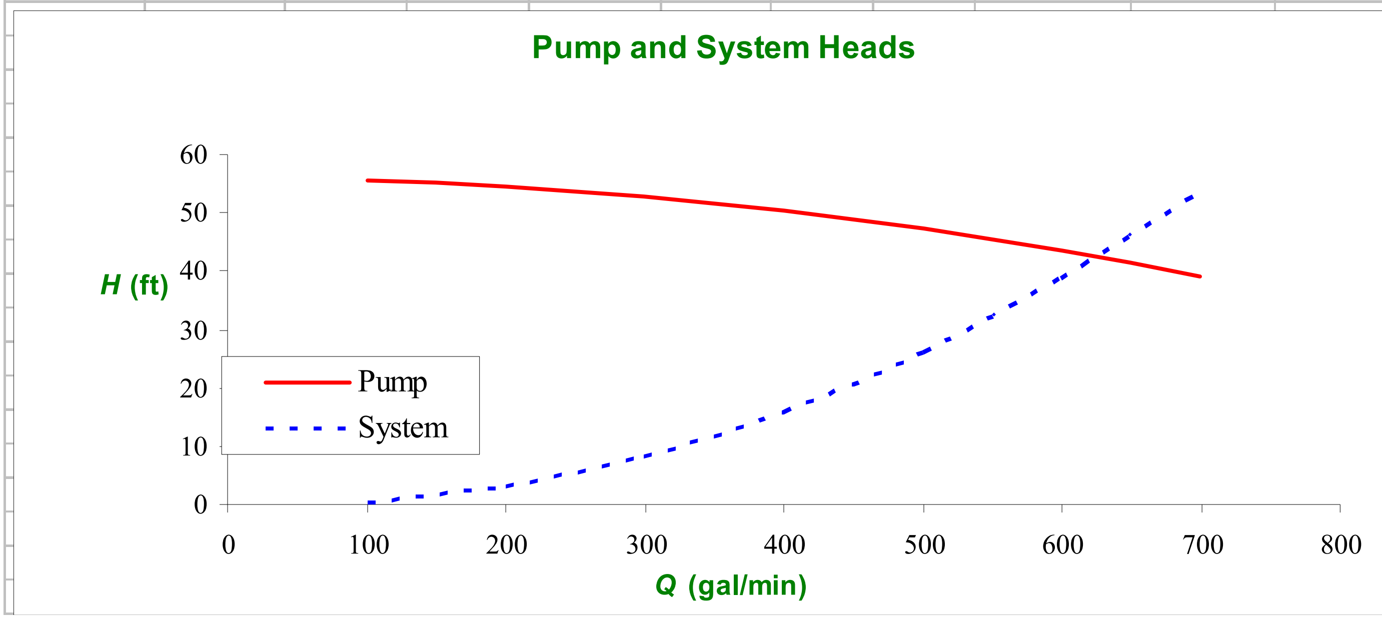
Given or available data:

L_1 =	3000	ft	ν =	1.23E-05	ft^2/s (Table A.7)
D_1 =	9	in	K_{ent} =	0.5	(Fig. 8.14)
L_2 =	1000	ft	K_{exp} =	1	
D_2 =	6	in	Q_{loss} =	75	gpm
e =	0.00085	ft (Table 8.1)			

The system and pump heads are computed and plotted below.
To find the operating condition, *Goal Seek* is used to vary Q_1
so that the error between the two heads is zero.

Q_1 (gpm)	Q_2 (gpm)	V_1 (ft/s)	V_2 (ft/s)	Re_1	Re_2	f_1	f_2	H_{lT} (ft)	H_{pump} (ft)
100	25	0.504	0.284	30753	11532	0.0262	0.0324	0.498	55.6
200	125	1.01	1.42	61506	57662	0.0238	0.0254	3.13	54.5
300	225	1.51	2.55	92260	103792	0.0228	0.0242	8.27	52.8
400	325	2.02	3.69	123013	149922	0.0222	0.0237	15.9	50.4
500	425	2.52	4.82	153766	196052	0.0219	0.0234	26.0	47.3
600	525	3.03	5.96	184519	242182	0.0216	0.0233	38.6	43.5
700	625	3.53	7.09	215273	288312	0.0215	0.0231	53.6	39.0

Q_1 (gpm)	Q_2 (gpm)	V_1 (ft/s)	V_2 (ft/s)	Re_1	Re_2	f_1	f_2	H_{lT} (ft)	H_{pump} (ft)	Error)
627	552	3.162	6.263	192785	254580	0.0216	0.0232	42.4	42.4	0%



Problem 10.60 [Difficulty: 3]

10.60 Performance data for a pump are

H (ft)	90	87	81	70	59	43	22
Q (cfm)	0	50	100	150	200	250	300

The pump is to be used to move water between two open reservoirs with an elevation increase of 24 ft. The connecting pipe system consists of 1750 ft of commercial steel pipe containing two 90° elbows and an open gate valve. Find the flow rate if we use (a) 8-in., (b) 10-in., and (c) 12-in. (nominal) pipe.

Given: Pump and reservoir/pipe system

Find: Flow rate using different pipe sizes

Solution:

Basic equations:
$$\left(\frac{p_1}{\rho} + \alpha_1 \cdot \frac{V_1^{\,2}}{2} + g \cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \cdot \frac{V_2^{\,2}}{2} + g \cdot z_2\right) = h_{lT} - h_p$$

$$h_{lT} = f \cdot \frac{L}{D} \cdot \frac{V^2}{2} + \Sigma \cdot f \cdot \frac{L_e}{D} \cdot \frac{V^2}{2} + \Sigma \cdot K \cdot \frac{V^2}{2}$$ L_e for the elbows, and K for the square entrance and exit

and also $H = \frac{h}{g}$

Assumptions: 1) $p_1 = p_2 = p_{atm}$ 2) $V_1 = V_2 = 0$ 3) $\alpha = 1$ 4) $z_1 = 0,\ z_2 = 24 \cdot ft$ 4) $K = K_{ent} + K_{exp}$ 5) $\frac{L_e}{D}$ is for two elbows

Hence
$$h_{lT} = f \cdot \frac{L}{D} \cdot \frac{V^2}{2} + f \cdot \frac{L_e}{D} \cdot \frac{V^2}{2} + K \cdot \frac{V^2}{2}$$ and also $-z_2 = H_{lT} - H_p$ or $H_{lT} = H_p - z_2$

We want to find a flow that satisfies these equations, rewritten as energy/weight rather than energy/mass

$$H_{lT} = \left[f \cdot \left(\frac{L}{D} + \frac{L_e}{D}\right) + K\right] \cdot \frac{V^2}{2 \cdot g} \qquad H_{lT} + z_2 = H_p$$

Here are the results calculated in *Excel*:

Given or available data **(Note: final results will vary depending on fluid data selected)**:

L =	1750	ft	K_{ent} =	0.5	(Fig. 8.14)
e =	0.00015	ft (Table 8.1)	K_{exp} =	1	
D =	7.981	in	L_e/D_{elbow} =	60	(Two)
ν =	1.06E-05	ft^2/s (Table A.8)	L_e/D_{valve} =	8	(Table 8.4)
z_2 =	24	ft			

The pump data is curve-fitted to $H_{pump} = H_0 - AQ^2$.
The system and pump heads are computed and plotted below.
To find the operating condition, *Solver* is used to vary Q
so that the error between the two heads is minimized.
A plot of the pump and system heads is shown for the 8 in case - the others will look similar.

Q (cfm)	Q^2	H_p (ft)
0.000	0	90.0
50.000	2500	87.0
100.000	10000	81.0
150.000	22500	70.0
200.000	40000	59.0
250.000	62500	43.0
300.000	90000	22.0

V (ft/s)	Re	f
0.00	0	0.0000
2.40	150504	0.0180
4.80	301007	0.0164
7.20	451511	0.0158
9.59	602014	0.0154
11.99	752518	0.0152
14.39	903022	0.0150

H_p (fit)	$H_{lT} + z_2$ (ft)
89	24.0
87	28.5
81	40.4
72	59.5
59	85.8
42.3	119.1
21.9	159.5

$H_0 =$ 89 ft

$A =$ 7.41E-04 ft/(cfm)^2

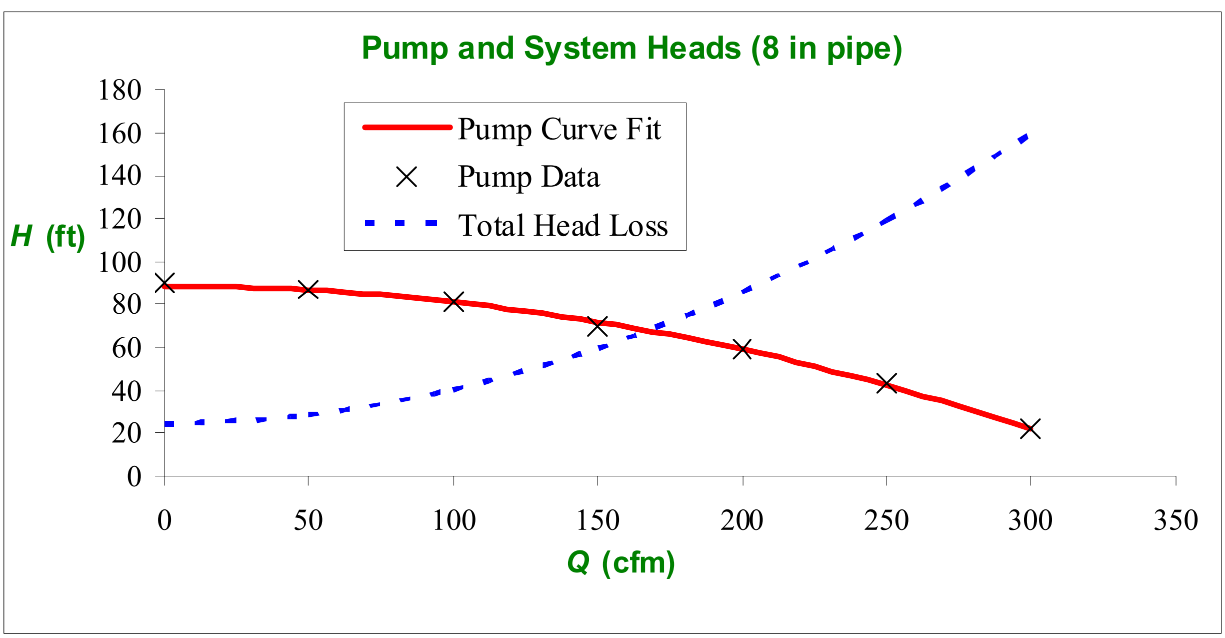
Q (cfm)	V (ft/s)	Re	f	H_p (fit)	$H_{lT} + z_2$ (ft)	Error)
167.5	8.03	504063	0.0157	67.9	67.9	0.00%

Repeating for: D = 10.02 in

Q (cfm)	V (ft/s)	Re	f	H_p (fit)	$H_{lT} + z_2$ (ft)	Error)
179.8	8.63	541345	0.0156	64.7	64.7	0.00%

Repeating for: D = 12 in

Q (cfm)	V (ft/s)	Re	f	H_p (fit)	$H_{lT} + z_2$ (ft)	Error)
189.4	9.09	570077	0.0155	62.1	62.1	0.00%



Problem 10.61

[Difficulty: 3]

10.61 Performance data for a pump are

H (ft)	179	176	165	145	119	84	43
Q (gpm)	0	500	1000	1500	2000	2500	3000

Estimate the delivery when the pump is used to move water between two open reservoirs, through 1200 ft of 12 in. commercial steel pipe containing two 90° elbows and an open gate valve, if the elevation increase is 50 ft. Determine the gate valve loss coefficient needed to reduce the volume flow rate by half.

Given: Data on pump and pipe system

Find: Delivery through system, valve position to reduce delivery by half

Solution:

Governing Equations:

For the pump and system

$$\left(\frac{p_1}{\rho}+\alpha_1\frac{\bar{V}_1^2}{2}+gz_1\right)-\left(\frac{p_2}{\rho}+\alpha_2\frac{\bar{V}_2^2}{2}+gz_2\right)=h_{l_T}-\Delta h_{\text{pump}} \qquad (8.49)$$

where the total head loss is comprised of major and minor losses

$$h_l = f\frac{L}{D}\frac{\bar{V}^2}{2} \qquad (8.34)$$

$$h_{l_m} = f\frac{L_e}{D}\frac{\bar{V}^2}{2} \qquad (8.40b)$$

$$h_{l_m} = K\frac{\bar{V}^2}{2} \qquad (8.40a)$$

Hence, applied between the two reservoir free surfaces ($p_1 = p_2 = 0$, $V_1 = V_2 = 0$, $z_1 - z_2 = \Delta z$) we have

$$g\cdot\Delta z = h_{lT} - \Delta h_{\text{pump}}$$

$$h_{lT} + g\cdot\Delta z = g\cdot H_{\text{system}} + g\cdot\Delta z = \Delta h_{\text{pump}} = g\cdot H_{\text{pump}}$$

or

$$H_{lT} + \Delta z = H_{\text{pump}}$$

where

$$H_{lT} = \left[f\cdot\left(\frac{L}{D} + 2\cdot\frac{L_e}{D_{\text{elbow}}} + \frac{L_e}{D_{\text{valve}}}\right) + K_{\text{ent}} + K_{\text{exit}}\right]\cdot\frac{V^2}{2\cdot g}$$

The calculations performed using *Excel* are shown on the next page:

Given or available data **(Note: final results will vary depending on fluid data selected)**:

L =	1200	ft	K_{ent} =	0.5	(Fig. 8.14)
D =	12	in	K_{exp} =	1	
e =	0.00015	ft (Table 8.1)	L_e/D_{elbow} =	30	
ν =	1.23E-05	ft^2/s (Table A.7)	L_e/D_{valve} =	8	(Table 8.4)
Hz =	-50	ft			

The pump data is curve-fitted to $H_{pump} = H_0 - AQ^2$.
The system and pump heads are computed and plotted below.
To find the operating condition, *Solver* is used to vary Q
so that the error between the two heads is minimized.

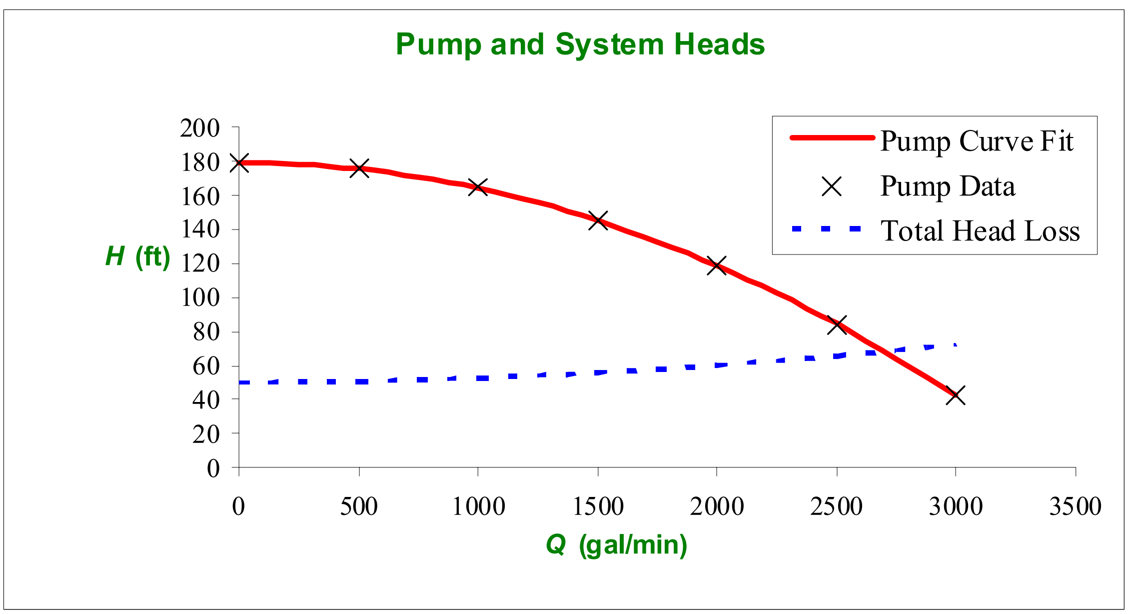
Q (gpm)	Q^2 (gpm)	H_{pump} (ft)
0	0	179
500	250000	176
1000	1000000	165
1500	2250000	145
2000	4000000	119
2500	6250000	84
3000	9000000	43

V (ft/s)	Re	f
0.00	0	0.0000
1.42	115325	0.0183
2.84	230649	0.0164
4.26	345974	0.0156
5.67	461299	0.0151
7.09	576623	0.0147
8.51	691948	0.0145

H_{pump} (fit)	H_{lT} + Hz (ft)
180	50.0
176	50.8
164	52.8
145	56.0
119	60.3
84.5	65.8
42.7	72.4

H_0 = 180 ft
A = 1.52E-05 ft/(gpm)^2

Q (gpm)	V (ft/s)	Re	f	H_{pump} (fit)	H_{lT} + Hz (ft)	Error)
2705	7.67	623829	0.0146	68.3	68.3	0%



For the valve setting to reduce the flow by half, use *Solver* to vary the value below to minimize the error.

L_e/D_{valve} = 26858

Q (gpm)	V (ft/s)	Re	f	H_{pump} (fit)	H_{lT} + Hz (ft)	Error)
1352	3.84	311914	0.0158	151.7	151.7	0%

Problem 10.62

[Difficulty: 3]

10.62 Consider again the pump and piping system of Problem 10.61. Determine the volume flow rate and gate valve loss coefficient for the case of two identical pumps installed in *series*.

Given:
Data on pump and pipe system

Find:
Delivery through series pump system; valve position to reduce delivery by half

Solution:

Governing Equations:

For the pumps and system

$$\left(\frac{p_1}{\rho}+\alpha_1\frac{\bar{V}_1^2}{2}+gz_1\right)-\left(\frac{p_2}{\rho}+\alpha_2\frac{\bar{V}_2^2}{2}+gz_2\right)=h_{l_T}-\Delta h_{\text{pump}} \qquad (8.49)$$

where the total head loss is comprised of major and minor losses

$$h_l=f\frac{L}{D}\frac{\bar{V}^2}{2} \qquad (8.34)$$

$$h_{l_m}=f\frac{L_e}{D}\frac{\bar{V}^2}{2} \qquad (8.40b)$$

$$h_{l_m}=K\frac{\bar{V}^2}{2} \qquad (8.40a)$$

Hence, applied between the two reservoir free surfaces ($p_1 = p_2 = 0$, $V_1 = V_2 = 0$, $z_1 - z_2 = \Delta z$) we have

$$g\cdot\Delta z = h_{lT} - \Delta h_{pump}$$

$$h_{lT} + g\cdot\Delta z = g\cdot H_{system} + g\cdot\Delta z = \Delta h_{pump} = g\cdot H_{pump}$$

or

$$H_{lT} + \Delta z = H_{pump}$$

where

$$H_{lT} = \left[f\cdot\left(\frac{L}{D} + 2\cdot\frac{L_e}{D_{elbow}} + \frac{L_e}{D_{valve}}\right) + K_{ent} + K_{exit}\right]\cdot\frac{V^2}{2\cdot g}$$

For pumps in series

$$H_{pump} = 2\cdot H_0 - 2\cdot A\cdot Q^2$$

where for a single pump

$$H_{pump} = H_0 - A\cdot Q^2$$

The calculations in *Excel* are shown on the next page.

Given or available data (**Note: final results will vary depending on fluid data selected**):

L =	1200	ft		K_{ent} =	0.5	(Fig. 8.14)
D =	12	in		K_{exp} =	1	
e =	0.00015	ft (Table 8.1)		L_e/D_{elbow} =	30	
ν =	1.23E-05	ft^2/s (Table A.7)		L_e/D_{valve} =	8	(Table 8.4)
Δz =	-50	ft				

The pump data is curve-fitted to $H_{pump} = H_0 - AQ^2$.
The system and pump heads are computed and plotted below.
To find the operating condition, *Solver* is used to vary Q
so that the error between the two heads is minimized.

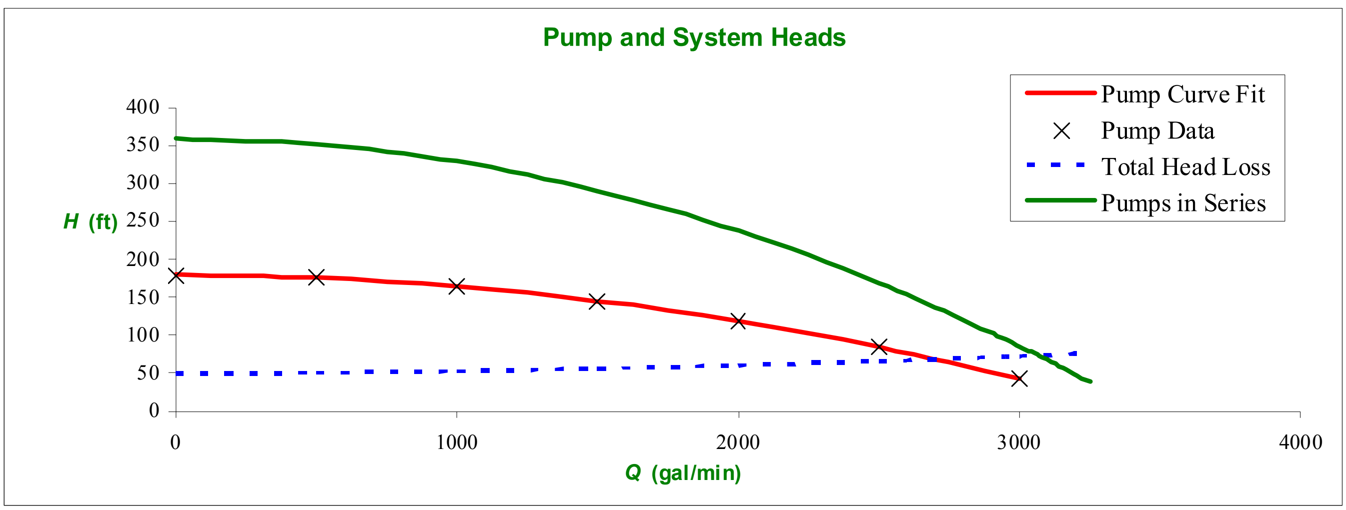
Q (gpm)	Q^2 (gpm)	H_{pump} (ft)	H_{pump} (fit)	V (ft/s)	Re	f
0	0	179	180	0.00	0	0.0000
500	250000	176	176	1.42	115325	0.0183
1000	1000000	165	164	2.84	230649	0.0164
1500	2250000	145	145	4.26	345974	0.0156
2000	4000000	119	119	5.67	461299	0.0151
2500	6250000	84	85	7.09	576623	0.0147
3000	9000000	43	43	8.51	691948	0.0145
3250				9.22	749610	0.0144

H_{pumps} (par)	$H_{lT} + \Delta z$ (ft)
359	50.0
351	50.8
329	52.8
291	56.0
237	60.3
169	65.8
85	72.4
38	76.1

H_0 = 180 ft

A = 1.52E-05 ft/(gpm)^2

Q (gpm)	V (ft/s)	Re	f	H_{pumps} (par)	$H_{lT} + \Delta z$ (ft)	Error)
3066	8.70	707124	0.0145	73.3	73.3	0%



For the valve setting to reduce the flow by half, use *Solver* to vary the value below to minimize the error.

L_e/D_{valve} = 50723

Q (gpm)	V (ft/s)	Re	f	H_{pumps} (par)	$H_{lT} + \Delta z$ (ft)	Error)
1533	4.35	353562	0.0155	287.7	287.7	0%

Problem 10.63

[Difficulty: 4]

10.63 The resistance of a given pipe increases with age as deposits form, increasing the roughness and reducing the pipe diameter (see Fig. 8.14). Typical multipliers to be applied to the friction factor are given in [15]:

Pipe Age (years)	Small Pipes, 4–10 in.	Large Pipes, 12–60 in.
New	1.00	1.00
10	2.20	1.60
20	5.00	2.00
30	7.25	2.20
40	8.75	2.40
50	9.60	2.86
60	10.0	3.70
70	10.1	4.70

Consider again the pump and piping system of Problem 10.61. Estimate the percentage reductions in volume flow rate that occur after (a) 20 years and (b) 40 years of use, if the pump characteristics remain constant. Repeat the calculation if the pump head is reduced 10 percent after 20 years of use and 25 percent after 40 years.

Given: Data on pump and pipe system, and their aging

Find: Reduction in delivery through system after 20 and 40 years (aging and non-aging pumps)

Solution:

Given or available data **(Note: final results will vary depending on fluid data selected)** :

L =	1200	ft		K_{ent} =	0.5	(Fig. 8.14)
D =	12	in		K_{exp} =	1	
e =	0.00015	ft (Table 8.1)		L_e/D_{elbow} =	30	
ν =	1.23E-05	ft^2/s (Table A.7)		L_e/D_{valve} =	8	(Table 8.4)
Δz =	-50	ft				

Governing Equations:

For the pump and system

$$\left(\frac{p_1}{\rho}+\alpha_1\frac{\bar{V}_1^2}{2}+gz_1\right)-\left(\frac{p_2}{\rho}+\alpha_2\frac{\bar{V}_2^2}{2}+gz_2\right)=h_{l_T}-\Delta h_{pump} \qquad (8.49)$$

where the total head loss is comprised of major and minor losses

$$h_l = f\frac{L}{D}\frac{\bar{V}^2}{2} \qquad (8.34)$$

$$h_{l_m} = f\frac{L_e}{D}\frac{\bar{V}^2}{2} \qquad (8.40b)$$

$$h_{l_m} = K\frac{\bar{V}^2}{2} \qquad (8.40a)$$

Hence, applied between the two reservoir free surfaces ($p_1 = p_2 = 0$, $V_1 = V_2 = 0$, $z_1 - z_2 = \Delta z$) we have

$$g\cdot\Delta z = h_{lT} - \Delta h_{pump}$$

$$h_{lT} + g\cdot\Delta z = g\cdot H_{system} + g\cdot\Delta z = \Delta h_{pump} = g\cdot H_{pump}$$

or

$$H_{lT} + \Delta z = H_{pump}$$

where

$$H_{lT} = \left[f\cdot\left(\frac{L}{D}+2\cdot\frac{L_e}{D_{elbow}}+\frac{L_e}{D_{valve}}\right)+K_{ent}+K_{exit}\right]\cdot\frac{V^2}{2\cdot g}$$

The pump data is curve-fitted to $H_{pump} = H_0 - AQ^2$.
The system and pump heads are computed and plotted below.
To find the operating condition, *Solver* is used to vary Q
so that the error between the two heads is minimized.

New System:

Q (gpm)	Q^2 (gpm)	H_{pump} (ft)
0	0	179
500	250000	176
1000	1000000	165
1500	2250000	145
2000	4000000	119
2500	6250000	84
3000	9000000	43

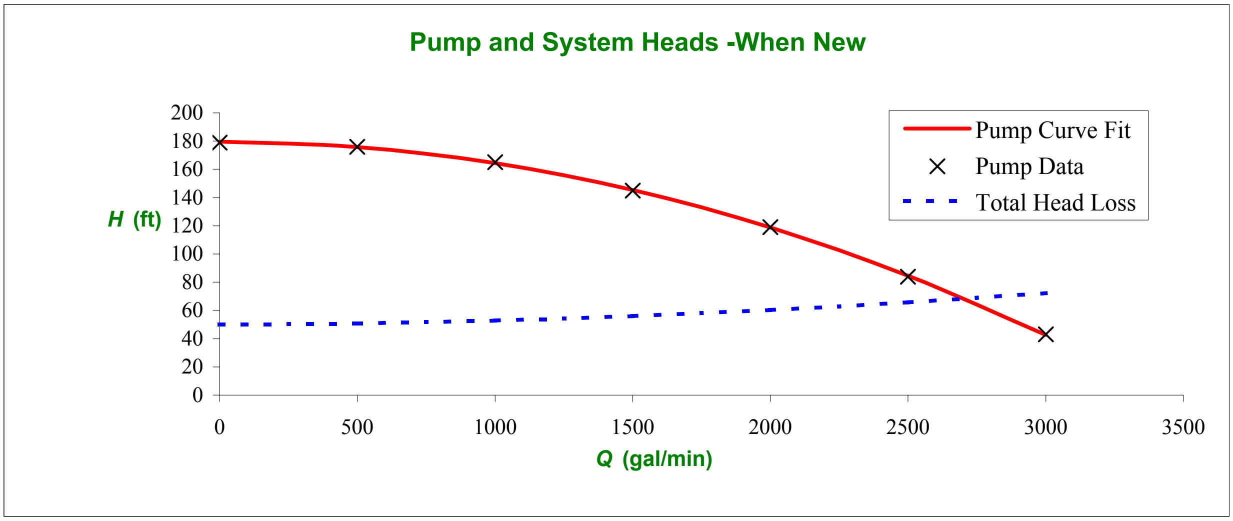
V (ft/s)	Re	f
0.00	0	0.0000
1.42	115325	0.0183
2.84	230649	0.0164
4.26	345974	0.0156
5.67	461299	0.0151
7.09	576623	0.0147
8.51	691948	0.0145

H_{pump} (fit)	$H_{lT} + \Delta z$ (ft)
180	50.0
176	50.8
164	52.8
145	56.0
119	60.3
84.5	65.8
42.7	72.4

$H_0 =$ 180 ft

$A =$ 1.52E-05 ft/(gpm)^2

Q (gpm)	V (ft/s)	Re	f	H_{pump} (fit)	$H_{lT} + \Delta z$ (ft)	Error)
2705	7.67	623829	0.0146	68.3	68.3	0%



20-Year Old System:

$f = 2.00\, f_{new}$

Q (gpm)	V (ft/s)	Re	f	H_{pump} (fit)	$H_{lT} + \Delta z$ (ft)	Error)
2541	7.21	586192	0.0295	81.4	81.4	0%

Flow reduction:
163 gpm
6.0% Loss

40-Year Old System:

$f = 2.40\, f_{new}$

Q (gpm)	V (ft/s)	Re	f	H_{pump} (fit)	$H_{lT} + \Delta z$ (ft)	Error)
2484	7.05	572843	0.0354	85.8	85.8	0%

Flow reduction:
221 gpm
8.2% Loss

20-Year Old System and Pump:

$f = 2.00\, f_{new}$ $H_{pump} = 0.90\, H_{new}$

Q (gpm)	V (ft/s)	Re	f	H_{pump} (fit)	$H_{lT} + \Delta z$ (ft)	Error)
2453	6.96	565685	0.0296	79.3	79.3	0%

Flow reduction:
252 gpm
9.3% Loss

40-Year Old System and Pump:

$f = 2.40\, f_{new}$ $H_{pump} = 0.75\, H_{new}$

Q (gpm)	V (ft/s)	Re	f	H_{pump} (fit)	$H_{lT} + \Delta z$ (ft)	Error)
2214	6.28	510754	0.0358	78.8	78.8	0%

Flow reduction:
490 gpm
18.1% Loss

Problem 10.64

[Difficulty: 3]

10.64 Consider again the pump and piping system of Problem 10.61. Determine the volume flow rate and gate valve loss coefficient for the case of two identical pumps installed in *parallel*.

Given:

Data on pump and pipe system

Find:

Delivery through parallel pump system; valve position to reduce delivery by half

Solution:

Governing Equations:

For the pumps and system

$$\left(\frac{p_1}{\rho} + \alpha_1 \frac{\bar{V}_1^2}{2} + gz_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \frac{\bar{V}_2^2}{2} + gz_2\right) = h_{l_T} - \Delta h_{\text{pump}} \qquad (8.49)$$

where the total head loss is comprised of major and minor losses

$$h_l = f\frac{L}{D}\frac{\bar{V}^2}{2} \qquad (8.34)$$

$$h_{l_m} = f\frac{L_e}{D}\frac{\bar{V}^2}{2} \qquad (8.40b)$$

$$h_{l_m} = K\frac{\bar{V}^2}{2} \qquad (8.40a)$$

Hence, applied between the two reservoir free surfaces ($p_1 = p_2 = 0$, $V_1 = V_2 = 0$, $z_1 - z_2 = \Delta z$) we have

$$g\cdot\Delta z = h_{lT} - \Delta h_{\text{pump}}$$

$$h_{lT} + g\cdot\Delta z = g\cdot H_{\text{system}} + g\cdot\Delta z = \Delta h_{\text{pump}} = g\cdot H_{\text{pump}}$$

or

$$H_{lT} + \Delta z = H_{\text{pump}}$$

where

$$H_{lT} = \left[f\cdot\left(\frac{L}{D} + 2\cdot\frac{L_e}{D_{\text{elbow}}} + \frac{L_e}{D_{\text{valve}}}\right) + K_{\text{ent}} + K_{\text{exit}}\right]\cdot\frac{V^2}{2\cdot g}$$

For pumps in parallel

$$H_{\text{pump}} = H_0 - \frac{1}{4}\cdot A\cdot Q^2$$

where for a single pump

$$H_{\text{pump}} = H_0 - A\cdot Q^2$$

The calculations performed using *Excel* are shown on the next page.

Given or available data **(Note: final results will vary depending on fluid data selected)**:

L =	1200	ft	K_{ent} =	0.5	(Fig. 8.14)
D =	12	in	K_{exp} =	1	
e =	0.00015	ft (Table 8.1)	L_e/D_{elbow} =	30	
ν =	1.23E-05	ft^2/s (Table A.7)	L_e/D_{valve} =	8	(Table 8.4)
Δz =	-50	ft			

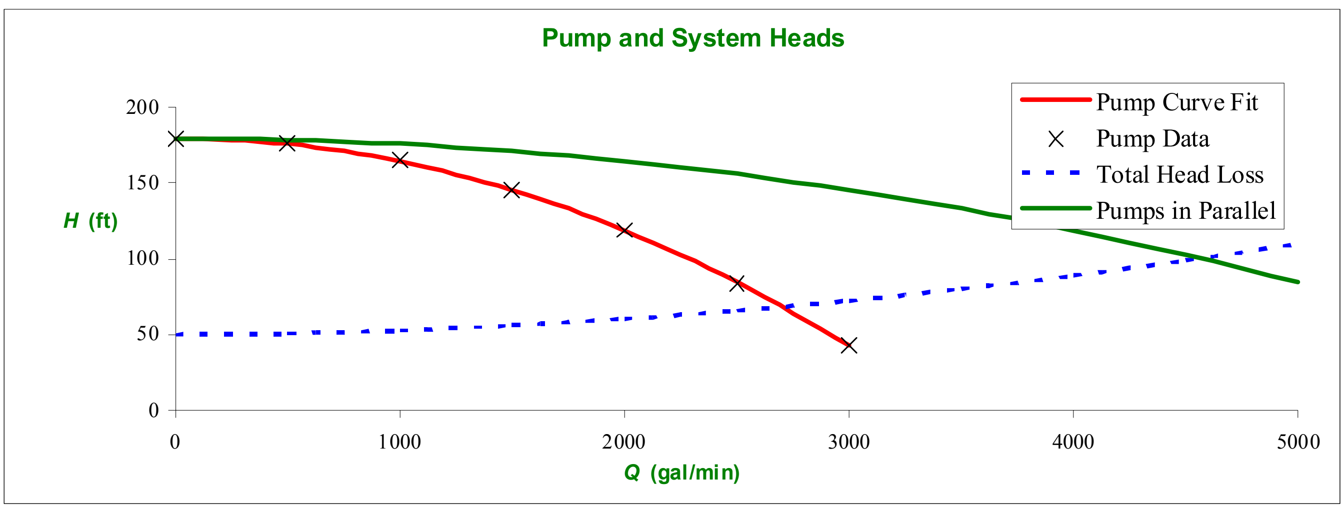
The pump data is curve-fitted to $H_{pump} = H_0 - AQ^2$.
The system and pump heads are computed and plotted below.
To find the operating condition, *Solver* is used to vary Q
so that the error between the two heads is minimized.

Q (gpm)	Q^2 (gpm)	H_{pump} (ft)	H_{pump} (fit)	V (ft/s)	Re	f
0	0	179	180	0.00	0	0.0000
500	250000	176	176	1.42	115325	0.0183
1000	1000000	165	164	2.84	230649	0.0164
1500	2250000	145	145	4.26	345974	0.0156
2000	4000000	119	119	5.67	461299	0.0151
2500	6250000	84	85	7.09	576623	0.0147
3000	9000000	43	43	8.51	691948	0.0145
3500				9.93	807273	0.0143
4000				11.35	922597	0.0142
4500				12.77	1037922	0.0141
5000				14.18	1153247	0.0140

H_{pumps} (par)	$H_{lT} + \Delta z$ (ft)
180	50.0
179	50.8
176	52.8
171	56.0
164	60.3
156	65.8
145	72.4
133	80.1
119	89.0
103	98.9
85	110.1

H_0 = 180 ft
A = 1.52E-05 ft/(gpm)^2

Q (gpm)	V (ft/s)	Re	f	H_{pumps} (par)	$H_{lT} + \Delta z$ (ft)	Error)
4565	12.95	1053006	0.0141	100.3	100.3	0%



For the valve setting to reduce the flow by half, use *Solver* to vary the value below to minimize the error.

L_e/D_{valve} = 9965

Q (gpm)	V (ft/s)	Re	f	H_{pumps} (par)	$H_{lT} + \Delta z$ (ft)	Error)
2283	6.48	526503	0.0149	159.7	159.7	0%

Problem 10.65

[Difficulty: 4]

10.65 Consider again the pump and piping system of Problem 10.64. Estimate the percentage reductions in volume flow rate that occur after (a) 20 years and (b) 40 years of use, if the pump characteristics remain constant. Repeat the calculation if the pump head is reduced 10 percent after 20 years of use and 25 percent after 40 years. (Use the data of Problem 10.63 for increase in pipe friction factor with age.)

Given: Data on pump and pipe system

Find: Delivery through parallel pump system; reduction in delivery after 20 and 40 years

Solution:

Given or available data **(Note: final results will vary depending on fluid data selected)** :

L =	1200	ft	K_{ent} =	0.5	(Fig. 8.14)
D =	12	in	K_{exp} =	1	
e =	0.00015	ft (Table 8.1)	L_e/D_{elbow} =	30	
ν =	1.23E-05	ft^2/s (Table A.7)	L_e/D_{valve} =	8	(Table 8.4)
Δz =	-50	ft			

Governing Equations:

For the pumps and system

$$\left(\frac{p_1}{\rho}+\alpha_1\frac{\bar{V}_1^2}{2}+gz_1\right)-\left(\frac{p_2}{\rho}+\alpha_2\frac{\bar{V}_2^2}{2}+gz_2\right)=h_{l_T}-\Delta h_{pump} \qquad (8.49)$$

where the total head loss is comprised of major and minor losses

$$h_l = f\frac{L}{D}\frac{\bar{V}^2}{2} \qquad (8.34)$$

$$h_{l_m} = f\frac{L_e}{D}\frac{\bar{V}^2}{2} \qquad (8.40b)$$

$$h_{l_m} = K\frac{\bar{V}^2}{2} \qquad (8.40a)$$

Hence, applied between the two reservoir free surfaces ($p_1 = p_2 = 0$, $V_1 = V_2 = 0$, $z_1 - z_2 = \Delta z$) we have

$$g\cdot\Delta z = h_{lT} - \Delta h_{pump}$$

$$h_{lT} + g\cdot\Delta z = g\cdot H_{system} + g\cdot\Delta z = \Delta h_{pump} = g\cdot H_{pump}$$

or

$$H_{lT} + \Delta z = H_{pump}$$

where

$$H_{lT} = \left[f\cdot\left(\frac{L}{D}+2\cdot\frac{L_e}{D_{elbow}}+\frac{L_e}{D_{valve}}\right)+K_{ent}+K_{exit}\right]\cdot\frac{V^2}{2\cdot g}$$

For pumps in parallel

$$H_{pump} = H_0 - \frac{1}{4}\cdot A\cdot Q^2$$

where for a single pump

$$H_{pump} = H_0 - A\cdot Q^2$$

The pump data is curve-fitted to $H_{pump} = H_0 - AQ^2$.
The system and pump heads are computed and plotted below.
To find the operating condition, *Solver* is used to vary Q
so that the error between the two heads is minimized.

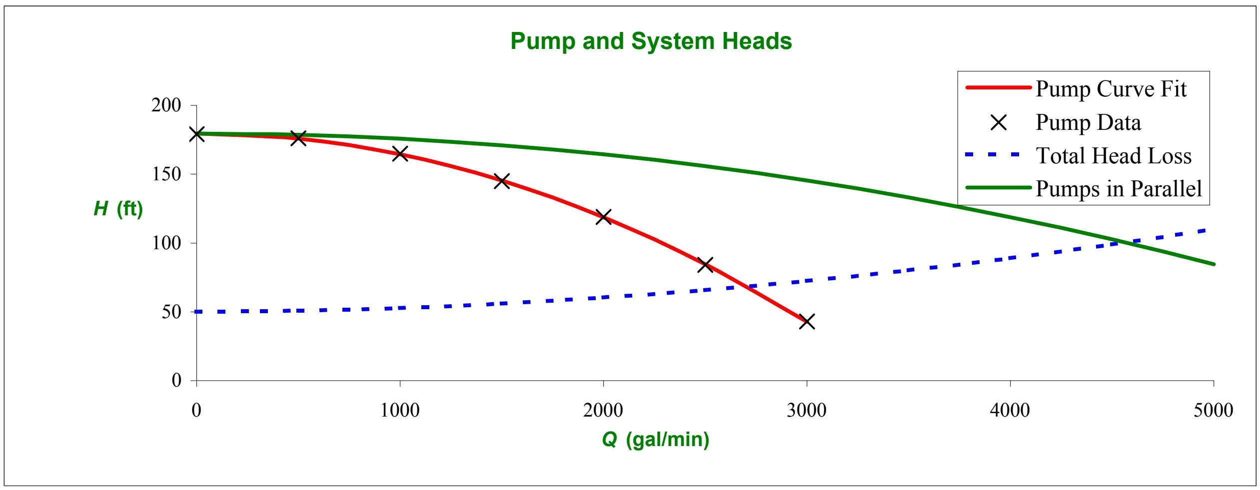
Q (gpm)	Q^2 (gpm)	H_{pump} (ft)	H_{pump} (fit)	V (ft/s)	Re	f
0	0	179	180	0.00	0	0.0000
500	250000	176	176	1.42	115325	0.0183
1000	1000000	165	164	2.84	230649	0.0164
1500	2250000	145	145	4.26	345974	0.0156
2000	4000000	119	119	5.67	461299	0.0151
2500	6250000	84	85	7.09	576623	0.0147
3000	9000000	43	43	8.51	691948	0.0145
3500				9.93	807273	0.0143
4000				11.35	922597	0.0142
4500				12.77	1037922	0.0141
5000				14.18	1153247	0.0140

H_{pumps} (par)	$H_{IT} + \Delta z$ (ft)
180	50.0
179	50.8
176	52.8
171	56.0
164	60.3
156	65.8
145	72.4
133	80.1
119	89.0
103	98.9
85	110.1

$H_0 =$ 180 ft

$A =$ 1.52E-05 ft/(gpm)^2

Q (gpm)	V (ft/s)	Re	f	H_{pumps} (par)	$H_{IT} + \Delta z$ (ft)	Error)
4565	12.95	1053006	0.0141	100.3	100.3	0%



20-Year Old System:

$f = 2.00\, f_{new}$

Q (gpm)	V (ft/s)	Re	f	H_{pumps} (par)	$H_{IT} + \Delta z$ (ft)	Error)
3906	11.08	900891	0.0284	121.6	121.6	0%

Flow reduction:

660 gpm

14.4% Loss

40-Year Old System:

$f = 2.40\, f_{new}$

Q (gpm)	V (ft/s)	Re	f	H_{pump} (fit)	$H_{IT} + \Delta z$ (ft)	Error)
3710	10.52	855662	0.0342	127.2	127.2	0%

Flow reduction:

856

18.7%

20-Year Old System and Pumps:

$f = 2.00\, f_{new}$ $\quad H_{pump} = 0.90\, H_{new}$

Q (gpm)	V (ft/s)	Re	f	H_{pump} (fit)	$H_{IT} + \Delta z$ (ft)	Error)
3705	10.51	854566	0.0285	114.6	114.6	0%

Flow reduction:

860 gpm

18.8% Loss

40-Year Old System and Pumps:

$f = 2.40\, f_{new}$ $\quad H_{pump} = 0.75\, H_{new}$

Q (gpm)	V (ft/s)	Re	f	H_{pump} (fit)	$H_{IT} + \Delta z$ (ft)	Error)
3150	8.94	726482	0.0347	106.4	106.4	0%

Flow reduction:

1416

31.0%

Problem 10.66

[Difficulty: 4]

10.66 Consider again the pump and piping system of Problem 10.62. Estimate the percentage reductions in volume flow rate that occur after (a) 20 years and (b) 40 years of use, if the pump characteristics remain constant. Repeat the calculation if the pump head is reduced 10 percent after 20 years of use and 25 percent after 40 years. (Use the data of Problem 10.63 for increase in pipe friction factor with age.)

Given:
Data on pump and pipe system

Find:
Delivery through series pump system; reduction after 20 and 40 years

Solution:

Given or available data **(Note: final results will vary depending on fluid data selected)** :

L =	1200	ft	K_{ent} =	0.5	(Fig. 8.14)
D =	12	in	K_{exp} =	1	
e =	0.00015	ft (Table 8.1)	L_e/D_{elbow} =	30	
ν =	1.23E-05	ft^2/s (Table A.7)	L_e/D_{valve} =	8	(Table 8.4)
Δz =	-50	ft			

Governing Equations:

For the pumps and system

$$\left(\frac{p_1}{\rho}+\alpha_1\frac{\bar{V}_1^2}{2}+gz_1\right)-\left(\frac{p_2}{\rho}+\alpha_2\frac{\bar{V}_2^2}{2}+gz_2\right)=h_{l_T}-\Delta h_{pump} \qquad (8.49)$$

where the total head loss is comprised of major and minor losses

$$h_l = f\frac{L}{D}\frac{\bar{V}^2}{2} \qquad (8.34)$$

$$h_{l_m} = f\frac{L_e}{D}\frac{\bar{V}^2}{2} \qquad (8.40b)$$

$$h_{l_m} = K\frac{\bar{V}^2}{2} \qquad (8.40a)$$

Hence, applied between the two reservoir free surfaces ($p_1 = p_2 = 0$, $V_1 = V_2 = 0$, $z_1 - z_2 = \Delta z$) we have

$$g\cdot\Delta z = h_{lT} - \Delta h_{pump}$$

$$h_{lT} + g\cdot\Delta z = g\cdot H_{system} + g\cdot\Delta z = \Delta h_{pump} = g\cdot H_{pump}$$

or

$$H_{lT} + \Delta z = H_{pump}$$

where

$$H_{lT} = \left[f\cdot\left(\frac{L}{D}+2\cdot\frac{L_e}{D_{elbow}}+\frac{L_e}{D_{valve}}\right)+K_{ent}+K_{exit}\right]\cdot\frac{V^2}{2\cdot g}$$

For pumps in series

$$H_{pump} = 2\cdot H_0 - 2\cdot A\cdot Q^2$$

where for a single pump

$$H_{pump} = H_0 - A\cdot Q^2$$

The pump data is curve-fitted to $H_{pump} = H_0 - AQ^2$.
The system and pump heads are computed and plotted below.
To find the operating condition, *Solver* is used to vary Q
so that the error between the two heads is minimized.

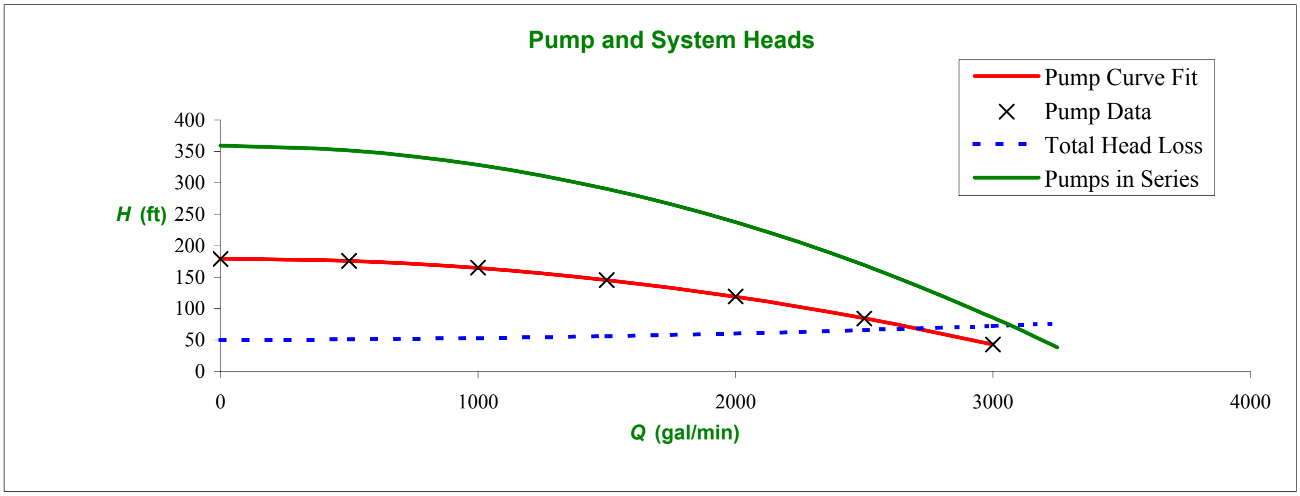
Q (gpm)	Q^2 (gpm)	H_{pump} (ft)	H_{pump} (fit)	V (ft/s)	Re	f
0	0	179	180	0.00	0	0.0000
500	250000	176	176	1.42	115325	0.0183
1000	1000000	165	164	2.84	230649	0.0164
1500	2250000	145	145	4.26	345974	0.0156
2000	4000000	119	119	5.67	461299	0.0151
2500	6250000	84	85	7.09	576623	0.0147
3000	9000000	43	43	8.51	691948	0.0145
3250				9.22	749610	0.0144

H_{pumps} (par)	$H_{lT} + \Delta z$ (ft)
359	50.0
351	50.8
329	52.8
291	56.0
237	60.3
169	65.8
85	72.4
38	76.1

$H_0 =$ 180 ft

$A =$ 1.52E-05 ft/(gpm)^2

Q (gpm)	V (ft/s)	Re	f	H_{pumps} (par)	$H_{lT} + \Delta z$ (ft)	Error)
3066	8.70	707124	0.0145	73.3	73.3	0%



20-Year Old System:

$f = 2.00\, f_{new}$

Q (gpm)	V (ft/s)	Re	f	H_{pumps} (par)	$H_{lT} + \Delta z$ (ft)	Error)
2964	8.41	683540	0.0291	92.1	92.1	0%

Flow reduction:
102 gpm
3.3% Loss

40-Year Old System:

$f = 2.40\, f_{new}$

Q (gpm)	V (ft/s)	Re	f	H_{pump} (fit)	$H_{lT} + \Delta z$ (ft)	Error)
2925	8.30	674713	0.0349	98.9	98.9	0%

Flow reduction:
141 gpm
4.6% Loss

20-Year Old System and Pumps:

$f = 2.00\, f_{new}$ $H_{pump} = 0.90\, H_{new}$

Q (gpm)	V (ft/s)	Re	f	H_{pump} (fit)	$H_{lT} + \Delta z$ (ft)	Error)
2915	8.27	672235	0.0291	90.8	90.8	0%

Flow reduction:
151 gpm
4.9% Loss

40-Year Old System and Pumps:

$f = 2.40\, f_{new}$ $H_{pump} = 0.75\, H_{new}$

Q (gpm)	V (ft/s)	Re	f	H_{pump} (fit)	$H_{lT} + \Delta z$ (ft)	Error)
2772	7.86	639318	0.0351	94.1	94.1	0%

Flow reduction:
294 gpm
9.6% Loss

Problem 10.67

[Difficulty: 3]

10.67 The city of Englewood, Colorado, diverts water for municipal use from the South Platte River at elevation 1610 m [54]. The water is pumped to storage reservoirs at 1620-m elevation. The inside diameter of the steel water line is 68.5 cm; its length is 1770 m. The facility is designed for an initial capacity (flow rate) of 3200 m^3/hr, with an ultimate capacity of 3900 m^3/hr. Calculate and plot the system resistance curve. Ignore entrance losses. Specify an appropriate pumping system. Estimate the pumping power required for steady-state operation, at both the initial and ultimate flow rates.

Given: Water supply for Englewood, CO

Find:
(a) system resistance curve
(b) specify appropriate pumping system
(c) estimate power required for steady-state operation at two specified flow rates

Solution:

Basic equations:

$$\left(\frac{p_1}{\rho} + \alpha_1 \cdot \frac{V_1^2}{2} + g \cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \cdot \frac{V_2^2}{2} + g \cdot z_2\right) = h_{lT} - h_p$$

$$h_{lT} = f \cdot \frac{L}{D} \cdot \frac{V^2}{2} + \Sigma \cdot f \cdot \frac{L_e}{D} \cdot \frac{V^2}{2} + \Sigma \cdot K \cdot \frac{V^2}{2} \qquad H = \frac{h}{g} \qquad W_p = \frac{\rho \cdot Q \cdot g \cdot H_p}{\eta_p}$$

Assumptions: 1) $p_1 = p_2 = p_{atm}$ 2) $V_1 = V_2 = 0$ 3) $K_{ent} = 0$ 4) $K_{exit} = 1$ 5) $L_e/D = 0$

Hence $$g \cdot (z_1 - z_2) = \left(f \cdot \frac{L}{D} + 1\right) \cdot \frac{V^2}{2} - h_p \quad \text{or} \quad H_p = (z_2 - z_1) + \left(f \cdot \frac{L}{D} + 1\right) \cdot \frac{V^2}{2 \cdot g}$$

The results calculated using Excel are shown below:

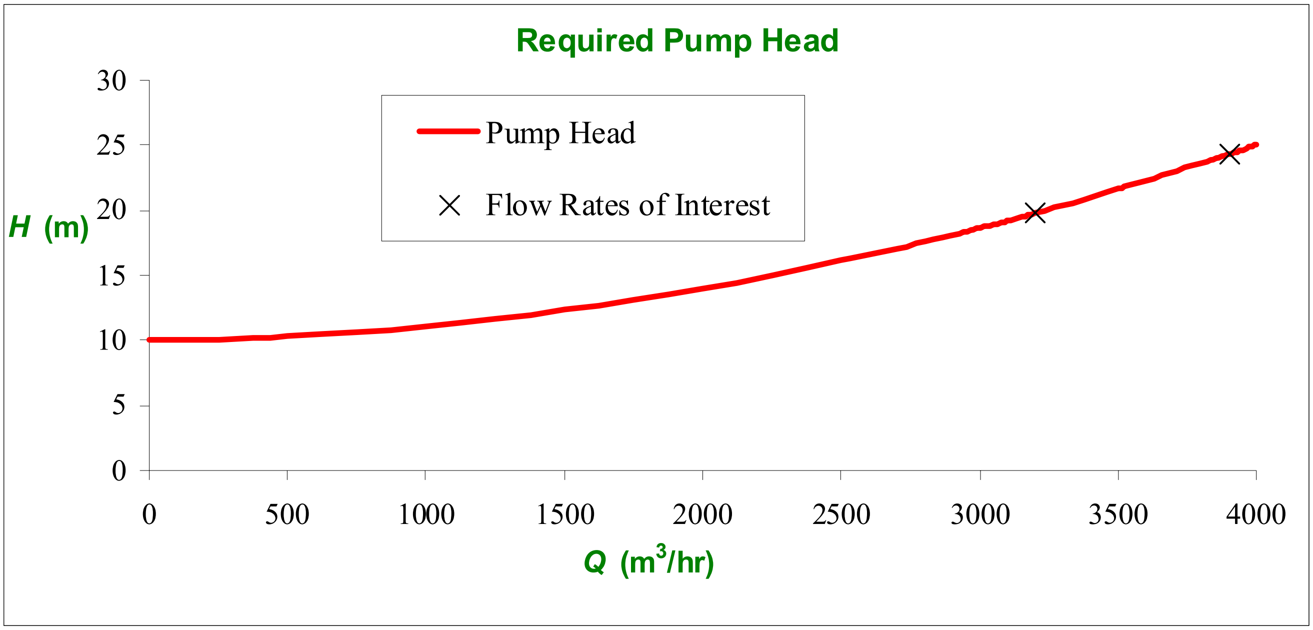
Given or available data **(Note: final results will vary depending on fluid data selected)**:

L = 1770 m
e = 0.046 mm (Table 8.1)
D = 68.5 cm
ν = 1.01E-06 m^2/s (Table A.8)

z_1 = 1610 m
z_2 = 1620 m
ρ = 998 kg/m^3

The required pump head is computed and plotted below.

Q (m^3/hr)	V (m/s)	Re	f	H_p (m)
0	0.00	0.00E+00	0.0000	10.0
500	0.38	2.56E+05	0.0155	10.3
1000	0.75	5.11E+05	0.0140	11.1
1500	1.13	7.67E+05	0.0133	12.3
2000	1.51	1.02E+06	0.0129	14.0
2500	1.88	1.28E+06	0.0126	16.1
3000	2.26	1.53E+06	0.0124	18.6
3200	2.41	1.64E+06	0.0124	19.8
3500	2.64	1.79E+06	0.0123	21.6
3900	2.94	1.99E+06	0.0122	24.3
4000	3.01	2.04E+06	0.0122	25.0



The maximum flow rate is:	17172	gpm
The associated head is:	80	ft

Based on these data and the data of Figures D.1 and D.2, we could choose two 16A 18B pumps in parallel, or three 10AE14 (G) pumps in parallel. The efficiency will be approximately 90%

Therefore, the required power would be:	191.21	kW at Q =	3200	m^3/hr
	286.47	kW at Q =	3900	m^3/hr

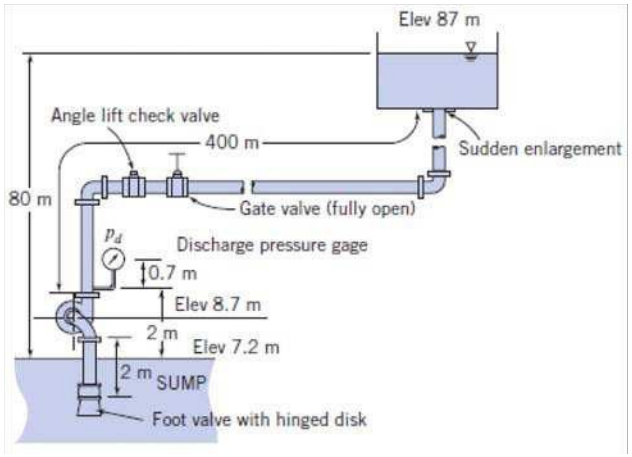
Problem 10.68

[Difficulty: 3]

10.68 A pump in the system shown draws water from a sump and delivers it to an open tank through 400 m of new, 10-cm-diameter steel pipe. The vertical suction pipe is 2 m long and includes a foot valve with hinged disk and a 90° standard elbow. The discharge line includes two 90° standard elbows, an angle lift check valve, and a fully open gate valve. The design flow rate is 800 L/min. Find the head losses in the suction and discharge lines. Calculate the *NPSHA*. Select a pump suitable for this application.

Given: System shown, design flow rate

Find: Head losses for suction and discharge lines, NPSHA, select a suitable pump



Solution:

We will apply the energy equation for steady, incompressible pipe flow.

Basic equations:

$$\left(\frac{p_1}{\rho} + \alpha_1 \cdot \frac{V_1^2}{2} + g \cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \cdot \frac{V_2^2}{2} + g \cdot z_2\right) = h_{lT} - h_p$$

$$h_{lT} = f \cdot \frac{L}{D} \cdot \frac{V^2}{2} + \Sigma \cdot f \cdot \frac{L_e}{D} \cdot \frac{V^2}{2} + \Sigma \cdot K \cdot \frac{V^2}{2} \qquad H = \frac{h}{g}$$

Assumptions: 1) $p_{ent} = p_{exit} = p_{atm}$ 2) $V_{ent} = V_{exit} = 0$

The given or available data is $Q = 800 \cdot \frac{L}{min}$ $\quad D = 10 \cdot cm$ $\quad e = 0.046 \cdot mm$ $\quad p_{atm} = 101.3 \cdot kPa$

From Table A.8 at 20°C $\nu = 1.01 \times 10^{-6} \cdot \frac{m^2}{s}$ $\quad p_v = 2.34 \cdot kPa$ $\quad \rho = 998 \cdot \frac{kg}{m^3}$

At the specified flow rate, the speed of the water is: $V = \frac{Q}{A} = \frac{4 \cdot Q}{\pi \cdot D^2}$ $\quad V = 1.698 \frac{m}{s}$ $\quad Re = \frac{V \cdot D}{\nu} = 1.681 \times 10^5$

$\frac{e}{D} = 4.6 \times 10^{-4}$ Therefore we can calculate the friction factor: $f = \left[-1.8 \cdot \log\left[\left(\frac{e}{3.7 \cdot D}\right)^{1.11} + \frac{6.9}{Re}\right]\right]^{-2} = 0.019$

At the inlet: $g \cdot z_1 - \left(\frac{p_2}{\rho} + \alpha_2 \cdot \frac{V_2^2}{2} + g \cdot z_2\right) = f \cdot \left(\frac{L}{D} + \frac{L_e}{D}\right) \cdot \frac{V^2}{2} + \Sigma \cdot K \cdot \frac{V^2}{2}$ In this case: $L_e = 75 \cdot D$ $\quad K = 0.78$ $\quad L = 2 \cdot m$

$z_2 = 8.7 \cdot m$ $\quad z_1 = 7.2 \cdot m$

Solving for total pressure at 2: $p_{2t} = -\rho \cdot \left[g \cdot (z_2 - z_1) + f \cdot \left(\frac{L}{D} + \frac{L_e}{D}\right) \cdot \frac{V^2}{2} + K \cdot \frac{V^2}{2}\right]$ $\quad p_{2t} = -18.362 \cdot kPa$ (gage)

The NPSHA can be calculated: $NPSHA = \frac{p_{2tabs} - p_v}{\rho \cdot g}$ $\quad NPSHA = \frac{p_{2t} + p_{atm} - p_v}{\rho \cdot g}$ $\quad NPSHA = 8.24\,m$

For the entire system: $g\cdot\left(z_1 - z_2\right) = f\cdot\frac{L}{D}\cdot\frac{V^2}{2} + f\cdot\frac{L_e}{D}\cdot\frac{V^2}{2} + K\cdot\frac{V^2}{2} - h_p$

In this case: $z_1 = 7.2\cdot m$ $z_2 = 88\cdot m$ $L = 2\cdot m + 400\cdot m = 402\,m$ $L_e = (75 + 55 + 8 + 2 \times 30)\cdot D$ $K = 0.78 + 1$

Solving for the required head at the pump: $H_p = \left(z_2 - z_1\right) + \left[\left[f\cdot\left(\frac{L}{D} + \frac{L_e}{D}\right) + K\right]\cdot\frac{V^2}{2\cdot g}\right]$ $H_p = 92.7\,m$

In U.S. Customary units: $Q = 211\cdot gpm$ $H_p = 304\cdot ft$

A pump would be selected by finding one for which the NPSHR is less than the NPSHA. Based on these data and the information in Appendix D, a 2AE11 or a 4AE12 pump would be capable of supplying the required head at the given flow rate. The pump should be operated at a speed between 1750 and 3500 rpm, but the efficiency may not be acceptable. One should consult a complete catalog to make a better selection.

Problem 10.69

[Difficulty: 3]

10.69 Consider the flow system described in Problem 8.175. Select a pump appropriate for this application. Check the *NPSHR* versus the *NPSHA* for this system.

Solution: Apply the energy equation for steady, incompressible pipe flow.

Computing equation: $\frac{p_1}{\rho g}+\frac{\alpha_1 \bar{V}_1^2}{2g}+z_1+H_p=\frac{p_2}{\rho g}+\frac{\alpha_2 \bar{V}_2^2}{2g}+z_2+\frac{h_{lT}}{g}$; $h_{lT}=\left[f\left(\frac{L}{D}+\frac{L_e}{D}\right)+K\right]\frac{\bar{V}^2}{2}$ ($\bar{V}_1 \approx 0$ (1))

Assumptions: (1) $\bar{V}_1 \approx 0$, (2) $K_{ent}=0.5$, (3) $\frac{L_e}{D}=2(8)+1(150)+7(30)=376$; $D=2.47$ in.

(4) Galvanized pipe, $e=0.0005$ ft; $\frac{e}{D}=\frac{0.0005\text{ ft}}{2.47\text{ in.}}\times\frac{12\text{ in.}}{\text{ft}}=0.00243$

Then

$$H_p=\frac{p_2-p_1}{\rho g}+\frac{\alpha_2\bar{V}_2^2}{2g}+z_2-z_1+\left[f\left(\frac{L}{D}+\frac{L_e}{D}\right)+K\right]\frac{\bar{V}^2}{2g};\ \bar{V}=\frac{Q}{A}=13.2\text{ ft/s};\ Re=\frac{\bar{V}D}{\nu}=2.54\times10^5;\ f=0.02$$

$$H_p=(0-20)\frac{\text{lbf}}{\text{in.}^2}\times 144\frac{\text{in.}^2}{\text{ft}^2}\times\frac{\text{ft}^3}{62.4\text{ lbf}}+\frac{1}{2}\times(13.2)^2\frac{\text{ft}^2}{\text{s}^2}\times\frac{\text{s}^2}{32.2\text{ ft}}+(94.0-50.0)\text{ ft}$$

$$+\left[0.025\left(\frac{290}{2.47/12}+376\right)+0.5\right]\frac{1}{2}\times(13.2)^2\frac{\text{ft}^2}{\text{s}^2}\cdot\frac{\text{s}^2}{32.2\text{ ft}}=123\text{ ft}$$

H_p

The pump requirement is $Q=197$ gpm at $H=123$ ft. This could be supplied by a Peerless Type 4AE12 pump, with impeller $D=11$ in., operating at 1750 rpm.

(This pump may be slightly too large, since this operating point is at a flow rate below that for best efficiency.)

The NPSHR for this pump at $Q=197$ gpm is about 5 ft.

The NPSHA is $\frac{p_1}{\rho g}+\frac{p_{atm}}{\rho g}+\frac{\alpha_1\bar{V}_1^2}{2g}-p_v=46.2+33.9+2.71-0.782\text{ ft}=82.0\text{ ft}$

Thus NPSHA >> NPSHR

NPSH

Cavitation-free operation is assured.

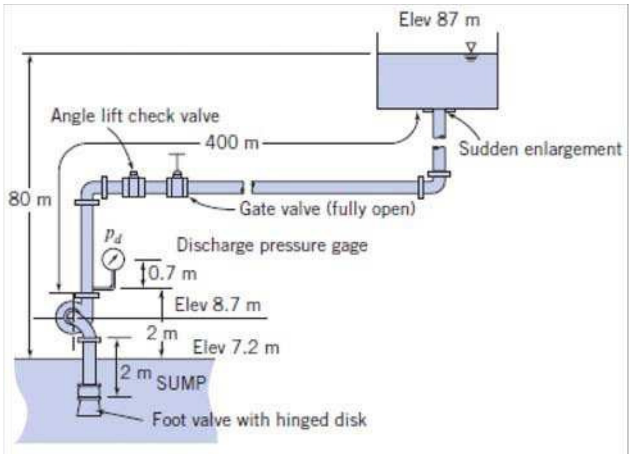
Problem 10.70

[Difficulty: 3]

10.70 Consider the flow system and data of Problem 10.68 and the data for pipe aging given in Problem 10.63. Select pump(s) that will maintain the system flow at the desired rate for (a) 10 years and (b) 20 years. Compare the delivery produced by these pumps with the delivery by the pump sized for new pipes only.

Given: Flow system and data of Problem 10.68; data for pipe aging from Problem 10.63

Find: Pumps to maintain system flow rates; compare delivery to that with pump sized for new pipes only



Solution:

We will apply the energy equation for steady, incompressible pipe flow.

Basic equations:

$$\left(\frac{p_1}{\rho} + \alpha_1 \cdot \frac{V_1^2}{2} + g \cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \cdot \frac{V_2^2}{2} + g \cdot z_2\right) = h_{lT} - h_p$$

$$h_{lT} = f \cdot \frac{L}{D} \cdot \frac{V^2}{2} + \Sigma \cdot f \cdot \frac{L_e}{D} \cdot \frac{V^2}{2} + \Sigma \cdot K \cdot \frac{V^2}{2} \qquad H = \frac{h}{g}$$

Assumptions: 1) $p_{ent} = p_{exit} = p_{atm}$ 2) $V_{ent} = V_{exit} = 0$

The given or available data is $Q = 800 \cdot \frac{L}{min}$ $D = 10 \cdot cm$ $e = 0.046\,mm$ $p_{atm} = 101.3\,kPa$

From Table A.8 at 20ºC $\nu = 1.01 \times 10^{-6} \cdot \frac{m^2}{s}$ $p_v = 2.34\,kPa$ $\rho = 998 \cdot \frac{kg}{m^3}$

At the specified flow rate, the speed of the water is: $V = \frac{Q}{A} = \frac{4 \cdot Q}{\pi \cdot D^2}$ $V = 1.698 \frac{m}{s}$ $Re = \frac{V \cdot D}{\nu} = 1.681 \times 10^5$

$\frac{e}{D} = 4.6 \times 10^{-4}$ Therefore we can calculate the friction factor: $f = \left[-1.8 \cdot \log\left[\left(\frac{e}{3.7 \cdot D}\right)^{1.11} + \frac{6.9}{Re}\right]\right]^{-2} = 0.019$

For the entire system: $g \cdot (z_1 - z_2) = f \cdot \frac{L}{D} \cdot \frac{V^2}{2} + f \cdot \frac{L_e}{D} \cdot \frac{V^2}{2} + K \cdot \frac{V^2}{2} - h_p$

In this case: $z_1 = 7.2 \cdot m$ $z_2 = 87 \cdot m$ $L = 2 \cdot m + 400 \cdot m = 402\,m$ $L_e = (75 + 55 + 8 + 2 \times 30) \cdot D$ $K = 0.78 + 1$

Solving for the required head at the pump: $H_p = (z_2 - z_1) + \left[\left[f \cdot \left(\frac{L}{D} + \frac{L_e}{D}\right) + K\right] \cdot \frac{V^2}{2 \cdot g}\right]$

For old pipes, we apply the multipliers from Problem 10.63: $f_{20} = 5.00 \cdot f_{new}$ $f_{40} = 8.75 \cdot f_{new}$

The results of the analysis, computed in *Excel*, are shown on the next page.

The required pump head is computed and plotted below.

				New		New	20 yo	40 yo	
Q (L/min)	V (m/s)	Re	f	H_p (m)	Q (gpm)	H_p (ft)	H_p (ft)	H_p (ft)	Pump (ft)
0	0.000	0.00E+00	0.0000	79.80	0.00	261.81	261.81	261.81	856.54
200	0.424	4.20E+04	0.0231	80.71	52.84	264.81	276.57	287.59	840.48
400	0.849	8.40E+04	0.0207	83.07	105.68	272.52	314.52	353.89	792.30
600	1.273	1.26E+05	0.0196	86.77	158.52	284.67	374.19	458.10	712.00
800	1.698	1.68E+05	0.0189	91.80	211.36	301.17	455.19	599.58	599.58
922	1.957	1.94E+05	0.0187	95.52	243.64	313.38	515.10	704.21	515.10
1000	2.122	2.10E+05	0.0185	98.14	264.20	321.99	557.37	778.03	455.04
1136	2.410	2.39E+05	0.0183	103.20	300.08	338.59	638.8	920.2	338.59
1200	2.546	2.52E+05	0.0182	105.80	317.04	347.12	680.64	993.31	278.38
1400	2.971	2.94E+05	0.0180	114.77	369.88	376.54	824.95	1245.3	69.59
1600	3.395	3.36E+05	0.0178	125.04	422.72	410.25	990.27	1534.0	-171.31
1800	3.820	3.78E+05	0.0177	136.62	475.56	448.24	1176.6	1859.4	-444.33
2000	4.244	4.20E+05	0.0176	149.51	528.40	490.51	1383.9	2221.4	-749.47

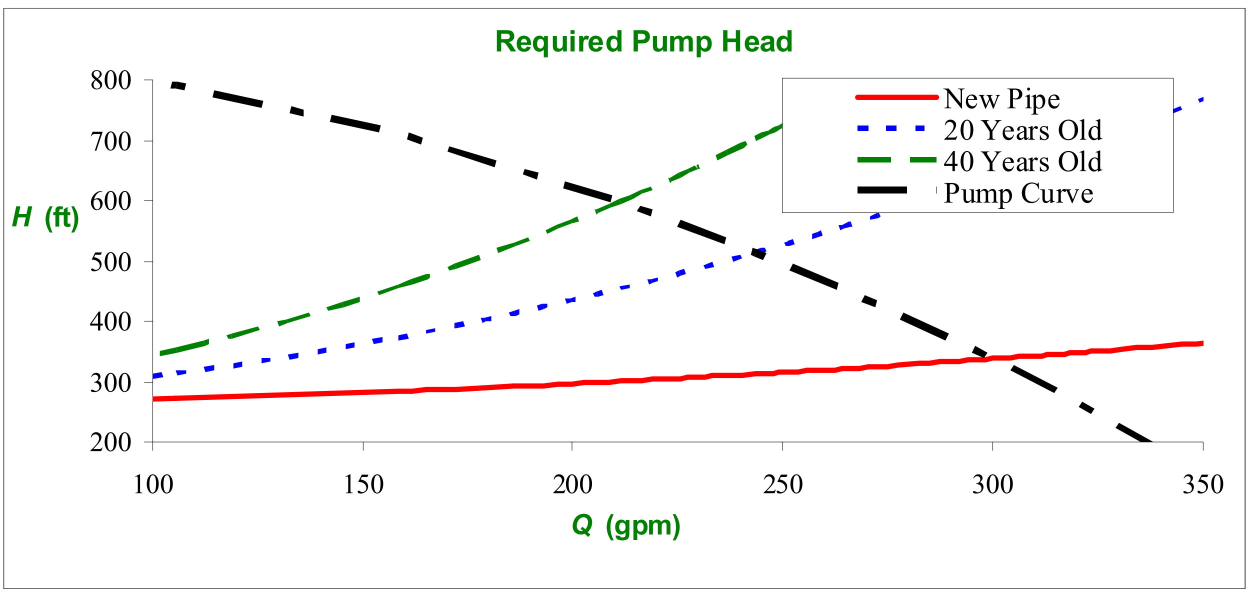
If we assume that the head at 800 L/min for 40 year old pipe is 70% of the maximum head for the pump, and that the pump curve has the form $H = H_0 - AQ^2$:

$H_{800} =$ 599.58 ft

$H_0 =$ 856.54 ft

$A =$ 0.005752 ft/gpm^2

We plot the pump curve along with the head loss on the graph below:



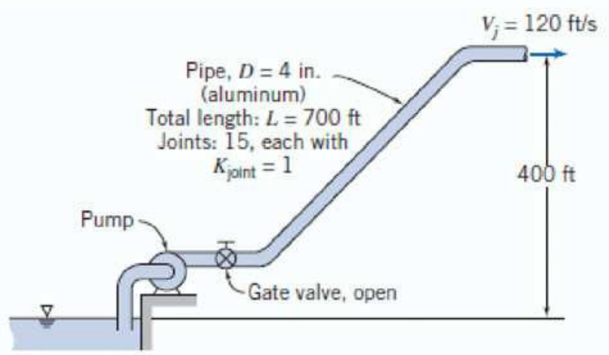
Sizing the pump for 800 L/min for at 40 years would (assuming no change in the pump characteristics) produce 922 L/min at 20 years and 1136 L/min for new pipe.

Since the head increases by a factor of two, the extra head could be obtained by placing a second identical pump in series with the pump of Problem 10.68.

Problem 10.71

[Difficulty: 3]

10.71 Consider the flow system shown in Problem 8.176. Select an appropriate pump for this application. Check the pump efficiency and power requirement compared with those in the problem statement.



Solution: Apply the energy equation for steady, incompressible pipe flow.

Computing equation: $\frac{p_1}{\rho g} + \alpha_1 \frac{\bar{V}_1^2}{2g} + z_1 + H_p = \frac{p_2}{\rho g} + \alpha_2 \frac{\bar{V}_2^2}{2g} + z_2 + \frac{h_{lT}}{g}$; $h_{lT} = \left[f\left(\frac{L}{D} + \frac{L_e}{D}\right) + K\right]\frac{\bar{V}^2}{2}$ ($\bar{V}_1 \approx 0$ (2))

Assumptions: (1) $p_1 = p_2 = p_{atm}$, (2) $\bar{V}_1 \approx 0$, (3) Neglect elbow and nozzle losses, (4) $\frac{L_e}{D}$(valve) $= 8$

$$\bar{V} = \frac{Q}{A} = 600\,\frac{gal}{min} \times \frac{4}{\pi}\left(\frac{12}{4}\right)^2 \frac{1}{ft^2} \times \frac{ft^3}{7.48\,gal} \times \frac{min}{60\,s} = 15.3\,\frac{ft}{s}\;;\quad \frac{\bar{V}^2}{2g} = \frac{1}{2} \times \frac{(15.3)^2 ft^2}{s^2} \times \frac{s^2}{32.2\,ft} = 3.63\,ft$$

$$Re = \frac{\bar{V}D}{\nu} = 15.3\,\frac{ft}{s} \times \left(\frac{4}{12}\right) ft \times \frac{s}{1.08 \times 10^{-5}\,ft^2} = 4.77 \times 10^5\,;\ \text{smooth};\ f = 0.013 \quad (T = 68°F)$$

$$H_p = z_2 - z_1 + \frac{\alpha_2 \bar{V}_2^2}{2g} + \frac{h_{lT}}{g} = z_2 - z_1 + \frac{\bar{V}_2^2}{2g} + \left[f\left(\frac{L}{D} + \frac{L_e}{D}\right) + 15 K_j\right]\frac{\bar{V}^2}{2g}$$

$$H_p = 400\,ft + \frac{1}{2} \times \frac{(120)^2 ft^2}{s^2} \times \frac{s^2}{32.2\,ft} + \left[0.013\left(\frac{700}{4/12} + 8\right) + 15\right] 3.63\,ft = 778\,ft$$

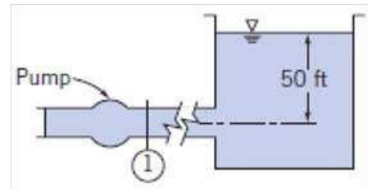
Thus the pump requirement is $H_p \approx 778$ ft at $Q = 600$ gpm. This head is too great to be developed by a single-stage pump (Fig. D.1). From Fig. D.12, the flow could be supplied by a 5TUT168 3-stage pump, driven at 1750 rpm.

Single-stage pumps have peak efficiencies of 86 percent at 1750 rpm (Fig. D.8). Thus 70 percent efficiency might be reasonable for a 3-stage pump, since $(0.86)^3 = 0.636$.

Problem 10.72

[Difficulty: 3]

10.72 Consider the flow system shown in Problem 8.124. Assume the minimum *NPSHR* at the pump inlet is 15 ft of water. Select a pump appropriate for this application. Use the data for increase in friction factor with pipe age given in Problem 10.65 to determine and compare the system flow rate after 10 years of operation.



Given: Flow from pump to reservoir

Find: Select a pump to satisfy NPSHR

Solution:

Basic equations $\left(\frac{p_1}{\rho} + \alpha\cdot\frac{V_1^{\,2}}{2} + g\cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha\cdot\frac{V_2^{\,2}}{2} + g\cdot z_2\right) = h_{lT} - h_p \qquad h_{lT} = h_l + h_{lm} = f\cdot\frac{L}{D}\cdot\frac{V_1^{\,2}}{2} + K_{exit}\cdot\frac{V_1^{\,2}}{2}$

Assumptions: 1) Steady flow 2) Incompressible flow 3) α at 1 is approximately 1 4) $V_2 << V_1$

Note that we compute head per unit weight, H, not head per unit mass, h, so the energy equation between Point 1 and the free surface (Point 2) becomes

$$\left(\frac{p_1}{\rho\cdot g} + \frac{V^2}{2\cdot g}\right) - \left(z_2\right) = f\cdot\frac{L}{D}\cdot\frac{V^2}{2\cdot g} + K_{exit}\cdot\frac{V^2}{2\cdot g} - H_p$$

Solving for H_p $\qquad H_p = z_2 - \frac{p_1}{\rho\cdot g} - \frac{V^2}{2\cdot g} + f\cdot\frac{L}{D}\cdot\frac{V^2}{2\cdot g} + K_{exit}\cdot\frac{V^2}{2\cdot g}$

From Table A.7 (68°F) $\qquad \rho = 1.94\cdot\frac{slug}{ft^3} \qquad \nu = 1.08\times 10^{-5}\cdot\frac{ft^2}{s} \qquad Re = \frac{V\cdot D}{\nu} \qquad Re = 6.94\times 10^5$

For commercial steel pipe $\qquad e = 0.00015\cdot ft$ (Table 8.1) so $\frac{e}{D} = 0.0002$

Flow is turbulent: Given $\qquad \frac{1}{\sqrt{f}} = -2.0\cdot\log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re\cdot\sqrt{f}}\right) \qquad f = 0.0150$

For the exit $\qquad K_{exit} = 1.0$ so we find $\qquad H_p = z_2 - \frac{p_1}{\rho\cdot g} + f\cdot\frac{L}{D}\cdot\frac{V^2}{2\cdot g}$

Note that for an NPSHR of 15 ft this means $\qquad \frac{p_1}{\rho\cdot g} = 15\cdot ft \qquad H_p = z_2 - \frac{p_1}{\rho\cdot g} + f\cdot\frac{L}{D}\cdot\frac{V^2}{2\cdot g} \qquad H_p = 691\cdot ft$

Note that $\qquad Q = \frac{\pi\cdot D^2}{4}\cdot V \qquad Q = 4.42\cdot\frac{ft^3}{s} \qquad Q = 1983\cdot gpm$

For this combination of Q and Hp, from Fig. D.11 the best pump appears to be a Peerless two-stage 10TU22C operating at 1750 rpm

After 10 years, from Problem 10.63, the friction factor will have increased by a factor of 2.2 $\qquad f = 2.2\times 0.150 \qquad f = 0.330$

We now need to solve $\qquad H_p = z_2 - \frac{p_1}{\rho\cdot g} + f\cdot\frac{L}{D}\cdot\frac{V^2}{2\cdot g}$ for the new velocity V

$$V = \sqrt{\frac{2\cdot D\cdot g}{f\cdot L}\cdot\left(H_p - z_2 + \frac{p_1}{\rho\cdot g}\right)} \qquad V = 2.13\cdot\frac{ft}{s}$$

and f will still be 2.2×0.150

$$Q = \frac{\pi\cdot D^2}{4}\cdot V \qquad Q = 0.94\cdot\frac{ft^3}{s} \qquad Q = 423\cdot gpm$$

Much less!

Problem 10.73

[Difficulty: 3]

10.73 Consider the pipe network of Problem 8.189. Select a pump suitable to deliver a total flow rate of 300 gpm through the pipe network.

Given:

Water pipe system

Find:

Pump suitable for 300 gpm

Solution:

$$\left(\frac{p_1}{\rho} + \alpha_1 \cdot \frac{V_1^2}{2} + g \cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2 \cdot \frac{V_2^2}{2} + g \cdot z_2\right) = h_l \qquad h_{lT} = f \cdot \frac{L}{D} \cdot \frac{V^2}{2}$$

$$f = \frac{64}{Re} \quad \text{(Laminar)} \qquad \frac{1}{\sqrt{f}} = -2.0 \cdot \log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re \cdot \sqrt{f}}\right) \quad \text{(Turbulent)}$$

The energy equation can be simplified to $\Delta p = \rho \cdot f \cdot \frac{L}{D} \cdot \frac{V^2}{2}$

This can be written for each pipe section

Pipe A (first section) $\Delta p_A = \rho \cdot f_A \cdot \frac{L_A}{D_A} \cdot \frac{V_A^2}{2}$ (1)

Pipe B (1.5 in branch) $\Delta p_B = \rho \cdot f_B \cdot \frac{L_B}{D_B} \cdot \frac{V_B^2}{2}$ (2)

Pipe C (1 in branch) $\Delta p_C = \rho \cdot f_C \cdot \frac{L_C}{D_C} \cdot \frac{V_C^2}{2}$ (3)

Pipe D (last section) $\Delta p_D = \rho \cdot f_D \cdot \frac{L_D}{D_D} \cdot \frac{V_D^2}{2}$ (4)

In addition we have the following contraints

$Q_A = Q_D = Q$ (5)

$Q = Q_B + Q_C$ (6)

$\Delta p = \Delta p_A + \Delta p_B + \Delta p_D$ (7)

$\Delta p_B = \Delta p_C$ (8)

We have 2 unknown flow rates (or, equivalently, velocities); We solve the above eight equations simultaneously

Once we compute the flow rates and pressure drops, we can compute data for the pump

$\Delta p_{pump} = \Delta p$ and $Q_{pump} = Q_A$ $\qquad W_{pump} = \Delta p_{pump} \cdot Q_{pump}$

The calculations, performed in *Excel*, are shown on the next page.

Pipe Data:

Pipe	L (ft)	D (in)	e (ft)
A	150	1.5	0.00085
B	150	1.5	0.00085
C	150	1	0.00085
D	150	1.5	0.00085

Fluid Properties:

$\rho =$ 1.94 slug/ft^3

$\mu =$ 2.10E-05 lbf-s/ft^2

Flow Rate:

$Q =$ 300 gpm

$=$ 0.668 ft^3/s

Flows:

Q_A (ft^3/s)	Q_B (ft^3/s)	Q_C (ft^3/s)	Q_D (ft^3/s)
0.668	0.499	0.169	0.668

V_A (ft/s)	V_B (ft/s)	V_C (ft/s)	V_D (ft/s)
54.47	40.67	31.04	54.47

Re_A	Re_B	Re_C	Re_D
6.29E+05	4.70E+05	2.39E+05	6.29E+05

f_A	f_B	f_C	f_D
0.0335	0.0336	0.0384	0.0335

Heads:

Δp_A (psi)	Δp_B (psi)	Δp_C (psi)	Δp_D (psi)
804.0	448.8	448.8	804.0

Constraints:

(6) $Q = Q_B + Q_C$	(8) $\Delta p_B = \Delta p_C$
0.00%	**0.00%**

Error: **0.00%**

Vary Q_B and Q_C using *Solver* to minimize total error

For the pump:

Δp (psi)	Q (gpm)	$\mathscr{P}$ (hp)
2057	300	360

This is a very high pressure; a sequence of pumps would be needed

Problem 10.74

[Difficulty: 3]

10.74 A fire nozzle is supplied through 300 ft of 3-in.-diameter canvas hose (with $e=0.001$ ft). Water from a hydrant is supplied at 50 psig to a booster pump on board the pumper truck. At design operating conditions, the pressure at the nozzle inlet is 100 psig, and the pressure drop along the hose is 33 psi per 100 ft of length. Calculate the design flow rate and the maximum nozzle exit speed. Select a pump appropriate for this application, determine its efficiency at this operating condition, and calculate the power required to drive the pump.

Solution: Apply the energy equation for pipe flow:

Computing equation: $\left(\frac{p_1}{\rho}+\alpha_1\cancel{\frac{\bar{V}_1^2}{2}}+\cancel{gz_1}\right)-\left(\frac{p_2}{\rho}+\alpha_2\cancel{\frac{\bar{V}_2^2}{2}}+\cancel{gz_2}\right)=h_{l_T}=h_l+\cancel{h_{lm}}$

$h_l=f\frac{L}{D}\frac{\bar{V}^2}{2}$

Assumptions: (1) $\bar{V}_1=\bar{V}_2$; (2) $z_1=z_2$, (3) $h_{lm}=0$

Then $\frac{p_1}{\rho}-\frac{p_2}{\rho}=\frac{\Delta p}{\rho}=f\frac{L}{D}\frac{\bar{V}^2}{2}$; $\bar{V}=\left[\frac{2D\Delta p}{f\rho L}\right]^{\frac{1}{2}}$; $\frac{e}{D}=\frac{0.001\,ft}{3\,in.}\times 12\frac{in.}{ft}=0.004$; $f=0.028$ (fully rough)

$$\bar{V}=\left[\frac{2}{0.028}\times 3\,in.\times 3(33)\frac{lbf}{in.^2}\times\frac{ft^3}{1.94\,slug}\times\frac{1}{300\,ft}\times 12\frac{in.}{ft}\times\frac{slug\cdot ft}{lbf\cdot s^2}\right]^{\frac{1}{2}}=20.9\,ft/s$$

$Q=\bar{V}A$; $A=\frac{\pi D^2}{4}=\frac{\pi}{4}\left(\frac{3}{12}\right)^2 ft^2=0.0491\,ft^2$

$$Q=20.9\frac{ft}{s}\times 0.0491\,ft^2\times 7.48\frac{gal}{ft^3}\times\frac{60\,s}{min}=461\,gpm \text{ (design flow rate)}$$ Q

Apply Bernoulli to nozzle.

$$\frac{p_2}{\rho}+\frac{V_2^2}{2}+\cancel{gz_2}=\frac{\cancelto{0}{p_{atm}}}{\rho}+\frac{V_n^2}{2}+\cancel{gz_n}\ ;\ V_n=\left[\frac{2p_2}{\rho}+V_2^2\right]^{\frac{1}{2}}$$

$$V_n=\left[2\times 100\frac{lbf}{in.^2}\times\frac{ft^3}{1.94\,slug}\times 144\frac{in.^2}{ft^2}+(20.9)^2\frac{ft^2}{s^2}\right]^{\frac{1}{2}}=124\,ft/s$$ V_n

The pump head requirement (neglecting V and z) will be

$$H_p=\frac{p_1-p_0}{\rho g}=[100+3(33)-50]\frac{lbf}{in.^2}\times\frac{ft^3}{62.4\,lbf}\times 144\frac{in.}{ft}=344\,ft$$ H_p

From the pump selector chart (Fig. D.1) choose 3AE9G or 4AE10 pump, 3500 rpm.

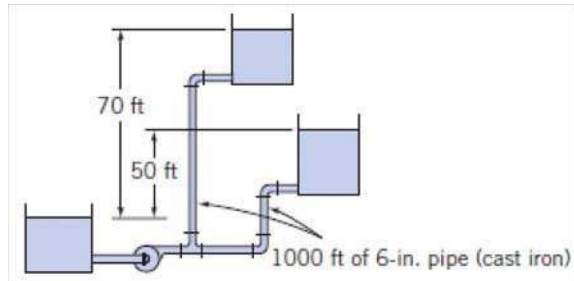
Based on 4AE12 at 3550 rpm (Fig. D.5), expect $\eta\approx 0.75$ η

$$P=\frac{Q\Delta p}{\eta}=\frac{Q\rho g\Delta h}{\eta}=\frac{1}{0.75}\times 20.9\frac{ft}{s}\times 0.0491\,ft^2\times 62.4\frac{lbf}{ft^3}\times 344\,ft\times\frac{hp\cdot s}{550\,ft\cdot lbf}=53.4\,hp$$ P

Problem 10.75

[Difficulty: 4]

10.75 A pumping system with two different static lifts is shown. Each reservoir is supplied by a line consisting of 1000 ft of 6-in. cast-iron pipe. Evaluate and plot the system head versus flow curve. Explain what happens when the pump head is less than the height of the upper reservoir. Calculate the flow rate delivered at a pump head of 85 ft.



Given: Pump and supply pipe system

Find: Head versus flow curve; Flow for a head of 85 ft

Solution:

Basic equations:
$$\left(\frac{p_1}{\rho}+\alpha_1\cdot\frac{V_1^{\,2}}{2}+g\cdot z_1\right)-\left(\frac{p_2}{\rho}+\alpha_2\cdot\frac{V_2^{\,2}}{2}+g\cdot z_2\right)=h_{lT}-h_{pump} \qquad h_{lT}=f\cdot\frac{L}{D}\cdot\frac{V^2}{2}+f\cdot\frac{L_e}{D}\cdot\frac{V^2}{2}+K\cdot\frac{V^2}{2}$$

Applying to the 70 ft branch (branch a)
$$-g\cdot H_a=f\cdot\frac{L}{D}\cdot\frac{V_a^{\,2}}{2}+f\cdot\frac{L_e}{D}\cdot\frac{V_a^{\,2}}{2}+K\cdot\frac{V_a^{\,2}}{2}-g\cdot H_{pump}$$

where $H_a = 70\cdot ft$ and $\frac{L_{ea}}{D}$ is due to a standard T branch (= 60) and a standard elbow (= 30) from Table 8.4, and

$K = K_{ent} + K_{exit} = 1.5$ from Fig. 8.14
$$H_{pump}=H_a+\left[f\cdot\left(\frac{L}{D}+\frac{L_{ea}}{D}\right)+K\right]\cdot\frac{V_a}{2\cdot g} \qquad (1)$$

Applying to the 50 ft branch (branch b)
$$H_{pump}=H_b+\left[f\cdot\left(\frac{L}{D}+\frac{L_{eb}}{D}\right)+K\right]\cdot\frac{V_b}{2\cdot g} \qquad (2)$$

where $H_b = 50\cdot ft$ and $\frac{L_{eb}}{D}$ is due to a standard T run (= 20) and two standard elbows (= 60), and $K = K_{ent} + K_{exit} = 1.5$

Here are the calculations, performed in *Excel*:

Given data:

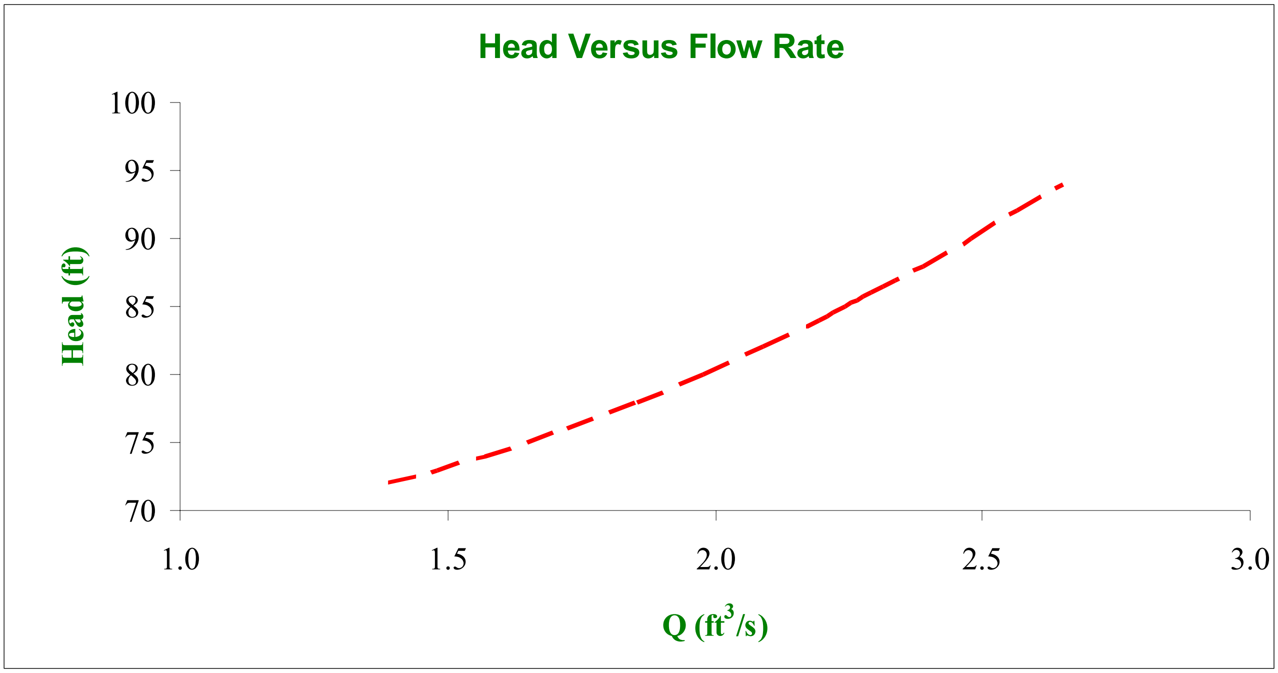
L =	1000	ft
e =	0.00085	ft
D =	6.065	in
K =	1.5	
L_{ea}/D =	90	
L_{eb}/D =	80	
H_a =	70	ft
H_b =	50	ft
ρ =	1.94	slug/ft^3
ν =	1.06E-05	ft^2/s

Computed results: Set up Solver so that it varies all flow rates to make the total head error zero

H_{pump} (ft)	Q (ft^3/s)	Q_a (ft^3/s)	V_a (ft/s)	Re_a	f_a	H_{pump} (Eq. 1)	Q_b (ft^3/s)	V_b (ft/s)	Re_b	f_b	H_{pump} (Eq. 2)	H (Errors)
72.0	1.389	0.313	1.561	7.44E+04	0.0248	72.0	1.076	5.364	2.56E+05	0.0232	72.0	0.00
74.0	1.574	0.449	2.237	1.07E+05	0.0241	74.0	1.125	5.607	2.67E+05	0.0231	74.0	0.00
76.0	1.724	0.553	2.756	1.31E+05	0.0238	76.0	1.171	5.839	2.78E+05	0.0231	76.0	0.00
78.0	1.857	0.641	3.195	1.52E+05	0.0237	78.0	1.216	6.063	2.89E+05	0.0231	78.0	0.00
80.0	1.978	0.718	3.581	1.71E+05	0.0235	80.0	1.260	6.279	2.99E+05	0.0231	80.0	0.00
82.0	2.090	0.789	3.931	1.87E+05	0.0234	82.0	1.302	6.487	3.09E+05	0.0230	82.0	0.00
84.0	2.195	0.853	4.252	2.03E+05	0.0234	84.0	1.342	6.690	3.19E+05	0.0230	84.0	0.00
85.0	2.246	0.884	4.404	2.10E+05	0.0233	85.0	1.362	6.789	3.24E+05	0.0230	85.0	0.00
86.0	2.295	0.913	4.551	2.17E+05	0.0233	86.0	1.382	6.886	3.28E+05	0.0230	86.0	0.00
88.0	2.389	0.970	4.833	2.30E+05	0.0233	88.0	1.420	7.077	3.37E+05	0.0230	88.0	0.00
90.0	2.480	1.023	5.099	2.43E+05	0.0232	90.0	1.457	7.263	3.46E+05	0.0230	90.0	0.00
92.0	2.567	1.074	5.352	2.55E+05	0.0232	92.0	1.494	7.445	3.55E+05	0.0230	92.0	0.00
94.0	2.651	1.122	5.593	2.67E+05	0.0231	94.0	1.529	7.622	3.63E+05	0.0229	94.0	0.00

For the pump head less than the upper reservoir head flow will be out of the reservoir (into the lower one)

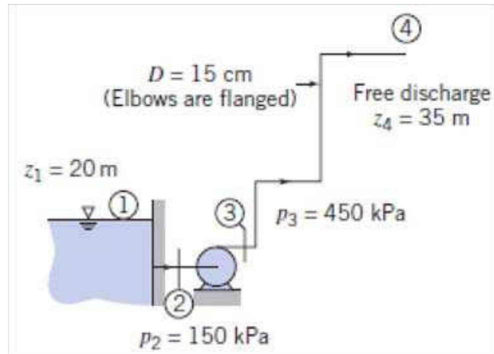
Total Error: 0.00


Head Versus Flow Rate
100
95
90
85
80
75
70
Head (ft)
1.0
1.5
2.0
2.5
3.0
Q (ft3/s)

Problem 10.76

[Difficulty: 4]

10.76 Consider the flow system shown in Problem 8.90. Evaluate the *NPSHA* at the pump inlet. Select a pump appropriate for this application. Use the data on pipe aging from Problem 10.63 to estimate the reduction in flow rate after 10 years of operation.



Given: Data on flow from reservoir/pump

Find: Appropriate pump; Reduction in flow after 10 years

Solution:

Basic equation:
$$\left(\frac{p_1}{\rho\cdot g}+\alpha\cdot\frac{V_1^{\ 2}}{2\cdot g}+z_1\right)-\left(\frac{p_4}{\rho\cdot g}+\alpha\cdot\frac{V_4^{\ 2}}{2\cdot g}+z_4\right)=H_{lT}-H_p$$
for flow from 1 to 4

$$H_{lT}=f\cdot\frac{L}{D}\cdot\frac{V^2}{2\cdot g}+f\cdot\frac{L_e}{D}\cdot\frac{V^2}{2\cdot g}+K\cdot\frac{V^2}{2\cdot g}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) α at 1 and 2 is approximately 1 4) $V_2 = V_3 = V_4$ (constant area pipe)

Given or available data $\rho = 1000\cdot\frac{kg}{m^3}$ $\nu = 1.01\times10^{-6}\cdot\frac{m^2}{s}$ $p_v = 2.34\cdot kPa$ (Table A.8)

$p_2 = 150\cdot kPa$ $p_3 = 450\cdot kPa$ $D = 15\cdot cm$ $e = 0.046\cdot mm$ $Q = 0.075\cdot\frac{m^3}{s}$

$z_1 = 20\cdot m$ $z_4 = 35\cdot m$ $V = \frac{4\cdot Q}{\pi\cdot D^2}$ $V = 4.24\frac{m}{s}$

For minor losses we have Four elbows: $\frac{L_e}{D} = 4\times12 = 48$ (Fig. 8.16) Square inlet: $K_{ent} = 0.5$

At the pump inlet $NPSHA = \frac{p_2+\frac{1}{2}\cdot\rho\cdot V^2-p_v}{\rho\cdot g}$ $NPSHA = 16.0\,m$

The head rise through the pump is $H_p = \frac{p_3-p_2}{\rho\cdot g}$ $H_p = 30.6\,m$

Hence for a flow rate of $Q = 0.075\frac{m^3}{s}$ or $Q = 1189\cdot gpm$ and $H_p = 30.6\,m$ or $H_p = 100\cdot ft$, from Appendix D. Fig. D3 a Peerless4AE11 would suffice

We do not know the pipe length L! Solving the energy equation for i $z_1 - z_4 = H_{lT} - H_p = f\cdot\frac{L}{D}\cdot\frac{V^2}{2\cdot g}+f\cdot\frac{L_e}{D}\cdot\frac{V^2}{2\cdot g}+K_{ent}\cdot\frac{V^2}{2\cdot g}-H_p$

For f $Re = \frac{V\cdot D}{\nu}$ $Re = 6.303\times10^5$ and $\frac{e}{D} = 3.07\times10^{-4}$

Given $$\frac{1}{\sqrt{f}} = -2.0\cdot\log\left(\frac{\frac{e}{D}}{3.7}+\frac{2.51}{Re\cdot\sqrt{f}}\right)$$ $f = 0.0161$

Hence, substituting values $L = \frac{2 \cdot g \cdot D}{f \cdot V^2} \cdot \left(z_1 - z_4 + H_p\right) - D \cdot \left(\frac{L_e}{D}\right) - \frac{K_{ent} \cdot D}{f}$ $\quad L = 146\,m$

From Problem 10.63, for a pipe $D = 0.15\,m$ or $D = 5.91 \cdot in$, the aging over 10 years leads to $\quad f_{worn} = 2.2 \cdot f$

We need to solve the energy equation for a new V

$$V_{worn} = \sqrt{\frac{2 \cdot g \cdot \left(z_1 - z_4 + H_p\right)}{f_{worn} \cdot \left(\frac{L}{D} + \frac{L_e}{D}\right) + K_{ent}}} \qquad V_{worn} = 2.88\,\frac{m}{s}$$

Hence $\quad Q_{worn} = \frac{\pi \cdot D^2}{4} \cdot V_{worn} \qquad Q_{worn} = 0.0510 \cdot \frac{m^3}{s}$

$\Delta Q = Q_{worn} - Q \qquad \Delta Q = -0.0240 \cdot \frac{m^3}{s} \qquad \frac{\Delta Q}{Q} = -32.0 \cdot \%$

Check f $\quad Re_{worn} = \frac{V_{worn} \cdot D}{\nu}$ $\quad$ Given $\quad \frac{1}{\sqrt{f}} = -2.0 \cdot \log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re_{worn} \cdot \sqrt{f}}\right) \qquad f = 0.0165$

Hence using 2.2 x 0.0161 is close enough to using 2.2 x 0.0165

10.77 Consider the gasoline pipeline flow of Problem 8.142. Select pumps that, combined in parallel, supply the total flow requirement. Calculate the power required for 4 pumps in parallel. Also calculate the volume flow rates and power required when only 1, 2, or 3 of these pumps operates.

Solution: Apply the energy equation for steady, incompressible pipe flow.

Computing equation: $\frac{p_i}{\rho g} + \frac{\alpha_i \bar{V}_i^2}{2g} + z_i + H_p = \frac{p_j}{\rho g} + \frac{\alpha_j \bar{V}_j^2}{2g} + z_j + \frac{h_{lT}}{g}$; $h_{lT} = \left[f\left(\frac{L}{D} + \frac{L_e}{D}\right) + K\right]\frac{\bar{V}^2}{2}$

Assumptions: (1) $\bar{V}_1 = \bar{V}_2$, $\alpha_1 = \alpha_2$, (2) $z_1 = z_2$, (3) Neglect minor losses, $\frac{L_e}{D} \approx 0$, $K \approx 0$

Find flow rate to size pump. From 1 to 2, $H_p = 0$, so $D = 0.6\,m \times \frac{ft}{0.305\,m} = 1.97\,ft$

$\frac{p_1}{\rho g} = \frac{p_2}{\rho g} + \frac{h_{lT}}{g}$; $\Delta p = \rho h_{lT} = \rho f \frac{L}{D}\frac{\bar{V}^2}{2}$; $\bar{V} = \left[\frac{2 \Delta p D}{\rho f L}\right]^{\frac{1}{2}}$ $\frac{e}{D} = \frac{0.00015\,m}{0.6\,m} = 2.5 \times 10^{-4}$

But $f = f(Re, e/D)$; Re is not known. Choose f from fully-rough region, $f = 0.014$.

$$\Delta p = 1.4 \times 10^6\,Pa \times \frac{14.7\,psi}{101 \times 10^3\,Pa} = 204\,psi\,;\ \rho = 0.72\rho_{H_2O} = 0.72 \times 1.94\,\frac{slug}{ft^3} = 1.40\,slug/ft^3$$

$$\bar{V} = \left[2 \times 204\,\frac{lbf}{in.^2} \times 1.97\,ft \times \frac{ft^3}{1.40\,slug} \times \frac{1}{0.014} \times \frac{1}{42{,}600\,ft} \times \frac{slug \cdot ft}{lbf \cdot s^2} \times 144\,\frac{in.^2}{ft^2}\right]^{\frac{1}{2}} = 11.8\,ft/s$$

Check: $Re = \frac{\bar{V}D}{\nu} = \frac{11.8\,ft}{s} \times 1.97\,ft \times \frac{s}{8.6 \times 10^{-6}\,ft^2} = 2.70 \times 10^6$; $f = 0.0146$ (See Note below.)

$\bar{V} = \left[\frac{0.014}{0.0146}\right]^{\frac{1}{2}} 11.8\,\frac{ft}{s} = 11.6\,ft/s$, $Q = \bar{V}A = 35.3\,\frac{ft^3}{s} = 15{,}700\,gpm$ Q

For parallel operation with four pumps, each must supply $\frac{Q}{4} = 3930\,gpm$.

The head requirement is $H_p = \frac{p_1 - p_2}{\rho g} = 204\,\frac{lbf}{in.^2} \times \frac{ft^3}{(0.72)62.4\,lbf} \times \frac{144\,in.^2}{ft^2} = 654\,ft$ (gasoline)

This combination of head and flow rate cannot be supplied by a single-stage pump. From Fig. D.11, the two-stage Peerless Type 10 TU 22C pump may be chosen. Pump

The input power requirement is $\dot{W}_m = \frac{\rho Q g H}{\eta_p}$. Assuming $\eta_p = 0.65$,

$$\dot{W}_m = \frac{1}{0.65} \times 15{,}700\,\frac{gal}{min} \times 204\,\frac{lbf}{in.^2} \times \frac{ft^3}{7.48\,gal} \times \frac{min}{60\,s} \times 144\,\frac{in.^2}{ft^2} \times \frac{hp \cdot s}{550\,ft \cdot lbf} = 2870\,hp\ (total)$$ $\dot{W}_m$

Flow rates with fewer pumps operating may be found from a plot. The pump characteristic may be approximated as (assume $H = 0.7\,H_0$):

$\hat{H}_p \approx 934 - 1.81 \times 10^{-5} Q^2$ $\left\{H_0 = \frac{654}{0.7} = 934;\ B = \frac{H_0 - H}{Q^2} = \frac{(0.3)934}{(3930)^2} = 1.81 \times 10^{-5}\right\}$

{Note: gasoline is between octane and heptane, Fig. A.3. For $T = 15°C$, $\nu \approx 8 \times 10^{-7}\,\frac{m^2}{s} = 8.6 \times 10^{-6}\,\frac{ft^2}{s}$.}

Problem 10.77

For two pumps in parallel, $\hat{H} = H_0 - A\left(\frac{Q}{2}\right)^2 \approx 934 - 4.54 \times 10^{-6} Q^2$

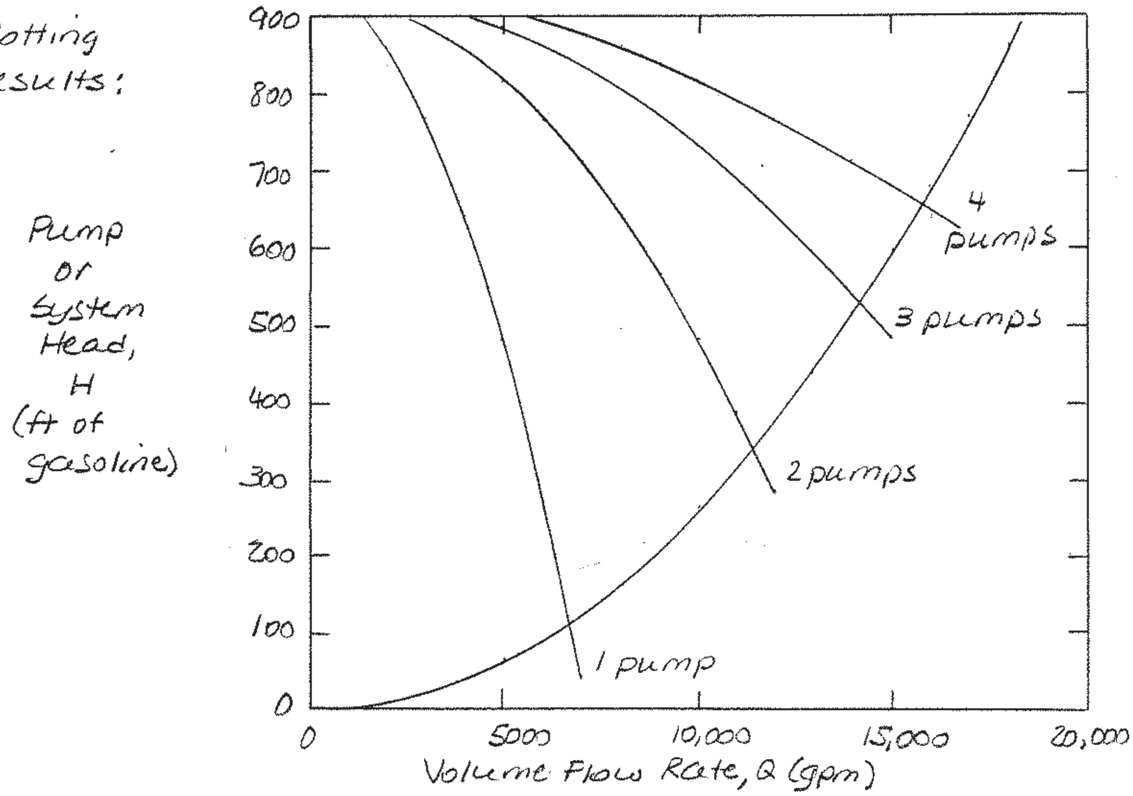
For three pumps in parallel, $\hat{H} = H_0 - A\left(\frac{Q}{3}\right)^2 \approx 934 - 2.02 \times 10^{-6} Q^2$

For four pumps in parallel, $\hat{H} = H_0 - A\left(\frac{Q}{4}\right)^2 \approx 934 - 1.13 \times 10^{-6} Q^2$

The pipe system characteristic is approximately given by

$$\hat{H}_s \approx BQ^2 \,;\; B = \frac{\hat{H}_s}{Q^2} = \frac{654}{(15{,}700)^2} = 2.65 \times 10^{-6}\ \text{ft/gpm}^2,\ \text{so}\ \hat{H}_s = 2.65 \times 10^{-6} Q^2$$

Plotting results:



The approximate volume flow rates, heads, and power requirements (assuming $\eta_p = 0.65$) are:

Number of Pumps	1	2	3	4
Flow Rate (gpm)	6710	11,400	14,200	15,700
Head (ft gasoline)	119	345	531	654
Power (hp)	224	1100	2110	2880

10.78 Consider the chilled water circulation system of Problem 8.178. Select pumps that may be combined in parallel to supply the total flow requirement. Calculate the power required for 3 pumps in parallel. Also calculate the volume flow rates and power required when only 1 or 2 of these pumps operates.

Solution: Apply the energy equation for steady, incompressible pipe flow.

Computing equation: $\frac{p_1}{\rho g} + \alpha_1\frac{\bar{V}_1^2}{2g} + z_1 + H_a = \frac{p_2}{\rho g} + \alpha_2\frac{\bar{V}_2^2}{2g} + z_2 + \frac{h_{lT}}{g}$; $h_{lT} = \left[f\left(\frac{L}{D}+\frac{L_e}{D}\right)+K\right]\frac{\bar{V}^2}{2}$

Assumptions: (1) $p_1 = p_2$, (2) $\alpha_1\bar{V}_1^2 = \alpha_2\bar{V}_2^2$, (3) $z_1 = z_2$, (4) Neglect minor losses, $\frac{L_e}{D}\approx 0$, $K\approx 0$

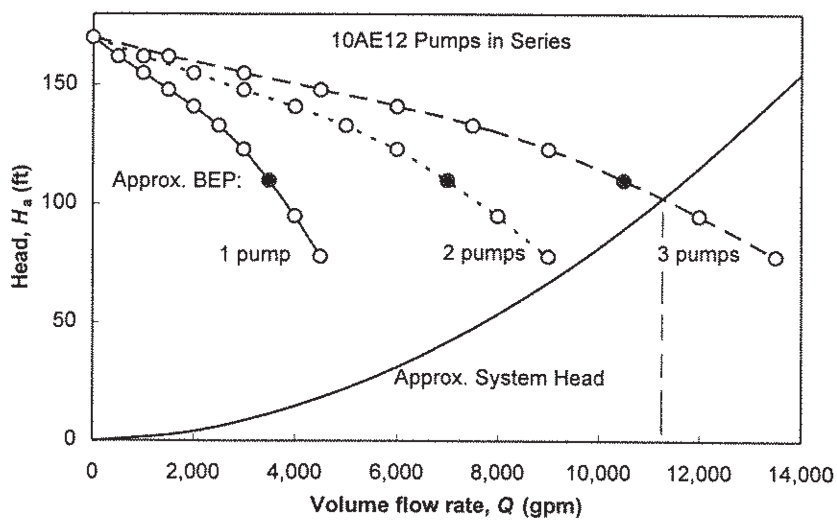
$$\bar{V} = \frac{Q}{A} = 11{,}200\,\frac{gal}{min}\times\frac{4}{\pi(2)^2\,ft^2}\times\frac{ft^3}{7.48\,gal}\times\frac{min}{60\,sec} = 7.94\,ft/s;\quad \frac{\bar{V}^2}{2g} = \frac{1}{2}\times(7.94)^2\frac{ft^2}{s^2}\times\frac{s^2}{32.2\,ft} = 0.979\,ft$$

Assume $T = 40\,F$, so $\nu = 1.64\times10^{-5}\,ft^2/s$; $Re = \frac{\bar{V}D}{\nu} = 9.68\times10^5$; $\frac{e}{D} = 7.5\times10^{-5}$; $f = 0.013$

$$H_a = f\frac{L}{D}\frac{\bar{V}^2}{2g} = 0.013\times\frac{3(5280)\,ft}{2\,ft}\times 0.979\,ft = 101\,ft$$ H_p

For three pumps in parallel, each will operate at $Q/3 = 3730$ gpm. The requirement for each pump is $H = 101$ ft at $Q = 3730$ gpm. This can be supplied by Peerless Type 10AE12 pumps with impellers of $D = 12$ in. diameter, operating at $N = 1760$ nominal rpm. The efficiency at this operating point is $\eta \approx 0.85$. 3 pump.

Find operating points graphically for 1, 2, and 3 pumps:



The graphical solution is shown

Q_1 = not satisfactory, $Q_2 = 9400$ gpm (marginal), $Q_3 = 11{,}200$ gpm (OK) Q

Assuming $\eta_p \approx 0.7$, then $\dot{W}_{m_1} = \frac{\rho Q g H}{\eta} \approx 78$ hp, $\dot{W}_{m_2} \approx 241$ hp, and $\dot{W}_{m_3} \approx 409$ hp

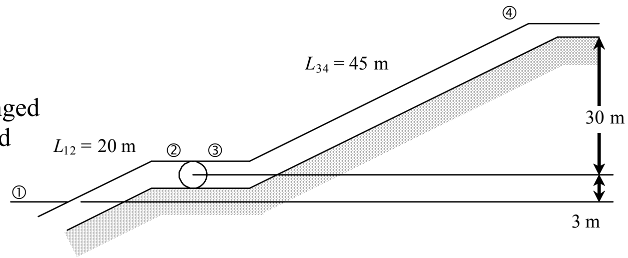
Problem 10.79

[Difficulty: 4]

10.79 Water for the sprinkler system at a lakeside summer home is to be drawn from the adjacent lake. The home is located on a bluff 33 m above the lake surface. The pump is located on level ground 3 m above the lake surface. The sprinkler system requires 40 L/min at 300 kPa (gage). The piping system is to be 2-cm-diameter galvanized iron. The inlet section (between the lake and pump inlet) includes a reentrant inlet, one standard 45° elbow, one standard 90° elbow, and 20 m of pipe. The discharge section (between the pump outlet and the sprinkler connection) includes two standard 45° elbows and 45 m of pipe. Evaluate the head loss on the suction side of the pump. Calculate the gage pressure at the pump inlet. Determine the hydraulic power requirement of the pump. If the pipe diameter were increased to 4 cm., would the power requirement of the pump increase, decrease, or stay the same? What difference would it make if the pump were located halfway up the hill?

Given: Sprinkler system for lakeside home

Find:
(a) Head loss on suction side of pump
(b) Gage pressure at pump inlet
(c) Hydraulic power requirement for the pump
(d) Change in power requirement if pipe diameter is changed
(e) Change in power requirement if the pump were moved



Solution:

We will apply the energy equation for steady, incompressible pipe flow.

Basic equations:

$$\left(\frac{p_1}{\rho}+\alpha_1\cdot\frac{V_1^2}{2}+g\cdot z_1\right)-\left(\frac{p_2}{\rho}+\alpha_2\cdot\frac{V_2^2}{2}+g\cdot z_2\right)=h_{lT}-h_p$$

$$h_{lT}=f\cdot\frac{L}{D}\cdot\frac{V^2}{2}+\Sigma\cdot f\cdot\frac{L_e}{D}\cdot\frac{V^2}{2}+\Sigma\cdot K\cdot\frac{V^2}{2}\qquad H=\frac{h}{g}$$

Assumptions: 1) $p_1 = p_{atm}$ 2) $V_1 = 0$

The given or available data is $Q = 40\cdot\frac{L}{min}$ $D = 2\cdot cm$ $e = 0.15\cdot mm$ $p_{atm} = 101.3\cdot kPa$ $p_4 = 300\cdot kPa$ (gage)

$z_1 = 0\cdot m$ $z_2 = 3\cdot m$ $z_3 = z_2$ $z_4 = 33\cdot m$ $L_{12} = 20\cdot m$ $L_{34} = 45\cdot m$

From Table A.8 at 20°C $\nu = 1.01\times 10^{-6}\cdot\frac{m^2}{s}$ $p_v = 2.34\cdot kPa$ $\rho = 998\cdot\frac{kg}{m^3}$

At the specified flow rate, the speed of the water is: $V=\frac{Q}{A}=\frac{4\cdot Q}{\pi\cdot D^2}$ $V = 2.122\,\frac{m}{s}$ $Re=\frac{V\cdot D}{\nu}=4.202\times 10^4$

$\frac{e}{D}=7.5\times 10^{-3}$ Therefore we can calculate the friction factor: $f=\left[-1.8\cdot\log\left[\left(\frac{e}{3.7\cdot D}\right)^{1.11}+\frac{6.9}{Re}\right]\right]^{-2}=0.036$

Between 1 and 2: $-\left(\frac{p_2}{\rho}+\alpha_2\cdot\frac{V^2}{2}+g\cdot z_2\right)=f\cdot\left(\frac{L_{12}}{D}+\frac{L_e}{D}\right)\cdot\frac{V^2}{2}+K\cdot\frac{V^2}{2}$ In this case: $L_e = (30+16)\cdot D$ $K = 0.78$

The head loss before the pump is: $H_{lT12} = f\cdot\left(\frac{L_{12}}{D} + \frac{L_e}{D}\right)\cdot\frac{V^2}{2\cdot g} + K\cdot\frac{V^2}{2\cdot g}$ $\quad H_{lT12} = 8.844\,\text{m}$

Solving for pressure at 2: $p_2 = \rho\cdot\left[\frac{V^2}{2} + g\cdot z_2 - f\cdot\left(\frac{L_{12}}{D} + \frac{L_e}{D}\right)\cdot\frac{V^2}{2} - K\cdot\frac{V^2}{2}\right]$ $\quad p_2 = -54.946\cdot\text{kPa}$ (gage)

To find the pump power, we need to analyze between 3 and 4:

$$\left(\frac{p_3}{\rho} + g\cdot z_3\right) - \left(\frac{p_4}{\rho} + g\cdot z_4\right) = f\cdot\left(\frac{L_{34}}{D} + \frac{L_e}{D}\right)\cdot\frac{V^2}{2} + K\cdot\frac{V^2}{2}$$

In this case: $L_e = (16 + 16)\cdot D \quad K = 0$

$p_3 = p_4 + \rho\cdot\left[g\cdot(z_4 - z_3) + f\cdot\left(\frac{L_{34}}{D} + \frac{L_e}{D}\right)\cdot\frac{V^2}{2}\right]$ $\quad p_3 = 778.617\cdot\text{kPa}$ Thus the pump head is: $H_p = \frac{p_3 - p_2}{\rho\cdot g} = 85.17\,\text{m}$

Now we can calculate the power: $W_p = \rho\cdot g\cdot Q\cdot H_p$ $\quad W_p = 556\,\text{W}$

Changing to 4 centimeter pipe would reduce the mean velocity and hence the head loss and minor loss: $D = 4\cdot\text{cm}$

$V = \frac{Q}{A} = \frac{4\cdot Q}{\pi\cdot D^2}$ $\quad V = 0.531\,\frac{\text{m}}{\text{s}}$ $\quad Re = \frac{V\cdot D}{\nu} = 2.101\times 10^4$ $\quad \frac{e}{D} = 3.75\times 10^{-3}$ $\quad f = \left[-1.8\cdot\log\left[\left(\frac{e}{3.7\cdot D}\right)^{1.11} + \frac{6.9}{Re}\right]\right]^{-2} = 0.032$

$L_e = (30 + 16)\cdot D \quad K = 0.78 \quad p_2 = \rho\cdot\left[\frac{V^2}{2} + g\cdot z_2 - f\cdot\left(\frac{L_{12}}{D} + \frac{L_e}{D}\right)\cdot\frac{V^2}{2} - K\cdot\frac{V^2}{2}\right]$ $\quad p_2 = 26.922\cdot\text{kPa}$ (gage)

$L_e = (16 + 16)\cdot D \quad K = 0 \quad p_3 = p_4 + \rho\cdot\left[g\cdot(z_4 - z_3) + f\cdot\left(\frac{L_{34}}{D} + \frac{L_e}{D}\right)\cdot\frac{V^2}{2}\right]$ $\quad p_3 = 778.617\cdot\text{kPa}$ (gage)

$H_p = \frac{p_3 - p_2}{\rho\cdot g} = 58.44\,\text{m}$ $\quad W_{pnew} = \rho\cdot g\cdot Q\cdot H_p = 381.283\,\text{W}$ $\quad \Delta W_p = \frac{W_{pnew} - W_p}{W_p} = -31\cdot\%$

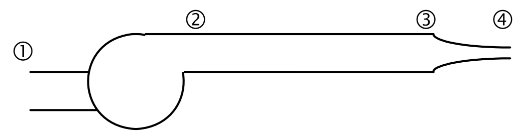
The pump should not be moved up the hill. The NPSHA is: $NPSHA = \frac{p_2 + p_{atm} + \rho\cdot\frac{V^2}{2} - p_v}{\rho\cdot g} = 4.512\,\text{m}$ for 2-cm pipe.

If anything, the pump should be moved down the hill to increase the NPSHA.

Problem 10.80

[Difficulty: 4]

10.80 Consider the fire hose and nozzle of Problem 8.179. Specify an appropriate pump to supply four such hoses simultaneously. Calculate the power input to the pump.



Given: Fire nozzle/pump system

Find: Appropriate pump; Impeller diameter; Pump power input needed

Solution:

Basic equations $\left(\frac{p_2}{\rho} + \alpha\cdot\frac{V_2^2}{2} + g\cdot z_2\right) - \left(\frac{p_3}{\rho} + \alpha\cdot\frac{V_3^2}{2} + g\cdot z_3\right) = h_l \quad h_l = f\cdot\frac{L}{D}\cdot\frac{V_2^2}{2}$ for the hose

Assumptions: 1) Steady flow 2) Incompressible flow 3) α at 2 and 3 is approximately 1 4) No minor loss

$$\left(\frac{p_2}{\rho} + \alpha\cdot\frac{V_2^2}{2} + g\cdot z_2\right) - \left(\frac{p_1}{\rho} + \alpha\cdot\frac{V_1^2}{2} + g\cdot z_1\right) = h_{pump} \quad \text{for the pump}$$

Assumptions: 1) Steady flow 2) Incompressible flow 3) α at 1 and 2 is approximately 1 4) No minor loss

The first thing we need is the flow rate. Below we repeat Problem 8.179 calculations

Hence for the hose $\frac{\Delta p}{\rho} = \frac{p_2 - p_3}{\rho} = f\cdot\frac{L}{D}\cdot\frac{V^2}{2}$ or $V = \sqrt{\frac{2\cdot\Delta p\cdot D}{\rho\cdot f\cdot L}}$

We need to iterate to solve this for V because f is unknown until Re is known. This can be done using *Excel*'s Solver, but here:

$\Delta p = 750\cdot kPa$ $L = 100\cdot m$ $e = 0$ $D = 3.5\cdot cm$ $\rho = 1000\cdot\frac{kg}{m^3}$ $\nu = 1.01\times 10^{-6}\cdot\frac{m^2}{s}$

Make a guess for f $f = 0.01$ $V = \sqrt{\frac{2\cdot\Delta p\cdot D}{\rho\cdot f\cdot L}}$ $V = 7.25\,\frac{m}{s}$ $Re = \frac{V\cdot D}{\nu}$ $Re = 2.51\times 10^5$

Given $\frac{1}{\sqrt{f}} = -2.0\cdot\log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re\cdot\sqrt{f}}\right)$ $f = 0.0150$

$V = \sqrt{\frac{2\cdot\Delta p\cdot D}{\rho\cdot f\cdot L}}$ $V = 5.92\,\frac{m}{s}$ $Re = \frac{V\cdot D}{\nu}$ $Re = 2.05\times 10^5$

Given $\frac{1}{\sqrt{f}} = -2.0\cdot\log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re\cdot\sqrt{f}}\right)$ $f = 0.0156$

$V = \sqrt{\frac{2\cdot\Delta p\cdot D}{\rho\cdot f\cdot L}}$ $V = 5.81\,\frac{m}{s}$ $Re = \frac{V\cdot D}{\nu}$ $Re = 2.01\times 10^5$

Given $\frac{1}{\sqrt{f}} = -2.0\cdot\log\left(\frac{\frac{e}{D}}{3.7} + \frac{2.51}{Re\cdot\sqrt{f}}\right)$ $f = 0.0156$

$V = \sqrt{\frac{2\cdot\Delta p\cdot D}{\rho\cdot f\cdot L}}$ $V = 5.80\,\frac{m}{s}$ $Re = \frac{V\cdot D}{\nu}$ $Re = 2.01\times 10^5$

$$Q = \frac{\pi \cdot D^2}{4} \cdot V \qquad Q = 5.578 \times 10^{-3} \frac{m^3}{s} \qquad Q = 0.335 \cdot \frac{m^3}{min}$$

We have $\quad p_1 = 350 \cdot kPa \qquad p_2 = 700 \cdot kPa + 750 \cdot kPa \qquad p_2 = 1450 \cdot kPa$

For the pump

$$\left(\frac{p_2}{\rho} + \alpha \cdot \frac{V_2^{\,2}}{2} + g \cdot z_2\right) - \left(\frac{p_1}{\rho} + \alpha \cdot \frac{V_1^{\,2}}{2} + g \cdot z_1\right) = h_{pump}$$

so $\quad h_{pump} = \dfrac{p_2 - p_1}{\rho} \quad$ or $\quad H_{pump} = \dfrac{p_2 - p_1}{\rho \cdot g} \qquad H_{pump} = 112\,m$

We need a pump that can provide a flow of $Q = 0.335 \cdot \dfrac{m^3}{min}$ or $Q = 88.4 \cdot gpm$, with a head of $H_{pump} = 112\,m$ or $H_{pump} = 368 \cdot ft$

From Appendix D, Fig. D.1 we see that a Peerless 2AE11 can provide this kind of flow/head combination; it could also handle four such hoses (the flow rate would be $4 \cdot Q = 354 \cdot gpm$). An impeller diameter could be chosen from proprietary curves.

The required power input is $\quad W_m = \dfrac{W_h}{\eta_p} \qquad$ where we choose $\eta_p = 75 \cdot \%$ from Fig. 10.15

$W_m = \dfrac{\rho \cdot Q \cdot g \cdot H_{pump}}{\eta_p} \qquad W_m = 8.18 \cdot kW \quad$ for one hose or $\quad 4 \cdot W_m = 32.7 \cdot kW \quad$ for four

$P_{required} = \dfrac{P_{pump}}{\eta} \qquad P_{required} = \dfrac{6.14 \cdot kW}{70 \cdot \%} \qquad P_{required} = 8.77 \cdot kW \quad$ or $\quad 4 \cdot P_{required} = 35.1 \cdot kW \quad$ for four

Problem 10.81

[Difficulty: 4]

10.81 Manufacturer's data for a submersible utility pump are

Discharge height (ft)	0.3	0.7	1.5	3.0	4.5	6.0	8.0
Water flow rate (L/min)	77.2	75	71	61	51	26	0

The owner's manual also states, "Note: These ratings are based on discharge into 25-mm pipe with friction loss neglected. Using 20-mm garden hose adaptor, performance will be reduced approximately 15 percent." Plot a performance curve for the pump. Develop a curve-fit equation for the performance curve; show the curve-fit on the plot. Calculate and plot the pump delivery versus discharge height through a 15-m length of smooth 20-mm garden hose. Compare with the curve for delivery into 25-mm pipe.

Given: Manufacturer data for a pump

Find: (a) Plot performance and develop curve-fit equation.
(b) Calculate pump delivery vs discharge height for length of garden hose

Solution:

Basic equations: $h_{lT} = f\cdot\frac{L}{D}\cdot\frac{V^2}{2} + f\cdot\frac{L_e}{D}\cdot\frac{V^2}{2} + K\cdot\frac{V^2}{2}$ $\qquad H = \frac{h}{g}$ $\qquad H_p = H_0 - A\cdot Q^2$

For this case, $L_e = K = 0$, therefore: $h_{lT} = f\cdot\frac{L}{D}\cdot\frac{V^2}{2}$ $\qquad$ Here are the results calculated in *Excel*:

Given data:

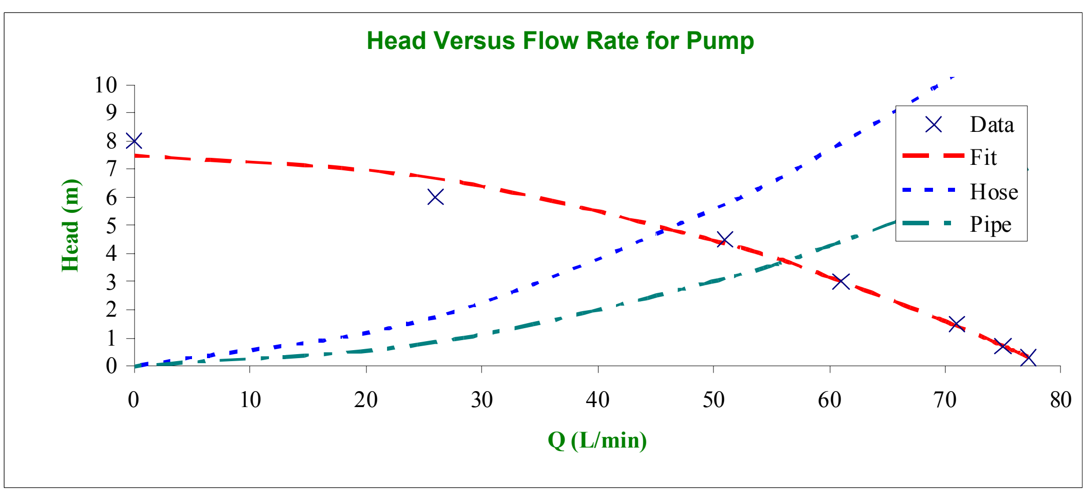
L = 15 m
e = 0 ft
D = 20 mm
ρ = 998 kg/m^3
ν = 1.01E-06 m^2/s

H_0 = 7.48727 m
A = 0.0012 m/(L/min)^2

Here are the data for the head generated by the pump, as well as the head losses for the hose and the pipe:

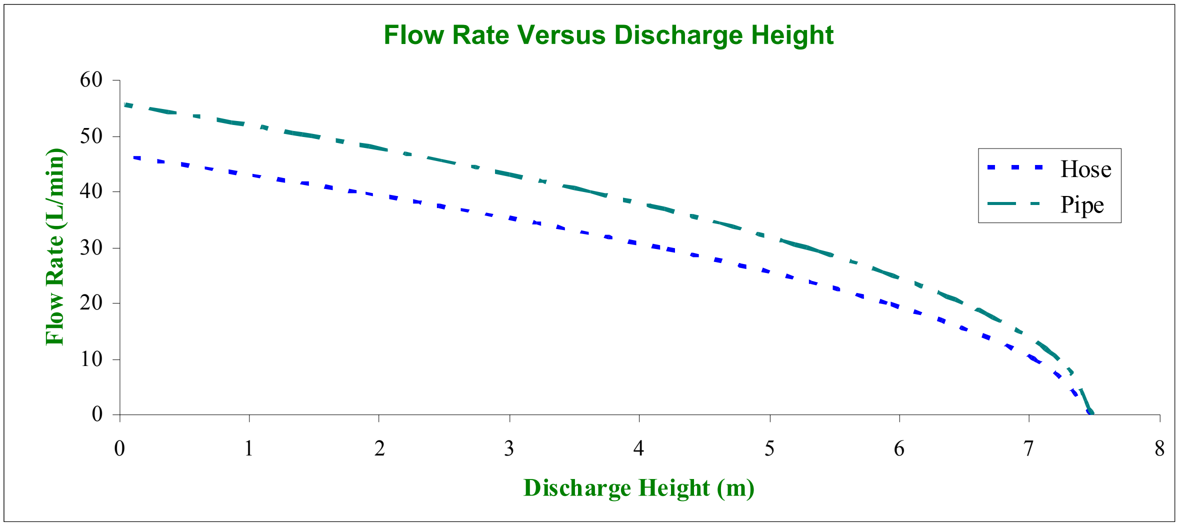
Hose: D = 20 mm, e = 0 mm $\qquad$ Pipe: D = 25 mm, e = 0.15 mm

z (m)	Q (L/min)	Q^2	z_{fit} (m)	V (m/s)	Re_a	f_a	H_L (m)	V (m/s)	Re_a	f_a	H_L (m)
0.3	77.2	5959.840	0.320	4.096	8.11E+04	0.0188	12.1	2.621	6.49E+04	0.0334	7.0
0.7	75.0	5625.000	0.722	3.979	7.88E+04	0.0189	11.4	2.546	6.30E+04	0.0334	6.6
1.5	71.0	5041.000	1.425	3.767	7.46E+04	0.0191	10.4	2.411	5.97E+04	0.0335	6.0
3.0	61.0	3721.000	3.012	3.236	6.41E+04	0.0198	7.9	2.071	5.13E+04	0.0337	4.4
4.5	51.0	2601.000	4.359	2.706	5.36E+04	0.0206	5.8	1.732	4.29E+04	0.0340	3.1
6.0	26.0	676.000	6.674	1.379	2.73E+04	0.0240	1.7	0.883	2.19E+04	0.0356	0.8
8.0	0.0	0.000	7.487	0.000	0.00E+00	0.0000	0.0	0.000	0.00E+00	0.0000	0.0



To determine the discharge heights for the hose and the pipe, we subtract the head loss from the head generated by the pump.

	For the hose:					For the pipe:					
Q (L/min)	V (m/s)	Re_a	f_a	H_L (m)	Disch (m)	V (m/s)	Re_a	f_a	H_L (m)	Disch (m)	% Diff
0.0	0.000	0.00E+00	0.0000	0.000	7.487	0.000	0.00E+00	0.0000	0.000	7.487	0%
10.0	0.531	1.05E+04	0.0305	0.328	7.039	0.340	8.40E+03	0.0398	0.140	7.227	-3%
20.0	1.061	2.10E+04	0.0256	1.101	5.906	0.679	1.68E+04	0.0364	0.514	6.492	-9%
30.0	1.592	3.15E+04	0.0232	2.248	4.157	1.019	2.52E+04	0.0351	1.115	5.290	-21%
40.0	2.122	4.20E+04	0.0217	3.740	1.823	1.358	3.36E+04	0.0345	1.943	3.620	-50%
50.0	2.653	5.25E+04	0.0207	5.558	-1.077	1.698	4.20E+04	0.0340	2.998	1.483	
60.0	3.183	6.30E+04	0.0199	7.689	-4.531	2.037	5.04E+04	0.0337	4.279	-1.122	



The results show that the 15% performance loss is an okay "ball park" guess at the lower flow rates, but not very good at flow rates above 30 L/min.

Problem 10.82

[Difficulty: 4]

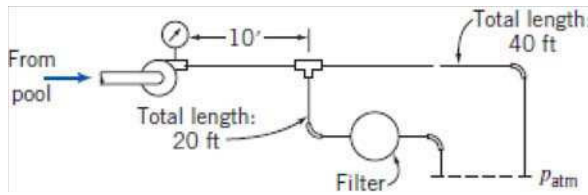
10.82 Consider the swimming pool filtration system of Problem 8.190. Assume the pipe used is 20-mm PVC (smooth plastic). Specify the speed and impeller diameter and estimate the efficiency of a suitable pump.

Given: Swimming pool filtration system, filter pressure drop is $\Delta p = 0.6Q^2$, with Δp in psi and Q in gpm

Find: Speed and impeller diameter of suitable pump; estimate efficiency

Solution:

We will apply the energy equation for steady, incompressible pipe flow.



Basic equations:

$$\left(\frac{p_1}{\rho} + \alpha_1\cdot\frac{V_1^2}{2} + g\cdot z_1\right) - \left(\frac{p_2}{\rho} + \alpha_2\cdot\frac{V_2^2}{2} + g\cdot z_2\right) = h_{lT} - h_p \qquad h_{lT} = f\cdot\frac{L}{D}\cdot\frac{V^2}{2} + \Sigma\cdot f\cdot\frac{L_e}{D}\cdot\frac{V^2}{2} + \Sigma\cdot K\cdot\frac{V^2}{2} \qquad H = \frac{h}{g}$$

The given or available data are: $Q = 30\cdot gpm \quad Q = 1.893\times 10^{-3}\,\frac{m^3}{s} \quad \nu = 1.06\times 10^{-5}\cdot\frac{ft^2}{s} \quad \nu = 9.848\times 10^{-7}\,\frac{m^2}{s}$

$$\rho = 1.93\cdot\frac{slug}{ft^3} \quad \rho = 995\,\frac{kg}{m^3} \quad D = 20\cdot mm \quad e = 0\cdot mm$$

Setting state 1 at the pump discharge, state 2 at the tee, state 3a downstream of the filter, and state 3b after the 40 ft pipe, we can look at the pressure drop between 1 and 2:

$$V_1 = V_2 \qquad \frac{e}{D} = 0 \qquad V = \frac{Q}{A} = \frac{4\cdot Q}{\pi\cdot D^2} \qquad V = 6.025\,\frac{m}{s} \qquad Re = \frac{V\cdot D}{\nu} = 1.224\times 10^5$$

$\frac{e}{D} = 0$ Therefore we can calculate the friction factor: $f = \left[-1.8\cdot\log\left[\left(\frac{e}{3.7\cdot D}\right)^{1.11} + \frac{6.9}{Re}\right]\right]^{-2} = 0.017$

Since this is a straight run of pipe: $L_e = 0 \quad K = 0$ and therefore the pressure drop is: $\Delta p_{12} = \rho\cdot f\cdot\frac{V^2}{2}\cdot\frac{L}{D} \qquad \Delta p_{12} = 47.04\cdot kPa$

Since both legs exhaust to the same pressure, the pressure drops between the two must be equal, and the flow rates must equal the total flow rate of the system. This requires an iterative solution, using *Solver* in Excel. The result is:

$Q_a = 1.094\times 10^{-3}\cdot\frac{m^3}{s} \qquad Q_b = 7.99\times 10^{-4}\cdot\frac{m^3}{s}$ The resulting pressure drop is $\Delta p_{23} = 42.96\cdot kPa$

Neglecting any pressure at the pump inlet, the pump must supply: $\Delta p_{pump} = \Delta p_{12} + \Delta p_{23} = 90.0\cdot kPa$

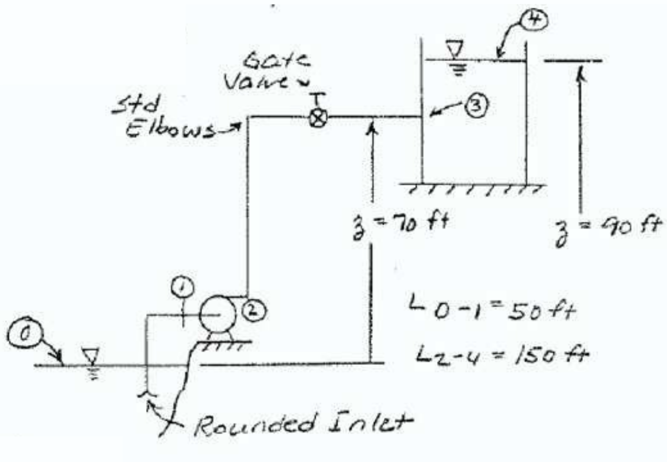
The resulting head is: $H_{pump} = \frac{\Delta p_{pump}}{\rho\cdot g} = 9.226\,m$ in U.S. units: $H_{pump} = 30.269\cdot ft$

This head is too low for any of the pumps in Fig. D.1. Therefore, assuming a speed of 3500 rpm: $N = \frac{\omega\cdot\sqrt{Q}}{\left(g\cdot H_{pump}\right)^{0.75}} = 0.544$

In customary units: $N_{cu} = 2733\cdot N = 1485$ So from Figure 10.9 we can estimate the efficiency: $\eta = 65\cdot\%$

The pump power is: $W_p = \frac{\rho\cdot Q\cdot g\cdot H_{pump}}{\eta} = 262.056\,W \qquad W_p = 262.1\,W$

10.83 Water is pumped from a lake (at $z = 0$) to a large storage tank located on a bluff above the lake. The pipe is 3-in.-diameter galvanized iron. The inlet section (between the lake and the pump) includes one rounded inlet, one standard 90° elbow, and 50 ft of pipe. The discharge section (between the pump outlet and the discharge to the open tank) includes two standard 90° elbows, one gate valve, and 150 ft of pipe. The pipe discharge (into the side of the tank) is at $z = 70$ ft. Calculate the system flow curve. Estimate the system operating point. Determine the power input to the pump if its efficiency at the operating point is 80 percent. Sketch the system curve when the water level in the upper tank reaches $z = 90$ ft. If the water level in the upper tank is at $z = 75$ ft and the valve is partially closed to reduce the flow rate to 0.1 ft^3/s, sketch the system curve for this operating condition. Would you expect the pump efficiency to be higher for the first or second operating condition? Why?



Solution: Apply the energy equation for pipe flow. The pump must overcome the gravity lift plus the head losses in the pipe and fittings.

Assume: (1) Nominal speed is $\bar{V} = 12$ ft/s, $T = 60°F$, $\nu = 1.21\times10^{-5}$ ft^2/s (Table A.7)
(2) Flow in fully rough zone ($e = 0.0005$ ft (Table 8.1), $e/D = 0.002$, $f \approx 0.024$)
(3) Cases: (1) Water in tank below ③ } Valve open
(2) Water in tank at $z = 90$ ft }
(3) Valve closed so $Q = 0.1$ ft^3/s, valve part closed.

Then

$$H_{\ell T} = \frac{h_{\ell T}}{g} = \left[K_{ent} + f\left(\frac{L}{D} + 3\frac{L_e}{D}(\text{elbow}) + \frac{L_e}{D}(\text{gate valve})\right) + K_{exit}\right]\frac{\bar{V}^2}{2g}$$

$$H_{\ell T} = \left[0.04 + 0.024\left(\frac{200\text{ ft}}{3.068\text{ in.}}\times 12\frac{\text{in.}}{\text{ft}} + 3(30) + 8\right) + 1\right]\frac{\bar{V}^2}{2g} = 22.2\,\frac{\bar{V}^2}{2g}$$

and $H_s = z_{end} + 22.2\dfrac{\bar{V}^2}{2g}$ — H_s

Assume $\bar{V} = 12$ ft/s, $Q = 276$ gpm, $\dfrac{\bar{V}^2}{2g} = 2.24$ ft, $H_s = 70 + 22.2(2.24) = 120$ ft

Case 1: $z_{end} = 70$ ft Operating point: $Q = 276$ gpm, $H_p = H_s = 120$ ft — Oper. Pt.

$$P = \frac{\rho g Q H}{\eta_p} = 62.4\,\frac{\text{lbf}}{\text{ft}^3}\times 12.0\,\frac{\text{ft}}{\text{s}}\times 0.0513\text{ ft}^2\times 120\text{ ft}\times\frac{1}{0.8}\times\frac{\text{hp}\cdot\text{s}}{550\text{ ft}\cdot\text{lbf}} = 10.5\text{ hp}$$ — P

Case 2: $z_{end} = 90$ ft ; $H_s = 90 + 22.2(2.24) = 140$ ft

$$H_s = 90 + 50.0\frac{[Q(\text{gpm})]^2}{(276)^2(\text{gpm})^2} = 90 + 6.56\times10^{-4}[Q(\text{gpm})]^2$$ — H_s

Case 3: $Q = 0.1$ ft^3/s $= 44.9$ gpm ; $H_s = H_p$; Assume $H_{BEP} = 0.7\,H_0$

$$H_p = H_0 + \frac{(H_0 - H_{op})}{Q_{op}^2}Q^2 = \frac{120}{0.7} - \frac{(120/0.7 - 120)}{(276)^2}Q^2 = 169 - 6.75\times10^{-4}Q^2$$

$H_p = 168$ ft at $Q = 44.9$ gpm — H_p

over

Problem 10.83

Water pumped from lake to storage tank on bluff:

Input Data:

Friction factor:	$f =$	0.024	(---)
Pipe diameter:	$D =$	3.068	in.

Calculated Results:

Pipe area:	$A =$	0.0513	ft^2

System Curves for Various Conditions:

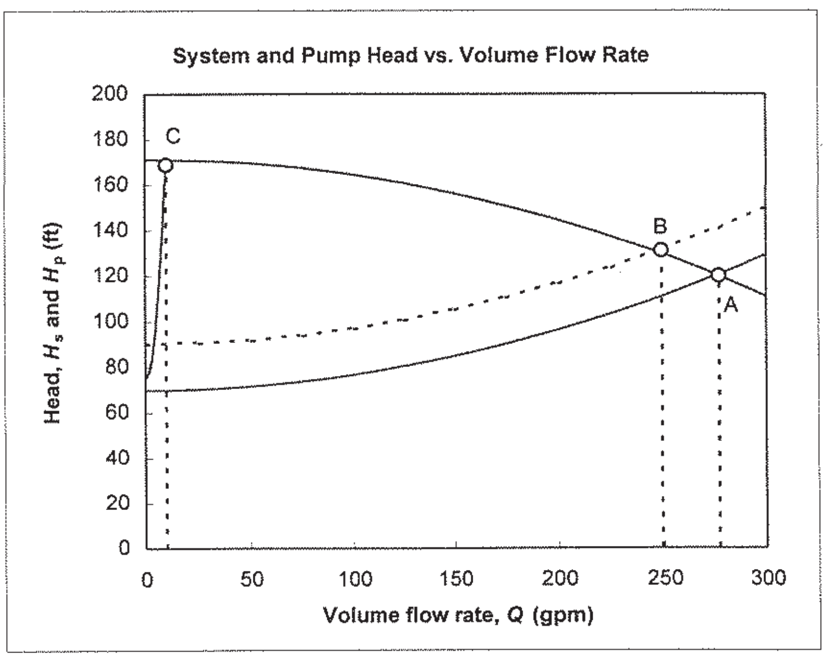
Q (gpm)	V (ft/s)	$V^2/2g$ (ft)	Case 1: H_s (z_3 = 70 ft) (ft)	Case 2: H_s (z_3 = 90 ft) (ft)
0	0	0.00	70.0	90.0
25	1.09	0.02	70.4	90.4
50	2.17	0.07	71.7	91.7
75	3.26	0.16	73.7	93.7
100	4.34	0.29	76.6	96.6
125	5.43	0.46	80.3	100
150	6.51	0.66	84.9	105
175	7.60	0.90	90.2	110
200	8.68	1.17	96.4	116
225	9.77	1.48	103	123
249.3	10.8	1.82	111	131
277	12.0	2.24	121	141
300	13.0	2.63	129	149

Case 3: Valve partially closed

Q (gpm)	H_s (z_3 = 75 ft) (ft)
0	75.0
2	78.8
4	90.0
6	109
8	135
10	169

Pump Head Curve:

Q (gpm)	H_p (ft)
0	171
50	170
100	165
150	156
200	145
250	130
277	120
300	111



Problem 10.84

[Difficulty: 3]

10.84 Performance data for a centrifugal fan of 3-ft diameter, tested at 750 rpm, are

Volume flow rate Q (ft^3/s)	106	141	176	211	246	282
Static pressure rise, Δp (psi)	0.075	0.073	0.064	0.050	0.033	0.016
Power output $\mathcal{P}$ (hp)	2.75	3.18	3.50	3.51	3.50	3.22

Plot the performance data versus volume flow rate. Calculate static efficiency, and show the curve on the plot. Find the best efficiency point, and specify the fan rating at this point.

Given: Data on centrifugal fan

Find: Plot of performance curves; Best effiiciency point

Solution:

Basic equations:
$$\eta_p = \frac{W_h}{W_m} \qquad W_h = Q \cdot \Delta p \qquad \Delta p = \rho_w \cdot g \cdot \Delta h$$
(Note: Software cannot render a dot!)

Here are the results, calculated using *Excel*:

$\rho_w =$ 1.94 slug/ft^3

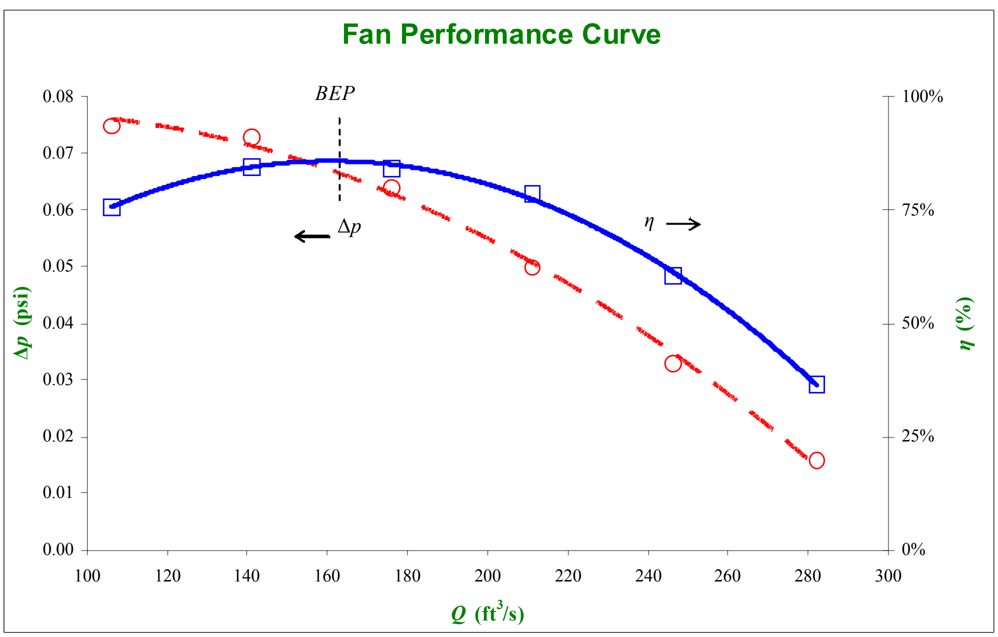
Q (ft^3/s)	Δp (psi)	$\mathcal{P}_m$ (hp)	$\mathcal{P}_h$ (hp)	η (%)
106	0.075	2.75	2.08	75.7%
141	0.073	3.18	2.69	84.7%
176	0.064	3.50	2.95	84.3%
211	0.050	3.51	2.76	78.7%
246	0.033	3.50	2.13	60.7%
282	0.016	3.22	1.18	36.7%

Fitting a 2nd order polynomial to each set of data we find

$\Delta p = -1.51\text{x}10^{-6} Q^2 + 2.37\text{x}10^{-4} Q + 0.0680$
$\eta = -3.37\text{x}10^{-5} Q^2 + 0.0109 Q - 0.0151$

Finally, we use Solver to maximize η by varying Q:

Q (ft^3/s)	Δp (psi)	η (%)
161.72	0.0668	86.6%



Problem 10.85

[Difficulty: 3]

10.85 Using the fan of Problem 10.84, determine the minimum size square sheet-metal duct that will carry a flow of 200 ft³/s over a distance of 50 ft. Estimate the increase in delivery if the fan speed is increased to 1000 rpm.

Given: Data on centrifugal fan and square metal duct

Find: Minimum duct geometry for flow required; Increase if fan speed is increased

Solution:

Basic equations: $\eta_p = \frac{W_h}{W_m}$ $\quad W_h = Q \cdot \Delta p$ $\quad \Delta p = \rho_w \cdot g \cdot \Delta h$ (Note: Software cannot render a dot!)

and for the duct $\Delta p = \rho_{air} \cdot f \cdot \frac{L}{D_h} \cdot \frac{V^2}{2}$ $\quad D_h = \frac{4 \cdot A}{P} = \frac{4 \cdot H^2}{4 \cdot H} = H$

and fan scaling $Q = 200 \cdot \frac{ft^3}{s}$ $\quad \omega = 750 \cdot rpm$ $\quad \omega' = 1000 \cdot rpm$ $\quad Q' = \frac{\omega'}{\omega} \cdot Q$ $\quad Q' = 266.67 \cdot \frac{ft^3}{s}$

Here are the results, calculated using *Excel*:

$\rho_w =$ 1.94 slug/ft³

$\rho_{air} =$ 0.00237 slug/ft³

$\nu_{air} =$ 1.58E-04 ft²/s

$L =$ 50 ft

Assume smooth ducting

Note: Efficiency curve not needed for this problem.

We use the data to get a relationship for pressure increase.

Q (ft³/s)	Δp (psi)	$\mathscr{P}_m$ (hp)	$\mathscr{P}_h$ (hp)	η (%)
106	0.075	2.75	2.08	75.7%
141	0.073	3.18	2.69	84.7%
176	0.064	3.50	2.95	84.3%
211	0.050	3.51	2.76	78.7%
246	0.033	3.50	2.13	60.7%
282	0.016	3.22	1.18	36.7%

Fitting a 2nd order polynomial to each set of data we find

$\Delta p = -1.51\times10^{-6} Q^2 + 2.37\times10^{-4} Q + 0.0680$

Now we need to match the pressure loss in the duct with the pressure rise across the fan. To do this, we use Solver to vary H so the error in δp is zero

Q (ft³/s)	Fan Δp (psi)
266.67	0.0238

H (ft)	V (ft/s)	Re	f	Duct Δp (psi)
1.703	91.94	9.91.E+05	0.0117	0.0238

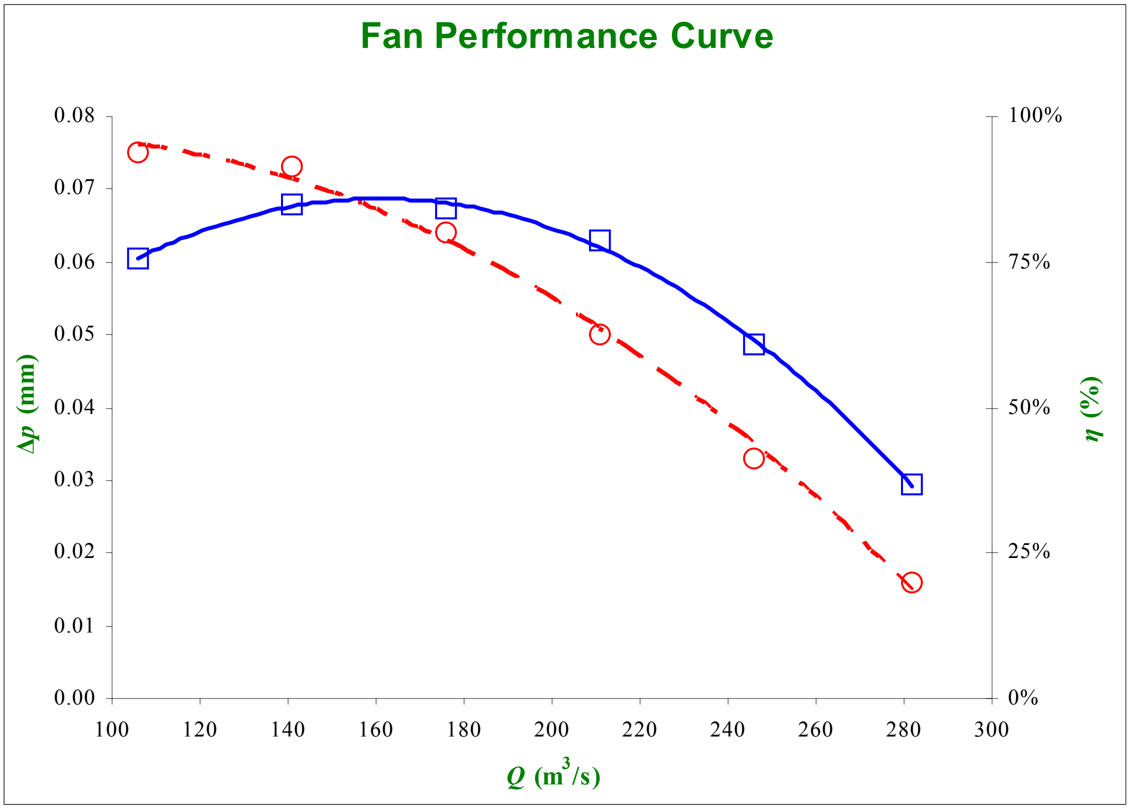
Error in Δp 0.00%

Answers:

Q (ft³/s)	H (ft)
200.00	1.284

Q (ft³/s)	H (ft)
266.67	1.703

A plot of the performance curve is shown on the next page.



Fan Performance Curve
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0.00
Δp (mm)
100%
75%
50%
25%
0%
η (%)
100
120
140
160
180
200
220
240
260
280
300
Q (m3/s)

Problem 10.86

[Difficulty: 3]

10.86 The performance data of Problem 10.84 are for a 36-in.-diameter fan wheel. The fan also is manufactured with 42-, 48-, 54-, and 60-in. diameter wheels. Pick a standard fan to deliver 600 ft^3/s against a 1-in. H_2O static pressure rise. Determine the required fan speed and input power required.

Given: Data on centrifugal fan and various sizes

Find: Suitable fan; Fan speed and input power

Solution:

Basic equations:

$$\frac{Q'}{Q} = \left(\frac{\omega'}{\omega}\right)\cdot\left(\frac{D'}{D}\right)^3 \qquad \frac{h'}{h} = \left(\frac{\omega'}{\omega}\right)^2\cdot\left(\frac{D'}{D}\right)^2 \qquad \frac{P'}{P} = \left(\frac{\omega'}{\omega}\right)^3\cdot\left(\frac{D'}{D}\right)^5$$

We choose data from the middle of the table above as being in the region of the best efficiency

$$Q = 176\cdot\frac{ft^3}{s} \quad \Delta p = 0.064\cdot psi \quad P = 3.50\cdot hp \quad \text{and} \quad \omega = 750\cdot rpm \quad D = 3\cdot ft \quad \rho_W = 1.94\cdot\frac{slug}{ft^3}$$

The flow and head are $Q' = 600\cdot\frac{ft^3}{s}$ $\quad h' = 1\cdot in$ $\quad$ At best efficiency point: $h = \frac{\Delta p}{\rho_W\cdot g} = 1.772\cdot in$

These equations are the scaling laws for scaling from the table data to the new fan. Solving for scaled fan speed, and diameter using the first two equations

$$\omega' = \omega\cdot\left(\frac{Q}{Q'}\right)^{\frac{1}{2}}\cdot\left(\frac{h'}{h}\right)^{\frac{3}{4}} \qquad \omega' = 265\cdot rpm \qquad D' = D\cdot\left(\frac{Q'}{Q}\right)^{\frac{1}{2}}\cdot\left(\frac{h}{h'}\right)^{\frac{1}{4}} \qquad D' = 76.69\cdot in$$

This size is too large; choose (by trial and error)

$$Q = 246\cdot\frac{ft^3}{s} \qquad h = \frac{0.033\cdot psi}{\rho_W\cdot g} = 0.914\cdot in \qquad P = 3.50\cdot hp$$

$$\omega' = \omega\cdot\left(\frac{Q}{Q'}\right)^{\frac{1}{2}}\cdot\left(\frac{h'}{h}\right)^{\frac{3}{4}} \qquad \omega' = 514\cdot rpm \qquad D' = D\cdot\left(\frac{Q'}{Q}\right)^{\frac{1}{2}}\cdot\left(\frac{h}{h'}\right)^{\frac{1}{4}} \qquad D' = 54.967\cdot in$$

Hence it looks like the 54-inch fan will work; it must run at about 500 rpm. Note that it will NOT be running at best efficiency. The power will be

$$P' = P\cdot\left(\frac{\omega'}{\omega}\right)^3\cdot\left(\frac{D'}{D}\right)^5 \qquad P' = 9.34\cdot hp$$

Problem 10.87

[Difficulty: 3]

10.87 Consider the fan and performance data of Problem 10.84. At $Q = 200\ \text{ft}^3/\text{s}$, the dynamic pressure is equal to 0.25 in. of water. Evaluate the fan outlet area. Plot total pressure rise and input horsepower for this fan versus volume flow rate. Calculate the fan total efficiency, and show the curve on the plot. Find the best efficiency point, and specify the fan rating at this point.

Given:
Data on centrifugal fan

Find:
Fan outlet area; Plot total pressure rise and power; Best effiiciency point

Solution:

Basic equations: $\eta_p = \dfrac{W_h}{W_m}$ $\quad W_h = Q \cdot \Delta p_t \quad \Delta p = \rho_w \cdot g \cdot \Delta h_t$ (Note: Software cannot render a dot!)

$$p_{dyn} = \frac{1}{2} \cdot \rho_{air} \cdot V^2$$

At $Q = 200 \cdot \dfrac{ft^3}{s}$ we have $h_{dyn} = 0.25in$ $\qquad Q = V \cdot A \qquad$ and $\qquad h_{dyn} = \dfrac{p_{dyn}}{\rho_w \cdot g} = \dfrac{\rho_{air}}{\rho_w} \cdot \dfrac{V^2}{2}$

Hence $\qquad V = \sqrt{\dfrac{\rho_w}{\rho_{air}} \cdot 2 \cdot g \cdot h_{dyn}} \qquad$ and $\qquad A = \dfrac{Q}{V}$

The velocity V is directly proportional to Q, so the dynamic pressure at any flow rate Q is $\quad h_{dyn} = 0.25 \cdot in \cdot \left(\dfrac{Q}{200 \cdot \dfrac{ft^3}{s}} \right)^2$

The total pressure Δh_t will then be $\qquad \Delta h_t = \Delta h + h_{dyn} \qquad$ Δh is the tabulated static pressure rise

Here are the results, generated in *Excel*:

At Q = 200 ft^3/s
h_{dyn} = 0.25 in $\qquad$ Hence $\qquad$ V = 33.13 ft/s
A = 6.03749 ft^2

ρ_w = 1.94 $slug/ft^3$
ρ_{air} = 0.00237 $slug/ft^3$

Q (ft^3/s)	Δp (psi)	$\mathscr{P}_m$ (hp)	h_{dyn} (in)	h_t (in)	$\mathscr{P}_h$ (hp)	η (%)
106	0.075	2.75	0.07	2.15	2.15	78.2%
141	0.073	3.18	0.12	2.15	2.86	90.0%
176	0.064	3.50	0.19	1.97	3.27	93.5%
211	0.050	3.51	0.28	1.66	3.32	94.5%
246	0.033	3.50	0.38	1.29	3.01	85.9%
282	0.016	3.22	0.50	0.94	2.51	77.9%

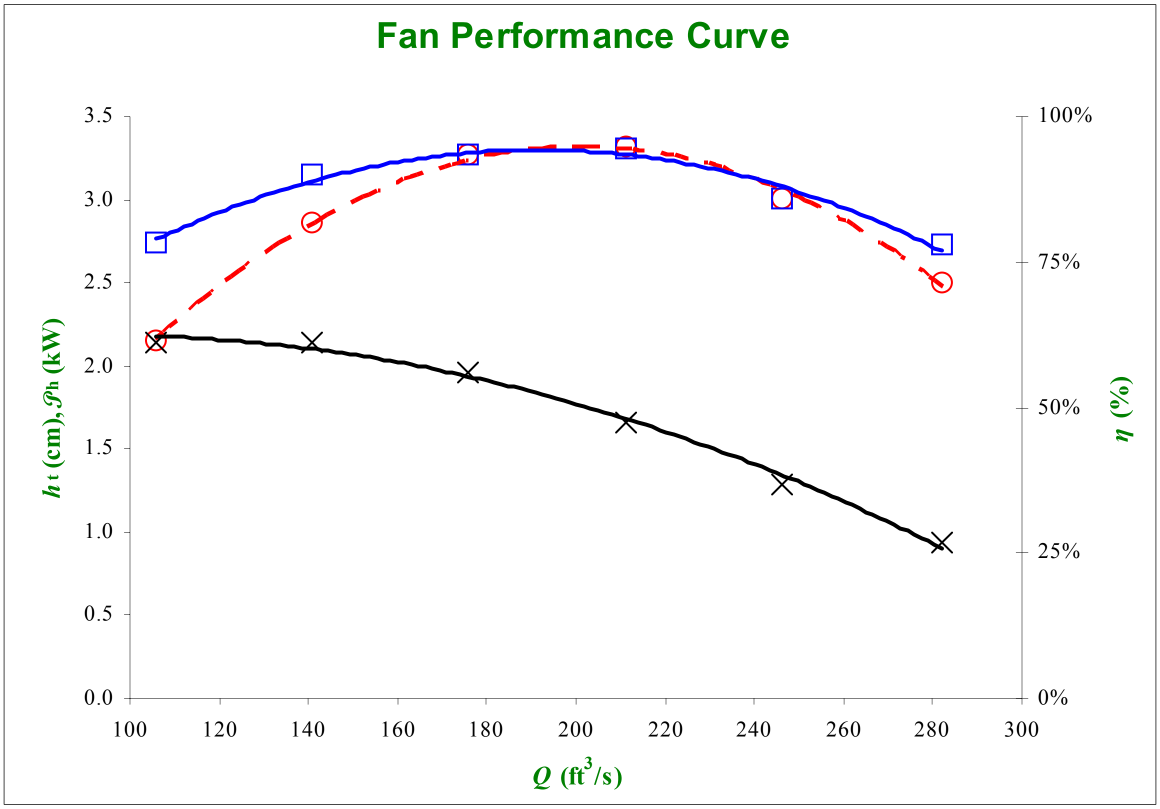
Fitting a 2nd order polynomial to each set of data we find

$$h_t = -3.56\text{x}10^{-5}Q^2 + 6.57\text{x}10^{-3}Q + 1.883$$
$$\mathscr{P}_h = -1.285\text{x}10^{-4}Q^2 + 0.0517Q - 1.871$$
$$\eta = -3.37\text{x}10^{-5}Q^2 + 0.0109Q - 0.0151$$

Finally, we use Solver to maximize η by varying Q:

Q (ft^3/s)	h_t (in)	$\mathscr{P}_h$ (hp)	η (%)
161.72	2.01	3.13	86.6%

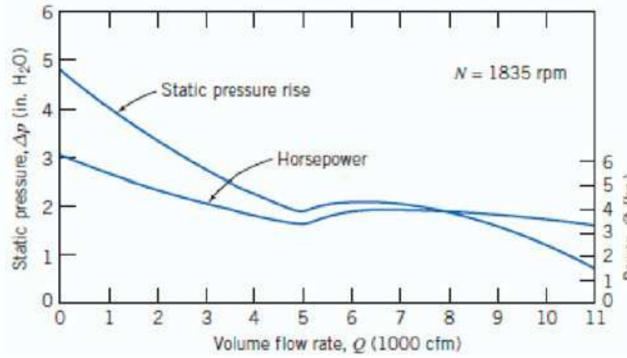
A plot of the performance curves is shown on the next page.


Fan Performance Curve
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
100%
75%
50%
25%
0%
100
120
140
160
180
200
220
240
260
280
300
h t (cm), 𝒫h (kW)
η (%)
Q (ft3/s)

Problem 10.88

[Difficulty: 4]

10.88 Performance characteristics of a Howden Buffalo axial flow fan are presented below. The fan is used to power a wind tunnel with 1-ft square test section. The tunnel consists of a smooth inlet contraction, two screens (each with loss coefficient $K=0.12$), the test section, and a diffuser where the cross section is expanded to 24 in. diameter at the fan inlet. Flow from the fan is discharged back to the room. Calculate and plot the system characteristic curve of pressure loss versus volume flow rate. Estimate the maximum air flow speed available in this wind tunnel test section.



Solution: Apply energy

Computing equation: $\frac{p_0}{\rho} + \overset{\approx 0(2)}{\frac{V_0^2}{2}} + \cancel{g z_0} = \frac{p_1}{\rho} + \alpha_1\frac{\bar{V}_1^2}{2} + \cancel{g z_1} + h_{lT}$; $h_{lT} = \left[\cancel{f\left(\frac{L}{D} + \frac{L_e}{D}\right)} + K\right]\frac{V^2}{2}$

Assumptions: (1) $p_0 = p_{atm}$, (2) $V_0 \approx 0$, $\alpha_1 \approx 1$, (3) $z_0 = z_1$, (4) Losses in diffuser, screens

$$\frac{\Delta p_{fan}}{\rho} = \frac{p_{atm} - p_1}{\rho} = \frac{V_1^2}{2} + h_{lT} = \frac{V_1^2}{2} + (2K_{screen} + K_{diffuser})\frac{V^2}{2} = \left[2K_s + K_d + \left(\frac{A}{A_1}\right)^2\right]\frac{Q^2}{2A^2}$$

From continuity, $V_1 A_1 = VA$; $V_1^2 = V^2\left(\frac{A}{A_1}\right)^2$; $V = \frac{Q}{A}$; $V^2 = \frac{Q^2}{A^2}$; $A = 1\ ft^2$; $A_1 = \frac{\pi}{4}D_1^2 = 3.14\ ft^2$

From Fig. 8.19, $K_d = C_{p_i} - C_p = 1 - \left(\frac{1}{AR}\right)^2 - 0.70 = 1 - \left(\frac{1}{3.14}\right)^2 - 0.70 = 0.199$

$$\Delta p_{fan} = [2(0.12) + 0.199 + 0.101]\frac{1}{2} \times Q^2\frac{ft^6}{min^2} \times \frac{1}{(1)^2\,ft^4} \times 0.00238\frac{slug}{ft^3} \times \frac{lbf \cdot s^2}{slug \cdot ft} \times \frac{min^2}{3600\ s^2}$$

$$\Delta h_{fan} = \frac{\Delta p}{\rho g} = 1.79 \times 10^{-7}\left[Q(cfm)^2\right]\frac{lbf}{ft^2} \times \frac{ft^3}{62.4\ lbf} \times 12\frac{in.}{ft} = 3.43 \times 10^{-8}\left[Q(cfm)^2\right]$$

The resulting curve is plotted above; computed values are tabulated below.

The system will operate where the fan curve and system curve cross. The approximate operating point is $Q = 7400$ cfm at $h = 1.9$ in. H_2O.

The test section speed is

$$V = \frac{Q}{A} = 7400\frac{ft^3}{min} \times \frac{1}{1\ ft^2} \times \frac{min}{60\ s} = 123\ ft/s$$

V

Q (1000 cfm)	1	2	3	4	5	6	7	8	9	10
Δp (in. H_2O)	0.03	0.14	0.31	0.55	0.86	1.24	1.68	2.20	2.78	3.43

Problem 10.89

[Difficulty: 4]

10.89 Consider again the axial-flow fan and wind tunnel of Problem 10.88. Scale the performance of the fan as it varies with operating speed. Develop and plot a "calibration curve" showing test section flow speed (in m/s) versus fan speed (in rpm).

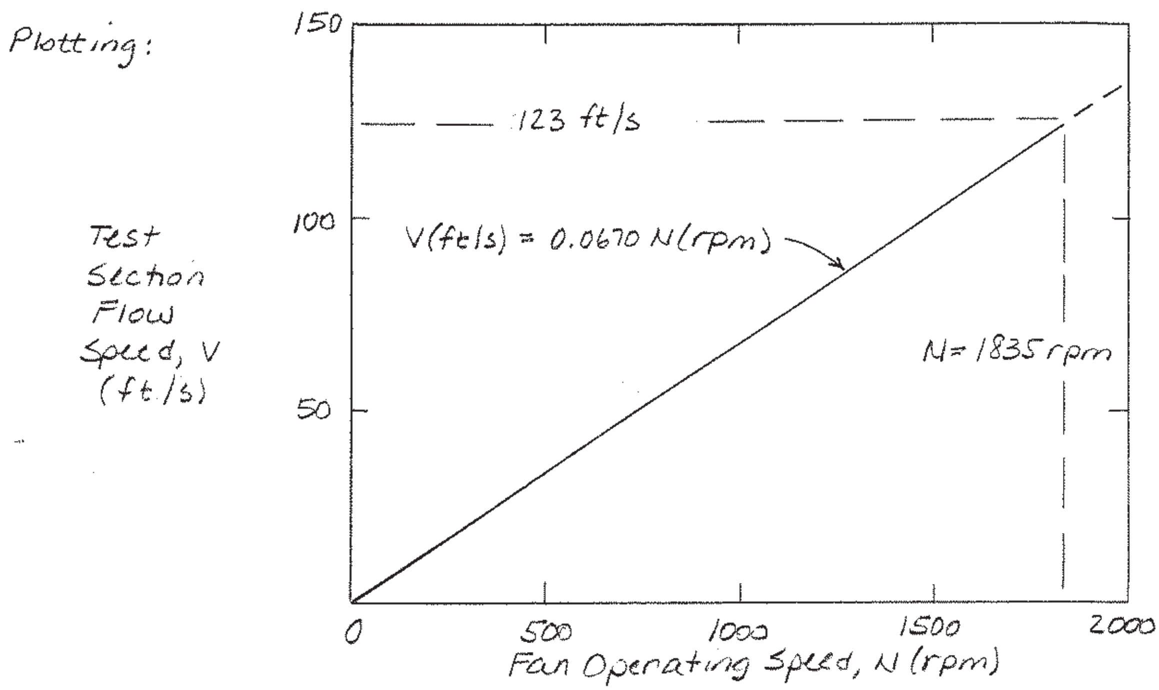
Solution: From the solution to Problem 10.88, $\Delta h_{fan} \propto Q^2 \propto V^2$.

The scaling laws for varying fan speed suggest $Q \propto \omega$ and $p \propto \omega^2$.

Thus $\Delta h_{fan} \propto Q^2 \propto p \propto \omega^2$ or $Q \propto \omega$. The volume flow rate (and the test section flow speed) should vary directly with ω.

From the results of Problem 10.88, $V = 123$ ft/s when $N = 1835$ rpm.

Plotting:



The slope of the linear relationship is $\frac{V}{N} = 123\,\frac{ft}{s} \times \frac{1}{1835\ rpm} = 0.0670$ ft/s/rpm.

Thus $V(ft/s) = 0.0670\, N(rpm)$

V

Problem 10.90

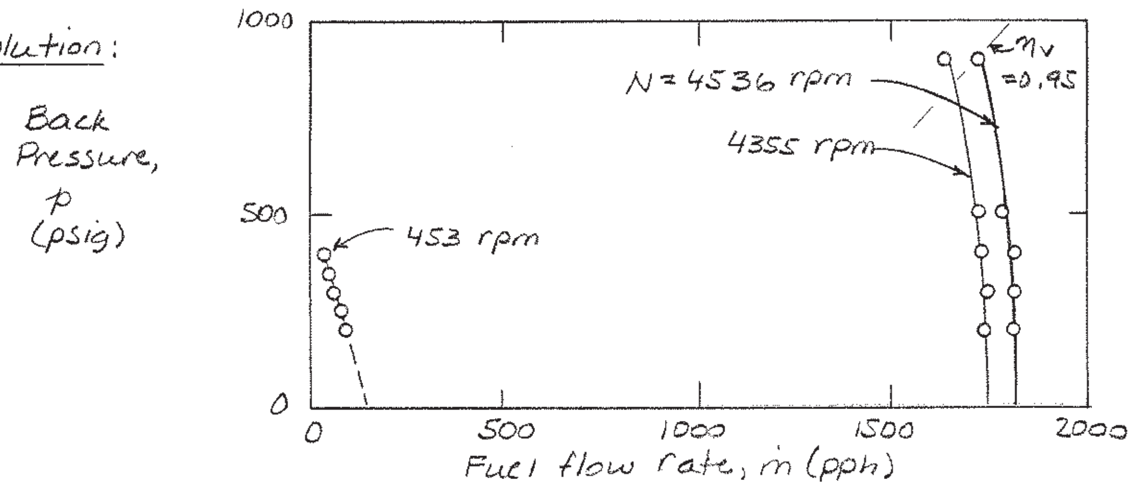
[Difficulty: 3]

10.90 Experimental test data for an aircraft engine fuel pump are presented below. This gear pump is required to supply jet fuel at 450 pounds per hour and 150 psig to the engine fuel controller. Tests were conducted at 10, 96, and 100 percent of the rated pump speed of 4536 rpm. At each constant speed, the back pressure on the pump was set, and the flow rate measured. On one graph, plot curves of pressure versus delivery at the three constant speeds. Estimate the pump displacement volume per revolution. Calculate the volumetric efficiency at each test point and sketch contours of constant η_v. Evaluate the energy loss caused by valve throttling at 100 percent speed and full delivery to the engine.

Pump Speed (rpm)	Back Pressure (psig)	Fuel Flow (pph*)	Pump Speed (rpm)	Back Pressure (psig)	Fuel Flow (pph)	Pump Speed (rpm)	Back Pressure (psig)	Fuel Flow (pph)
	200	1810		200	1730		200	89
4536	300	1810	4355	300	1750	453	250	73
(100%)	400	1810	(96%)	400	1735	(10%)	300	58.5
	500	1790		500	1720		350	45
	900	1720		900	1635		400	30

* Fuel flow rate measured in pounds per hour (pph).

Solution:



For the pump, $\dot{m} = \rho \forall N$, so $\forall = \dfrac{\dot{m}}{\rho N}$. Analyzing the 4536 rpm case,

$$\forall \approx 1810 \frac{\text{lbm}}{\text{hr}} \times \frac{\text{gal}}{6.8\ \text{lbm}} \times \frac{\text{min}}{4536\ \text{rev}} \times \frac{\text{ft}^3}{7.48\ \text{gal}} \times 1728 \frac{\text{in.}^3}{\text{ft}^3} \times \frac{\text{hr}}{60\ \text{min}} = 0.226\ \text{in.}^3/\text{rev}$$ $\forall$

At constant speed, $\eta_v = \dfrac{\forall_{actual}}{\forall_{geometric}} = \dfrac{\dot{m}}{\dot{m}(p=0)}$. Calculation shows η_v decreases as speed is reduced, see below.

Energy loss is
$$\dot{W}_L = \left(\frac{\dot{m}_p - \dot{m}_L}{\rho}\right) p_L = (1810-450)\frac{\text{lbm}}{\text{hr}} \times \frac{\text{gal}}{6.8\ \text{lbm}} \times 150 \frac{\text{lbf}}{\text{in.}^2} \times \frac{\text{ft}^3}{7.48\ \text{gal}} \times 144 \frac{\text{in.}^2}{\text{ft}^2} \times \frac{\text{hp}\cdot\text{s}}{550\ \text{ft}\cdot\text{lbf}} \times \frac{\text{hr}}{3600\ \text{s}}$$

$$\dot{W}_L = 0.292\ \text{hp}$$ $\dot{W}_L$

At 453 rpm, the best volumetric efficiency is

$$\eta_v \approx \frac{\dot{m}}{\dot{m}(p=0)} \times \frac{4536}{453} \approx \frac{89\ \text{pph}}{1810\ \text{pph}} \times \frac{4536}{453} = 0.0492, \text{ or about } 5\%$$ η_v

At 4355 rpm,

$$\eta_v \approx \frac{1730\ \text{pph}}{1810\ \text{pph}} \times \frac{4536}{4355} = 0.996, \text{ or more than } 99\% \text{ (this is doubtful).}$$ η_v

Problem 10.91

[Difficulty: 3]

10.91 A hydraulic turbine is designed to produce 36,000 hp at 95 rpm under 50 ft of head. Laboratory facilities are available to provide 15 ft of head and to absorb 50 hp from the model turbine. Determine (a) the appropriate model test speed and scale ratio and (b) volume flow rate, assuming a model efficiency of 86 percent.

Given: Data on turbine system

Find: Model test speed; Scale; Volume flow rate

Solution:

Basic equations: $W_h = \rho \cdot Q \cdot g \cdot H$ $\qquad \eta = \dfrac{W_{mech}}{W_h}$ $\qquad N_S = \dfrac{\omega \cdot P^{\frac{1}{2}}}{\rho^{\frac{1}{2}} \cdot h^{\frac{5}{4}}}$

The given or available data is

$$\rho = 1.94 \cdot \frac{slug}{ft^3} \qquad W_p = 36000 \cdot hp \qquad H_p = 50 \cdot ft \qquad \omega_p = 95 \cdot rpm \qquad H_m = 15 \cdot ft \qquad W_m = 50 \cdot hp$$

where sub p stands for prototype and sub m stands for model

Note that we need h (energy/mass), not H (energy/weight) $h_p = H_p \cdot g$ $\quad h_p = 1609 \cdot \dfrac{ft^2}{s^2}$ $\quad h_m = H_m \cdot g$ $\quad h_m = 482.6 \cdot \dfrac{ft^2}{s^2}$

Hence for the prototype $\qquad N_S = \dfrac{\omega_p \cdot W_p^{\frac{1}{2}}}{\rho^{\frac{1}{2}} \cdot h_p^{\frac{5}{4}}} \qquad N_S = 3.12$

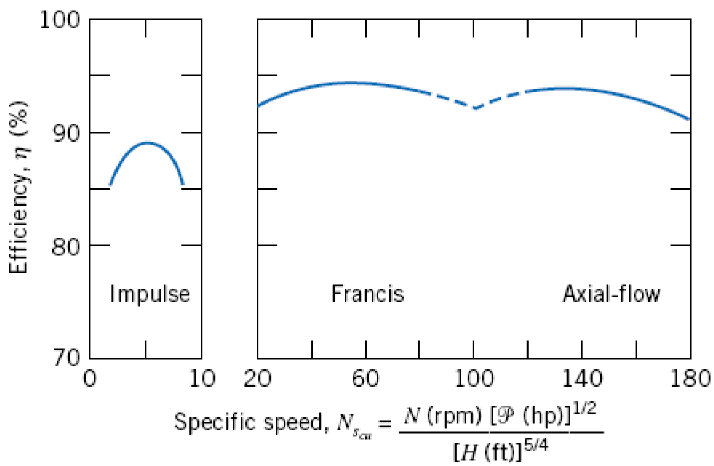
Then for the model $\qquad N_S = \dfrac{\omega_m \cdot W_m^{\frac{1}{2}}}{\rho^{\frac{1}{2}} \cdot h_m^{\frac{5}{4}}} \qquad \omega_m = N_S \cdot \dfrac{\rho^{\frac{1}{2}} \cdot h_m^{\frac{5}{4}}}{W_m^{\frac{1}{2}}} \qquad \omega_m = 59.3 \cdot \dfrac{rad}{s} \qquad \omega_m = 566 \cdot rpm$

For dynamically similar conditions $\qquad \dfrac{H_p}{\omega_p^{\,2} \cdot D_p^{\,2}} = \dfrac{H_m}{\omega_m^{\,2} \cdot D_m^{\,2}} \qquad$ so $\qquad \dfrac{D_m}{D_p} = \dfrac{\omega_p}{\omega_m} \cdot \sqrt{\dfrac{H_m}{H_p}} = 0.092$

Also $\qquad \dfrac{Q_p}{\omega_p \cdot D_p^{\,3}} = \dfrac{Q_m}{\omega_m \cdot D_m^{\,3}} \qquad$ so $\qquad Q_m = Q_p \cdot \dfrac{\omega_m}{\omega_p} \cdot \left(\dfrac{D_m}{D_p}\right)^3$

To find Q_p we need efficiency. At $W_p = 36000 \cdot hp$ and $H_p = 50 \cdot ft$ from F ig. 10.17 we find (see below), for

$$N_{Scu} = \frac{N(rpm) \cdot P(hp)^{\frac{1}{2}}}{H(ft)^{\frac{5}{4}}} = 135.57 \qquad \eta = 93 \cdot \%$$



Hence from $\eta = \dfrac{W_{mech}}{W_h} = \dfrac{W_{mech}}{\rho \cdot Q \cdot g \cdot H}$ $\qquad Q_p = \dfrac{W_p}{\rho \cdot g \cdot H_p \cdot \eta}$ $\qquad Q_p = 3.06 \times 10^6 \cdot \text{gpm}$

and also $Q_m = \dfrac{W_m}{\rho \cdot g \cdot H_m \cdot \eta}$ $\qquad Q_m = 1.418 \times 10^4 \cdot \text{gpm}$

Problem 10.92

[Difficulty: 2]

10.92 Preliminary calculations for a hydroelectric power generation site show a net head of 2350 ft is available at a water flow rate of 75 ft^3/s. Compare the geometry and efficiency of Pelton wheels designed to run at (a) 450 rpm and (b) 600 rpm.

Solution: Apply specific speed equation to classify performance.

Computing equation: $N_{S_{cu}} = \dfrac{N\,\mathscr{P}^{1/2}}{H^{5/4}}$ (rpm, hp, and ft units)

From Fig. 10.17, $\eta_{max} \approx 0.89$ at $N_{S_{cu}} = 5$. The output power (used to define $N_{S_{cu}}$) is

$$\mathscr{P}_{out} = \eta \rho Q g H = 0.89 \times 62.4 \frac{lbf}{ft^3} \times 75 \frac{ft^3}{s} \times 2350\, ft \times \frac{hp \cdot s}{550\, ft \cdot lbf} = 17{,}800\ hp$$

At $N = 450$ rpm

$$N_{S_{cu}} = \frac{450\, rpm\, (17{,}800\, hp)^{\frac{1}{2}}}{(2350)^{5/4}} = 3.67, \text{ so } \eta \approx 0.88$$

Neglect nozzle losses and elevation above the tailrace. Then

$$V_j \approx \sqrt{2gH} = \left[2 \times 32.2 \frac{ft}{s^2} \times 2350\, ft\right]^{\frac{1}{2}} = 389\ ft/s$$

From Fig. 10.10, $U = R\omega \approx 0.47 V_j = 183$ ft/s. Thus

$$D = 2R = \frac{2(0.47 V_j)}{\omega} = 2 \times 183 \frac{ft}{s} \times \frac{s}{47.1\, rad} = 7.77\ ft$$

The jet diameter is found from $Q = V_j A_j = \pi V_j D_j^2/4$, so

$$D_j = \sqrt{\frac{4Q}{\pi V_j}} = \left[\frac{4}{\pi} \times 75 \frac{ft^3}{s} \times \frac{s}{389\, ft}\right]^{\frac{1}{2}} = 0.495\ ft\ (5.95\ in.)$$

The ratio of jet diameter to wheel diameter is

$$r = \frac{D_j}{D} = \frac{0.495\, ft}{7.77\, ft} = 0.0637 \quad \text{or } 1:15.7 \quad \text{(this is reasonable)}$$

Results from similar computations at $N = 600$ rpm are:

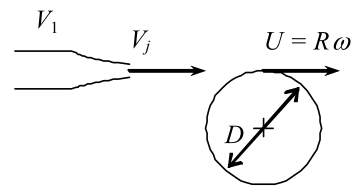
N (rpm)	$N_{S_{cu}}$ (USCS)	D (ft)	D_j/D (---)	η (---)
450	3.67	7.77	1:15.7	0.88
600	4.89	5.83	1:11.8	0.89

The unit operating at 600 rpm is closer to $N_{S_{cu}} = 5$, where peak hydraulic efficiency is expected.

Problem 10.93

[Difficulty: 2]

10.93 Conditions at the inlet to the nozzle of a Pelton wheel are $p = 700$ psig and $V = 15$ mph. The jet diameter is $d = 7.5$ in. and the nozzle loss coefficient is $K_{nozzle} = 0.04$. The wheel diameter is $D = 8$ ft. At this operating condition, $\eta = 0.86$. Calculate (a) the power output, (b) the normal operating speed, (c) the approximate runaway speed, (d) the torque at normal operating speed, and (e) the approximate torque at zero speed.



Given: Pelton turbine

Find: 1) Power 2) Operating speed 3) Runaway speed 4) Torque 5) Torque at zero speed

Solution:

Basic equations
$$\left(\frac{p_1}{\rho \cdot g} + \alpha \cdot \frac{V_1^2}{2 \cdot g} + z_1\right) - \left(\frac{p_j}{\rho \cdot g} + \alpha \cdot \frac{V_j^2}{2 \cdot g} + z_j\right) = \frac{h_{lT}}{g} \qquad h_{lT} = h_l + h_{lm} = K \cdot \frac{V^2}{2}$$

and from Example 10.5
$$T_{ideal} = \rho \cdot Q \cdot R \cdot \left(V_j - U\right) \cdot (1 - \cos(\theta)) \qquad \theta = 165 \cdot deg$$

Assumptions: 1) $p_j = p_{amt}$ 2) Incompressible flow 3) α at 1 and j is approximately 1 4) Only minor loss at nozzle 5) $z_1 = z_j$

Given data
$$p_{1g} = 700 \cdot psi \qquad V_1 = 15 \cdot mph \qquad V_1 = 22 \cdot \frac{ft}{s} \qquad \eta = 86 \cdot \%$$
$$d = 7.5 \cdot in \qquad D = 8 \cdot ft \qquad R = \frac{D}{2} \qquad K = 0.04 \qquad \rho = 1.94 \cdot \frac{slug}{ft^3}$$

Then
$$\frac{p_{1g}}{\rho \cdot g} + \frac{V_1^2}{2 \cdot g} - \frac{V_j^2}{2 \cdot g} = \frac{K}{g} \cdot \frac{V_j^2}{2} \quad \text{or} \quad V_j = \sqrt{\frac{2 \cdot \left(\frac{p_{1g}}{\rho} + \frac{V_1^2}{2}\right)}{1 + K}} \qquad V_j = 317 \cdot \frac{ft}{s}$$

and
$$Q = V_j \cdot \frac{\pi \cdot d^2}{4} \qquad Q = 97.2 \cdot \frac{ft^3}{s} \qquad H = \frac{p_{1g}}{\rho \cdot g} + \frac{V_1^2}{2 \cdot g} \qquad H = 1622 \cdot ft$$

Hence
$$P = \eta \cdot \rho \cdot Q \cdot g \cdot H \qquad P = 15392 \cdot hp$$

From Fig. 10.10, normal operating speed is around
$$U = 0.47 \cdot V_j \qquad U = 149 \cdot \frac{ft}{s} \qquad \omega = \frac{U}{R} \qquad \omega = 37.2 \cdot \frac{rad}{s} \qquad \omega = 356 \cdot rpm$$

At runaway
$$U_{run} = V_j \qquad \omega_{run} = \frac{U_{run}}{\left(\frac{D}{2}\right)} \qquad \omega_{run} = 79.2 \cdot \frac{rad}{s} \qquad \omega_{run} = 756 \cdot rpm$$

From Example 10.5
$$T_{ideal} = \rho \cdot Q \cdot R \cdot \left(V_j - U\right) \cdot (1 - \cos(\theta)) \qquad T_{ideal} = 2.49 \times 10^5 \cdot ft \cdot lbf$$

Hence
$$T = \eta \cdot T_{ideal} \qquad T = 2.14 \times 10^5 \cdot ft \cdot lbf$$

Stall occurs when
$$U = 0 \qquad T_{stall} = \eta \cdot \rho \cdot Q \cdot R \cdot V_j \cdot (1 - \cos(\theta)) \qquad T_{stall} = 4.04 \times 10^5 \cdot ft \cdot lbf$$

Problem 10.94

[Difficulty: 2]

10.94 The reaction turbines at Niagara Falls are of the Francis type. The impeller outside diameter is 4.5 m. Each turbine produces 54 MW at 107 rpm, with 93.8 percent efficiency under 65 m of net head. Calculate the specific speed of these units. Evaluate the volume flow rate to each turbine. Estimate the penstock size if it is 400 m long and the net head is 83 percent of the gross head.

Given: Data on Francis turbines at Niagra Falls

Find: Specific speed, volume flow rate to each turbine, penstock size

Solution:

Basic equations: $W_h = \rho \cdot Q \cdot g \cdot H$ $\quad \eta = \dfrac{W_{mech}}{W_h}$ $\quad N_S = \dfrac{\omega \cdot P^{\frac{1}{2}}}{\rho^{\frac{1}{2}} \cdot h^{\frac{5}{4}}}$ $\quad h = g \cdot H$ $\quad h_{lT} = f \cdot \dfrac{L}{D} \cdot \dfrac{V^2}{2}$

The given or available data is

$\rho = 998 \cdot \dfrac{kg}{m^3}$ $\quad W_{mech} = 54 \cdot MW$ $\quad \omega = 107 \cdot rpm$ $\quad \eta = 93.8\%$ $\quad H = 65 \cdot m$ $\quad L_{penstock} = 400 \cdot m$ $\quad H_{net} = H \cdot 83\%$

The specific energy of the turbine is: $h = g \cdot H = 637.4 \dfrac{m^2}{s^2}$ The specific speed is: $N_S = \dfrac{\omega \cdot W_{mech}^{\frac{1}{2}}}{\rho^{\frac{1}{2}} \cdot h^{\frac{5}{4}}}$ $\quad N_S = 0.814$

Solving for the flow rate of the turbine: $Q = \dfrac{W_{mech}}{\rho \cdot h \cdot \eta} = 90.495 \dfrac{m^3}{s}$ $\quad Q = 90.5 \dfrac{m^3}{s}$

Based on the head loss: $h_{lT} = g \cdot (H - H_{net}) = 108.363 \dfrac{m^2}{s^2}$ Since $V = \dfrac{Q}{A} = \dfrac{4 \cdot Q}{\pi \cdot D^2}$ into the head loss equation:

$h_{lT} = f \cdot \dfrac{L}{D} \cdot \dfrac{1}{2} \cdot \left(\dfrac{4 \cdot Q}{\pi \cdot D^2}\right)^2 = \dfrac{8}{\pi^2} \cdot \dfrac{f \cdot L \cdot Q^2}{D^5}$ Solving for the diameter: $D = \left(\dfrac{8 \cdot f \cdot L \cdot Q^2}{\pi^2 \cdot h_{lT}}\right)^{\frac{1}{5}}$ This will require an iterative solution.

Assuming concrete-lined penstocks: $e = 3 \cdot mm$ If we assume a diameter of 2 m, we can iterate to find the actual diameter:

D (m)	*V* (m/s)	Re	*e* /*D*	*f*	*D* (m)
2.000	28.807	5.70E+07	0.001500	0.02173	3.510
3.510	9.354	3.25E+07	0.000855	0.01892	3.414
3.414	9.888	3.34E+07	0.000879	0.01904	3.418
3.418	9.862	3.34E+07	0.000878	0.01904	3.418

$D = 3.42 \cdot m$

Problem 10.95

[Difficulty: 2]

10.95 Francis turbine Units 19, 20, and 21, installed at the Grand Coulee Dam on the Columbia River, are *very* large [55]. Each runner is 32.6 ft in diameter and contains 550 tons of cast steel. At rated conditions, each turbine develops 820,000 hp at 72 rpm under 285 ft of head. Efficiency is nearly 95 percent at rated conditions. The turbines operate at heads from 220 to 355 ft. Calculate the specific speed at rated operating conditions. Estimate the maximum water flow rate through each turbine.

Solution: Apply definitions of specific speed and efficiency.

Computing equations: $N_{s_{cu}} = \dfrac{N \mathcal{P}^{1/2}}{H^{5/4}}$ $\quad \eta = \dfrac{\mathcal{P}}{\rho Q g H}$

Thus

$$N_{s_{cu}} = \frac{72 \text{ rpm } (820{,}000 \text{ hp})^{\frac{1}{2}}}{(285 \text{ ft})^{5/4}} = 55.7$$

$N_{s_{cu}}$

From η,

$$Q = \frac{\mathcal{P}}{\eta \rho g H}$$

so Q is maximum at minimum head. Assuming $\eta = 0.95$, the

$$Q \approx \frac{1}{0.95} \times 820{,}000 \text{ hp} \times \frac{\text{ft}^3}{62.4 \text{ lbf}} \times \frac{1}{220 \text{ ft}} \times 550 \frac{\text{ft} \cdot \text{lbf}}{\text{hp} \cdot \text{s}} = 34{,}600 \text{ ft}^3/\text{s} \quad (\max)$$

Q

{ This is an estimate because η may not be constant, nor may it be possible to develop full power at $H = 220$ ft. }

10.96 Measured data for performance of the reaction turbines at Shasta Dam near Redding, California, are shown in Fig. 10.39. Each turbine is rated at 103,000 hp when operating at 138.6 rpm under a net head of 380 ft. Evaluate the specific speed and compute the shaft torque developed by each turbine at rated operating conditions. Calculate and plot the water flow rate per turbine required to produce rated output power as a function of head.

Solution: Apply the definitions of specific speed and efficiency, use data from Fig. 10.39:

Computing equations: $N_{s_{cu}} = \dfrac{N \mathcal{P}^{\frac{1}{2}}}{H^{\frac{5}{4}}}$ $\qquad \eta = \dfrac{\mathcal{P}}{\rho Q g H}$ $\qquad \mathcal{P} = \omega T$

At rated conditions, $N_{s_{cu}} = \dfrac{(138.6\,\text{rpm})(103{,}000\,\text{hp})^{\frac{1}{2}}}{(380\,\text{ft})^{\frac{5}{4}}} = 26.5$ N_s

$$T = \frac{\mathcal{P}}{\omega} = 103{,}000\,\text{hp} \times \frac{\text{s}}{14.5\,\text{rad}} \times 550\,\frac{\text{ft}\cdot\text{lbf}}{\text{hp}\cdot\text{s}} = 3.91 \times 10^6\,\text{ft}\cdot\text{lbf}$$

Find Q from definition of η; at rated conditions, $\eta \approx 0.93$ (Fig. 10.39):

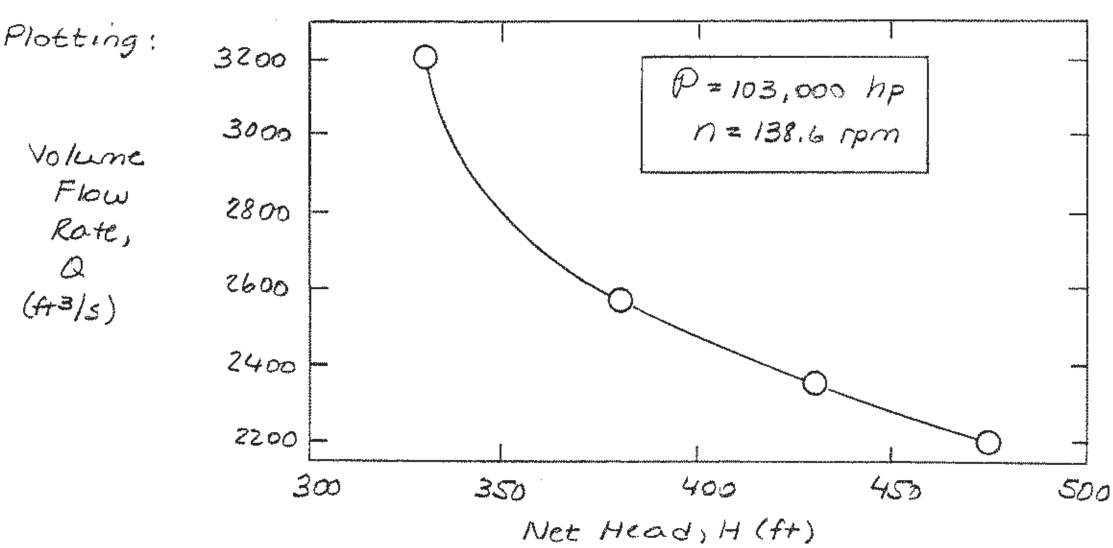
$$Q = \frac{\mathcal{P}}{\eta \rho g H} = \frac{1}{0.93} \times 103{,}000\,\text{hp} \times \frac{\text{ft}^3}{62.4\,\text{lbf}} \times \frac{1}{380\,\text{ft}} \times 550\,\frac{\text{ft}\cdot\text{lbf}}{\text{hp}\cdot\text{s}} = 2570\,\text{ft}^3/\text{s}$$

Tabulating similar calculations:

H (ft)	$\mathcal{P}$ (hp)	η (---)	Q (ft³/s)
238	*	–	–
280	*	–	–
330	103,000	0.86	3200
380	103,000	0.93	2570
430	103,000	0.90	2350
475	103,000	0.87	2200

* cannot produce rated power at this head

Plotting:



Problem 10.97

[Difficulty: 3]

10.97 Figure 10.37 contains data for the efficiency of a large Pelton waterwheel installed in the Tiger Creek Power House of Pacific Gas & Electric Company near Jackson, California. This unit is rated at 26.8 MW when operated at 225 rpm under a net head of 360 m of water. Assume reasonable flow angles and nozzle loss coefficient, and water at 15°C. Determine the rotor radius, and estimate the jet diameter and the mass flow rate of water.

Given: Data on Pelton wheel

Find: Rotor radius, jet diameter, water flow rate.

Solution:

The given or available data is

$$\rho = 999 \cdot \frac{kg}{m^3} \quad W_{mech} = 26.8 \cdot MW \quad \omega = 225 \cdot rpm \quad H = 360 \cdot m \quad \nu = 1.14 \times 10^{-6} \cdot \frac{m^2}{s}$$

From Bernoulli, the jet velocity is: $V_i = \sqrt{2 \cdot g \cdot H}$ Assuming a velocity coefficient of $C_V = 0.98$ (4% loss in the nozzle):

$$V_j = C_V \cdot \sqrt{2 \cdot g \cdot H} = 82.35 \frac{m}{s}$$

From Fig. 10.36, at maximum efficiency: $U = R \cdot \omega = 0.47 \cdot V_j$ So the radius can be calculated:

$$R = 0.47 \cdot \frac{V_j}{\omega} = 1.643 m$$

From Fig. 10.37 the efficiency at full load is $\eta = 86\%$ Thus: $\eta = \frac{W_{mech}}{Q \cdot \rho \cdot g \cdot H}$ Solving for the flow rate:

$$Q = \frac{W_{mech}}{\eta \cdot \rho \cdot g \cdot H} = 8.836 \frac{m^3}{s}$$

We can now calculate the jet velocity: $A_j = \frac{\pi}{4} \cdot D_j^2 = \frac{Q}{V_j}$ Therefore, $D_j = 2 \cdot \sqrt{\frac{Q}{\pi \cdot V_j}} = 0.37\, m$ $\quad D_j = 37.0 \cdot cm$

$$m_{rate} = \rho \cdot Q = 8.83 \times 10^3 \frac{kg}{s}$$

Problem 10.98

[Difficulty: 3]

10.98 An impulse turbine is to develop 15 MW from a single wheel at a location where the net head is 350 m. Determine the appropriate speed, wheel diameter, and jet diameter for single- and multiple-jet operation. Compare with a double-overhung wheel installation. Estimate the required water consumption.

Given: Impulse turbine requirements

Find: 1) Operating speed 2) Wheel diameter 4) Jet diameter 5) Compare to multiple-jet and double-overhung

Solution:

Basic equations: $V_j = \sqrt{2 \cdot g \cdot H}$ $\qquad N_S = \dfrac{\omega \cdot P^{\frac{1}{2}}}{\rho^{\frac{1}{2}} \cdot h^{\frac{5}{4}}}$ $\qquad \eta = \dfrac{P}{\rho \cdot Q \cdot g \cdot H}$ $\qquad Q = V_j \cdot A_j$

Model as optimum. This means. from Fig. 10.10 $U = 0.47 \cdot V_j$ and from Fig. 10.17 $N_{Scu} = 5$ with $\eta = 89 \cdot \%$

Given or available data $H = 350 \cdot m$ $\qquad P = 15 \cdot MW$ $\qquad \rho = 1.94 \cdot \dfrac{slug}{ft^3}$

Then $V_j = \sqrt{2 \cdot g \cdot H}$ $\qquad V_j = 82.9 \dfrac{m}{s}$ $\qquad U = 0.47 \cdot V_j$ $\qquad U = 38.9 \dfrac{m}{s}$

We need to convert from N_{Scu} (from Fig. 10.17) to N_S (see discussion after Eq. 10.18b). $N_S = \dfrac{N_{Scu}}{43.46}$ $\qquad N_S = 0.115$

The water consumption is $Q = \dfrac{P}{\eta \cdot \rho \cdot g \cdot H}$ $\qquad Q = 4.91 \dfrac{m^3}{s}$

For a single jet $\omega = N_S \cdot \dfrac{\rho^{\frac{1}{2}} \cdot (g \cdot H)^{\frac{5}{4}}}{P^{\frac{1}{2}}}$ (1) $\qquad \omega = 236 \cdot rpm$ $\qquad D_j = \sqrt{\dfrac{4 \cdot Q}{\pi \cdot V_j}}$ (2) $\qquad D_j = 0.275\,m$

The wheel radius is $D = \dfrac{2 \cdot U}{\omega}$ (3) $\qquad D = 3.16\,m$

For multiple (n) jets, we use the power and flow per jet

From Eq 1 $\omega_n = \omega \cdot \sqrt{n}$ $\qquad$ From Eq. 2 $D_{jn} = \dfrac{D_j}{\sqrt{n}}$ an d $D_n = \dfrac{D}{\sqrt{n}}$ from Eq. 3

Results:

$n =$	$\omega_n(n) =$ (rpm)	$D_{jn}(n) =$ (m)	$D_n(n) =$ (m)
1	236	0.275	3.16
2	333	0.194	2.23
3	408	0.159	1.82
4	471	0.137	1.58
5	527	0.123	1.41

A double-hung wheel is equivalent to having a single wheel with two jets

Problem 10.99

[Difficulty: 2]

10.99 An impulse turbine under a net head of 33 ft was tested at a variety of speeds. The flow rate and the brake force needed to set the impeller speed were recorded:

Wheel Speed (rpm)	Flow rate (cfm)	Brake Force (lbf) (R = 0.5 ft)
0	7.74	2.63
1000	7.74	2.40
1500	7.74	2.22
1900	7.44	1.91
2200	7.02	1.45
2350	5.64	0.87
2600	4.62	0.34
2700	4.08	0.09

Calculate and plot the machine power output and efficiency as a function of water turbine speed.

Given:
Data on impulse turbine

Find:
Plot of power and efficiency curves

Solution:

Basic equations: $T = F \cdot R$ $\qquad P = \omega \cdot T$ $\qquad \eta = \dfrac{P}{\rho \cdot Q \cdot g \cdot H}$

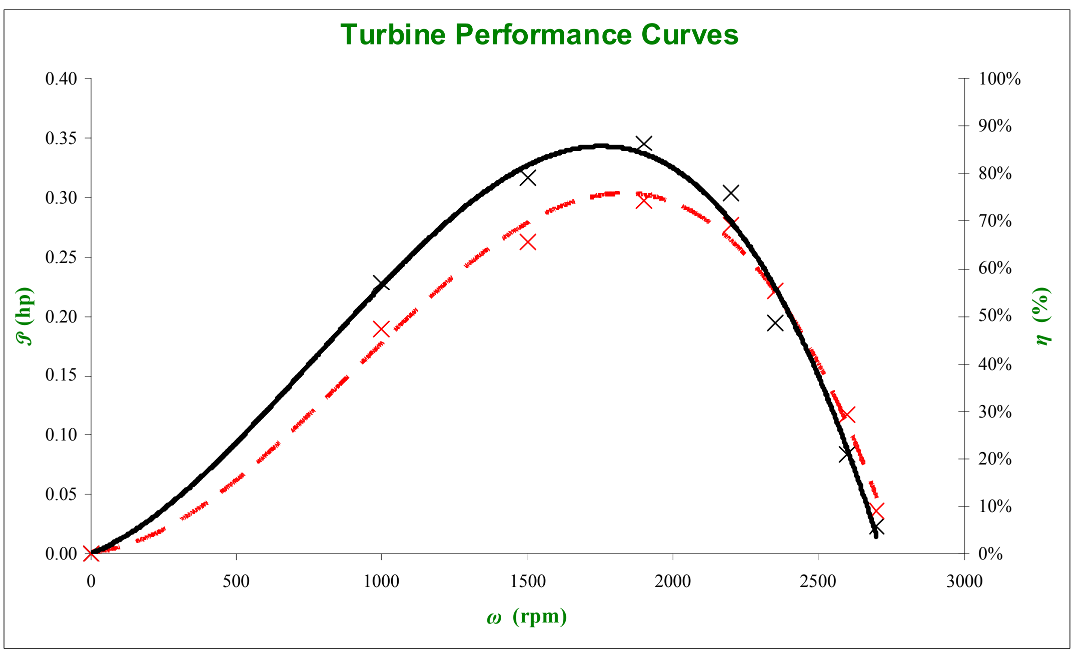
Here are the results calculated in *Excel*:

H = 33 ft

ρ = 1.94 slug/ft^3

R = 0.50 ft

ω (rpm)	Q (cfm)	F (lbf)	T (ft-lbf)	$\mathscr{P}$ (hp)	η (%)
0	7.74	2.63	1.32	0.000	0.0%
1000	7.74	2.40	1.20	0.228	47.3%
1500	7.74	2.22	1.11	0.317	65.6%
1900	7.44	1.91	0.96	0.345	74.4%
2200	7.02	1.45	0.73	0.304	69.3%
2350	5.64	0.87	0.44	0.195	55.3%
2600	4.62	0.34	0.17	0.084	29.2%
2700	4.08	0.09	0.05	0.023	9.1%



Problem 10.100

[Difficulty: 4]

10.100 In U.S. customary units, the common definition of specific speed for a hydraulic turbine is given by Eq. 10.13b. Develop a conversion between this definition and a truly dimensionless one in SI units. Evaluate the specific speed of an impulse turbine, operating at 400 rpm under a net head of 1190 ft with 86 percent efficiency, when supplied by a single 6-in.-diameter jet. Use both U.S. customary and SI units. Estimate the wheel diameter.

Solution: Apply definitions of specific speed and efficiency.

Computing equations: $N_{s_{cu}} = \dfrac{N\mathcal{P}^{1/2}}{H^{5/4}}$ $\quad \eta \equiv \dfrac{\mathcal{P}}{\rho Q g H}$ $\quad N_s = \dfrac{\omega \mathcal{P}^{1/2}}{\rho^{1/2} h^{5/4}}$

From dimensional analysis, recall that for pumps, $N_s = \dfrac{NQ^{1/2}}{H^{3/4}}$ was dimensionless only when H is expressed as gH, energy per unit mass. Thus the dimensions are

$$[N_s] = \left[\frac{NQ^{1/2}}{(gH)^{3/4}}\right] = \frac{1}{t}\left(\frac{L^3}{t}\right)^{\frac{1}{2}}\left(\frac{t^2}{L^2}\right)^{\frac{3}{4}} = \frac{1}{t}\,\frac{L^{3/2}}{t^{1/2}}\,\frac{t^{3/2}}{L^{3/2}} = 1 \quad \checkmark\checkmark$$

To form the Ns for turbines, Q must be multiplied by $\rho g H$ to obtain power. Thus for turbines, the dimensionless specific speed is

$$[N_s] = \left[\frac{NQ^{1/2}}{(gH)^{3/4}} \times \frac{(\rho g H)^{1/2}}{(\rho g H)^{1/2}}\right] = \left[\frac{N\mathcal{P}^{1/2}}{\rho^{1/2}(gH)^{5/4}}\right] = \frac{1}{t}\left(\frac{FL}{t} \times \frac{ML}{Ft^2}\right)^{\frac{1}{2}}\left(\frac{L^3}{M}\right)^{\frac{1}{2}}\left(\frac{t^2}{L^2}\right)^{\frac{5}{4}} = \frac{M^{\frac{1}{2}}L^{\frac{5}{2}}t^{\frac{5}{2}}}{M^{\frac{1}{2}}L^{5/2}t^{5/2}} = 1 \quad \checkmark\checkmark$$

The simplest way to convert is to evaluate each specific speed, then take the ratio.

The jet speed will be approximately $V_j = \sqrt{2gH} = \left[2 \times 32.2\,\frac{ft}{s^2} \times 1190\,ft\right]^{\frac{1}{2}} = 277\,ft/s$

and $Q = V_j \dfrac{\pi D_j^2}{4} = 277\,\dfrac{ft}{s} \times \dfrac{\pi}{4}\left(\dfrac{1}{2}\right)^2 ft^2 = 54.4\,ft^3/s.$ Thus

$$\mathcal{P} = \eta \rho Q g H = 0.86 \times 62.4\,\frac{lbf}{ft^3} \times 54.4\,\frac{ft^3}{s} \times 1190\,ft \times \frac{hp \cdot s}{550\,ft \cdot lbf} = 6320\,hp\ (4.71\ MW)$$

For the wheel, $U = 0.47V_j = R\omega$; $R = 3.10$ ft; $D = 6.20$ ft D

In U.S. units, $N_{s_{cu}} = \dfrac{400\,rpm\,(6320\,hp)^{\frac{1}{2}}}{(1190\,ft)^{5/4}} = 4.55$ Ns

In S.I. units, $\omega = 41.9$ rad/s and $gH = 3560\,m^2/s^2$, so

$$N_s = \frac{\omega \mathcal{P}^{1/2}}{\rho^{1/2}(gH)^{5/4}} = 41.9\,\frac{rad}{s} \times (4.71 \times 10^6\,W)^{\frac{1}{2}} \times \left(\frac{m^3}{999\,kg}\right)^{\frac{1}{2}}\left(\frac{s^2}{3560\,m^2}\right)^{\frac{5}{4}} = 0.105$$ Ns

The conversion is $\dfrac{N_{s\,cu}}{N_s} = \dfrac{4.55}{0.105} = 43.5$ Ratio

Problem 10.101

[Difficulty: 3]

10.101 According to a spokesperson for Pacific Gas & Electric Company, the Tiger Creek plant, located east of Jackson, California, is one of 71 PG&E hydroelectric powerplants. The plant has 373 m of gross head, consumes 21 m^3/s of water, is rated at 60 MW, and operates at 58 MW. The plant is claimed to produce 0.785 kW·hr/(m^2·m) of water and 336.4×10^6 kW·hr/yr of operation. Estimate the net head at the site, the turbine specific speed, and its efficiency. Comment on the internal consistency of these data.

Given: Published data for the Tiger Creek Power Plant

Find: (a) Estimate net head at the site, turbine specific speed, and turbine efficiency
(b) Comment on consistency of the published data

Solution:

Basic Equations: $N_{Scu} = \dfrac{N\cdot\sqrt{P}}{H^{\frac{5}{4}}}$ $\quad N_S = \dfrac{N\cdot\sqrt{P}}{\sqrt{\rho}\cdot(g\cdot H)^{\frac{5}{4}}}$ $\quad \eta = \dfrac{P}{\rho\cdot Q\cdot g\cdot H_{net}}$

The given or available data is

$\rho = 999\cdot\dfrac{kg}{m^3}$ $\quad P = 58\cdot MW$ $\quad Q = 21\cdot\dfrac{m^3}{s}$ $\quad H_{gross} = 373\cdot m$ $\quad \nu = 1.14\times 10^{-6}\cdot\dfrac{m^2}{s}$

Using data from Fig. 10.37, we will assume $\eta = 87\%$ We can take this to estimate the net head: $H_{net} = \dfrac{P}{\rho\cdot Q\cdot g\cdot\eta} = 324\,m$

Therefore: $\dfrac{H_{net}}{H_{gross}} = 86.875\cdot\%$ This is close to 87%, so the assumption for the efficiency was a good one.

From the same figure, we will assume $N_{Scu} = 5$ Therefore the dimensionless specific speed is $N_S = \dfrac{N_{Scu}}{43.46} = 0.115$

We may then calculate the rotational speed for the turbine: $N = \dfrac{N_S\cdot\sqrt{\rho}\cdot\left(g\cdot H_{net}\right)^{\frac{5}{4}}}{\sqrt{P}} = 108.8\cdot rpm$

The power output seems low for a turbine used for electricity generation; several turbines are probably used in this one plant.

To check the claims: $58\cdot MW\times\dfrac{24\cdot hr}{1\cdot day}\times\dfrac{365\cdot day}{yr} = 5.081\times 10^8\cdot\dfrac{kW\cdot hr}{yr}$ This number is 50% higher than the claim.

$58\cdot MW\times\dfrac{s}{21\cdot m^3}\times\dfrac{hr}{3600\cdot s} = 0.767\cdot\dfrac{kW\cdot hr}{m^2\cdot m}$ This is in excellent agreement with the claim.

Problem 10.102

[Difficulty: 4]

10.102 Design the piping system to supply a water turbine from a mountain reservoir. The reservoir surface is 320 m above the turbine site. The turbine efficiency is 83 percent, and it must produce 30 kW of mechanical power. Define the minimum standard-size pipe required to supply water to the turbine and the required volume flow rate of water. Discuss the effects of turbine efficiency, pipe roughness, and installing a diffuser at the turbine exit on the performance of the installation.

Given: Hydraulic turbine site

Find: Minimum pipe size; Fow rate; Discuss

Solution:

Basic equations: $H_l = \dfrac{h_l}{g} = f \cdot \dfrac{L}{D} \cdot \dfrac{V^2}{2 \cdot g}$ and also, from Example 10.15 the optimum is when $H_l = \dfrac{\Delta z}{3}$

As in Fig. 10.41 we assume $L = 2 \cdot \Delta z$ and $f = 0.02$

Then, for a given pipe diameter D $\quad V = \sqrt{\dfrac{2 \cdot g \cdot D \cdot H_l}{f \cdot L}} = \sqrt{\dfrac{g \cdot D}{3 \cdot f}}$

Also $\quad Q = V \cdot \dfrac{\pi \cdot D^2}{4} \qquad P_h = \rho \cdot Q \cdot \dfrac{V^2}{2} \qquad P_m = \eta \cdot P_h$ $\quad$ Here are the results in *Excel*:

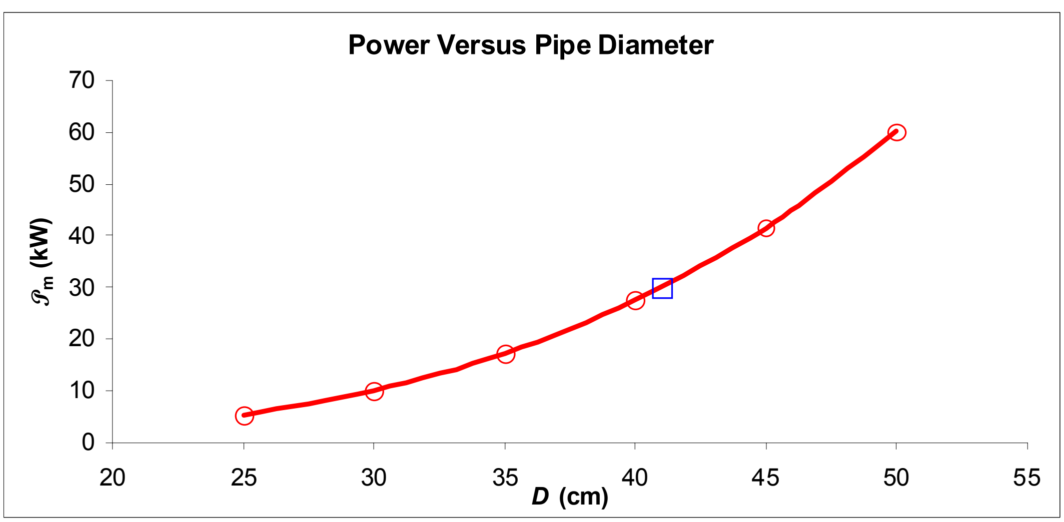
f = 0.02
ρ = 998.00 kg/m^3
η = 83%

D (cm)	V (m/s)	Q (m^3/s)	$\mathscr{P}_h$ (kW)	$\mathscr{P}_m$ (kW)
25	6.39	0.314	6.40	5.31
30	7.00	0.495	12.12	10.06
35	7.56	0.728	20.78	17.25
40	8.09	1.016	33.16	27.53
45	8.58	1.364	50.09	41.57
50	9.04	1.775	72.42	60.11

Turbine efficiency varies with specific speed
Pipe roughness appears to the 1/2 power, so has a secondary effect.
A 20% error in f leads to a 10% change in water speed
and 30% change in power.
A Pelton wheel is an impulse turbine that does not flow full of water;
it directs the stream with open buckets.
A diffuser could not be used with this system.

41.0	8.19	1.081	36.14	30.00

Use Goal Seek or Solver to vary D to make $\mathscr{P}_m$ 30 kW!



Problem 10.103

[Difficulty: 4]

10.103 A small hydraulic impulse turbine is supplied with water through a penstock with diameter D and length L; the jet diameter is d. The elevation difference between the reservoir surface and nozzle centerline is Z. The nozzle head loss coefficient is K_{nozzle} and the loss coefficient from the reservoir to the penstock entrance is $K_{entrance}$. Determine the water jet speed, the volume flow rate, and the hydraulic power of the jet, for the case where $Z = 300$ ft, $L = 1000$ ft, $D = 6$ in., $K_{entrance} = 0.5$, $K_{nozzle} = 0.04$, and $d = 2$ in., if the pipe is made from commercial steel. Plot the jet power as a function of jet diameter to determine the optimum jet diameter and the resulting hydraulic power of the jet. Comment on the effects of varying the loss coefficients and pipe roughness.

Solution: Apply the energy equation for steady, incompressible pipe flow.

Computing equation: $$\frac{p_1}{\rho g} + \alpha_1 \frac{\bar{V}_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \alpha_2\frac{\bar{V}_2^2}{2g} + z_2 + \frac{h_{lT}}{g}\ ; \quad h_{lT} = \left[f\left(\frac{L}{D} + \frac{L_e}{D}\right) + K\right]\frac{\bar{V}^2}{2}$$ ($\bar{V}_1 \approx 0(2)$)

Assumptions: (1) $p_1 = p_2 = p_{atm}$, (2) $\bar{V}_1 \approx 0$, $\alpha_2 = 1$, (3) $L_e/D = 0$, (4) K_{nozzle} based on V_j^2

Then
$$H = \frac{V_j^2}{2g} + \left(f\frac{L}{D} + K_{entrance}\right)\frac{\bar{V}^2}{2g} + K_{nozzle}\frac{V_j^2}{2g}$$

From continuity, $\bar{V}A = V_jA_j$, so $\bar{V} = V_j\, A_j/A = V_j(d/D)^2$; $\bar{V}^2 = V_j^2(d/D)^4$, and
$$H = \left[\left(f\frac{L}{D} + K_{ent}\right)\left(\frac{d}{D}\right)^4 + 1 + K_{nozzle}\right]\frac{V_j^2}{2g}\ ; \quad V_j = \left[\frac{2gH}{\left(f\frac{L}{D} + K_{ent}\right)\left(\frac{d}{D}\right)^4 + 1 + K_n}\right]^{\frac{1}{2}}$$

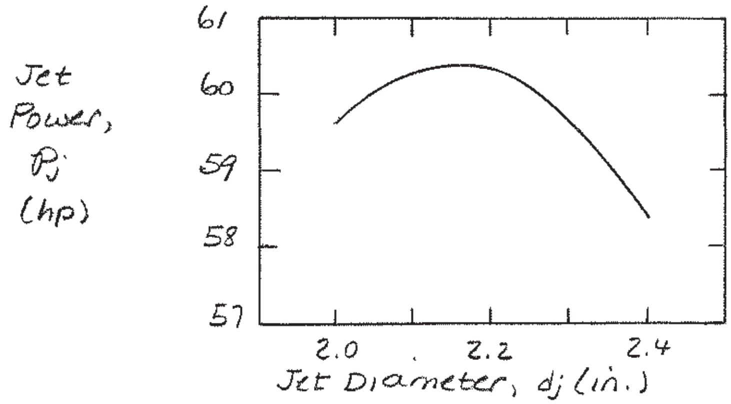
Assume $e = 0.00015$ ft (Table 8.1), so $e/D = 0.0003$. From Fig. 8.13, in the fully rough zone, $f = 0.015$. Then for $d_j = 2$ in.

$$V_j = \left[2 \times 32.2\,\frac{ft}{s^2} \times 300\,ft \times \frac{1}{\left(0.015\,\frac{1000\,ft}{0.5\,ft} + 0.5\right)\left(\frac{2}{6}\right)^4 + 1 + 0.04}\right]^{\frac{1}{2}} = 117\ ft/sec$$

($\bar{V} = 13.0$ ft/s, $Re = \bar{V}D/\nu = 6.05 \times 10^5$, so $f = 0.016$, which makes $V_j = 116$ ft/s.)

The jet flow rate is $Q = V_jA_j = 116\,\frac{ft}{s} \times \frac{\pi}{4}\left(\frac{2}{12}\right)^2 ft^2 = 2.53\ ft^3/s$, and the jet power is
$$\mathcal{P}_h = \rho Q\frac{V_j^2}{2} = \frac{1}{2} \times 1.94\,\frac{slug}{ft^3} \times (116)^2\frac{ft^2}{s^2} \times 2.53\,\frac{ft^3}{s} \times \frac{lbf\cdot s^2}{slug\cdot ft} \times \frac{hp\cdot s}{550\,ft\cdot lbf} = 60.0\ hp$$

Repeating these calculations using a computer program gives:



Peak power, $\mathcal{P}_j \approx 60.3$ hp, occurs for $2.15 < d < 2.20$ in. Opt

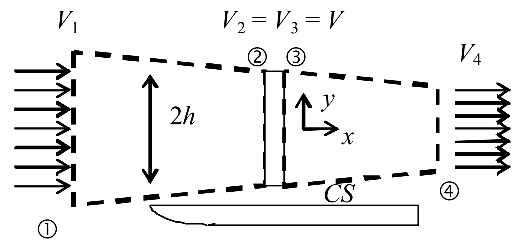
Loss coefficients have a minor effect. Making both K_{ent} and K_n zero increases $\mathcal{P}_j$ by 4.8 percent.

Pipe roughness causes larger changes; $\mathcal{P}_j$ increased 12.8 percent with $e = 0$ (smooth).

Problem 10.104

[Difficulty: 3]

10.104 The propeller on a fanboat used in the Florida Everglades moves air at the rate of 50 kg/s. When at rest, the speed of the slipstream behind the propeller is 45 m/s at a location where the pressure is atmospheric. Calculate (a) the propeller diameter, (b) the thrust produced at rest, and (c) the thrust produced when the fanboat is moving ahead at 15 m/s if the mass flow rate through the propeller remains constant.



Given: Data on boat and propeller

Find: Propeller diameter; Thrust at rest; Thrust at 15 m/s

Solution:

Basic equation: $$\vec{F} = \vec{F}_S + \vec{F}_B = \frac{\partial}{\partial t}\int_{CV} \vec{V}_{xyz}\,\rho\, d\forall + \int_{CS} \vec{V}_{xyz}\,\rho\vec{V}_{xyz}\cdot d\vec{A} \qquad (4.26)$$

Assumption: 1) Atmospheric pressure on CS 2) Horizontal 3) Steady w.r.t. the CV 4) Use velocities relative to CV

The x-momentum is then $T = u_1\cdot(-m_{rate}) + u_4\cdot(m_{rate}) = (V_4 - V_1)\cdot m_{rate}$ where $m_{rate} = 50\cdot\frac{kg}{s}$ is the mass flow rate

It can be shown (see Example 10.13) that $V = \frac{1}{2}\cdot(V_4 + V_1)$

For the static case $V_1 = 0\cdot\frac{m}{s}$ $\quad V_4 = 45\cdot\frac{m}{s}$ $\quad$ so $\quad V = \frac{1}{2}\cdot(V_4 + V_1)$ $\quad V = 22.5\cdot\frac{m}{s}$

From continuity $m_{rate} = \rho\cdot V\cdot A = \rho\cdot V\cdot\frac{\pi\cdot D^2}{4}$ with $\rho = 1.23\cdot\frac{kg}{m^3}$

Hence $D = \sqrt{\frac{4\cdot m_{rate}}{\rho\cdot\pi\cdot V}}$ $\quad D = 1.52\cdot m$

For $V_1 = 0$ $\quad T = m_{rate}\cdot(V_4 - V_1)$ $\quad T = 2250\cdot N$

When in motion $V_1 = 15\cdot\frac{m}{s}$ and $V = \frac{1}{2}\cdot(V_4 + V_1)$ so $V_4 = 2\cdot V - V_1$ $\quad V_4 = 30\cdot\frac{m}{s}$

Hence for $V_1 = 15$ m/s $\quad T = m_{rate}\cdot(V_4 - V_1)$ $\quad T = 750\cdot N$

Problem 10.105

[Difficulty: 3]

10.105 A fanboat in the Florida Everglades is powered by a propeller, with $D = 1.5$ m, driven at maximum speed, $N = 1800$ rpm, by a 125 kW engine. Estimate the maximum thrust produced by the propeller at (a) standstill and (b) $V = 12.5$ m/s.

Given: Data on fanboat and propeller

Find: Thrust at rest; Thrust at 12.5 m/s

Solution:

Assume the aircraft propeller coefficients in Fi.g 10.40 are applicable to this propeller.

At V = 0, J = 0. Extrapolating from Fig. 10.40b $\quad C_F = 0.16$

We also have $\quad D = 1.5 \cdot m \quad n = 1800 \cdot rpm \quad n = 30 \cdot \frac{rev}{s} \quad$ and $\quad \rho = 1.225 \cdot \frac{kg}{m^3}$

The thrust at standstill (J = 0) is found from $\quad F_T = C_F \cdot \rho \cdot n^2 \cdot D^4 \quad$ (Note: n is in rev/s) $\quad F_T = 893 \cdot N$

At a speed $V = 12.5 \cdot \frac{m}{s} \quad J = \frac{V}{n \cdot D} \quad J = 0.278 \quad$ and so from Fig. 10.40b $\quad C_P = 0.44 \quad$ and $\quad C_F = 0.145$

The thrust and power at this speed can be found $\quad F_T = C_F \cdot \rho \cdot n^2 \cdot D^4 \quad F_T = 809 \cdot N \quad P = C_P \cdot \rho \cdot n^3 \cdot D^5 \quad P = 111 \cdot kW$

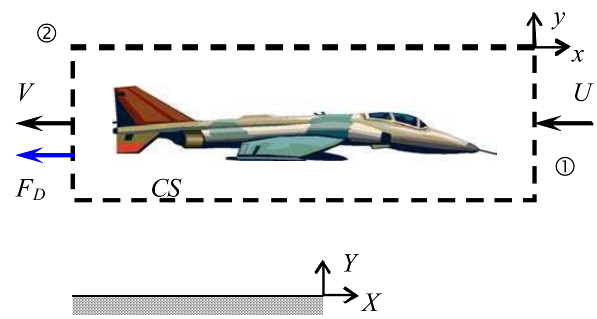
Problem 10.106

[Difficulty: 3]

10.106 A jet-propelled aircraft traveling at 225 m/s takes in 50 kg/s of air. If the propulsive efficiency (defined as the ratio of the useful work output to the mechanical energy input to the fluid) of the aircraft is 45 percent, determine the speed at which the exhaust is discharged relative to the aircraft.

Given: Data on jet-propelled aircraft

Find: Propulsive efficiency



Solution:

Basic equation:

$$\vec{F} = \vec{F}_s + \vec{F}_B = \frac{\partial}{\partial t}\int_{CV} \vec{V}_{xyz}\,\rho\, d\mathcal{V} + \int_{CS} \vec{V}_{xyz}\,\rho \vec{V}_{xyz}\cdot d\vec{A} \quad (4.26)$$

$$\dot{Q} - \dot{W}_s - \dot{W}_{shear} - \dot{W}_{other} = \frac{\partial}{\partial t}\int_{CV} e\,\rho\, d\mathcal{V} + \int_{CS}\left(u + pv + \frac{V^2}{2} + gz\right)\rho\vec{V}\cdot d\vec{A} \quad (4.56)$$

Assumption: 1) Atmospheric pressure on CS 2) Horizontal 3) Steady w.r.t. the CV 4) Use velocities relative to CV

The x-momentum is then $\quad -F_D = u_1\cdot\left(-m_{rate}\right) + u_2\cdot\left(m_{rate}\right) = (-U)\cdot\left(-m_{rate}\right) + (-V)\cdot\left(m_{rate}\right)$

or $\quad F_D = m_{rate}\cdot(V - U) \quad$ where $m_{rate} = 50\cdot\frac{kg}{s}$ is the mass flow rate

The useful work is then $\quad F_D\cdot U = m_{rate}\cdot(V - U)\cdot U$

The energy equation simplifies to $\quad -W = \left(\frac{U^2}{2}\right)\cdot\left(-m_{rate}\right) + \left(\frac{V^2}{2}\right)\cdot\left(m_{rate}\right) = \frac{m_{rate}}{2}\cdot\left(V^2 - U^2\right)$

Hence

$$\eta = \frac{m_{rate}\cdot(V - U)\cdot U}{\frac{m_{rate}}{2}\cdot\left(V^2 - U^2\right)} = \frac{2\cdot(V - U)\cdot U}{\left(V^2 - U^2\right)} = \frac{2}{1 + \frac{V}{U}}$$

With $\quad U = 225\cdot\frac{m}{s} \quad$ and $\quad \eta = 45\% \qquad V = U\cdot\left(\frac{2}{\eta} - 1\right) \qquad V = 775\,\frac{m}{s}$

Problem 10.107

[Difficulty: 4]

10.107 Drag data for model and prototype guided missile frigates are presented in Figs. 7.2 and 7.3. Dimensions of the prototype vessel are given in Problem 9.89. Use these data, with the propeller performance characteristics of Fig. 10.44, to size a single propeller to power the full-scale vessel. Calculate the propeller size, operating speed, and power input, if the propeller operates at maximum efficiency when the vessel travels at its maximum speed, $V = 37.6$ knots.

Solution: From Problem 9.89, $L = 409$ ft and $A = 19{,}500\ \text{ft}^2$. At maximum speed,

$$V = 37.6\ \frac{\text{nm}}{\text{hr}} \times 6076\ \frac{\text{ft}}{\text{nm}} \times \frac{\text{hr}}{3600\ \text{s}} = 63.5\ \text{ft/s}$$

The Froude number is

$$Fr = \frac{V}{\sqrt{gL}} = 63.5\ \frac{\text{ft}}{\text{s}} \times \left[\frac{\text{s}^2}{32.2\ \text{ft}} \times \frac{1}{409\ \text{ft}}\right]^{\frac{1}{2}} = 0.553$$

From Fig. 7.2, $C_D \approx 0.0054$. The definition is $C_D = \dfrac{F_D}{\frac{1}{2}\rho V^2 A}$, so

$$F_D = C_D A \tfrac{1}{2}\rho V^2 \ ; \ \tfrac{1}{2}\rho V^2 = \tfrac{1}{2} \times (1.025)\, 1.94\ \frac{\text{slug}}{\text{ft}^3} \times (63.5)^2 \frac{\text{ft}^2}{\text{s}^2} \times \frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}} = 4010\ \text{lbf/ft}^2$$

$$F_D = 0.0054 \times 19{,}500\ \text{ft}^2 \times 4010\ \frac{\text{lbf}}{\text{ft}^2} = 422{,}000\ \text{lbf}$$

From Fig. 10.40(a), the maximum efficiency is $\eta = 0.67$ at $J = 0.85$. Then

$$nD = \frac{V}{J} = 63.5\ \frac{\text{ft}}{\text{s}} \times \frac{1}{0.85} = 74.7\ \text{ft/s} \qquad (1)$$

Since $C_F = 0.11 = \dfrac{F_D}{\rho n^2 D^4} = \dfrac{F_D}{\rho (n^2 D^2) D^2} = \dfrac{F_D}{\rho (nD)^2 D^2}$, then

$$D = \left[\frac{F_D}{\rho (nD)^2 C_F}\right]^{\frac{1}{2}} = \left[422{,}000\ \text{lbf} \times \frac{\text{ft}^3}{(1.025)\, 1.94\ \text{slug}} \times \frac{\text{s}^2}{(74.7)^2 \text{ft}^2} \times \frac{1}{0.11} \times \frac{\text{slug}\cdot\text{ft}}{\text{lbf}\cdot\text{s}^2}\right]^{\frac{1}{2}} = 18.6\ \text{ft}$$ D

From Eq. 1,

$$n = \frac{nD}{D} = 74.7\ \frac{\text{ft}}{\text{s}} \times \frac{1}{18.6\ \text{ft}} = 4.02\ \text{rev/s} \quad (241\ \text{rpm})$$ n

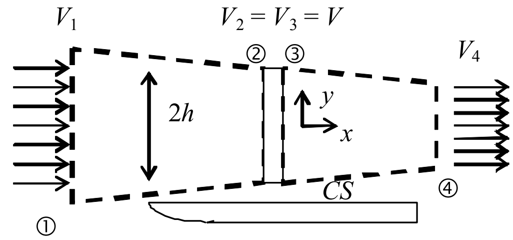
The input power would be

$$\mathcal{P}_{in} = \frac{\mathcal{P}_{out}}{\eta} = \frac{F_D V}{\eta} = 422{,}000\ \text{lbf} \times 63.5\ \frac{\text{ft}}{\text{s}} \times \frac{1}{0.67} \times \frac{\text{hp}\cdot\text{s}}{550\ \text{ft}\cdot\text{lbf}} = 72{,}700\ \text{hp}$$ $\mathcal{P}_{in}$

Problem 10.108

[Difficulty: 4]

10.108 The propulsive efficiency, η, of a propeller is defined as the ratio of the useful work produced to the mechanical energy input to the fluid. Determine the propulsive efficiency of the moving fanboat of Problem 10.104. What would be the efficiency if the boat were not moving?



Given: Definition of propulsion efficiency η

Find: η for moving and stationary boat

Solution:

Assumption: 1) Atmospheric pressure on CS 2) Horizontal 3) Steady w.r.t. the CV 4) Use velocities relative to CV

The x-momentum (Example 10.3): $T = u_1 \cdot (-m_{rate}) + u_4 \cdot (m_{rate}) = m_{rate} \cdot (V_4 - V_1)$

Applying the energy equation to steady, incompressible, uniform flow through the moving CV gives the minimum power input requirement

$$P_{min} = m_{rate} \cdot \left(\frac{V_4^2}{2} - \frac{V_1^2}{2} \right)$$

On the other hand, useful work is done at the rate of

$$P_{useful} = V_1 \cdot T = V_1 \cdot m_{rate} \cdot (V_4 - V_1)$$

Combining these expressions

$$\eta = \frac{V_1 \cdot m_{rate} \cdot (V_4 - V_1)}{m_{rate} \cdot \left(\frac{V_4^2}{2} - \frac{V_1^2}{2} \right)} = \frac{V_1 \cdot (V_4 - V_1)}{\frac{1}{2} \cdot (V_4 - V_1) \cdot (V_4 + V_1)}$$

or

$$\eta = \frac{2 \cdot V_1}{V_1 + V_4}$$

When in motion $V_1 = 30 \cdot mph$ and $V_4 = 90 \cdot mph$ $\eta = \frac{2 \cdot V_1}{V_1 + V_4}$ $\eta = 50 \cdot \%$

For the stationary case $V_1 = 0 \cdot mph$ $\eta = \frac{2 \cdot V_1}{V_1 + V_4}$ $\eta = 0 \cdot \%$

Problem 10.109

[Difficulty: 4]

10.109 The propeller for the Gossamer Condor human-powered aircraft has $D = 12$ ft and rotates at $N = 107$ rpm. Additional details on the aircraft are given in Problem 9.174. Estimate the dimensionless performance characteristics and efficiency of this propeller at cruise conditions. Assume the pilot expends 70 percent of maximum power at cruise. (See Reference [56] for more information on human-powered flight.)

Solution: From the solution to Problem 9.174, minimum power to propel the aircraft occurs at $V = 10.7$ mph (16.0 ft/s). Assume this is the cruise condition.

From the given data, at 12 mph (17.6 ft/s), $F_D = 6$ lbf

$$\frac{1}{2}\rho V^2 = \frac{1}{2} \times 0.00238 \frac{\text{slug}}{\text{ft}^3} \times (17.6)^2 \frac{\text{ft}^2}{\text{s}^2} \times \frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}} = 0.369 \text{ lbf/ft}^2$$

$$C_L = \frac{F_L}{\frac{1}{2}\rho V^2 A} = \frac{W}{qA} = \frac{W/A}{q} = \frac{0.4 \text{ lbf/ft}^2}{0.369 \text{ lbf/ft}^2} = 1.08$$

$$C_D = C_L \frac{F_D}{F_L} = 1.08 \frac{6 \text{ lbf}}{200 \text{ lbf}} = 0.0324$$

$$C_{D,0} = C_D - C_{D,i} = C_D - \frac{C_L^2}{\pi ar} = 0.0324 - \frac{(1.08)^2}{\pi(17)} = 0.0106$$

At $V = 10.7$ mph (16.0 ft/s), $q = 0.305$ lbf/ft²

$$C_L = \frac{W}{qA} = \frac{W/A}{q} = 0.4 \frac{\text{lbf}}{\text{ft}^2} \times \frac{\text{ft}^2}{0.305 \text{ lbf}} = 1.31; \quad C_{D,i} = \frac{C_L^2}{\pi ar} = 0.0321$$

$$C_D = C_{D,0} + C_{D,i} = 0.0106 + 0.0321 = 0.0427; \quad F_D = F_L \frac{C_D}{C_L} = 200 \text{ lbf} \frac{0.0427}{1.31} = 6.52 \text{ lbf}$$

For the propeller,

$$J = \frac{V}{nD} = 16.0 \frac{\text{ft}}{\text{s}} \times \left(\frac{60}{107}\right) \frac{\text{s}}{\text{rev}} \times \frac{1}{12.0 \text{ ft}} = 0.748$$

J

$$C_F = \frac{F_D}{\rho n^2 D^4} = 6.52 \text{ lbf} \times \frac{\text{ft}^3}{0.00238 \text{ slug}} \times \left(\frac{60}{107}\right)^2 \frac{\text{s}^2}{\text{rev}^2} \times \frac{1}{(12.0)^4 \text{ ft}^4} \times \frac{\text{slug}\cdot\text{ft}}{\text{lbf}\cdot\text{s}^2} = 0.0415$$

C_F

Assume a 30 percent reserve for climbing and maneuvers. Then if $\eta_d = 0.9$,

$$\eta_p = \frac{\mathcal{P}_{out}}{\mathcal{P}_{in}\eta_d} = \frac{F_D V}{(0.7)\,0.39 \text{ hp}\,(0.9)} = \frac{1}{0.246 \text{ hp}} \times 6.52 \text{ lbf} \times 16.0 \frac{\text{ft}}{\text{s}} \times \frac{\text{hp}\cdot\text{s}}{550 \text{ ft}\cdot\text{lbf}} = 0.771$$

η

Finally, $\mathcal{P}_{prop} = \eta_d \mathcal{P}_{in} = 0.246 \text{ hp} = \omega T$; $T = \frac{0.246 \text{ hp}}{\omega} = 12.1$ ft·lbf

$$C_T = \frac{T}{\rho n^2 D^5} = 12.1 \text{ ft}\cdot\text{lbf} \times \frac{\text{ft}^3}{0.00238 \text{ slug}} \times \left(\frac{60}{107}\right)^2 \frac{\text{s}^2}{\text{rev}^2} \times \frac{1}{(12.0)^5 \text{ ft}^5} \times \frac{\text{slug}\cdot\text{ft}}{\text{lbf}\cdot\text{s}^2} = 0.00642$$

C_T

$$C_P = \frac{C_T}{n} = 0.00642 \times \frac{60}{107} = 0.0036$$

C_P

Problem 10.110

[Difficulty: 5]

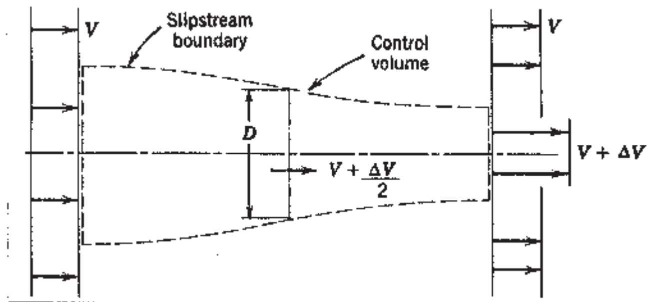
10.110 Equations for the thrust, power, and efficiency of propulsion devices were derived in Section 10.6. Show that these equations may be combined for the condition of constant thrust to obtain

$$\eta = \frac{2}{1 + \left(1 + \dfrac{F_T}{\dfrac{\rho V^2}{2}\dfrac{\pi D^2}{4}}\right)^{1/2}}$$

Interpret this result physically.

Solution: Apply 1-D forms of momentum and energy to CV of Fig. 10.38:

Computing equations:



Continuity $\dot{m} = \rho\left(V + \frac{\Delta V}{2}\right)\frac{\pi D^2}{4}$

Momentum $F_T = \dot{m}\,\Delta V$ (10.28)

Energy $\mathcal{P}_{in} = \dot{m} V \Delta V\left(1 + \frac{\Delta V}{2V}\right)$ (10.29)

Assumptions: (1) Steady flow, (2) Incompressible flow, (3) Uniform flow, (4) Frictionless flow

Propulsion efficiency is $\eta_p = \frac{\mathcal{P}_{out}}{\mathcal{P}_{in}} = \frac{F_T V}{\dot{m} V \Delta V\left(1 + \frac{\Delta V}{2V}\right)} = \frac{\dot{m} V \Delta V}{\dot{m} V \Delta V\left(1 + \frac{\Delta V}{2V}\right)} = \frac{1}{1 + \frac{\Delta V}{2V}}$ (1)

F_T may be written using continuity as

$$F_T = \dot{m}\,\Delta V = \rho\left(V + \frac{\Delta V}{2}\right)\frac{\pi D^2}{4}\Delta V = 2\rho V^2 \frac{\pi D^2}{4}\left(1 + \frac{\Delta V}{2V}\right)\left(\frac{\Delta V}{2V}\right) = 2\rho V^2 \frac{\pi D^2}{4}(1+\lambda)\lambda$$

where $\lambda = \frac{\Delta V}{2V}$. For constant F_T,

$$\lambda^2 + \lambda - \frac{F_T}{2\rho V^2 \frac{\pi D^2}{4}} = 0$$

Solving via the quadratic equation, and choosing the positive root

$$\lambda = \frac{-1 \pm \sqrt{1 + 4\dfrac{F_T}{2\rho V^2 \frac{\pi D^2}{4}}}}{2} = \frac{1}{2}\left\{-1 + \sqrt{1 + \frac{F_T}{\frac{\rho V^2}{2}\frac{\pi D^2}{4}}}\right\}$$

From Eq. 1, $\eta_p = \frac{1}{1+\lambda} = \frac{1}{1 + \frac{1}{2}\{\ \}} = \frac{2}{2 + \{\ \}} = \frac{2}{1 + \left(1 + \dfrac{F_T}{\frac{\rho V^2}{2}\frac{\pi D^2}{4}}\right)^{\frac{1}{2}}}$ η_p

The ratio, $F_T / \frac{\pi D^2}{4}$, may be interpreted as the disk loading: the force developed per unit area of the actuator disk. Note $\eta_p \to 1$ as $\frac{\pi D^2}{4}$ increases.

Problem 10.111

[Difficulty: 2]

10.111 The National Aeronautics & Space Administration (NASA) and the U.S. Department of Energy (DOE) cosponsor a large demonstration wind turbine generator at Plum Brook, near Sandusky, Ohio [47]. The turbine has two blades, with a radius of 63 ft, and delivers maximum power when the wind speed is above $V = 16$ knots. It is designed to produce 135 hp with powertrain efficiency of 74 percent. The rotor is designed to operate at a constant speed of 45 rpm in winds over 5 knots by controlling system load and adjusting blade angles. For the maximum power condition, estimate the rotor tip speed and power coefficient.

Given: NASA-DOE wind turbine generator

Find: Estimate rotor tip speed and power coefficient at maximum power condition

Solution:

Basic equations: $C_P = \dfrac{P_m}{\frac{1}{2}\cdot\rho\cdot V^3\cdot\pi\cdot R^2}$ $\quad X = \dfrac{\omega\cdot R}{V}$ $\quad U = \omega\cdot R$ $\quad \eta = \dfrac{P_m}{P_{ideal}}$

and we have $\rho = 0.00237\cdot\dfrac{slug}{ft^3}$ $\quad \omega = 45\cdot rpm = 4.712\cdot\dfrac{rad}{s}$ $\quad R = 63\cdot ft$ $\quad V = 16\cdot knot = 27.005\cdot\dfrac{ft}{s}$ $\quad P = 135\cdot hp$ $\quad \eta = 74\%$

The blade tip speed is: $U = \omega\cdot R = 297\cdot\dfrac{ft}{s}$

The tip speed ratio is: $X = \dfrac{\omega\cdot R}{V} = 10.994$ (X will decrease at the wind speed increases.)

The mechanical work out is: $P_m = \dfrac{P}{\eta} = 182.4\cdot hp$ From this we can calculate the power coefficient:

$$C_P = \frac{P_m}{\frac{1}{2}\cdot\rho\cdot V^3\cdot\pi\cdot R^2} = 0.345$$

Problem 10.112

[Difficulty: 3]

10.112 A typical American multiblade farm windmill has $D=7$ ft and is designed to produce maximum power in winds with $V=15$ mph. Estimate the rate of water delivery, as a function of the height to which the water is pumped, for this windmill.

Solution: Assume the efficiency trends shown in Fig. 10.45.

Computing equations: $C_P = \dfrac{\mathcal{P}}{\frac{1}{2}\rho V^3 \pi R^2}$ $\quad X = \omega R / V$

From Fig. 10.45, $C_P\,\text{max} \approx 0.3$ at $X = 0.8$. $V = 15$ mph (22.0 ft/s). Then the power developed is

$$\mathcal{P} = \frac{\pi}{2} \times 0.3 \times 0.00238 \frac{\text{slug}}{\text{ft}^3} \times (22.0)^3 \frac{\text{ft}^3}{\text{s}^3} \times \left(\frac{7}{2}\right)^2 \text{ft}^2 \times \frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}} \times \frac{\text{hp}\cdot\text{s}}{550\,\text{ft}\cdot\text{lbf}} = 0.266\ \text{hp}$$

Converting this mechanical power to pumping gives hydraulic power as

$$\mathcal{P}_h = \rho Q g h = \eta \mathcal{P}_m$$

Thus $Qh = \dfrac{\eta \mathcal{P}_m}{\rho g} = 0.7 \times 0.266\ \text{hp} \times \dfrac{\text{ft}^3}{62.4\ \text{lbf}} \times 550 \dfrac{\text{ft}\cdot\text{lbf}}{\text{hp}\cdot\text{s}} \times 7.48 \dfrac{\text{gal}}{\text{ft}^3} \times 60 \dfrac{\text{s}}{\text{min}}$

$$Qh = 737\ \text{gpm}\cdot\text{ft}$$ Qh

Q varies inversely with the distance lifted, h. The volume flow rate actually delivered would be less, due to suction lift, pipe friction, and minor losses.

Problem 10.113

[Difficulty: 2]

10.113 A model of an American multiblade farm windmill is to be built for display. The model, with $D = 1$ m, is to develop full power at $V = 10$ m/s wind speed. Calculate the angular speed of the model for optimum power generation. Estimate the power output.

Given: Model of farm windmill

Find: Angular speed for optimum power; Power output

Solution:

Basic equations: $C_P = \dfrac{P}{\frac{1}{2}\cdot\rho\cdot V^3\cdot\pi\cdot R^2}$ $\quad X = \dfrac{\omega\cdot R}{V}$ $\quad$ and we have $\rho = 1.225\cdot\dfrac{kg}{m^3}$

From Fig. 10.45 $\quad C_{Pmax} = 0.3$ $\quad$ at $\quad X = 0.8$ $\quad$ and $\quad D = 1\cdot m$ $\quad R = \dfrac{D}{2}$ $\quad R = 0.5\,m$

Hence, for $\quad V = 10\cdot\dfrac{m}{s}$ $\quad \omega = \dfrac{X\cdot V}{R}$ $\quad \omega = 16\cdot\dfrac{rad}{s}$ $\quad \omega = 153\cdot rpm$

Also $\quad P = C_{Pmax}\cdot\dfrac{1}{2}\cdot\rho\cdot V^3\cdot\pi\cdot R^2$ $\quad P = 144\,W$

Problem 10.114

[Difficulty: 2]

10.114 A large Darrieus vertical axis wind turbine was built by the U.S. Department of Energy near Sandia, New Mexico [48]. This machine is 18 m tall and has a 5-m radius; the area swept by the rotor is over 110 m^2. If the rotor is constrained to rotate at 70 rpm, plot the power this wind turbine can produce in kilowatts for wind speeds between 5 and 50 knots.

Given: NASA-DOE wind turbine generator

Find: Estimate rotor tip speed and power coefficient at maximum power condition

Solution:

Basic equations:
$$C_P = \frac{P_m}{\frac{1}{2}\cdot\rho\cdot V^3\cdot\pi\cdot R^2} \qquad X = \frac{\omega\cdot R}{V} \qquad U = \omega\cdot R \qquad \eta = \frac{P_m}{P_{ideal}}$$

and we have $\rho = 1.23\cdot\frac{kg}{m^3}$ $\quad \omega = 70\cdot rpm \quad R = 5\cdot m \quad H = 18\cdot m \quad A = 110\cdot m^2 \quad U = \omega\cdot R = 36.652\frac{m}{s}$

From Fig. 10.45: $C_P = 0.34$ when $X = 5.3$ (maximum power condition) If we replace the $\pi\cdot R^2$ term in the power coefficient with the swept area we will get: $P = \frac{1}{2}\cdot C_P\cdot\rho\cdot V^3\cdot A$

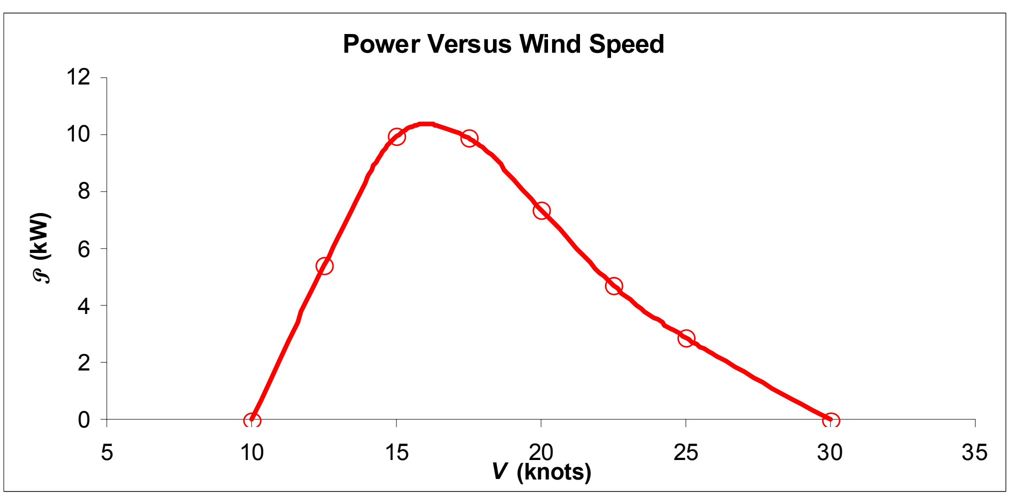
Here are the results, calculated using *Excel*:

A = 110.00 m^2 Power coefficient data were taken from Fig. 10.45

ρ = 1.23 kg/m^3

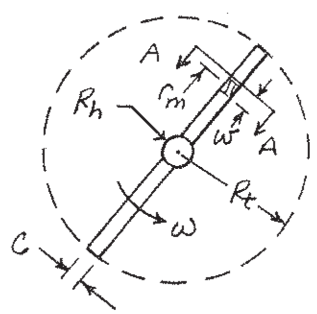
U = 36.65 m/s

V (kt)	V (m/s)	X	C_P	$\mathscr{P}$ (kW)
10.0	5.14	7.125	0.00	0.00
12.5	6.43	5.700	0.30	5.40
15.0	7.72	4.750	0.32	9.95
17.5	9.00	4.071	0.20	9.87
20.0	10.29	3.562	0.10	7.37
22.5	11.57	3.167	0.05	4.72
25.0	12.86	2.850	0.02	2.88
30.0	15.43	2.375	0.00	0.00

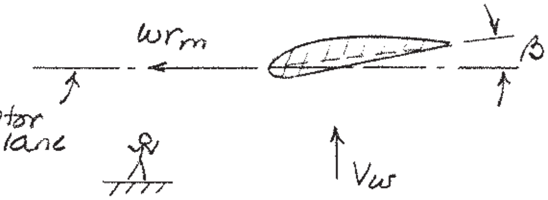


10.115 Lift and drag data for the NACA 23015 airfoil section are presented in Fig. 9.17. Consider a two-blade horizontal-axis propeller wind turbine with NACA 23015 blade section. Analyze the air flow relative to a blade element of the rotating wind turbine. Develop a numerical model for the blade element. Calculate the power coefficient developed by the blade element as a function of tip-speed ratio. Compare your result with the general trend of power output for high-speed two-bladed turbine rotors shown in Fig. 10.50.

Solution: Front view of rotor; blade element shown cross-hatched:

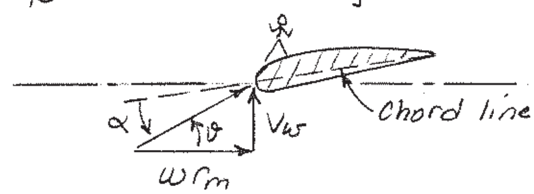


Section A-A:
(absolute velocities)

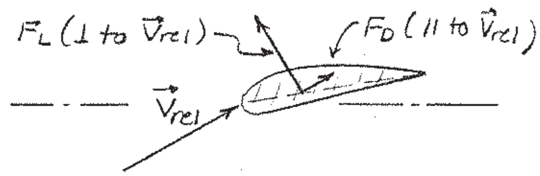


β = blade twist angle

(relative velocities)



α = angle of attack



Resolve to relative velocity:

$$\vec{V}_{abs} = \vec{V}_{blade\ element} + \vec{V}_{rel}$$

$$\vec{V}_{rel} = \vec{V}_w - \vec{V}_{blade} = \vec{V}_w + (-\vec{V}_{blade})$$

Computing equations:

{Note: $\beta > 0$ is required for windmill to self start.}

$F_L = C_L \frac{1}{2}\rho V_r^2 A_p$; V_r = relative velocity

$F_D = C_D \frac{1}{2}\rho V_r^2 A_p$; A_p = planform area

$T = n_b r_m (F_L \sin\theta - F_D \cos\theta)$; n_b = number of blades (2)

$V_{rel} = \left[(\omega r_m)^2 + V_w^2\right]^{\frac{1}{2}}$; $\theta = \tan^{-1}(V_w/\omega r_m)$; $\alpha = \theta - \beta$

Both C_L and C_D must be modeled as functions of angle of attack, α. From Fig. 9.17, satisfactory representations are:

$0 \le \alpha < 12°$ $\quad C_L = 0.12 + 0.107\alpha$ $\qquad 0 \le \alpha < 4°$ $\quad C_D = 0.0065 + 4.55\times10^{-5}\alpha$

$12° \le \alpha < 18°$ $\quad C_L = 0.12 + 0.107\alpha - 0.00852(\alpha-12)^2$ $\qquad 4° \le \alpha < 16°$ $\quad C_D$ = above $+ 7.72\times10^{-5}(\alpha-4)^2$

$18° \le \alpha$ $\quad C_L = 0.2$ $\qquad 16° \le \alpha$ $\quad C_D = 0.02$

{Obviously both models are crude for $C_L(\alpha > 18°)$, $C_D(\alpha > 16°)$.} Choose $R_t = 10$ ft, $r_m = 5.5$ ft, $c = 6$ in., $w = 1$ ft, $V_w = 20$ mph (29.4 ft/s), $X = 5$, and $\beta = 5°$. Then

$$\omega = \frac{X V_w}{R_t} = 5.0 \times 29.4\,\frac{ft}{s} \times \frac{1}{10\ ft} = 14.7\ rad/s \quad (140\ rpm);\ \omega r_m = 80.9\ ft/s$$

Problem 10.115

$$V_{rel} = \left[(80.9)^2 + (29.4)^2\right]^{\frac{1}{2}} = 86.1\ ft/s; \quad q = \frac{1}{2}\rho V_{rel}^2 = 8.82\ lbf/ft^2;\ A_p = wc = 0.5\ ft^2$$

$$\theta = \tan^{-1}(29.4/80.9) = 20.0^\circ;\ \alpha = \theta - \beta = 20.0 - 5.0 = 15.0^\circ;\ C_L = 1.65;\ C_D = 0.017$$

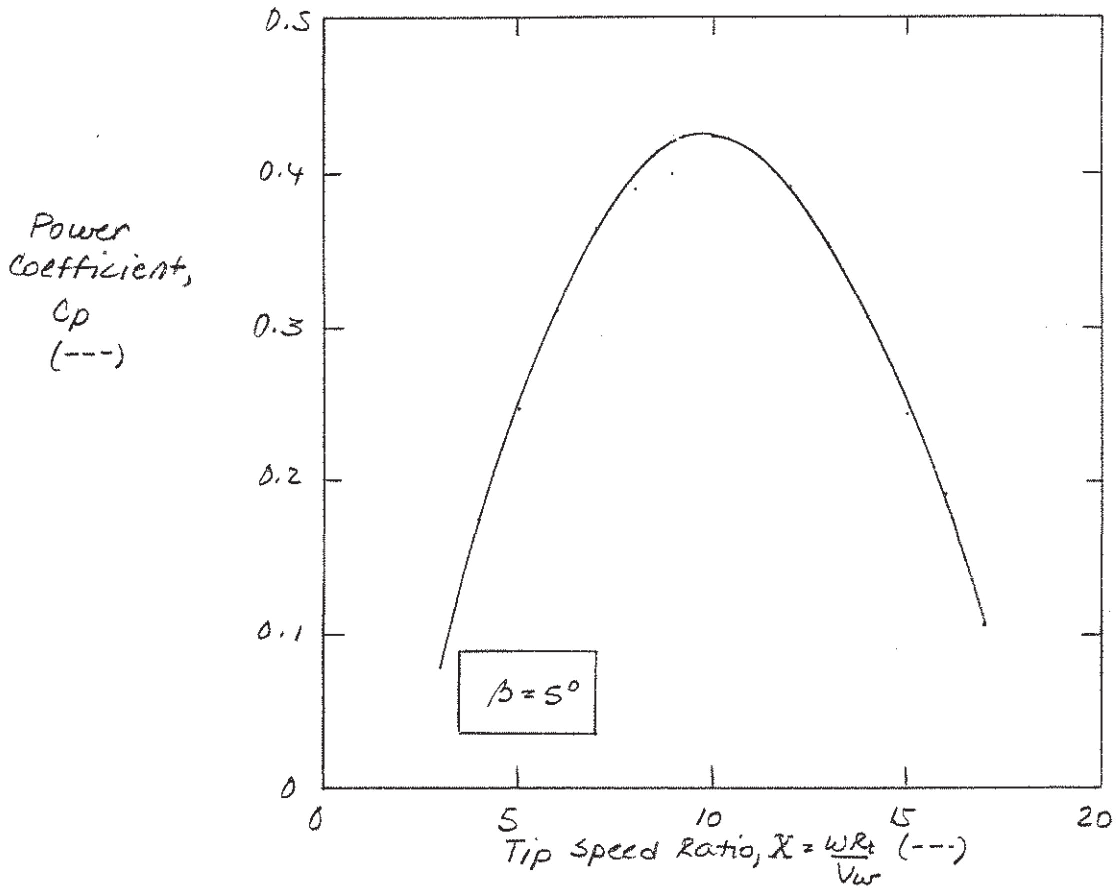
$$F_L = 1.65 \times 8.82\ \frac{lbf}{ft^2} \times 0.5\ ft^2 = 7.28\ lbf;\ F_D = 0.017 \times 8.82\ \frac{lbf}{ft^2} \times 0.5\ ft^2 = 0.075\ lbf$$

$$T = 2 \times 5.5\ ft\ (7.28 \times \sin 20^\circ - 0.075 \cos 20^\circ)\ lbf = 26.6\ ft \cdot lbf$$

Similar calculations for the other blade elements show that torque for the complete propeller is $T_p = 159\ ft \cdot lbf$. The power coefficient is

$$C_P = \frac{\mathcal{P}}{\frac{1}{2}\rho V_w^3 \pi R_t^2} = \frac{\omega T_p}{\frac{1}{2}\rho V_w^3 \pi R_t^2} = \frac{2}{\pi} \times 14.7\ \frac{rad}{s} \times 159\ ft \cdot lbf \times \frac{ft^3}{0.00238\ slug} \times \frac{s^3}{(29.4)^3 ft^3} \times \frac{1}{(10)^2 ft^2} \times \frac{slug \cdot ft}{lbf \cdot s^2} = 0.246$$

Calculated results are tabulated on the next page, plotted and discussed below:



Trends shown are similar to Fig. 10.45. At small X, the blade is entirely stalled so useful output is low. At large X, α becomes negative near the tips, reducing output.

This model does not include: (1) axial interference that reduces normal velocity below V_w as loading increases, or (2) swirl introduce by blade drag. Both these effects reduce performance. For more details, see Division L, Section XI of [30].

Computed results:

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*****************************************************
*                                                   *
*                                                   *
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*    West Lafayette, Indiana 47907                  *
*                                                   *
*****************************************************
```

Airfoil: NACA 23015 Section; Chord, c = 6 in.
Tip radius, Rt = 10 ft
Twist angle, beta = 5 degrees

Blade element: Delta r, dr = 1.00 ft

Input data: Tip speed ratio, X = 5.0 (---)
Wind speed, Vw = 20 mph (29.4 ft/s)

Calculated: Rotor speed, omega = 140.4 rpm

Rm (ft)	Vrel (ft/s)	alpha (deg)	Cl (--)	Cd (--)	Fl (lbf)	Fd (lbf)	T (ft-lbf)
1.50	22	48.13	0.20	0.020	0.06	0.006	0.13
2.50	37	33.66	0.20	0.020	0.16	0.016	0.44
3.50	51	24.74	0.20	0.020	0.32	0.032	0.90
4.50	66	18.96	0.20	0.020	0.52	0.052	1.48
5.50	81	14.98	1.65	0.017	6.41	0.067	23.39
6.50	96	12.10	1.41	0.013	7.69	0.069	28.53
7.50	110	9.93	1.18	0.010	8.55	0.074	31.99
8.50	125	8.24	1.00	0.009	9.31	0.081	34.90
9.50	140	6.89	0.86	0.008	9.95	0.091	37.25

Torque for complete propeller: T = 159.0 ft-lbf

Power coefficient for windmill: Cp = 0.246 (---)

10.116 Aluminum extrusions, patterned after NACA symmetric airfoil sections, frequently are used to form Darrieus wind turbine "blades." Below are section lift and drag coefficient data [57] for a NACA 0012 section, tested at $Re = 6 \times 10^6$ with standard roughness (the section stalled for $\alpha > 12°$):

Angle of attack, α (deg)	0	2	4	6	8	10	12
Lift coefficient, C_L (—)	0	0.23	0.45	0.68	0.82	0.94	1.02
Drag coefficient, C_D (—)	0.0098	0.0100	0.0119	0.0147	0.0194	—	—

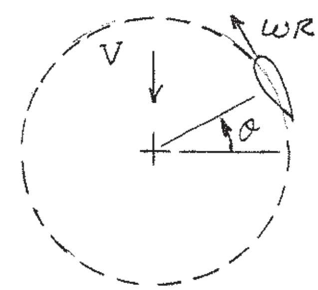
Analyze the air flow relative to a blade element of a Darrieus wind turbine rotating about its troposkien axis. Develop a numerical model for the blade element. Calculate the power coefficient developed by the blade element as a function of tip-speed ratio. Compare your result with the general trend of power output for Darrieus rotors shown in Fig. 10.50.

<u>Solution:</u> Consider plan view of rotor element, absolute velocities:

Computing equations: $F_L = C_L \frac{1}{2}\rho V_r^2 A_p$, V_r = relative velocity

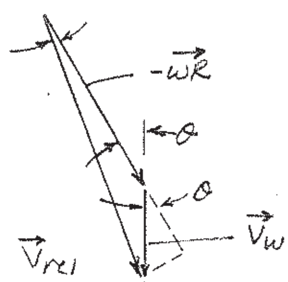
$F_D = C_D \frac{1}{2}\rho V_r^2 A_p$ A_p = planform, A_s = swept

$C_p = \dfrac{\mathcal{P}}{\frac{1}{2}\rho V^3 A_s}$ V = wind velocity



Resolve to relative velocity, for position shown:

$$\vec{V}_{abs} = \vec{V}_{blade} + \vec{V}_{rel} \; ; \; \vec{V}_{rel} = \vec{V}_{abs} - \vec{V}_{blade}$$



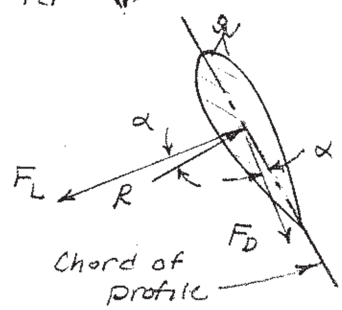
To compute V_{rel}, resolve into components along (a) and transverse (t) to the airfoil chord:

$$V_{rel})_a = \omega R + V_w \cos\theta$$

$$V_{rel})_t = V_w \sin\theta$$

$$V_{rel} = \left[V_{rel})_a^2 + V_{rel})_t^2 \right]^{\frac{1}{2}}$$

$$\alpha = \tan^{-1}[V_{rel})_t / V_{rel})_a]$$



Lift force (F_L) is normal to $\vec{V}_{rel}$ and drag force (F_D) is parallel to $\vec{V}_{rel}$. Thus

$$T = R(F_L \sin\alpha - F_D \cos\alpha) \quad (\text{torque, } T>0 \text{ when } F_L/F_D > \cot\alpha)$$

Both C_L and C_D must be modeled as functions of angle of attack, α. From a graph of C_L and C_D versus α, a satisfactory representation is

$$C_L = 0.12\alpha - 0.0026|\alpha|\alpha, \quad -12 < \alpha < 12 \text{ degrees}; \; C_L = 0, \; |\alpha| > 12 \text{ degrees}$$

$$C_D = 0.00952 + 1.52\times10^{-4}\alpha^2; \quad -12 < \alpha < 12 \text{ degrees}; \; C_D = 0.0314, \; |\alpha| > 12 \text{ degrees}$$

(The models in the stalled region, $|\alpha| > 12$ degrees, obviously are crude.)

Sample calculation: Choose $R = 10$ ft, $c = 0.5$ ft, $w = 1$ ft, $X = 5$, $V_w = 20$ mph

At $\theta = 30°$, with $V_w = 20$ mph (29.3 ft/s)

$$X = \omega R / V_w \; ; \; \omega = \frac{X V_w}{R} = 5 \times 29.3 \frac{ft}{s} \cdot \frac{1}{10\,ft} = 14.7 \text{ rad/s} \quad (N = 140 \text{ rpm})$$

$$\omega R = 14.7 \frac{rad}{s} \times 10\,ft = 147 \text{ ft/s}$$

$$V_{rel})_a = \omega R + V_w \cos\theta = 147 + 29.3 \cos 30° = 172 \text{ ft/s}$$

$$V_{rel})_t = V_w \sin\theta = 29.3 \sin\theta = 14.7 \text{ ft/s}$$

$$V_{rel} = \left[V_{rel})_a^2 + V_{rel})_t^2 \right]^{\frac{1}{2}} = \left[(172)^2 + (14.7)^2 \right]^{\frac{1}{2}} = 173 \text{ ft/s}$$

$$\alpha = \tan^{-1}\left[V_{rel})_t / V_{rel})_a \right] = \tan^{-1}(14.7/172) = 4.88 \text{ degrees}$$

$$q = \frac{1}{2}\rho V_{rel}^2 = \frac{1}{2} \times 0.00238 \frac{slug}{ft^3} \times (173)^2 \frac{ft^2}{s^2} \times \frac{lbf \cdot s^2}{slug \cdot ft} = 35.6 \text{ lbf/ft}^2$$

A_p (projected area of airfoil section) $= cw = 0.5\,ft \times 1\,ft = 0.5\,ft^2$

$$C_L = 0.12\alpha - 0.0026|\alpha|\alpha = 0.12 \times 4.88 - 0.0026|4.88|4.88 = 0.524$$

$$C_D = 0.00952 + 1.52 \times 10^{-4}\alpha^2 = 0.00952 + 1.52 \times 10^{-4}(4.88)^2 = 0.0131$$

$$\left.\begin{aligned} F_L &= C_L q A_p = 0.524 \times 35.6 \frac{lbf}{ft^2} \times 0.5\,ft^2 = 9.33 \text{ lbf} \\ F_D &= C_D q A_p = 0.0131 \times 35.6 \frac{lbf}{ft^2} \times 0.5\,ft^2 = 0.233 \text{ lbf} \end{aligned}\right\} F_L/F_D = 40.0$$

$$T = R(F_L \sin\alpha - F_D \cos\alpha) = 10\,ft\,(9.33 \sin(4.88°) - 0.233 \cos(4.88°))\,lbf = 5.62 \text{ ft} \cdot \text{lbf}$$

$$\mathcal{P} = \omega T = 14.7 \frac{rad}{s} \cdot 5.62\,ft \cdot lbf = 82.6 \frac{ft \cdot lbf}{s} \quad (0.150 \text{ hp})$$

$$C_p = \frac{\mathcal{P}}{\frac{1}{2}\rho V_w^3 A_s} \; ; \; A_s = \text{area swept by element} = 2Rw = 2 \times 10\,ft \times 1\,ft = 20\,ft^2$$

$$C_p = 82.6 \frac{ft \cdot lbf}{s} \times \frac{ft^3}{(\frac{1}{2})\,0.00238\,slug} \times \frac{s^3}{(29.3)^3 ft^3} \times \frac{1}{20\,ft^2} \times \frac{slug \cdot ft}{lbf \cdot s^2} = 0.138 \quad (\text{at } \theta = 15°)$$

Obtain $\bar{C}_p$ for a complete rotor revolution by integrating numerically. Such results are presented on the next page, and plotted versus tip speed ratio, $X = \omega R / V_w$.

From the plot, $\bar{C}_p$ is small at low X. It increases as X is raised, then peaks and decreases again. Comparison with Fig. 10.45 shows the trends are similar, but the model predicts useful power at larger X than observed experimentally. Blade elements at smaller radii on the rotor would produce less power, since ω = constant along rotor. $\bar{C}_p$ at large X is also sensitive to C_D.

Low $\bar{C}_p$ at small X occurs because the airfoil is stalled.

Computed results:

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Airfoil: NACA 0012 Section; Chord, c = 6 in.

Blade element: Span, w = 1 ft; Radius, R = 10 ft

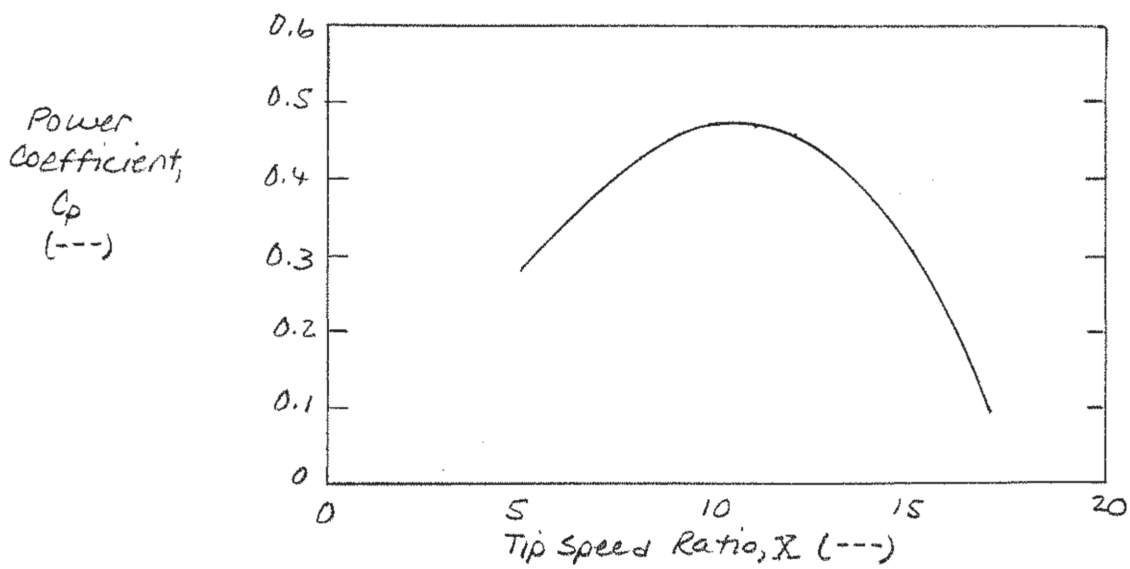
Input data: Tip speed ratio, X = 5.0 (---)
Wind speed, Vw = 20 mph (29.3 ft/sec)

Calculated: Rotor speed, omega = 140.4 rpm

theta (deg)	Vrel (ft/s)	alpha (deg)	Cl (--)	Cd (--)	Fl (lbf)	Fd (lbf)	T (ft-lbf)	Cp (--)
0	176	0.00	0.00	0.010	0.0	0.176	-1.8	-0.043
30	173	4.87	0.52	0.013	9.3	0.233	5.6	0.136
60	164	8.95	0.87	0.022	13.8	0.343	18.1	0.439
90	150	11.31	1.02	0.029	13.7	0.384	23.1	0.562
120	135	10.89	1.00	0.027	10.8	0.295	17.5	0.425
150	122	6.90	0.70	0.017	6.3	0.148	6.1	0.147
180	118	-0.00	-0.00	0.010	-0.0	0.078	-0.8	-0.019
210	122	-6.90	-0.70	0.017	-6.3	0.148	6.1	0.147
240	135	-10.89	-1.00	0.027	-10.8	0.295	17.5	0.425
270	150	-11.31	-1.02	0.029	-13.7	0.384	23.1	0.562
300	164	-8.95	-0.87	0.022	-13.8	0.343	18.1	0.439
330	173	-4.87	-0.52	0.013	-9.3	0.233	5.6	0.136
360	176	0.00	0.00	0.010	0.0	0.176	-1.8	-0.043

Average power coefficient for complete revolution: Cp,bar = 0.280

Plotting results of similar calculations at various tip speed ratios give:



Problem 10.117

[Difficulty: 2]

10.117 A prototype air compressor with a compression ratio of 7 is designed to take 8.9 kg/s air at 1 atmosphere and 20°C. The design point speed, power requirement, and efficiency are 600 rpm, 5.6 MW, and 80 percent, respectively. A 1:5-scale model of the prototype is built to help determine operability for the prototype. If the model takes in air at identical conditions to the prototype design point, what will the mass flow and power requirement be for operation at 80 percent efficiency?

Given: Prototype air compressor, 1/5 scale model to be built

Find: Mass flow rate and power requirements for operation at equivalent efficiency

Solution:

Basic equations: $\eta = f_1\left(\dfrac{M\cdot\sqrt{R\cdot T_{01}}}{p_{01}\cdot D^2}, \dfrac{\omega\cdot D}{c_{01}}\right)$ $\qquad \dfrac{W_c}{\rho_{01}\cdot\omega^3\cdot D^5} = f_2\left(\dfrac{M\cdot\sqrt{R\cdot T_{01}}}{p_{01}\cdot D^2}, \dfrac{\omega\cdot D}{c_{01}}\right)$ $\qquad \dfrac{D_m}{D_p} = \dfrac{1}{5}$

Given data: $M_p = 8.9\cdot\dfrac{kg}{s}$ $\quad \omega_p = 600\cdot rpm$ $\quad W_{cp} = 5.6\cdot MW$

Since the efficiencies are the same for the prototype and the model, it follows that:

$$\frac{M_m\cdot\sqrt{R_m\cdot T_{01m}}}{p_{01m}\cdot D_m^2} = \frac{M_p\cdot\sqrt{R_p\cdot T_{01p}}}{p_{01p}\cdot D_p^2} \qquad \frac{\omega_m\cdot D_m}{c_{01m}} = \frac{\omega_p\cdot D_p}{c_{01p}} \qquad \frac{W_{cm}}{\rho_{01m}\cdot\omega_m^3\cdot D_m^5} = \frac{W_{cp}}{\rho_{01p}\cdot\omega_p^3\cdot D_p^5}$$

Given identical entrance conditions for model and prototype and since the working fluid for both is air:

$\dfrac{M_m}{D_m^2} = \dfrac{M_p}{D_p^2}$ Solving for the mass flow rate of the model: $M_m = M_p\cdot\left(\dfrac{D_m}{D_p}\right)^2$ $\qquad M_m = 0.356\,\dfrac{kg}{s}$

$\omega_m\cdot D_m = \omega_p\cdot D_p$ Solving for the speed of the model: $\omega_m = \omega_p\cdot\dfrac{D_p}{D_m} = 3000\cdot rpm$

$\dfrac{W_{cm}}{\omega_m^3\cdot D_m^5} = \dfrac{W_{cp}}{\omega_p^3\cdot D_p^5}$ Solving for the power requirement for the model: $W_{cm} = W_{cp}\cdot\left(\dfrac{\omega_m}{\omega_p}\right)^3\cdot\left(\dfrac{D_m}{D_p}\right)^5$

$W_{cm} = 0.224\cdot MW$

Problem 10.118

[Difficulty: 3]

10.118 A compressor has been designed for entrance conditions of 14.7 psia and 70°F. To economize on the power required, it is being tested with a throttle in the entry duct to reduce the entry pressure. The characteristic curve for its normal design speed of 3200 rpm is being obtained on a day when the ambient temperature is 58°F. At what speed should the compressor be run? At the point on the characteristic curve at which the mass flow would normally be 125 lbm/s, the entry pressure is 8.0 psia. Calculate the actual mass flow rate during the test.

Given: Prototype air compressor equipped with throttle to control entry pressure

Find: Speed and mass flow rate of compressor at off-design entrance conditions

Solution:

Basic equations: $\eta = f_1\left(\frac{M\cdot\sqrt{T_{01}}}{p_{01}}, \frac{\omega}{\sqrt{T_{01}}}\right)$ $\qquad \frac{\Delta T_{01}}{T_{01}} = f_2\left(\frac{M\cdot\sqrt{T_{01}}}{p_{01}}, \frac{\omega}{\sqrt{T_{01}}}\right)$

Given data: $p_{01d} = 14.7\cdot psi$ $\quad T_{01d} = 70\,°F$ $\quad \omega_d = 3200\cdot rpm$ $\quad T_{01} = 58\,°F$ $\quad M_d = 125\cdot\frac{lbm}{s}$ $\quad p_{01} = 8.0\cdot psi$

Since the normalized speed is equal to that of the design point, it follows that: $\frac{\omega}{\sqrt{T_{01}}} = \frac{\omega_d}{\sqrt{T_{01d}}}$

Solving for the required speed: $\omega = \omega_d\cdot\sqrt{\frac{T_{01}}{T_{01d}}}$ $\qquad \omega = 3164\cdot rpm$

At similar conditions: $\frac{M\cdot\sqrt{T_{01}}}{p_{01}} = \frac{M_d\cdot\sqrt{T_{01d}}}{p_{01d}}$ Solving for the actual mass flow rate: $M = M_d\cdot\sqrt{\frac{T_{01d}}{T_{01}}}\cdot\frac{p_{01}}{p_{01d}}$ $\qquad M = 68.8\cdot\frac{lbm}{s}$

Problem 10.119

[Difficulty: 3]

10.119 The turbine for a new jet engine was designed for entrance conditions of 160 psia and 1700°F, ingesting 500 lbm/s at a speed of 500 rpm, and exit conditions of 80 psia and 1350°F. If the altitude and fueling for the engine were changed such that the entrance conditions were now 140 psia and 1600°F, calculate the new operating speed, mass flow rate, and exit conditions for similar operation, i.e., equal efficiency.

Given: Design conditions for jet turbine, off-design actual conditions

Find: New operating speed, mass flow rate, and exit conditions for similar operation

Solution:

Basic equations:
$$\eta = f_1\left(\frac{M\cdot\sqrt{T_{01}}}{p_{01}}, \frac{\omega}{\sqrt{T_{01}}}\right) \qquad \frac{\Delta T_0}{T_{01}} = f_2\left(\frac{M\cdot\sqrt{T_{01}}}{p_{01}}, \frac{\omega}{\sqrt{T_{01}}}\right) \qquad \frac{p_{01}}{p_{02}} = f_3\left(\frac{M\cdot\sqrt{T_{01}}}{p_{01}}, \frac{\omega}{\sqrt{T_{01}}}\right)$$

Given data: $p_{01d} = 160\cdot\text{psi}$ $\quad T_{01d} = 1700\,°\text{F}$ $\quad \omega_d = 500\cdot\text{rpm}$ $\quad M_d = 500\cdot\frac{\text{lbm}}{\text{s}}$ $\quad p_{02d} = 80\cdot\text{psi}$ $\quad T_{02d} = 1350\,°\text{F}$

$p_{01} = 140\cdot\text{psi}$ $\quad T_{01} = 1600\,°\text{F}$

At similar conditions: $\frac{\omega}{\sqrt{T_{01}}} = \frac{\omega_d}{\sqrt{T_{01d}}}$ Solving for the required speed: $\omega = \omega_d\cdot\sqrt{\frac{T_{01}}{T_{01d}}}$ $\qquad \omega = 488\cdot\text{rpm}$

$\frac{M\cdot\sqrt{T_{01}}}{p_{01}} = \frac{M_d\cdot\sqrt{T_{01d}}}{p_{01d}}$ Solving for the actual mass flow rate: $M = M_d\cdot\sqrt{\frac{T_{01d}}{T_{01}}}\cdot\frac{p_{01}}{p_{01d}}$ $\qquad M = 448\cdot\frac{\text{lbm}}{\text{s}}$

$\frac{\Delta T_0}{T_{01}} = \frac{\Delta T_{0d}}{T_{01d}}$ Solving for the temperature drop: $\Delta T_0 = \Delta T_{0d}\cdot\frac{T_{01}}{T_{01d}}$ Substituting in temperatures:

$$T_{01} - T_{02} = \left(T_{01d} - T_{02d}\right)\cdot\frac{T_{01}}{T_{01d}} \qquad T_{02} = T_{01} - \left(T_{01d} - T_{02d}\right)\cdot\frac{T_{01}}{T_{01d}} \qquad T_{02} = 1266\cdot°\text{F}$$

$\frac{p_{01}}{p_{02}} = \frac{p_{01d}}{p_{02d}}$ Solving for the exit pressure: $p_{02} = p_{01}\cdot\frac{p_{02d}}{p_{01d}}$ $\qquad p_{02} = 70\cdot\text{psi}$

Problem 10.120

[Difficulty: 4]

10.120 We have seen many examples in Chapter 7 of replacing working fluids in order to more easily achieve similitude between models and prototypes. Describe the effects of testing an air compressor using helium as the working fluid on the dimensionless and dimensional parameters we have discussed for compressible flow machines.

Discussion: When we change the working fluid, we need to be sure that we use the correct similitude relationships. Specifically, we would need to keep fluid-specific parameters (gas constant and specific heat ratio) in the relationships. The functional relationships are:

$$\frac{\Delta h_{0s}}{(ND)^2}, \eta, \frac{P}{\rho_{01} N^3 D^5} = f_1\left(\frac{\dot{m}}{\rho_{01} N D^3}, \frac{\rho_{01} N D^2}{\mu}, \frac{ND}{c_{01}}, k\right)$$

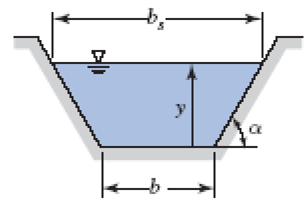
So these dimensionless groups need to be considered. When we replace air with helium, both the gas constant R and the specific heat ratio k will increase. Given a fixed inflow pressure and temperature and a fixed geometry, the effect would be to decrease density and increase sound speed. Therefore, replacing air with helium should result in decreased mass flow rate and power, and an increased operating speed.

When considering dimensional parameters, the important thing to remember is that the operability maps for compressors and/or turbines were constructed for a single working fluid. Therefore, to be safe, an engineer should reconstruct an operability map for a new working fluid.

Problem 11.1

[Difficulty: 2]

11.1 Verify the equation given in Table 11.1 for the hydraulic radius of a trapezoidal channel. Plot the ratio R/y for $b = 2$ m with side slope angles of 30° and 60° for 0.5 m $< y <$ 3 m.



Given: Trapezoidal channel

Find: Derive expression for hydraulic radius; Plot R/y versus y for two different side slopes

Solution:

Available data $\quad b = 2 \cdot m \quad \alpha_1 = 30 \cdot deg \quad \alpha_2 = 60 \cdot deg$

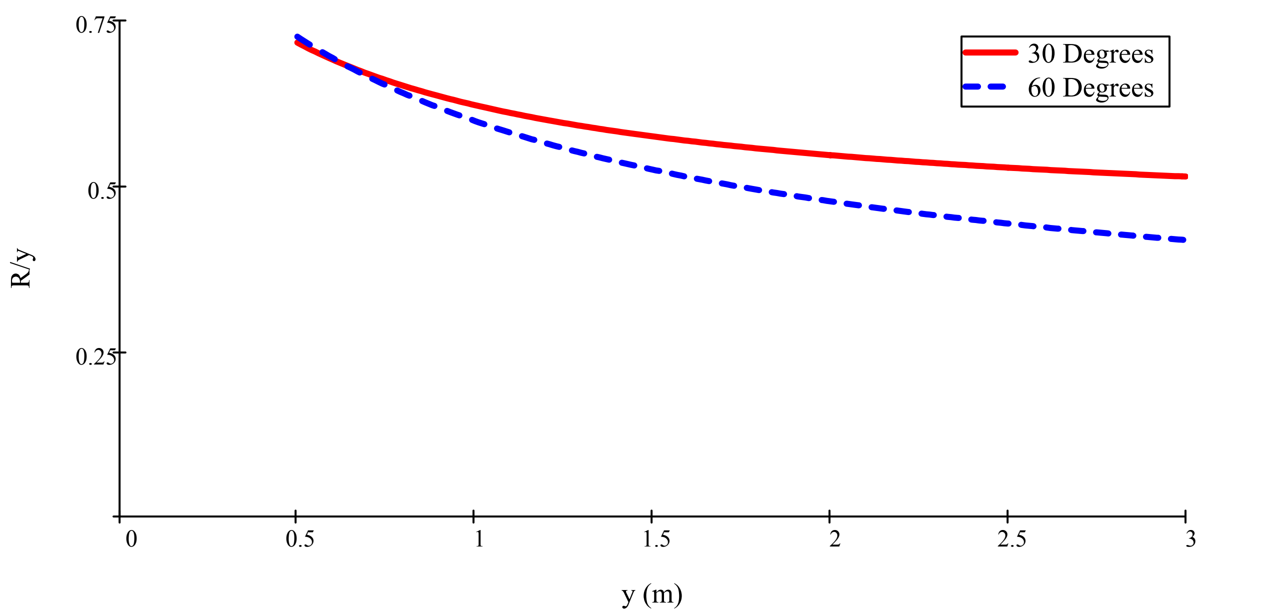
The area is (from simple geometry of a rectangle and triangles) $\quad A = b \cdot y + 2 \cdot \frac{1}{2} \cdot y \cdot y \cdot \cot(\alpha) = y \cdot (b + y \cdot \cot(\alpha))$

The wetted perimeter is (from simple geometry) $\quad P = b + 2 \cdot \frac{y}{\sin(\alpha)}$

Hence the hydraulic radius is $\quad R = \frac{A}{P} = \frac{y \cdot (b + y \cdot \cot(\alpha))}{b + 2 \cdot \frac{y}{\sin(\alpha)}}$ which is the same as that listed in Table 11.1

We are to plot $\quad \frac{R}{y} = \frac{b + y \cdot \cot(\alpha)}{b + 2 \cdot \frac{y}{\sin(\alpha)}}$ with $\quad b = 2 \cdot m \quad$ for $\alpha = 30^o$ and 60^o, and $0.5 < y < 3$ m.

The graph is shown below; it can be plotted in *Excel*.



As the depth increases, the hydraulic radius becomes smaller relative to depth y - wetted perimeter becomes dominant over area

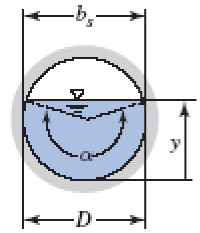
Problem 11.2

[Difficulty: 2]

11.2 Verify the equation given in Table 11.1 for the hydraulic radius of a circular channel. Evaluate and plot the ratio R/D, for liquid depths between 0 and D.

Given: Circular channel

Find: Derive expression for hydraulic radius; Plot R/D versus D for a range of depths



Solution:

The area is (from simple geometry - a segment of a circle plus two triangular sections)

$$A = \frac{D^2}{8}\cdot\alpha + 2\cdot\frac{1}{2}\cdot\frac{D}{2}\cdot\sin\left(\pi - \frac{\alpha}{2}\right)\cdot\frac{D}{2}\cdot\cos\left(\pi - \frac{\alpha}{2}\right) = \frac{D^2}{8}\cdot\alpha + \frac{D^2}{4}\cdot\sin\left(\pi - \frac{\alpha}{2}\right)\cdot\cos\left(\pi - \frac{\alpha}{2}\right)$$

$$A = \frac{D^2}{8}\cdot\alpha + \frac{D^2}{8}\cdot\sin(2\cdot\pi - \alpha) = \frac{D^2}{8}\cdot\alpha - \frac{D^2}{8}\cdot\sin(\alpha) = \frac{D^2}{8}\cdot(\alpha - \sin(\alpha))$$

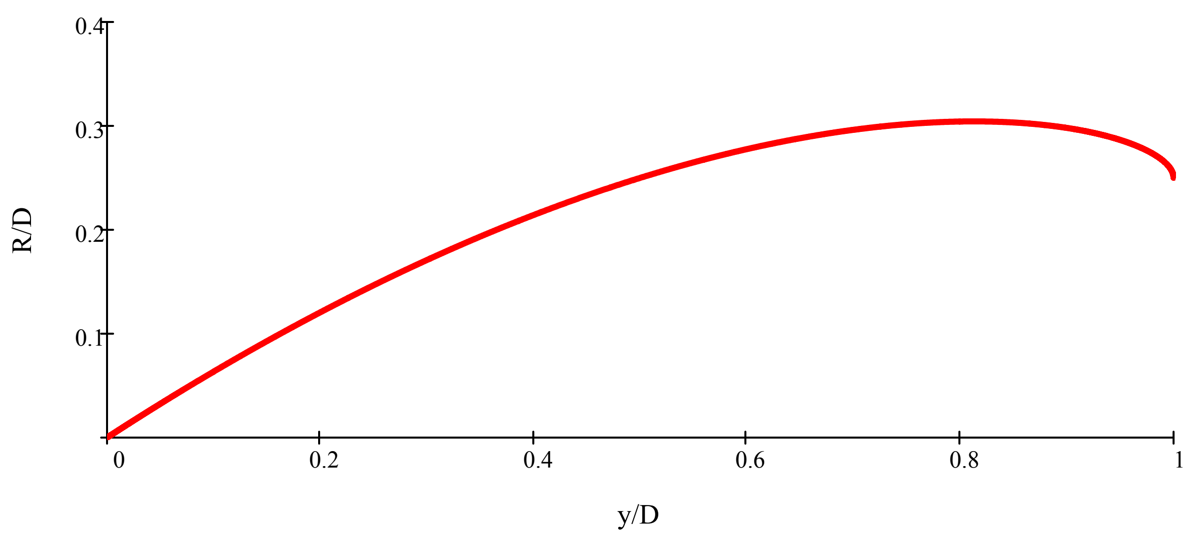
The wetted perimeter is (from simple geometry) $\quad P = \frac{D}{2}\cdot\alpha$

Hence the hydraulic radius is $\quad R = \frac{A}{P} = \frac{\frac{D^2}{8}\cdot(\alpha - \sin(\alpha))}{\frac{D}{2}\cdot\alpha} = \frac{1}{4}\cdot\left(1 - \frac{\sin(\alpha)}{\alpha}\right)\cdot D \quad$ which is the same as that listed in Table 11.1

We are to plot $\quad \frac{R}{D} = \frac{1}{4}\cdot\left(1 - \frac{\sin(\alpha)}{\alpha}\right)$

We will need y as a function of α: $\quad y = \frac{D}{2} + \frac{D}{2}\cdot\cos\left(\pi - \frac{\alpha}{2}\right) = \frac{D}{2}\cdot\left(1 - \cos\left(\frac{\alpha}{2}\right)\right) \quad$ or $\quad \frac{y}{D} = \frac{1}{2}\cdot\left(1 - \cos\left(\frac{\alpha}{2}\right)\right)$

The graph can be plotted in *Excel*.



Problem 11.3

[Difficulty: 1]

11.3 A wave from a passing boat in a lake is observed to travel at 10 mph. Determine the approximate water depth at this location.

Given: Wave from a passing boat

Find: Estimate of water depth

Solution:

Basic equation $c = \sqrt{g \cdot y}$

Available data $c = 10 \cdot mph$ or $c = 14.7 \frac{ft}{s}$

We assume a shallow water wave (long wave compared to water depth)

$c = \sqrt{g \cdot y}$ so $y = \frac{c^2}{g}$ $y = 6.69\, ft$

Problem 11.4

[Difficulty: 1]

11.4 A pebble is dropped into a stream of water that flows in a rectangular channel at 2 m depth. In one second, a ripple caused by the stone is carried 7 m downstream. What is the speed of the flowing water?

Given: Pebble dropped into flowing stream

Find: Estimate of water speed

Solution:

Basic equation $c = \sqrt{g \cdot y}$ and relative speeds will be $V_{wave} = V_{stream} + c$

Available data $y = 2 \cdot m$ and $V_{wave} = \frac{7 \cdot m}{1 \cdot s}$ $V_{wave} = 7 \frac{m}{s}$

We assume a shallow water wave (long wave compared to water depth)

$c = \sqrt{g \cdot y}$ so $c = 4.43 \frac{m}{s}$

Hence $V_{stream} = V_{wave} - c$ $V_{stream} = 2.57 \frac{m}{s}$

Problem 11.5

[Difficulty: 2]

11.5 A pebble is dropped into a stream of water of uniform depth. A wave is observed to travel upstream 5 ft in 1 s, and 13 ft downstream in the same time. Determine the flow speed and depth.

Given: Pebble dropped into flowing stream

Find: Estimate of water depth and speed

Solution:

Basic equation $c = \sqrt{g \cdot y}$ and relative speeds will be $V_{wave} = V_{stream} + c$

Available data $V_{waveupstream} = \frac{-5 \cdot ft}{1 \cdot s}$ $V_{waveupstream} = -5 \frac{ft}{s}$

$V_{wavedownstream} = \frac{13 \cdot ft}{1 \cdot s}$ $V_{wavedownstream} = 13 \frac{ft}{s}$

But we have $V_{wavedownstream} = V_{stream} + c$ and $V_{waveupstream} = V_{stream} - c$

Adding $V_{stream} = \frac{V_{wavedownstream} + V_{waveupstream}}{2}$ $V_{stream} = 4 \frac{ft}{s}$

Subtracting $c = \frac{V_{wavedownstream} - V_{waveupstream}}{2}$ $c = 9 \frac{ft}{s}$

We assume a shallow water wave (long wave compared to water depth)

Hence $c = \sqrt{g \cdot y}$ so $y = \frac{c^2}{g}$ $y = 2.52 \cdot ft$

Problem 11.6

[Difficulty: 3]

11.6 Solution of the complete differential equations for wave motion without surface tension shows that wave speed is given by

$$c = \sqrt{\frac{g\lambda}{2\pi}\tanh\left(\frac{2\pi y}{\lambda}\right)}$$

where λ is the wave wavelength and y is the liquid depth. Show that when $\lambda/y \ll 1$, wave speed becomes proportional to $\sqrt{\lambda}$. In the limit as $\lambda/y \to \infty$, $V_w = \sqrt{gy}$. Determine the value of λ/y for which $V_w > 0.99\sqrt{gy}$.

Given:

Speed of surface waves with no surface tension

Find:

Speed when λ/y approaches zero or infinity; Value of λ/y for which speed is 99% of this latter value

Solution:

Basic equation $\quad c = \sqrt{\frac{g\cdot\lambda}{2\cdot\pi\cdot\tanh\left(\frac{2\cdot\pi\cdot y}{\lambda}\right)}} \quad (1)$

For $\lambda/y \ll 1$ $\quad \tanh\left(\frac{2\cdot\pi\cdot y}{\lambda}\right)$ approaches 1 $\quad \tanh(\infty) \to 1$ $\quad$ so $\quad c = \sqrt{\frac{g\cdot\lambda}{2\cdot\pi}}$

Hence c is proportional to $\sqrt{\lambda}$ $\quad$ so as λ/y approaches ∞ $\quad c = \sqrt{g\cdot y}$

We wish to find λ/y when $\quad c = 0.99\cdot\sqrt{g\cdot y}$

Combining this with Eq 1 $\quad 0.99\cdot\sqrt{g\cdot y} = \sqrt{\frac{g\cdot\lambda}{2\cdot\pi\cdot\tanh\left(\frac{2\cdot\pi\cdot y}{\lambda}\right)}}$ $\quad$ or $\quad 0.99^2\cdot g\cdot y = \frac{g\cdot\lambda}{2\cdot\pi\cdot\tanh\left(\frac{2\cdot\pi\cdot y}{\lambda}\right)}$

Hence $\quad 0.99^2\cdot 2\cdot\pi\cdot\tanh\left(\frac{2\cdot\pi\cdot y}{\lambda}\right) = \frac{\lambda}{y}$ $\quad$ Letting $\lambda/y = x$ $\quad$ we find $\quad 0.99^2\cdot 2\cdot\pi\cdot\tanh\left(\frac{2\cdot\pi}{x}\right) = x$

This is a nonlinear equation in x that can be solved by iteration or using *Excel*'s *Goal Seek* or *Solver*

$x = 1$	$x = 0.99^2\cdot 2\cdot\pi\cdot\tanh\left(\frac{2\cdot\pi}{x}\right)$	$x = 6.16$	$x = 0.99^2\cdot 2\cdot\pi\cdot\tanh\left(\frac{2\cdot\pi}{x}\right)$	$x = 4.74$
$x = 4.74$	$x = 0.99^2\cdot 2\cdot\pi\cdot\tanh\left(\frac{2\cdot\pi}{x}\right)$	$x = 5.35$	$x = 0.99^2\cdot 2\cdot\pi\cdot\tanh\left(\frac{2\cdot\pi}{x}\right)$	$x = 5.09$
$x = 5.09$	$x = 0.99^2\cdot 2\cdot\pi\cdot\tanh\left(\frac{2\cdot\pi}{x}\right)$	$x = 5.2$	$x = 0.99^2\cdot 2\cdot\pi\cdot\tanh\left(\frac{2\cdot\pi}{x}\right)$	$x = 5.15$
$x = 5.15$	$x = 0.99^2\cdot 2\cdot\pi\cdot\tanh\left(\frac{2\cdot\pi}{x}\right)$	$x = 5.17$	$x = 0.99^2\cdot 2\cdot\pi\cdot\tanh\left(\frac{2\cdot\pi}{x}\right)$	$x = 5.16$
$x = 5.16$	$x = 0.99^2\cdot 2\cdot\pi\cdot\tanh\left(\frac{2\cdot\pi}{x}\right)$	$x = 5.17$	$x = 0.99^2\cdot 2\cdot\pi\cdot\tanh\left(\frac{2\cdot\pi}{x}\right)$	$x = 5.16$

Hence $\quad \frac{\lambda}{y} = 5.16$

Problem 11.7

[Difficulty: 1]

11.7 Capillary waves (ripples) are small amplitude and wavelength waves, commonly seen, for example, when an insect or small particle hits the water surface. They are waves generated due to the interaction of the inertia force of the fluid ρ and the fluid surface tension σ. The wave speed is

$$c = 2\pi\sqrt{\frac{\sigma}{\rho g}}$$

Find the speed of capillary waves in water and mercury.

Given: Expression for capillary wave length

Find: Length of water and mercury waves

Solution:

Basic equation $\lambda = 2\cdot\pi\cdot\sqrt{\dfrac{\sigma}{\rho\cdot g}}$

Available data Table A.2 (20ºC) $SG_{Hg} = 13.55$ $SG_w = 0.998$ $\rho = 1000\cdot\dfrac{kg}{m^3}$

Table A.4 (20ºC) $\sigma_{Hg} = 484\times 10^{-3}\cdot\dfrac{N}{m}$ $\sigma_w = 72.8\times 10^{-3}\cdot\dfrac{N}{m}$

Hence $\lambda_{Hg} = 2\cdot\pi\cdot\sqrt{\dfrac{\sigma_{Hg}}{SG_{Hg}\cdot\rho\cdot g}}$ $\lambda_{Hg} = 12\,mm$ $\lambda_{Hg} = 0.472\,in$

$\lambda_w = 2\cdot\pi\cdot\sqrt{\dfrac{\sigma_w}{SG_w\cdot\rho\cdot g}}$ $\lambda_w = 17.1\,mm$ $\lambda_w = 0.675\,in$

Problem 11.8

[Difficulty: 2]

11.8 Solution of the complete differential equations for wave motion in quiescent liquid, including the effects of surface tension, shows that wave speed is given by

$$c = \sqrt{\left(\frac{g\lambda}{2\pi} + \frac{2\pi\sigma}{\rho\lambda}\right)\tanh\left(\frac{2\pi y}{\lambda}\right)}$$

where λ is the wave wavelength, y is the liquid depth, and σ is the surface tension. Plot wave speed versus wavelength for the range 1 mm < λ < 100 mm for (a) water and (b) mercury. Assume $y = 7$ mm for both liquids.

Given: Expression for surface wave speed

Find: Plot speed versus wavelength for water and mercury waves

Solution:

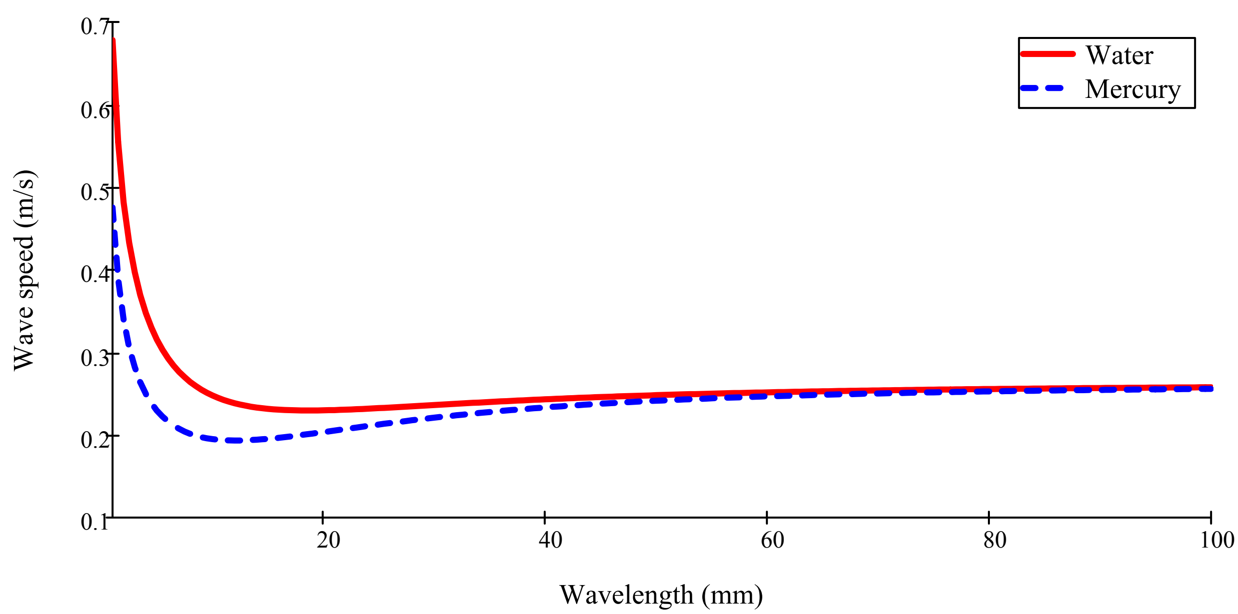
Basic equation $\quad c = \sqrt{\left(\frac{g\cdot\lambda}{2\cdot\pi} + \frac{2\cdot\pi\cdot\sigma}{\rho\cdot\lambda}\right)\cdot\tanh\left(\frac{2\cdot\pi\cdot y}{\lambda}\right)}$

Available data

Table A.2 (20ºC) $\quad SG_{Hg} = 13.55 \quad SG_w = 0.998 \quad \rho = 1000\cdot\frac{kg}{m^3}$

Table A.4 (20ºC) $\quad \sigma_{Hg} = 484\times10^{-3}\cdot\frac{N}{m} \quad \sigma_w = 72.8\times10^{-3}\cdot\frac{N}{m} \quad y = 7\cdot mm$

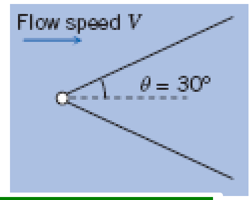
Hence $\quad c_w(\lambda) = \sqrt{\left(\frac{g\cdot\lambda}{2\cdot\pi} + \frac{2\cdot\pi\cdot\sigma_w}{SG_w\cdot\rho\cdot\lambda}\right)\cdot\tanh\left(\frac{2\cdot\pi\cdot y}{\lambda}\right)} \quad c_{Hg}(\lambda) = \sqrt{\left(\frac{g\cdot\lambda}{2\cdot\pi} + \frac{2\cdot\pi\cdot\sigma_{Hg}}{SG_{Hg}\cdot\rho\cdot\lambda}\right)\cdot\tanh\left(\frac{2\cdot\pi\cdot y}{\lambda}\right)}$



Problem 11.9

[Difficulty: 1]

11.9 Surface waves are caused by a sharp object that just touches the free surface of a stream of flowing water, forming the wave pattern shown. The stream depth is 150 mm. Determine the flow speed and Froude number. Note that the wave travels at speed c (Eq. 11.6) normal to the wave front, as shown in the velocity diagram.



Given:

Sharp object causing waves

Find:

Flwo speed and Froude number

Solution:

Basic equation $c = \sqrt{g \cdot y}$

Available data $y = 150 \cdot mm$ $\theta = 30 \cdot deg$

We assume a shallow water wave (long wave compared to water depth)

$$c = \sqrt{g \cdot y} \qquad \text{so} \qquad c = 1.21 \frac{m}{s}$$

From geometry

Hence $\sin(\theta) = \frac{c}{V}$ so $V = \frac{c}{\sin(\theta)}$ $V = 2.43 \frac{m}{s}$

Also $Fr = \frac{V}{c}$ $Fr = 2$ or $Fr = \frac{1}{\sin(\theta)}$ $Fr = 2$

Problem 11.10

[Difficulty: 2]

11.10 The Froude number characterizes flow with a free surface. Plot on a log-log scale the speed versus depth for $0.1 \text{ m/s} < V < 3 \text{ m/s}$ and $0.001 < y < 1 \text{ m}$; plot the line $Fr = 1$, and indicate regions that correspond to tranquil and rapid flow.

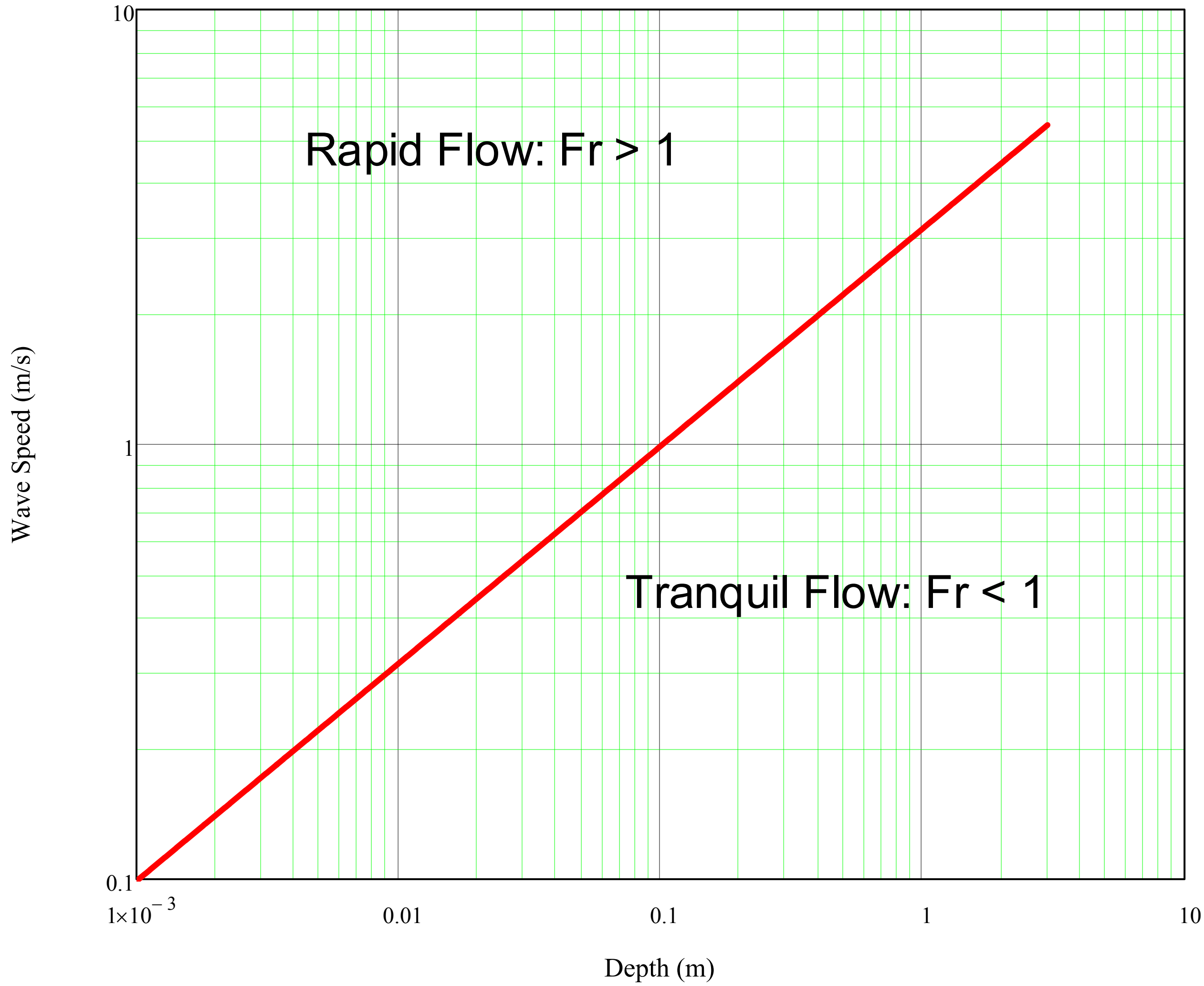
Given: Shallow water waves

Find: Speed versus depth

Solution:

Basic equation $\quad c(y) = \sqrt{g \cdot y}$

We assume a shallow water wave (long wave compared to water depth)



Problem 11.11

[Difficulty: 2]

11.11 A submerged body traveling horizontally beneath a liquid surface at a Froude number (based on body length) about 0.5 produces a strong surface wave pattern if submerged less than half its length. (The wave pattern of a surface ship also is pronounced at this Froude number.) On a log-log plot of speed versus body (or ship) length for 1 m/s < V < 30 m/s and 1 m < x < 300 m, plot the line $Fr = 0.5$.

Given: Motion of sumerged body

Find: Speed versus ship length

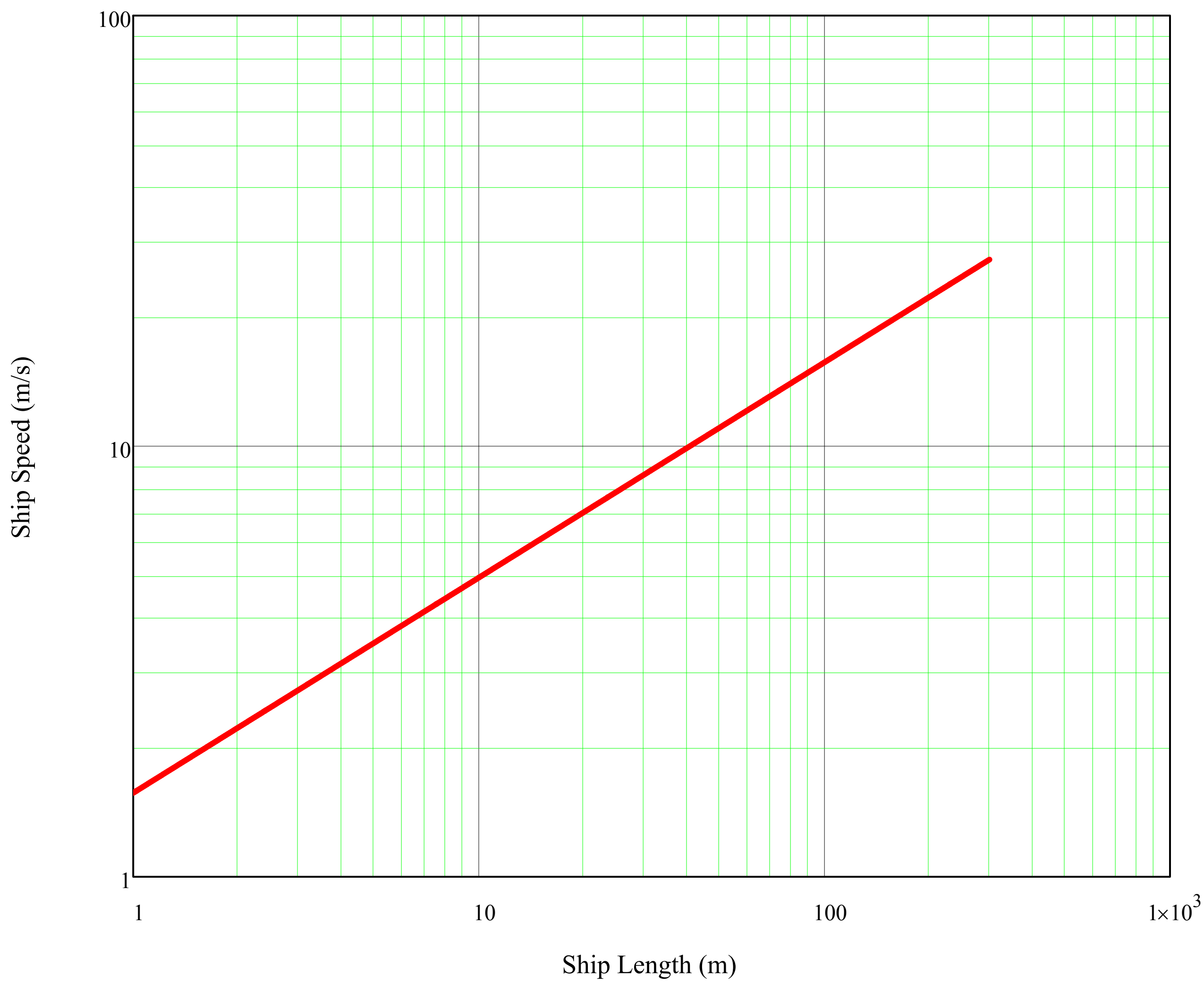
Solution:

Basic equation $\quad c = \sqrt{g \cdot y}$

We assume a shallow water wave (long wave compared to water depth)

In this case we want the Froude number to be 0.5, with $\quad Fr = 0.5 = \dfrac{V}{c}$ $\quad$ and $\quad c = \sqrt{g \cdot x}$ $\quad$ where x is the ship length

Hence $\quad V = 0.5 \cdot c = 0.5 \cdot \sqrt{g \cdot x}$



Problem 11.12

[Difficulty: 1]

11.12 Water flows in a rectangular channel at a depth of 750 mm. If the flow speed is (a) 1 m/s and (b) 4 m/s, compute the corresponding Froude numbers.

Given: Flow in a rectangular channel

Find: Froude numbers

Solution:

Basic equation $$Fr = \frac{V}{\sqrt{g \cdot y}}$$

Available data $y = 750 \cdot mm$ $\quad V_1 = 1 \cdot \frac{m}{s}$ $\quad V_2 = 4 \cdot \frac{m}{s}$

Hence $Fr_1 = \frac{V_1}{\sqrt{g \cdot y}}$ $\quad Fr_1 = 0.369$ $\quad$ Subcritical flow

$Fr_2 = \frac{V_2}{\sqrt{g \cdot y}}$ $\quad Fr_2 = 1.47$ $\quad$ Supercritical flow

Problem 11.12

[Difficulty: 2]

11.13 A long rectangular channel 10 ft wide is observed to have a wavy surface at a depth of about 6 ft. Estimate the rate of discharge.

Given: Flow in a rectangular channel with wavy surface

Find: Froude numbers

Solution:

Basic equation $\quad Fr = \dfrac{V}{\sqrt{g \cdot y}}$

Available data $\quad b = 10 \cdot ft \qquad y = 6 \cdot ft$

A "wavy" surface indicates an unstable flow, which suggests critical flow $\quad Fr = 1$

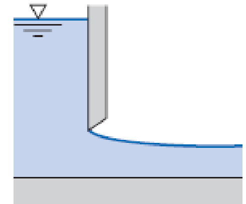
Hence $\quad V = Fr \cdot \sqrt{g \cdot y} \qquad V = 13.9 \dfrac{ft}{s}$

Then $\quad Q = V \cdot b \cdot y \qquad Q = 834 \dfrac{ft^3}{s} \qquad Q = 3.74 \times 10^5 \, gpm$

Problem 11.14

[Difficulty: 2]

11.14 A partially open sluice gate in a 5-m-wide rectangular channel carries water at 10 m^3/s. The upstream depth is 2.5 m. Find the downstream depth and Froude number.



Given: Data on sluice gate

Find: Downstream depth; Froude number

Solution:

Basic equation: $$\frac{p_1}{\rho \cdot g} + \frac{V_1^2}{2 \cdot g} + y_1 = \frac{p_2}{\rho \cdot g} + \frac{V_2^2}{2 \cdot g} + y_2 + h$$ The Bernoulli equation applies because we have steady, incompressible, frictionless flow.

Noting that $p_1 = p_2 = p_{atm}$, (1 = upstream, 2 = downstream) the Bernoulli equation becomes

$$\frac{V_1^2}{2 \cdot g} + y_1 = \frac{V_2^2}{2 \cdot g} + y_2$$

The given data is $b = 5 \cdot m$ $\quad y_1 = 2.5 \cdot m$ $\quad Q = 10 \cdot \frac{m^3}{s}$

For mass flow $Q = V \cdot A$ so $V_1 = \frac{Q}{b \cdot y_1}$ and $V_2 = \frac{Q}{b \cdot y_2}$

Using these in the Bernoulli equation $$\frac{\left(\frac{Q}{b \cdot y_1}\right)^2}{2 \cdot g} + y_1 = \frac{\left(\frac{Q}{b \cdot y_2}\right)^2}{2 \cdot g} + y_2 \qquad (1)$$

The only unknown on the right is y_2. The left side evaluates to $\frac{\left(\frac{Q}{b \cdot y_1}\right)^2}{2 \cdot g} + y_1 = 2.53\,m$

To find y_2 we need to solve the non-linear equation. We must do this numerically; we may use the Newton method or similar, or *Excel*'s *Solver* or *Goal Seek*. Here we interate manually, starting with an arbitrary value less than y_1.

For $y_2 = 0.25 \cdot m$ $\quad \frac{\left(\frac{Q}{b \cdot y_2}\right)^2}{2 \cdot g} + y_2 = 3.51\,m$ $\quad$ For $y_2 = 0.3 \cdot m$ $\quad \frac{\left(\frac{Q}{b \cdot y_2}\right)^2}{2 \cdot g} + y_2 = 2.57\,m$

For $y_2 = 0.305 \cdot m$ $\quad \frac{\left(\frac{Q}{b \cdot y_2}\right)^2}{2 \cdot g} + y_2 = 2.50\,m$ $\quad$ For $y_2 = 0.302 \cdot m$ $\quad \frac{\left(\frac{Q}{b \cdot y_2}\right)^2}{2 \cdot g} + y_2 = 2.54\,m$

Hence $y_2 = 0.302\,m$ is the closest to three figs.

Then $V_2 = \frac{Q}{b \cdot y_2}$ $\quad V_2 = 6.62\,\frac{m}{s}$ $\quad Fr_2 = \frac{V_2}{\sqrt{g \cdot y_2}}$ $\quad Fr_2 = 3.85$

Problem 11.15

[Difficulty: 3]

11.15 For a rectangular channel of width $b = 20$ ft, construct a family of specific energy curves for $Q = 0$, 25, 75, 125, and 200 ft^3/s. What are the minimum specific energies for these curves?

Given: Rectangular channel

Find: Plot of specific energy curves; Critical depths; Critical specific energy

Solution:

Given data: $b =$ 20 ft

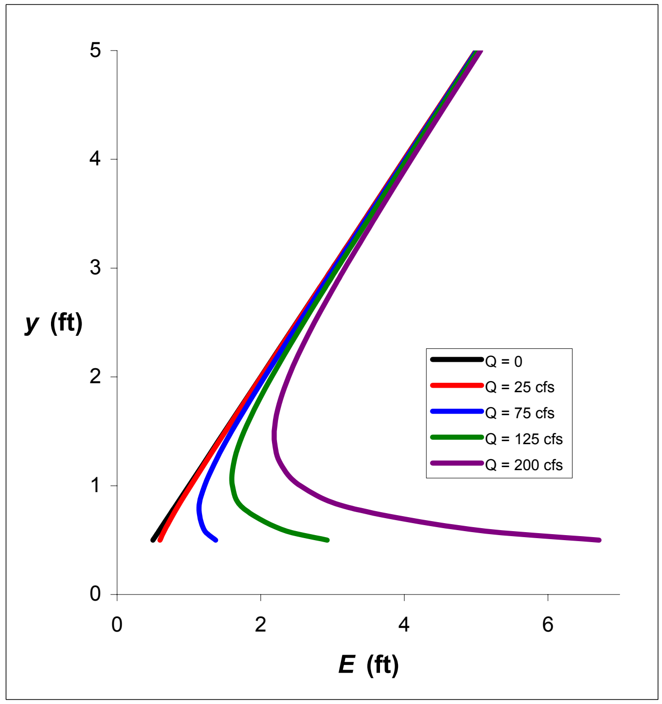
Specific energy: $$E = y + \left(\frac{Q^2}{2gb^2}\right)\frac{1}{y^2}$$

Critical depth: $$y_c = \left(\frac{Q^2}{gb^2}\right)^{\frac{1}{3}}$$

Specific Energy, E (ft·lb/lb)

y (ft)	Q = 0	Q = 25	Q = 75	Q = 125	Q = 200
0.5	0.50	0.60	1.37	2.93	6.71
0.6	0.60	0.67	1.21	2.28	4.91
0.8	0.80	0.84	1.14	1.75	3.23
1.0	1.00	1.02	1.22	1.61	2.55
1.2	1.20	1.22	1.35	1.62	2.28
1.4	1.40	1.41	1.51	1.71	2.19
1.6	1.60	1.61	1.69	1.84	2.21
1.8	1.80	1.81	1.87	1.99	2.28
2.0	2.00	2.01	2.05	2.15	2.39
2.2	2.20	2.21	2.25	2.33	2.52
2.4	2.40	2.40	2.44	2.51	2.67
2.6	2.60	2.60	2.63	2.69	2.83
2.8	2.80	2.80	2.83	2.88	3.00
3.0	3.00	3.00	3.02	3.07	3.17
3.5	3.50	3.50	3.52	3.55	3.63
4.0	4.00	4.00	4.01	4.04	4.10
4.5	4.50	4.50	4.51	4.53	4.58
5.0	5.00	5.00	5.01	5.02	5.06

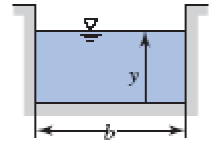
y_c (ft)	0.365	0.759	1.067	1.46
E_c (ft)	0.547	1.14	1.60	2.19



Problem 11.16

[Difficulty: 1]

11.16 Find the critical depth for flow at 3 m^3/s in a rectangular channel of width 2.5 m.



Given: Rectangular channel flow

Find: Critical depth

Solution:

Basic equations: $$y_c = \left(\frac{Q^2}{g \cdot b^2}\right)^{\frac{1}{3}}$$

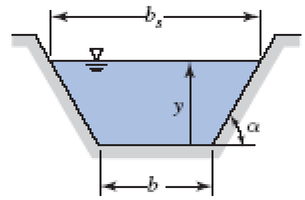
Given data: $b = 2.5 \cdot m$ $\qquad Q = 3 \cdot \frac{m^3}{s}$

Hence $$y_c = \left(\frac{Q^2}{g \cdot b^2}\right)^{\frac{1}{3}} \qquad y_c = 0.528\,m$$

Problem 11.17

[Difficulty: 3]

11.17 A trapezoidal channel with a bottom width of 20 ft, side slopes of 1 to 2, channel bottom slope of 0.0016, and a Manning's *n* of 0.025 carries a discharge of 400 cfs. Compute the critical depth and velocity of this channel.



Given: Data on trapezoidal channel

Find: Critical depth and velocity

Solution:

Basic equation: $E = y + \dfrac{V^2}{2 \cdot g}$

The given data is: $b = 20 \cdot ft$ $\quad \alpha = \text{atan}(2)$ $\quad \alpha = 63.4\text{deg}$ $\quad S_0 = 0.0016$ $\quad n = 0.025$ $\quad Q = 400 \cdot \dfrac{ft^3}{s}$

In terms of flow rate $E = y + \dfrac{Q^2}{2 \cdot A^2 \cdot g}$ where (Table 11.1) $A = y \cdot (b + y \cdot \cot(\alpha))$

Hence in terms of y $E = y + \dfrac{Q^2}{2 \cdot (b + y \cdot \cot(\alpha))^2 \cdot y^2 \cdot g}$

For critical conditions $\dfrac{dE}{dy} = 0 = 1 - \dfrac{Q^2}{g \cdot y^3 \cdot (b + y \cdot \cot(\alpha))^2} - \dfrac{Q^2 \cdot \cot(\alpha)}{g \cdot y^2 \cdot (b + y \cdot \cot(\alpha))^3} = 1 - \dfrac{Q^2 \cdot (b + 2 \cdot y \cdot \cot(\alpha))}{g \cdot y^3 \cdot (b + y \cdot \cot(\alpha))^3}$

Hence $g \cdot y^3 \cdot (b + y \cdot \cot(\alpha))^3 - Q^2 \cdot (b + 2 \cdot y \cdot \cot(\alpha)) = 0$

Let $f(y) = g \cdot y^3 \cdot (b + y \cdot \cot(\alpha))^3 - Q^2 \cdot (b + 2 \cdot y \cdot \cot(\alpha))$

We can iterate or use *Excel*'s *Goal Seek* or Solver to find y when f(y) = 0

Guess $y = 2 \cdot ft$ $\quad f(y) = -1.14 \times 10^6 \dfrac{ft^7}{s^2}$ $\quad y = 2.25 \cdot ft$ $\quad f(y) = -1.05 \times 10^5 \dfrac{ft^7}{s^2}$ $\quad y = 2.35 \cdot ft$ $\quad f(y) = 3.88 \times 10^5 \dfrac{ft^7}{s^2}$

The solution is somewhere between y = 2.25 ft and y = 2.35 ft, as the sign of f(y) changes here.

$y = 2.3 \cdot ft$ $\quad f(y) = 1.36 \times 10^5 \dfrac{ft^7}{s^2}$ $\quad y = 2.275 \cdot ft$ $\quad f(y) = 1.38 \times 10^4 \dfrac{ft^7}{s^2}$ $\quad y = 2.272 \cdot ft$ $\quad f(y) = -657 \dfrac{ft^7}{s^2}$

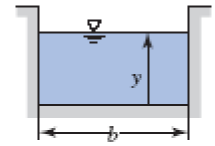
Hence critical depth is $y = 2.27 \cdot ft$ and $A = y \cdot (b + y \cdot \cot(\alpha))$ $\quad A = 48.0\, ft^2$

and critical speed is $V = \dfrac{Q}{A}$ $\quad V = 8.34 \cdot \dfrac{ft}{s}$

Problem 11.18

[Difficulty: 2]

11.18 A rectangular channel carries a discharge of 10 ft^3/s per foot of width. Determine the minimum specific energy possible for this flow. Compute the corresponding flow depth and speed.



Given: Data on rectangular channel

Find: Minimum specific energy; Flow depth; Speed

Solution:

Basic equation: $$E = y + \frac{V^2}{2\cdot g}$$

In Section 11-2 we prove that the minimum specific energy is when we have critical flow; here we rederive the minimum energy point

For a rectangular channel $$Q = V\cdot b\cdot y \quad \text{or} \quad V = \frac{Q}{b\cdot y} \quad \text{with} \quad \frac{Q}{b} = 10\cdot\frac{\frac{ft^3}{s}}{ft} = \text{constant}$$

Hence, using this in the basic equation $$E = y + \left(\frac{Q}{b\cdot y}\right)^2\cdot\frac{1}{2\cdot g} = y + \left(\frac{Q^2}{2\cdot b^2\cdot g}\right)\cdot\frac{1}{y^2}$$

E is a minimum when $$\frac{dE}{dy} = 1 - \left(\frac{Q^2}{b^2\cdot g}\right)\cdot\frac{1}{y^3} = 0 \quad \text{or} \quad y = \left(\frac{Q^2}{b^2\cdot g}\right)^{\frac{1}{3}} \qquad y = 1.46\cdot ft$$

The speed is then given by $$V = \frac{Q}{b\cdot y} \qquad V = 6.85\cdot\frac{ft}{s}$$

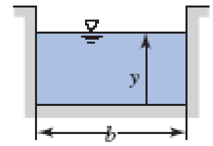
Note that from Eq. 11.22 we also have $$V_c = \left(\frac{g\cdot Q}{b}\right)^{\frac{1}{3}} \qquad V_c = 6.85\cdot\frac{ft}{s}$$ which agrees with the above

The minimum energy is then $$E_{min} = y + \frac{V^2}{2\cdot g} \qquad E_{min} = 2.19\cdot ft$$

Problem 11.19

[Difficulty: 3]

11.19 Flow in the channel of Problem 11.18 ($E_{min} = 2.19$ ft) is to be at twice the minimum specific energy. Compute the alternate depths for this E.



Given: Data on rectangular channel

Find: Depths for twice the minimum energy

Solution:

Basic equation: $$E = y + \frac{V^2}{2 \cdot g}$$

For a rectangular channel $\quad Q = V \cdot b \cdot y \quad$ or $\quad V = \frac{Q}{b \cdot y} \quad$ with $\quad \frac{Q}{b} = 10 \cdot \frac{\frac{ft^3}{s}}{ft} = \text{constant}$

Hence, using this in the basic eqn. $\quad E = y + \left(\frac{Q}{b \cdot y}\right)^2 \cdot \frac{1}{2 \cdot g} = y + \left(\frac{Q^2}{2 \cdot b^2 \cdot g}\right) \cdot \frac{1}{y^2} \quad$ and $\quad E = 2 \times 2.19 \cdot ft \quad E = 4.38 \cdot ft$

We have a nonlinear implicit equation for y $\quad y + \left(\frac{Q^2}{2 \cdot b^2 \cdot g}\right) \cdot \frac{1}{y^2} = E$

This is a nonlinear implicit equation for y and must be solved numerically. We can use one of a number of numerical root finding techniques, such as Newton's method, or we can use *Excel*'s *Solver* or *Goal Seek*, or we can manually iterate, as below. We start with a y larger than the critical, and evaluate the left side of the equation so that it is equal to $E = 4.38 \cdot ft$

For $y = 2 \cdot ft \quad y + \left(\frac{Q^2}{2 \cdot b^2 \cdot g}\right) \cdot \frac{1}{y^2} = 2.39 \cdot ft$ $\quad$ For $y = 4 \cdot ft \quad y + \left(\frac{Q^2}{2 \cdot b^2 \cdot g}\right) \cdot \frac{1}{y^2} = 4.10 \cdot ft$

For $y = 4.5 \cdot ft \quad y + \left(\frac{Q^2}{2 \cdot b^2 \cdot g}\right) \cdot \frac{1}{y^2} = 4.58 \cdot ft$ $\quad$ For $y = 4.30 \cdot ft \quad y + \left(\frac{Q^2}{2 \cdot b^2 \cdot g}\right) \cdot \frac{1}{y^2} = 4.38 \cdot ft$

Hence $y = 4.30 \cdot ft$

For the shallow depth

For $y = 1 \cdot ft \quad y + \left(\frac{Q^2}{2 \cdot b^2 \cdot g}\right) \cdot \frac{1}{y^2} = 2.55 \cdot ft$ $\quad$ For $y = 0.5 \cdot ft \quad y + \left(\frac{Q^2}{2 \cdot b^2 \cdot g}\right) \cdot \frac{1}{y^2} = 6.72 \cdot ft$

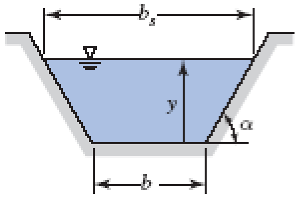
For $y = 0.6 \cdot ft \quad y + \left(\frac{Q^2}{2 \cdot b^2 \cdot g}\right) \cdot \frac{1}{y^2} = 4.92 \cdot ft$ $\quad$ For $y = 0.65 \cdot ft \quad y + \left(\frac{Q^2}{2 \cdot b^2 \cdot g}\right) \cdot \frac{1}{y^2} = 4.33 \cdot ft$

For $y = 0.645 \cdot ft \quad y + \left(\frac{Q^2}{2 \cdot b^2 \cdot g}\right) \cdot \frac{1}{y^2} = 4.38 \cdot ft$ $\quad$ Hence $y = 0.645 \cdot ft$

Problem 11.20

[Difficulty: 2]

11.20 For a channel of nonrectangular cross section, critical depth occurs at minimum specific energy. Obtain a general equation for critical depth in a trapezoidal section in terms of Q, g, b, and θ. It will be implicit in y_c!



Given: Trapezoidal channel

Find: Critcal depth

Solution:

Basic equation: $$E = y + \frac{V^2}{2\cdot g}$$

The critical depth occurs when the specific energy is minimized

For a trapezoidal channel (Table 11.1) $$A = y\cdot(b + \cot(\alpha)\cdot y)$$

Hence for V $$V = \frac{Q}{A} = \frac{Q}{y\cdot(b + \cot(\alpha)\cdot y)}$$

Using this in Eq. 11.14 $$E = y + \left[\frac{Q}{y\cdot(b + \cot(\alpha)\cdot y)}\right]^2\cdot\frac{1}{2\cdot g}$$

E is a minimum when $$\frac{dE}{dy} = 1 - \frac{Q^2\cdot\cot(\alpha)}{g\cdot y^2\cdot(b + y\cdot\cot(\alpha))^3} - \frac{Q^2}{g\cdot y^3\cdot(b + y\cdot\cot(\alpha))^2} = 0$$

Hence we obtain for y $$\frac{Q^2\cdot\cot(\alpha)}{g\cdot y^2\cdot(b + y\cdot\cot(\alpha))^3} + \frac{Q^2}{g\cdot y^3\cdot(b + y\cdot\cot(\alpha))^2} = 1$$

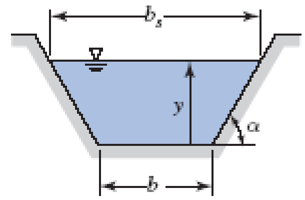
This can be simplified to $$\frac{Q^2\cdot(b + 2\cdot y\cdot\cot(\alpha))}{g\cdot y^3\cdot(b + y\cdot\cot(\alpha))^3} = 1$$

This expression is the simplest one for y; it is implicit

Problem 11.21

[Difficulty: 3]

11.21 Water flows at 400 ft^3/s in a trapezoidal channel with bottom width of 10 ft. The sides are sloped at 3:1. Find the critical depth for this channel.



Given: Data on trapezoidal channel

Find: Critical depth

Solution:

Basic equation: $E = y + \frac{V^2}{2 \cdot g}$

In Section 11-2 we prove that the minimum specific energy is when we have critical flow; here we rederive the minimum energy point

For a trapezoidal channel (Table 11.1) $A = (b + \cot(\alpha) \cdot y) \cdot y$ and $b = 10 \cdot ft$ $\quad \alpha = \operatorname{atan}\left(\frac{3}{1}\right)$ $\quad \alpha = 71.6\,deg$

Hence for V $\quad V = \frac{Q}{A} = \frac{Q}{(b + \cot(\alpha) \cdot y) \cdot y}$ and $Q = 400 \cdot \frac{ft^3}{s}$

Using this in the basic equation $\quad E = y + \left[\frac{Q}{(b + \cot(\alpha) \cdot y) \cdot y}\right]^2 \cdot \frac{1}{2 \cdot g}$

E is a minimum when $\quad \frac{dE}{dy} = 1 - \frac{Q^2 \cdot \cot(\alpha)}{g \cdot y^2 \cdot (b + y \cdot \cot(\alpha))^3} - \frac{Q^2}{g \cdot y^3 \cdot (b + y \cdot \cot(\alpha))^2} = 0$

Hence we obtain for y $\quad \frac{Q^2 \cdot \cot(\alpha)}{g \cdot y^2 \cdot (b + y \cdot \cot(\alpha))^3} + \frac{Q^2}{g \cdot y^3 \cdot (b + y \cdot \cot(\alpha))^2} = 1$ or $\frac{Q^2 \cdot (b + 2 \cdot y \cdot \cot(\alpha))}{g \cdot y^3 \cdot (b + y \cdot \cot(\alpha))^3} = 1$

This is a nonlinear implicit equation for y and must be solved numerically. We can use one of a number of numerical root finding techniques, such as Newton's method, or we can use *Excel*'s *Solver* or *Goal Seek*, or we can manually iterate, as below, to make the left side equal unity

$y = 5 \cdot ft \quad \frac{Q^2 \cdot (b + 2 \cdot y \cdot \cot(\alpha))}{g \cdot y^3 \cdot (b + y \cdot \cot(\alpha))^3} = 0.3$ $\qquad y = 4 \cdot ft \quad \frac{Q^2 \cdot (b + 2 \cdot y \cdot \cot(\alpha))}{g \cdot y^3 \cdot (b + y \cdot \cot(\alpha))^3} = 0.7$

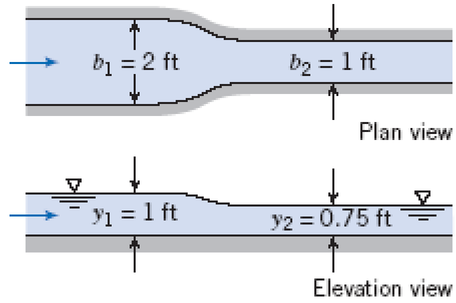
$y = 3.5 \cdot ft \quad \frac{Q^2 \cdot (b + 2 \cdot y \cdot \cot(\alpha))}{g \cdot y^3 \cdot (b + y \cdot \cot(\alpha))^3} = 1.03$ $\qquad y = 3.55 \cdot ft \quad \frac{Q^2 \cdot (b + 2 \cdot y \cdot \cot(\alpha))}{g \cdot y^3 \cdot (b + y \cdot \cot(\alpha))^3} = 0.98$

$y = 3.53 \cdot ft \quad \frac{Q^2 \cdot (b + 2 \cdot y \cdot \cot(\alpha))}{g \cdot y^3 \cdot (b + y \cdot \cot(\alpha))^3} = 1.00$

The critical depth is $\quad y = 3.53 \cdot ft$

Problem 11.22

11.22 Consider the Venturi flume shown. The bed is horizontal, and flow may be considered frictionless. The upstream depth is 1 ft, and the downstream depth is 0.75 ft. The upstream breadth is 2 ft, and the breadth of the throat is 1 ft. Estimate the flow rate through the flume.



Given: Data on venturi flume

Find: Flow rate

Solution:

Basic equation: $$\frac{p_1}{\rho \cdot g} + \frac{V_1^{\,2}}{2 \cdot g} + y_1 = \frac{p_2}{\rho \cdot g} + \frac{V_2^{\,2}}{2 \cdot g} + y_2$$ The Bernoulli equation applies because we have steady, incompressible, frictionless flow

At each section $Q = V \cdot A = V \cdot b \cdot y$ or $V = \dfrac{Q}{b \cdot y}$

The given data is $b_1 = 2 \cdot ft$ $y_1 = 1 \cdot ft$ $b_2 = 1 \cdot ft$ $y_2 = 0.75 \cdot ft$

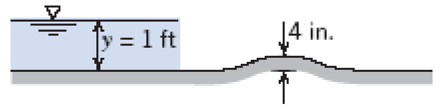
Hence the Bernoulli equation becomes (with $p_1 = p_2 = p_{atm}$) $$\frac{\left(\frac{Q}{b_1 \cdot y_1}\right)^2}{2 \cdot g} + y_1 = \frac{\left(\frac{Q}{b_2 \cdot y_2}\right)^2}{2 \cdot g} + y_2$$

Solving for Q $$Q = \sqrt{\frac{2 \cdot g \cdot (y_1 - y_2)}{\left(\frac{1}{b_2 \cdot y_2}\right)^2 - \left(\frac{1}{b_1 \cdot y_1}\right)^2}}$$ $$Q = 3.24 \cdot \frac{ft^3}{s}$$

Problem 11.23

[Difficulty: 3]

11.23 A rectangular channel 10 ft wide carries 100 cfs on a horizontal bed at 1.0 ft depth. A smooth bump across the channel rises 4 in. above the channel bottom. Find the elevation of the liquid free surface above the bump.



Given: Data on rectangular channel and a bump

Find: Elevation of free surface above the bump

Solution:

Basic equation: $\frac{p_1}{\rho \cdot g} + \frac{V_1^2}{2 \cdot g} + y_1 = \frac{p_2}{\rho \cdot g} + \frac{V_2^2}{2 \cdot g} + y_2 + h$ The Bernoulli equation applies because we have steady, incompressible, frictionless flow. Note that at location 2 (the bump), the potential is $y_2 + h$, where h is the bump height

Recalling the specific energy $E = \frac{V^2}{2 \cdot g} + y$ and noting that $p_1 = p_2 = p_{atm}$, the Bernoulli equation becomes $E_1 = E_2 + h$

At each section $Q = V \cdot A = V \cdot b \cdot y$ or $V = \frac{Q}{b \cdot y}$

The given data is $b = 10 \cdot ft$ $y_1 = 1 \cdot ft$ $h = 4 \cdot in$ $Q = 100 \cdot \frac{ft^3}{s}$

Hence we find $V_1 = \frac{Q}{b \cdot y_1}$ $V_1 = 10 \cdot \frac{ft}{s}$

and $E_1 = \frac{V_1^2}{2 \cdot g} + y_1$ $E_1 = 2.554 \cdot ft$

Hence $E_1 = E_2 + h = \frac{V_2^2}{2 \cdot g} + y_2 + h = \frac{Q^2}{2 \cdot g \cdot b^2 \cdot y_2^2} + y_2 + h$ or $\frac{Q^2}{2 \cdot g \cdot b^2 \cdot y_2^2} + y_2 = E_1 - h$

This is a nonlinear implicit equation for y_2 and must be solved numerically. We can use one of a number of numerical root finding techniques, such as Newton's method, or we can use *Excel*'s *Solver* or *Goal Seek*, or we can manually iterate, as below. We select y_2 so the left side of the equation equals $E_1 - h = 2.22 \cdot ft$

For $y_2 = 1 \cdot ft$ $\frac{Q^2}{2 \cdot g \cdot b^2 \cdot y_2^2} + y_2 = 2.55 \cdot ft$ For $y_2 = 1.5 \cdot ft$ $\frac{Q^2}{2 \cdot g \cdot b^2 \cdot y_2^2} + y_2 = 2.19 \cdot ft$

For $y_2 = 1.4 \cdot ft$ $\frac{Q^2}{2 \cdot g \cdot b^2 \cdot y_2^2} + y_2 = 2.19 \cdot ft$ For $y_2 = 1.3 \cdot ft$ $\frac{Q^2}{2 \cdot g \cdot b^2 \cdot y_2^2} + y_2 = 2.22 \cdot ft$

Hence $y_2 = 1.30 \cdot ft$

Note that $V_2 = \frac{Q}{b \cdot y_2}$ $V_2 = 7.69 \cdot \frac{ft}{s}$

so we have $Fr_1 = \frac{V_1}{\sqrt{g \cdot y_1}}$ $Fr_1 = 1.76$ and $Fr_2 = \frac{V_2}{\sqrt{g \cdot y_2}}$ $Fr_2 = 1.19$

Problem 11.24

[Difficulty: 3]

11.24 A rectangular channel 10 ft wide carries a discharge of 20 ft³/s at 1.0 ft depth. A smooth bump 0.25 ft high is placed on the floor of the channel. Estimate the local change in flow depth caused by the bump.

Given: Data on rectangular channel and a bump

Find: Local change in flow depth caused by the bump

Solution:

Basic equation: $\frac{p_1}{\rho \cdot g} + \frac{V_1^2}{2 \cdot g} + y_1 = \frac{p_2}{\rho \cdot g} + \frac{V_2^2}{2 \cdot g} + y_2 + h$ The Bernoulli equation applies because we have steady, incompressible, frictionless flow. Note that at location 2 (the bump), the potential is $y_2 + h$, where h is the bump height

Recalling the specific energy $E = \frac{V^2}{2 \cdot g} + y$ and noting that $p_1 = p_2 = p_{atm}$, the Bernoulli equation becomes $E_1 = E_2 + h$

At each section $Q = V \cdot A = V \cdot b \cdot y$ or $V = \frac{Q}{b \cdot y}$

The given data is $b = 10 \cdot ft$ $y_1 = 1 \cdot ft$ $h = 0.25 \cdot ft$ $Q = 20 \cdot \frac{ft^3}{s}$

Hence we find $V_1 = \frac{Q}{b \cdot y_1}$ $V_1 = 2 \cdot \frac{ft}{s}$

and $E_1 = \frac{V_1^2}{2 \cdot g} + y_1$ $E_1 = 1.062 \cdot ft$

Hence $E_1 = E_2 + h = \frac{V_2^2}{2 \cdot g} + y_2 + h = \frac{Q^2}{2 \cdot g \cdot b^2 \cdot y_2^2} + y_2 + h$ or $\frac{Q^2}{2 \cdot g \cdot b^2 \cdot y_2^2} + y_2 = E_1 - h$

This is a nonlinear implicit equation for y_2 and must be solved numerically. We can use one of a number of numerical root finding techniques, such as Newton's method, or we can use *Excel*'s *Solver* or *Goal Seek*, or we can manually iterate, as below. We select y_2 so the left side of the equation equals $E_1 - h = 0.812 \cdot ft$

For $y_2 = 0.75 \cdot ft$ $\frac{Q^2}{2 \cdot g \cdot b^2 \cdot y_2^2} + y_2 = 0.861 \cdot ft$ For $y_2 = 0.7 \cdot ft$ $\frac{Q^2}{2 \cdot g \cdot b^2 \cdot y_2^2} + y_2 = 0.827 \cdot ft$

For $y_2 = 0.65 \cdot ft$ $\frac{Q^2}{2 \cdot g \cdot b^2 \cdot y_2^2} + y_2 = 0.797 \cdot ft$ For $y_2 = 0.676 \cdot ft$ $\frac{Q^2}{2 \cdot g \cdot b^2 \cdot y_2^2} + y_2 = 0.812 \cdot ft$

Hence $y_2 = 0.676 \cdot ft$ and $\frac{y_2 - y_1}{y_1} = -32.4 \cdot \%$

Note that $V_2 = \frac{Q}{b \cdot y_2}$ $V_2 = 2.96 \cdot \frac{ft}{s}$

so we have $Fr_1 = \frac{V_1}{\sqrt{g \cdot y_1}}$ $Fr_1 = 0.353$ and $Fr_2 = \frac{V_2}{\sqrt{g \cdot y_2}}$ $Fr_2 = 0.634$

Problem 11.25

[Difficulty: 3]

11.25 At a section of a 10-ft-wide rectangular channel, the depth is 0.3 ft for a discharge of 20 ft^3/s. A smooth bump 0.1 ft high is placed on the floor of the channel. Determine the local change in flow depth caused by the bump.

Given: Data on rectangular channel and a bump

Find: Local change in flow depth caused by the bump

Solution:

Basic equation: $$\frac{p_1}{\rho \cdot g} + \frac{V_1^2}{2 \cdot g} + y_1 = \frac{p_2}{\rho \cdot g} + \frac{V_2^2}{2 \cdot g} + y_2 + h$$

The Bernoulli equation applies because we have steady, incompressible, frictionless flow. Note that at location 2 (the bump), the potential is $y_2 + h$, where h is the bump height

Recalling the specific energy $E = \frac{V^2}{2 \cdot g} + y$ and noting that $p_1 = p_2 = p_{atm}$, the Bernoulli equation becomes $E_1 = E_2 + h$

At each section $Q = V \cdot A = V \cdot b \cdot y$ or $V = \frac{Q}{b \cdot y}$

The given data is $b = 10 \cdot ft$ $\quad y_1 = 0.3 \cdot ft$ $\quad h = 0.1 \cdot ft$ $\quad Q = 20 \cdot \frac{ft^3}{s}$

Hence we find $V_1 = \frac{Q}{b \cdot y_1}$ $\quad V_1 = 6.67 \cdot \frac{ft}{s}$

and $E_1 = \frac{V_1^2}{2 \cdot g} + y_1$ $\quad E_1 = 0.991 \cdot ft$

Hence $$E_1 = E_2 + h = \frac{V_2^2}{2 \cdot g} + y_2 + h = \frac{Q^2}{2 \cdot g \cdot b^2 \cdot y_2^2} + y_2 + h \quad \text{or} \quad \frac{Q^2}{2 \cdot g \cdot b^2 \cdot y_2^2} + y_2 = E_1 - h$$

This is a nonlinear implicit equation for y_2 and must be solved numerically. We can use one of a number of numerical root finding techniques, such as Newton's method, or we can use *Excel*'s *Solver* or *Goal Seek*, or we can manually iterate, as below. We select y_2 so the left side of the equation equals $E_1 - h = 0.891 \cdot ft$

For $y_2 = 0.3 \cdot ft$ $\quad \frac{Q^2}{2 \cdot g \cdot b^2 \cdot y_2^2} + y_2 = 0.991 \cdot ft$ $\quad$ For $y_2 = 0.35 \cdot ft$ $\quad \frac{Q^2}{2 \cdot g \cdot b^2 \cdot y_2^2} + y_2 = 0.857 \cdot ft$

For $y_2 = 0.33 \cdot ft$ $\quad \frac{Q^2}{2 \cdot g \cdot b^2 \cdot y_2^2} + y_2 = 0.901 \cdot ft$ $\quad$ For $y_2 = 0.334 \cdot ft$ $\quad \frac{Q^2}{2 \cdot g \cdot b^2 \cdot y_2^2} + y_2 = 0.891 \cdot ft$

Hence $y_2 = 0.334 \cdot ft$ and $\frac{y_2 - y_1}{y_1} = 11.3 \cdot \%$

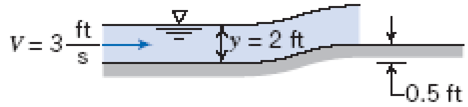
Note that $V_2 = \frac{Q}{b \cdot y_2}$ $\quad V_2 = 5.99 \cdot \frac{ft}{s}$

so we have $Fr_1 = \frac{V_1}{\sqrt{g \cdot y_1}}$ $\quad Fr_1 = 2.15$ $\quad$ and $\quad Fr_2 = \frac{V_2}{\sqrt{g \cdot y_2}}$ $\quad Fr_2 = 1.83$

Problem 11.26

[Difficulty: 3]

11.26 Water, at 3 ft/s and 2 ft depth, approaches a smooth rise in a wide channel. Estimate the stream depth after the 0.5 ft rise.



Given: Data on wide channel

Find: Stream depth after rise

Solution:

Basic equation: $\frac{p_1}{\rho \cdot g} + \frac{V_1^2}{2 \cdot g} + y_1 = \frac{p_2}{\rho \cdot g} + \frac{V_2^2}{2 \cdot g} + y_2 + h$

The Bernoulli equation applies because we have steady, incompressible, frictionless flow. Note that at location 2 (the bump), the potential is $y_2 + h$, where h is the bump height

Recalling the specific energy $E = \frac{V^2}{2 \cdot g} + y$ and noting that $p_1 = p_2 = p_{atm}$, the Bernoulli equation becomes $E_1 = E_2 + h$

At each section $Q = V \cdot A = V_1 \cdot b \cdot y_1 = V_2 \cdot b \cdot y_2$ $\quad V_2 = V_1 \cdot \frac{y_1}{y_2}$

The given data is $y_1 = 2 \cdot ft$ $\quad V_1 = 3 \cdot \frac{ft}{s}$ $\quad h = 0.5 \cdot ft$

Hence $E_1 = \frac{V_1^2}{2 \cdot g} + y_1$ $\quad E_1 = 2.14 \cdot ft$

Then $E_1 = E_2 + h = \frac{V_2^2}{2 \cdot g} + y_2 + h = \frac{V_1^2 \cdot y_1^2}{2 \cdot g} \cdot \frac{1}{y_2^2} + y_2 + h$ or $\frac{V_1^2 \cdot y_1^2}{2 \cdot g} \cdot \frac{1}{y_2^2} + y_2 = E_1 - h$

This is a nonlinear implicit equation for y_2 and must be solved numerically. We can use one of a number of numerical root finding techniques, such as Newton's method, or we can use *Excel*'s *Solver* or *Goal Seek*, or we can manually iterate, as below. We select y_2 so the left side of the equation equals $E_1 - h = 1.64 \cdot ft$

For $y_2 = 2 \cdot ft$ $\quad \frac{V_1^2 \cdot y_1^2}{2 \cdot g} \cdot \frac{1}{y_2^2} + y_2 = 2.14 \cdot ft$ $\quad$ For $y_2 = 1.5 \cdot ft$ $\quad \frac{V_1^2 \cdot y_1^2}{2 \cdot g} \cdot \frac{1}{y_2^2} + y_2 = 1.75 \cdot ft$

For $y_2 = 1.3 \cdot ft$ $\quad \frac{V_1^2 \cdot y_1^2}{2 \cdot g} \cdot \frac{1}{y_2^2} + y_2 = 1.63 \cdot ft$ $\quad$ For $y_2 = 1.31 \cdot ft$ $\quad \frac{V_1^2 \cdot y_1^2}{2 \cdot g} \cdot \frac{1}{y_2^2} + y_2 = 1.64 \cdot ft$

Hence $y_2 = 1.31 \cdot ft$

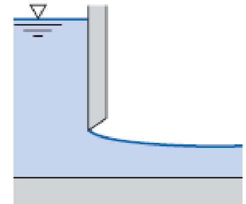
Note that $V_2 = V_1 \cdot \frac{y_1}{y_2}$ $\quad V_2 = 4.58 \cdot \frac{ft}{s}$

so we have $Fr_1 = \frac{V_1}{\sqrt{g \cdot y_1}}$ $\quad Fr_1 = 0.37$ $\quad$ and $\quad Fr_2 = \frac{V_2}{\sqrt{g \cdot y_2}}$ $\quad Fr_2 = 0.71$

Problem 11.27

[Difficulty: 2]

11.27 Water issues from a sluice gate at 1.25 m depth. The discharge per unit width is 10 $m^3/s/m$. Estimate the water level far upstream where the flow speed is negligible. Calculate the maximum rate of flow per unit width that could be delivered through the sluice gate.



Given: Data on sluice gate

Find: Water level upstream; Maximum flow rate

Solution:

Basic equation:

$$\frac{p_1}{\rho \cdot g} + \frac{V_1^2}{2 \cdot g} + y_1 = \frac{p_2}{\rho \cdot g} + \frac{V_2^2}{2 \cdot g} + y_2 + h$$

The Bernoulli equation applies because we have steady, incompressible, frictionless flow.

Noting that $p_1 = p_2 = p_{atm}$, and V_1 is approximately zero (1 = upstream, 2 = downstream) the Bernoulli equation becomes

$$y_1 = \frac{V_2^2}{2 \cdot g} + y_2$$

The given data is $\frac{Q}{b} = 10 \cdot \frac{m^2}{s}$ $\quad y_2 = 1.25 \cdot m$

Hence $Q = V_2 \cdot A_2 = V_2 \cdot b \cdot y_2$ or $V_2 = \frac{Q}{b \cdot y_2}$ $\quad V_2 = 8\,\frac{m}{s}$

Then upstream $y_1 = \left(\frac{V_2^2}{2 \cdot g} + y_2\right)$ $\quad y_1 = 4.51\,m$

The maximum flow rate occurs at critical conditions (see Section 11-2), for constant specific energy

In this case $V_2 = V_c = \sqrt{g \cdot y_c}$

Hence we find $y_1 = \frac{V_c^2}{2 \cdot g} + y_c = \frac{g \cdot y_c}{2 \cdot g} + y_c = \frac{3}{2} \cdot y_c$

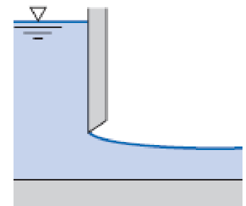
Hence $y_c = \frac{2}{3} \cdot y_1$ $\quad y_c = 3.01\,m$ $\quad V_c = \sqrt{g \cdot y_c}$ $\quad V_c = 5.43\,\frac{m}{s}$

$\frac{Q}{b} = V_c \cdot y_c$ $\quad \frac{Q}{b} = 16.3 \cdot \frac{\frac{m^3}{s}}{m}$ (Maximum flow rate)

Problem 11.28

[Difficulty: 2]

11.28 A horizontal rectangular channel 3 ft wide contains a sluice gate. Upstream of the gate the depth is 6 ft; the depth downstream is 0.9 ft. Estimate the volume flow rate in the channel.



Given: Data on sluice gate

Find: Flow rate

Solution:

Basic equation: $$\frac{p_1}{\rho \cdot g} + \frac{V_1^2}{2 \cdot g} + y_1 = \frac{p_2}{\rho \cdot g} + \frac{V_2^2}{2 \cdot g} + y_2$$

The Bernoulli equation applies because we have steady, incompressible, frictionless flow.

Noting that $p_1 = p_2 = p_{atm}$, (1 = upstream, 2 = downstream) the Bernoulli equation becomes

$$\frac{V_1^2}{2 \cdot g} + y_1 = \frac{V_2^2}{2 \cdot g} + y_2$$

The given data is $b = 3 \cdot ft$ $\quad y_1 = 6 \cdot ft$ $\quad y_2 = 0.9 \cdot ft$

Also $Q = V \cdot A$ so $V_1 = \frac{Q}{b \cdot y_1}$ and $V_2 = \frac{Q}{b \cdot y_2}$

Using these in the Bernoulli equation $$\frac{\left(\frac{Q}{b \cdot y_1}\right)^2}{2 \cdot g} + y_1 = \frac{\left(\frac{Q}{b \cdot y_2}\right)^2}{2 \cdot g} + y_2$$

Solving for Q $$Q = \sqrt{\frac{2 \cdot g \cdot b^2 \cdot y_1^2 \cdot y_2^2}{y_1 + y_2}} \qquad Q = 49.5 \cdot \frac{ft^3}{s}$$

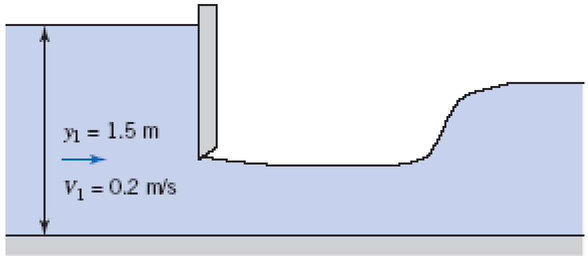
Note that $V_1 = \frac{Q}{b \cdot y_1}$ $\quad V_1 = 2.75 \cdot \frac{ft}{s}$ $\quad Fr_1 = \frac{V_1}{\sqrt{g \cdot y_1}}$ $\quad Fr_1 = 0.198$

$V_2 = \frac{Q}{b \cdot y_2}$ $\quad V_2 = 18.3 \cdot \frac{ft}{s}$ $\quad Fr_2 = \frac{V_2}{\sqrt{g \cdot y_2}}$ $\quad Fr_2 = 3.41$

Problem 11.29

[Difficulty: 2]

11.29 Flow through a sluice gate is shown. Estimate the water depth and velocity after the gate (well before the hydraulic jump).



Given: Data on sluice gate

Find: Water depth and velocity after gate

Solution:

Basic equation: $$E_1 = \frac{V_1^2}{2\cdot g} + y_1 = \frac{p_2}{\rho\cdot g} + \frac{V_2^2}{2\cdot g} = E_2$$ For the gate

$$\frac{y_3}{y_2} = \frac{1}{2}\cdot\left(-1 + \sqrt{1 + 8\cdot Fr_2^2}\right)$$ For the jump (state 2 before, state 3 after)

The given data is $y_1 = 1.5\cdot m$ $\quad V_1 = 0.2\cdot\frac{m}{s}$

Hence $q = y_1\cdot V_1$ $\quad q = 0.3\,\frac{m^2}{s}$ $\quad E_1 = \frac{V_1^2}{2\cdot g} + y_1$ $\quad E_1 = 1.50\,m$

Then we need to solve $\frac{V_2^2}{2\cdot g} + y_2 = E_1$ or $\frac{q^2}{2\cdot g\cdot y_2^2} + y_2 = E_1$ with $E_1 = 1.50\,m$

We can solve this equation iteratively (or use *Excel*'s *Goal Seek* or *Solver*)

For $y_2 = 0.5\cdot m$ $\quad \frac{\left(\frac{q}{y_2}\right)^2}{2\cdot g} + y_2 = 0.518\,m$ $\qquad$ For $y_2 = 0.05\cdot m$ $\quad \frac{\left(\frac{q}{y_2}\right)^2}{2\cdot g} + y_2 = 1.89\,m$

For $y_2 = 0.055\cdot m$ $\quad \frac{\left(\frac{q}{y_2}\right)^2}{2\cdot g} + y_2 = 1.57\,m$ $\qquad$ For $y_2 = 0.057\cdot m$ $\quad \frac{\left(\frac{q}{y_2}\right)^2}{2\cdot g} + y_2 = 1.47\,m$

For $y_2 = 0.0563\cdot m$ $\quad \frac{\left(\frac{q}{y_2}\right)^2}{2\cdot g} + y_2 = 1.50\,m$ $\qquad$ Hence $y_2 = 0.056\,m$ is the closest to three figs.

Then $V_2 = \frac{q}{y_2}$ $\quad V_2 = 5.33\,\frac{m}{s}$ $\qquad$ Note that $Fr_2 = \frac{V_2}{\sqrt{g\cdot y_2}}$ $\quad Fr_2 = 7.17$

Problem 11.30

[Difficulty: 2]

11.30 Rework Example 11.4 for a 350-mm-high bump and a side wall constriction that reduces the channel width to 1.5 m.

Given: Rectangular channel flow with hump and/or side wall restriction

Find: Whether critical flow occurs

Solution:

Basic equations: $y_c = \left(\frac{Q^2}{g \cdot b^2}\right)^{\frac{1}{3}}$ $\quad E = y + \frac{Q^2}{2 \cdot g \cdot A^2}$ $\quad A = b \cdot y$ $\quad E_{min} = \frac{3}{2} \cdot y_c$ (From Example 11.4)

Given data: $b = 2 \cdot m$ $\quad y = 1 \cdot m$ $\quad h = 350 \cdot mm$ $\quad Q = 2.4 \cdot \frac{m^3}{s}$

(a) For a hump with $h = 35 \cdot cm$ $\quad E_1 = y + \frac{Q^2}{2 \cdot g \cdot b^2} \cdot \frac{1}{y^2}$ $\quad E_1 = 1.07\,m$

Then for the bump $E_{bump} = E_1 - h$ $\quad E_{bump} = 0.723\,m$ (1)

For the minimum specific energy $y_c = \left[\frac{\left(\frac{Q}{b}\right)^2}{g}\right]^{\frac{1}{3}}$ $\quad y_c = 0.528\,m$ $\quad E_{min} = \frac{3}{2} \cdot y_c$ $\quad E_{min} = 0.791\,m$ (2)

Comparing Eqs. 1 and 2 we see that the bump IS sufficient for critical flow

(b) For the sidewall restriction with $b_{const} = 1.5 \cdot m$ as in Example 11.4 we have $E_{const} = E_1$ $\quad E_{const} = 1.073\,m$ (3)

With b_{const}: $y_c = \left[\frac{\left(\frac{Q}{b_{const}}\right)^2}{g}\right]^{\frac{1}{3}}$ $\quad y_c = 0.639\,m$ $\quad E_{minconst} = \frac{3}{2} \cdot y_c$ $\quad E_{minconst} = 0.959\,m$ (4)

Comparing Eqs. 3 and 4 we see that the constriction is NOT sufficient for critical flow

(c) For both, following Example 11.4 $E_{both} = E_1 - h$ $\quad E_{both} = 0.723\,m$ (5)

$E_{minboth} = E_{minconst}$ $\quad E_{minboth} = 0.959\,m$ (6)

Comparing Eqs. 5 and 6 we see that the bump AND constriction ARE sufficient for critical flow (not surprising, as the bump alone is sufficient!)

Problem 11.31

[Difficulty: 2]

11.31 Find the rate at which energy is being consumed (kW) by the hydraulic jump of Example 11.5. Is this sufficient to produce a significant temperature rise in the water?

Given: Hydaulic jump data

Find: Energy consumption; temperature rise

Solution:

Basic equations: $P = \rho \cdot g \cdot H_l \cdot Q$ (1)

H_l is the head loss in m of fluid); multiplying by ρg produces energy/vol; multiplying by Q produces energy/time, or power

$$U_{rate} = \rho \cdot Q \cdot c_{H2O} \cdot \Delta T \qquad (2)$$

U_{rate} is the rate of increase of internal energy of the flow; $c_{H20}\Delta T$ is the energy increase per unit mass due to a ΔT temperature rise; multiplying by ρQ converts to energy rise of the entire flow/time

Given data: From Example 11.5 $Q = 9.65 \cdot \frac{m^3}{s}$ $H_l = 0.258 \cdot m$ $\rho = 999 \cdot \frac{kg}{m^3}$ and $c_{H2O} = 1 \cdot \frac{kcal}{kg \cdot K}$

From Eq. 1 $P = \rho \cdot g \cdot H_l \cdot Q$ $P = 24.4\,kW$ a significant energy consumption

Equating Eqs. 1 and 2 $\rho \cdot g \cdot H_l \cdot Q = \rho \cdot Q \cdot c_{H2O} \cdot \Delta T$ or $\Delta T = \frac{g \cdot H_l}{c_{H2O}}$ $\Delta T = 6.043 \times 10^{-4}\,\Delta°C$

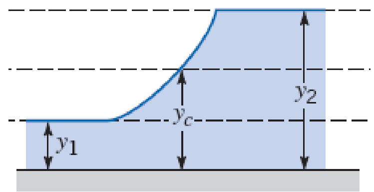
The power consumed by friction is quite large, but the flow is very large, so the rise in temperature is insignificant.
In English units:

$P = 32.7\,hp$ $Q = 1.53 \times 10^5\,gpm$ $\Delta T = 1.088 \times 10^{-3}\,\Delta°F$

Problem 11.32

[Difficulty: 2]

11.32 A hydraulic jump occurs in a rectangular channel 4.0 m wide. The water depth before the jump is 0.4 m and 1.7 m after the jump. Compute the flow rate in the channel, the critical depth, and the head loss in the jump.



Given: Data on rectangular channel and hydraulic jump

Find: Flow rate; Critical depth; Head loss

Solution:

Basic equations: $\frac{y_2}{y_1} = \frac{1}{2}\cdot\left(-1 + \sqrt{1 + 8\cdot Fr_1^{\,2}}\right)$ $\quad H_l = E_1 - E_2 = \left(y_1 + \frac{V_1^{\,2}}{2\cdot g}\right) - \left(y_2 + \frac{V_2^{\,2}}{2\cdot g}\right)$ $\quad y_c = \left(\frac{Q^2}{g\cdot b^2}\right)^{\frac{1}{3}}$

The given data is $\quad b = 4\cdot m \quad y_1 = 0.4\cdot m \quad y_2 = 1.7\cdot m$

We can solve for Fr_1 from the basic equation $\quad \sqrt{1 + 8\cdot Fr_1^{\,2}} = 1 + 2\cdot\frac{y_2}{y_1}$

$$Fr_1 = \sqrt{\frac{\left(1 + 2\cdot\frac{y_2}{y_1}\right)^2 - 1}{8}} \qquad Fr_1 = 3.34 \qquad \text{and} \qquad Fr_1 = \frac{V_1}{\sqrt{g\cdot y_1}}$$

Hence $\quad V_1 = Fr_1\cdot\sqrt{g\cdot y_1} \quad V_1 = 6.62\,\frac{m}{s}$

Then $\quad Q = V_1\cdot b\cdot y_1 \quad Q = 10.6\cdot\frac{m^3}{s}$

The critical depth is $\quad y_c = \left(\frac{Q^2}{g\cdot b^2}\right)^{\frac{1}{3}} \quad y_c = 0.894\,m$

Also $\quad V_2 = \frac{Q}{b\cdot y_2} \quad V_2 = 1.56\,\frac{m}{s} \quad Fr_2 = \frac{V_2}{\sqrt{g\cdot y_2}} \quad Fr_2 = 0.381$

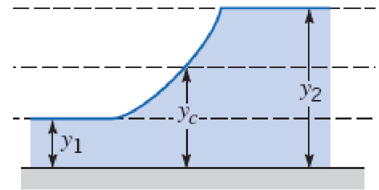
The energy loss is $\quad H_l = \left(y_1 + \frac{V_1^{\,2}}{2\cdot g}\right) - \left(y_2 + \frac{V_2^{\,2}}{2\cdot g}\right) \quad H_l = 0.808\,m$

Note that we could used $\quad H_l = \frac{(y_2 - y_1)^3}{4\cdot y_1\cdot y_2} \quad H_l = 0.808\,m$

Problem 11.33

[Difficulty: 2]

11.33 A wide channel carries 10 m^3/s per foot of width at a depth of 1 m at the toe of a hydraulic jump. Determine the depth of the jump and the head loss across it.



Given: Data on wide channel and hydraulic jump

Find: Jump depth; Head loss

Solution:

Basic equations:
$$\frac{y_2}{y_1} = \frac{1}{2}\cdot\left(-1 + \sqrt{1 + 8\cdot Fr_1^{\,2}}\right) \qquad H_l = E_1 - E_2 = \left(y_1 + \frac{V_1^{\,2}}{2\cdot g}\right) - \left(y_2 + \frac{V_2^{\,2}}{2\cdot g}\right)$$

The given data is
$$\frac{Q}{b} = 10\cdot\frac{\frac{m^3}{s}}{m} \qquad y_1 = 1\cdot m$$

Also
$$Q = V\cdot A = V\cdot b\cdot y$$

Hence
$$V_1 = \frac{Q}{b\cdot y_1} \qquad V_1 = 10.0\frac{m}{s} \qquad Fr_1 = \frac{V_1}{\sqrt{g\cdot y_1}} \qquad Fr_1 = 3.19$$

Then
$$y_2 = \frac{y_1}{2}\cdot\left(-1 + \sqrt{1 + 8\cdot Fr_1^{\,2}}\right) \qquad y_2 = 4.04\,m$$

$$V_2 = \frac{Q}{b\cdot y_2} \qquad V_2 = 2.47\frac{m}{s} \qquad Fr_2 = \frac{V_2}{\sqrt{g\cdot y_2}} \qquad Fr_2 = 0.393$$

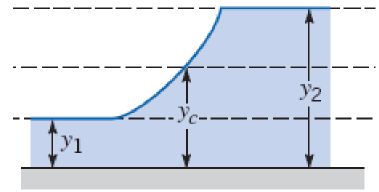
The energy loss is
$$H_l = \left(y_1 + \frac{V_1^{\,2}}{2\cdot g}\right) - \left(y_2 + \frac{V_2^{\,2}}{2\cdot g}\right) \qquad H_l = 1.74\,m$$

Note that we could use
$$H_l = \frac{\left(y_2 - y_1\right)^3}{4\cdot y_1\cdot y_2} \qquad H_l = 1.74\,m$$

Problem 11.34

[Difficulty: 1]

11.34 A hydraulic jump occurs in a wide horizontal channel. The discharge is 2 m^3/s per meter of width. The upstream depth is 750 mm. Determine the depth of the jump.



Given: Data on wide channel and hydraulic jump

Find: Jump depth

Solution:

Basic equations:

$$\frac{y_2}{y_1} = \frac{1}{2}\cdot\left(-1 + \sqrt{1 + 8\cdot Fr_1^{\,2}}\right)$$

The given data is

$$\frac{Q}{b} = 2\cdot\frac{\frac{m^3}{s}}{m} \qquad y_1 = 500\cdot mm$$

Also

$$Q = V\cdot A = V\cdot b\cdot y$$

Hence

$$V_1 = \frac{Q}{b\cdot y_1} \qquad V_1 = 4.00\frac{m}{s} \qquad Fr_1 = \frac{V_1}{\sqrt{g\cdot y_1}} \qquad Fr_1 = 1.806$$

Then

$$y_2 = \frac{y_1}{2}\cdot\left(-1 + \sqrt{1 + 8\cdot Fr_1^{\,2}}\right) \qquad y_2 = 1.05\cdot m$$

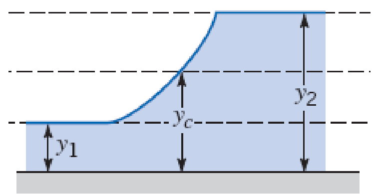
Note:

$$V_2 = \frac{Q}{b\cdot y_2} \qquad V_2 = 6.24\cdot\frac{ft}{s} \qquad Fr_2 = \frac{V_2}{\sqrt{g\cdot y_2}} \qquad Fr_2 = 0.592$$

Problem 11.35

[Difficulty: 2]

11.35 A hydraulic jump occurs in a rectangular channel. The flow rate is 200 ft^3/s, and the depth before the jump is 1.2 ft. Determine the depth behind the jump and the head loss, if the channel is 10 ft wide.



Given: Data on wide channel and hydraulic jump

Find: Jump depth; Head loss

Solution:

Basic equations:
$$\frac{y_2}{y_1} = \frac{1}{2}\cdot\left(-1 + \sqrt{1 + 8\cdot Fr_1^2}\right) \qquad H_l = E_1 - E_2 = \left(y_1 + \frac{V_1^2}{2\cdot g}\right) - \left(y_2 + \frac{V_2^2}{2\cdot g}\right)$$

The given data is
$$Q = 200\cdot\frac{ft^3}{s} \qquad b = 10\cdot ft \qquad y_1 = 1.2\cdot ft$$

Also
$$Q = V\cdot A = V\cdot b\cdot y$$

Hence
$$V_1 = \frac{Q}{b\cdot y_1} \qquad V_1 = 16.7\cdot\frac{ft}{s} \qquad Fr_1 = \frac{V_1}{\sqrt{g\cdot y_1}} \qquad Fr_1 = 2.68$$

Then
$$y_2 = \frac{y_1}{2}\cdot\left(-1 + \sqrt{1 + 8\cdot Fr_1^2}\right) \qquad y_2 = 3.99\cdot ft$$

$$V_2 = \frac{Q}{b\cdot y_2} \qquad V_2 = 5.01\cdot\frac{ft}{s} \qquad Fr_2 = \frac{V_2}{\sqrt{g\cdot y_2}} \qquad Fr_2 = 0.442$$

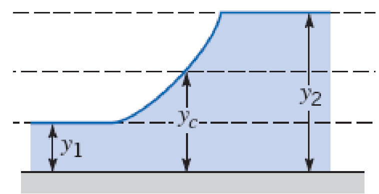
The energy loss is
$$H_l = \left(y_1 + \frac{V_1^2}{2\cdot g}\right) - \left(y_2 + \frac{V_2^2}{2\cdot g}\right) \qquad H_l = 1.14\cdot ft$$

Note that we could use
$$H_l = \frac{(y_2 - y_1)^3}{4\cdot y_1\cdot y_2} \qquad H_l = 1.14\cdot ft$$

Problem 11.36

[Difficulty: 2]

11.36 The hydraulic jump may be used as a crude flow meter. Suppose that in a horizontal rectangular channel 5 ft wide the observed depths before and after a hydraulic jump are 0.66 and 3.0 ft. Find the rate of flow and the head loss.



Given: Data on wide channel and hydraulic jump

Find: Flow rate; Head loss

Solution:

Basic equations: $$\frac{y_2}{y_1} = \frac{1}{2}\cdot\left(-1 + \sqrt{1 + 8\cdot Fr_1^{\,2}}\right) \qquad H_l = E_1 - E_2 = \left(y_1 + \frac{V_1^{\,2}}{2\cdot g}\right) - \left(y_2 + \frac{V_2^{\,2}}{2\cdot g}\right)$$

The given data is $b = 5\cdot ft$ $\quad y_1 = 0.66\cdot ft$ $\quad y_2 = 3.0\cdot ft$

We can solve for Fr_1 from the basic equation $$\sqrt{1 + 8\cdot Fr_1^{\,2}} = 1 + 2\cdot\frac{y_2}{y_1}$$

$$Fr_1 = \sqrt{\frac{\left(1 + 2\cdot\frac{y_2}{y_1}\right)^2 - 1}{8}} \qquad Fr_1 = 3.55 \qquad \text{and} \qquad Fr_1 = \frac{V_1}{\sqrt{g\cdot y_1}}$$

Hence $\quad V_1 = Fr_1\cdot\sqrt{g\cdot y_1} \qquad V_1 = 16.4\cdot\frac{ft}{s}$

Then $\quad Q = V_1\cdot b\cdot y_1 \qquad Q = 54.0\cdot\frac{ft^3}{s}$

Also $\quad V_2 = \frac{Q}{b\cdot y_2} \qquad V_2 = 3.60\cdot\frac{ft}{s} \qquad Fr_2 = \frac{V_2}{\sqrt{g\cdot y_2}} \qquad Fr_2 = 0.366$

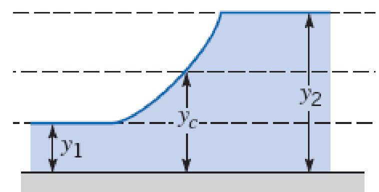
The energy loss is $$H_l = \left(y_1 + \frac{V_1^{\,2}}{2\cdot g}\right) - \left(y_2 + \frac{V_2^{\,2}}{2\cdot g}\right) \qquad H_l = 1.62\cdot ft$$

Note that we could use $$H_l = \frac{(y_2 - y_1)^3}{4\cdot y_1\cdot y_2} \qquad H_l = 1.62\cdot ft$$

Problem 11.37

[Difficulty: 2]

11.37 A hydraulic jump occurs on a horizontal apron downstream from a wide spillway, at a location where depth is 0.9 m and speed is 25 m/s. Estimate the depth and speed downstream from the jump. Compare the specific energy downstream of the jump to that upstream.



Given: Data on wide spillway flow

Find: Depth after hydraulic jump; Specific energy change

Solution:

Basic equations:
$$\frac{y_2}{y_1} = \frac{1}{2}\cdot\left(-1 + \sqrt{1 + 8\cdot Fr_1^{\,2}}\right) \qquad H_l = E_1 - E_2 = \left(y_1 + \frac{V_1^{\,2}}{2\cdot g}\right) - \left(y_2 + \frac{V_2^{\,2}}{2\cdot g}\right)$$

The given data is
$$y_1 = 0.9\cdot m \qquad V_1 = 25\frac{m}{s}$$

Then Fr_1 is
$$Fr_1 = \frac{V_1}{\sqrt{g\cdot y_1}} \qquad Fr_1 = 8.42$$

Hence
$$y_2 = \frac{y_1}{2}\cdot\left(-1 + \sqrt{1 + 8\cdot Fr_1^{\,2}}\right) \qquad y_2 = 10.3\,m$$

Then
$$Q = V_1\cdot b\cdot y_1 = V_2\cdot b\cdot y_2 \qquad V_2 = V_1\cdot\frac{y_1}{y_2} \qquad V_2 = 2.19\frac{m}{s}$$

For the specific energies
$$E_1 = y_1 + \frac{V_1^{\,2}}{2\cdot g} \qquad E_1 = 32.8\,m$$

$$E_2 = y_2 + \frac{V_2^{\,2}}{2\cdot g} \qquad E_2 = 10.5\,m \qquad \frac{E_2}{E_1} = 0.321$$

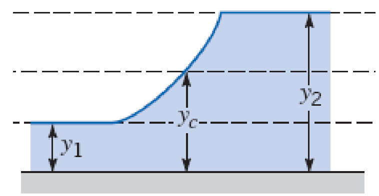
The energy loss is
$$H_l = E_1 - E_2 \qquad H_l = 22.3\,m$$

Note that we could use
$$H_l = \frac{\left(y_2 - y_1\right)^3}{4\cdot y_1\cdot y_2} \qquad H_l = 22.3\cdot m$$

Problem 11.38

[Difficulty: 2]

11.38 A hydraulic jump occurs in a rectangular channel. The flow rate is 50 m^3/s, and the depth before the jump is 2 m. Determine the depth after the jump and the head loss, if the channel is 1 m wide.



Given: Data on rectangular channel flow

Find: Depth after hydraulic jump; Specific energy change

Solution:

Basic equations:
$$\frac{y_2}{y_1} = \frac{1}{2}\cdot\left(-1 + \sqrt{1 + 8\cdot Fr_1^{\ 2}}\right) \qquad H_l = E_1 - E_2 = \left(y_1 + \frac{V_1^{\ 2}}{2\cdot g}\right) - \left(y_2 + \frac{V_2^{\ 2}}{2\cdot g}\right)$$

The given data is
$$y_1 = 0.4\cdot m \qquad b = 1\cdot m \qquad Q = 6.5\frac{m^3}{s}$$

Then
$$Q = V_1\cdot b\cdot y_1 = V_2\cdot b\cdot y_2 \qquad V_1 = \frac{Q}{b\cdot y_1} \qquad V_1 = 16.3\frac{m}{s}$$

Then Fr_1 is
$$Fr_1 = \frac{V_1}{\sqrt{g\cdot y_1}} \qquad Fr_1 = 8.20$$

Hence
$$y_2 = \frac{y_1}{2}\cdot\left(-1 + \sqrt{1 + 8\cdot Fr_1^{\ 2}}\right) \qquad y_2 = 4.45\,m$$

and
$$V_2 = \frac{Q}{b\cdot y_2} \qquad V_2 = 1.46\frac{m}{s}$$

For the specific energies
$$E_1 = y_1 + \frac{V_1^{\ 2}}{2\cdot g} \qquad E_1 = 13.9\,m$$

$$E_2 = y_2 + \frac{V_2^{\ 2}}{2\cdot g} \qquad E_2 = 4.55\,m$$

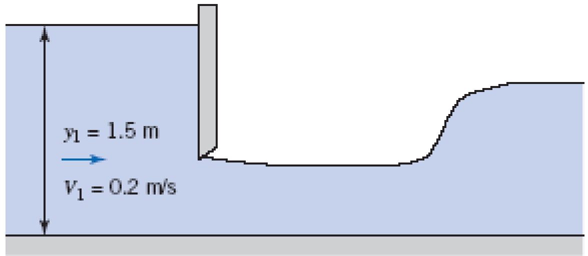
The energy loss is
$$H_l = E_1 - E_2 \qquad H_l = 9.31\,m$$

Note that we could use
$$H_l = \frac{\left(y_2 - y_1\right)^3}{4\cdot y_1\cdot y_2} \qquad H_l = 9.31\cdot m$$

Problem 11.39

[Difficulty: 3]

11.39 Estimate the depth of water before and after the jump for the hydraulic jump downstream of the sluice gate of Fig. P11.29.



Given: Data on sluice gate

Find: Water depth before and after the jump

Solution:

Basic equation: $$E_1 = \frac{V_1^2}{2\cdot g} + y_1 = \frac{p_2}{\rho\cdot g} + \frac{V_2^2}{2\cdot g} = E_2$$ For the gate

$$\frac{y_3}{y_2} = \frac{1}{2}\cdot\left(-1 + \sqrt{1 + 8\cdot Fr_2^2}\right)$$ For the jump (state 2 before, state 3 after)

The given data is $y_1 = 1.5\cdot m$ $\quad V_1 = 0.2\cdot\frac{m}{s}$

Hence $q = y_1\cdot V_1$ $\quad q = 0.3\,\frac{m^2}{s}$ $\quad E_1 = \frac{V_1^2}{2\cdot g} + y_1$ $\quad E_1 = 1.50\,m$

Then we need to solve $\frac{V_2^2}{2\cdot g} + y_2 = E_1$ or $\frac{q^2}{2\cdot g\cdot y_2^2} + y_2 = E_1$ with $E_1 = 1.50\,m$

We can solve this equation iteratively (or use *Excel*'s *Goal Seek* or *Solver*)

For $y_2 = 0.5\cdot m$ $\quad \frac{\left(\frac{q}{y_2}\right)^2}{2\cdot g} + y_2 = 0.518\,m$ For $y_2 = 0.05\cdot m$ $\quad \frac{\left(\frac{q}{y_2}\right)^2}{2\cdot g} + y_2 = 1.89\,m$

For $y_2 = 0.055\cdot m$ $\quad \frac{\left(\frac{q}{y_2}\right)^2}{2\cdot g} + y_2 = 1.57\,m$ For $y_2 = 0.057\cdot m$ $\quad \frac{\left(\frac{q}{y_2}\right)^2}{2\cdot g} + y_2 = 1.47\,m$

For $y_2 = 0.0563\cdot m$ $\quad \frac{\left(\frac{q}{y_2}\right)^2}{2\cdot g} + y_2 = 1.50\,m$ Hence $y_2 = 0.056\,m$ is the closest to three figs.

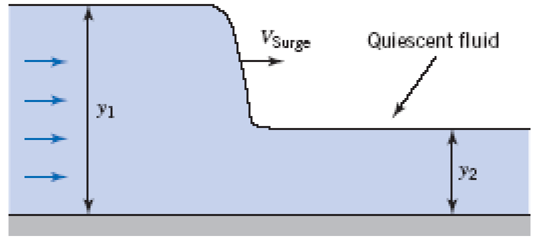
Then $V_2 = \frac{q}{y_2}$ $\quad V_2 = 5.33\,\frac{m}{s}$ Note that $Fr_2 = \frac{V_2}{\sqrt{g\cdot y_2}}$ $\quad Fr_2 = 7.17$

For the jump (States 2 to 3) $$y_3 = \frac{y_2}{2}\cdot\left(-1 + \sqrt{1 + 8\cdot Fr_2^2}\right)$$ $y_3 = 0.544\,m$

Problem 11.40

[Difficulty: 3]

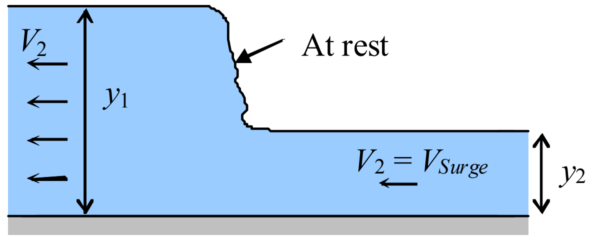
11.40 A positive surge wave, or moving hydraulic jump, can be produced in the laboratory by suddenly opening a sluice gate. Consider a surge of depth y_2 advancing into a quiescent channel of depth y_1. Obtain an expression for surge speed in terms of y_1 and y_2.



Given: Surge wave

Find: Surge speed

Solution:



Basic equations:

$$\frac{V_1^2 \cdot y_1}{g} + \frac{y_1^2}{2} = \frac{V_2^2 \cdot y_2}{g} + \frac{y_2^2}{2}$$

(This is the basic momentum equation for the flow)

$$V_1 \cdot y_1 = V_2 \cdot y_2 \qquad \text{or} \qquad \frac{V_1}{V_2} = \frac{y_2}{y_1}$$

Then

$$y_2^2 - y_1^2 = \frac{2}{g} \cdot \left(V_1^2 \cdot y_1 - V_2^2 \cdot y_2\right) = \frac{2 \cdot V_2^2}{g} \cdot \left[\left(\frac{V_1}{V_2}\right)^2 \cdot y_1 - y_2\right] = \frac{2 \cdot V_2^2}{g} \cdot \left[\left(\frac{y_2}{y_1}\right)^2 \cdot y_1 - y_2\right]$$

$$y_2^2 - y_1^2 = \frac{2 \cdot V_2^2}{g} \cdot \left(\frac{y_2^2}{y_1} - y_2\right) = \frac{2 \cdot V_2^2 \cdot y_2}{g} \cdot \frac{(y_2 - y_1)}{y_1}$$

Dividing by $(y_2 - y_1)$

$$y_2 + y_1 = 2 \cdot \frac{V_2^2}{g} \cdot \frac{y_2}{y_1} \qquad \text{or} \qquad V_2^2 = \frac{g}{2} \cdot y_1 \cdot \frac{(y_2 + y_1)}{y_2}$$

$$V_2 = \sqrt{\frac{g \cdot y_1}{2} \cdot \left(1 + \frac{y_1}{y_2}\right)}$$

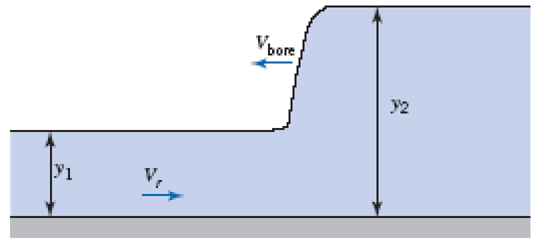
But

$$V_2 = V_{Surge} \qquad \text{so} \qquad V_{Surge} = \sqrt{\frac{g \cdot y_1}{2} \cdot \left(1 + \frac{y_1}{y_2}\right)}$$

Problem 11.41

[Difficulty: 3]

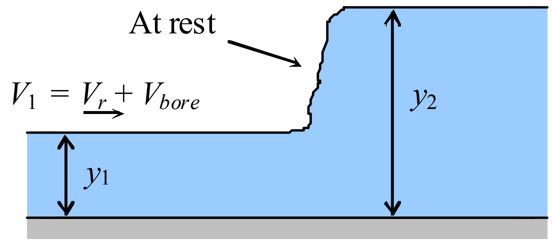
11.41 A tidal bore (an abrupt translating wave or moving hydraulic jump) often forms when the tide flows into the wide estuary of a river. In one case, a bore is observed to have a height of 12 ft above the undisturbed level of the river that is 8 ft deep. The bore travels upstream at $V_{bore} = 18$ mph. Determine the approximate speed V_r of the current of the undisturbed river.



Given: Tidal bore

Find: Speed of undisturbed river

Solution:



Basic equations:
$$\frac{V_2^2 \cdot y_2}{g} + \frac{y_2^2}{2} = \frac{V_1^2 \cdot y_1}{g} + \frac{y_1^2}{2}$$

(This is the basic momentum equation for the flow)

$$V_2 \cdot y_2 = V_1 \cdot y_1 \qquad \text{or} \qquad \frac{V_2}{V_1} = \frac{y_1}{y_2}$$

Given data $\quad V_{bore} = 18 \cdot mph \quad V_{bore} = 26.4 \cdot \frac{ft}{s} \quad y_1 = 8 \cdot ft \quad y_2 = y_1 + 12 \cdot ft \quad y_2 = 20 \cdot ft$

Then
$$y_1^2 - y_2^2 = \frac{2}{g} \cdot \left(V_2^2 \cdot y_2 - V_1^2 \cdot y_1\right) = \frac{2 \cdot V_1^2}{g} \cdot \left[\left(\frac{V_2}{V_1}\right)^2 \cdot y_2 - y_1\right] = \frac{2 \cdot V_1^2}{g} \cdot \left[\left(\frac{y_1}{y_2}\right)^2 \cdot y_2 - y_1\right]$$

$$y_1^2 - y_2^2 = \frac{2 \cdot V_1^2}{g} \cdot \left(\frac{y_1^2}{y_2} - y_1\right) = \frac{2 \cdot V_1^2 \cdot y_1}{g} \cdot \frac{(y_1 - y_2)}{y_2}$$

Dividing by $(y_2 - y_1)$
$$y_1 + y_2 = 2 \cdot \frac{V_1^2}{g} \cdot \frac{y_1}{y_2} \qquad \text{or} \qquad V_1^2 = \frac{g}{2} \cdot y_2 \cdot \frac{(y_1 + y_2)}{y_1}$$

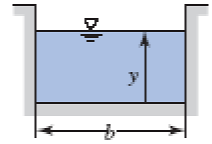
$$V_1 = \sqrt{\frac{g \cdot y_2}{2} \cdot \left(1 + \frac{y_2}{y_1}\right)} \qquad V_1 = 33.6 \cdot \frac{ft}{s} \qquad V_1 = 22.9 \cdot mph$$

But
$$V_1 = V_r + V_{bore} \qquad \text{or} \qquad V_r = V_1 - V_{bore} \qquad V_r = 7.16 \cdot \frac{ft}{s} \qquad V_r = 4.88 \cdot mph$$

Problem 11.42

[Difficulty: 1]

11.42 A 2-m-wide rectangular channel with a bed slope of 0.0005 has a depth of flow of 1.5 m. Manning's roughness coefficient is 0.015. Determine the steady uniform discharge in the channel.



Given: Rectangular channel flow

Find: Discharge

Solution:

Basic equation: $$Q = \frac{1}{n} \cdot A \cdot R_h^{\frac{2}{3}} \cdot S_b^{\frac{1}{2}}$$

Note that this is an "engineering" equation, to be used without units!

For a rectangular channel of width $b = 2 \cdot m$ and depth $y = 1.5 \cdot m$ we find from Table 11.1

$$A = b \cdot y \qquad A = 3.00 \cdot m^2 \qquad R_h = \frac{b \cdot y}{b + 2 \cdot y} \qquad R_h = 0.600 \cdot m$$

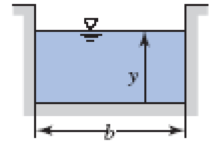
Manning's roughness coefficient is $n = 0.015$ and $S_b = 0.0005$

$$Q = \frac{1.49}{n} \cdot A \cdot R_h^{\frac{2}{3}} \cdot S_b^{\frac{1}{2}} \qquad Q = 3.18 \cdot \frac{m^3}{s}$$

Problem 11.43

[Difficulty: 3]

11.43 Determine the uniform flow depth in a rectangular channel 2.5 m wide with a discharge of 3 m^3/s. The slope is 0.0004 and Manning's roughness factor is 0.015.



Given: Data on rectangular channel

Find: Depth of flow

Solution:

Basic equation: $$Q = \frac{1}{n} \cdot A \cdot R_h^{\frac{2}{3}} \cdot S_b^{\frac{1}{2}}$$

Note that this is an "engineering" equation, to be used without units!

For a rectangular channel of width $b = 2.5 \cdot m$ and flow rate $Q = 3 \cdot \frac{m^3}{s}$ we find from Table 11.1 $A = b \cdot y$ $R = \frac{b \cdot y}{b + 2 \cdot y}$

Manning's roughness coefficient is $n = 0.015$ and $S_b = 0.0004$

Hence the basic equation becomes $$Q = \frac{1}{n} \cdot b \cdot y \cdot \left(\frac{b \cdot y}{b + 2 \cdot y}\right)^{\frac{2}{3}} \cdot S_b^{\frac{1}{2}}$$

Solving for y $$y \cdot \left(\frac{b \cdot y}{b + 2 \cdot y}\right)^{\frac{2}{3}} = \frac{Q \cdot n}{b \cdot S_b^{\frac{1}{2}}}$$

This is a nonlinear implicit equation for y and must be solved numerically. We can use one of a number of numerical root finding techniques, such as Newton's method, or we can use *Excel*'s *Solver* or *Goal Seek*, or we can manually iterate, as below, to make the left side evaluate to $\frac{Q \cdot n}{b \cdot S_b^{\frac{1}{2}}} = 0.900$.

For $y = 1$ (m) $y \cdot \left(\frac{b \cdot y}{b + 2 \cdot y}\right)^{\frac{2}{3}} = 0.676$ For $y = 1.2$ (m) $y \cdot \left(\frac{b \cdot y}{b + 2 \cdot y}\right)^{\frac{2}{3}} = 0.865$

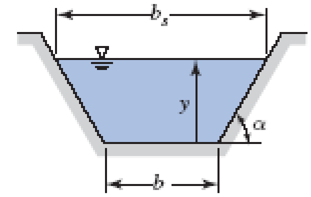
For $y = 1.23$ (m) $y \cdot \left(\frac{b \cdot y}{b + 2 \cdot y}\right)^{\frac{2}{3}} = 0.894$ For $y = 1.24$ (m) $y \cdot \left(\frac{b \cdot y}{b + 2 \cdot y}\right)^{\frac{2}{3}} = 0.904$

The solution to three figures is $y = 1.24$ (m)

Problem 11.44

[Difficulty: 3]

11.44 Determine the uniform flow depth in a trapezoidal channel with a bottom width of 8 ft and side slopes of 1 vertical to 2 horizontal. The discharge is 100 ft^3/s. Manning's roughness factor is 0.015 and the channel bottom slope is 0.0004.



Given: Data on trapzoidal channel

Find: Depth of flow

Solution:

Basic equation: $$Q = \frac{1.49}{n} \cdot A \cdot R_h^{\frac{2}{3}} \cdot S_b^{\frac{1}{2}}$$

Note that this is an "engineering" equation, to be used without units!

For the trapezoidal channel we have $b = 8 \cdot ft$ $\quad \alpha = \operatorname{atan}\left(\frac{1}{2}\right)$ $\quad \alpha = 26.6deg$ $\quad Q = 100 \cdot \frac{ft^3}{s}$ $\quad S_0 = 0.0004$

$n = 0.015$

Hence from Table 11.1 $$A = y \cdot (b + y \cdot \cot(\alpha)) = y \cdot (8 + 2 \cdot y)$$ $$R_h = \frac{y \cdot (b + y \cdot \cot(\alpha))}{b + \frac{2 \cdot y}{\sin(\alpha)}} = \frac{y \cdot (8 + 2 \cdot y)}{8 + 2 \cdot y \cdot \sqrt{5}}$$

Hence $$Q = \frac{1.49}{n} \cdot A \cdot R_h^{\frac{2}{3}} \cdot S_b^{\frac{1}{2}} = \frac{1.49}{0.015} \cdot y \cdot (8 + 2 \cdot y) \cdot y \cdot \left[\frac{y \cdot (8 + 2 \cdot y)}{8 + 2 \cdot y \cdot \sqrt{5}}\right]^{\frac{2}{3}} \cdot 0.0004^{\frac{1}{2}} = 100$$ (Note that we don't use units!)

Solving for y $$\frac{[y \cdot (8 + 2 \cdot y)]^{\frac{5}{3}}}{\left(8 + 2 \cdot y \cdot \sqrt{5}\right)^{\frac{2}{3}}} = 50.3$$

This is a nonlinear implicit equation for y and must be solved numerically. We can use one of a number of numerical root finding techniques, such as Newton's method, or we can use *Excel*'s *Solver* or *Goal Seek*, or we can manually iterate, as below.

For $y = 2$ (ft) $$\frac{[y \cdot (8 + 2 \cdot y)]^{\frac{5}{3}}}{\left(8 + 2 \cdot y \cdot \sqrt{5}\right)^{\frac{2}{3}}} = 30.27$$

For $y = 3$ (ft) $$\frac{[y \cdot (8 + 2 \cdot y)]^{\frac{5}{3}}}{\left(8 + 2 \cdot y \cdot \sqrt{5}\right)^{\frac{2}{3}}} = 65.8$$

For $y = 2.6$ (ft) $$\frac{[y \cdot (8 + 2 \cdot y)]^{\frac{5}{3}}}{\left(8 + 2 \cdot y \cdot \sqrt{5}\right)^{\frac{2}{3}}} = 49.81$$

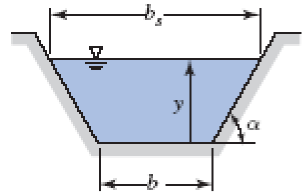
For $y = 2.61$ (ft) $$\frac{[y \cdot (8 + 2 \cdot y)]^{\frac{5}{3}}}{\left(8 + 2 \cdot y \cdot \sqrt{5}\right)^{\frac{2}{3}}} = 50.18$$

The solution to three figures is $y = 2.61$ (ft)

Problem 11.45

[Difficulty: 3]

11.45 Determine the uniform flow depth in a trapezoidal channel with a bottom width of 2.5 m and side slopes of 1 vertical to 2 horizontal with a discharge of 3 m³/s. The slope is 0.0004 and Manning's roughness factor is 0.015.



Given: Data on trapezoidal channel

Find: Depth of flow

Solution:

Basic equation: $$Q = \frac{1}{n}\cdot A\cdot R_h^{\frac{2}{3}}\cdot S_b^{\frac{1}{2}}$$

Note that this is an "engineering" equation, to be used without units!

For the trapezoidal channel we have $b = 2.5\cdot m$ $\quad \alpha = \operatorname{atan}\left(\frac{1}{2}\right)$ $\quad \alpha = 26.6\,deg$ $\quad Q = 3\cdot\frac{m^3}{s}$ $\quad S_0 = 0.0004$

$n = 0.015$

Hence from Table 11.1 $\quad A = y\cdot(b + \cot(\alpha)\cdot y) = y\cdot(8 + 2\cdot y)$ $\quad R = \dfrac{y\cdot(b + y\cdot\cot(\alpha))}{b + \dfrac{2\cdot y}{\cot(\alpha)}} = \dfrac{y\cdot(2.5 + 2\cdot y)}{2.5 + 2\cdot y\cdot\sqrt{5}}$

Hence $$Q = \frac{1}{n}\cdot A\cdot R_h^{\frac{2}{3}}\cdot S_b^{\frac{1}{2}} = \frac{1}{0.015}\cdot y\cdot(2.5 + 2\cdot y)\cdot\left[\frac{(2.5 + 2\cdot y)\cdot y}{2.5 + 2\cdot y\cdot\sqrt{5}}\right]^{\frac{2}{3}}\cdot 0.0004^{\frac{1}{2}} = 3$$ (Note that we don't use units!)

Solving for y $$\frac{[y\cdot(2.5 + 2\cdot y)]^{\frac{5}{3}}}{\left(2.5 + 2\cdot y\cdot\sqrt{5}\right)^{\frac{2}{3}}} = 2.25$$

This is a nonlinear implicit equation for y and must be solved numerically. We can use one of a number of numerical root finding techniques, such as Newton's method, or we can use *Excel*'s *Solver* or *Goal Seek*, or we can manually iterate, as below.

For $y = 1$ (m) $$\frac{[y\cdot(2.5 + 2\cdot y)]^{\frac{5}{3}}}{\left(2.5 + 2\cdot y\cdot\sqrt{5}\right)^{\frac{2}{3}}} = 3.36$$

For $y = 0.8$ (m) $$\frac{[y\cdot(2.5 + 2\cdot y)]^{\frac{5}{3}}}{\left(2.5 + 2\cdot y\cdot\sqrt{5}\right)^{\frac{2}{3}}} = 2.17$$

For $y = 0.81$ (m) $$\frac{[y\cdot(2.5 + 2\cdot y)]^{\frac{5}{3}}}{\left(2.5 + 2\cdot y\cdot\sqrt{5}\right)^{\frac{2}{3}}} = 2.23$$

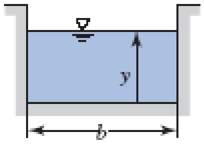
For $y = 0.815$ (m) $$\frac{[y\cdot(2.5 + 2\cdot y)]^{\frac{5}{3}}}{\left(2.5 + 2\cdot y\cdot\sqrt{5}\right)^{\frac{2}{3}}} = 2.25$$

The solution to three figures is $\quad y = 0.815$ (m)

Problem 11.46

[Difficulty: 1]

11.46 A rectangular flume built of concrete, with 1 ft per 1000 ft slope, is 6 ft wide. Water flows at a normal depth of 3 ft. Compute the discharge.



Given: Data on flume

Find: Discharge

Solution:

Basic equation: $Q = \frac{1.49}{n} \cdot A \cdot R_h^{\frac{2}{3}} \cdot S_b^{\frac{1}{2}}$

Note that this is an "engineering" equation, to be used without units!

For a rectangular channel of width $b = 6 \cdot ft$ and depth $y = 3 \cdot ft$ we find from Table 11.1

$A = b \cdot y$ $\quad A = 18 \cdot ft^2$ $\quad R_h = \frac{b \cdot y}{b + 2 \cdot y}$ $\quad R_h = 1.50 \cdot ft$

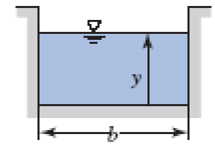
For concrete (Table 11.2) $\quad n = 0.013$ and $\quad S_b = \frac{1 \cdot ft}{1000 \cdot ft}$ $\quad S_b = 0.001$

$Q = \frac{1.49}{n} \cdot A \cdot R_h^{\frac{2}{3}} \cdot S_b^{\frac{1}{2}}$ $\quad Q = 85.5 \cdot \frac{ft^3}{s}$

Problem 11.47

[Difficulty: 1]

11.47 A rectangular flume built of timber is 3 ft wide. The flume is to handle a flow of 90 ft^3/s at a normal depth of 6 ft. Determine the slope required.



Given: Data on flume

Find: Slope

Solution:

Basic equation: $$Q = \frac{1.49}{n} \cdot A \cdot R_h^{\frac{2}{3}} \cdot S_b^{\frac{1}{2}}$$

Note that this is an "engineering" equation, to be used without units!

For a rectangular channel of width $b = 3 \cdot ft$ and depth $y = 6 \cdot ft$ we find

$$A = b \cdot y \qquad A = 18 \cdot ft^2 \qquad R_h = \frac{b \cdot y}{b + 2 \cdot y} \qquad R_h = 1.20 \cdot ft$$

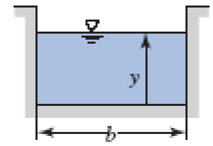
For wood (not in Table 11.2) a Google search finds n = 0.012 to 0.017; we use $n = 0.0145$ with $Q = 90 \cdot \frac{ft^3}{s}$

$$S_b = \left(\frac{n \cdot Q}{1.49 \cdot A \cdot R_h^{\frac{2}{3}}} \right)^2 \qquad S_b = 1.86 \times 10^{-3}$$

Problem 11.48

[Difficulty: 2]

11.48 A channel with square cross section is to carry 20 m^3/s of water at normal depth on a slope of 0.003. Compare the dimensions of the channel required for (a) concrete and (b) soil cement.



Given:
Data on square channel

Find:
Dimensions for concrete and soil cement

Solution:

Basic equation: $$Q = \frac{1}{n}\cdot A\cdot R_h^{\frac{2}{3}}\cdot S_b^{\frac{1}{2}}$$

Note that this is an "engineering" equation, to be used without units!

For a square channel of width b we find $A = b^2$ $$R = \frac{b\cdot y}{b + 2\cdot y} = \frac{b^2}{b + 2\cdot b} = \frac{b}{3}$$

Hence $$Q = \frac{1}{n}\cdot b^2\cdot\left(\frac{b}{3}\right)^{\frac{2}{3}}\cdot S_b^{\frac{1}{2}} = \frac{S_b^{\frac{1}{2}}}{n\cdot 3^{\frac{2}{3}}}\cdot b^{\frac{8}{3}}$$ or $$b = \left(\frac{3^{\frac{2}{3}}\cdot Q}{S_b^{\frac{1}{2}}}\cdot n\right)^{\frac{3}{8}}$$

The given data is $Q = 20\cdot\frac{m^3}{s}$ $S_b = 0.003$

For concrete, from Table 11.2 (assuming large depth) $n = .013$

$b = 2.36\,m$

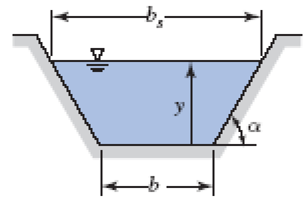
For soil cement from Table 11.2 (assuming large depth) $n = .020$

$b = 2.77\,m$

Problem 11.49

[Difficulty: 1]

11.49 Water flows in a trapezoidal channel at a normal depth of 1.2 m. The bottom width is 2.4 m and the sides slope at 1:1 (45°). The flow rate is 7.1 m^3/s. The channel is excavated from bare soil. Find the bed slope.



Given: Data on trapezoidal channel

Find: Bed slope

Solution:

Basic equation: $Q = \frac{1}{n} \cdot A \cdot R_h^{\frac{2}{3}} \cdot S_b^{\frac{1}{2}}$

Note that this is an "engineering" equation, to be used without units!

For the trapezoidal channel we have $b = 2.4 \cdot m$ $\quad \alpha = 45 \cdot deg$ $\quad y = 1.2 \cdot m$ $\quad Q = 7.1 \cdot \frac{m^3}{s}$

For bare soil (Table 11.2) $n = 0.020$

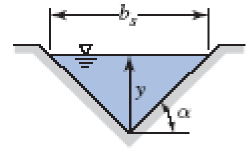
Hence from Table 11.1 $A = y \cdot (b + \cot(\alpha) \cdot y)$ $\quad A = 4.32\, m^2$ $\quad R_h = \dfrac{y \cdot (b + y \cdot \cot(\alpha))}{b + \dfrac{2 \cdot y}{\sin(\alpha)}}$ $\quad R_h = 0.746\, m$

Hence $S_b = \left(\dfrac{Q \cdot n}{A \cdot R_h^{\frac{2}{3}}} \right)^2$ $\quad S_b = 1.60 \times 10^{-3}$

Problem 11.50

[Difficulty: 1]

11.50 A triangular channel with side angles of 45° is to carry 10 m^3/s at a slope of 0.001. The channel is concrete. Find the required dimensions.



Given: Data on triangular channel

Find: Required dimensions

Solution:

Basic equation: $$Q = \frac{1}{n}\cdot A\cdot R_h^{\frac{2}{3}}\cdot S_b^{\frac{1}{2}}$$

Note that this is an "engineering" equation, to be used without units!

For the triangular channel we have $\alpha = 45\cdot deg$ $\quad S_b = 0.001$ $\quad Q = 10\cdot\frac{m^3}{s}$

For concrete (Table 11.2) $n = 0.013$ (assuming y > 60 cm: verify later)

Hence from Table 11.1 $A = y^2\cdot\cot(\alpha) = y^2$ $\quad R_h = \frac{y\cdot\cos(\alpha)}{2} = \frac{y}{2\cdot\sqrt{2}}$

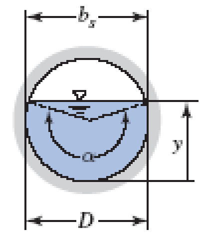
Hence $$Q = \frac{1}{n}\cdot A\cdot R_h^{\frac{2}{3}}\cdot S_b^{\frac{1}{2}} = \frac{1}{n}\cdot y^2\cdot\left(\frac{y}{2\cdot\sqrt{2}}\right)^{\frac{2}{3}}\cdot S_b = \frac{1}{n}\cdot y^{\frac{8}{3}}\cdot\left(\frac{1}{8}\right)^{\frac{1}{3}}\cdot S_b^{\frac{1}{2}} = \frac{1}{2\cdot n}\cdot y^{\frac{8}{3}}\cdot S_b^{\frac{1}{2}}$$

Solving for y $$y = \left(\frac{2\cdot n\cdot Q}{\sqrt{S_b}}\right)^{\frac{3}{8}}$$ $y = 2.20\,m$ (The assumption that y > 60 cm is verified)

Problem 11.51

[Difficulty: 2]

11.51 A semicircular trough of corrugated steel, with diameter $D = 1$ m, carries water at depth $y = 0.25$ m. The slope is 0.01. Find the discharge.



Given:
Data on semicircular trough

Find:
Discharge

Solution:

Basic equation: $Q = \frac{1}{n} \cdot A \cdot R_h^{\frac{2}{3}} \cdot S_b^{\frac{1}{2}}$

Note that this is an "engineering" equation, to be used without units!

For the semicircular channel $D = 1 \cdot m$ $\quad y = 0.25 \cdot m$ $\quad S_b = 0.01$

Hence, from geometry $\alpha = 2 \cdot \text{asin}\left(\frac{y - \frac{D}{2}}{\frac{D}{2}}\right) + 180 \cdot deg$ $\quad \alpha = 120 \cdot deg$

For corrugated steel, a Google search leads to $n = 0.022$

Hence from Table 11.1 $A = \frac{1}{8} \cdot (\alpha - \sin(\alpha)) \cdot D^2$ $\quad A = 0.154\,m^2$

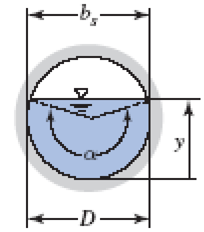
$R_h = \frac{1}{4} \cdot \left(1 - \frac{\sin(\alpha)}{\alpha}\right) \cdot D$ $\quad R_h = 0.147\,m$

Then the discharge is $Q = \frac{1}{n} \cdot A \cdot R_h^{\frac{2}{3}} \cdot S_b^{\frac{1}{2}} \cdot \frac{m^3}{s}$ $\quad Q = 0.194\,\frac{m^3}{s}$

Problem 11.52

[Difficulty: 1]

11.52 Find the discharge at which the channel of Problem 11.51 flows full.



Given: Data on semicircular trough

Find: Discharge

Solution:

Basic equation: $$Q = \frac{1}{n} \cdot A \cdot R_h^{\frac{2}{3}} \cdot S_b^{\frac{1}{2}}$$

Note that this is an "engineering" equation, to be used without units!

For the semicircular channel $D = 1 \cdot m$ $\quad \alpha = 180 \cdot deg$ $\quad S_b = 0.01$

For corrugated steel, a Google search leads to (Table 11.2) $n = 0.022$

Hence from Table 11.1 $A = \frac{1}{8} \cdot (\alpha - \sin(\alpha)) \cdot D^2$ $\quad A = 0.393\,m^2$

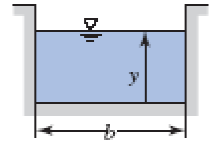
$$R_h = \frac{1}{4} \cdot \left(1 - \frac{\sin(\alpha)}{\alpha}\right) \cdot D \qquad R_h = 0.25\,m$$

Then the discharge is $Q = \frac{1}{n} \cdot A \cdot R_h^{\frac{2}{3}} \cdot S_b^{\frac{1}{2}} \cdot \frac{m^3}{s}$ $\quad Q = 0.708\,\frac{m^3}{s}$

Problem 11.53

[Difficulty: 3]

11.53 The flume of Problem 11.46 is fitted with a new plastic film liner ($n = 0.010$). Find the new depth of flow if the discharge remains constant at 85.5 ft^3/s.



Given: Data on flume with plastic liner

Find: Depth of flow

Solution:

Basic equation: $$Q = \frac{1.49}{n} \cdot A \cdot R_h^{\frac{2}{3}} \cdot S_b^{\frac{1}{2}}$$

Note that this is an "engineering" equation, to be used without units!

For a rectangular channel of width $b = 6 \cdot ft$ and depth y we find from Table 11.1

$$A = b \cdot y = 6 \cdot y \qquad R = \frac{b \cdot y}{b + 2 \cdot y} = \frac{6 \cdot y}{6 + 2 \cdot y}$$

and also $n = 0.010$ and $S_b = \frac{1 \cdot ft}{1000 \cdot ft}$ $\quad S_b = 0.001$

Hence $$Q = \frac{1.49}{n} \cdot A \cdot R_h^{\frac{2}{3}} \cdot S_b^{\frac{1}{2}} = \frac{1.49}{0.010} \cdot 6 \cdot y \cdot \left(\frac{6 \cdot y}{6 + 2 \cdot y}\right)^{\frac{2}{3}} \cdot 0.001^{\frac{1}{2}} = 85.5$$ (Note that we don't use units!)

Solving for y $$\frac{y^{\frac{5}{3}}}{(6 + 2 \cdot y)^{\frac{2}{3}}} = \frac{85.5 \cdot 0.010}{1.49 \cdot .001^{\frac{1}{2}} \cdot 6 \cdot 6^{\frac{2}{3}}} \quad \text{or} \quad \frac{y^{\frac{5}{3}}}{(6 + 2 \cdot y)^{\frac{2}{3}}} = 0.916$$

This is a nonlinear implicit equation for y and must be solved numerically. We can use one of a number of numerical root finding techniques, such as Newton's method, or we can use *Excel*'s *Solver* or *Goal Seek*, or we can manually iterate, as below. We start with Problem 11.46's depth

For $y = 3$ (feet) $\quad \frac{y^{\frac{5}{3}}}{(6 + 2 \cdot y)^{\frac{2}{3}}} = 1.191$ $\qquad$ For $y = 2$ (feet) $\quad \frac{y^{\frac{5}{3}}}{(6 + 2 \cdot y)^{\frac{2}{3}}} = 0.684$

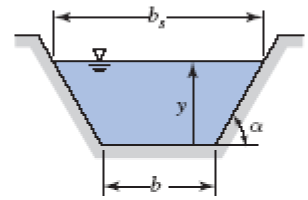
For $y = 2.5$ (feet) $\quad \frac{y^{\frac{5}{3}}}{(6 + 2 \cdot y)^{\frac{2}{3}}} = 0.931$ $\qquad$ For $y = 2.45$ (feet) $\quad \frac{y^{\frac{5}{3}}}{(6 + 2 \cdot y)^{\frac{2}{3}}} = 0.906$

For $y = 2.47$ (feet) $\quad \frac{y^{\frac{5}{3}}}{(6 + 2 \cdot y)^{\frac{2}{3}}} = 0.916$ $\qquad$ $y = 2.47$ (feet)

Problem 11.54

[Difficulty: 3]

11.54 Discharge through the channel of Problem 11.49 is increased to 10 m^3/s. Find the corresponding normal depth if the bed slope is 0.00193.



Given: Data on trapzoidal channel

Find: New depth of flow

Solution:

Basic equation: $Q = \frac{1}{n} \cdot A \cdot R_h^{\frac{2}{3}} \cdot S_b^{\frac{1}{2}}$

Note that this is an "engineering" equation, to be used without units!

For the trapezoidal channel we have $b = 2.4 \cdot m$ $\quad \alpha = 45 \cdot deg$ $\quad Q = 10 \cdot \frac{m^3}{s}$ $\quad S_b = 0.00193$

For bare soil (Table 11.2) $n = 0.020$

Hence from Table 11.1 $A = y \cdot (b + \cot(\alpha) \cdot y) = y \cdot (2.4 + y)$ $\quad R = \dfrac{y \cdot (b + y \cdot \cot(\alpha))}{b + \dfrac{2 \cdot y}{\sin(\alpha)}} = \dfrac{y \cdot (2.4 + y)}{2.4 + 2 \cdot y \cdot \sqrt{2}}$

Hence $$Q = \frac{1}{n} \cdot A \cdot R_h^{\frac{2}{3}} \cdot S_b^{\frac{1}{2}} = \frac{1}{0.020} \cdot y \cdot (2.4 + y) \cdot \left[\frac{y \cdot (2.4 + y)}{2.4 + 2 \cdot y \cdot \sqrt{2}}\right]^{\frac{2}{3}} \cdot 0.00193^{\frac{1}{2}} = 10$$ (Note that we don't use units!)

Solving for y $$\frac{[y \cdot (2.4 + y)]^{\frac{5}{3}}}{\left(2.4 + 2 \cdot y \cdot \sqrt{2}\right)^{\frac{2}{3}}} = 4.55$$

This is a nonlinear implicit equation for y and must be solved numerically. We can use one of a number of numerical root finding techniques, such as Newton's method, or we can use *Excel*'s *Solver* or *Goal Seek*, or we can manually iterate, as below. We start with a larger depth than Problem 11.49's.

For $y = 1.5$ (m) $\quad \dfrac{[y \cdot (2.4 + y)]^{\frac{5}{3}}}{\left(2.4 + 2 \cdot y \cdot \sqrt{2}\right)^{\frac{2}{3}}} = 5.37$ $\qquad$ For $y = 1.4$ (m) $\quad \dfrac{[y \cdot (2.4 + y)]^{\frac{5}{3}}}{\left(2.4 + 2 \cdot y \cdot \sqrt{2}\right)^{\frac{2}{3}}} = 4.72$

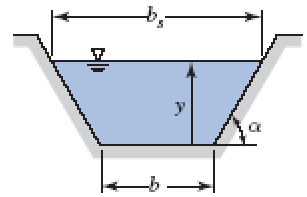
For $y = 1.35$ (m) $\quad \dfrac{[y \cdot (2.4 + y)]^{\frac{5}{3}}}{\left(2.4 + 2 \cdot y \cdot \sqrt{2}\right)^{\frac{2}{3}}} = 4.41$ $\qquad$ For $y = 1.37$ (m) $\quad \dfrac{[y \cdot (2.4 + y)]^{\frac{5}{3}}}{\left(2.4 + 2 \cdot y \cdot \sqrt{2}\right)^{\frac{2}{3}}} = 4.536$

The solution to three figures is $y = 1.37$ (m)

Problem 11.55

[Difficulty: 3]

11.55 The channel of Problem 11.49 has 0.00193 bed slope. Find the normal depth for the given discharge after a new plastic liner ($n = 0.010$) is installed.



Given: Data on trapzoidal channel

Find: New depth of flow

Solution:

Basic equation: $Q = \frac{1}{n} \cdot A \cdot R_h^{\frac{2}{3}} \cdot S_b^{\frac{1}{2}}$

Note that this is an "engineering" equation, to be used without units!

For the trapezoidal channel we have $b = 2.4 \cdot m$ $\quad \alpha = 45 \cdot deg$ $\quad Q = 7.1 \cdot \frac{m^3}{s}$ $\quad S_b = 0.00193$

For bare soil (Table 11.2) $n = 0.010$

Hence from Table 11.1 $A = y \cdot (b + y \cdot \cot(\alpha)) = y \cdot (2.4 + y)$ $\quad R_h = \frac{y \cdot (b + y \cdot \cot(\alpha))}{b + \frac{2 \cdot y}{\sin(\alpha)}} = \frac{y \cdot (2.4 + y)}{2.4 + 2 \cdot y \cdot \sqrt{2}}$

Hence $$Q = \frac{1}{n} \cdot A \cdot R_h^{\frac{2}{3}} \cdot S_b^{\frac{1}{2}} = \frac{1}{0.010} \cdot y \cdot (2.4 + y) \cdot \left[\frac{y \cdot (2.4 + y)}{2.4 + 2 \cdot y \cdot \sqrt{2}} \right]^{\frac{2}{3}} \cdot 0.00193^{\frac{1}{2}} = 7.1$$ (Note that we don't use units!)

Solving for y $$\frac{[y \cdot (2.4 + y)]^{\frac{5}{3}}}{\left(2.4 + 2 \cdot y \cdot \sqrt{2}\right)^{\frac{2}{3}}} = 1.62$$

This is a nonlinear implicit equation for y and must be solved numerically. We can use one of a number of numerical root finding techniques, such as Newton's method, or we can use *Excel*'s *Solver* or *Goal Seek*, or we can manually iterate, as below. We start with a shallower depth than that of Problem 11.49.

For $y = 1$ (m) $\quad \frac{[y \cdot (2.4 + y)]^{\frac{5}{3}}}{\left(2.4 + 2 \cdot y \cdot \sqrt{2}\right)^{\frac{2}{3}}} = 2.55$ $\qquad$ For $y = 0.75$ (m) $\quad \frac{[y \cdot (2.4 + y)]^{\frac{5}{3}}}{\left(2.4 + 2 \cdot y \cdot \sqrt{2}\right)^{\frac{2}{3}}} = 1.53$

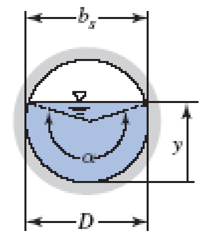
For $y = 0.77$ (m) $\quad \frac{[y \cdot (2.4 + y)]^{\frac{5}{3}}}{\left(2.4 + 2 \cdot y \cdot \sqrt{2}\right)^{\frac{2}{3}}} = 1.60$ $\qquad$ For $y = 0.775$ (m) $\quad \frac{[y \cdot (2.4 + y)]^{\frac{5}{3}}}{\left(2.4 + 2 \cdot y \cdot \sqrt{2}\right)^{\frac{2}{3}}} = 1.62$

The solution to three figures is $y = 0.775$ (m)

Problem 11.56

[Difficulty: 4]

11.56 Consider again the semicircular channel of Problem 11.51. Find the normal depth that corresponds to a discharge of 0.5 m^3/s.



Given: Data on semicircular trough

Find: New depth of flow

Solution:

Basic equation: $$Q = \frac{1}{n} \cdot A \cdot R_h^{\frac{2}{3}} \cdot S_b^{\frac{1}{2}}$$

Note that this is an "engineering" equation, to be used without units!

For the semicircular channel $D = 1 \cdot m$ $\quad S_b = 0.01$ $\quad Q = 0.5 \cdot \frac{m^3}{s}$

For corrugated steel, a Google search leads to (Table 11.2) $\quad n = 0.022$

From Table 11.1 $$A = \frac{1}{8} \cdot (\alpha - \sin(\alpha)) \cdot D^2 = \frac{1}{8} \cdot (\alpha - \sin(\alpha)) \qquad R_h = \frac{1}{4} \cdot \left(1 - \frac{\sin(\alpha)}{\alpha}\right) \cdot D = \frac{1}{4} \cdot \left(1 - \frac{\sin(\alpha)}{\alpha}\right)$$

Hence $$Q = \frac{1}{n} \cdot A \cdot R_h^{\frac{2}{3}} \cdot S_b^{\frac{1}{2}} = \frac{1}{0.022} \cdot \left[\frac{1}{8} \cdot (\alpha - \sin(\alpha))\right] \cdot \left[\frac{1}{4} \cdot \left(1 - \frac{\sin(\alpha)}{\alpha}\right)\right]^{\frac{2}{3}} \cdot 0.01^{\frac{1}{2}} = 0.5$$ (Note that we don't use units!)

Solving for α $$\alpha^{-\frac{2}{3}} \cdot (\alpha - \sin(\alpha))^{\frac{5}{3}} = 2.21$$

This is a nonlinear implicit equation for α and must be solved numerically. We can use one of a number of numerical root finding techniques, such as Newton's method, or we can use *Excel*'s *Solver* or *Goal Seek*, or we can manually iterate, as below. We start with a half-full channel

For $\alpha = 180 \cdot deg$ $\quad \alpha^{-\frac{2}{3}} \cdot (\alpha - \sin(\alpha))^{\frac{5}{3}} = 3.14$ $\qquad$ For $\alpha = 160 \cdot deg$ $\quad \alpha^{-\frac{2}{3}} \cdot (\alpha - \sin(\alpha))^{\frac{5}{3}} = 2.25$

For $\alpha = 159 \cdot deg$ $\quad \alpha^{-\frac{2}{3}} \cdot (\alpha - \sin(\alpha))^{\frac{5}{3}} = 2.20$ $\qquad$ For $\alpha = 159.2 \cdot deg$ $\quad \alpha^{-\frac{2}{3}} \cdot (\alpha - \sin(\alpha))^{\frac{5}{3}} = 2.212$

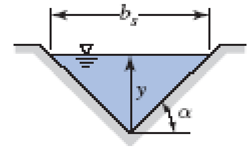
The solution to three figures is $\alpha = 159 \cdot deg$

From geometry $$y = \frac{D}{2} \cdot \left(1 - \cos\left(\frac{\alpha}{2}\right)\right) \qquad y = 0.410\,m$$

Problem 11.57

[Difficulty: 3]

11.57 Consider a symmetric open channel of triangular cross section. Show that for a given flow area, the wetted perimeter is minimized when the sides meet at a right angle.



Given: Triangular channel

Find: Proof that wetted perimeter is minimized when sides meet at right angles

Solution:

From Table 11.1 $\quad A = y^2 \cdot \cot(\alpha) \qquad P = \dfrac{2 \cdot y}{\sin(\alpha)}$

We need to vary z to minimize P while keeping A constant, which means that $\quad y = \sqrt{\dfrac{A}{\cot(\alpha)}} \quad$ with A = constant

Hence we eliminate y in the expression for P $\quad P = 2 \cdot \sqrt{\dfrac{A}{\cot(\alpha)}} \cdot \dfrac{1}{\sin(\alpha)}$

For optimizing P $\quad \dfrac{dP}{d\alpha} = -\dfrac{2 \cdot (A \cdot \cos(\alpha) - A \cdot \sin(\alpha) \cdot \tan(\alpha))}{\sin(2 \cdot \alpha) \cdot \sqrt{A \cdot \tan(\alpha)}} = 0$

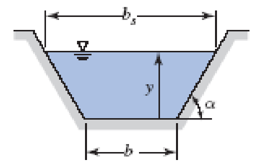
or $\quad A \cdot \cos(\alpha) - A \cdot \sin(\alpha) \cdot \tan(\alpha) = 0 \qquad \dfrac{1}{\tan(\alpha)} = \tan(\alpha) \qquad \tan(\alpha) = 1 \qquad \alpha = 45 \cdot \text{deg}$

For $\alpha = 45^o$ we find from the figure that we have the case where the sides meet at 90^o. Note that we have only proved that this is a minimum OR maximum of P! It makes sense that it's the minimum, as, for constant A, we get a huge P if we set α to a large number (almost vertical walls); hence we can't have a maximum value at $\alpha = 45^o$.

Problem 11.58

[Difficulty: 3]

11.58 Compute the normal depth and velocity of the channel of Problem 11.17.



Given: Data on trapezoidal channel

Find: Normal depth and velocity

Solution:

Basic equation: $$Q = \frac{1.49}{n}\cdot A\cdot R_h^{\frac{2}{3}}\cdot S_b^{\frac{1}{2}}$$

Note that this is an "engineering" equation, to be used without units!

For the trapezoidal channel we have $b = 20\cdot ft$ $\quad \alpha = atan(2)$ $\quad \alpha = 63.4\,deg$ $\quad Q = 400\cdot\frac{ft^3}{s}$ $\quad S_b = 0.0016$ $\quad n = 0.025$

Hence from Table 11.2 $$A = y\cdot(b + y\cdot cot(\alpha)) = y\cdot\left(20 + \frac{1}{2}\cdot y\right) \qquad R_h = \frac{y\cdot(b + y\cdot cot(\alpha))}{b + \frac{2\cdot y}{sin(\alpha)}} = \frac{y\cdot\left(20 + \frac{1}{2}\cdot y\right)}{20 + y\cdot\sqrt{5}}$$

Hence $$Q = \frac{1}{n}\cdot A\cdot R_h^{\frac{2}{3}}\cdot S_b^{\frac{1}{2}} = \frac{1}{0.025}\cdot y\cdot\left(20 + \frac{1}{2}\cdot y\right)\cdot\left[\frac{y\cdot\left(20 + \frac{1}{2}\cdot y\right)}{20 + y\cdot\sqrt{5}}\right]^{\frac{2}{3}}\cdot 0.0016^{\frac{1}{2}} = 400$$ (Note that we don't use units!)

Solving for y $$\frac{\left[y\cdot\left(20 + \frac{1}{2}\cdot y\right)\right]^{\frac{5}{3}}}{\left(20 + y\cdot\sqrt{5}\right)^{\frac{2}{3}}} = 250$$

This is a nonlinear implicit equation for y and must be solved numerically. We can use one of a number of numerical root finding techniques, such as Newton's method, or we can use *Excel*'s *Solver* or *Goal Seek*, or we can manually iterate, as below. We start with an arbitrary depth

For $y = 5$ (ft) $$\frac{\left[y\cdot\left(20 + \frac{1}{2}\cdot y\right)\right]^{\frac{5}{3}}}{\left(20 + y\cdot\sqrt{5}\right)^{\frac{2}{3}}} = 265$$

For $y = 4.9$ (ft) $$\frac{\left[y\cdot\left(20 + \frac{1}{2}\cdot y\right)\right]^{\frac{5}{3}}}{\left(20 + y\cdot\sqrt{5}\right)^{\frac{2}{3}}} = 256$$

For $y = 4.85$ (ft) $$\frac{\left[y\cdot\left(20 + \frac{1}{2}\cdot y\right)\right]^{\frac{5}{3}}}{\left(20 + y\cdot\sqrt{5}\right)^{\frac{2}{3}}} = 252$$

For $y = 4.83$ (ft) $$\frac{\left[y\cdot\left(20 + \frac{1}{2}\cdot y\right)\right]^{\frac{5}{3}}}{\left(20 + y\cdot\sqrt{5}\right)^{\frac{2}{3}}} = 250$$

The solution to three figures is $y = 4.83\cdot ft$

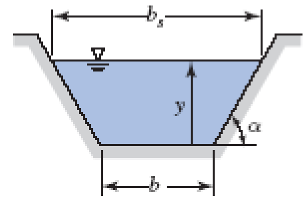
Then $A = (b + y\cdot cot(\alpha))\cdot y$ $\qquad A = 108\cdot ft^2$

Finally, the normal velocity is $V = \frac{Q}{A}$ $\qquad V = 3.69\cdot\frac{ft}{s}$

Problem 11.59

[Difficulty: 5]

11.59 Determine the cross section of the greatest hydraulic efficiency for a trapezoidal channel with side slope of 1 vertical to 2 horizontal if the design discharge is 250 m^3/s. The channel slope is 0.001 and Manning's roughness factor is 0.020.



Given: Data on trapezoidal channel

Find: Geometry for greatest hydraulic efficiency

Solution:

Basic equation: $$Q = \frac{1}{n} \cdot A \cdot R_h^{\frac{2}{3}} \cdot S_b^{\frac{1}{2}}$$

Note that this is an "engineering" equation, to be used without units!

For the trapezoidal channel we have $\alpha = \text{atan}\left(\frac{1}{2}\right)$ $\quad \alpha = 26.6 \cdot \text{deg}$ $\quad Q = 250 \cdot \frac{m^3}{s}$ $\quad S_b = 0.001$ $\quad n = 0.020$

From Table 11.1 $\quad A = y \cdot (b + y \cdot \cot(\alpha))$ $\quad P = b + \frac{2 \cdot y}{\sin(\alpha)}$

We need to vary b and y to obtain optimum conditions. These are when the area and perimeter are optimized. Instead of two independent variables b and y, we eliminate b by doing the following

$$b = \frac{A}{y} - y \cdot \cot(\alpha) \qquad \text{and so} \qquad P = \frac{A}{y} - y \cdot \cot(\alpha) + \frac{2 \cdot y}{\sin(\alpha)}$$

Taking the derivative w.r.t. y $$\frac{\partial}{\partial y} P = \frac{1}{y} \cdot \frac{\partial}{\partial y} A - \frac{A}{y^2} - \cot(\alpha) + \frac{2}{\sin(\alpha)}$$

But at optimum conditions $$\frac{\partial}{\partial y} P = 0 \qquad \text{and} \qquad \frac{\partial}{\partial y} A = 0$$

Hence $$0 = -\frac{A}{y^2} - \cot(\alpha) + \frac{2}{\sin(\alpha)} \qquad \text{or} \qquad A = \frac{2 \cdot y^2}{\sin(\alpha)} - y^2 \cdot \cot(\alpha)$$

Comparing to $$A = y \cdot (b + y \cdot \cot(\alpha)) \qquad \text{we find} \qquad A = y \cdot (b + y \cdot \cot(\alpha)) = \frac{2 \cdot y^2}{\sin(\alpha)} - y^2 \cdot \cot(\alpha)$$

Hence $$b = \frac{2 \cdot y}{\sin(\alpha)} - 2 \cdot y \cdot \cot(\alpha)$$

Then $$A = y \cdot (b + y \cdot \cot(\alpha)) = y \cdot \left(\frac{2 \cdot y}{\sin(\alpha)} - 2 \cdot y \cdot \cot(\alpha) + y \cdot \cot(\alpha)\right) = y^2 \cdot \left(\frac{2}{\sin(\alpha)} - \cot(\alpha)\right)$$

$$P = b + \frac{2 \cdot y}{\sin(\alpha)} = \frac{4 \cdot y}{\sin(\alpha)} - 2 \cdot y \cdot \cot(\alpha) = 2 \cdot y \cdot \left(\frac{2}{\sin(\alpha)} - \cot(\alpha)\right)$$

and $$R_h = \frac{A}{P} = \frac{y^2\cdot\left(\frac{2}{\sin(\alpha)} - \cot(\alpha)\right)}{2\cdot y\cdot\left(\frac{2}{\sin(\alpha)} - \cot(\alpha)\right)} = \frac{y}{2}$$

Hence $$Q = \frac{1}{n}\cdot A\cdot R_h^{\frac{2}{3}}\cdot S_b^{\frac{1}{2}} = \frac{1}{n}\cdot\left[y^2\cdot\left(\frac{2}{\sin(\alpha)} - \cot(\alpha)\right)\right]\cdot\left(\frac{y}{2}\right)^{\frac{2}{3}}\cdot S_b^{\frac{1}{2}}$$

$$Q = \left(\frac{2}{\sin(\alpha)} - \cot(\alpha)\right)\cdot\frac{y^{\frac{8}{3}}\cdot S_b^{\frac{1}{2}}}{n\cdot 2^{\frac{2}{3}}}$$

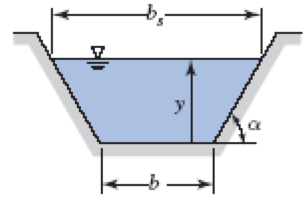
Solving for y $$y = \left[\frac{2^{\frac{2}{3}}\cdot n\cdot Q}{\left(\frac{2}{\sin(\alpha)} - \cot(\alpha)\right)\cdot S_b^{\frac{1}{2}}}\right]^{\frac{3}{8}}$$ $y = 5.66$ (m)

Finally $$b = \frac{2\cdot y}{\sin(\alpha)} - 2\cdot y\cdot\cot(\alpha)$$ $b = 2.67$ (m)

Problem 11.60

[Difficulty: 3]

11.60 For a trapezoidal shaped channel ($n = 0.014$ and slope $S_b = 0.0002$ with a 20-ft bottom width and side slopes of 1 vertical to 1.5 horizontal), determine the normal depth for a discharge of 1000 cfs.



Given: Data on trapezoidal channel

Find: Normal depth

Solution:

Basic equation: $$Q = \frac{1.49}{n}\cdot A\cdot R_h^{\frac{2}{3}}\cdot S_b^{\frac{1}{2}}$$

Note that this is an "engineering" equation, to be used without units!

For the trapezoidal channel we have $b = 20\cdot ft$ $\quad \alpha = atan\left(\frac{1}{1.5}\right)$ $\quad \alpha = 33.7\,deg$ $\quad Q = 1000\cdot\frac{ft^3}{s}$

$S_0 = 0.0002 \quad n = 0.014$

Hence from Table 11.1 $$A = y\cdot(b + y\cdot cot(\alpha)) = y\cdot(20 + 1.5\cdot y) \qquad R_h = \frac{y\cdot(b + y\cdot cot(\alpha))}{b + \frac{2\cdot y}{sin(\alpha)}} = \frac{y\cdot(20 + 1.5\cdot y)}{20 + 2\cdot y\cdot\sqrt{3.25}}$$

Hence $$Q = \frac{1.49}{n}\cdot A\cdot R_h^{\frac{2}{3}}\cdot S_b^{\frac{1}{2}} = \frac{1.49}{0.014}\cdot y\cdot(20 + 1.5\cdot y)\cdot\left[\frac{y\cdot(20 + 1.5\cdot y)}{20 + 2\cdot y\cdot\sqrt{3.25}}\right]^{\frac{2}{3}}\cdot 0.0002^{\frac{1}{2}} = 1000$$ (Note that we don't use units!)

Solving for y $$\frac{[(20 + 1.5\cdot y)\cdot y]^{\frac{5}{3}}}{\left(20 + 2\cdot y\cdot\sqrt{3.25}\right)^{\frac{2}{3}}} = 664$$

This is a nonlinear implicit equation for y and must be solved numerically. We can use one of a number of numerical root finding techniques, such as Newton's method, or we can use *Excel*'s *Solver* or *Goal Seek*, or we can manually iterate, as below.

For $y = 7.5$ (ft) $$\frac{[(20 + 1.5\cdot y)\cdot y]^{\frac{5}{3}}}{\left(20 + 2\cdot y\cdot\sqrt{3.25}\right)^{\frac{2}{3}}} = 684$$ For $y = 7.4$ (ft) $$\frac{[(20 + 1.5\cdot y)\cdot y]^{\frac{5}{3}}}{\left(20 + 2\cdot y\cdot\sqrt{3.25}\right)^{\frac{2}{3}}} = 667$$

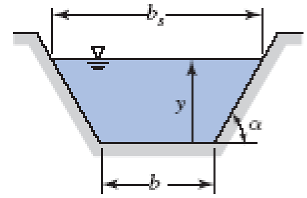
For $y = 7.35$ (ft) $$\frac{[(20 + 1.5\cdot y)\cdot y]^{\frac{5}{3}}}{\left(20 + 2\cdot y\cdot\sqrt{3.25}\right)^{\frac{2}{3}}} = 658$$ For $y = 7.38$ (ft) $$\frac{[(20 + 1.5\cdot y)\cdot y]^{\frac{5}{3}}}{\left(20 + 2\cdot y\cdot\sqrt{3.25}\right)^{\frac{2}{3}}} = 663$$

The solution to three figures is $y = 7.38$ (ft)

Problem 11.61

[Difficulty: 5]

11.61 Show that the best hydraulic trapezoidal section is one-half of a hexagon.



Given: Trapezoidal channel

Find: Geometry for greatest hydraulic efficiency

Solution:

From Table 11.1 $\quad A = y\cdot(b + y\cdot\cot(\alpha)) \quad P = b + \dfrac{2\cdot y}{\sin(\alpha)}$

We need to vary b and y (and then α!) to obtain optimum conditions. These are when the area and perimeter are optimized. Instead of two independent variables b and y, we eliminate b by doing the following

$$b = \frac{A}{y} - y\cdot\cot(\alpha) \qquad \text{and so} \qquad P = \frac{A}{y} - y\cdot\cot(\alpha) + \frac{2\cdot y}{\sin(\alpha)}$$

Taking the derivative w.r.t. y $\quad \dfrac{\partial}{\partial y}P = \dfrac{1}{y}\cdot\dfrac{\partial}{\partial y}A - \dfrac{A}{y^2} - \cot(\alpha) + \dfrac{2}{\sin(\alpha)}$

But at optimum conditions $\quad \dfrac{\partial}{\partial y}P = 0 \quad$ and $\quad \dfrac{\partial}{\partial y}A = 0$

Hence $\quad 0 = -\dfrac{A}{y^2} - \cot(\alpha) + \dfrac{2}{\sin(\alpha)} \quad$ or $\quad A = \dfrac{2\cdot y^2}{\sin(\alpha)} - y^2\cdot\cot(\alpha) \qquad (1)$

Now we optimize A w.r.t. α $\quad \dfrac{\partial}{\partial \alpha}A = -\dfrac{2\cdot y^2\cdot\cos(\alpha)}{\sin(\alpha)^2} - y^2\cdot\left(-1 - \cot(\alpha)^2\right) = 0 \quad$ or $\quad -\dfrac{2\cdot\cos(\alpha)}{\sin(\alpha)^2} + \left(\cot(\alpha)^2 + 1\right) = 0$

But $\quad \cot(\alpha)^2 + 1 = \dfrac{\cos(\alpha)^2}{\sin(\alpha)^2} + 1 = \dfrac{\sin(\alpha)^2 + \cos(\alpha)^2}{\sin(\alpha)^2} = \dfrac{1}{\sin(\alpha)^2}$

Hence $\quad -2\cdot\cos(\alpha) = -1 \qquad \alpha = \operatorname{acos}\left(\dfrac{1}{2}\right) \qquad \alpha = 60\,\text{deg}$

We can now evaluate A from Eq 1 $\quad A = \dfrac{2\cdot y^2}{\sin(\alpha)} - y^2\cdot\cot(\alpha) = \dfrac{2\cdot y^2}{\frac{\sqrt{3}}{2}} - \dfrac{1}{\sqrt{3}}\cdot y^2 = \left(\dfrac{4}{\sqrt{3}} - \dfrac{1}{\sqrt{3}}\right)\cdot y^2 = \sqrt{3}\cdot y^2$

But for a trapezoid $\quad A = y\cdot(b + y\cdot\cot(\alpha)) = y\cdot\left(b + \dfrac{1}{\sqrt{3}}\cdot y\right)$

Comparing the two A expressions $\quad A = \left(b + \dfrac{1}{\sqrt{3}}\cdot y\right)\cdot y = \sqrt{3}\cdot y^2 \quad$ we find $\quad b = \left(\sqrt{3} - \dfrac{1}{\sqrt{3}}\right)\cdot y = \dfrac{2}{\sqrt{3}}\cdot y$

But the perimeter is $\quad P = b + \dfrac{2\cdot y}{\sin(\alpha)} = b + 2\cdot y\cdot\dfrac{2}{\sqrt{3}} = b + \dfrac{4}{\sqrt{3}}\cdot y = b + 2\cdot b = 3\cdot b$

In summary we have $\quad \alpha = 60\,\text{deg}$

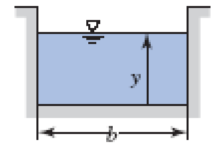
and $\quad b = \dfrac{1}{3}\cdot P \quad$ so each of the symmetric sides is $\quad \dfrac{P - \dfrac{1}{3}\cdot P}{2} = \dfrac{1}{3}\cdot P$

We have proved that the optimum shape is equal side and bottom lengths, with 60 angles i.e., half a hexagon!

Problem 11.62

[Difficulty: 1]

11.62 Compute the critical depth for the channel in Problem 11.41.



Given: Rectangular channel flow

Find: Critical depth

Solution:

Basic equations: $y_c = \left(\dfrac{Q^2}{g \cdot b^2}\right)^{\frac{1}{3}}$ $\quad Q = \dfrac{1.49}{n} \cdot A \cdot R_h^{\frac{2}{3}} \cdot S_b^{\frac{1}{2}}$

For a rectangular channel of width $b = 2 \cdot m$ and depth $y = 1.5 \cdot m$ we find from Table 11.1

$A = b \cdot y$ $\quad A = 3.00 \cdot m^2$ $\quad R_h = \dfrac{b \cdot y}{b + 2 \cdot y}$ $\quad R_h = 0.600 \cdot m$

Manning's roughness coefficient is $n = 0.015$ and $S_b = 0.0005$

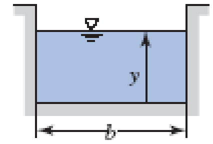
$Q = \dfrac{1.49}{n} \cdot A \cdot R_h^{\frac{2}{3}} \cdot S_b^{\frac{1}{2}}$ $\quad Q = 3.18 \cdot \dfrac{m^3}{s}$

Hence $y_c = \left(\dfrac{Q^2}{g \cdot b^2}\right)^{\frac{1}{3}}$ $\quad y_c = 0.637\,m$

Problem 11.63

[Difficulty: 4]

11.63 Consider a 2.45-m-wide rectangular channel with a bed slope of 0.0004 and a Manning's roughness factor of 0.015. A weir is placed in the channel, and the depth upstream of the weir is 1.52 m for a discharge of 5.66 m³/s. Determine whether a hydraulic jump forms upstream of the weir.



Given: Data on rectangular channel and weir

Find: If a hydraulic jump forms upstream of the weir

Solution:

Basic equations: $$Q = \frac{1}{n}\cdot A\cdot R_h^{\frac{2}{3}}\cdot S_b^{\frac{1}{2}} \qquad y_c = \left(\frac{Q^2}{g\cdot b^2}\right)^{\frac{1}{3}}$$

Note that the Q equation is an "engineering" equation, to be used without units!

For a rectangular channel of width $b = 2.45\cdot m$ and depth y we find from Table 11.1

$$A = b\cdot y = 2.45\cdot y \qquad R_h = \frac{b\cdot y}{b + 2\cdot y} = \frac{2.45\cdot y}{2.45 + 2\cdot y} \quad \text{and also} \quad n = 0.015 \quad \text{and} \quad S_b = 0.0004 \qquad Q = 5.66\cdot\frac{m^3}{s}$$

Hence $$Q = \frac{1}{n}\cdot A\cdot R_h^{\frac{2}{3}}\cdot S_b^{\frac{1}{2}} = \frac{1}{0.015}\cdot 2.45\cdot y\cdot\left(\frac{2.45\cdot y}{2.45 + 2\cdot y}\right)^{\frac{2}{3}}\cdot 0.0004^{\frac{1}{2}} = 5.66$$ (Note that we don't use units!)

Solving for y $$\frac{y^{\frac{5}{3}}}{(2.45 + 2\cdot y)^{\frac{2}{3}}} = \frac{5.66\cdot 0.015}{.0004^{\frac{1}{2}}\cdot 2.54\cdot 2.54^{\frac{2}{3}}} \quad \text{or} \quad \frac{y^{\frac{5}{3}}}{(2.54 + 2\cdot y)^{\frac{2}{3}}} = 0.898$$

This is a nonlinear implicit equation for y and must be solved numerically. We can use one of a number of numerical root finding techniques, or we can use *Excel*'s *Solver* or *Goal Seek*, or we can manually iterate, as below. We start with the given depth

For $y = 1.52$ (m) $$\frac{y^{\frac{5}{3}}}{(2.54 + 2\cdot y)^{\frac{2}{3}}} = 0.639$$ For $y = 2$ (m) $$\frac{y^{\frac{5}{3}}}{(2.54 + 2\cdot y)^{\frac{2}{3}}} = 0.908$$

For $y = 1.95$ (m) $$\frac{y^{\frac{5}{3}}}{(2.54 + 2\cdot y)^{\frac{2}{3}}} = 0.879$$ For $y = 1.98$ (m) $$\frac{y^{\frac{5}{3}}}{(2.54 + 2\cdot y)^{\frac{2}{3}}} = 0.896$$

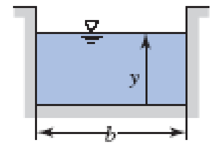
$y = 1.98$ (m) This is the normal depth. We also have the critical depth: $$y_c = \left(\frac{Q^2}{g\cdot b^2}\right)^{\frac{1}{3}} \qquad y_c = 0.816\,m$$

Hence the given depth is 1.52 m > y_c, but 1.52 m < y_n, the normal depth. This implies the flow is subcritical (far enough upstream it is depth 1.98 m), and that it draws down to 1.52 m as it gets close to the wier. There is no jump.

Problem 11.64

[Difficulty: 2]

11.64 An above-ground rectangular flume is to be constructed of timber. For a drop of 10 ft/mile, what will be the depth and width for the most economical flume if it is to discharge 40 cfs?



Given:
Data on rectangular flume

Find:
Optimum geometry

Solution:

Basic equations: $Q = \frac{1.49}{n} \cdot A \cdot R_h^{\frac{2}{3}} \cdot S_b^{\frac{1}{2}}$ and from Table 11.3, for optimum geometry $b = 2 \cdot y_n$

Note that the Q equation is an "engineering" equation, to be used without units!

Available data $S_b = 10 \cdot \frac{ft}{mile}$ $S_b = 0.00189$ $Q = 40 \cdot \frac{ft^3}{s}$

For wood (unplaned), a Google seach gives $n = 0.013$

Hence $A = b \cdot y_n = 2 \cdot y_n^2$ $R_h = \frac{A}{P} = \frac{2 \cdot y_n^2}{y_n + 2 \cdot y_n + y_n} = \frac{y_n}{2}$

Then $$Q = \frac{1.49}{n} \cdot A \cdot R_h^{\frac{2}{3}} \cdot S_b^{\frac{1}{2}} = \frac{1.49}{n} \cdot 2 \cdot y_n^2 \cdot \left(\frac{y_n}{2}\right)^{\frac{2}{3}} \cdot S_b^{\frac{1}{2}}$$

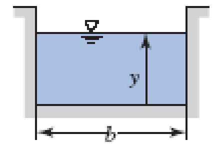
Solving for y_n $$y_n = \left(\frac{Q \cdot n \cdot 2^{\frac{2}{3}}}{4 \cdot 1.49 \cdot S_b^{\frac{1}{2}}}\right)^{\frac{3}{5}}$$ $y_n = 2.00$ (ft) $b = 2y_n$ $b = 4.01$ (ft)

Problem 11.65

[Difficulty: 2]

11.65 Consider flow in a rectangular channel. Show that, for flow at critical depth and optimum aspect ratio ($b = 2y$), the volume flow rate and bed slope are given by the expressions:

$$Q = 62.6 y_c^{5/2} \quad \text{and} \quad S_c = 24.7 \frac{n^2}{y_c^{1/3}}$$



Given: Data on rectangular channel

Find: Expressions valid for critical depth at optimum geometry

Solution:

Basic equations: $Q = \frac{1}{n} \cdot A \cdot R_h^{\frac{2}{3}} \cdot S_b^{\frac{1}{2}}$ and from Table 11.3, for optimum geometry $b = 2 \cdot y_n$

Note that the Q equation is an "engineering" equation, to be used without units!

Hence $A = b \cdot y_n = 2 \cdot y_n^2$ $\qquad R_h = \frac{A}{P} = \frac{2 \cdot y_n^2}{y_n + 2 \cdot y_n + y_n} = \frac{y_n}{2}$

Then $Q = \frac{1}{n} \cdot A \cdot R_h^{\frac{2}{3}} \cdot S_b^{\frac{1}{2}} = \frac{1}{n} \cdot 2 \cdot y_n^2 \cdot \left(\frac{y_n}{2}\right)^{\frac{2}{3}} \cdot S_b^{\frac{1}{2}}$ or $Q = \frac{2^{\frac{1}{3}}}{n} \cdot y_n^{\frac{8}{3}} \cdot S_b^{\frac{1}{2}}$

We can write the Froude number in terms of Q

$$Fr = \frac{V}{\sqrt{g \cdot y}} = \frac{Q}{A \cdot \sqrt{g \cdot y}} = \frac{Q}{2 \cdot y_n^2 \cdot \sqrt{g} \cdot y_n^{\frac{1}{2}}} \quad \text{or} \quad Fr = \frac{Q}{2 \cdot \sqrt{g} \cdot y_n^{\frac{5}{2}}}$$

Hence for critical flow, $Fr = 1$ and $y_n = y_c$, so $1 = \frac{Q}{2 \cdot \sqrt{g} \cdot y_c^{\frac{5}{2}}}$ or $Q = 2 \cdot \sqrt{g} \cdot y_c^{\frac{5}{2}}$ $\qquad Q = 6.26 \cdot y_c^{\frac{5}{2}}$

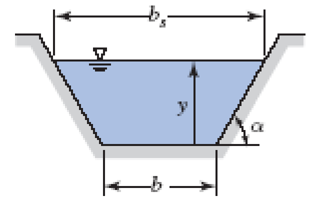
To find S_c, equate the expressions for Q and set $S_b = S_c$

$$Q = \frac{2^{\frac{1}{3}}}{n} \cdot y_c^{\frac{8}{3}} \cdot S_c^{\frac{1}{2}} = 2 \cdot \sqrt{g} \cdot y_c^{\frac{5}{2}} \quad \text{or} \quad S_c = 2^{\frac{4}{3}} \cdot g \cdot n^2 \cdot y_c^{-\frac{1}{3}} \qquad S_c = \frac{24.7 \cdot n^2}{y_c^{\frac{1}{3}}}$$

Problem 11.66

[Difficulty: 3]

11.66 A trapezoidal canal lined with brick has side slopes of 2:1 and bottom width of 10 ft. It carries 600 ft^3/s at critical speed. Determine the critical slope (the slope at which the depth is critical).



Given: Data on trapezoidal canal

Find: Critical slope

Solution:

Basic equations: $Q = \frac{1.49}{n} \cdot A \cdot R_h^{\frac{2}{3}} \cdot S_b^{\frac{1}{2}}$ and $A = y \cdot b + y \cdot \cot(\alpha)$ $\quad R_h = \frac{y \cdot (b + y \cdot \cot(\alpha))}{b + \frac{2 \cdot y}{\sin(\alpha)}}$

Note that the Q equation is an "engineering" equation, to be used without units!

Available data $\quad b = 10 \cdot ft \quad \alpha = \operatorname{atan}\left(\frac{2}{1}\right) \quad \alpha = 63.4 \deg \quad Q = 600 \cdot \frac{ft^3}{s}$

For brick, a Google search gives $\quad n = 0.015$

For critical flow $\quad y = y_c \quad V_c = \sqrt{g \cdot y_c}$

so $\quad Q = A \cdot V_c = \left(y_c \cdot b + y_c \cdot \cot(\alpha)\right) \cdot \sqrt{g \cdot y_c} \quad \left(y_c \cdot b + y_c \cdot \cot(\alpha)\right) \cdot \sqrt{g \cdot y_c} = Q$ with $Q = 600 \cdot \frac{ft^3}{s}$

This is a nonlinear implicit equation for y_c and must be solved numerically. We can use one of a number of numerical root finding techniques, or we can use *Excel*'s *Solver* or *Goal Seek*, or we can manually iterate, as below. We start with the given depth

For $y_c = 5$ (ft) $\left(y_c \cdot b + y_c \cdot \cot(\alpha)\right) \cdot \sqrt{g \cdot y_c} = 666$ For $y_c = 4.5$ (ft) $\left(y_c \cdot b + y_c \cdot \cot(\alpha)\right) \cdot \sqrt{g \cdot y_c} = 569$

For $y_c = 4.7$ (ft) $\left(y_c \cdot b + y_c \cdot \cot(\alpha)\right) \cdot \sqrt{g \cdot y_c} = 607$ For $y_c = 4.67$ (ft) $\left(y_c \cdot b + y_c \cdot \cot(\alpha)\right) \cdot \sqrt{g \cdot y_c} = 601$

Hence $y_c = 4.67$ (ft) and $A_{crit} = y_c \cdot b + y_c \cdot \cot(\alpha)$ $\quad A_{crit} = 49.0$ (ft2)

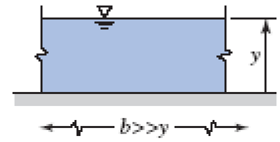
$$R_{hcrit} = \frac{y_c \cdot \left(b + y_c \cdot \cot(\alpha)\right)}{b + \frac{2 \cdot y_c}{\sin(\alpha)}} \qquad R_{hcrit} = 2.818 \quad \text{(ft)}$$

Solving the basic equation for S_c $\quad Q = \frac{1.49}{n} \cdot A \cdot R_h^{\frac{2}{3}} \cdot S_b^{\frac{1}{2}}$ $\quad S_{bcrit} = \left(\frac{n \cdot Q}{1.49 \cdot A_{crit} \cdot R_{hcrit}^{\frac{2}{3}}}\right)^2$ $\quad S_{bcrit} = 0.00381$

Problem 11.67

[Difficulty: 2]

11.67 A wide flat unfinished concrete channel discharges water at 20 ft^3/s per foot of width. Find the critical slope (the slope at which depth is critical).



Given: Data on wide channel

Find: Critical slope

Solution:

Basic equations: $Q = \frac{1.49}{n} \cdot A \cdot R_h^{\frac{2}{3}} \cdot S_b^{\frac{1}{2}}$ and $A = b \cdot y$ $\qquad R_h = y$

Note that the Q equation is an "engineering" equation, to be used without units!

Available data $\quad q = 20 \cdot \frac{\frac{ft^3}{s}}{ft}$

From Table 11.2 $\quad n = 0.015$

For critical flow $\quad y = y_c \qquad V_c = \sqrt{g \cdot y_c}$

so $\quad Q = A \cdot V_c = b \cdot y_c \cdot \sqrt{g \cdot y_c}$ $\quad$ or $\quad y_c = \left(\frac{Q}{b \cdot \sqrt{g}}\right)^{\frac{2}{3}}$ $\qquad y_c = \left(\frac{q}{\sqrt{g}}\right)^{\frac{2}{3}}$

Hence $\quad y_c = 2.316$ (ft)

Solving the basic equation for S_c $\quad Q = \frac{1.49}{n} \cdot A \cdot R_h^{\frac{2}{3}} \cdot S_b^{\frac{1}{2}} = \frac{1.49}{n} \cdot b \cdot y_c \cdot y_c^{\frac{2}{3}} \cdot S_b^{\frac{1}{2}}$

$$S_{bcrit} = \left(\frac{n \cdot Q}{1.49 \cdot b \cdot y_c \cdot y_c^{\frac{2}{3}}}\right)^2 \qquad S_{bcrit} = \left(\frac{n \cdot q}{1.49 \cdot y_c^{\frac{5}{3}}}\right)^2 \qquad S_{bcrit} = 0.00247$$

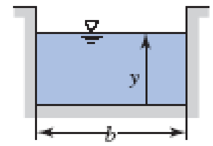
Note from Table 11.2 that a better roughness is $\quad n = 0.013$

and then $\quad S_{bcrit} = \left(\frac{n \cdot q}{1.49 \cdot y_c^{\frac{5}{3}}}\right)^2 \qquad S_{bcrit} = 0.00185$

Problem 11.68

[Difficulty: 2]

11.68 An optimum rectangular storm sewer channel made of unfinished concrete is to be designed to carry a maximum flow rate of 100 ft³/s, at which the flow is at critical condition. Determine the channel width and slope.



Given: Data on optimum rectangular channel

Find: Channel width and slope

Solution:

Basic equations: $$Q = \frac{1.49}{n}\cdot A\cdot R_h^{\frac{2}{3}}\cdot S_b^{\frac{1}{2}}$$ and from Table 11.3, for optimum geometry $b = 2\cdot y_n$

Note that the Q equation is an "engineering" equation, to be used without units!

Available data $Q = 100\cdot\frac{ft^3}{s}$ $n = 0.015$ (Table 11.2)

Hence $$A = b\cdot y_n = 2\cdot y_n^2 \qquad R_h = \frac{A}{P} = \frac{2\cdot y_n^2}{y_n + 2\cdot y_n + y_n} = \frac{y_n}{2}$$

We can write the Froude number in terms of Q

$$Fr = \frac{V}{\sqrt{g\cdot y}} = \frac{Q}{A\cdot\sqrt{g\cdot y}} = \frac{Q}{2\cdot y_n^2\cdot\sqrt{g}\cdot y_n^{\frac{1}{2}}} \qquad \text{or} \qquad Fr = \frac{Q}{2\cdot\sqrt{g}\cdot y_n^{\frac{5}{2}}}$$

Hence for critical flow, Fr = 1 and $y_n = y_c$, so $$1 = \frac{Q}{2\cdot\sqrt{g}\cdot y_c^{\frac{5}{2}}} \qquad \text{or} \qquad Q = 2\cdot\sqrt{g}\cdot y_c^{\frac{5}{2}}$$

Hence $$y_c = \left(\frac{Q}{2\cdot\sqrt{g}}\right)^{\frac{2}{5}}$$ $y_c = 2.39$ (ft) and $b = 2\cdot y_c$ $b = 4.78$ (ft)

Then $$Q = \frac{1.49}{n}\cdot A\cdot R_h^{\frac{2}{3}}\cdot S_b^{\frac{1}{2}} = \frac{1.49}{n}\cdot 2\cdot y_c^2\cdot\left(\frac{y_c}{2}\right)^{\frac{2}{3}}\cdot S_c^{\frac{1}{2}} \qquad \text{or} \qquad Q = \frac{1.49\cdot 2^{\frac{1}{3}}}{n}\cdot y_c^{\frac{8}{3}}\cdot S_c^{\frac{1}{2}}$$

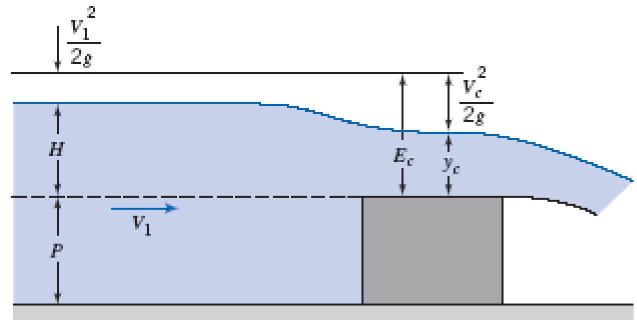
Hence $$S_c = \left(\frac{n\cdot Q}{1.49\cdot 2^{\frac{1}{3}}\cdot y_c^{\frac{8}{3}}}\right)^2$$ $S_c = 0.00615$

Using (from Table 11.2) $n = 0.013$ $$S_c = \left(\frac{n\cdot Q}{1.49\cdot 2^{\frac{1}{3}}\cdot y_c^{\frac{8}{3}}}\right)^2$$ $S_c = 0.00462$

Problem 11.69

[Difficulty: 1]

11.69 The crest of a broad-crested weir is 1 ft below the level of an upstream reservoir, where the water depth is 8 ft. For $C_w \approx 3.4$, what is the maximum flow rate per unit width that could pass over the weir?



Given: Data on broad-crested wier

Find: Maximum flow rate/width

Solution:

Basic equation: $Q = C_w \cdot b \cdot H^{\frac{3}{2}}$

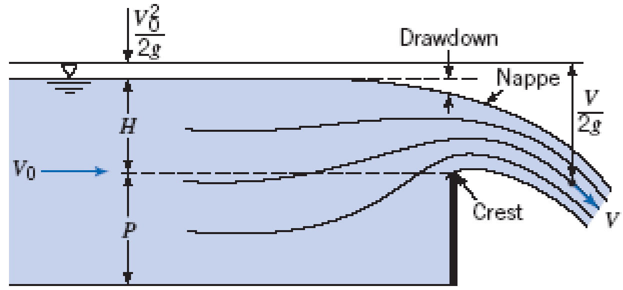
Available data $H = 1 \cdot ft$ $\quad P = 8 \cdot ft - 1 \cdot ft$ $\quad P = 7 \cdot ft$ $\quad C_w = 3.4$

Then $\dfrac{Q}{b} = q = C_w \cdot H^{\frac{3}{2}} = 3.4 \cdot \dfrac{\frac{ft^3}{s}}{ft}$

Problem 11.70

[Difficulty: 3]

11.70 A rectangular, sharp-crested weir with end contraction is 1.6 m long. How high should it be placed in a channel to maintain an upstream depth of 2.5 m for 0.5 m^3/s flow rate?



Given: Data on rectangular, sharp-crested weir

Find: Required weir height

Solution:

Basic equations: $Q = C_d \cdot \frac{2}{3} \cdot \sqrt{2 \cdot g} \cdot b' \cdot H^{\frac{3}{2}}$ where $C_d = 0.62$ and $b' = b - 0.1 \cdot n \cdot H$ with $n = 2$

Given data: $b = 1.6 \cdot m$ $\quad Q = 0.5 \cdot \frac{m^3}{s}$

Hence we find

$$Q = C_d \cdot \frac{2}{3} \cdot \sqrt{2 \cdot g} \cdot b' \cdot H^{\frac{3}{2}} = C_d \cdot \frac{2}{3} \cdot \sqrt{2 \cdot g} \cdot (b - 0.1 \cdot n \cdot H) \cdot H^{\frac{3}{2}}$$

Rearranging $\quad (b - 0.1 \cdot n \cdot H) \cdot H^{\frac{3}{2}} = \frac{3 \cdot Q}{2 \cdot \sqrt{2 \cdot g} \cdot C_d}$

This is a nonlinear implicit equation for H and must be solved numerically. We can use one of a number of numerical root finding techniques, such as Newton's method, or we can use *Excel*'s *Solver* or *Goal Seek*, or we can manually iterate, as below.

The right side evaluates to $\frac{3 \cdot Q}{2 \cdot \sqrt{2 \cdot g} \cdot C_d} = 0.273 \cdot m^{\frac{5}{2}}$

For $H = 1 \cdot m$ $\quad (b - 0.1 \cdot n \cdot H) \cdot H^{\frac{3}{2}} = 1.40 \cdot m^{\frac{5}{2}}$ $\quad$ For $H = 0.5 \cdot m$ $\quad (b - 0.1 \cdot n \cdot H) \cdot H^{\frac{3}{2}} = 0.530 \cdot m^{\frac{5}{2}}$

For $H = 0.3 \cdot m$ $\quad (b - 0.1 \cdot n \cdot H) \cdot H^{\frac{3}{2}} = 0.253 \cdot m^{\frac{5}{2}}$ $\quad$ For $H = 0.35 \cdot m$ $\quad (b - 0.1 \cdot n \cdot H) \cdot H^{\frac{3}{2}} = 0.317 \cdot m^{\frac{5}{2}}$

For $H = 0.31 \cdot m$ $\quad (b - 0.1 \cdot n \cdot H) \cdot H^{\frac{3}{2}} = 0.265 \cdot m^{\frac{5}{2}}$ $\quad$ For $H = 0.315 \cdot m$ $\quad (b - 0.1 \cdot n \cdot H) \cdot H^{\frac{3}{2}} = 0.272 \cdot m^{\frac{5}{2}}$

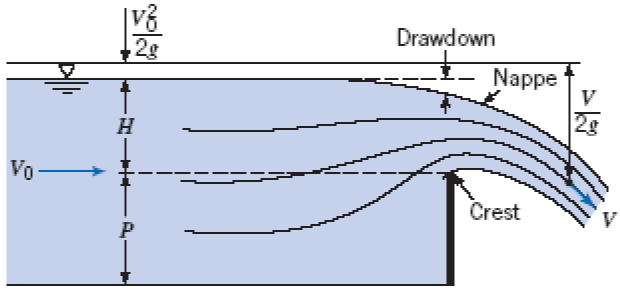
For $H = 0.316 \cdot m$ $\quad (b - 0.1 \cdot n \cdot H) \cdot H^{\frac{3}{2}} = 0.273 \cdot m^{\frac{5}{2}}$ $\quad H = 0.316\, m$

But from the figure $\quad H + P = 2.5 \cdot m$ $\quad P = 2.5 \cdot m - H$ $\quad P = 2.18\, m$

Problem 11.71

[Difficulty: 1]

11.71 For a sharp-crested suppressed weir ($C_w \approx 3.33$) of length $B = 8.0$ ft, $P = 2.0$ ft, and $H = 1.0$ ft, determine the discharge over the weir. Neglect the velocity of approach head.



Given: Data on rectangular, sharp-crested weir

Find: Discharge

Solution:

Basic equation: $Q = C_w \cdot b \cdot H^{\frac{3}{2}}$ where $C_w = 3.33$ and $b = 8 \cdot ft$ $P = 2 \cdot ft$ $H = 1 \cdot ft$

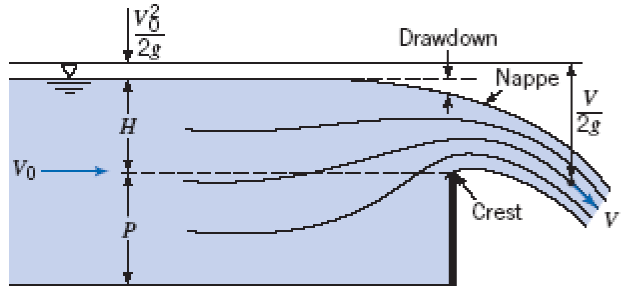
Note that this is an "engineering" equation, to be used without units!

$$Q = C_w \cdot b \cdot H^{\frac{3}{2}} \qquad Q = 26.6 \frac{ft^3}{s}$$

Problem 11.72

[Difficulty: 3]

11.72 A rectangular sharp-crested weir with end contractions is 1.5 m long. How high should the weir crest be placed in a channel to maintain an upstream depth of 2.5 m for 0.5 m^3/s flow rate?



Given: Data on rectangular, sharp-crested weir

Find: Required weir height

Solution:

Basic equations: $Q = C_d \cdot \frac{2}{3} \cdot \sqrt{2 \cdot g} \cdot b' \cdot H^{\frac{3}{2}}$ where $C_d = 0.62$ and $b' = b - 0.1 \cdot n \cdot H$ with $n = 2$

Given data: $b = 1.5 \cdot m$ $\quad Q = 0.5 \cdot \frac{m^3}{s}$

Hence we find

$$Q = C_d \cdot \frac{2}{3} \cdot \sqrt{2 \cdot g} \cdot b' \cdot H^{\frac{3}{2}} = C_d \cdot \frac{2}{3} \cdot \sqrt{2 \cdot g} \cdot (b - 0.1 \cdot n \cdot H) \cdot H^{\frac{3}{2}}$$

Rearranging $\quad (b - 0.1 \cdot n \cdot H) \cdot H^{\frac{3}{2}} = \frac{3 \cdot Q}{2 \cdot \sqrt{2 \cdot g} \cdot C_d}$

This is a nonlinear implicit equation for H and must be solved numerically. We can use one of a number of numerical root finding techniques, such as Newton's method, or we can use *Excel*'s *Solver* or *Goal Seek*, or we can manually iterate, as below.

The right side evaluates to $\frac{3 \cdot Q}{2 \cdot \sqrt{2 \cdot g} \cdot C_d} = 0.273 \cdot m^{\frac{5}{2}}$

For $H = 1 \cdot m$ $\quad (b - 0.1 \cdot n \cdot H) \cdot H^{\frac{3}{2}} = 1.30 \cdot m^{\frac{5}{2}}$ $\qquad$ For $H = 0.5 \cdot m$ $\quad (b - 0.1 \cdot n \cdot H) \cdot H^{\frac{3}{2}} = 0.495 \cdot m^{\frac{5}{2}}$

For $H = 0.3 \cdot m$ $\quad (b - 0.1 \cdot n \cdot H) \cdot H^{\frac{3}{2}} = 0.237 \cdot m^{\frac{5}{2}}$ $\qquad$ For $H = 0.35 \cdot m$ $\quad (b - 0.1 \cdot n \cdot H) \cdot H^{\frac{3}{2}} = 0.296 \cdot m^{\frac{5}{2}}$

For $H = 0.34 \cdot m$ $\quad (b - 0.1 \cdot n \cdot H) \cdot H^{\frac{3}{2}} = 0.284 \cdot m^{\frac{5}{2}}$ $\qquad$ For $H = 0.33 \cdot m$ $\quad (b - 0.1 \cdot n \cdot H) \cdot H^{\frac{3}{2}} = 0.272 \cdot m^{\frac{5}{2}}$

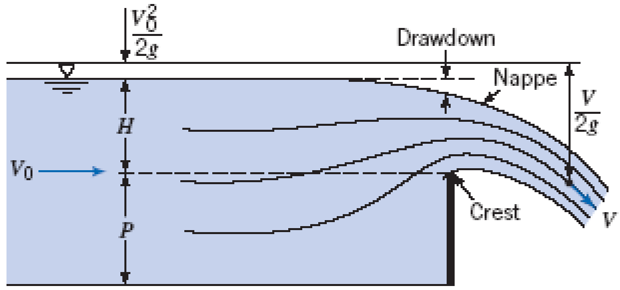
For $H = 0.331 \cdot m$ $\quad (b - 0.1 \cdot n \cdot H) \cdot H^{\frac{3}{2}} = 0.273 \cdot m^{\frac{5}{2}}$ $\qquad H = 0.331\ m$

But from the figure $\quad H + P = 2.5 \cdot m$ $\qquad P = 2.5 \cdot m - H$ $\qquad P = 2.17\ m$

Problem 11.73

[Difficulty: 1]

11.73 Determine the head on a 60° V-notch weir for a discharge of 150 L/s. Take $C_d \approx 0.58$.



Given: Data on V-notch weir

Find: Flow head

Solution:

Basic equation:

$$Q = C_d \cdot \frac{8}{15} \cdot \sqrt{2 \cdot g} \cdot \tan\left(\frac{\theta}{2}\right) \cdot H^{\frac{5}{2}}$$

where $C_d = 0.58$ $\quad \theta = 60 \cdot \text{deg}$ $\quad Q = 150 \cdot \frac{L}{s}$

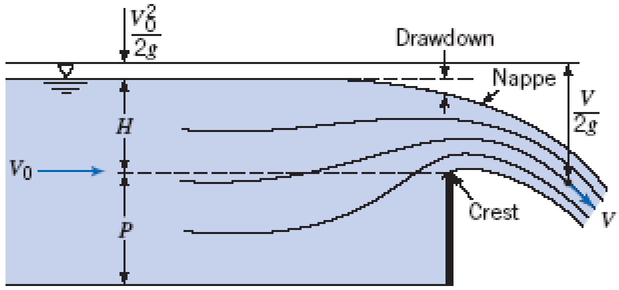
$$H = \left(\frac{Q}{C_d \cdot \frac{8}{15} \cdot \sqrt{2 \cdot g} \cdot \tan\left(\frac{\theta}{2}\right)}\right)^{\frac{2}{5}}$$

$H = 0.514\text{m}$

Problem 11.74

[Difficulty: 1]

11.74 The head on a 90° V-notch weir is 1.5 ft. Determine the discharge.



Given: Data on V-notch weir

Find: Discharge

Solution:

Basic equation: $Q = C_w \cdot H^{\frac{5}{2}}$ where $H = 1.5 \cdot ft$ $C_w = 2.50$ for $\theta = 90 \cdot deg$

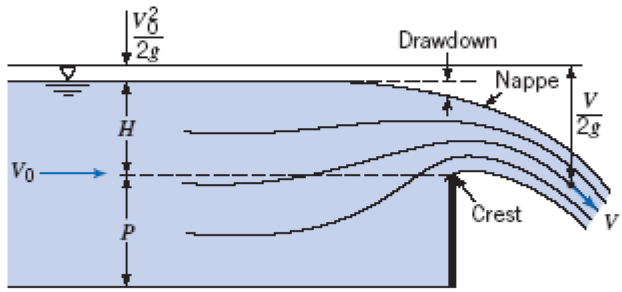
Note that this is an "engineering" equation in which we ignore units!

$Q = C_w \cdot H^{\frac{5}{2}}$ $\qquad Q = 6.89 \frac{ft^3}{s}$

Problem 11.75

[Difficulty: 1]

11.75 Determine the weir coefficient of a 90° V-notch weir for a head of 180 mm for a flow rate of 20 L/s.



Given: Data on V-notch weir

Find: Weir coefficient

Solution:

Basic equation: $Q = C_w \cdot H^{\frac{5}{2}}$ where $H = 180 \cdot mm$ $Q = 20 \cdot \frac{L}{s}$

Note that this is an "engineering" equation in which we ignore units!

$$C_w = \frac{Q}{H^{\frac{5}{2}}} \qquad C_w = 1.45$$

Problem 12.1

[Difficulty: 2]

12.1 An air flow in a duct passes through a thick filter. What happens to the pressure, temperature, and density of the air as it does so? *Hint*: This is a throttling process.

Given: Air flow through a filter

Find: Change in p, T and ρ

Solution:

Basic equations: $h_2 - h_1 = c_p \cdot (T_2 - T_1)$ $\quad p = \rho \cdot R \cdot T$

Assumptions: 1) Ideal gas 2) Throttling process

In a throttling process enthalpy is constant. Hence $h_2 - h_1 = 0$ so $T_2 - T_1 = 0$ or $T = \text{constant}$

The filter acts as a resistance through which there is a pressure drop (otherwise there would be no flow. Hence $p_2 < p_1$

From the ideal gas equation $\frac{p_1}{p_2} = \frac{\rho_1 \cdot T_1}{\rho_2 \cdot T_2}$ so $\rho_2 = \rho_1 \cdot \left(\frac{T_1}{T_2}\right) \cdot \left(\frac{p_2}{p_1}\right) = \rho_1 \cdot \left(\frac{p_2}{p_1}\right)$ Hence $\rho_2 < \rho_1$

The governing equation for entropy is $\Delta s = c_p \cdot \ln\left(\frac{T_2}{T_1}\right) - R \cdot \ln\left(\frac{p_2}{p_1}\right)$

Hence $\Delta s = -R \cdot \ln\left(\frac{p_2}{p_1}\right)$ and $\frac{p_2}{p_1} < 1$ so $\Delta s > 0$

Entropy increases because throttling is an irreversible adiabatic process

Problem 12.2

[Difficulty: 2]

12.2 Air is expanded in a steady flow process through a turbine. Initial conditions are 1300°C and 2.0 MPa (abs). Final conditions are 500°C and atmospheric pressure. Show this process on a *Ts* diagram. Evaluate the changes in internal energy, enthalpy, and specific entropy for this process.

Solution:

$$\Delta u = u_2 - u_1 = c_v(T_2 - T_1) = 717.4 \frac{J}{kg\cdot K}(500-1300)K = -574 \text{ kJ/kg} \qquad u_2 - u_1$$

$$\Delta h = h_2 - h_1 = c_p(T_2 - T_1) = 1004 \frac{J}{kg\cdot K}(-800K) = -803 \text{ kJ/kg} \qquad h_2 - h_1$$

To calculate the entropy change, we use the Tds equation

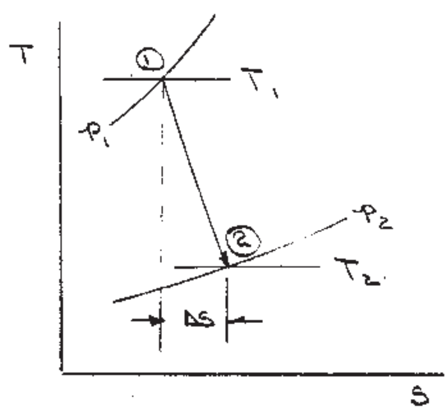
$$T ds = dh - v\,dp = c_p dT - RT\frac{dp}{p}$$

$$\therefore \quad ds = c_p \frac{dT}{T} - R\frac{dp}{p}$$

$$s_2 - s_1 = c_p \ln\frac{T_2}{T_1} - R\ln\frac{p_2}{p_1}$$

$$= 1004 \frac{J}{kg\cdot K}\ln\frac{500+273}{1300+273} - 286.9\frac{J}{kg\cdot K}\ln\frac{0.101}{2.0}$$

$$s_2 - s_1 = (-713.3 + 856.6)\text{ J/kg}\cdot K = 143 \text{ J/kg}\cdot K \qquad s_2 - s_1$$



Problem 12.3

[Difficulty: 2]

12.3 A vendor claims that an adiabatic air compressor takes in air at atmosphere pressure and 50°F and delivers the air at 150 psig and 200°F. Is this possible? Justify your answer by calculation. Sketch the process on a *Ts* diagram.

Given: Data on an air compressor

Find: Whether or not the vendor claim is feasible

Solution:

Basic equation: $$\Delta s = c_p \cdot \ln\left(\frac{T_2}{T_1}\right) - R \cdot \ln\left(\frac{p_2}{p_1}\right)$$

The data provided, or available in the Appendices, is:

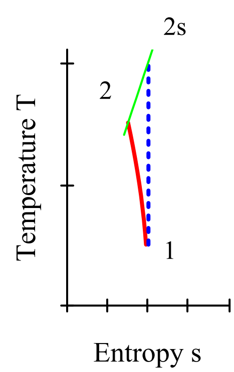
$$p_1 = 14.7 \cdot psi \qquad T_1 = (50 + 460) \cdot R$$

$$p_2 = (150 + 14.7) \cdot psi \qquad T_2 = (200 + 460) \cdot R$$

$$c_p = 0.2399 \cdot \frac{BTU}{lb \cdot R} \qquad R_{gas} = 53.33 \cdot \frac{ft \cdot lbf}{lb \cdot R} = 0.0685 \cdot \frac{BTU}{lb \cdot R}$$

Then $$\Delta s = c_p \cdot \ln\left(\frac{T_2}{T_1}\right) - R_{gas} \cdot \ln\left(\frac{p_2}{p_1}\right) \qquad \Delta s = -0.1037 \cdot \frac{BTU}{lb \cdot R}$$

We have plotted the actual process in red (1-2) on this temperature-entropy diagram, and the ideal compression (isentropic) in blue (1-2s). The line of constant pressure equal to 150 psig is shown in green. However, can this process actually occur? The second law of thermodynamics states that, for an adiabatic process



$$\Delta s \geq 0 \qquad \text{or for all real processes} \qquad \Delta s > 0$$

Hence the process is NOT feasible!

Problem 12.4

[Difficulty: 2]

12.4 A turbine manufacturer claims that an adiabatic gas turbine can take flow at 10 atmospheres and 2200°F and exhaust to atmospheric pressure at a temperature of 850°F. Sketch the process on a *Ts* diagram, and prove whether the manufacturer's claims are possible. Assume that the gas has the same properties as air.

Given:
Data on turbine inlet and exhaust

Find:
Whether or not the vendor claim is feasible

Solution:

Basic equation: $$\Delta s = c_p \cdot \ln\left(\frac{T_2}{T_1}\right) - R \cdot \ln\left(\frac{p_2}{p_1}\right)$$

The data provided, or available in the Appendices, is:

$$p_1 = 10 \cdot \text{atm} = 146.959 \cdot \text{psi} \qquad T_1 = (2200 + 460) \cdot R \qquad T_1 = 1.478 \times 10^3 \, K$$

$$p_2 = 1 \cdot \text{atm} = 14.696 \cdot \text{psi} \qquad T_2 = (850 + 460) \cdot R \qquad T_2 = 727.778 \, K$$

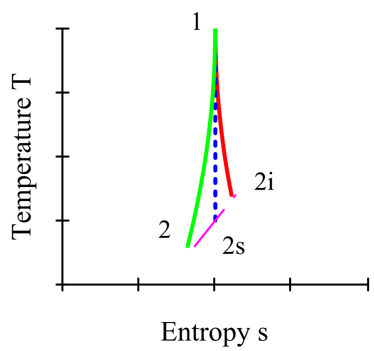
$$c_p = 0.2399 \cdot \frac{BTU}{lb \cdot R} \qquad R_{gas} = 53.33 \cdot \frac{ft \cdot lbf}{lb \cdot R} = 0.0685 \cdot \frac{BTU}{lb \cdot R}$$

Then $$\Delta s = c_p \cdot \ln\left(\frac{T_2}{T_1}\right) - R_{gas} \cdot \ln\left(\frac{p_2}{p_1}\right) \qquad \Delta s = -0.0121 \cdot \frac{BTU}{lb \cdot R}$$

An example of this type of process is plotted in green on the graph. Also plotted are an isentropic process (blue - 1-2s) and one with an increase in entropy (red: 1-2i). All three processes expand to the same pressure. The constant pressure curve is drawn in purple.The second law of thermodynamics states that, for an adiabatic process

$\Delta s \geq 0$ or for all real processes $\Delta s > 0$

Hence the process is NOT feasible!



Problem 12.5

[Difficulty: 2]

12.5 Air initially at 50 psia and 660°R expands to atmospheric pressure. The process by which this expansion occurs is defined by the expression $pV^{1.3}$ = constant. Calculate the final temperature and the change in entropy through this process.

Given: Air before and after expansion; process

Find: Final temperature and change in entropy

Solution:

Basic equations: $\Delta s = c_p \cdot \ln\left(\frac{T_2}{T_1}\right) - R \cdot \ln\left(\frac{p_2}{p_1}\right)$ $\qquad p \cdot V = m \cdot R \cdot T$

The data provided, or available in the Appendices, is:

$p_1 = 50 \cdot psi \qquad T_1 = 660 \cdot R$

$p_2 = 1 \cdot atm = 14.696 \cdot psi$

$c_p = 0.2399 \cdot \frac{Btu}{lb \cdot R} \qquad R_{gas} = 53.33 \cdot \frac{ft \cdot lbf}{lb \cdot R} = 0.0685 \cdot \frac{Btu}{lb \cdot R}$

From the process given: $p_1 \cdot V_1^{1.3} = p_2 \cdot V_2^{1.3}$ From the ideal gas equation of state: $\frac{p_2 \cdot V_2}{p_1 \cdot V_1} = \frac{T_2}{T_1} \qquad \frac{V_1}{V_2} = \frac{p_2}{p_1} \cdot \frac{T_1}{T_2}$

When we combine these two equations we get: $\frac{p_2}{p_1} = \left(\frac{V_1}{V_2}\right)^{1.3} = \left(\frac{p_2}{p_1} \cdot \frac{T_1}{T_2}\right)^{1.3}$ Solving for temperature ratio: $\frac{T_1}{T_2} = \left(\frac{p_2}{p_1}\right)^{\frac{1}{1.3}-1}$

So the final temperature is: $T_2 = T_1 \cdot \left(\frac{p_1}{p_2}\right)^{\frac{1}{1.3}-1} \qquad T_2 = 497.5 \cdot R$

Then $\qquad \Delta s = c_p \cdot \ln\left(\frac{T_2}{T_1}\right) - R_{gas} \cdot \ln\left(\frac{p_2}{p_1}\right) \qquad \Delta s = 0.0161 \cdot \frac{Btu}{lb \cdot R}$

Problem 12.6

[Difficulty: 2]

12.6 What is the lowest possible delivery temperature generated by an adiabatic air compressor, starting with standard atmosphere conditions and delivering the air at 500 kPa (gage)? Sketch the process on a *Ts* diagram.

Given: Adiabatic air compressor

Find: Lowest delivery temperature; Sketch the process on a Ts diagram

Solution:

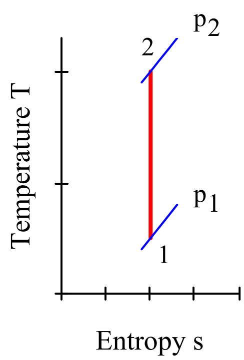
Basic equation: $$\Delta s = c_p \cdot \ln\left(\frac{T_2}{T_1}\right) - R \cdot \ln\left(\frac{p_2}{p_1}\right)$$

The lowest temperature implies an ideal (reversible) process; it is also adiabatic, so $\Delta s = 0$, and $$T_2 = T_1 \cdot \left(\frac{p_1}{p_2}\right)^{\frac{1-k}{k}}$$

The data provided, or available in the Appendices, is: $p_1 = 101 \cdot kPa$ $\quad p_2 = (500 + 101) \cdot kPa$ $\quad T_1 = 288.2 \cdot K$ $\quad k = 1.4$

Hence $$T_2 = T_1 \cdot \left(\frac{p_1}{p_2}\right)^{\frac{1-k}{k}}$$ $\qquad T_2 = 864 \cdot R$

The process is



Problem 12.7

[Difficulty: 2]

12.7 Air expands without heat transfer through a turbine from a pressure of 10 bars and a temperature of 1400 K to a pressure of 1 bar. If the turbine has an efficiency of 80 percent, determine the exit temperature, and the changes in enthalpy and entropy across the turbine. If the turbine is generating 1 MW of power, what is the mass flow rate of air through the turbine?

Given: Data on turbine inlet and exhaust

Find: Whether or not the vendor claim is feasible

Solution:

Basic equations: $\Delta s = c_p \cdot \ln\left(\frac{T_2}{T_1}\right) - R \cdot \ln\left(\frac{p_2}{p_1}\right)$ $\qquad \Delta h = c_p \cdot \Delta T \qquad \frac{T_2}{T_1} = \left(\frac{p_2}{p_2}\right)^{\frac{k-1}{k}}$ when s = constant

$$\eta = \frac{h_1 - h_2}{h_1 - h_{2s}} = \frac{T_1 - T_2}{T_1 - T_{2s}}$$

The data provided, or available in the Appendices, is:

$p_1 = 10 \cdot bar = 1 \times 10^3 \cdot kPa \qquad T_1 = 1400 \cdot K \qquad \eta = 80 \cdot \% \qquad P = 1 \cdot MW$

$p_2 = 1 \cdot bar = 100 \cdot kPa$

$c_p = 1004 \cdot \frac{J}{kg \cdot K} \qquad R_{gas} = 287 \cdot \frac{J}{kg \cdot K} \qquad k = 1.4$

If the expansion were isentropic, the exit temperature would be: $T_{2s} = T_1 \cdot \left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}} = 725.126\,K$

Since the turbine is not isentropic, the final temperature is higher: $T_2 = T_1 - \eta \cdot (T_1 - T_{2s}) = 860.101\,K$

Then $\Delta h = c_p \cdot (T_1 - T_2) = 542.058 \cdot \frac{kJ}{kg} \qquad \Delta s = c_p \cdot \ln\left(\frac{T_2}{T_1}\right) - R_{gas} \cdot \ln\left(\frac{p_2}{p_1}\right) \qquad \Delta s = 171.7157 \cdot \frac{J}{kg \cdot K}$

The mass flow rate is: $m = \frac{P}{\Delta h} = 1.845\,\frac{kg}{s}$

Problem 12.8

[Difficulty: 2]

12.8 A test chamber is separated into two equal chambers by a rubber diaphragm. One contains air at 20°C and 200 kPa (absolute), and the other has a vacuum. If the diaphragm is punctured, find the pressure and temperature of the air after it expands to fill the chamber. *Hint*: This is a rapid, violent event, so is irreversible but adiabatic.

Given: Test chamber with two chambers

Find: Pressure and temperature after expansion

Solution:

Basic equation: $p = \rho \cdot R \cdot T$ $\quad \Delta u = q - w$ (First law - closed system) $\quad \Delta u = c_V \cdot \Delta T$

Assumptions: 1) Ideal gas 2) Adiabatic 3) No work

For no work and adiabatic the first law becomes $\Delta u = 0$ or for an Ideal gas $\Delta T = 0$ $\quad T_2 = T_1$

We also have $M = \rho \cdot Vol = const$ and $Vol_2 = 2 \cdot Vol_1$ so $\rho_2 = \frac{1}{2} \cdot \rho_1$

From the ideal gas equation $\frac{p_2}{p_1} = \frac{\rho_2}{\rho_1} \cdot \frac{T_2}{T_1} = \frac{1}{2}$ so $p_2 = \frac{1}{2} \cdot p_1$

Hence $T_2 = 20\,°F$ $\quad p_2 = \frac{200 \cdot kPa}{2}$ $\quad p_2 = 100 \cdot kPa$

Note that $\Delta s = c_p \cdot \ln\left(\frac{T_2}{T_1}\right) - R \cdot \ln\left(\frac{p_2}{p_1}\right) = -R \cdot \ln\left(\frac{1}{2}\right) = 0.693 \cdot R$ so entropy increases (irreversible adiabatic)

Problem 12.9

[Difficulty: 2]

12.9 An automobile supercharger is a device that pressurizes the air that is used by the engine for combustion to increase the engine power (how does it differ from a turbocharger?). A supercharger takes in air at 70°F and atmospheric pressure and boosts it to 200 psig, at an intake rate of 0.5 ft^3/s. What are the pressure, temperature, and volume flow rate at the exit? (The relatively high exit temperature is the reason an inter-cooler is also used.) Assuming a 70 percent efficiency, what is the power drawn by the supercharger? *Hint*: The efficiency is defined as the ratio of the isentropic power to actual power.

Given: Supercharger

Find: Pressure, temperature and flow rate at exit; power drawn

Solution:

Basic equation: $p = \rho\cdot R_{air}\cdot T$ $\qquad$ $\Delta s = c_p\cdot\ln\left(\frac{T_2}{T_1}\right) - R\cdot\ln\left(\frac{p_2}{p_1}\right)$

$\Delta h = q - w$ (First law - open system) $\qquad$ $\Delta h = c_p\cdot\Delta T$

Assumptions: 1) Ideal gas 2) Adiabatic

In an ideal process (reversible and adiabatic) the first law becomes $\Delta h = w$ or for an ideal gas $w_{ideal} = c_p\cdot\Delta T$

For an isentropic process $\Delta s = 0 = c_p\cdot\ln\left(\frac{T_2}{T_1}\right) - R\cdot\ln\left(\frac{p_2}{p_1}\right)$ or $\frac{T_2}{T_1} = \left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}}$

The given or available data is $T_1 = (70 + 460)\cdot R$ $\qquad$ $p_1 = 14.7\cdot psi$ $\qquad$ $p_2 = (200 + 14.7)\cdot psi$ $\qquad$ $\eta = 70\cdot\%$

$Q_1 = 0.5\cdot\frac{ft^3}{s}$ $\qquad$ $k = 1.4$ $\qquad$ $c_p = 0.2399\cdot\frac{Btu}{lbm\cdot R}$ $\qquad$ $R_{air} = 53.33\cdot\frac{ft\cdot lbf}{lbm\cdot R}$

Hence $T_2 = \left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}}\cdot T_1$ $\qquad$ $T_2 = 1140\cdot R$ $\qquad$ $T_2 = 681\cdot°F$ $\qquad$ $p_2 = 215\cdot psi$

We also have $m_{rate} = \rho_1\cdot Q_1 = \rho_2\cdot Q_2$ $\qquad$ $Q_2 = Q_1\cdot\frac{\rho_1}{\rho_2}$ $\qquad$ $Q_2 = Q_1\cdot\frac{p_1}{p_2}\cdot\frac{T_2}{T_1}$ $\qquad$ $Q_2 = 0.0737\cdot\frac{ft^3}{s}$

For the power we use $P_{ideal} = m_{rate}\cdot w_{ideal} = \rho_1\cdot Q_1\cdot c_p\cdot\Delta\cdot T$

From the ideal gas equation $\rho_1 = \frac{p_1}{R_{air}\cdot T_1}$ $\qquad$ $\rho_1 = 0.00233\cdot\frac{slug}{ft^3}$ or $\rho_1 = 0.0749\cdot\frac{lbm}{ft^3}$

Hence $P_{ideal} = \rho_1\cdot Q_1\cdot c_p\cdot\left(T_2 - T_1\right)$ $\qquad$ $P_{ideal} = 5.78\cdot kW$

The actual power needed is $P_{actual} = \frac{P_{ideal}}{\eta}$ $\qquad$ $P_{actual} = 8.26\cdot kW$

A supercharger is a pump that forces air into an engine, but generally refers to a pump that is driven directly by the engine, as opposed to a turbocharger that is driven by the pressure of the exhaust gases.

Problem 12.10

[Difficulty: 2]

12.10 Five kilograms of air is cooled in a closed tank from 250 to 50°C. The initial pressure is 3 MPa. Compute the changes in entropy, internal energy, and enthalpy. Show the process state points on a Ts diagram.

Given: Cooling of air in a tank

Find: Change in entropy, internal energy, and enthalpy

Solution:

Basic equation: $p = \rho \cdot R \cdot T$ $\qquad \Delta s = c_p \cdot \ln\left(\frac{T_2}{T_1}\right) - R \cdot \ln\left(\frac{p_2}{p_1}\right)$

$\Delta u = c_v \cdot \Delta T$ $\qquad \Delta h = c_p \cdot \Delta T$

Assumptions: 1) Ideal gas 2) Constant specific heats

Given or available data $M = 5 \cdot kg$ $\quad T_1 = (250 + 273) \cdot K$ $\quad T_2 = (50 + 273) \cdot K$ $\quad p_1 = 3 \cdot MPa$

$c_p = 1004 \cdot \frac{J}{kg \cdot K}$ $\quad c_v = 717.4 \cdot \frac{J}{kg \cdot K}$ $\quad k = \frac{c_p}{c_v}$ $\quad k = 1.4$ $\quad R = c_p - c_v$ $\quad R = 287 \cdot \frac{J}{kg \cdot K}$

For a constant volume process the ideal gas equation gives $\frac{p_2}{p_1} = \frac{T_2}{T_1}$ $\quad p_2 = \frac{T_2}{T_1} \cdot p_1$ $\quad p_2 = 1.85 \cdot MPa$

Then $\Delta s = c_p \cdot \ln\left(\frac{T_2}{T_1}\right) - R \cdot \ln\left(\frac{p_2}{p_1}\right)$ $\quad \Delta s = -346 \cdot \frac{J}{kg \cdot K}$

$\Delta u = c_v \cdot (T_2 - T_1)$ $\quad \Delta u = -143 \cdot \frac{kJ}{kg}$

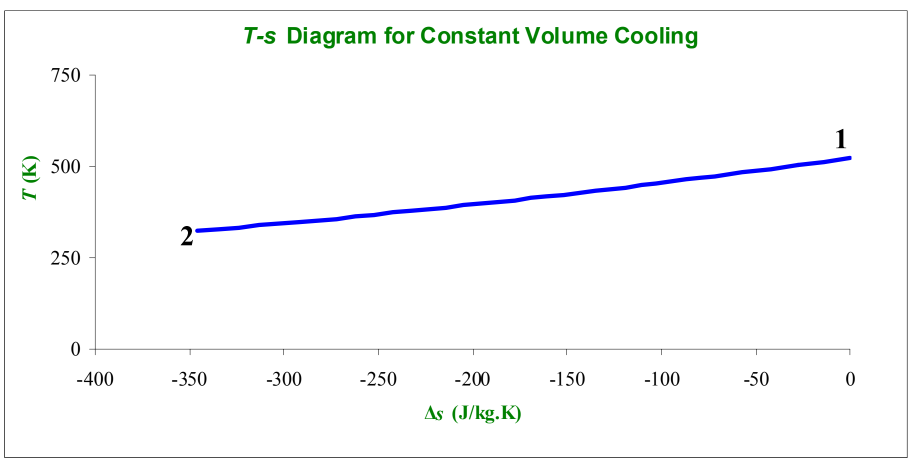
$\Delta h = c_p \cdot (T_2 - T_1)$ $\quad \Delta h = -201 \cdot \frac{kJ}{kg}$

Total amounts are $\Delta S = M \cdot \Delta s$ $\quad \Delta S = -1729 \cdot \frac{J}{K}$

$\Delta U = M \cdot \Delta u$ $\quad \Delta U = -717 \cdot kJ$

$\Delta H = M \cdot \Delta h$ $\quad \Delta H = -1004 \cdot kJ$

Here is a plot of the T-s diagram:



Problem 12.11

[Difficulty: 3]

12.11 Air is contained in a piston-cylinder device. The temperature of the air is 100°C. Using the fact that for a reversible process the heat transfer $q = \int T ds$, compare the amount of heat (J/kg) required to raise the temperature of the air to 1200°C at (a) constant pressure and (b) constant volume. Verify your results using the first law of thermodynamics. Plot the processes on a *Ts* diagram.

Given: Air in a piston-cylinder

Find: Heat to raise temperature to 1200ºC at a) constant pressure and b) constant volume

Solution:

The data provided, or available in the Appendices, is:

$T_1 = (100 + 273) \cdot K \qquad T_2 = (1200 + 273) \cdot K \qquad R = 287 \cdot \frac{J}{kg \cdot K} \qquad c_p = 1004 \cdot \frac{J}{kg \cdot K} \qquad c_v = c_p - R \qquad c_v = 717 \cdot \frac{J}{kg \cdot K}$

a) For a constant pressure process we start with $\quad T \cdot ds = dh - v \cdot dp$

Hence, for *p* = const. $\quad ds = \frac{dh}{T} = c_p \cdot \frac{dT}{T}$

But $\quad \delta q = T \cdot ds$

Hence $\quad \delta q = c_p \cdot dT \qquad q = \int c_p\, dT \qquad q = c_p \cdot (T_2 - T_1) \qquad q = 1104 \cdot \frac{kJ}{kg}$

b) For a constant volume process we start $\quad T \cdot ds = du + p \cdot dv$

Hence, for v = const. $\quad ds = \frac{du}{T} = c_v \cdot \frac{dT}{T}$

But $\quad \delta q = T \cdot ds$

Hence $\quad \delta q = c_v \cdot dT \qquad q = \int c_v\, dT \qquad q = c_v \cdot (T_2 - T_1) \qquad q = 789 \cdot \frac{kJ}{kg}$

Heating to a higher temperature at constant pressure requires more heat than at constant volume: some of the heat is used to do work in expanding the gas; hence for constant pressure less of the heat is available for raising the temperature.

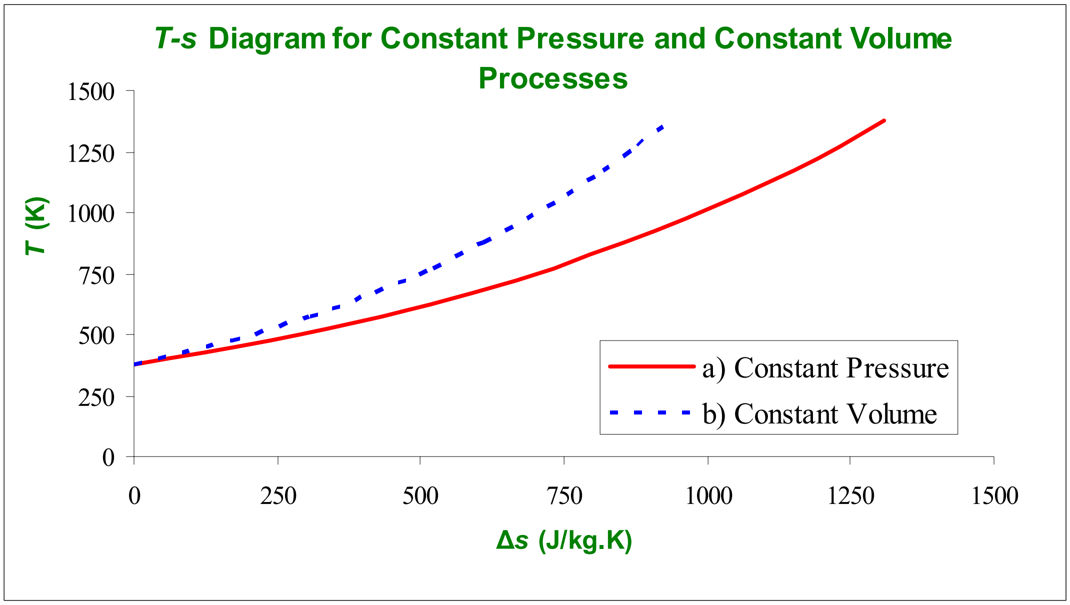
From the first law: Constant pressure: $\quad q = \Delta u + w \qquad$ Constant volume: $\quad q = \Delta u$

The two processes can be plotted using Eqs. 11.11b and 11.11a, simplified for the case of constant pressure and constant volume.

a) For constant pressure $\quad s_2 - s_1 = c_p \cdot \ln\left(\frac{T_2}{T_1}\right) - R \cdot \ln\left(\frac{p_2}{p_1}\right) \quad$ so $\quad \Delta s = c_p \cdot \ln\left(\frac{T_2}{T_1}\right)$

b) For constant volume $\quad s_2 - s_1 = c_v \cdot \ln\left(\frac{T_2}{T_1}\right) + R \cdot \ln\left(\frac{v_2}{v_1}\right) \quad$ so $\quad \Delta s = c_v \cdot \ln\left(\frac{T_2}{T_1}\right)$

The processes are plotted in *Excel* and shown on the next page



T-s Diagram for Constant Pressure and Constant Volume Processes
T (K)
1500
1250
1000
750
500
250
0
0
250
500
750
1000
1250
1500
Δs (J/kg.K)
a) Constant Pressure
b) Constant Volume

Problem 12.12

[Difficulty: 4]

12.12 The four-stroke Otto cycle of a typical automobile engine is sometimes modeled as an ideal air-standard closed system. In this simplified system the combustion process is modeled as a heating process, and the exhaust-intake process as a cooling process of the working fluid (air). The cycle consists of: isentropic compression from state ① ($p_1 = 100$ kPa (abs), $T_1 = 20°C$, $V_1 = 500$ cc) to state ② ($V_2 = V_1/8.5$); isometric (constant volume) heat addition to state ③ ($T_3 = 2750°C$); isentropic expansion to state ④ ($V_4 = V_1$); and isometric cooling back to state ①. Plot the pV and Ts diagrams for this cycle, and find the efficiency, defined as the net work (the cycle area in pV space) divided by the heat added.

Given: Data on Otto cycle

Find: Plot of pV and Ts diagrams; efficiency

Solution:

The data provided, or available in the Appendices, is:

$c_p = 1004 \cdot \frac{J}{kg \cdot K}$ $\quad R = 287 \cdot \frac{J}{kg \cdot K}$ $\quad c_v = c_p - R$ $\quad c_v = 717 \cdot \frac{J}{kg \cdot K}$ $\quad k = \frac{c_p}{c_v}$ $\quad k = 1.4$

$p_1 = 100 \cdot kPa$ $\quad T_1 = (20 + 273) \cdot K$ $\quad T_3 = (2750 + 273) \cdot K$ $\quad V_1 = 500 \cdot cc$ $\quad V_2 = \frac{V_1}{8.5}$ $\quad V_2 = 58.8 \cdot cc$

$V_4 = V_1$

Computed results: $\quad M = \frac{p_1 \cdot V_1}{R \cdot T_1}$ $\quad M = 5.95 \times 10^{-4} \, kg$

For process 1-2 we have isentropic behavior $\quad T \cdot v^{k-1} = \text{constant}$ $\quad p \cdot v^k = \text{constant}$ $\quad$ (12.12 a and 12.12b)

Hence $\quad T_2 = T_1 \cdot \left(\frac{V_1}{V_2}\right)^{k-1}$ $\quad T_2 = 690 \, K$ $\quad p_2 = p_1 \cdot \left(\frac{V_1}{V_2}\right)^k$ $\quad p_2 = 2002 \cdot kPa$

The process from 1 -2 is $\quad p(V) = p_1 \cdot \left(\frac{V_1}{V}\right)^k$ $\quad$ and $\quad s = \text{constant}$

The work is $\quad W_{12} = \left(\int_{V_1}^{V_2} p(V) \, dV = \frac{p_1 \cdot V_1 - p_2 \cdot V_2}{k - 1}\right)$ $\quad W_{12} = -169 \, J$ $\quad Q_{12} = 0 \cdot J$ $\quad$ (Isentropic)

For process 2 - 3 we have constant volume $\quad V_3 = V_2$ $\quad V_3 = 58.8 \cdot cc$

Hence $\quad p_3 = p_2 \cdot \frac{T_3}{T_2}$ $\quad p_3 = 8770 \cdot kPa$

The process from 2 -3 is $V = V_2 = \text{constant}$ and $\Delta s = c_v \cdot \ln\left(\frac{T}{T_2}\right)$ $W_{23} = 0 \cdot J$

(From 12.11a)

$Q_{23} = M \cdot \Delta u = M \cdot \int c_v \, dT$ $Q_{23} = M \cdot c_v \cdot (T_3 - T_2)$ $Q_{23} = 995 \, J$

For process 3 - 4 we again have isentropic behavior

Hence $T_4 = T_3 \cdot \left(\frac{V_3}{V_4}\right)^{k-1}$ $T_4 = 1284 \, K$ $p_4 = p_3 \cdot \left(\frac{V_3}{V_4}\right)^{k}$ $p_4 = 438 \cdot kPa$

The process from 3 - 4 is $p(V) = p_3 \cdot \left(\frac{V_3}{V}\right)^{k}$ and $s = \text{constant}$

The work is $W_{34} = \frac{p_3 \cdot V_3 - p_4 \cdot V_4}{k - 1}$ $W_{34} = 742 \, J$ $Q_{34} = 0 \cdot J$

For process 4-1 we again have constant volume

The process from 4 -1 is $V = V_4 = \text{constant}$ and $\Delta s = c_v \cdot \ln\left(\frac{T}{T_4}\right)$ $W_{41} = 0 \cdot J$

(From 12.11a)

$Q_{41} = M \cdot c_v \cdot (T_1 - T_4)$ $Q_{41} = -422 \, J$

The net work is $W_{net} = W_{12} + W_{23} + W_{34} + W_{41}$ $W_{net} = 572 \, J$

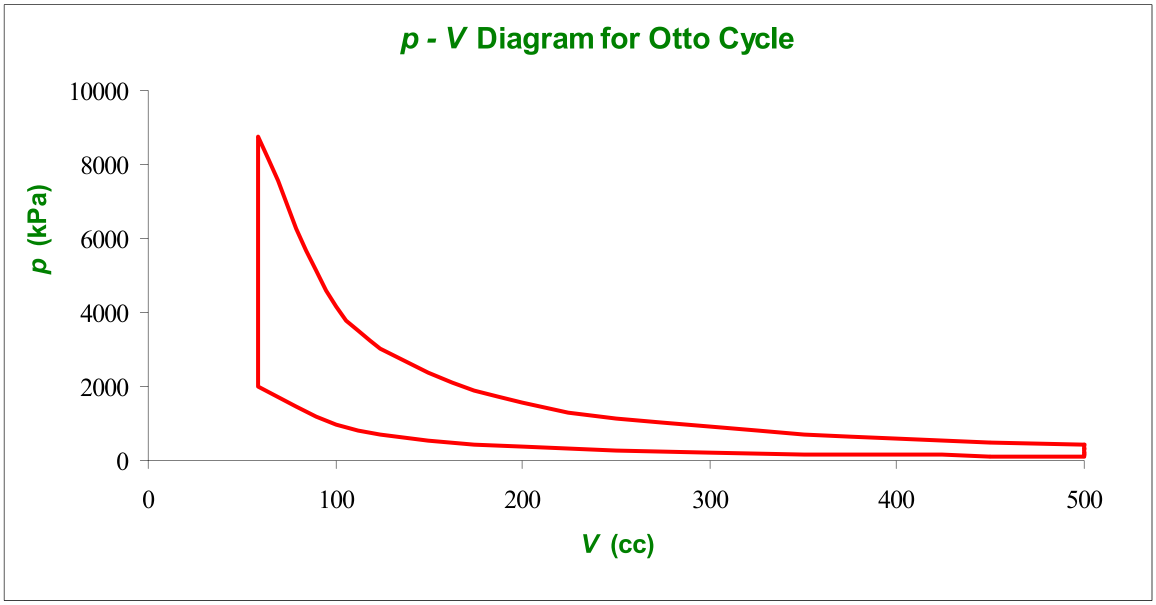
The efficiency is $\eta = \frac{W_{net}}{Q_{23}}$ $\eta = 57.5 \cdot \%$

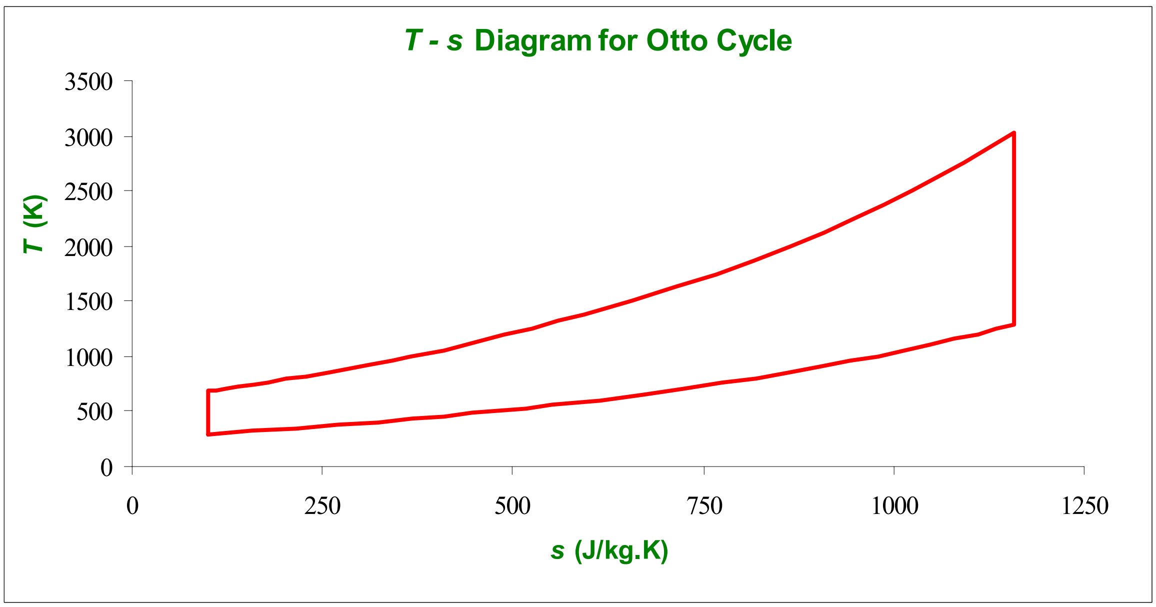
This is consistent with the expression for the Otto efficiency $\eta_{Otto} = 1 - \frac{1}{r^{k-1}}$

where *r* is the compression ratio $r = \frac{V_1}{V_2}$ $r = 8.5$

$\eta_{Otto} = 57.5 \cdot \%$

Plots of the cycle in *pV* and *Ts* space, generated using *Excel*, are shown on the next page.


p - V Diagram for Otto Cycle
10000
8000
6000
4000
2000
0
p (kPa)
0
100
200
300
400
500
V (cc)


T - s Diagram for Otto Cycle
3500
3000
2500
2000
1500
1000
500
0
T (K)
0
250
500
750
1000
1250
s (J/kg.K)

Problem 12.13

[Difficulty: 4]

12.13 The four-stroke cycle of a typical diesel engine is sometimes modeled as an ideal air-standard closed system. In this simplified system the combustion process is modeled as a heating process, and the exhaust-intake process as a cooling process of the working fluid (air). The cycle consists of: isentropic compression from state ① ($p_1 = 100$ kPa (abs), $T_1 = 20°$C, $V_1 = 500$ cc) to state ② ($V_2 = V_1/12.5$); isometric (constant volume) heat addition to state ③ ($T_3 = 3000°$C); isobaric heat addition to state ④ ($V_4 = 1.75V_3$); isentropic expansion to state ⑤; and isometric cooling back to state ①. Plot the pV and Ts diagrams for this cycle, and find the efficiency, defined as the net work (the cycle area in pV space) divided by the heat added.

Given: Data on diesel cycle

Find: Plot of pV and Ts diagrams; efficiency

Solution:

The data provided, or available in the Appendices, is:

$$c_p = 1004\cdot\frac{J}{kg\cdot K} \qquad R = 287\cdot\frac{J}{kg\cdot K} \qquad c_v = c_p - R \qquad c_v = 717\cdot\frac{J}{kg\cdot K} \qquad k = \frac{c_p}{c_v} \qquad k = 1.4$$

$$p_1 = 100\cdot kPa \qquad T_1 = (20 + 273)\cdot K \qquad T_3 = (3000 + 273)\cdot K \qquad V_1 = 500\cdot cc$$

$$V_2 = \frac{V_1}{12.5} \qquad V_2 = 40\cdot cc \qquad V_3 = V_2 \qquad V_4 = 1.75\cdot V_3 \qquad V_4 = 70\cdot cc \qquad V_5 = V_1$$

Computed results: $M = \frac{p_1\cdot V_1}{R\cdot T_1}$ $\qquad M = 5.95\times 10^{-4}\,kg$

For process 1-2 we have isentropic behavior $\quad T\cdot v^{k-1} = \text{constant}$ (12.12a) $\qquad p\cdot v^k = \text{constant}$ (12.12c)

Hence $\quad T_2 = T_1\cdot\left(\frac{V_1}{V_2}\right)^{k-1} \qquad T_2 = 805\,K \qquad p_2 = p_1\cdot\left(\frac{V_1}{V_2}\right)^k \qquad p_2 = 3435\cdot kPa$

The process from 1 -2 is $\quad p(V) = p_1\cdot\left(\frac{V_1}{V}\right)^k \quad$ and $\quad s = \text{constant}$

The work is $\quad W_{12} = \int_{V_1}^{V_2} p(V)\,dV = \frac{p_1\cdot V_1 - p_2\cdot V_2}{k-1} \qquad W_{12} = -218\,J \qquad Q_{12} = 0\cdot J$ (Isentropic)

For process 2 - 3 we have constant volume $\quad V_3 = V_2 \qquad V_3 = 40\cdot cc$

Hence $\quad p_3 = p_2\cdot\frac{T_3}{T_2} \qquad p_3 = 13963\cdot kPa$

The process from 2 -3 is $V = V_2 = \text{constant}$ and $\Delta s = c_v \cdot \ln\left(\frac{T}{T_2}\right)$ $W_{23} = 0 \cdot J$

(From Eq. 12.11a)

$Q_{23} = M \cdot \Delta u = M \cdot \int c_v \, dT$ $Q_{23} = M \cdot c_v \cdot (T_3 - T_2)$ $Q_{23} = 1052\,J$

For process 3 - 4 we have constant pressure $p_4 = p_3$ $p_4 = 13963 \cdot kPa$ $T_4 = T_3 \cdot \left(\frac{V_4}{V_3}\right)$ $T_4 = 5728\,K$

The process from 3 - 4 is $p = p_3 = \text{constant}$ and $\Delta s = c_p \cdot \ln\left(\frac{T}{T_3}\right)$

(From Eq. 12.11b)

$W_{34} = p_3 \cdot (V_4 - V_3)$ $W_{34} = 419\,J$ $Q_{34} = M \cdot c_p \cdot (T_4 - T_3)$ $Q_{34} = 1465\,J$

For process 4 - 5 we again have isentropic behavior $T_5 = T_4 \cdot \left(\frac{V_4}{V_5}\right)^{k-1}$ $T_5 = 2607 \cdot K$

Hence $p_5 = p_4 \cdot \left(\frac{V_4}{V_5}\right)^k$ $p_5 = 890 \cdot kPa$

The process from 4 - 5 is $p(V) = p_4 \cdot \left(\frac{V_4}{V}\right)^k$ and $s = \text{constant}$

The work is $W_{45} = \frac{p_4 \cdot V_4 - p_5 \cdot V_5}{k - 1}$ $W_{45} = 1330\,J$ $Q_{45} = 0 \cdot J$

For process 5-1 we again have constant volume

The process from 5 -1 is $V = V_5 = \text{constant}$ and $\Delta s = c_v \cdot \ln\left(\frac{T}{T_5}\right)$

(From Eq. 12.11a)

$Q_{51} = M \cdot c_v \cdot (T_1 - T_5)$ $Q_{51} = -987\,J$ $W_{51} = 0 \cdot J$

The net work is $W_{net} = W_{12} + W_{23} + W_{34} + W_{45} + W_{51}$ $W_{net} = 1531\,J$

The heat added is $Q_{added} = Q_{23} + Q_{34}$ $Q_{added} = 2517\,J$

The efficiency is $\eta = \frac{W_{net}}{Q_{added}}$ $\eta = 60.8 \cdot \%$

This is consistent with the expression from thermodynamics for the diesel efficiency

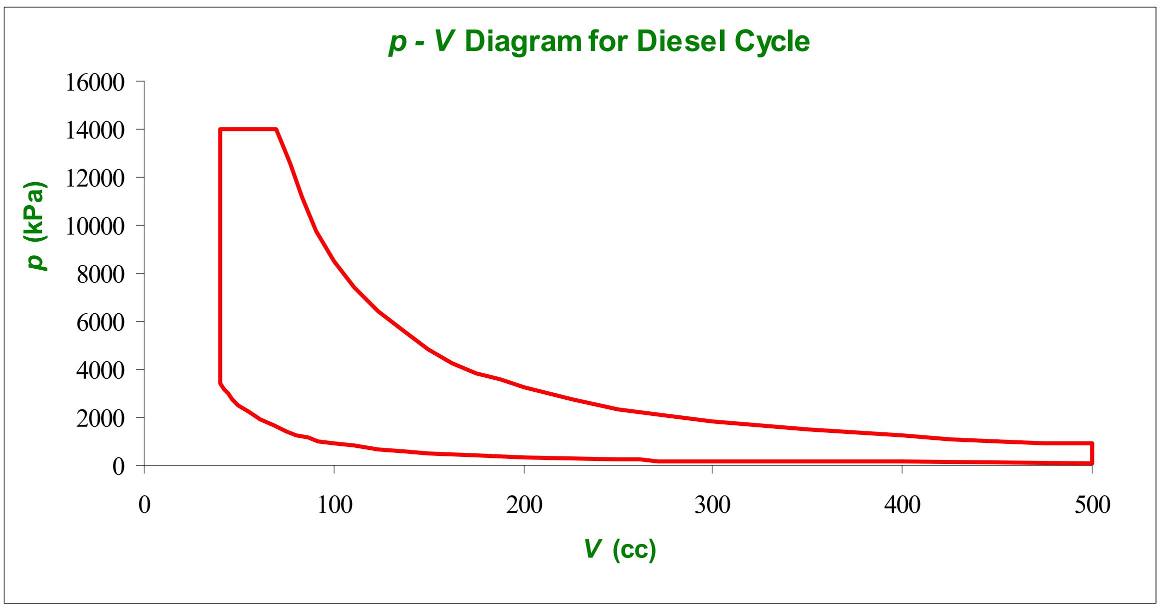
$$\eta_{diesel} = 1 - \frac{1}{r^{k-1}} \cdot \left[\frac{r_c{}^k - 1}{k \cdot (r_c - 1)}\right]$$

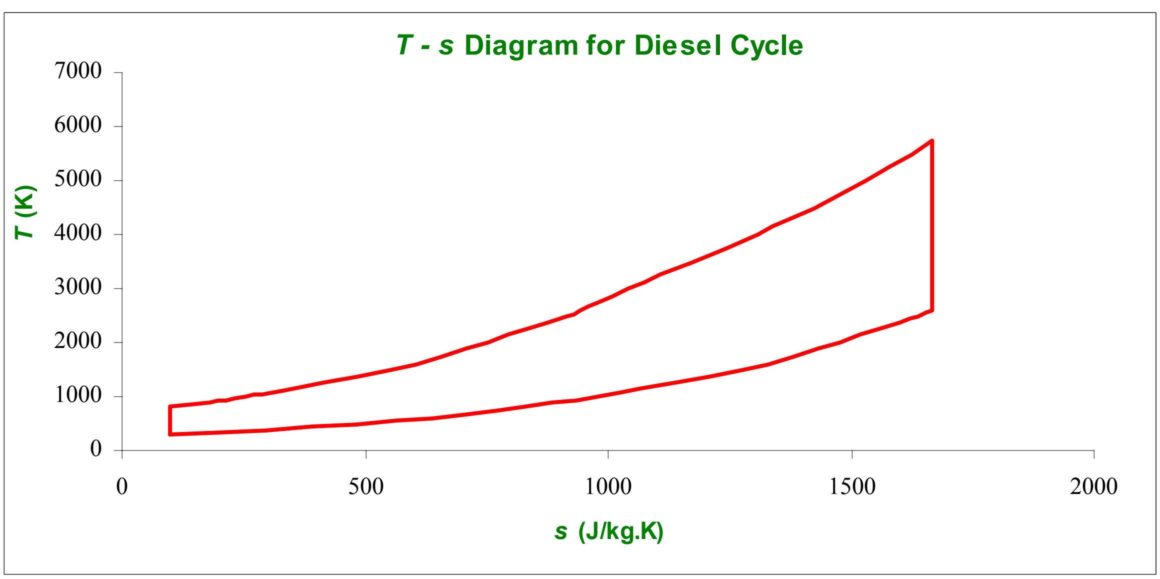
where r is the compression ratio $\quad r = \frac{V_1}{V_2} \qquad r = 12.5$

and r_c is the cutoff ratio $\quad r_c = \frac{V_4}{V_3} \qquad r_c = 1.75$

$$\eta_{diesel} = 58.8 \cdot \%$$

The plots of the cycle in pV and Ts space, generated using *Excel*, are shown here:





Problem 12.14

[Difficulty: 3]

12.14 A 1-m^3 tank contains air at 0.1 MPa (abs) and 20°C. The tank is pressurized to 2 MPa. Assuming that the tank is filled adiabatically and reversibly, calculate the final temperature of the air in the tank. Now assuming that the tank is filled isothermally, how much heat is lost by the air in the tank during filling? Which process (adiabatic or isothermal) results in a greater mass of air in the tank?

Given: Air is compressed from standard conditions to fill a tank

Find:
(a) Final temperature of air if tank is filled adiabatically and reversibly
(b) Heat lost if tank is filled isothermally
(c) Which process results in a greater mass of air in the tank

Solution:

The data provided, or available in the Appendices, is:

$$c_p = 1004 \cdot \frac{J}{kg \cdot K} \qquad R = 287 \cdot \frac{J}{kg \cdot K} \qquad c_v = c_p - R \qquad c_v = 717 \cdot \frac{J}{kg \cdot K} \qquad k = \frac{c_p}{c_v} \qquad k = 1.4$$

$$V = 1 \cdot m^3 \qquad p_1 = 0.1 \cdot MPa \qquad T_1 = (20 + 273) \cdot K \qquad p_2 = 2 \cdot MPa$$

Adiabatic, reversible process is isentropic: $T_{2s} = T_1 \cdot \left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}}$ $\qquad T_{2s} = 689.9\,K$

For the isothermal process, we look at the first law: $\Delta u = q - w = c_v \cdot \Delta T$ but $\Delta T = 0$ so: $\Delta u = 0$ and $q = w$

The work is equal to: $w = \int p\,dv = \int \frac{R \cdot T_1}{v}\,dv = R \cdot T_1 \cdot \int_{v_1}^{v_2} \frac{1}{v}\,dv = R \cdot T_1 \cdot \ln\left(\frac{v_2}{v_1}\right)$

From Boyle's law: $p_1 \cdot v_1 = p_2 \cdot v_2$ $\quad \frac{v_2}{v_1} = \frac{p_1}{p_2}$ substituting this into the above equation: $w = R \cdot T_1 \cdot \ln\left(\frac{p_1}{p_2}\right)$

$w = -252 \cdot \frac{kJ}{kg}$ $\qquad$ Therefore the heat transfer is $q = w = -252 \cdot \frac{kJ}{kg}$ (The negative sign indicates heat loss)

The mass of the air can be calculated from the ideal gas equation of state: $p \cdot V = M \cdot R \cdot T$ $\quad M = \frac{p_2 \cdot V}{R \cdot T_1} = 23.8\,kg$

So the actual heat loss is equal to: $Q = M \cdot q$ $\qquad Q = -5.99 \times 10^3 \cdot kJ$

The mass in the tank after compression isothermally is: $M_t = 23.8\,kg$

For the isentropic compression: $M = \frac{p_2 \cdot V}{R \cdot T_{2s}} = 10.1\,kg$ $\qquad$ Therefore the isothermal compression results in more mass in the tank.

Problem 12.15

[Difficulty: 3]

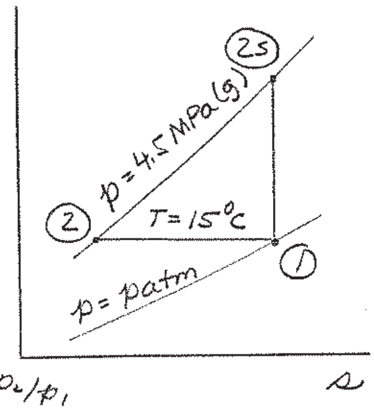
12.15 A tank of volume $V = 10\,\mathrm{m}^3$ contains compressed air at 15°C. The gage pressure in the tank is 4.50 MPa. Evaluate the work required to fill the tank by compressing air from standard atmosphere conditions for (a) isothermal compression and (b) isentropic compression followed by cooling at constant pressure. What is the peak temperature of the isentropic compression process? Calculate the energy removed during cooling for process (b). Assume ideal gas behavior and reversible processes. Label state points on a Ts diagram and a pV diagram for each process.

Solution: Apply ideal gas, energy, isentropic process equations.

Computing equations: $Q - W = \Delta E = m(u_2 - u_1)$, $W = -\int p\,dV$

$p/\rho^k = \text{constant}$, $p = \rho R T$

The tank contains $m = \rho V$



$$\rho = \frac{p}{RT} = (4.5 + 0.101)10^6 \frac{N}{m^2} \times \frac{kg \cdot K}{287\,N \cdot m} \times \frac{1}{(273+15)\,K} = 55.7\,\frac{kg}{m^3}$$

$$m = \rho V = 55.7\,\frac{kg}{m^3} \times 10\,m^3 = 557\,kg$$

For process $1 \to 2$,

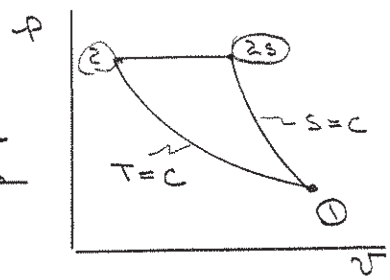
$$-W = -\int p\,dV = -m\int p\,dv = -m\int RT\frac{dp}{p} = -mRT\ln p_2/p_1$$

$$W_{12} = -557\,kg \times 287\,\frac{J}{kg \cdot K} \times (273+15)\,K \times \ln\left(\frac{4.5+0.101}{0.101}\right) = 176\,MJ$$ W_{12}

For process $2 \to 2_s$, $\frac{p_{2s}}{p_1} = \left(\frac{\rho_{2s}}{\rho_1}\right)^k \to \frac{T_{2s}}{T_1} = \left(\frac{p_{2s}}{p_1}\right)^{\frac{k-1}{k}} = \left(\frac{4.601}{0.101}\right)^{0.286} = 2.98$

$T_{2s} = 2.98\,T_1 = 2.98(273+15)K = 858\,K$ T_{2s}

$$W_{12s} = m(u_{2s} - u_1) - \cancel{Q_{12s}}^{\,0} = m c_v (T_{2s} - T_1)$$



$$W_{12s} = 557\,kg \times 0.717\,\frac{kJ}{kg \cdot K}(858-288)K = 228\,MJ$$ W_{12s}

Process $2s \to 2$ is at constant pressure

$$Q_{2s2} = m(u_2 - u_{2s}) - W_{2s2} = m(u_2 - u_{2s}) - mp(v_2 - v_{2s}) = m(h_2 - h_{2s})$$

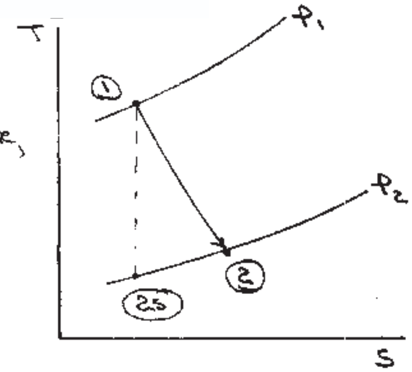
$$= m c_p (T_2 - T_{2s})$$

$$Q_{2s2} = 557\,kg \times 1.00\,\frac{kJ}{kg \cdot K} \times (288-858)K = -317\,MJ$$ (out of air) Q_{2s2}

Problem 12.16

[Difficulty: 2]

12.16 Air enters a turbine in steady flow at 0.5 kg/s with negligible velocity. Inlet conditions are 1300°C and 2.0 MPa (abs). The air is expanded through the turbine to atmospheric pressure. If the actual temperature and velocity at the turbine exit are 500°C and 200 m/s, determine the power produced by the turbine. Label state points on a *Ts* diagram for this process.



Solution:

For an isentropic expansion through the turbine,

$$T_{2s} = T_1\left(\frac{p_2}{p_1}\right)^{(k-1)/k}$$

$$T_{2s} = (1300+273)\,K\left(\frac{0.101\ MPa}{2.0\ MPa}\right)^{\frac{1.4-1}{1.4}}$$

$$T_{2s} = 670\,K\ (397^\circ C)$$

Writing the first law of thermodynamics between the turbine inlet and outlet

$$\dot{W} + \cancel{\dot{Q}} = \dot{m}\left[\left(h_2 + \frac{V_2^2}{2}\right) - \left(h_1 + \cancel{\frac{V_1^2}{2}}^{\,\approx 0}\right)\right]$$

(Assume $\dot{Q} = 0$)

For an ideal gas with constant specific heats, $h_2 - h_1 = c_p(T_2 - T_1)$

$$\therefore \dot{W} = \dot{m}\left[c_p(T_2 - T_1) - \frac{V_2^2}{2}\right]$$

$$= 0.5\,\frac{kg}{s}\left[1004\,\frac{N\cdot m}{kg\cdot K}\,(773-1573)\,K - \frac{1}{2}(200)^2\,\frac{m^2}{s^2} \times \frac{N\cdot s^2}{kg\cdot m}\right]$$

$$\dot{W} = -392\times10^3\ \frac{N\cdot m}{s}$$ (negative sign indicates work out).

$$\therefore \dot{W}_{out} = 392\ kW$$ $\dot{W}_{out}$

Problem 12.17

[Difficulty: 3]

12.17 Natural gas, with the thermodynamic properties of methane, flows in an underground pipeline of 0.6 m diameter. The gage pressure at the inlet to a compressor station is 0.5 MPa; outlet pressure is 8.0 MPa (gage). The gas temperature and speed at inlet are 13°C and 32 m/s, respectively. The compressor efficiency is $\eta = 0.85$. Calculate the mass flow rate of natural gas through the pipeline. Label state points on a Ts diagram for compressor inlet and outlet. Evaluate the gas temperature and speed at the compressor outlet and the power required to drive the compressor.

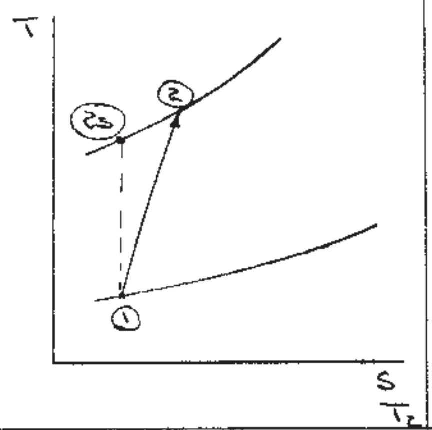
Solution:

The mass flow rate is given by $\dot{m} = \rho V A$ where $\rho = \frac{p}{RT}$

$$\therefore \dot{m} = \frac{p_1}{RT_1} V_1 \frac{\pi D^2}{4} = (500+101)\times 10^3 \frac{N}{m^2} \times \frac{kg\cdot K}{518.3\, N\cdot m} \times \frac{1}{286\,K} \times 32\,\frac{m}{s} \times \frac{\pi}{4}(0.6)^2 m^2$$

$\dot{m} = 36.7$ kg/s ⟵ $\dot{m}$

For an isentropic compression



$$T_{2s} = T_1 \left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}} = 286\,K \left(\frac{8.101\,MPa}{0.601\,MPa}\right)^{\frac{1.31-1}{1.31}} = 529\,K$$

$$\eta_c = \frac{T_{2s} - T_1}{T_2 - T_1} \qquad \therefore T_2 - T_1 = \frac{T_{2s} - T_1}{\eta_c}$$

$$T_2 = T_1 + \frac{T_{2s} - T_1}{\eta_c} = 286\,K + \frac{(529-286)K}{0.85}$$

$T_2 = 572$ K ⟵ T_2

From continuity, $\dot{m} = \rho_1 V_1 A_1 = \rho_2 V_2 A_2$. Assuming $A_1 = A_2$, then

$$V_2 = \frac{\rho_1}{\rho_2} V_1 = \frac{p_1}{p_2}\frac{T_2}{T_1} V_1 = \frac{0.601}{8.101} \times \frac{572}{286} \times 32\,\frac{m}{s} = 4.75\ m/s$$ ⟵ V_2

Writing the first law of thermodynamics between compressor inlet & outlet

$$\dot{W} + \cancel{\dot{Q}}^{\,\approx 0} = \dot{m}\left[\left(h_2 + \frac{V_2^2}{2}\right) - \left(h_1 + \frac{V_1^2}{2}\right)\right] \qquad (\text{Assume } \dot{Q} = 0)$$

$$\dot{W} = \dot{m}\left[(h_2 - h_1) + \frac{1}{2}(V_2^2 - V_1^2)\right] = \dot{m}\left[c_p(T_2 - T_1) + \frac{1}{2}(V_2^2 - V_1^2)\right]$$

$$\dot{W} = 36.7\,\frac{kg}{s}\left[2190\,\frac{N\cdot m}{kg\cdot K}(572-286)K + \frac{1}{2}\left\{(4.75)^2 - (32)^2\right\}\frac{m^2}{s^2} \times \frac{N\cdot s^2}{kg\cdot m}\right]$$

$$\dot{W} = 36.7\left[626\times 10^3 - 501\right]\ \frac{N\cdot m}{s}$$

$\dot{W} = 23$ MW ⟵ $\dot{W}_{in}$

Problem 12.18

[Difficulty: 3]

12.18 Over time the efficiency of the compressor of Problem 12.17 drops. At what efficiency will the power required to attain 8.0 MPa (gage) exceed 30 MW? Plot the required power and the gas exit temperature as functions of efficiency.

Given: Data on flow through compressor

Find: Efficiency at which power required is 30 MW; plot required efficiency and exit temperature as functions of efficiency

Solution:

The data provided, or available in the Appendices, is:

$$R = 518.3\cdot\frac{J}{kg\cdot K} \qquad c_p = 2190\cdot\frac{J}{kg\cdot K} \qquad c_v = c_p - R \qquad c_v = 1672\cdot\frac{J}{kg\cdot K} \qquad k = \frac{c_p}{c_v} \qquad k = 1.31$$

$$T_1 = (13 + 273)\cdot K \qquad p_1 = 0.5\cdot MPa + 101\cdot kPa \qquad V_1 = 32\cdot\frac{m}{s}$$

$$p_2 = 8\cdot MPa + 101\cdot kPa \qquad W_{comp} = 30\cdot MW \qquad D = 0.6\cdot m$$

The governing equation is the first law of thermodynamics for the compressor

$$M_{flow}\cdot\left[\left(h_2 + \frac{V_2^{\ 2}}{2}\right) - \left(h_1 + \frac{V_1^{\ 2}}{2}\right)\right] = W_{comp} \qquad \text{or} \qquad W_{comp} = M_{flow}\cdot\left[c_p\cdot\left(T_2 - T_1\right) + \frac{V_2^{\ 2} - V_1^{\ 2}}{2}\right]$$

We need to find the mass flow rate and the temperature and velocity at the exit

$$M_{flow} = \rho_1\cdot A_1\cdot V_1 = \frac{p_1}{R\cdot T_1}\cdot\frac{\pi}{4}\cdot D^2\cdot V_1 \qquad M_{flow} = \frac{p_1}{R\cdot T_1}\cdot\frac{\pi}{4}\cdot D^2\cdot V_1 \qquad M_{flow} = 36.7\,\frac{kg}{s}$$

The exit velocity is then given by $\quad M_{flow} = \dfrac{p_2}{R\cdot T_2}\cdot\dfrac{\pi}{4}\cdot D^2\cdot V_2 \qquad V_2 = \dfrac{4\cdot M_{flow}\cdot R\cdot T_2}{\pi\cdot p_2\cdot D^2} \qquad (1)$

The exit velocity cannot be computed until the exit temperature is determined!

Using Eq. 1 in the first law

$$W_{comp} = M_{flow}\cdot\left[c_p\cdot\left(T_2 - T_1\right) + \frac{\left(\dfrac{4\cdot M_{flow}\cdot R\cdot T_2}{\pi\cdot p_2\cdot D^2}\right)^2 - V_1^{\ 2}}{2}\right]$$

In this complicated expression the only unknown is T_2, the exit temperature. The equation is a quadratic, so is solvable explicitly for T_2, but instead we use *Excel*'s *Goal Seek* to find the solution (the second solution is mathematically correct but physically unrealistic - a very large negative absolute temperature). The exit temperature is $\quad T_2 = 660\cdot K$

If the compressor was ideal (isentropic), the exit temperature would be given by $\quad T\cdot p^{\frac{1-k}{k}} = \text{constant} \qquad (12.12b)$

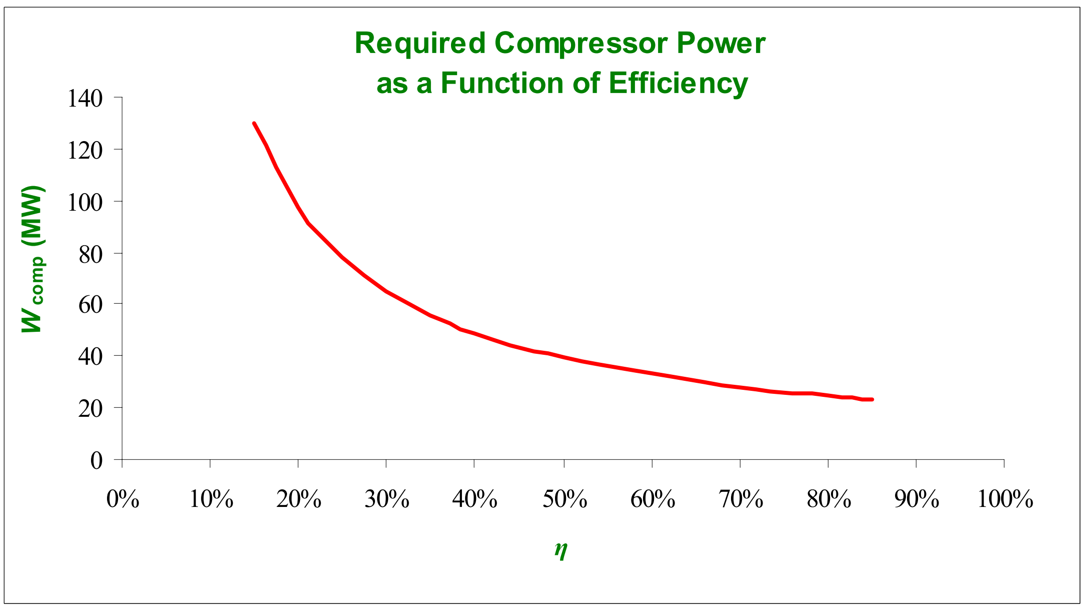
Hence $\quad T_{2s} = T_1 \cdot \left(\frac{p_1}{p_2}\right)^{\frac{1-k}{k}} \qquad T_{2s} = 529\,K$

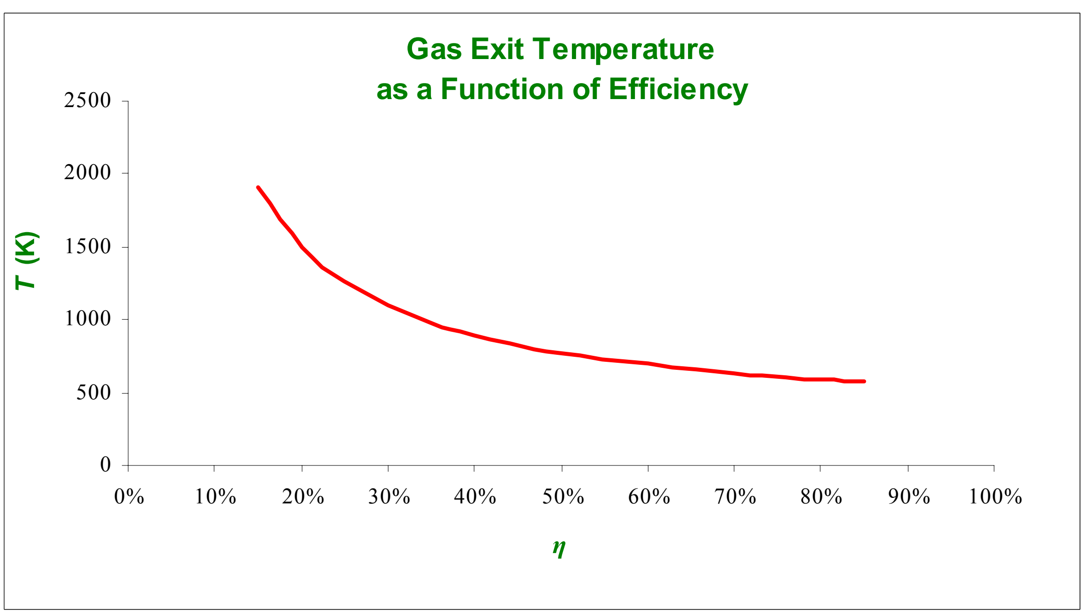
For a compressor efficiency η, we have $\quad \eta = \frac{h_{2s} - h_1}{h_2 - h_1} \quad$ or $\quad \eta = \frac{T_{2s} - T_1}{T_2 - T_1} \qquad \eta = 65.1 \cdot \%$

To plot the exit temperature and power as a function of efficiency we use $\quad T_2 = T_1 + \frac{T_{2s} - T_1}{\eta}$

with $\quad V_2 = \frac{4 \cdot M_{flow} \cdot R \cdot T_2}{\pi \cdot p_2 \cdot D^2} \quad$ and $\quad W_{comp} = M_{flow} \cdot \left[c_p \cdot \left(T_2 - T_1\right) + \frac{V_2^{\ 2} - V_1^{\ 2}}{2}\right]$

The dependencies of T_2 and W_{comp} on efficiency are plotted in *Excel* and shown here:





Problem 12.19

[Difficulty: 3]

12.19 Improper maintenance on the turbine of Problem 12.7 has resulted in a gradual decrease in its efficiency over time. Assuming that the efficiency drops by 1 percent per year, how long would it take for the power output of the turbine to drop to 950 kW, assuming that entrance conditions, flow rate, and exhaust pressure were all kept constant?

Given: Data on performance degradation of turbine

Find: Time necessary for power output to drop to 950 kW

Solution:

The data provided, or available in the Appendices, is:

$p_1 = 10 \cdot \text{bar} = 1 \times 10^3 \cdot \text{kPa}$ $\quad T_1 = 1400 \cdot \text{K}$ $\quad \eta_{initial} = 80 \cdot \%$ $\quad P_{initial} = 1 \cdot \text{MW}$

$p_2 = 1 \cdot \text{bar} = 100 \cdot \text{kPa}$ $\quad P_{final} = 950 \cdot \text{kW}$

$c_p = 1004 \cdot \frac{\text{J}}{\text{kg} \cdot \text{K}}$ $\quad R_{gas} = 287 \cdot \frac{\text{J}}{\text{kg} \cdot \text{K}}$ $\quad k = 1.4$

If the turbine expansion were isentropic, the actual output would be: $P_{ideal} = \frac{P_{initial}}{\eta_{initial}} = 1.25 \cdot \text{MW}$

So when the power output drops to 950 kW, the new efficiency is: $\eta_{final} = \frac{P_{final}}{P_{ideal}} = 76 \cdot \%$

Since the efficiency drops by 1% per year, the time elapsed is: $\Delta t = 4 \cdot \text{yr}$

Problem 12.20

[Difficulty: 4]

12.20 In an isothermal process, 0.1 cubic feet of standard air per minute (SCFM) is pumped into a balloon. Tension in the rubber skin of the balloon is given by $\sigma = kA$, where $k = 200\ \text{lbf/ft}^3$, and A is the surface area of the balloon in ft^2. Compute the time required to increase the balloon radius from 5 to 7 in.

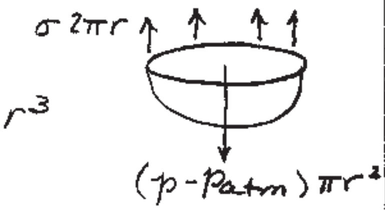
Solution: The mass flow rate is $\dot{m} = \rho_{std} Q$ = constant, so

Computing equation: $\Delta t = \dfrac{\Delta m}{\dot{m}}$ $\qquad p = \rho R T$

Assume: (1) Standard air, $\rho = 0.0765\ \text{lbm/ft}^3$; (2) Ideal gas

Then $\dot{m} = \rho Q = 0.0765 \dfrac{\text{lbm}}{\text{ft}^3} \times 0.10 \dfrac{\text{ft}^3}{\text{min}} \times \dfrac{\text{min}}{60\ \text{s}} = 1.28 \times 10^{-4}\ \text{lbm/s}$

From a force balance on the balloon:



$$(p - p_{atm})\pi r^2 = \sigma 2\pi r = k(4\pi r^2) 2\pi r = 8\pi^2 k r^3$$

or $p = p_{atm} + 8\pi k r$

For $r = 5$ in., $p = 14.7 + 8\pi \times 200 \dfrac{\text{lbf}}{\text{ft}^3} \times 5\ \text{in.} \times \dfrac{\text{ft}^3}{1728\ \text{in.}^3} = 29.2\ \text{psia}$

$$\rho = \frac{p}{RT} = 29.2 \frac{\text{lbf}}{\text{in.}^2} \times \frac{\text{lbm} \cdot {}^\circ\text{R}}{53.3\ \text{ft} \cdot \text{lbf}} \times \frac{1}{519\,{}^\circ\text{R}} \times 144 \frac{\text{in.}^2}{\text{ft}^2} = 0.152\ \text{lbm/ft}^3$$

$$\forall = \frac{4}{3}\pi r^3 = \frac{4\pi}{3} \times (5)^3\ \text{in.}^3 \times \frac{\text{ft}^3}{1728\ \text{in.}^3} = 0.303\ \text{ft}^3$$

$$m = \rho \forall = 0.152 \frac{\text{lbm}}{\text{ft}^3} \times 0.303\ \text{ft}^3 = 0.0461\ \text{lbm}$$

For $r = 7$ in., $p = 14.7 \dfrac{\text{lbf}}{\text{in.}^2} + 8\pi \times 200 \dfrac{\text{lbf}}{\text{ft}^3} \times 7\ \text{in.} \times \dfrac{\text{ft}^3}{1728\ \text{in.}^3} = 35.1\ \text{psia}$

Tabulating,

r (in.)	p (psia)	ρ (lbm/ft^3)	$\forall$ (ft^3)	m (lbm)
5	29.2	0.152	0.303	0.0461
7	35.1	0.183	0.831	0.152

Then $\Delta m = m_7 - m_5 = 0.152 - 0.0461\ \text{lbm} = 0.106\ \text{lbm}$

and $\Delta t = 0.106\ \text{lbm} \times \dfrac{\text{s}}{1.28 \times 10^{-4}\ \text{lbm}} = 828\ \text{s} \quad (\simeq 14\ \text{min})$

Δt

Problem 12.21

[Difficulty: 3]

12.21 For the balloon process of Problem 12.20 we could define a "volumetric ratio" as the ratio of the volume of standard air supplied to the volume increase of the balloon, per unit time. Plot this ratio over time as the balloon radius is increased from 5 to 7 in.

Given:
Data on flow rate and balloon properties

Find:
"Volumetric ratio" over time

Solution:

The given or available data are: $R_{air} = 53.3 \cdot \frac{ft \cdot lbf}{lbm \cdot R}$ $\quad T_{atm} = 519 \cdot R$ $\quad p_{atm} = 14.7 \cdot psi$ $\quad k = 200 \cdot \frac{lbf}{ft^3}$ $\quad V_{rate} = 0.1 \cdot \frac{ft^3}{min}$

Basic equation:

Standard air density $\quad \rho_{air} = \frac{p_{atm}}{R_{air} \cdot T_{atm}} = 0.0765 \frac{lbm}{ft^3}$

Mass flow rate $\quad M_{rate} = V_{rate} \cdot \rho_{air} = 1.275 \times 10^{-4} \frac{lbm}{s}$

From a force balance on each hemisphere $\quad \left(p - p_{atm}\right) \cdot \pi \cdot r^2 = \sigma \cdot 2 \cdot \pi \cdot r$ $\quad$ where $\quad \sigma = k \cdot A = k \cdot 4 \cdot \pi \cdot r^2$

Hence $\quad p = p_{atm} + \frac{2 \cdot \sigma}{r}$ $\quad$ or $\quad p = p_{atm} + 8 \cdot \pi \cdot k \cdot r$

Density in balloon $\quad \rho = \frac{p}{R_{air} \cdot T_{air}}$

The instantaneous volume is $\quad V_{ball} = \frac{4}{3} \cdot \pi \cdot r^3$

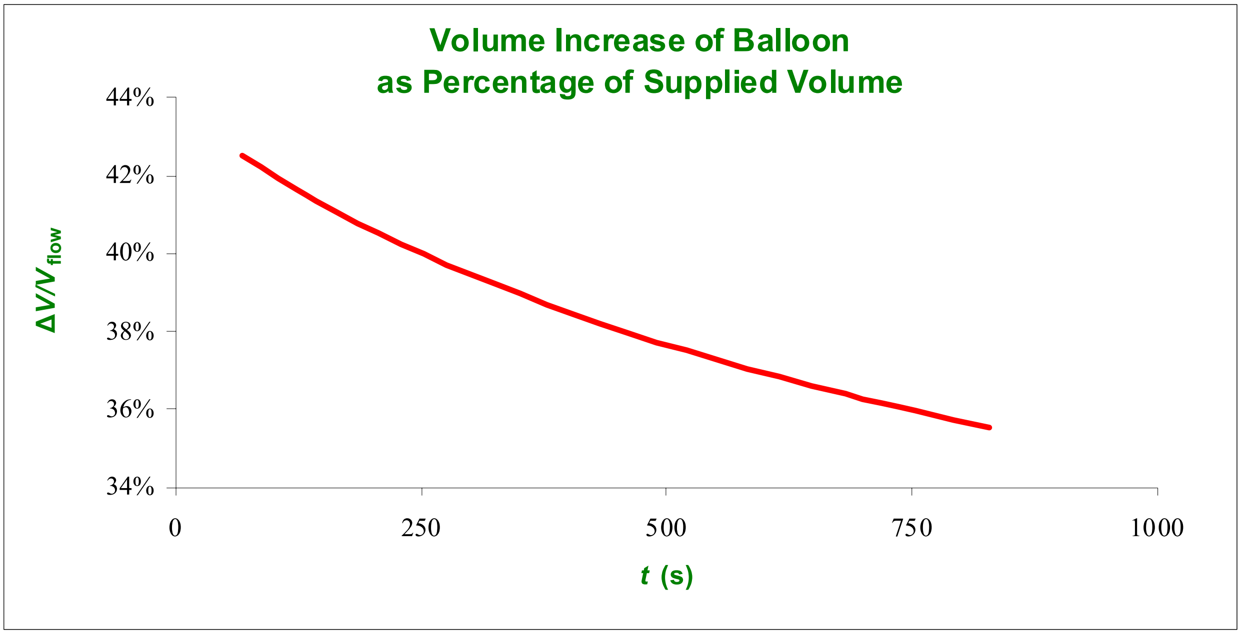
The instantaneous mass is $\quad M_{ball} = V_{ball} \cdot \rho$

The time to fill to radius *r* from *r* = 5 in is $\quad t = \frac{M_{ball}(r) - M_{ball}(r = 5in)}{M_{rate}}$

The volume change between time steps Δt is $\quad \Delta V = V_{ball}(t + \Delta t) - V_{ball}(t)$

The results, calculated using *Excel*, are shown on the next page:

r (in)	p (psi)	ρ (lb/ft^3)	V_{ball} (ft^3)	M_{ball} (lb)	t (s)	$\Delta V/V_{rate}$
5.00	29.2	0.152	0.303	0.0461	0.00	0.00
5.25	30.0	0.156	0.351	0.0547	67.4	42.5%
5.50	30.7	0.160	0.403	0.0645	144	41.3%
5.75	31.4	0.164	0.461	0.0754	229	40.2%
6.00	32.2	0.167	0.524	0.0876	325	39.2%
6.25	32.9	0.171	0.592	0.101	433	38.2%
6.50	33.6	0.175	0.666	0.116	551	37.3%
6.75	34.3	0.179	0.746	0.133	683	36.4%
7.00	35.1	0.183	0.831	0.152	828	35.5%



Problem 12.22

[Difficulty: 3]

12.22 A sound pulse level above about 20 Pa can cause permanent hearing damage. Assuming such a sound wave travels through air at 20°C and 100 kPa, estimate the density, temperature, and velocity change immediately after the sound wave passes.

Given:
Sound wave

Find:
Estimate of change in density, temperature, and velocity after sound wave passes

Solution:

Basic equation: $p = \rho \cdot R \cdot T$ $\qquad \Delta s = c_p \cdot \ln\left(\frac{T_2}{T_1}\right) - R \cdot \ln\left(\frac{p_2}{p_1}\right)$

$du = c_v \cdot dT \qquad dh = c_p \cdot dT$

Assumptions: 1) Ideal gas 2) Constant specific heats 3) Isentropic process 4) infinitesimal changes

Given or available data

$T_1 = (20 + 273) \cdot K \qquad p_1 = 100 \cdot kPa \qquad dp = 20 \cdot Pa \qquad k = 1.4 \qquad R = 286.9 \frac{J}{kg \cdot K}$

$c = \sqrt{k \cdot R \cdot T_1} \qquad c = 343 \frac{m}{s}$

For small changes, from Section 11-2 $\quad dp = c^2 \cdot d\rho \quad$ so $\quad d\rho = \frac{dp}{c^2} \qquad d\rho = 1.70 \times 10^{-4} \cdot \frac{kg}{m^3}$ a very small change!

The air density is $\rho_1 = \frac{p_1}{R \cdot T_1} \qquad \rho_1 = 1.19 \frac{kg}{m^3}$

Then $\quad dV_x = \frac{1}{\rho_1 \cdot c} \cdot dp \qquad dV_x = 0.049 \frac{m}{s}$ $\qquad$ This is the velocity of the air after the sound wave!

For the change in temperature we start with the ideal gas equation $\quad p = \rho \cdot R \cdot T \quad$ and differentiate $\quad dp = d\rho \cdot R \cdot T + \rho \cdot R \cdot dT$

Dividing by the ideal gas equation we find $\quad \frac{dp}{p} = \frac{d\rho}{\rho} + \frac{dT}{T}$

Hence $\quad dT = T_1 \cdot \left(\frac{dp}{p_1} - \frac{d\rho}{\rho_1}\right) \qquad dT = 0.017\,K \qquad dT = 0.030 \cdot \Delta°F \qquad$ a very small change!

Problem 12.23

[Difficulty: 3]

12.23 Calculate the speed of sound at 20°C for (a) hydrogen, (b) helium, (c) methane, (d) nitrogen, and (e) carbon dioxide.

Given: Five different gases at specified temperature

Find: Sound speeds for each gas at that temperature

Solution: Basic equation: $c = \sqrt{k \cdot R \cdot T}$

The data provided, or available in the Appendices, is: $T = (20 + 273) \cdot K$

$k_{H2} = 1.41 \quad R_{H2} = 4124 \cdot \frac{J}{kg \cdot K} \qquad k_{He} = 1.66 \quad R_{He} = 2077 \cdot \frac{J}{kg \cdot K}$

$k_{CH4} = 1.31 \quad R_{CH4} = 518.3 \cdot \frac{J}{kg \cdot K} \qquad k_{N2} = 1.40 \quad R_{N2} = 296.8 \cdot \frac{J}{kg \cdot K}$

$k_{CO2} = 1.29 \quad R_{CO2} = 188.9 \cdot \frac{J}{kg \cdot K}$

$c_{H2} = \sqrt{k_{H2} \cdot R_{H2} \cdot T} \qquad c_{H2} = 1305 \frac{m}{s}$

$c_{He} = \sqrt{k_{He} \cdot R_{He} \cdot T} \qquad c_{He} = 1005 \frac{m}{s}$

$c_{CH4} = \sqrt{k_{CH4} \cdot R_{CH4} \cdot T} \qquad c_{CH4} = 446 \frac{m}{s}$

$c_{N2} = \sqrt{k_{N2} \cdot R_{N2} \cdot T} \qquad c_{N2} = 349 \frac{m}{s}$

$c_{CO2} = \sqrt{k_{CO2} \cdot R_{CO2} \cdot T} \qquad c_{CO2} = 267 \frac{m}{s}$

Problem 12.24

[Difficulty: 3]

12.24 The bulk modulus E_v of a material indicates how hard it is to compress the material; a large E_v indicates the material requires a large pressure to compress. Is air "stiffer" when suddenly or slowly compressed? To answer this, find expressions in terms of instantaneous pressure p for the bulk modulus of air (kPa) when it is (a) rapidly compressed and (b) slowly compressed. *Hint*: Rapid compression is approximately isentropic (it is adiabatic because it is too quick for heat transfer to occur), and slow compression is isothermal (there is plenty of time for the air to equilibrate to ambient temperature).

Given: Sound wave

Find: Estimate of change in density, temperature, and velocity after sound wave passes

Solution:

Basic equations: $p = \rho \cdot R \cdot T$ $\qquad E_V = \dfrac{dp}{\dfrac{d\rho}{\rho}}$

Assumptions: 1) Ideal gas 2) Constant specific heats 3) Infinitesimal changes

To find the bulk modulus we need $\dfrac{dp}{d\rho}$ in $E_V = \dfrac{dp}{\dfrac{d\rho}{\rho}} = \rho \cdot \dfrac{dp}{d\rho}$

For rapid compression (isentropic) $\dfrac{p}{\rho^k} = \text{const}$ and so $\dfrac{dp}{d\rho} = k \cdot \dfrac{p}{\rho}$

Hence $E_V = \rho \cdot \left(k \cdot \dfrac{p}{\rho} \right)$ $\qquad E_V = k \cdot p$

For gradual compression (isothermal) we can use the ideal gas equation $p = \rho \cdot R \cdot T$ so $dp = d\rho \cdot R \cdot T$

Hence $E_V = \rho \cdot (R \cdot T) = p$ $\qquad E_V = p$

We conclude that the "stiffness" (E_V) of air is equal to kp when rapidly compressed and p when gradually compressed. To give an idea of values:

For water $E_V = 2.24 \cdot \text{GPa}$

For air ($k = 1.4$) at $p = 101 \cdot \text{kPa}$ Rapid compression $E_V = k \cdot p$ $\qquad E_V = 141 \cdot \text{kPa}$

Gradual compression $E_V = p$ $\qquad E_V = 101 \cdot \text{kPa}$

Problem 12.25

[Difficulty: 2]

12.25 You have designed a device for determining the bulk modulus, E_v, of a material. It works by measuring the time delay between sending a sound wave into a sample of the material and receiving the wave after it travels through the sample and bounces back. As a test, you use a 1-m rod of steel ($E_v \approx 200$ GN/m^2). What time delay should your device indicate? You now test a 1-m rod (1 cm diameter) of an unknown material and find a time delay of 0.5 ms. The mass of the rod is measured to be 0.25 kg. What is this material's bulk modulus?

Given: Device for determining bulk modulus

Find: Time delay; Bulk modulus of new material

Solution:

Basic equation: $c = \sqrt{\dfrac{E_v}{\rho}}$

Hence for given data $E_v = 200 \cdot \dfrac{GN}{m^2}$ $\quad L = 1 \cdot m$ $\quad$ and for steel $\quad SG = 7.83$ $\quad \rho_w = 1000 \cdot \dfrac{kg}{m^3}$

For the steel $c = \sqrt{\dfrac{E_v}{SG \cdot \rho_w}}$ $\quad c = 5054 \dfrac{m}{s}$

Hence the time to travel distance L is $\Delta t = \dfrac{L}{c}$ $\quad \Delta t = 1.98 \times 10^{-4} \cdot s$ $\quad \Delta t = 0.198 \cdot ms$ $\quad \Delta t = 198 \cdot \mu s$

For the unknown material $M = 0.25 \cdot kg$ $\quad D = 1 \cdot cm$ $\quad \Delta t = 0.5 \cdot ms$

The density is then $\rho = \dfrac{M}{L \cdot \dfrac{\pi \cdot D^2}{4}}$ $\quad \rho = 3183 \dfrac{kg}{m^3}$

The speed of sound in it is $c = \dfrac{L}{\Delta t}$ $\quad c = 2000 \dfrac{m}{s}$

Hence th bulk modulus is $E_v = \rho \cdot c^2$ $\quad E_v = 12.7 \cdot \dfrac{GN}{m^2}$

Problem 12.26

[Difficulty: 2]

12.26 Dolphins often hunt by listening for sounds made by their prey. They "hear" with the lower jaw, which conducts the sound vibrations to the middle ear via a fat-filled cavity in the lower jaw bone. If the prey is half a mile away, how long after a sound is made does a dolphin hear it? Assume the seawater is at 68°F.

Given: Hunting dolphin

Find: Time delay before it hears prey at 1/2 mile

Solution:

Basic equation: $c = \sqrt{\dfrac{E_V}{\rho}}$

Given (and Table A.2) data $L = 0.5\cdot\text{mi} = 2.64\times10^{3}\cdot\text{ft}$ $\quad SG = 1.025$ $\quad E_V = 3.20\times10^{5}\cdot\text{psi}$ $\quad \rho_w = 1.94\cdot\dfrac{\text{slug}}{\text{ft}^3}$

For the seawater $c = \sqrt{\dfrac{E_V}{SG\cdot\rho_w}}$ $\qquad c = 4814\cdot\dfrac{\text{ft}}{\text{s}}$

Hence the time for sound to travel distance L is $\Delta t = \dfrac{L}{c}$ $\qquad \Delta t = 0.548\cdot\text{s}$ $\qquad \Delta t = 548\cdot\text{ms}$

Problem 12.27

[Difficulty: 2]

12.27 A submarine sends a sonar signal to detect the enemy. The reflected wave returns after 3.25 s. Estimate the separation between the submarines. (As an approximation, assume the seawater is at 20°C.)

Given: Submarine sonar

Find: Separation between submarines

Solution:

Basic equation: $c = \sqrt{\dfrac{E_v}{\rho}}$

Given (and Table A.2) data $\Delta t = 3.25 \cdot s$ $\quad SG = 1.025$ $\quad E_v = 2.42 \cdot \dfrac{GN}{m^2}$ $\quad \rho_w = 1000 \cdot \dfrac{kg}{m^3}$

For the seawater $c = \sqrt{\dfrac{E_v}{SG \cdot \rho_w}}$ $\quad c = 1537 \dfrac{m}{s}$

Hence the distance sound travels in time Δt is $L = c \cdot \Delta t$ $\quad L = 5 \cdot km$

The distance between submarines is half of this $x = \dfrac{L}{2}$ $\quad x = 2.5 \cdot km$

Problem 12.28

[Difficulty: 1]

12.28 An airplane flies at 550 km/hr at 1500 m altitude on a standard day. The plane climbs to 15,000 m and flies at 1200 km/h. Calculate the Mach number of flight in both cases.

Given: Airplane cruising at two different elevations

Find: Mach numbers

Solution:

Basic equation: $c = \sqrt{k \cdot R \cdot T}$ $\quad M = \frac{V}{c}$

Available data $R = 286.9 \frac{J}{kg \cdot K}$ $\quad k = 1.4$

At $z = 1500 \cdot m$ $\quad T = 278.4 \cdot K$ from Table A.3

Hence $c = \sqrt{k \cdot R \cdot T}$ $\quad c = 334 \frac{m}{s}$ $\quad c = 1204 \cdot \frac{km}{hr}$ and we have $V = 550 \cdot \frac{km}{hr}$

The Mach number is $M = \frac{V}{c}$ $\quad M = 0.457$

Repeating at $z = 15000 \cdot m$ $\quad T = 216.7 \cdot K$

Hence $c = \sqrt{k \cdot R \cdot T}$ $\quad c = 295 \frac{m}{s}$ $\quad c = 1062 \cdot \frac{km}{hr}$ and we have $V = 1200 \cdot \frac{km}{hr}$

The Mach number is $M = \frac{V}{c}$ $\quad M = 1.13$

Problem 12.29

[Difficulty: 2]

12.29 Next-generation missiles will use scramjet engines to travel at Mach numbers as high as 7. If a scramjet-powered missile travels at Mach 7 at an altitude of 85,000 ft, how long will it take for the missile to travel 600 nautical miles? Assume standard atmospheric conditions. (*Note:* This is the range for the Tomahawk missile, which uses a conventional propulsion system, but it takes 90 min to cover that same distance.)

Given: Scramjet-powered missile traveling at fixed Mach number and altitude

Find: Time necessary to cover specified range

Solution:

Basic equation: $c = \sqrt{k \cdot R \cdot T}$ $\quad M = \dfrac{V}{c}$

Available data $R = 286.9 \dfrac{J}{kg \cdot K}$ $\quad k = 1.4$ $\quad M = 7$ $\quad \Delta x = 600 \cdot nmi = 3.65 \times 10^{6} \cdot ft$

At $z = 85000 \cdot ft$ $\quad z = 25908\,m$ interpolating from Table A.3 $T = 220.6 \cdot K + (222.5 \cdot K - 220.6 \cdot K) \cdot \dfrac{25908 - 24000}{26000 - 24000}$

$T = 222\,K$

Hence $c = \sqrt{k \cdot R \cdot T}$ $\quad c = 299 \dfrac{m}{s}$ $\quad c = 981 \cdot \dfrac{ft}{s}$ and we have $V = M \cdot c = 6864 \cdot \dfrac{ft}{s}$

The time needed to cover the range is: $\Delta t = \dfrac{\Delta x}{V} = 531\,s$ $\quad \Delta t = 8.85 \cdot min$ This is about ten times as fast as the Tomahawk!

Problem 12.30

[Difficulty: 1]

12.30 Actual performance characteristics of the Lockheed SR-71 "Blackbird" reconnaissance aircraft never were released. However, it was thought to cruise at $M = 3.3$ at 85,000 ft altitude. Evaluate the speed of sound and flight speed for these conditions. Compare to the muzzle speed of a 30-06 rifle bullet (700 m/s).

Solution:

At altitude, $z = 85{,}000\ \text{ft} \times 0.3048\ \frac{\text{m}}{\text{ft}} = 25.9\ \text{km}$

From Table A.3, $T = 222\ \text{K}$

$$\therefore\ c = \sqrt{kRT} = \left[1.4 \times 287\ \frac{\text{N}\cdot\text{m}}{\text{kg}\cdot\text{K}} \times 222\,\text{K} \times \frac{\text{kg}\cdot\text{m}}{\text{N}\cdot\text{s}^2}\right]^{1/2} = 299\ \text{m/s} \qquad c$$

$$V = Mc = 3.3 \times 299\ \text{m/s} = 987\ \text{m/s} \qquad V$$

$$\frac{V}{V_{bullet}} = \frac{987}{700} = 1.41$$

Problem 12.31

[Difficulty: 1]

12.31 The Boeing 727 aircraft of Example 9.8 cruises at 520 mph at 33,000 ft altitude on a standard day. Calculate the cruise Mach number of the aircraft. If the maximum allowable operating Mach number for the aircraft is 0.9, what is the corresponding flight speed?

Solution:

At 33,000 ft, $z = 10.06$ km. From Table A.3, $T = 223$ K.

Then, $c = \sqrt{kRT} = \left[1.4 \times 287 \frac{N \cdot m}{kg \cdot K} \times 223 K \times \frac{kg \cdot m}{N \cdot s^2}\right]^{1/2} = 299$ m/s

$$V = 520 \frac{mi}{hr} \times 5280 \frac{ft}{mi} \times \frac{hr}{3600 s} \times 0.3048 \frac{m}{ft} = 232 \text{ m/s}$$

$$M = \frac{V}{c} = \frac{232 \text{ m/s}}{299 \text{ m/s}} = 0.776$$

M_{cr}

At $M = 0.90$

$$V = Mc = 0.90 \times 299 \text{ m/s} = 269 \text{ m/s} \quad (603 \text{ mph})$$

V

Problem 12.32

[Difficulty: 2]

12.32 Investigate the effect of altitude on Mach number by plotting the Mach number of a 500 mph airplane as it flies at altitudes ranging from sea level to 10 km.

Given: Airplane cruising at 550 mph

Find: Mach number versus altitude

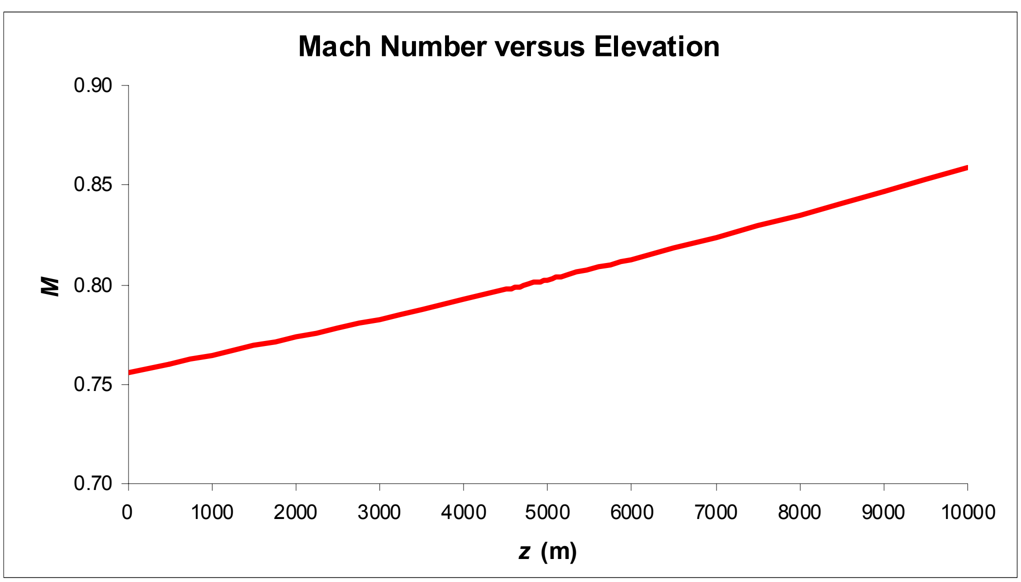
Solution:

Basic equation: $c = \sqrt{k \cdot R \cdot T}$ $\qquad M = \dfrac{V}{c}$ $\qquad$ Here are the results, generated using *Excel*:

$V =$ 500 mph
$R =$ 286.90 J/kg-K (Table A.6)
$k =$ 1.40

Data on temperature versus height obtained from Table A.3

z (m)	T (K)	c (m/s)	c (mph)	M
0	288.2	340	661	0.756
500	284.9	338	658	0.760
1000	281.7	336	654	0.765
1500	278.4	334	650	0.769
2000	275.2	332	646	0.774
2500	271.9	330	642	0.778
3000	268.7	329	639	0.783
3500	265.4	326	635	0.788
4000	262.2	325	631	0.793
4500	258.9	322	627	0.798
5000	255.7	320	623	0.803
6000	249.2	316	615	0.813
7000	242.7	312	607	0.824
8000	236.2	308	599	0.835
9000	229.7	304	590	0.847
10000	223.3	299	582	0.859



Problem 12.33

[Difficulty: 2]

12.33 You are watching a July 4th fireworks display from a distance of one mile. How long after you see an explosion do you hear it? You also watch New Year's fireworks (same place and distance). How long after you see an explosion do you hear it? Assume it's 75°F in July and 5°F in January.

Given: Fireworks displays!

Find: How long after seeing them do you hear them?

Solution:

Basic equation: $c = \sqrt{k \cdot R \cdot T}$

Assumption: Speed of light is essentially infinite (compared to speed of sound)

The given or available data is $T_{July} = (75 + 460) \cdot R$ $\quad L = 1 \cdot mi$ $\quad k = 1.4$ $\quad R_{air} = 53.33 \cdot \frac{ft \cdot lbf}{lbm \cdot R}$

Hence $c_{July} = \sqrt{k \cdot R_{air} \cdot T_{July}}$ $\quad c_{July} = 1134 \cdot \frac{ft}{s}$

Then the time is $\Delta t_{July} = \frac{L}{c_{July}}$ $\quad \Delta t_{July} = 4.66\,s$

In January $T_{Jan} = (5 + 460) \cdot R$

Hence $c_{Jan} = \sqrt{k \cdot R_{air} \cdot T_{Jan}}$ $\quad c_{Jan} = 1057 \cdot \frac{ft}{s}$

Then the time is $\Delta t_{Jan} = \frac{L}{c_{Jan}}$ $\quad \Delta t_{Jan} = 5.00\,s$

Problem 12.34

[Difficulty: 2]

12.34 The X-15 North American rocket plane held the record for the fastest manned flight. In 1967, the X-15 flew at a speed of 7270 km/h at an altitude of 58.4 km. At what Mach number did the X-15 fly?

Given: X-15 rocket plane speed and altitude

Find: Mach number

Solution:

Basic equation: $c = \sqrt{k \cdot R \cdot T}$ $\qquad M = \dfrac{V}{c}$

Available data $\quad R = 286.9 \dfrac{J}{kg \cdot K} \qquad k = 1.4 \qquad V = 7270 \cdot \dfrac{km}{hr}$

At $\quad z = 58400 \cdot m \quad$ interpolating from Table A.3 $\quad T = 270.7 \cdot K + (255.8 \cdot K - 270.7 \cdot K) \cdot \dfrac{58400 - 50000}{60000 - 50000}$

$T = 258\,K$

Hence $\quad c = \sqrt{k \cdot R \cdot T} \qquad c = 322 \dfrac{m}{s} \qquad c = 1159 \cdot \dfrac{km}{hr}$ and we have $\quad M = \dfrac{V}{c} = 6.27$

Problem 12.35

[Difficulty: 2]

12.35 You need to estimate the speed of a hypersonic aircraft traveling at Mach 7 and 120,000 ft. Not having a table of atmospheric tables handy, you remember that through the stratosphere (approximately 36,000 ft to 72,000 ft) the temperature of air is nearly constant at 390°R, and you assume this temperature for your calculation. Later, you obtain the appropriate data and recalculate the speed. What was the percentage error? What would the percentage error have been if you used the air temperature at sea level?

Given: Mach number and altitude of hypersonic aircraft

Find: Speed assuming stratospheric temperature, actual speed, speed assuming sea level static temperature

Solution:

Basic equation: $c = \sqrt{k \cdot R \cdot T}$ $\qquad M = \dfrac{V}{c}$

Available data $\quad R_{air} = 286.9 \dfrac{J}{kg \cdot K} \quad k = 1.4 \quad M = 7$

Assuming $\quad T = 390 \cdot R = 217\,K$

Hence $\quad c = \sqrt{k \cdot R_{air} \cdot T} \qquad c = 295 \dfrac{m}{s}$ and we have $\quad V_{strat} = M \cdot c = 2065 \dfrac{m}{s}$

At $\quad z = 120000 \cdot ft \quad z = 36576\,m$ interpolating from Table A.3 $\quad T = 226.5 \cdot K + (250.4 \cdot K - 226.5 \cdot K) \cdot \dfrac{36576 - 30000}{40000 - 30000}$

$T = 242\,K$

Hence $\quad c = \sqrt{k \cdot R_{air} \cdot T} \qquad c = 312 \dfrac{m}{s}$ and we have $\quad V_{actual} = M \cdot c = 2183 \dfrac{m}{s}$

The error is: $\quad \dfrac{V_{strat} - V_{actual}}{V_{actual}} = -5.42 \cdot \%$

Assuming $\quad T = 288.2 \cdot K$

Hence $\quad c = \sqrt{k \cdot R_{air} \cdot T} \qquad c = 340 \dfrac{m}{s}$ and we have $\quad V_{sls} = M \cdot c = 2382 \dfrac{m}{s}$

The error is: $\quad \dfrac{V_{sls} - V_{actual}}{V_{actual}} = 9.08 \cdot \%$

Problem 12.36

[Difficulty: 2]

12.36 The grandstand at the Kennedy Space Center is located 3.5 mi away from the Space Shuttle Launch Pad. On a day when the air temperature is 80°F, how long does it take the sound from a blastoff to reach the spectators? If the launch was early on a winter morning, the temperature may be as low as 50°F. How long would the sound take to reach the spectators under those conditions?

Given: Shuttle launch

Find: How long after seeing it do you hear it?

Solution:

Basic equation: $c = \sqrt{k \cdot R \cdot T}$

Assumption: Speed of light is essentially infinite (compared to speed of sound)

The given or available data is $T = (80 + 460) \cdot R$ $\quad L = 3.5 \cdot mi$ $\quad k = 1.4$ $\quad R_{air} = 53.33 \cdot \frac{ft \cdot lbf}{lbm \cdot R}$

Hence $c = \sqrt{k \cdot R_{air} \cdot T}$ $\quad c = 1139 \cdot \frac{ft}{s}$

Then the time is $\Delta t = \frac{L}{c}$ $\quad \Delta t = 16.23\, s$

In the winter: $T = (50 + 460) \cdot R$

Hence $c = \sqrt{k \cdot R_{air} \cdot T}$ $\quad c = 1107 \cdot \frac{ft}{s}$

Then the time is $\Delta t = \frac{L}{c}$ $\quad \Delta t = 16.7\, s$

Problem 12.37

[Difficulty: 2]

12.37 While working on a pier on a mountain lake, you notice that the sounds of your hammering are echoing from the mountains in the distance. If the temperature is 25°C and the echoes reach you 3 seconds after the hammer strike, how far away are the mountains?

Given: Echo heard while hammering near mountain lake, time delay of echo is known

Find: How far away are the mountains

Solution:

Basic equation: $c = \sqrt{k \cdot R \cdot T}$

Assumption: Speed of light is essentially infinite (compared to speed of sound)

The given or available data is $T = (25 + 273) \cdot K$ $\quad k = 1.4$ $\quad R_{air} = 287 \cdot \frac{J}{kg \cdot K}$ $\quad \Delta t = 3 \cdot s$

Hence $c = \sqrt{k \cdot R_{air} \cdot T}$ $\quad c = 346 \frac{m}{s}$

The distance covered by the sound is: $L = c \cdot \Delta t$ $\quad L = 1038\,m$ but the distance to the mountains is half that distance:

$$\frac{L}{2} = 519\,m$$

Problem 12.38

[Difficulty: 2]

12.38 Use data for specific volume to calculate and plot the speed of sound in saturated liquid water over the temperature range from 0 to 200°C.

Given: Data on water specific volume

Find: Speed of sound over temperature range

Solution:

Basic equation: $c = \sqrt{\frac{\partial}{\partial \rho} p}$ at isentropic conditions

As an approximation for a liquid $c = \sqrt{\frac{\Delta p}{\Delta \rho}}$ using available data.

We use compressed liquid data at adjacent pressures of 5 MPa and 10 MPa, and estimate the change in density between these pressures from the corresponding specific volume changes

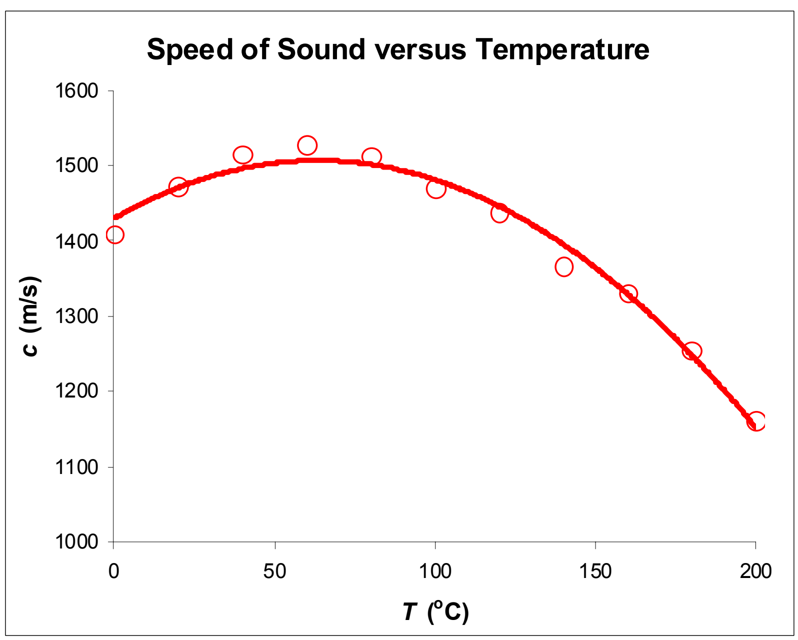
$$\Delta p = p_2 - p_1 \qquad \Delta\rho = \frac{1}{v_2} - \frac{1}{v_1} \qquad \text{and} \qquad c = \sqrt{\frac{\Delta p}{\Delta \rho}}$$ at each temperature

Here are the results, calculated using *Excel*:

$p_2 =$ 10 MPa
$p_1 =$ 5 MPa
$\eta p =$ 5 MPa

Data on specific volume versus temperature can be obtained fro any good thermodynamics text (try the Web!)

	p_1	p_2		
T (°C)	v (m³/kg)	v (m³/kg)	$\Delta\rho$ (kg/m³)	c (m/s)
0	0.0009977	0.0009952	2.52	1409
20	0.0009996	0.0009973	2.31	1472
40	0.0010057	0.0010035	2.18	1514
60	0.0010149	0.0010127	2.14	1528
80	0.0010267	0.0010244	2.19	1512
100	0.0010410	0.0010385	2.31	1470
120	0.0010576	0.0010549	2.42	1437
140	0.0010769	0.0010738	2.68	1366
160	0.0010988	0.0010954	2.82	1330
180	0.0011240	0.0011200	3.18	1254
200	0.0011531	0.0011482	3.70	1162



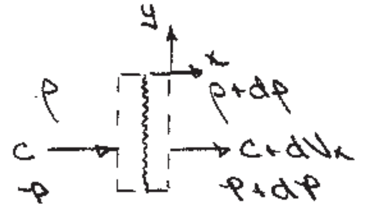
Problem 12.39

[Difficulty: 3]

12.39 Re-derive the equation for sonic speed (Eq. 12.18) assuming that the direction of fluid motion behind the sound wave is dV_x to the *right*. Show that the result is identical to that given by Eq. 12.18.

Solution:

(a) Propagating Wave



(b) Inertial CV moving with wave

Apply continuity to CV (b)

$$0 = \{-|\rho c A|\} + \{|(\rho+d\rho)(c+dV_x)A|\}$$

$$0 = \cancel{-\rho c A} + \cancel{\rho c A} + \rho dV_x A + cA d\rho + \cancel{d\rho\, dV_x}^{\approx 0}$$

then

$$\rho dV_x + c\, d\rho = 0 \quad \text{or} \quad dV_x = -\frac{c}{\rho} d\rho \quad \ldots\ldots(1)$$

Applying x-momentum equation to same CV gives

$$\sum F_{s_x} = pA - (p+dp)A = \int_{CS} V_x \rho \vec{V}_{xyz} \cdot d\vec{A} = c\{-|\rho A c|\} + (c+dV_x)\{|\rho A c|\}$$

$\{\dot{m}_{in} = \dot{m}_{out}\}$.

$$\therefore -A dp = \rho A c\, dV_x \text{ , and}$$

$$dV_x = -\frac{dp}{\rho c} \quad \ldots\ldots\ldots(2)$$

Combining equations (1) and (2) we obtain

$$-\frac{c}{\rho} d\rho = -\frac{dp}{\rho c}$$

or

$$c^2 = \frac{dp}{d\rho}$$

This is the same result as obtained in the derivation of Section 12-2 with the direction of the fluid motion behind the wave to the left

Problem 12.40

[Difficulty: 2]

12.40 Compute the speed of sound at sea level in standard air. By scanning data from Table A.3 into your PC (or using Fig. 3.3), evaluate the speed of sound and plot for altitudes to 90 km.

Given: Data on atmospheric temperature variation with altitude

Find: Sound of speed at sea level; plot speed as function of altitude

Solution

The given or available data is:

$R =$ 286.9 J/kg.K
$k =$ 1.4

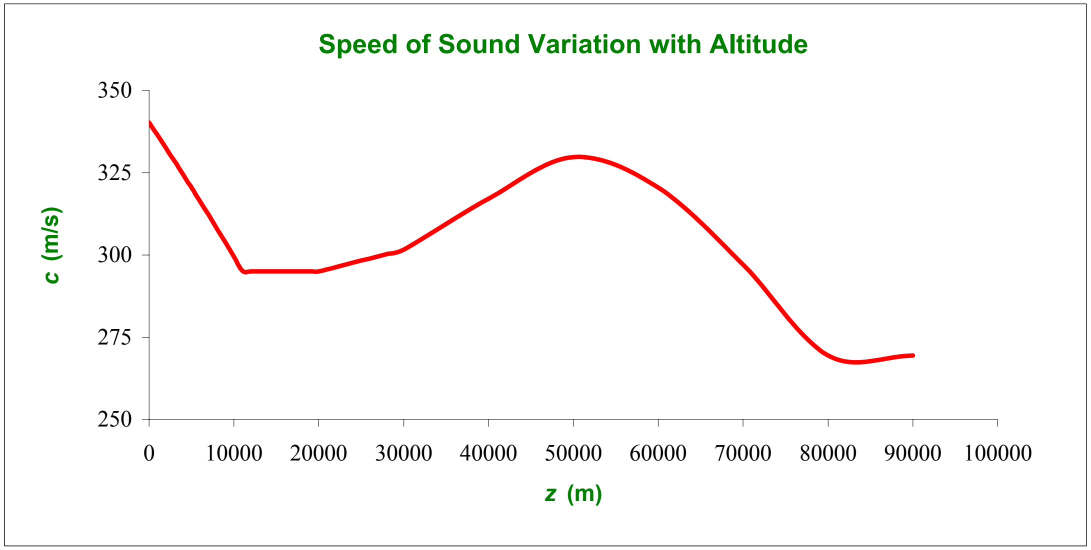
Computing equation:

$$c = \sqrt{kRT}$$

Computed results:

(Only partial data is shown in table)

z (m)	T (K)	c (m/s)
0	288.2	340
500	284.9	338
1000	281.7	336
1500	278.4	334
2000	275.2	332
2500	271.9	330
3000	268.7	329
3500	265.4	326
4000	262.2	325
4500	258.9	322
5000	255.7	320
6000	249.2	316
7000	242.7	312
8000	236.2	308
9000	229.7	304
10000	223.3	299



Problem 12.41

[Difficulty: 3]

12.41 The temperature varies linearly from sea level to approximately 11 km altitude in the standard atmosphere. Evaluate the *lapse rate*—the rate of decrease of temperature with altitude—in the standard atmosphere. Derive an expression for the rate of change of sonic speed with altitude in an ideal gas under standard atmospheric conditions. Evaluate and plot from sea level to 10 km altitude.

Given: Data on atmospheric temperature variation with altitude

Find: Lapse rate; plot rate of change of sonic speed with altitude

Solution:

The given or available data is: $R_{air} = 286.9\cdot\frac{J}{kg\cdot K}$ $\quad k = 1.4 \quad T_0 = 288.2\cdot K \quad T_{10k} = 223.3\cdot K \quad z = 10000\cdot m$

For a linear temperature variation $\quad T = T_0 + m\cdot z$

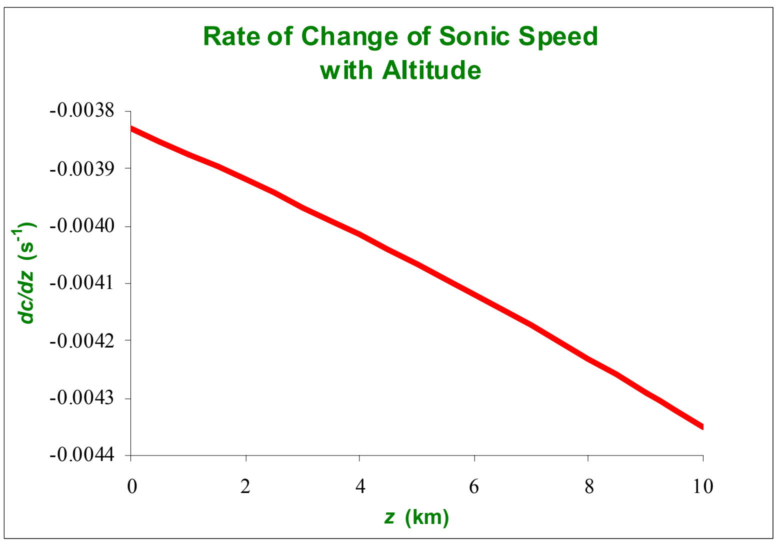
$$\frac{dT}{dz} = m = \frac{T - T_0}{z}$$ which can be evaluated at z = 10 km

$$m = \frac{T_{10k} - T_0}{z} = -6.49 \times 10^{-3}\,\frac{K}{m}$$

For an ideal gas $\quad c = \sqrt{k\cdot R\cdot T} = \sqrt{k\cdot R\cdot\left(T_0 + m\cdot z\right)}$

Hence $\quad \frac{dc}{dz} = \frac{m\cdot k\cdot R}{2\cdot c}$ $\quad$ Here are the results, calculated using *Excel*:

z (km)	*T* (K)	*dc/dz* (s^{-1})
0	288.2	-0.00383
1	281.7	-0.00387
2	275.2	-0.00392
3	268.7	-0.00397
4	262.2	-0.00402
5	255.8	-0.00407
6	249.3	-0.00412
7	242.8	-0.00417
8	236.3	-0.00423
9	229.8	-0.00429
10	223.3	-0.00435



Problem 12.42

[Difficulty: 1]

12.42 Air at 77°F flows at $M = 1.9$. Determine the air speed and the Mach angle.

Given:
Air flow at M = 1.9

Find:
Air speed; Mach angle

Solution:

Basic equations: $c = \sqrt{k \cdot R \cdot T}$ $\quad M = \dfrac{V}{c}$ $\quad \alpha = \mathrm{asin}\left(\dfrac{1}{M}\right)$

The given or available data is $T = (77 + 460) \cdot R$ $\quad M = 1.9$ $\quad k = 1.4$ $\quad R_{air} = 53.33 \cdot \dfrac{ft \cdot lbf}{lbm \cdot R}$

Hence $c = \sqrt{k \cdot R_{air} \cdot T}$ $\quad c = 1136 \cdot \dfrac{ft}{s}$

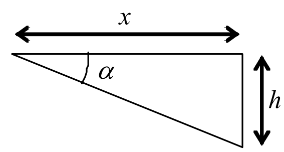
Then the air speed is $V = M \cdot c$ $\quad V = 2158 \cdot \dfrac{ft}{s}$ $\quad V = 1471 \cdot mph$

The Mach angle is given by $\alpha = \mathrm{asin}\left(\dfrac{1}{M}\right)$ $\quad \alpha = 31.8 \cdot deg$

Problem 12.43

[Difficulty: 3]

12.43 Consider the hypersonic aircraft of Problem 12.35. How long would it take for an observer to hear the aircraft after it flies over the observer? In that elapsed time, how far did the aircraft travel?



Given: Hypersonic aircraft flying overhead

Find: Time at which airplane is heard, how far aircraft travelled

Solution:

Basic equations: $c = \sqrt{k \cdot R \cdot T}$ $\quad M = \dfrac{V}{c}$ $\quad \alpha = \operatorname{asin}\left(\dfrac{1}{M}\right)$

Given or available data $M = 7$ $\quad k = 1.4$ $\quad R = 286.9 \dfrac{J}{kg \cdot K}$

The time it takes to fly from directly overhead to where you hear it is $\Delta t = \dfrac{x}{V}$

If the temperature is constant then $x = \dfrac{h}{\tan(\alpha)}$

At $h = 120000 \cdot ft$ $\quad h = 36576\,m$ $\quad$ interpolating from Table A.3 $\quad T = 226.5 \cdot K + (250.4 \cdot K - 226.5 \cdot K) \cdot \dfrac{36576 - 30000}{40000 - 30000}$

$T = 242.2\,K$

Using this temperature $c = \sqrt{k \cdot R \cdot T}$ $\quad c = 312 \dfrac{m}{s}$ $\quad$ and $\quad V = M \cdot c$ $\quad V = 2183 \dfrac{m}{s}$

Hence $\alpha = \operatorname{asin}\left(\dfrac{1}{M}\right)$ $\quad \alpha = 8.2 \cdot deg$ $\quad x = \dfrac{h}{\tan(\alpha)}$ $\quad x = 253.4 \cdot km$

$\Delta t = \dfrac{x}{V}$ $\quad \Delta t = 116.06\,s$

Problem 12.44

[Difficulty: 3]

12.44 A projectile is fired into a gas (ratio of specific heats $k = 1.625$) in which the pressure is 450 kPa (abs) and the density is 4.5 kg/m^3. It is observed experimentally that a Mach cone emanates from the projectile with 25° total angle. What is the speed of the projectile with respect to the gas?

Given: Projectile fired into a gas, Mach cone formed

Find: Speed of projectile

Solution:

Basic equations: $c = \sqrt{k \cdot R \cdot T}$ $\quad M = \dfrac{V}{c}$ $\quad \alpha = \operatorname{asin}\left(\dfrac{1}{M}\right)$ $\quad p = \rho \cdot R \cdot T$

Given or available data $\quad p = 450 \cdot \text{kPa}$ $\quad \rho = 4.5 \cdot \dfrac{\text{kg}}{\text{m}^3}$ $\quad k = 1.625$ $\quad \alpha = \dfrac{25}{2} \cdot \text{deg} = 12.5 \cdot \text{deg}$

Combining ideal gas equation of state and the sonic speed: $\quad c = \sqrt{k \cdot \dfrac{p}{\rho}}$ $\quad c = 403.1 \dfrac{\text{m}}{\text{s}}$

From the Mach cone angle: $\quad M = \dfrac{1}{\sin(\alpha)}$ $\quad M = 4.62$ $\quad$ Therefore the speed is: $\quad V = M \cdot c$ $\quad V = 1862 \dfrac{\text{m}}{\text{s}}$

Problem 12.45

[Difficulty: 1]

12.45 A photograph of a bullet shows a Mach angle of 32°. Determine the speed of the bullet for standard air.

Solution:

Computing equations, $\sin\alpha = \frac{1}{M}$ $\quad c = \sqrt{kRT}$

Assumptions: (1) air behaves as an ideal gas
(2) constant specific heats.

$M = \frac{1}{\sin\alpha}$, $M = \frac{V}{c}$ $\therefore V = cM = \frac{c}{\sin\alpha}$.

Since $c = \sqrt{kRT}$, then

$$V = \frac{1}{\sin\alpha}(kRT)^{1/2} = \frac{1}{\sin 32°}\left(1.4 \times 287\,\frac{N\cdot m}{kg\cdot K} \times 288\,K \times \frac{kg\cdot m}{N\cdot s^2}\right)^{1/2}$$

$V = 642$ m/s $\quad$ (2110 ft/s) $\longleftarrow$ V

Problem 12.46

[Difficulty: 2]

12.46 The National Transonic Facility (NTF) is a high-speed wind tunnel designed to operate with air at cryogenic temperatures to reduce viscosity, thus raising the unit Reynolds number (Re/x) and reducing pumping power requirements. Operation is envisioned at temperatures of $-270°F$ and below. A schlieren photograph taken in the NTF shows a Mach angle of 57° where $T = -270°F$ and $p = 1.3$ psia. Evaluate the local Mach number and flow speed. Calculate the unit Reynolds number for the flow.

Solution:

$$\sin\alpha = \frac{1}{M} \quad \therefore \quad M = \frac{1}{\sin\alpha} = \frac{1}{\sin 57°} = 1.19$$ M

$$c = \sqrt{kRT} = \left[1.4 \times 53.3 \frac{ft \cdot lbf}{lbm \cdot °R} \times 32.2 \frac{lbm}{slug} \times 190°R \times \frac{slug \cdot ft}{lbf \cdot s^2}\right]^{1/2} = 676\ ft/s$$

$$V = Mc = 1.19\,(676\ ft/s) = 804\ ft/s$$ V

$$Re_x = \frac{\rho V x}{\mu}$$

$$\rho = \frac{p}{RT} = 1.3 \frac{lbf}{in^2} \times \frac{lbm \cdot °R}{53.3\ ft \cdot lbf} \times \frac{1}{190°R} \times \frac{144\ in^2}{ft^2} = 0.0185\ lbm/ft^3$$

From Eq. A.1 (Appendix A)

$$\mu = \frac{bT^{1/2}}{1 + S/T} \qquad b = 1.458 \times 10^{-6}\ kg/m \cdot s \cdot K^{1/2}$$

$S = 110.4\ K$

T in K

$$T = -270F = -167C = 106K.$$

$$\mu = 1.458 \times 10^{-6} \frac{kg}{m \cdot s \cdot K^{1/2}} (106K)^{1/2} \times \frac{1}{1 + \frac{110.4}{106}} = 7.35 \times 10^{-6}\ kg/m \cdot s$$

$$\mu = 7.35 \times 10^{-6} \frac{kg}{m \cdot s} \times \frac{N \cdot s^2}{kg \cdot m} \times \frac{2.089 \times 10^{-2}\ lbf \cdot s/ft^2}{1\ N \cdot s}$$

$$\mu = 1.54 \times 10^{-7}\ lbf \cdot s/ft^2$$

$$\therefore \frac{Re}{x} = \frac{\rho V}{\mu} = 0.0185 \frac{lbm}{ft^3} \times 804 \frac{ft}{s} \times \frac{ft^2}{1.54 \times 10^{-7}\ lbf \cdot s} \times \frac{slug}{32.2\ lbm} \times \frac{lbf \cdot s^2}{slug \cdot ft}$$

$$\frac{Re}{x} = 3.00 \times 10^6\ ft^{-1} = 9.84 \times 10^6\ m^{-1}$$ Re/x

Problem 12.47

[Difficulty: 2]

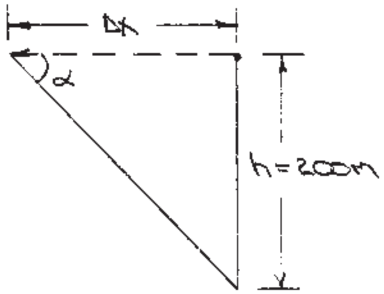
12.47 An F-4 aircraft makes a high-speed pass over an airfield on a day when $T = 35°C$. The aircraft flies at $M = 1.4$ and 200 m altitude. Calculate the speed of the aircraft. How long after it passes directly over point A on the ground does its Mach cone pass over point A?

Solution: Assume T = constant over 200m elevation.

$T = 35\ C = 308\ K$

$$c = \sqrt{kRT} = \left[1.4 \times 287\ \frac{N\cdot m}{kg\cdot K} \times 308\ K \times \frac{kg\cdot m}{N\cdot s^2}\right]^{1/2} = 352\ m/s$$

$$V = Mc = 1.4 \times 352\ m/s = 493\ m/s \quad \longleftarrow V$$



From the instant the aircraft is directly overhead until the Mach cone reaches the ground, the plane travels a distance Δx at speed $V = 493\ m/s$.

$$\sin\alpha = \frac{1}{M} = \frac{1}{1.4} = 0.7143$$

$$\alpha = 45.6°$$

$$\frac{h}{\Delta x} = \tan\alpha \qquad \therefore\ \Delta x = \frac{h}{\tan\alpha} = \frac{200\ m}{\tan 45.6°} = 196\ m$$

Since the plane moves at constant speed V

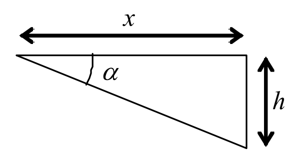
$$\Delta x = V\,\Delta t \quad \text{and} \quad \Delta t = \frac{\Delta x}{V} = \frac{196\ m}{493\ m/s}$$

$$\Delta t = 0.398\ s \quad \longleftarrow \Delta t$$

Problem 12.48

[Difficulty: 2]

12.48 While jogging on the beach (it's a warm summer day, about 25°C) a high-speed jet flies overhead. You guesstimate that it's at an altitude of about 3000 m, and count off about 7.5 s before you hear it. Estimate the speed and Mach number of the jet.



Given: High-speed jet flying overhead

Find: Estimate speed and Mach number of jet

Solution:

Basic equations: $c = \sqrt{k \cdot R \cdot T}$ $\quad M = \frac{V}{c}$ $\quad \alpha = \text{asin}\left(\frac{1}{M}\right)$

Given or available data $\quad T = (25 + 273) \cdot K \quad h = 3000 \cdot m \quad k = 1.4 \quad R = 286.9 \frac{J}{kg \cdot K}$

The time it takes to fly from directly overhead to where you hear it is $\quad \Delta t = 7.5 \cdot s$

The distance traveled, moving at speed V, is $\quad x = V \cdot \Delta t$

The Mach angle is related to height h and distance x by

$$\tan(\alpha) = \frac{\sin(\alpha)}{\cos(\alpha)} = \frac{h}{x} = \frac{h}{V \cdot \Delta t} \qquad (1)$$

and also we have

$$\sin(\alpha) = \frac{1}{M} = \frac{c}{V} \qquad (2)$$

Dividing Eq. 2 by Eq 1

$$\cos(\alpha) = \frac{c}{V} \cdot \frac{V \cdot \Delta t}{h} = \frac{c \cdot \Delta t}{h}$$

Note that we could have written this equation from geometry directly!

We have $\quad c = \sqrt{k \cdot R \cdot T} \quad c = 346 \frac{m}{s}$ $\quad$ so $\quad \alpha = \text{acos}\left(\frac{c \cdot \Delta t}{h}\right) \quad \alpha = 30.1 \cdot deg$

Hence $\quad M = \frac{1}{\sin(\alpha)} \quad M = 1.99$

Then the speed is $\quad V = M \cdot c \quad V = 689 \frac{m}{s}$

Note that we assume the temperature of the air is uniform. In fact the temperature will vary over 3000 m, so the Mach cone will be curved. This speed and Mach number are only rough estimates.

Problem 12.49

[Difficulty: 2]

12.49 An aircraft passes overhead at 3 km altitude. The aircraft flies at $M = 1.5$; assume air temperature is constant at 20°C. Find the air speed of the aircraft. A headwind blows at 30 m/s. How long after the aircraft passes directly overhead does its sound reach a point on the ground?

Solution:

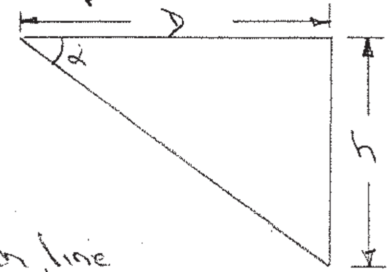
$T = \text{constant} = 20°C = 293\ K$

$c = (kRT)^{1/2} = \left(1.4 \times 287 \frac{N \cdot m}{kg \cdot K} \times 293\ K \times \frac{kg \cdot m}{N \cdot s^2}\right)^{1/2} = 343\ m/s$

$V = Mc = 1.5 \times 343 \frac{m}{sec} = 515\ m/s \longleftarrow$ airspeed

The airspeed is the velocity of the plane relative to the air

The ground speed is then $\vec{V}_p = \vec{V}_{air} + \vec{V}_{pla}$ or $V_p = 485\ m/sec$



From the instant the aircraft is directly overhead until the Mach cone reaches the ground, the plane travels a distance, D, at speed $V_p = 485\ m/s$.

The value of the time, t, is then $t = D/V_p$

Since the air temperature is constant, the Mach line is straight and $D = h/\tan\alpha$, where $\alpha = \sin^{-1}(1/M)$

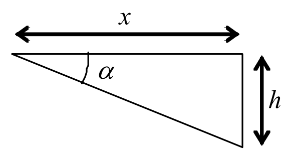
$\alpha = \sin^{-1}\left(\frac{1}{M}\right) = \sin^{-1}\left(\frac{1}{1.5}\right) = 41.8°$

Then, $t = \frac{D}{V_p} = \frac{h}{\tan\alpha}\frac{1}{V_p} = \frac{3000\ m}{\tan 41.8} \times \frac{s}{485\ m} = 6.92\ s \longleftarrow t$

Problem 12.50

[Difficulty: 3]

12.50 A supersonic aircraft flies at 3 km altitude at a speed of 1000 m/s on a standard day. How long after passing directly above a ground observer is the sound of the aircraft heard by the ground observer?



Given:
Supersonic aircraft flying overhead

Find:
Time at which airplane heard

Solution:

Basic equations: $c = \sqrt{k \cdot R \cdot T}$ $\quad M = \frac{V}{c}$ $\quad \alpha = \mathrm{asin}\left(\frac{1}{M}\right)$

Given or available data $V = 1000 \cdot \frac{m}{s}$ $\quad h = 3 \cdot km$ $\quad k = 1.4$ $\quad R = 286.9 \frac{J}{kg \cdot K}$

The time it takes to fly from directly overhead to where you hear it is $\Delta t = \frac{x}{V}$

If the temperature is constant then $x = \frac{h}{\tan(\alpha)}$

The temperature is not constant so the Mach line will not be straight. We can find a range of Δt by considering the temperature range

At $h = 3 \cdot km$ we find from Table A.3 that $T = 268.7 \cdot K$

Using this temperature $c = \sqrt{k \cdot R \cdot T}$ $\quad c = 329 \frac{m}{s}$ $\quad$ and $\quad M = \frac{V}{c}$ $\quad M = 3.04$

Hence $\alpha = \mathrm{asin}\left(\frac{1}{M}\right)$ $\quad \alpha = 19.2 \cdot deg$ $\quad x = \frac{h}{\tan(\alpha)}$ $\quad x = 8625 m$ $\quad \Delta t = \frac{x}{V}$ $\quad \Delta t = 8.62 s$

At sea level we find from Table A.3 that $T = 288.2 \cdot K$

Using this temperature $c = \sqrt{k \cdot R \cdot T}$ $\quad c = 340 \frac{m}{s}$ $\quad$ and $\quad M = \frac{V}{c}$ $\quad M = 2.94$

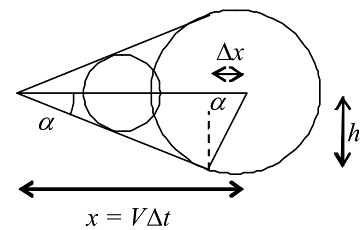
Hence $\alpha = \mathrm{asin}\left(\frac{1}{M}\right)$ $\quad \alpha = 19.9 \cdot deg$ $\quad x = \frac{h}{\tan(\alpha)}$ $\quad x = 8291 m$ $\quad \Delta t = \frac{x}{V}$ $\quad \Delta t = 8.29 s$

Thus we conclude that the time is somwhere between 8.62 and 8.29 s. Taking an average $\Delta t = 8.55 \cdot s$

Problem 12.51

[Difficulty: 3]

12.51 For the conditions of Problem 12.50, find the location at which the sound wave that first reaches the ground observer was emitted.



Given:
Supersonic aircraft flying overhead

Find:
Location at which first sound wave was emitted

Solution:

Basic equations: $c = \sqrt{k \cdot R \cdot T}$ $\quad M = \frac{V}{c}$ $\quad \alpha = \operatorname{asin}\left(\frac{1}{M}\right)$

Given or available data $V = 1000 \cdot \frac{m}{s}$ $\quad h = 3 \cdot km$ $\quad k = 1.4$ $\quad R = 286.9 \frac{J}{kg \cdot K}$

We need to find Δx as shown in the figure $\quad \Delta x = h \cdot \tan(\alpha)$

The temperature is not constant so the Mach line will not be straight (α is not constant). We can find a range of α and Δx by considering the temperature range

At $h = 3 \cdot km$ we find from Table A.3 that $\quad T = 268.7 \cdot K$

Using this temperature $\quad c = \sqrt{k \cdot R \cdot T}$ $\quad c = 329 \frac{m}{s}$ $\quad$ and $\quad M = \frac{V}{c}$ $\quad M = 3.04$

Hence $\quad \alpha = \operatorname{asin}\left(\frac{1}{M}\right)$ $\quad \alpha = 19.2 \cdot deg$ $\quad \Delta x = h \cdot \tan(\alpha)$ $\quad \Delta x = 1043\, m$

At sea level we find from Table A.3 that $\quad T = 288.2 \cdot K$

Using this temperature $\quad c = \sqrt{k \cdot R \cdot T}$ $\quad c = 340 \frac{m}{s}$ $\quad$ and $\quad M = \frac{V}{c}$ $\quad M = 2.94$

Hence $\quad \alpha = \operatorname{asin}\left(\frac{1}{M}\right)$ $\quad \alpha = 19.9 \cdot deg$ $\quad \Delta x = h \cdot \tan(\alpha)$ $\quad \Delta x = 1085\, m$

Thus we conclude that the distance is somwhere between 1043 and 1085 m. Taking an average $\quad \Delta x = 1064 \cdot m$

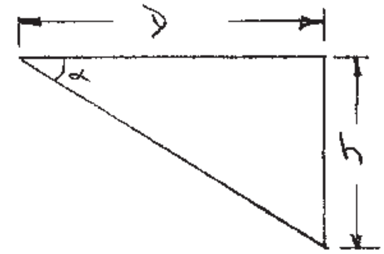
12.52 The Concorde supersonic transport cruised at $M = 2.2$ at 17 km altitude on a standard day. How long after the aircraft passed directly above a ground observer was the sound of the aircraft heard?

Solution:

At altitude, $T = 216.7\,K$ $\quad c = (kRT)^{1/2} = \left(1.4 \times 287 \frac{N\cdot m}{kg\cdot K} \times 216.7\,K \times \frac{kg\cdot m}{N\cdot s^2}\right)^{1/2} = 295\,m/s$

$V = Mc = 2.2 \times 295\,m/s = 649\,m/s.$

If the speed of sound were constant all the way to the ground, the Mach line would remain straight. The Mach angle α, would be constant with



$$\alpha = \sin^{-1}\left(\frac{1}{M}\right) = \sin^{-1}\left(\frac{1}{2.2}\right) = 27^\circ$$

Then, from the diagram $\quad D = \frac{h}{\tan\alpha}$

and $\quad t = \frac{D}{V} = \frac{h}{\tan\alpha\, V} = \frac{17000\,m}{\tan 27^\circ} \times \frac{1}{649}\frac{sec}{m} = 51.4\,s$

However, the speed of sound varies over the altitude because the temperature varies with altitude.

At sea level $\quad T = 288.2\,K$

$$c = (kRT)^{1/2} = \left(1.4 \times 287 \frac{N\cdot m}{kg\cdot K} \times 288.2\,K \times \frac{kg\cdot m}{N\cdot s^2}\right)^{1/2} = 340\,m/s$$

The corresponding value of Mach number for $V = 649\,m/s$ is

$$M = \frac{V}{c} = \frac{649}{340} = 1.91$$

$$\alpha = \sin^{-1}\left(\frac{1}{M}\right) = \sin^{-1}\left(\frac{1}{1.91}\right) = 31.6^\circ$$

Thus, if the speed of sound were constant (at the sea level) value over the entire altitude, then

$$t = \frac{D}{V} = \frac{h}{\tan\alpha\, V} = \frac{17000\,m}{\tan 31.6} \times \frac{s}{649\,m} = 42.6\,s$$

We can obtain a better approximate by considering the variation of temperature with altitude.

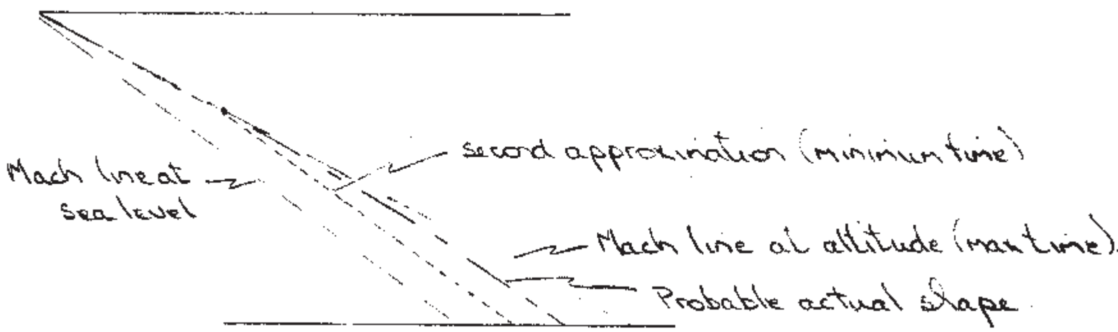
From Table A.3

$11\,km \le y < 20\,km \qquad T = 216.7\,K$

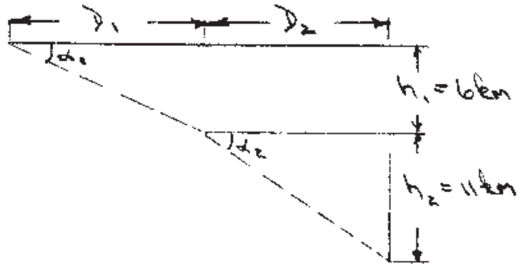
$0 < y \le 11\,km \qquad T$ varies linearly with y.

$T = T_0 - by \qquad T_0 = 288.2\,K$

$\qquad\qquad\qquad\qquad b = 6.50\,K/km$



Since T is constant for $y > y_0 = 11$ km, the second approximation which assumes the Mach line at sea level for $0 < y \le 11$ km gives the minimum time



$$D_1 = \frac{h_1}{\tan\alpha_1} = \frac{6\text{ km}}{\tan 27^\circ} = 11.77\text{ km}$$

$$D_2 = \frac{h_2}{\tan\alpha_2} = \frac{11\text{ km}}{\tan 31.6^\circ} = 17.88\text{ km}$$

$$D = D_1 + D_2$$

$$t = \frac{D}{V} = \frac{29.65\text{ km}}{649\text{ m/s}} = 45.7\text{ s}$$

Consequently, $45.7 \le t \le 51.4$ s

Since the two values are reasonably close, it is appropriate to take the average value and say $t \approx 48.5$ s ← t

Problem 12.53

[Difficulty: 2]

12.53 The airflow around an automobile is assumed to be incompressible. Investigate the validity of this assumption for an automobile traveling at 60 mph. (Relative to the automobile the minimum air velocity is zero, and the maximum is approximately 120 mph.)

Given:
Speed of automobile

Find:
Whether flow can be considered incompressible

Solution:

Consider the automobile at rest with 60 mph air flowing over it. Let state 1 be upstream, and point 2 the stagnation point on the automobile

The data provided, or available in the Appendices, is:

$$R = 287 \cdot \frac{J}{kg \cdot K} \qquad k = 1.4 \qquad V_1 = 60 \cdot mph \qquad p_1 = 101 \cdot kPa \qquad T_1 = (20 + 273) \cdot K$$

The basic equation for the density change is
$$\frac{\rho_0}{\rho} = \left[1 + \frac{(k-1)}{2} \cdot M^2\right]^{\frac{1}{k-1}} \qquad (12.20c)$$

or
$$\rho_0 = \rho_1 \cdot \left[1 + \frac{(k-1)}{2} \cdot M_1^{\,2}\right]^{\frac{1}{k-1}}$$

$$\rho_1 = \frac{p_1}{R \cdot T_1} \qquad \rho_1 = 1.201 \frac{kg}{m^3}$$

For the Mach number we need c
$$c_1 = \sqrt{k \cdot R \cdot T_1} \qquad c_1 = 343 \frac{m}{s}$$

$$V_1 = 26.8 \frac{m}{s} \qquad M_1 = \frac{V_1}{c_1} \qquad M_1 = 0.0782$$

$$\rho_0 = \rho_1 \cdot \left(1 + \frac{k-1}{2} \cdot M_1^{\,2}\right)^{\frac{1}{k-1}} \qquad \rho_0 = 1.205 \frac{kg}{m^3}$$

The percentage change in density is
$$\left|\frac{\rho_0 - \rho_1}{\rho_0}\right| = 0.305 \cdot \%$$

This is an insignificant change, so the flow can be considered incompressible. Note that $M < 0.3$, the usual guideline for incompressibility

For the maximum speed present
$$V_1 = 120 \cdot mph \qquad V_1 = 53.6 \frac{m}{s} \qquad M_1 = \frac{V_1}{c_1} \qquad M_1 = 0.156$$

$$\rho_0 = \rho_1 \cdot \left(1 + \frac{k-1}{2} \cdot M_1^{\,2}\right)^{\frac{1}{k-1}} \qquad \rho_0 = 1.216 \frac{kg}{m^3}$$

The percentage change in density is
$$\left|\frac{\rho_0 - \rho_1}{\rho_0}\right| = 1.21 \cdot \%$$

This is still an insignificant change, so the flow can be considered incompressible.

Problem 12.54

[Difficulty: 5]

12.54 Opponents of supersonic transport aircraft claim that sound waves can be refracted in the upper atmosphere and that, as a result, sonic booms can be heard several hundred miles away from the ground track of the aircraft. Explain the phenomenon of sound wave refraction.

Given: Supersonic transport aircraft

Find: Explanation of sound wave refraction

Solution:

A sound wave is refracted when the speed of sound varies with altitude in the atmosphere. (The variation in sound speed is caused by temperature variations in the atmosphere, as shown in Fig. 3.3)

Imagine a plane wave front that initially is vertical. When the wave encounters a region where the temperature increase with altitude (such as between 20.1 km and 47.3 km altitude in Fig. 3.3), the sound speed increases with elevation. Therefore the upper portion of the wave travels faster than the lower portion. The wave front turns gradually and the sound wave follows a curved path through the atmosphere. Thus a wave that initially is horizontal bends and follows a curved path, tending to reach the ground some distance from the source.

The curvature and the path of the sound could be calculated for any specific temperature variation in the atmosphere. However, the required analysis is beyond the scope of this text.

Problem 12.55

[Difficulty: 2]

12.55 Plot the percentage discrepancy between the density at the stagnation point and the density at a location where the Mach number is M, of a compressible flow, for Mach numbers ranging from 0.05 to 0.95. Find the Mach numbers at which the discrepancy is 1 percent, 5 percent, and 10 percent.

Given: Mach number range from 0.05 to 0.95

Find: Plot of percentage density change; Mach number for 1%, 5% and 10% density change

Solution:

The given or available data is: $k = 1.4$

Basic equation:

$$\frac{\rho_0}{\rho} = \left[1 + \frac{(k-1)}{2}\cdot M^2\right]^{\frac{1}{k-1}} \quad (12.20c) \qquad \text{Hence} \quad \frac{\Delta\rho}{\rho_0} = \frac{\rho_0 - \rho}{\rho_0} = 1 - \frac{\rho}{\rho_0} \quad \text{so} \quad \frac{\Delta\rho}{\rho_0} = 1 - \left[1 + \frac{(k-1)}{2}\cdot M^2\right]^{\frac{1}{1-k}}$$

Here are the results, generated using *Excel*:

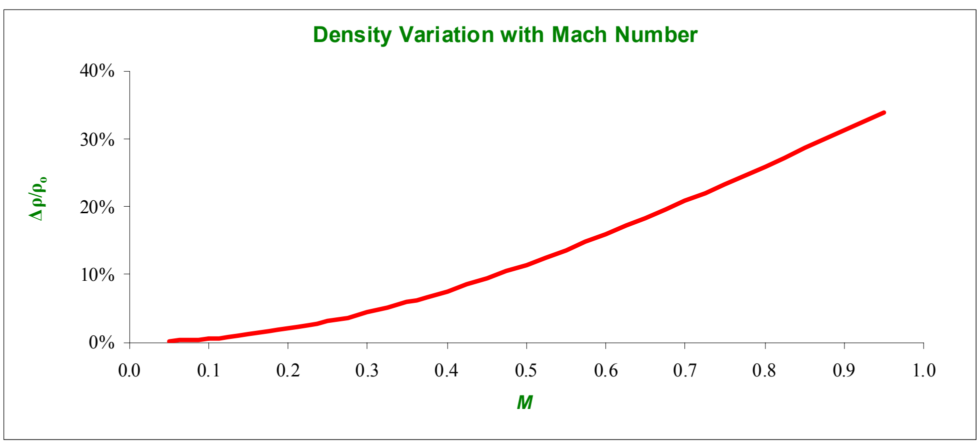
M	$\Delta\rho/\rho_o$
0.05	0.1%
0.10	0.5%
0.15	1.1%
0.20	2.0%
0.25	3.1%
0.30	4.4%
0.35	5.9%
0.40	7.6%
0.45	9.4%
0.50	11%
0.55	14%
0.60	16%
0.65	18%
0.70	21%
0.75	23%
0.80	26%
0.85	29%
0.90	31%
0.95	34%

To find *M* for specific density changes use *Goal Seek* repeatedly

M	$\Delta\rho/\rho_o$
0.142	1%
0.322	5%
0.464	10%

Note: Based on ρ (not ρ_o) the results are:

0.142	0.314	0.441



Problem 12.56

[Difficulty: 2]

12.56 Find the stagnation temperature at the nose of the missile described in Problem 12.29.

Given: Scramjet-powered missile traveling at fixed Mach number and altitude

Find: Stagnation temperature at the nose of the missile

Solution:

Basic equation: $T_0 = T\cdot\left(1 + \frac{k-1}{2}\cdot M^2\right)$

Available data $R = 286.9\,\frac{J}{kg\cdot K}$ $\quad k = 1.4 \quad M = 7$

At $z = 85000\cdot ft \quad z = 25908\,m$ interpolating from Table A.3 $T = 220.6\cdot K + (222.5\cdot K - 220.6\cdot K)\cdot\frac{25908 - 24000}{26000 - 24000}$

$T = 222\,K$

So the stagnation temperature is $T_0 = T\cdot\left(1 + \frac{k-1}{2}\cdot M^2\right)$ $\quad T_0 = 2402\,K$

Problem 12.57

[Difficulty: 2]

12.57 Find the stagnation temperature at the nose of the aircraft described in Problem 12.34.

Given: X-15 rocket plane traveling at fixed Mach number and altitude

Find: Stagnation temperature at the nose of the plane

Solution:

Basic equation: $T_0 = T\cdot\left(1 + \frac{k-1}{2}\cdot M^2\right)$ $\quad c = \sqrt{k\cdot R\cdot T}$ $\quad M = \frac{V}{c}$

Available data $R = 286.9\frac{J}{kg\cdot K}$ $\quad k = 1.4$ $\quad V = 7270\cdot\frac{km}{hr}$

At $z = 58400\cdot m$ interpolating from Table A.3 $T = 270.7\cdot K + (255.8\cdot K - 270.7\cdot K)\cdot\frac{58400 - 50000}{60000 - 50000}$

$T = 258\,K$

Hence $c = \sqrt{k\cdot R\cdot T}$ $\quad c = 322\frac{m}{s}$ $\quad c = 1159\cdot\frac{km}{hr}$ and we have $M = \frac{V}{c} = 6.27$

So the stagnation temperature is $T_0 = T\cdot\left(1 + \frac{k-1}{2}\cdot M^2\right)$ $\quad T_0 = 2289\,K$

Problem 12.58

[Difficulty: 1]

12.58 Find the ratio of static to total pressure for a car moving at 55 mph and for a Formula One race car traveling at 220 mph at sea level. Do you expect the flow over either car to experience compressibility effects?

Given: Car and F-1 race car traveling at sea level

Find: Ratio of static to total pressure in each case; are compressiblilty effects experienced?

Solution:

Basic equations: $c = \sqrt{k \cdot R \cdot T}$ $\quad M = \frac{V}{c}$ $\quad \frac{p_0}{p} = \left(1 + \frac{k-1}{2} \cdot M^2\right)^{\frac{k}{k-1}}$

Given or available data $\quad V_{car} = 55 \cdot mph \quad V_{car} = 80.7 \cdot \frac{ft}{s} \quad V_{F1} = 220 \cdot mph \quad V_{F1} = 323 \cdot \frac{ft}{s}$

$k = 1.4 \quad R_{air} = 53.33 \cdot \frac{ft \cdot lbf}{lbm \cdot R}$

At sea level, from Table A.3 $\quad T = 288.2 \cdot K$ or $T = 519 \cdot R \quad \rho = 0.002377 \cdot \frac{slug}{ft^3} \quad p = 14.696 \cdot psi$

Hence $\quad c = \sqrt{k \cdot R_{air} \cdot T} \quad c = 1116 \cdot \frac{ft}{s} \quad M_{car} = \frac{V_{car}}{c} \quad M_{car} = 0.0723$

The pressure ratio is $\quad \frac{p}{p_0} = \left(1 + \frac{k-1}{2} \cdot M_{car}^2\right)^{-\frac{k}{k-1}} = 0.996$

Note that the Bernoulli equation would give the same result! $\quad \frac{p}{p_0} = \left(1 + \frac{\rho \cdot V_{car}^2}{2 \cdot p}\right)^{-1} = 0.996$

For the Formula One car: $\quad M_{F1} = \frac{V_{F1}}{c} \quad M_{F1} = 0.289$

The pressure ratio is $\quad \frac{p}{p_0} = \left(1 + \frac{k-1}{2} \cdot M_{F1}^2\right)^{-\frac{k}{k-1}} = 0.944$

Note that the Bernoulli equation would give almost the same result: $\quad \frac{p}{p_0} = \left(1 + \frac{\rho \cdot V_{F1}^2}{2 \cdot p}\right)^{-1} = 0.945$

Incompressible flow can be assumed for both cases, but the F1 car gets very close to the Mach 0.3 rule of thumb for compressible vs. incompressible flow.

Problem 12.59

[Difficulty: 2]

12.59 Find the dynamic and stagnation pressures for the missile described in Problem 12.29.

Given: Scramjet-powered missile traveling at fixed Mach number and altitude

Find: Stagnation and dynamic pressures

Solution:

Basic equation: $c = \sqrt{k \cdot R \cdot T}$ $\quad M = \dfrac{V}{c}$ $\quad \dfrac{p_0}{p} = \left(1 + \dfrac{k-1}{2} \cdot M^2\right)^{\frac{k}{k-1}}$ $\quad p_{dyn} = \dfrac{1}{2} \cdot \rho \cdot V^2$

Available data $\quad R = 286.9 \dfrac{J}{kg \cdot K}$ $\quad k = 1.4$ $\quad M = 7$ $\quad p_{SL} = 14.696 \cdot psi$ $\quad \rho_{SL} = 0.2377 \cdot \dfrac{slug}{ft^3}$

At $\quad z = 85000 \cdot ft$ $\quad z = 25908\,m$ $\quad$ interpolating from Table A.3 $\quad T = 220.6 \cdot K + (222.5 \cdot K - 220.6 \cdot K) \cdot \dfrac{25908 - 24000}{26000 - 24000}$

$T = 222\,K$

Hence $\quad c = \sqrt{k \cdot R \cdot T}$ $\quad c = 299 \dfrac{m}{s}$ $\quad c = 981 \cdot \dfrac{ft}{s}$ $\quad$ and we have $\quad V = M \cdot c = 6864 \cdot \dfrac{ft}{s}$

The static pressure and density can be found by interpolation:

$p = p_{SL} \cdot \left[0.02933 + (0.02160 - 0.02933) \cdot \dfrac{25908 - 24000}{26000 - 24000}\right]$ $\quad p = 0.323 \cdot psi$ $\quad p_0 = p \cdot \left(1 + \dfrac{k-1}{2} \cdot M^2\right)^{\frac{k}{k-1}}$ $\quad p_0 = 1336 \cdot psi$

$\rho = \rho_{SL} \cdot \left[0.03832 + (0.02797 - 0.03832) \cdot \dfrac{25908 - 24000}{26000 - 24000}\right]$ $\quad \rho = 0.00676 \cdot \dfrac{slug}{ft^3}$ $\quad p_{dyn} = \dfrac{1}{2} \cdot \rho \cdot V^2$ $\quad p_{dyn} = 1106 \cdot psi$

Problem 12.60

[Difficulty: 2]

12.60 Find the dynamic and stagnation pressures for the aircraft described in Problem 12.34.

Given: X-15 rocket plane traveling at fixed Mach number and altitude

Find: Stagnation and dynamic pressures

Solution:

Basic equation: $c = \sqrt{k \cdot R \cdot T}$ $\quad M = \frac{V}{c}$ $\quad \frac{p_0}{p} = \left(1 + \frac{k-1}{2} \cdot M^2\right)^{\frac{k}{k-1}}$ $\quad p_{dyn} = \frac{1}{2} \cdot \rho \cdot V^2$

Available data $\quad R = 286.9 \frac{J}{kg \cdot K}$ $\quad k = 1.4$ $\quad V = 7270 \cdot \frac{km}{hr}$ $\quad p_{SL} = 101.3 \cdot kPa$ $\quad \rho_{SL} = 1.225 \cdot \frac{kg}{m^3}$

At $\quad z = 58400 \cdot m$ $\quad$ interpolating from Table A.3 $\quad T = 270.7 \cdot K + (255.8 \cdot K - 270.7 \cdot K) \cdot \frac{58400 - 50000}{60000 - 50000}$

$T = 258\,K$

Hence $\quad c = \sqrt{k \cdot R \cdot T}$ $\quad c = 322 \frac{m}{s}$ $\quad c = 1159 \cdot \frac{km}{hr}$ $\quad$ and we have $\quad M = \frac{V}{c} = 6.27$

The static pressure and density can be found by interpolation:

$$p = p_{SL} \cdot \left[0.0007874 + (0.0002217 - 0.0007874) \cdot \frac{58400 - 50000}{60000 - 50000}\right] \qquad p = 0.0316 \cdot kPa$$

$$p_0 = p \cdot \left(1 + \frac{k-1}{2} \cdot M^2\right)^{\frac{k}{k-1}} \qquad p_0 = 65.6 \cdot kPa$$

$$\rho = \rho_{SL} \cdot \left[0.0008383 + (0.0002497 - 0.0008383) \cdot \frac{58400 - 50000}{60000 - 50000}\right] \qquad \rho = 4.21 \times 10^{-4} \cdot \frac{kg}{m^3}$$

$$p_{dyn} = \frac{1}{2} \cdot \rho \cdot V^2 \qquad p_{dyn} = 0.86 \cdot kPa$$

Problem 12.61

[Difficulty: 1]

12.61 An aircraft flies at 250 m/s in air at 28 kPa and −50°C. Find the stagnation pressure at the nose of the aircraft.

Given: Aircraft flying at 250 m/s

Find: Stagnation pressure

Solution:

Basic equations: $c = \sqrt{k \cdot R \cdot T}$ $\quad M = \frac{V}{c}$ $\quad \frac{p_0}{p} = \left(1 + \frac{k-1}{2} \cdot M^2\right)^{\frac{k}{k-1}}$

Given or available data $V = 250 \cdot \frac{m}{s}$ $\quad T = (-50 + 273) \cdot K$ $\quad p = 28 \cdot kPa$ $\quad k = 1.4$ $\quad R = 286.9 \frac{J}{kg \cdot K}$

First we need $c = \sqrt{k \cdot R \cdot T}$ $\quad c = 299 \frac{m}{s}$ $\quad$ then $\quad M = \frac{V}{c}$ $\quad M = 0.835$

Finally we solve for p_0 $\quad p_0 = p \cdot \left(1 + \frac{k-1}{2} \cdot M^2\right)^{\frac{k}{k-1}}$ $\quad p_0 = 44.2 \cdot kPa$

Problem 12.62

[Difficulty: 2]

12.62 Compute the air density in the undisturbed air, and at the stagnation point, of Problem 12.61. What is the percentage increase in density? Can we approximate this as an incompressible flow?

Given:

Pressure data on aircraft in flight

Find:

Change in air density; whether flow can be considered incompressible

Solution:

The data provided, or available in the Appendices, is:

$$k = 1.4 \qquad p_0 = 48 \cdot \text{kPa} \qquad p = 27.6 \cdot \text{kPa} \qquad T = (-55 + 273) \cdot \text{K}$$

Governing equation (assuming isentropic flow):

$$\frac{p}{\rho^k} = \text{constant} \qquad (12.12c)$$

Hence

$$\frac{\rho}{\rho_0} = \left(\frac{p}{p_0}\right)^{\frac{1}{k}}$$

so

$$\frac{\Delta\rho}{\rho} = \frac{\rho_0 - \rho}{\rho} = \frac{\rho_0}{\rho} - 1 = \left(\frac{p_0}{p}\right)^{\frac{1}{k}} - 1 \qquad \frac{\Delta\rho}{\rho} = 48.5 \cdot \%$$

NOT an incompressible flow!

Problem 12.63

[Difficulty: 2]

12.63 For an aircraft traveling at $M = 2$ at an elevation of 12 km, find the dynamic and stagnation pressures.

Given: Aircraft flying at 12 km

Find: Dynamic and stagnation pressures

Solution:

Basic equations: $c = \sqrt{k \cdot R \cdot T}$ $\quad M = \frac{V}{c}$ $\quad \frac{p_0}{p} = \left(1 + \frac{k-1}{2} \cdot M^2\right)^{\frac{k}{k-1}}$ $\quad p_{dyn} = \frac{1}{2} \cdot \rho \cdot V^2$

Given or available data $\quad M = 2 \quad h = 12 \cdot km \quad k = 1.4 \quad R = 286.9 \cdot \frac{J}{kg \cdot K}$

$\rho_{SL} = 1.225 \cdot \frac{kg}{m^3} \quad p_{SL} = 101.3 \cdot kPa$

At $h = 12 \cdot km$, from Table A.3 $\quad \rho = 0.2546 \cdot \rho_{SL} \quad \rho = 0.312 \frac{kg}{m^3} \quad p = 0.1915 \cdot p_{SL} \quad p = 19.4 \cdot kPa \quad T = 216.7 \cdot K$

Hence $\quad p_0 = p \cdot \left(1 + \frac{k-1}{2} \cdot M^2\right)^{\frac{k}{k-1}} \quad p_0 = 152 \cdot kPa$

Also $\quad c = \sqrt{k \cdot R \cdot T} \quad c = 295 \frac{m}{s} \quad V = M \cdot c \quad V = 590 \frac{m}{s}$

Hence $\quad p_{dyn} = \frac{1}{2} \cdot \rho \cdot V^2 \quad p_{dyn} = 54.3 \cdot kPa$

Problem 12.64

[Difficulty: 1]

12.64 A body moves through standard air at 200 m/s. What is the stagnation pressure on the body? Assume (a) compressible flow and (b) incompressible flow.

Solution:

For standard air, $P = 101$ kPa, $T = 15°C$

Computing equations: $P_o = P + \frac{1}{2}\rho V^2$ (incompressible)

$$\frac{P_o}{P} = \left[1 + \frac{k-1}{2}M^2\right]^{k/k-1} \quad \text{(compressible)}.$$

(a) Incompressible flow

$$P_o = P + \frac{1}{2}\rho V^2 = 101\ \text{kPa} + \frac{1}{2} \times 1.225\ \frac{\text{kg}}{\text{m}^3} \times (200)^2\ \frac{\text{m}^2}{\text{s}^2} \times \frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}} \times \frac{\text{kPa}}{10^3}\ \frac{\text{m}^2}{\text{N}}$$

$P_o = 125.5$ kPa ← $P_{o\,inc}$

b. Compressible flow

$M = \frac{V}{c}$ $\quad c = (kRT)^{1/2} = \left(1.4 \times 287\ \frac{\text{N}\cdot\text{m}}{\text{kg}\cdot\text{K}} \times 288\text{K} \times \frac{\text{kg}\cdot\text{m}}{\text{N}\cdot\text{s}^2}\right)^{1/2} = 340$ m/s

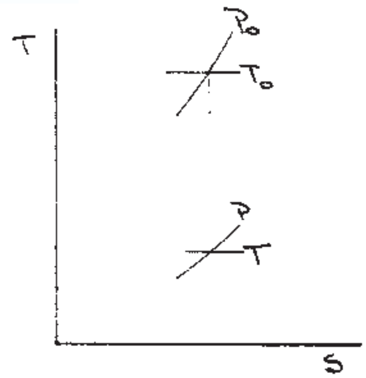
$$M = \frac{200}{340} = 0.588$$

$$P_o = P\left[1 + \frac{k-1}{2}M^2\right]^{k/k-1} = 101\ \text{kPa}\left[1 + 0.2(0.588)^2\right]^{3.5} = 127.6\ \text{kPa}$$ ← $P_{o\,com}$

Problem 12.65

[Difficulty: 1]

12.65 Consider flow of standard air at 600 m/s. What is the local isentropic stagnation pressure? The stagnation enthalpy? The stagnation temperature?



Solution:

Computing equations: $\dfrac{p_0}{p} = \left[1 + \dfrac{k-1}{2}M^2\right]^{k/k-1}$

$$\frac{T_0}{T} = 1 + \frac{k-1}{2}M^2$$

$$c = \sqrt{kRT}$$

Assumption: air behaves as an ideal gas, $k = 1.4$

$$c = (kRT)^{1/2} = \left(1.4 \times 287\,\frac{N\cdot m}{kg\cdot K} \times 288\,K \times \frac{kg\cdot m}{N\cdot s^2}\right)^{1/2} = 340\ m/s$$

$$M = \frac{V}{c} = \frac{600}{340} = 1.76$$

$$p_0 = p\left[1 + \frac{k-1}{2}M^2\right]^{k/k-1} = 101\ kPa\left[1 + 0.2(1.76)^2\right]^{3.5} = 546\ kPa$$ p_0

$$T_0 = T\left[1 + \frac{k-1}{2}M^2\right] = 288\ K\left[1 + 0.2(1.76)^2\right] = 466\ K$$ T_0

$dh = c_p\,dT$. For c_p = constant

$$h_0 - h = \int_h^{h_0} dh = \int_T^{T_0} c_p\,dT = c_p(T_0 - T) = 1000\,\frac{J}{kg\cdot K}(466 - 288)\,K$$

$$h_0 - h = 178\ kJ/kg$$ $h_0 - h$

Problem 12.66

[Difficulty: 1]

12.66 A DC-10 aircraft cruises at 12 km altitude on a standard day. A pitot-static tube on the nose of the aircraft measures stagnation and static pressures of 29.6 kPa and 19.4 kPa. Calculate (a) the flight Mach number of the aircraft, (b) the speed of the aircraft, and (c) the stagnation temperature that would be sensed by a probe on the aircraft.

Solution:

Computing equations: $\frac{p_0}{p} = \left(1 + \frac{k-1}{2}M^2\right)^{k/k-1}$ $\frac{T_0}{T} = 1 + \frac{k-1}{2}M^2$

Solving the first equation for M

$$M = \left\{\frac{2}{(k-1)}\left[\left(\frac{p_0}{p}\right)^{\frac{k-1}{k}} - 1\right]\right\} = \left\{\frac{2}{1.4-1}\left[\left(\frac{29.6}{19.4}\right)^{\frac{1.4-1}{1.4}} - 1\right]\right\}^{1/2} = 0.801$$ M

At $z = 12$ km, $T = 216.7$ K (Table A.3)

$$c = (kRT)^{1/2} = \left(1.4 \times 286.9 \frac{N \cdot m}{kg \cdot K} \times 216.7 K \times \frac{kg \cdot m}{N \cdot s^2}\right)^{1/2} = 295 \text{ m/s}$$

$$V = Mc = 0.801 \times 295 \text{ m/s} = 236 \text{ m/s}$$ V

$$T_0 = T\left(1 + \frac{k-1}{2}M^2\right) = 216.7K\left[1 + \frac{(1.4-1)}{2}(0.801)^2\right] = 245K$$ T_0

Problem 12.67

[Difficulty: 2]

12.67 An aircraft cruises at $M = 0.65$ at 10 km altitude on a standard day. The aircraft speed is deduced from measurement of the difference between the stagnation and static pressures. What is the value of this difference? Compute the air speed from this actual difference assuming (a) compressibility and (b) incompressibility. Is the discrepancy in airspeed computations significant in this case?

Given: Mach number of aircraft

Find: Pressure difference; air speed based on a) compressible b) incompressible assumptions

Solution:

The data provided, or available in the Appendices, is:

$$R = 287 \cdot \frac{J}{kg \cdot K} \qquad c_p = 1004 \cdot \frac{J}{kg \cdot K} \qquad k = 1.4 \qquad M = 0.65$$

From Table A.3, at 10 km altitude $\quad T = 223.3 \cdot K \quad p = 0.2615 \cdot 101 \cdot kPa \quad p = 26.4 \cdot kPa$

The governing equation for pressure change is:
$$\frac{p_0}{p} = \left(1 + \frac{k-1}{2} \cdot M^2\right)^{\frac{k}{k-1}} \qquad (12.20a)$$

Hence
$$p_0 = p \cdot \left(1 + \frac{k-1}{2} \cdot M^2\right)^{\frac{k}{k-1}} \qquad p_0 = 35.1 \cdot kPa$$

The pressure difference is $\quad p_0 - p = 8.67 \cdot kPa$

a) Assuming compressibility $\quad c = \sqrt{k \cdot R \cdot T} \quad c = 300 \frac{m}{s} \quad V = M \cdot c \quad V = 195 \frac{m}{s}$

b) Assuming incompressibility

Here the Bernoulli equation applies in the form
$$\frac{p}{\rho} + \frac{V^2}{2} = \frac{p_0}{\rho} \qquad \text{so} \qquad V = \sqrt{\frac{2 \cdot (p_0 - p)}{\rho}}$$

For the density $\quad \rho = \frac{p}{R \cdot T} \quad \rho = 0.412 \frac{kg}{m^3} \quad V = \sqrt{\frac{2 \cdot (p_0 - p)}{\rho}}$

Hence $\quad V = 205 \frac{m}{s}$

In this case the error at $M = 0.65$ in computing the speed of the aircraft using Bernoulli equation is $\frac{205 - 195}{195} = 5.13 \cdot \%$

Problem 12.68

[Difficulty: 1]

12.68 The Anglo-French Concorde supersonic transport cruised at $M = 2.2$ at 20 km altitude. Evaluate the speed of sound, aircraft flight speed, and Mach angle. What was the maximum air temperature at stagnation points on the aircraft structure?

Solution:

At $z = 20$ km, $T = 216.7\,K$ (Table A.3).

$$c = (kRT)^{1/2} = \left[1.4 \times 286.9 \frac{N \cdot m}{kg \cdot K} \times 216.7\,K \times \frac{kg \cdot m}{N \cdot s^2}\right]^{1/2} = 295\ m/s \qquad c$$

$$V = Mc = 2.2 \times 295\ m/s = 649\ m/s \qquad V$$

$$\alpha = \sin^{-1}\left(\frac{1}{M}\right) = \sin^{-1}\left(\frac{1}{2.2}\right) = 27.0^\circ \qquad \alpha$$

$$T_0 = T\left(1 + \frac{k-1}{2} M^2\right) = 216.7\,K\left[1 + \frac{(1.4-1)}{2}(2.2)^2\right] = 426\ K \qquad T_0$$

Problem 12.69

[Difficulty: 2]

12.69 Modern high-speed aircraft use "air data computers" to compute air speed from measurement of the difference between the stagnation and static pressures. Plot, as a function of actual Mach number M, for $M = 0.1$ to $M = 0.9$, the percentage error in computing the Mach number assuming incompressibility (i.e., using the Bernoulli equation), from this pressure difference. Plot the percentage error in speed, as a function of speed, of an aircraft cruising at 12 km altitude, for a range of speeds corresponding to the actual Mach number ranging from $M = 0.1$ to $M = 0.9$.

Given: Flight altitude of high-speed aircraft

Find: Mach number and aircraft speed errors assuming incompressible flow; plot

Solution:

The governing equation for pressure change is:

$$\frac{p_0}{p} = \left(1 + \frac{k-1}{2}\cdot M^2\right)^{\frac{k}{k-1}} \qquad (12.20a)$$

Hence

$$\Delta p = p_0 - p = p\cdot\left(\frac{p_0}{p} - 1\right) \qquad \Delta p = p\cdot\left[\left(1 + \frac{k-1}{2}\cdot M^2\right)^{\frac{k}{k-1}} - 1\right] \qquad (1)$$

For each Mach number the actual pressure change can be computed from Eq. 1

Assuming incompressibility, the Bernoulli equation applies in the form

$$\frac{p}{\rho} + \frac{V^2}{2} = \frac{p_0}{\rho} \quad \text{so} \quad V = \sqrt{\frac{2\cdot(p_0 - p)}{\rho}} = \sqrt{\frac{2\cdot\Delta p}{\rho}}$$

and the Mach number based on this is

$$M_{incomp} = \frac{V}{c} = \frac{\sqrt{\frac{2\cdot\Delta p}{\rho}}}{\sqrt{k\cdot R\cdot T}} = \sqrt{\frac{2\cdot\Delta p}{k\cdot\rho\cdot R\cdot T}}$$

Using Eq. 1

$$M_{incomp} = \sqrt{\frac{2}{k}\cdot\left[\left(1 + \frac{k-1}{2}\cdot M^2\right)^{\frac{k}{k-1}} - 1\right]}$$

The error in using Bernoulli to estimate the Mach number is

$$\frac{\Delta M}{M} = \frac{M_{incomp} - M}{M}$$

For errors in speed:

Actual speed:

$$V = M\cdot c \qquad V = M\cdot\sqrt{k\cdot R\cdot T}$$

Speed assuming incompressible flow:

$$V_{inc} = M_{incomp}\cdot\sqrt{k\cdot R\cdot T}$$

The error in using Bernoulli to estimate the speed from the pressure difference is

$$\frac{\Delta V}{V} = \frac{V_{incomp} - V}{V}$$

The computations and plots are shown below, generated using *Excel*:

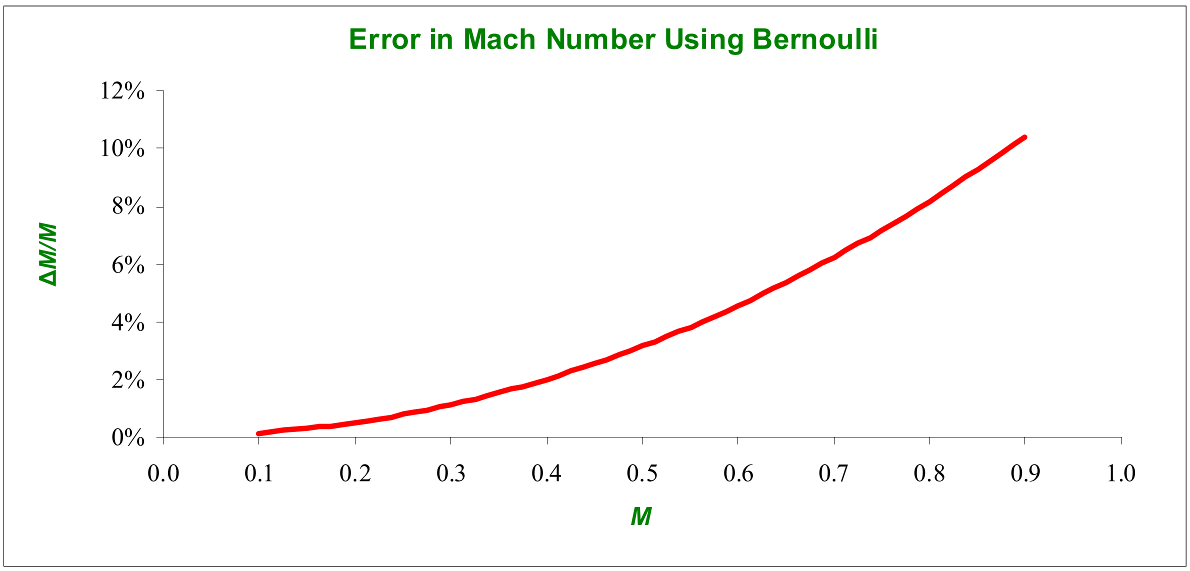
The given or available data is:

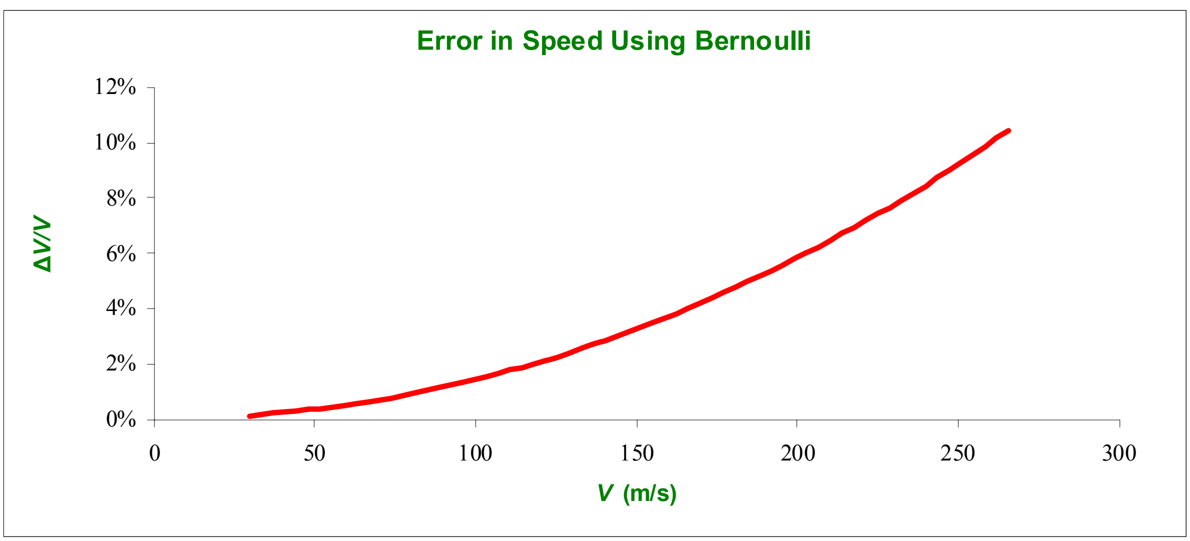
R = 286.9 J/kg.K
k = 1.4
T = 216.7 K (At 12 km, Table A.3)

Computed results:

c = 295 m/s

M	M_{incomp}	$\Delta M/M$	V (m/s)	V_{incomp} (m/s)	$\Delta V/V$
0.1	0.100	0.13%	29.5	29.5	0.13%
0.2	0.201	0.50%	59.0	59.3	0.50%
0.3	0.303	1.1%	88.5	89.5	1.1%
0.4	0.408	2.0%	118	120	2.0%
0.5	0.516	3.2%	148	152	3.2%
0.6	0.627	4.6%	177	185	4.6%
0.7	0.744	6.2%	207	219	6.2%
0.8	0.865	8.2%	236	255	8.2%
0.9	0.994	10.4%	266	293	10.4%





Problem 12.70

[Difficulty: 2]

12.70 A supersonic wind tunnel test section is designed to have $M = 2.5$ at 15°C and 35 kPa (abs). The fluid is air. Determine the required inlet stagnation conditions, T_0 and p_0. Calculate the required mass flow rate for a test section area of 0.175 m^2.

Given: Wind tunnel at M = 2.5

Find: Stagnation conditions; mass flow rate

Solution:

Basic equations: $c = \sqrt{k \cdot R \cdot T}$ $\quad M = \frac{V}{c}$ $\quad \frac{p_0}{p} = \left(1 + \frac{k-1}{2} \cdot M^2\right)^{\frac{k}{k-1}}$ $\quad \frac{T_0}{T} = 1 + \frac{k-1}{2} \cdot M^2$

Given or available data $\quad M = 2.5 \quad T = (15 + 273) \cdot K \quad p = 35 \cdot kPa \quad A = 0.175 \cdot m^2$

$k = 1.4 \quad R = 286.9 \cdot \frac{J}{kg \cdot K}$

Then $\quad T_0 = T \cdot \left(1 + \frac{k-1}{2} \cdot M^2\right) \quad T_0 = 648\,K \quad T_0 = 375 \cdot °C$

Also $\quad p_0 = p \cdot \left(1 + \frac{k-1}{2} \cdot M^2\right)^{\frac{k}{k-1}} \quad p_0 = 598 \cdot kPa$

The mass flow rate is given by $\quad m_{rate} = \rho \cdot A \cdot V$

We need $\quad c = \sqrt{k \cdot R \cdot T} \quad c = 340\,\frac{m}{s} \quad V = M \cdot c \quad V = 850\,\frac{m}{s}$

and also $\quad \rho = \frac{p}{R \cdot T} \quad \rho = 0.424\,\frac{kg}{m^3}$

Then $\quad m_{rate} = \rho \cdot A \cdot V \quad m_{rate} = 63.0\,\frac{kg}{s}$

Problem 12.71

[Difficulty: 2]

12.71 Air flows steadily through a length (① denotes inlet and ② denotes exit) of insulated constant-area duct. Properties change along the duct as a result of friction.

(a) Beginning with the control volume form of the first law of thermodynamics, show that the equation can be reduced to

$$h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2} = \text{constant}$$

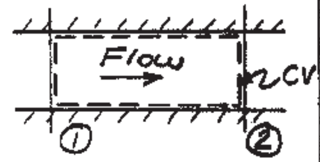
(b) Denoting the constant by h_0 (the stagnation enthalpy), show that for adiabatic flow of an ideal gas with friction

$$\frac{T_0}{T} = 1 + \frac{k-1}{2} M^2$$

(c) For this flow does $T_{0_1} = T_{0_2}$? $p_{0_1} = p_{0_2}$? Explain these results.

Solution: Apply the energy equation to the CV shown:

$$\text{BE} \quad \underbrace{\dot{Q}}_{=0(1)} + \underbrace{\dot{W}_s}_{=0(2)} + \underbrace{\dot{W}_{shear}}_{=0(3)} = \underbrace{\frac{\partial}{\partial t}\int_{CV} e\rho \, d\forall}_{=0(4)} + \int_{CS} (e + pv)\rho \vec{V} \cdot d\vec{A}$$



Assumptions: (1) $\dot{Q} = 0$ (adiabatic)
(2) $\dot{W}_s = 0$
(3) $\dot{W}_{shear} = 0$
(4) Steady flow
(5) Uniform flow at each section
(6) Neglect Δz

Then

$$0 = \left(u_1 + p_1 v_1 + \frac{V_1^2}{2}\right)\{-|\rho_1 V_1 A|\} + \left(u_2 + p_2 v_2 + \frac{V_2^2}{2}\right)\{|\rho_2 V_2 A|\}$$

But $h \equiv u + pv$, and $|\rho_1 V_1 A| = |\rho_2 V_2 A| = |\rho V A| = \dot{m}$, so

$$h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2} = h + \frac{V^2}{2} = h_0 = \text{Constant}$$ h_0

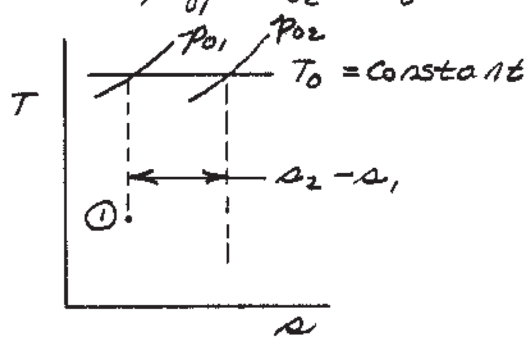
Assumption (7) Ideal gas; $h_0 - h = c_p(T_0 - T)$, $c_p = \frac{kR}{k-1}$, $c^2 = kRT$

Thus $$c_p T_0 = c_p T + \frac{V^2}{2}$$

$$\frac{T_0}{T} = 1 + \frac{V^2}{2 c_p T} = 1 + \frac{(k-1)V^2}{2kRT} = 1 + \frac{k-1}{2}\frac{V^2}{c^2} = 1 + \frac{k-1}{2}M^2$$ $\frac{T_0}{T}$

From the energy equation, $T_{0_1} = T_{0_2} = T_0 = \text{constant}$ T_0

The Ts diagram is



Since flow is frictional, $s_2 > s_1$. Therefore $p_{02} < p_{01}$ p_{02}

Problem 12.72

[Difficulty: 3]

12.72 A new design for a supersonic transport is tested in a wind tunnel at $M = 1.8$. Air is the working fluid. The stagnation temperature and pressure for the wind tunnel are 200 psia and 500°F, respectively. The model wing area is 100 in^2. The measured lift and drag are 12,000 lbf and 1600 lbf, respectively. Find the lift and drag coefficients.

Given: Wind tunnel test of supersonic transport

Find: Lift and drag coefficients

Solution:

Basic equations:

$$c = \sqrt{k \cdot R \cdot T} \qquad M = \frac{V}{c} \qquad \frac{p_0}{p} = \left(1 + \frac{k-1}{2} \cdot M^2\right)^{\frac{k}{k-1}} \qquad \frac{T_0}{T} = 1 + \frac{k-1}{2} \cdot M^2$$

$$C_L = \frac{F_L}{\frac{1}{2} \cdot \rho \cdot V^2 \cdot A} \qquad C_D = \frac{F_D}{\frac{1}{2} \cdot \rho \cdot V^2 \cdot A}$$

Given or available data

$$M = 1.8 \qquad T_0 = (500 + 460) \cdot R \qquad p_0 = 200 \cdot psi \qquad F_L = 12000 \cdot lbf \qquad F_D = 1600 \cdot lbf$$

$$A = 100 \cdot in^2 \qquad k = 1.4 \qquad R_{air} = 53.33 \cdot \frac{ft \cdot lbf}{lbm \cdot R}$$

We need local conditions

$$p = p_0 \cdot \left(1 + \frac{k-1}{2} \cdot M^2\right)^{-\frac{k}{k-1}} \qquad p = 34.8 \cdot psi$$

$$T = \frac{T_0}{1 + \frac{k-1}{2} \cdot M^2} \qquad T = 583 \cdot R \qquad T = 123 \cdot °F$$

Then

$$c = \sqrt{k \cdot R_{air} \cdot T} \qquad c = 1183 \cdot \frac{ft}{s} \qquad c = 807 \cdot mph$$

and

$$V = M \cdot c \qquad V = 2129 \cdot \frac{ft}{s} \qquad V = 1452 \cdot mph$$

We also need

$$\rho = \frac{p}{R_{air} \cdot T} \qquad \rho = 0.00501 \cdot \frac{slug}{ft^3}$$

Finally

$$C_L = \frac{F_L}{\frac{1}{2} \cdot \rho \cdot V^2 \cdot A} \qquad C_L = 1.52$$

$$C_D = \frac{F_D}{\frac{1}{2} \cdot \rho \cdot V^2 \cdot A} \qquad C_D = 0.203$$

Problem 12.73

[Difficulty: 2]

12.73 For aircraft flying at supersonic speeds, lift and drag coefficients are functions of Mach number only. A supersonic transport with wingspan of 75 m is to fly at 780 m/s at 20 km altitude on a standard day. Performance of the aircraft is to be measured from tests of a model with 0.9 m wingspan in a supersonic wind tunnel. The wind tunnel is to be supplied from a large reservoir of compressed air, which can be heated if desired. The static temperature of air in the test section is to be 10°C to avoid freezing of moisture. At what air speed should the wind tunnel tests be run to duplicate the Mach number of the prototype? What must be the stagnation temperature in the reservoir? What pressure is required in the reservoir if the test section pressure is to be 10 kPa (abs)?

Solution:

At $z = 20$ km, $T = 216.7$ K (Table A.3).

$$c = (kRT)^{1/2} = \left(1.4 \times 286.9 \frac{N \cdot m}{kg \cdot K} \times 216.7 K \times \frac{kg \cdot m}{N \cdot s^2}\right)^{1/2} = 295 \text{ m/s}$$

Thus for aircraft, $M = \frac{V}{c} = \frac{780}{295} = 2.64$

This Mach number must be duplicated in the model test. In the tunnel,

$$c = (kRT)^{1/2} = \left(1.4 \times 287 \frac{N \cdot m}{kg \cdot K} \times 283 K \times \frac{kg \cdot m}{N \cdot s^2}\right)^{1/2} = 337 \text{ m/s}$$

$$\therefore V = Mc = 2.64 (337 \text{ m/s}) = 890 \text{ m/s} \qquad V$$

$$\frac{T_0}{T} = 1 + \frac{k-1}{2} M^2 \qquad \therefore T_0 = T\left(1 + \frac{k-1}{2} M^2\right)$$

$$T_0 = 283 K\left[1 + \frac{(1.4-1)}{2} (2.64)^2\right] = 677 K \qquad T_0$$

$$\frac{p_0}{p} = \left[1 + \frac{k-1}{2} M^2\right]^{k/k-1} \qquad \therefore p_0 = p\left[1 + \frac{k-1}{2} M^2\right]^{k/k-1}$$

$$p_0 = 10 \text{ kPa}\left[1 + 0.2(2.64)^2\right]^{3.5} = 212 \text{ kPa} \qquad p_0$$

Problem 12.74

[Difficulty: 2]

12.74 Actual performance characteristics of the Lockheed SR-71 "Blackbird" reconnaissance aircraft were classified. However, it was thought to cruise at $M = 3.3$ at 26 km altitude. Calculate the aircraft flight speed for these conditions. Determine the local isentropic stagnation pressure. Because the aircraft speed is supersonic, a normal shock occurs in front of a total-head tube. The stagnation pressure decreases by 74.7 percent across the shock. Evaluate the stagnation pressure sensed by a probe on the aircraft. What is the maximum air temperature at stagnation points on the aircraft structure?

Solution:

At altitude $z = 26$ km, $T = 222.5$ K, $p/p_{SL} = 0.0216$ (Table A.3).

$$c = (kRT)^{1/2} = \left[1.4 \times 286.9 \frac{N\cdot m}{kg\cdot K} \times 222.5\,K \times \frac{kg\cdot m}{N\cdot s^2}\right]^{1/2} = 299 \text{ m/s}$$

$$V = Mc = 3.3 \times 299 \text{ m/s} = 987 \text{ m/s} \quad \leftarrow V$$

The stagnation pressure ahead of the shock (designated p_{0_1}) is given by

$$p_{0_1} = p\left[1 + \frac{k-1}{2}M^2\right]^{k/k-1} = 0.0216\, p_{SL}\left[1 + \frac{k-1}{2}M^2\right]^{k/k-1}$$

$$p_{0_1} = 0.0216 \times 101.3\text{ kPa}\left[1 + \frac{(1.4-1)}{2}(3.3)^2\right]^{\frac{1.4}{0.4}} = 125 \text{ kPa} \quad \leftarrow p_{0_1}$$

Designating the stagnation pressure behind the shock as p_{0_2}, then

$$\frac{p_{0_1} - p_{0_2}}{p_{0_1}} = 0.747 \quad \text{or} \quad p_{0_2} = p_{0_1} - 0.747\,p_{0_1} = 0.253\,p_{0_1}$$

$$p_{0_2} = 0.253 \times 125\text{ kPa} = 31.6 \text{ kPa} \quad \leftarrow p_{0_2}$$

The stagnation temperature does not change across a shock.

$$T_0 = T\left[1 + \frac{k-1}{2}M^2\right] = 222.5\,K\left[1 + \frac{(1.4-1)}{2}(3.3)^2\right] = 707\,K \quad \leftarrow T_0$$

Problem 12.75

[Difficulty: 2]

12.75 The NASA X-43A Hyper-X experimental vehicle traveled at $M = 9.68$ at an altitude of 110,000 ft. Calculate the flight speed for these conditions. Determine the local stagnation pressure. Because the aircraft speed is supersonic, a normal shock wave occurs in front of a total-head tube. However, the shock wave results in a stagnation pressure decrease of 99.6 percent. Evaluate the stagnation pressure sensed by a probe on the aircraft. What is the maximum air temperature at stagnation points on the aircraft structure?

Given: Data on air flow in a duct

Find: Stagnation pressures and temperatures; explain velocity increase; isentropic or not?

Solution:

The data provided, or available in the Appendices, is:

$$R_{air} = 287 \cdot \frac{J}{kg \cdot K} \qquad c_p = 1004 \cdot \frac{J}{kg \cdot K} \qquad k = 1.4 \qquad M = 9.68 \qquad p_{SL} = 101.3 \cdot kPa \qquad \rho_{SL} = 1.225 \cdot \frac{kg}{m^3}$$

At altitude: $z = 110000 \cdot ft$ $\quad z = 33528\,m$ $\quad T_1 = 226.5 \cdot K + (250.4 \cdot K - 226.5 \cdot K) \cdot \frac{33528 - 30000}{40000 - 30000}$ $\quad T_1 = 234.9\,K$

$$p_1 = p_{SL} \cdot \left[0.01181 + (0.002834 - 0.01181) \cdot \frac{33528 - 30000}{40000 - 30000}\right] \qquad p_1 = 0.8756 \cdot kPa$$

The sound speed is: $c = \sqrt{k \cdot R_{air} \cdot T_1} = 307.239 \frac{m}{s}$ so the flight speed is: $V = M \cdot c = 2974 \frac{m}{s}$ $\qquad V = 9757 \cdot \frac{ft}{s}$

So the stagnation temperature and pressure are: $T_{01} = T_1 \cdot \left(1 + \frac{k-1}{2} \cdot M^2\right)$ $\qquad T_{01} = 4638\,K \qquad T_{01} = 8348 \cdot R$

$$p_{01} = p_1 \cdot \left(1 + \frac{k-1}{2} \cdot M^2\right)^{\frac{k}{k-1}} \qquad p_{01} = 29.93 \cdot MPa$$

As the air passes through the shock wave, stagnation pressure decreases: $p_{02} = p_{01} \cdot (1 - 0.996)$

Therefore, the total head probe sees a pressure of $p_{02} = 119.7 \cdot kPa$

Since there is no heat transfer through the shock wave, the stagnation temperature remains the same: $T_{02} = T_{01}$

$$T_{02} = 8348 \cdot R$$

Problem 12.76

[Difficulty: 2]

12.76 Air flows in an insulated duct. At point ① the conditions are $M_1 = 0.1$, $T_1 = 20°C$, and $p_1 = 1.0$ MPa (abs). Downstream, at point ②, because of friction the conditions are $M_2 = 0.7$, $T_2 = -5.62°C$, and $p_2 = 136.5$ kPa (abs). (Four significant figures are given to minimize roundoff errors.) Compare the stagnation temperatures at points ① and ②, and explain the result. Compute the stagnation pressures at points ① and ②. Can you explain how it can be that the velocity *increases* for this frictional flow? Should this process be isentropic or not? Justify your answer by computing the change in entropy between points ① and ②. Plot static and stagnation state points on a *Ts* diagram.

Given: Data on air flow in a duct

Find: Stagnation pressures and temperatures; explain velocity increase; isentropic or not?

Solution:

The data provided, or available in the Appendices, is:

$$R = 287 \cdot \frac{J}{kg \cdot K} \qquad c_p = 1004 \cdot \frac{J}{kg \cdot K} \qquad k = 1.4$$

$$M_1 = 0.1 \qquad T_1 = (20 + 273) \cdot K \qquad p_1 = 1000 \cdot kPa \qquad M_2 = 0.7 \qquad T_2 = (-5.62 + 273) \cdot K \qquad p_2 = 136.5 \cdot kPa$$

For stagnation temperatures: $T_{01} = T_1 \cdot \left(1 + \frac{k-1}{2} \cdot M_1^{\ 2}\right)$ $\qquad T_{01} = 293.6\,K \qquad T_{01} = 20.6 \cdot C$

$T_{02} = T_2 \cdot \left(1 + \frac{k-1}{2} \cdot M_2^{\ 2}\right)$ $\qquad T_{02} = 293.6\,K \qquad T_{02} = 20.6 \cdot C$

(Because the stagnation temperature is constant, the process is adiabatic)

For stagnation pressures: $p_{01} = p_1 \cdot \left(1 + \frac{k-1}{2} \cdot M_1^{\ 2}\right)^{\frac{k}{k-1}}$ $\qquad p_{01} = 1.01 \cdot MPa$

$p_{02} = p_2 \cdot \left(1 + \frac{k-1}{2} \cdot M_2^{\ 2}\right)^{\frac{k}{k-1}}$ $\qquad p_{02} = 189 \cdot kPa$

The entropy change is: $\Delta s = c_p \cdot \ln\left(\frac{T_2}{T_1}\right) - R \cdot \ln\left(\frac{p_2}{p_1}\right)$ $\qquad \Delta s = 480 \cdot \frac{J}{kg \cdot K}$

Note that $V_1 = M_1 \cdot \sqrt{k \cdot R \cdot T_1}$ $\qquad V_1 = 34.3 \frac{m}{s} \qquad V_2 = M_2 \cdot \sqrt{k \cdot R \cdot T_2} \qquad V_2 = 229 \frac{m}{s}$

Although there is friction, suggesting the flow should decelerate, because the static pressure drops so much, the net effect is flow acceleration!

The entropy increases because the process is adiabatic but irreversible (friction).

From the second law of thermodynamics $ds \geq \frac{\delta q}{T}$: becomes $ds > 0$

Problem 12.77

[Difficulty: 2]

12.77 Air is cooled as it flows without friction at a rate of 0.05 kg/s in a duct. At point ① the conditions are $M_1 = 0.5$, $T_1 = 500°C$, and $p_1 = 500$ kPa (abs). Downstream, at point ②, the conditions are $M_2 = 0.2$, $T_2 = -18.57°C$, and $p_2 = 639.2$ kPa (abs). (Four significant figures are given to minimize roundoff errors.) Compare the stagnation temperatures at points ① and ②, and explain the result. Compute the rate of cooling. Compute the stagnation pressures at points ① and ②. Should this process be isentropic or not? Justify your answer by computing the change in entropy between points ① and ②. Plot static and stagnation state points on a *Ts* diagram.

Given: Data on air flow in a duct

Find: Stagnation temperatures; explain; rate of cooling; stagnation pressures; entropy change

Solution:

The data provided, or available in the Appendices, is: $R = 287 \cdot \frac{J}{kg \cdot K}$ $\quad c_p = 1004 \cdot \frac{J}{kg \cdot K}$ $\quad k = 1.4$

$T_1 = (500 + 273) \cdot K$ $\quad p_1 = 500 \cdot kPa$ $\quad T_2 = (-18.57 + 273) \cdot K$ $\quad p_2 = 639.2 \cdot kPa$

$M_1 = 0.5$ $\quad M_2 = 0.2$ $\quad M_{rate} = 0.05 \cdot \frac{kg}{s}$

For stagnation temperatures:
$$T_{01} = T_1 \cdot \left(1 + \frac{k-1}{2} \cdot M_1^2\right) \qquad T_{01} = 811.7\,K \qquad T_{01} = 539 \cdot C$$

$$T_{02} = T_2 \cdot \left(1 + \frac{k-1}{2} \cdot M_2^2\right) \qquad T_{02} = 256.5\,K \qquad T_{02} = -16.5 \cdot C$$

The fact that the stagnation temperature (a measure of total energy) decreases suggests cooling is taking place.

For the heat transfer:
$$Q = M_{rate} \cdot c_p \cdot (T_{02} - T_{01}) \qquad Q = -27.9 \cdot kW$$

For stagnation pressures:
$$p_{01} = p_1 \cdot \left(1 + \frac{k-1}{2} \cdot M_1^2\right)^{\frac{k}{k-1}} \qquad p_{01} = 593 \cdot kPa$$

$$p_{02} = p_2 \cdot \left(1 + \frac{k-1}{2} \cdot M_2^2\right)^{\frac{k}{k-1}} \qquad p_{02} = 657 \cdot kPa$$

The entropy change is:
$$\Delta s = c_p \cdot \ln\left(\frac{T_2}{T_1}\right) - R \cdot \ln\left(\frac{p_2}{p_1}\right) \qquad \Delta s = -1186 \cdot \frac{J}{kg \cdot K}$$

The entropy decreases because the process is a cooling process (*Q* is negative).

From the second law of thermodynamics: $ds \geq \frac{\delta q}{T}$ becomes $ds \geq -ve$

Hence, if the process is reversible, the entropy must decrease; if it is irreversible, it may increase or decrease

Problem 12.78

[Difficulty: 3]

12.78 Consider steady, adiabatic flow of air through a long straight pipe with $A = 0.05\ \text{m}^2$. At the inlet (section ①) the air is at 200 kPa (abs), 60°C, and 146 m/s. Downstream at section ②, the air is at 95.6 kPa (abs) and 280 m/s. Determine p_{0_1}, p_{0_2}, T_{0_1}, T_{0_2}, and the entropy change for the flow. Show static and stagnation state points on a Ts diagram.

Solution:

Computing equations: $\dfrac{T_0}{T} = 1 + \dfrac{k-1}{2}M^2 \qquad \dfrac{p_0}{p} = \left[1 + \dfrac{k-1}{2}M^2\right]^{k/k-1}$

Basic equations: $0 = \cancelto{0(4)}{\dfrac{\partial}{\partial t}\int_{CV} \rho\, d\forall} + \int_{CS} \rho \vec{V} \cdot d\vec{A}$

$\cancelto{0(1)}{\dot{Q}} + \cancelto{0(2)}{\dot{W}_s} + \cancelto{0(3)}{\dot{W}_{shear}} = \cancelto{0(4)}{\dfrac{\partial}{\partial t}\int_{CV} e\rho\, d\forall} + \int_{CS}\left(e + \dfrac{p}{\rho}\right)\rho\vec{V}\cdot d\vec{A}$

Assumptions:
(1) $\dot{Q} = 0$ (adiabatic flow)
(2) $\dot{W}_s = 0$
(3) $\dot{W}_{shear} = 0$
(4) steady flow
(5) uniform flow at each section
(6) $\Delta z = 0$
(7) ideal gas, $k = 1.4$
(8) $A_1 = A_2 = A = \text{constant}$

$M_1 = \dfrac{V_1}{c_1} \qquad c_1 = (kRT_1)^{1/2} = \left(1.4 \times 287\ \dfrac{N\cdot m}{kg\cdot K} \times 333 K \times \dfrac{kg\cdot m}{N\cdot s^2}\right)^{1/2} = 366\ m/s$

$M_1 = \dfrac{V_1}{c_1} = \dfrac{146}{366} = 0.399 \qquad T_{0_1} = T_1\left[1 + \dfrac{k-1}{2}M_1^2\right] = 333K\left[1 + 0.2(0.399)^2\right] = 344 K \longleftarrow T_{0_1}$

$p_{0_1} = p_1\left[1 + \dfrac{k-1}{2}M_1^2\right]^{k/k-1} = 200\ kPa\left[1 + 0.2(0.399)^2\right]^{3.5} = 223\ kPa \longleftarrow p_{0_1}$

From the energy equation, $0 = \left(u_1 + p_1 v_1 + \dfrac{V_1^2}{2}\right)\{-|\rho_1 V_1 A|\} + \left(u_2 + p_2 v_2 + \dfrac{V_2^2}{2}\right)\{|\rho_2 V_2 A|\}$

From continuity, $0 = -|\rho_1 V_1 A| + |\rho_2 V_2 A|$ or $\rho_1 V_1 = \rho_2 V_2$

Then, using $h = u + pv$,

$h_1 + \dfrac{V_1^2}{2} = h_{0_1} = h_2 + \dfrac{V_2^2}{2} = h_{0_2}$ or $h_{0_1} = h_{0_2}$

For an ideal gas with constant specific heats, $T_{0_2} = T_{0_1} = 344\ K \longleftarrow T_{0_2}$

From continuity, $\rho_2 = \rho_1 \dfrac{V_1}{V_2} = \dfrac{p_1}{RT_1}\dfrac{V_1}{V_2} = 200\times10^3\ \dfrac{N}{m^2} \times \dfrac{kg\cdot K}{287\ N\cdot m} \times \dfrac{1}{333 K} \times \dfrac{146}{280} = 1.09\ kg/m^3$

Then,

$T_2 = \dfrac{p_2}{\rho_2 R} = 95.6\times10^3\ \dfrac{N}{m^2} \times \dfrac{m^3}{1.09\ kg} \times \dfrac{kg\cdot K}{287\ N\cdot m} = 306 K \quad (33°C)$

$c_2 = (kRT_2)^{1/2} = \left(1.4 \times 287\ \dfrac{N\cdot m}{kg\cdot K} \times 306 K \times \dfrac{kg\cdot m}{N\cdot s^2}\right)^{1/2} = 351\ m/s$

$M_2 = \dfrac{V_2}{c_2} = \dfrac{280}{351} = 0.798$

$p_{0_2} = p_2\left[1 + \dfrac{k-1}{2}M_2^2\right]^{k/k-1} = 95.6\ kPa\left[1 + 0.2(0.798)^2\right]^{3.5}$

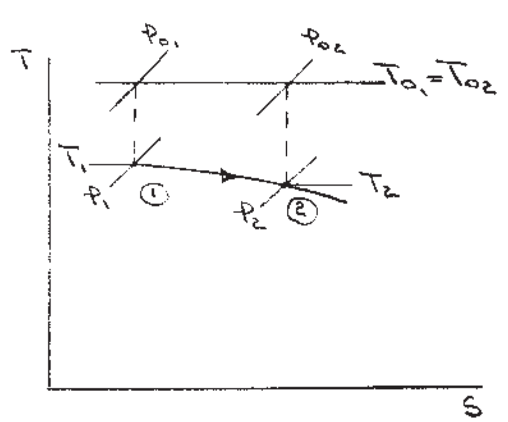
$p_{0_2} = 145\ kPa \longleftarrow p_{0_2}$

$T\,ds = dh - v\,dp = c_p\,dT - \dfrac{RT}{p}\,dp$

$ds = c_p\dfrac{dT}{T} - R\dfrac{dp}{p} = ds_0 = -R\dfrac{dp_0}{p_0}$

$s_{0_2} - s_{0_1} = s_2 - s_1 = -R\ln\dfrac{p_{0_2}}{p_{0_1}}$

$s_2 - s_1 = -287\ \dfrac{J}{kg\cdot K}\ln\dfrac{145}{223} = 0.124\ \dfrac{kJ}{kg\cdot K} \longleftarrow s_2 - s_1$



Problem 12.79

[Difficulty: 3]

12.79 Air flows steadily through a constant-area duct. At section ①, the air is at 400 kPa (abs), 325 K, and 150 m/s. As a result of heat transfer and friction, the air at section ② downstream is at 275 kPa (abs), 450 K. Calculate the heat transfer per kilogram of air between sections ① and ②, and the stagnation pressure at section ②.

Given: Air flow in duct with heat transfer and friction

Find: Heat transfer; Stagnation pressure at location 2

Solution:

Basic equations: $c = \sqrt{k \cdot R \cdot T}$ $\quad M = \frac{V}{c}$ $\quad \frac{p_0}{p} = \left(1 + \frac{k-1}{2} \cdot M^2\right)^{\frac{k}{k-1}}$

$\rho \cdot V \cdot A = \text{const}$ $\quad h_1 + \frac{V_1^2}{2} + \frac{\delta Q}{dm} = h_2 + \frac{V_2^2}{2}$

Given or available data $\quad p_1 = 400 \cdot \text{kPa}$ $\quad T_1 = 325 \cdot \text{K}$ $\quad V_1 = 150 \cdot \frac{m}{s}$

$p_2 = 275 \cdot \text{kPa}$ $\quad T_2 = 450 \cdot \text{K}$

$c_p = 1004 \cdot \frac{J}{kg \cdot K}$ $\quad k = 1.4$ $\quad R = 286.9 \cdot \frac{J}{kg \cdot K}$

Then $\quad \rho_1 = \frac{p_1}{R \cdot T_1}$ $\quad \rho_1 = 4.29 \frac{kg}{m^3}$ $\quad \rho_2 = \frac{p_2}{R \cdot T_2}$ $\quad \rho_2 = 2.13 \frac{kg}{m^3}$

and from $\quad \rho \cdot V \cdot A = \text{const}$ $\quad V_2 = V_1 \cdot \frac{\rho_1}{\rho_2}$ $\quad V_2 = 302 \frac{m}{s}$

Also $\quad \frac{\delta Q}{dm} = q = h_2 - h_1 + \frac{V_2^2 - V_1^2}{2}$

$q = c_p \cdot \left(T_2 - T_1\right) + \frac{V_2^2 - V_1^2}{2}$ $\quad q = 160 \cdot \frac{kJ}{kg}$

We also have $\quad c_2 = \sqrt{k \cdot R \cdot T_2}$ $\quad c_2 = 425 \frac{m}{s}$ $\quad$ so $\quad M_2 = \frac{V_2}{c_2}$ $\quad M_2 = 0.711$

Hence $\quad p_{02} = p_2 \cdot \left(1 + \frac{k-1}{2} \cdot M_2^2\right)^{\frac{k}{k-1}}$ $\quad p_{02} = 385 \cdot \text{kPa}$

Problem 12.80

[Difficulty: 2]

12.80 The combustion process in a ramjet engine is modeled as simple heat addition to air in a frictionless duct. Consider such a combustor, with air flowing at a rate of 0.1 lbm/s. At point ① the conditions are M_1 0.2, $T_1 = 600°F$, and $p_1 = 7$ psia. Downstream, at point ②, the conditions are $M_2 = 0.9$, $T_2 = 1890°F$, and $p_2 = 4.1$ psia. Compare the stagnation temperatures at points ① and ②, and explain the result. Compute the rate of heat addition to the flow. Compute the stagnation pressures at points ① and ②. Should this process be isentropic or not? Justify your answer by computing the change in entropy between points ① and ②. Plot static and stagnation state points on a *Ts* diagram.

Given: Data on air flow in a ramjet combustor

Find: Stagnation pressures and temperatures; isentropic or not?

Solution:

The data provided, or available in the Appendices, is:

$$R_{air} = 53.33\cdot\frac{ft\cdot lbf}{lbm\cdot R} \qquad c_p = 0.2399\cdot\frac{BTU}{lbm\cdot R} \qquad k = 1.4 \qquad M_{rate} = 0.1\cdot\frac{lbm}{s}$$

$$M_1 = 0.2 \qquad T_1 = (600 + 460)\cdot R \qquad p_1 = 7\cdot psi \qquad M_2 = 0.9 \qquad T_2 = (1890 + 460)\cdot R \qquad p_2 = 4.1\cdot psi$$

For stagnation temperatures:

$$T_{01} = T_1\cdot\left(1 + \frac{k-1}{2}\cdot M_1^2\right) \qquad T_{01} = 1068.5\cdot R \qquad T_{01} = 608.8\cdot°F$$

$$T_{02} = T_2\cdot\left(1 + \frac{k-1}{2}\cdot M_2^2\right) \qquad T_{02} = 2730.7\cdot R \qquad T_{02} = 2271\cdot°F$$

Since we are modeling heat addition, the stagnation temperature should increase.

The rate of heat addition is:

$$Q = M_{rate}\cdot c_p\cdot\left(T_{02} - T_{01}\right) \qquad Q = 39.9\cdot\frac{BTU}{s}$$

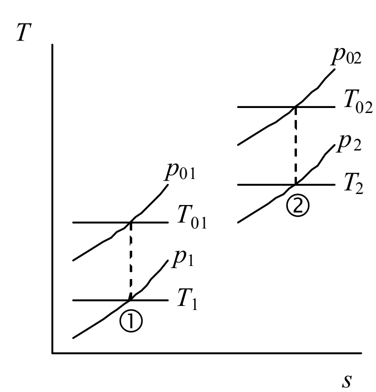
For stagnation pressures:

$$p_{01} = p_1\cdot\left(1 + \frac{k-1}{2}\cdot M_1^2\right)^{\frac{k}{k-1}} = 7.20\,psi \qquad p_{02} = p_2\cdot\left(1 + \frac{k-1}{2}\cdot M_2^2\right)^{\frac{k}{k-1}} = 6.93\,psi$$

The entropy change is:

$$\Delta s = c_p\cdot\ln\left(\frac{T_2}{T_1}\right) - R_{air}\cdot\ln\left(\frac{p_2}{p_1}\right) = 0.228\,\frac{BTU}{lbm\cdot R}$$

The entropy increases because heat is being added. Here is a *Ts* diagram of the process:



Problem 12.81

[Difficulty: 2]

12.81 Let us revisit the ramjet combustor in Problem 12.80. To more accurately model the flow, we now include the effects of friction in the duct. Once the effects of friction have been included, we find that the conditions at state (2) are now $M_2 = 0.9$, $T_2 = 1660°F$, and $p_2 = 1.6$ psia. Recalculate the heat transfer per pound of air between sections (1) and (2), and the stagnation pressure at section (2).

Given: Data on air flow in a ramjet combustor

Find: Stagnation pressures and temperatures; isentropic or not?

Solution:

The data provided, or available in the Appendices, is:

$$R_{air} = 53.33 \cdot \frac{ft \cdot lbf}{lbm \cdot R} \qquad c_p = 0.2399 \cdot \frac{BTU}{lbm \cdot R} \qquad k = 1.4 \qquad M_{rate} = 0.1 \cdot \frac{lbm}{s}$$

$$M_1 = 0.2 \qquad T_1 = (600 + 460) \cdot R \qquad p_1 = 7 \cdot psi \qquad M_2 = 0.9 \qquad T_2 = (1660 + 460) \cdot R \qquad p_2 = 1.6 \cdot psi$$

For stagnation temperatures:
$$T_{01} = T_1 \cdot \left(1 + \frac{k-1}{2} \cdot M_1^{\,2}\right) \qquad T_{01} = 1068.5 \cdot R \qquad T_{01} = 608.8 \cdot °F$$

$$T_{02} = T_2 \cdot \left(1 + \frac{k-1}{2} \cdot M_2^{\,2}\right) \qquad T_{02} = 2463.4 \cdot R \qquad T_{02} = 2003.8 \cdot °F$$

The rate of heat addition is:
$$Q = M_{rate} \cdot c_p \cdot (T_{02} - T_{01}) \qquad Q = 33.5 \cdot \frac{BTU}{s}$$

For stagnation pressures:
$$p_{01} = p_1 \cdot \left(1 + \frac{k-1}{2} \cdot M_1^{\,2}\right)^{\frac{k}{k-1}} \qquad p_{01} = 7.20 \cdot psi$$

$$p_{02} = p_2 \cdot \left(1 + \frac{k-1}{2} \cdot M_2^{\,2}\right)^{\frac{k}{k-1}} \qquad p_{02} = 2.71 \cdot psi$$

The entropy change is:
$$\Delta s = c_p \cdot \ln\left(\frac{T_2}{T_1}\right) - R_{air} \cdot \ln\left(\frac{p_2}{p_1}\right) \qquad \Delta s = 0.267 \cdot \frac{BTU}{lbm \cdot R}$$

The friction has increased the entropy increase across the duct, even though the heat addition has decreased.

Problem 12.82

[Difficulty: 2]

12.82 Air passes through a normal shock in a supersonic wind tunnel. Upstream conditions are $M_1 = 1.8$, $T_1 = 270$ K, and $p_1 = 10.0$ kPa (abs). Downstream conditions are $M_2 = 0.6165$, $T_2 = 413.6$ K, and $p_2 = 36.13$ kPa (abs). (Four significant figures are given to minimize roundoff errors.) Evaluate local isentropic stagnation conditions (a) upstream from, and (b) downstream from, the normal shock. Calculate the change in specific entropy across the shock. Plot static and stagnation state points on a Ts diagram.

Solution:

Computing equations: $\frac{T_0}{T} = 1 + \frac{k-1}{2}M^2$ $\qquad \frac{p_0}{p} = \left[1 + \frac{k-1}{2}M^2\right]^{k/k-1}$

$T_{0_1} = T_1\left[1 + \frac{k-1}{2}M_1^2\right] = 270\,\text{K}\left[1 + 0.2(1.8)^2\right] = 445\ \text{K}$ ——— T_{0_1}

$p_{0_1} = p_1\left[1 + \frac{k-1}{2}M_1^2\right]^{k/k-1} = 10.0\ \text{kPa}\left[1+0.2(1.8)^2\right]^{3.5} = 57.5\ \text{kPa (abs)}$ ——— p_{0_1}

$T_{0_2} = T_2\left[1 + \frac{k-1}{2}M_2^2\right] = 413.6\,\text{K}\left[1+0.2(0.6165)^2\right] = 445.\ \text{K}$ ——— T_{0_2}

(Flow through the shock is adiabatic, $T_{0_2} = T_{0_1}$)

$p_{0_2} = p_2\left[1 + \frac{k-1}{2}M_2^2\right]^{k/k-1} = 36.13\ \text{kPa}\left[1+0.2(0.6165)^2\right]^{3.5} = 46.7\ \text{kPa (abs)}$ ——— p_{0_2}

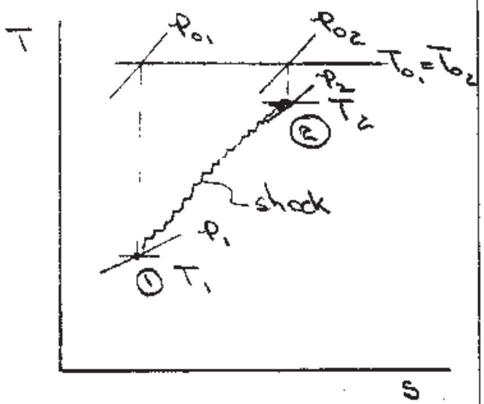
$T\,ds = dh - v\,dp = c_p\,dT - RT\frac{dp}{p}$

$ds = c_p\frac{dT}{T} - R\frac{dp}{p}$

$s_2 - s_1 = s_{0_2} - s_{0_1} = -R\ln\frac{p_{0_2}}{p_{0_1}}$

$= -287\,\frac{\text{J}}{\text{kg}\cdot\text{K}}\ln\frac{46.69}{57.46}$

$s_2 - s_1 = 59.6\ \text{J/kg}\cdot\text{K}$ ——— $s_2 - s_1$



Problem 12.83

[Difficulty: 2]

12.83 Air enters a turbine at $M_1 = 0.4$, $T_1 = 1250°C$, and $p_1 = 625$ kPa (abs). Conditions leaving the turbine are $M_2 = 0.8$, $T_2 = 650°C$, and $p_2 = 20$ kPa (abs). Evaluate local isentropic stagnation conditions (a) at the turbine inlet and (b) at the turbine outlet. Calculate the change in specific entropy across the turbine. Plot static and stagnation state points on a Ts diagram.

Given: Air flow through turbine

Find: Stagnation conditions at inlet and exit; change in specific entropy; Plot on Ts diagram

Solution:

Basic equations:

$$\frac{p_0}{p} = \left(1 + \frac{k-1}{2} \cdot M^2\right)^{\frac{k}{k-1}} \qquad \frac{T_0}{T} = 1 + \frac{k-1}{2} \cdot M^2 \qquad \Delta s = c_p \cdot \ln\left(\frac{T_2}{T_1}\right) - R \cdot \ln\left(\frac{p_2}{p_1}\right)$$

Given or available data

$$M_1 = 0.4 \qquad p_1 = 625 \cdot kPa \qquad T_1 = (1250 + 273) \cdot K$$

$$M_2 = 0.8 \qquad p_2 = 20 \cdot kPa \qquad T_2 = (650 + 273) \cdot K$$

$$c_p = 1004 \frac{J}{kg \cdot K} \qquad k = 1.4 \qquad R = 286.9 \frac{J}{kg \cdot K}$$

Then

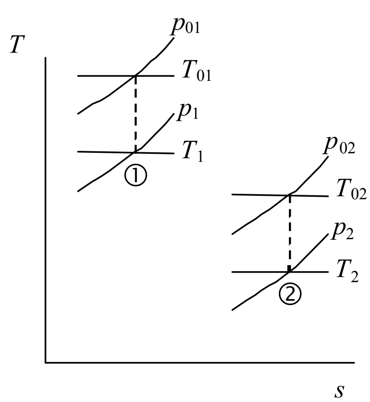
$$T_{01} = T_1 \cdot \left(1 + \frac{k-1}{2} \cdot M_1^2\right) \qquad T_{01} = 1572 K \qquad T_{01} = 1299 \cdot °C$$

$$p_{01} = p_1 \cdot \left(1 + \frac{k-1}{2} \cdot M_1^2\right)^{\frac{k}{k-1}} \qquad p_{01} = 698 \cdot kPa$$

$$T_{02} = T_2 \cdot \left(1 + \frac{k-1}{2} \cdot M_2^2\right) \qquad T_{02} = 1041 K \qquad T_{02} = 768 \cdot °C$$

$$p_{02} = p_2 \cdot \left(1 + \frac{k-1}{2} \cdot M_2^2\right)^{\frac{k}{k-1}} \qquad p_{02} = 30 \cdot kPa$$

$$\Delta s = c_p \cdot \ln\left(\frac{T_2}{T_1}\right) - R \cdot \ln\left(\frac{p_2}{p_1}\right) \qquad \Delta s = 485 \cdot \frac{J}{kg \cdot K}$$

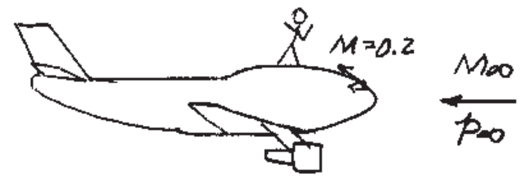


12.84 A Boeing 747 cruises at $M = 0.87$ at an altitude of 13 km on a standard day. A window in the cockpit is located where the external flow Mach number is 0.2 relative to the plane surface. The cabin is pressurized to an equivalent altitude of 2500 m in a standard atmosphere. Estimate the pressure difference across the window. Be sure to specify the direction of the net pressure force.

Solution: Apply isentropic stagnation relations.

Computing equation: $p_0 = p\left(1 + \frac{k-1}{2}M^2\right)^{k/k-1}$

Assumptions: (1) Ideal gas
(2) Isentropic flow



Consider observer on aircraft: air is decelerated isentropically from $M_\infty = 0.87$ to $M = 0.2$.

From Table A.3: Calculated: $p = \left(\frac{p}{p_0}\right) p_0$

$p_0 = 101.3$ kPa

Altitude (km)	p/p_0 (---)	p (kPa)
2.5	0.7372	74.7
13.0	0.1636	16.6

For isentropic stagnation:

$$p_0 = p_\infty\left(1 + \frac{k-1}{2}M_\infty^2\right)^{k/k-1} = 16.6\text{ kPa}\left(1 + \frac{1.4-1}{2}(0.87)^2\right)^{3.5} = 27.2\text{ kPa (abs)}$$

From stagnation to $M = 0.2$:

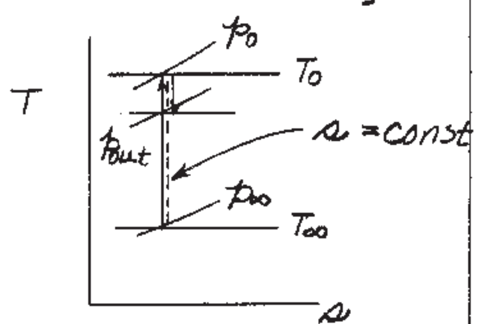
$$p_{out} = \frac{p_0}{\left(1 + \frac{k-1}{2}M^2\right)^{k/k-1}} = \frac{27.2\text{ kPa}}{\left(1 + 0.2(0.2)^2\right)^{3.5}} = 26.5\text{ kPa (abs)}$$

Pressure difference across window is:

$$\Delta p = p_{in} - p_{out} = (74.7 - 26.5)\text{ kPa} = 48.2\text{ kPa}$$ Δp

{Inside pressure is higher; window force is toward outside.}

The corresponding Ts diagram is:



Problem 12.85

[Difficulty: 2]

12.85 If a window of the cockpit in Problem 12.84 develops a tiny leak the air will start to rush out at critical speed. Find the mass flow rate if the leak area is 1 mm^2.

Given: Air flow leak in window of airplane

Find: Mass flow rate

Solution:

Basic equations:

$$m_{rate} = \rho \cdot V \cdot A \qquad V_{crit} = \sqrt{\frac{2 \cdot k}{k+1} \cdot R \cdot T_0} \qquad \frac{\rho_0}{\rho_{crit}} = \left(\frac{k+1}{2}\right)^{\frac{1}{k-1}}$$

The interior conditions are the stagnation conditions for the flow

Given or available data

$$T_0 = 271.9 \cdot K \qquad \rho_{SL} = 1.225 \cdot \frac{kg}{m^3} \qquad \rho_0 = 0.7812 \cdot \rho_{SL} \qquad \rho_0 = 0.957 \frac{kg}{m^3}$$

(Above data from Table A.3 at an altitude of 2500 m)

$$A = 1 \cdot mm^2 \qquad c_p = 1004 \cdot \frac{J}{kg \cdot K} \qquad k = 1.4 \qquad R = 286.9 \cdot \frac{J}{kg \cdot K}$$

Then

$$\rho_{crit} = \frac{\rho_0}{\left(\frac{k+1}{2}\right)^{\frac{1}{k-1}}} \qquad \rho_{crit} = 0.607 \frac{kg}{m^3} \qquad V_{crit} = \sqrt{\frac{2 \cdot k}{k+1} \cdot R \cdot T_0} \qquad V_{crit} = 302 \frac{m}{s}$$

The mass flow rate is

$$m_{rate} = \rho_{crit} \cdot V_{crit} \cdot A \qquad m_{rate} = 1.83 \times 10^{-4} \frac{kg}{s}$$

Problem 12.86

[Difficulty: 2]

12.86 Space debris impact is a real concern for spacecraft. If a piece of space debris were to create a hole of 0.001 in.^{2} area in the hull of the International Space Station (ISS), at what rate would air leak from the ISS? Assume that the atmosphere in the International Space Station (ISS) is air at a pressure of 14.7 psia and a temperature of 65°F.

Given: Air leak in ISS

Find: Mass flow rate

Solution:

Basic equations: $m_{rate} = \rho \cdot V \cdot A$ $\qquad V_{crit} = \sqrt{\frac{2 \cdot k}{k+1} \cdot R \cdot T_0}$ $\qquad \frac{\rho_0}{\rho_{crit}} = \left(\frac{k+1}{2}\right)^{\frac{1}{k-1}}$

The interior conditions are the stagnation conditions for the flow

Given or available data $\quad T_0 = (65 + 460) \cdot R \quad p_0 = 14.7 \cdot psi \quad R_{air} = 53.33 \cdot \frac{ft \cdot lbf}{lbm \cdot R} \quad k = 1.4 \quad A = 0.001 \cdot in^2$

The density of air inside the ISS would be: $\quad \rho_0 = \frac{p_0}{R_{air} \cdot T_0} \qquad \rho_0 = 2.35 \times 10^{-3} \cdot \frac{slug}{ft^3}$

Then $\quad \rho_{crit} = \frac{\rho_0}{\left(\frac{k+1}{2}\right)^{\frac{1}{k-1}}} \qquad \rho_{crit} = 1.49 \times 10^{-3} \cdot \frac{slug}{ft^3} \qquad V_{crit} = \sqrt{\frac{2 \cdot k}{k+1} \cdot R_{air} \cdot T_0} \qquad V_{crit} = 1025 \cdot \frac{ft}{s}$

The mass flow rate is $\quad m_{rate} = \rho_{crit} \cdot V_{crit} \cdot A \qquad m_{rate} = 1.061 \times 10^{-5} \cdot \frac{slug}{s} \qquad m_{rate} = 3.41 \times 10^{-4} \cdot \frac{lbm}{s}$

Problem 12.87

[Difficulty: 1]

12.87 A CO_2 cartridge is used to propel a toy rocket. Gas in the cartridge is pressurized to 45 MPa (gage) and is at 25°C. Calculate the critical conditions (temperature, pressure, and flow speed) that correspond to these stagnation conditions.

Solution:

Computing equations: $\frac{T_0}{T} = 1 + \frac{k-1}{2}M^2 \qquad \frac{p_0}{p} = \left(1 + \frac{k-1}{2}M^2\right)^{k/k-1}$

For CO_2, $k = 1.29$. At critical conditions, $M = 1$

$\frac{T_0^*}{T^*} = 1 + \frac{k-1}{2} = 1.145 \qquad \therefore T^* = \frac{T_0^*}{1.145} = \frac{298\text{K}}{1.145} = 260\text{ K}$ $\quad T^*$

$\frac{p_0^*}{p^*} = \left[1 + \frac{k-1}{2}\right]^{k/k-1} = \left[1.145\right]^{4.448} = 1.826$

$\therefore p^* = \frac{p_0^*}{1.826} = \frac{45.101\text{ MPa}}{1.826} = 24.7\text{ MPa (abs)}$ $\quad p^*$

$V^* = c^* = (kRT^*)^{1/2} = \left(1.29 \times 189\,\frac{\text{N}\cdot\text{m}}{\text{kg}\cdot\text{K}} \times 260\text{K} \times \frac{\text{kg}\cdot\text{m}}{\text{N}\cdot\text{s}^2}\right)^{1/2} = 252\text{ m/s}$ $\quad V^*$

Problem 12.88

[Difficulty: 1]

12.88 The gas storage reservoir for a high-speed wind tunnel contains helium at 3600°R and 725 psig. Calculate the critical conditions (temperature, pressure, and flow speed) that correspond to these stagnation conditions.

Given: Data on helium in reservoir

Find: Critical conditions

Solution:

The data provided, or available in the Appendices, is:

$$R_{He} = 386.1 \cdot \frac{ft \cdot lbf}{lbm \cdot R} \qquad k = 1.66 \qquad T_0 = 3600 \cdot R \qquad p_0 = (725 + 14.7) psi \qquad p_0 = 740 \cdot psi$$

For critical conditions

$$\frac{T_0}{T_{crit}} = \frac{k+1}{2} \qquad T_{crit} = \frac{T_0}{\frac{k+1}{2}} \qquad T_{crit} = 2707 \cdot R$$

$$\frac{p_0}{p_{crit}} = \left(\frac{k+1}{2}\right)^{\frac{k}{k-1}} \qquad p_{crit} = \frac{p_0}{\left(\frac{k+1}{2}\right)^{\frac{k}{k-1}}} \qquad p_{crit} = 361 \cdot psi \quad \text{absolute}$$

$$V_{crit} = \sqrt{k \cdot R_{He} \cdot T_{crit}} \qquad V_{crit} = 7471 \cdot \frac{ft}{s}$$

Problem 12.89

[Difficulty: 1]

12.89 Stagnation conditions in a solid propellant rocket motor are $T_0 = 3000$ K and $p_0 = 45$ MPa (gage). Critical conditions occur in the throat of the rocket nozzle where the Mach number is equal to one. Evaluate the temperature, pressure, and flow speed at the throat. Assume ideal gas behavior with $R = 323$ J/(kg·K) and $k = 1.2$.

Solution:

Computing equations: $\frac{T_0}{T} = 1 + \frac{k-1}{2} M^2$ $\quad \frac{p_0}{p} = \left(1 + \frac{k-1}{2} M^2\right)^{k/k-1}$

Assume flow to throat is isentropic.

At throat, $M = 1$

$\therefore T_{0t} = T_t\left(1 + \frac{k-1}{2}\right) = 1.1\, T_t \quad \therefore T_t = \frac{T_0}{1.1} = \frac{3000\text{K}}{1.10} = 2730\text{K}$ $\qquad T_t$

$\frac{p_{0t}}{p_t} = \left(1 + \frac{k-1}{2}\right)^{k/k-1} = (1.1)^{\frac{1.2}{0.2}} = 1.7716$

$\therefore p_t = \frac{p_0}{1.7716} = \frac{45.101 \text{ MPa}}{1.7716} = 25.5 \text{ MPa (abs)}$ $\qquad p_t$

$V_t = (kRT_t)^{1/2} = \left[1.2 \times 323 \frac{\text{N}\cdot\text{m}}{\text{kg}\cdot\text{K}} \times 2730\text{K} \times \frac{\text{kg}\cdot\text{m}}{\text{N}\cdot\text{s}^2}\right]^{1/2} = 1030 \text{ m/s}$ $\qquad V_t$

Problem 12.90

[Difficulty: 1]

12.90 The hot gas stream at the turbine inlet of a JT9-D jet engine is at 1500°C, 140 kPa (abs), and $M = 0.32$. Calculate the critical conditions (temperature, pressure, and flow speed) that correspond to these conditions. Assume the fluid properties of pure air.

Given: Data on hot gas stream

Find: Critical conditions

Solution:

The data provided, or available in the Appendices, is:

$R = 287 \cdot \frac{J}{kg \cdot K}$ $\quad k = 1.4$ $\quad T_0 = (1500 + 273) \cdot K$ $\quad T_0 = 1773 K$ $\quad p_0 = 140 \cdot kPa$

For critical conditions $\quad \frac{T_0}{T_{crit}} = \frac{k+1}{2}$ $\quad T_{crit} = \frac{T_0}{\frac{k+1}{2}}$ $\quad T_{crit} = 1478 K$

$$\frac{p_0}{p_{crit}} = \left(\frac{k+1}{2}\right)^{\frac{k}{k-1}} \qquad p_{crit} = \frac{p_0}{\left(\frac{k+1}{2}\right)^{\frac{k}{k-1}}} \qquad p_{crit} = 74.0 \cdot kPa \qquad \text{absolute}$$

$$V_{crit} = \sqrt{k \cdot R \cdot T_{crit}} \qquad V_{crit} = 770 \frac{m}{s}$$

Problem 12.91

[Difficulty: 1]

12.91 Certain high-speed wind tunnels use combustion air heaters to generate the extreme pressures and temperatures required to accurately simulate flow at high Mach numbers. In one set of tests, a combustion air heater supplied stagnation conditions of 1.7 MPa and 1010 K. Calculate the critical pressure and temperature corresponding to these stagnation conditions.

Given:
Data on air flow in a ramjet combustor

Find:
Critical temperature and pressure at nozzle exit

Solution:

The data provided, or available in the Appendices, is: $k = 1.4 \quad p_0 = 1.7 \cdot \text{MPa} \quad T_0 = 1010 \cdot K$

The critical temperature and pressure are:

$$\frac{T_0}{T_{crit}} = \frac{k+1}{2} \qquad T_{crit} = \frac{T_0}{\frac{k+1}{2}} \qquad T_{crit} = 841.7 \cdot K$$

$$\frac{p_0}{p_{crit}} = \left(\frac{k+1}{2}\right)^{\frac{k}{k-1}} \qquad p_{crit} = \frac{p_0}{\left(\frac{k+1}{2}\right)^{\frac{k}{k-1}}} \qquad p_{crit} = 0.898 \cdot \text{MPa}$$

Problem 12.92

[Difficulty: 1]

12.92 The ramjet combustor exhaust from Problem 12.81 is accelerated through a nozzle to critical conditions. Calculate the temperature, pressure, and flow velocity at the nozzle exit. Assume fluid properties of pure air.

Given: Data on air flow in a ramjet combustor

Find: Critical temperature and pressure at nozzle exit

Solution:

The data provided, or available in the Appendices, is: $k = 1.4$ $\quad M_2 = 0.9$ $\quad T_2 = (1660 + 460)\cdot R$ $\quad p_2 = 1.6\cdot psi$

Stagnation conditions are: $T_{02} = T_2\cdot\left(1 + \frac{k-1}{2}\cdot M_2{}^2\right)$ $\quad T_{02} = 2463.4\,R$ $\quad T_{02} = 2003.8\,°F$

$$p_{02} = p_2\cdot\left(1 + \frac{k-1}{2}\cdot M_2{}^2\right)^{\frac{k}{k-1}} \qquad p_{02} = 2.71\cdot psi$$

The critical temperature and pressure are:

$$\frac{T_{02}}{T_{crit2}} = \frac{k+1}{2} \qquad T_{crit2} = \frac{T_{02}}{\frac{k+1}{2}} \qquad T_{crit2} = 2052.9\cdot R \quad T_{crit2} = 1593.2\cdot °F$$

$$\frac{p_{02}}{p_{crit2}} = \left(\frac{k+1}{2}\right)^{\frac{k}{k-1}} \qquad p_{crit2} = \frac{p_{02}}{\left(\frac{k+1}{2}\right)^{\frac{k}{k-1}}} \qquad p_{crit2} = 1.430\cdot psi$$

Problem 13.1

[Difficulty: 2]

13.1 Air is extracted from a large tank in which the temperature and pressure are 70°C and 101 kPa (abs), respectively, through a nozzle. At one location in the nozzle the static pressure is 25 kPa and the diameter is 15 cm. What is the mass flow rate? Assume isentropic flow.

Given: Air extracted from a large tank

Find: Mass flow rate

Solution:

Basic equations: $m_{rate} = \rho\cdot V\cdot A$ $\quad h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2}$ $\quad \frac{p}{\rho^k} = \text{const}$ $\quad T\cdot p^{\frac{(1-k)}{k}} = \text{const}$

Given or available data $\quad T_0 = (70 + 273)\cdot K$ $\quad p_0 = 101\cdot kPa$ $\quad p = 25\cdot kPa$

$D = 15\cdot cm$ $\quad c_p = 1004\cdot\frac{J}{kg\cdot K}$ $\quad k = 1.4$ $\quad R = 286.9\cdot\frac{J}{kg\cdot K}$

The mass flow rate is given by $\quad m_{rate} = \rho\cdot A\cdot V$ $\quad A = \frac{\pi\cdot D^2}{4}$ $\quad A = 0.0177\,m^2$

We need the density and velocity at the nozzle. In the tank $\quad \rho_0 = \frac{p_0}{R\cdot T_0}$ $\quad \rho_0 = 1.026\,\frac{kg}{m^3}$

From the isentropic relation $\quad \rho = \rho_0\cdot\left(\frac{p}{p_0}\right)^{\frac{1}{k}}$ $\quad \rho = 0.379\,\frac{kg}{m^3}$

We can apply the energy equation between the tank (stagnation conditions) and the point in the nozzle to find the velocity

$$h_0 = h + \frac{V^2}{2} \qquad V = \sqrt{2\cdot(h_0 - h)} = \sqrt{2\cdot c_p\cdot(T_0 - T)}$$

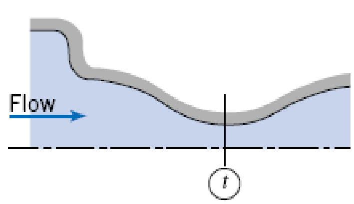
Fot T we again use insentropic relations $\quad T = T_0\cdot\left(\frac{p_0}{p}\right)^{\frac{(1-k)}{k}}$ $\quad T = 230.167\,K$ $\quad T = -43.0\cdot°C$

Then $\quad V = \sqrt{2\cdot c_p\cdot(T_0 - T)}$ $\quad V = 476\,\frac{m}{s}$

The mass flow rate is $\quad m_{rate} = \rho\cdot A\cdot V$ $\quad m_{rate} = 3.18\,\frac{kg}{s}$

Note that the flow is supersonic at this point $\quad c = \sqrt{k\cdot}$ $\; c = 304\,\frac{m}{s}$ $\quad M = \frac{V}{c}$ $\quad M = 1.57$

Hence we must have a converging-diverging nozzle



Problem 13.2

[Difficulty: 2]

13.2 Steam flows steadily and isentropically through a nozzle. At an upstream section where the speed is negligible, the temperature and pressure are 900°F and 900 psia. At a section where the nozzle diameter is 0.188 in., the steam pressure is 600 psia. Determine the speed and Mach number at this section and the mass flow rate of steam. Sketch the passage shape.

Solution:

Basic equations: $h_1 + \frac{V_1^2}{2} = h + \frac{V^2}{2} = h_0 = \text{constant}$

$\rho A V = \text{constant} = \dot{m}$

Assumptions:
(1) steady flow
(2) isentropic flow
(3) uniform flow at a section
(4) superheated steam can be treated as an ideal gas, $R = 85.8\ \text{ft}\cdot\text{lbf}/\text{lbm}\cdot{}^\circ\text{R}$, $k = 1.28$
(5) $\Delta z = 0$

From steam tables for superheated vapor with $T_0 = 900°\text{F}$, $p_0 = 900\ \text{psia}$,
$h_0 = 1451.8\ \text{Btu/lbm}$ $\quad s_0 = 1.6257\ \text{Btu/lbm}\cdot{}^\circ\text{R}$

From steam tables for superheated vapor with $s_1 = 1.6257\ \text{Btu/lbm}\cdot{}^\circ\text{R}$, and $p_1 = 600\ \text{psia}$,

$T_1 = 780°\text{F}$ $\quad h_1 = 1396.5\ \text{Btu/lbm}$, $\quad v_1 = 1.167\ \text{ft}^3/\text{lbm}$

From the first law,

$$V_1 = \left[2(h_0 - h_1)\right]^{1/2} = \left[2(1451.8 - 1396.5)\frac{\text{Btu}}{\text{lbm}} \times 778\frac{\text{ft}\cdot\text{lbf}}{\text{Btu}} \times 32.2\frac{\text{lbm}}{\text{slug}} \times \frac{\text{slug}\cdot\text{ft}}{\text{lbf}\cdot\text{s}^2}\right]^{1/2}$$

$$V_1 = 1660\ \text{ft/s} \qquad V_1$$

$$\dot{m} = \rho A V = \frac{1}{v}\pi\frac{D^2}{4}V = \frac{\text{lbm}}{1.167\ \text{ft}^3} \times \frac{\pi}{4} \times \left(\frac{0.188}{12}\right)^2 \text{ft}^2 \times 1660\frac{\text{ft}}{\text{s}} = 0.274\ \text{lbm/s} \qquad \dot{m}$$

$$M_1 = \frac{V_1}{c_1} \qquad c_1 = (kRT_1)^{1/2} = \left(1.30 \times 85.8\frac{\text{ft}\cdot\text{lbf}}{\text{lbm}\cdot{}^\circ\text{R}} \times 1240\ \text{R} \times 32.2\frac{\text{lbm}}{\text{slug}} \times \frac{\text{slug}\cdot\text{ft}}{\text{lbf}\cdot\text{s}^2}\right)^{1/2}$$

$$c_1 = 2110\ \text{ft/s}$$

$$M_1 = \frac{V_1}{c_1} = \frac{1660}{2110} = 0.787$$

Since $M_1 < 1.0$, passage converges as shown above

Problem 13.3

[Difficulty: 2]

13.3 Steam flows steadily and isentropically through a nozzle. At an upstream section where the speed is negligible, the temperature and pressure are 450°C and 6 MPa (abs). At a section where the nozzle diameter is 2 cm, the steam pressure is 2 MPa (abs). Determine the speed and Mach number at this section and the mass flow rate of steam. Sketch the passage shape.

Given: Steam flow through a nozzle

Find: Speed and Mach number; Mass flow rate; Sketch the shape

Solution:

Basic equations: $m_{rate} = \rho \cdot V \cdot A$ $\qquad h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2}$

Assumptions: 1) Steady flow 2) Isentropic 3) Uniform flow 4) Superheated steam can be treated as ideal gas

Given or available data $\quad T_0 = (450 + 273) \cdot K \qquad p_0 = 6 \cdot MPa \qquad p = 2 \cdot MPa$

$D = 2 \cdot cm \qquad k = 1.30 \qquad R = 461.4 \frac{J}{kg \cdot K}$ (Table A.6)

From the steam tables (try finding interactive ones on the Web!), at stagnation conditions

$s_0 = 6720 \cdot \frac{J}{kg \cdot K} \qquad h_0 = 3.302 \times 10^6 \cdot \frac{J}{kg}$

Hence at the nozzle section $\quad s = s_0 = 6720 \cdot \frac{J}{kg \cdot K}$ and $\quad p = 2 \cdot MPa$

From these values we find from the steam tables that $\quad T = 289\,°C \qquad h = 2.997 \times 10^6 \cdot \frac{J}{kg} \qquad v = 0.1225 \cdot \frac{m^3}{kg}$

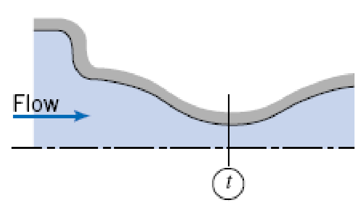
Hence the first law becomes $\quad V = \sqrt{2 \cdot (h_0 - h)} \qquad V = 781 \frac{m}{s}$

The mass flow rate is given by $\quad m_{rate} = \rho \cdot A \cdot V = \frac{A \cdot V}{v} \qquad A = \frac{\pi \cdot D^2}{4} \qquad A = 3.14 \times 10^{-4} m^2$

Hence $\quad m_{rate} = \frac{A \cdot V}{v} \qquad m_{rate} = 2.00 \frac{kg}{s}$

For the Mach number we need $\quad c = \sqrt{k \cdot R \cdot T} \qquad c = 581 \frac{m}{s} \qquad M = \frac{V}{c} \qquad M = 1.35$

The flow is supersonic starting from rest, so must be converging-diverging



Problem 13.4

[Difficulty: 2]

13.4 Nitrogen flows through a diverging section of duct with $A_1 = 0.15\ \text{m}^2$ and $A_2 = 0.45\ \text{m}^2$. If $M_1 = 0.7$ and $p_1 = 450\ \text{kPa}$, find M_2 and p_2.

Given: Data on flow in a passage

Find: Pressure and Mach number at downstream location

Solution:

The given or available data is:

$R =$	296.8	J/kg-K
$k =$	1.4	
$p_1 =$	450	kPa
$M_1 =$	0.7	
$A_1 =$	0.15	m^2
$A_2 =$	0.45	m^2

Equations and Computations:

From M_1 and p_1, and Eq. 13.7a
(using built-in function *Isenp* (M, k))

$$\frac{p_0}{p} = \left[1 + \frac{k-1}{2} M^2\right]^{k/(k-1)}$$

$p_{01} =$ 624 kPa

From M_1, and Eq. 13.7d
(using built-in function *IsenA* (M, k))

$$\frac{A}{A*} = \frac{1}{M}\left[\frac{1 + \frac{k-1}{2} M^2}{\frac{k+1}{2}}\right]^{(k+1)/2(k-1)}$$

$A^*_1 =$ 0.1371 m^2

For isentropic flow ($p_{01} = p_{02}$, $A^*_2 = A^*_1$)

$p_{02} =$	624	kPa
$A^*_2 =$	0.1371	m^2
$A_2/A^*_2 =$	3.2831	

From A_2/A^*_2, and Eq. 13.7d
(using built-in function *IsenMsubfromA* (M, k))
Since there is no throat, the flow stays subsonic

$M_2 =$ 0.1797

From M_2 and p_{02}, and Eq. 13.7a
(using built-in function *Isenp* (M, k))

$p_2 =$ 610 kPa

Problem 13.5

[Difficulty: 2]

13.5 Nitrogen flows through a diverging section of duct with $A_1 = 0.15\ \text{m}^2$ and $A_2 = 0.45\ \text{m}^2$. If $M_1 = 1.7$ and $T_1 = 30°\text{C}$, find M_2 and T_2.

Given: Data on flow in a passage

Find: Temperature and Mach number at downstream location

Solution:

The given or available data is:

$R =$	296.8	J/kg-K
$k =$	1.4	
$T_1 =$	30	°C
$T_1 =$	303	K
$M_1 =$	1.7	
$A_1 =$	0.15	m^2
$A_2 =$	0.45	m^2

Equations and Computations:

From M_1 and T_1, and Eq. 13.7b
(using built-in function *Isent* (M,k))

$$\frac{T_0}{T} = 1 + \frac{k-1}{2}M^2$$

$T_{01} =$ 478 K

From M_1, and Eq. 13.7d
(using built-in function *IsenA* (M,k))

$$\frac{A}{A*} = \frac{1}{M}\left[\frac{1 + \frac{k-1}{2}M^2}{\frac{k+1}{2}}\right]^{(k+1)/2(k-1)}$$

$A^*_1 =$ 0.1121 m^2

For isentropic flow ($T_{01} = T_{02}$, $A^*_2 = A^*_1$)

$T_{02} =$ 478 K

$A^*_2 =$ 0.1121 m^2

$A_2/A^*_2 =$ 4.0128

From A_2/A^*_2, and Eq. 13.7d
(using built-in function *IsenMsupfromA* (M,k))
Since there is no throat, the flow stays supersonic!

$M_2 =$ 2.94

From M_2 and T_{02}, and Eq. 13.7b
(using built-in function *Isent* (M,k))

$T_2 =$ 175 K

$T_2 =$ -98 °C

Problem 13.6

[Difficulty: 2]

13.6 At a section in a passage, the pressure is 150 kPa (abs), the temperature is 10°C, and the speed is 120 m/s. For isentropic flow of air, determine the Mach number at the point where the pressure is 50 kPa (abs). Sketch the passage shape.

Given: Air flow in a passage

Find: Mach number; Sketch shape

Solution:

Basic equations:

$$\frac{p_0}{p} = \left(1 + \frac{k-1}{2} \cdot M^2\right)^{\frac{k}{k-1}} \qquad c = \sqrt{k \cdot R \cdot T}$$

Given or available data

$$T_1 = (10 + 273) \cdot K \qquad p_1 = 150 \cdot kPa \qquad V_1 = 120 \cdot \frac{m}{s}$$

$$p_2 = 50 \cdot kPa \qquad k = 1.4 \qquad R = 286.9 \cdot \frac{J}{kg \cdot K}$$

The speed of sound at state 1 is $c_1 = \sqrt{k \cdot R \cdot T_1}$ $\qquad c_1 = 337 \frac{m}{s}$

Hence $M_1 = \frac{V_1}{c_1}$ $\qquad M_1 = 0.356$

For isentropic flow stagnation pressure is constant. Hence at state 2

$$\frac{p_0}{p_2} = \left(1 + \frac{k-1}{2} \cdot M_2^2\right)^{\frac{k}{k-1}}$$

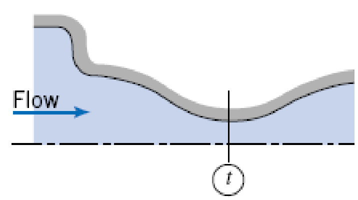
Hence

$$p_0 = p_1 \cdot \left(1 + \frac{k-1}{2} \cdot M_1^2\right)^{\frac{k}{k-1}} \qquad p_0 = 164 \cdot kPa$$

Solving for M_2

$$M_2 = \sqrt{\frac{2}{k-1} \cdot \left[\left(\frac{p_0}{p_2}\right)^{\frac{k-1}{k}} - 1\right]} \qquad M_2 = 1.42$$

Hence, as we go from subsonic to supersonic we must have a converging-diverging nozzle



Problem 13.7

[Difficulty: 2]

13.7 At a section in a passage, the pressure is 30 psia, the temperature is 100°F, and the speed is 1750 ft/s. At a section downstream the Mach number is 2.5. Determine the pressure at this downstream location for isentropic flow of air. Sketch the passage shape.

Given: Data on flow in a passage

Find: Pressure at downstream location

Solution:

The given or available data is:

$R =$	53.33	ft·lbf/lbm·°R
$k =$	1.4	
$T_1 =$	560	°R
$p_1 =$	30	psi
$V_1 =$	1750	ft/s
$M_2 =$	2.5	

Equations and Computations:

From T_1 and Eq. 12.18

$$c = \sqrt{kRT}$$

$c_1 =$ 1160 ft/s

Then

$M_1 =$ 1.51

From M_1 and p_1, and Eq. 13.7a
(using built-in function *Isenp* (M,k))

$$\frac{p_0}{p} = \left[1 + \frac{k-1}{2}M^2\right]^{k/(k-1)}$$

$p_{01} =$ 111 psi

For isentropic flow ($p_{01} = p_{02}$)

$p_{02} =$ 111 psi

From M_2 and p_{02}, and Eq. 13.7a
(using built-in function *Isenp* (M,k))

$p_2 =$ 6.52 psi

Problem 13.8

[Difficulty: 2]

13.8 Oxygen flows into an insulated duct with initial conditions of 200 kPa, 420 K, and 200 m/s. The area changes from $A_1 = 0.6\ \text{m}^2$ to $A_2 = 0.5\ \text{m}^2$. Compute M_1, p_{0_1}, and T_{0_1}. Is this duct a nozzle or a diffuser? Calculate the exit conditions (pressure, temperature, and Mach number) provided that there are no losses.

Given: Data on flow in a passage

Find: Stagnation conditions; whether duct is a nozzle or diffuser; exit conditions

Solution:

The given or available data is:

$R =$	259.8	J/kg-K
$k =$	1.4	
$p_1 =$	200	kPa
$T_1 =$	420	K
$V_1 =$	200	m/s
$A_1 =$	0.6	m^2
$A_2 =$	0.5	m^2

Equations and Computations:

From T_1 and Eq. 12.18

$$c = \sqrt{kRT}$$

$c_1 =$ 391 m/s

Then $M_1 =$ 0.512

From M_1 and T_1, and Eq. 13.7b
(using built-in function *Isent* (M, k))

$$\frac{T_0}{T} = 1 + \frac{k-1}{2}M^2$$

$T_{01} =$ 442 K

From M_1 and p_1, and Eq. 13.7a
(using built-in function *Isenp* (M, k))

$$\frac{p_0}{p} = \left[1 + \frac{k-1}{2}M^2\right]^{k/(k-1)}$$

$p_{01} =$ 239 kPa

Since the flow is subsonic and the area is decreasing, this duct is a nozzle.

From M_1, and Eq. 13.7d
(using built-in function *IsenA* (M, k))

$$\frac{A}{A*} = \frac{1}{M}\left[\frac{1 + \frac{k-1}{2}M^2}{\frac{k+1}{2}}\right]^{(k+1)/2(k-1)}$$

$A^*_1 =$ 0.4552 m^2

For isentropic flow ($p_{01} = p_{02}$, $T_{01} = T_{02}$, $A^*_2 = A^*_1$)

$p_{02} =$	239	kPa
$T_{02} =$	442	K
$A^*_2 =$	0.4552	m^2
$A_2/A^*_2 =$	1.0984	

From A_2/A^*_2, and Eq. 13.7d
(using built-in function *IsenMsubfromA* (M, k))
Since there is no throat, the flow stays subsonic!

$M_2 =$ 0.69

From M_2 and stagnation conditions:
(using built-in functions)

$p_2 =$	173	kPa
$T_2 =$	403	K

Problem 13.9

[Difficulty: 3]

13.9 Air is flowing in an adiabatic system at 20 lbm/s. At one section, the pressure is 30 psia, the temperature is 1200°F, and the area is 8 in^2. At a downstream section, $M_2 = 1.2$. Sketch the flow passage. Find the exit area provided the flow is reversible.

Given: Data on flow in a passage

Find: Shape of flow passage; exit area provided the flow is reversible

Solution:

The given or available data is:

$R =$	53.33	ft-lbf/lbm-°R
$k =$	1.4	
$m =$	20	lbm/s
$p_1 =$	30	psia
$T_1 =$	1200	°F
$T_1 =$	1660	°R
$A_1 =$	8	in^2
$M_2 =$	1.2	

Equations and Computations:

Using the ideal gas law we calculate the density at station 1:

$\rho_1 =$ 0.04880 lbm/ft^3

Now we can use the area and density to get the velocity from the mass flow rate:

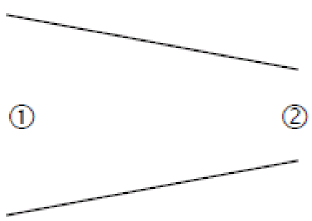
$V_1 =$ 7377 ft/s

From T_1 and Eq. 12.18 $\quad c = \sqrt{kRT}$

$c_1 =$ 1998 ft/s

Then $\quad M_1 =$ 3.69

Since the flow is supersonic and the velocity is decreasing, this duct is converging.



From M_1, and Eq. 13.7d
(using built-in function *IsenA* (M, k))

$$\frac{A}{A*} = \frac{1}{M}\left[\frac{1 + \frac{k-1}{2}M^2}{\frac{k+1}{2}}\right]^{(k+1)/2(k-1)}$$

$A^*_1 =$ 0.9857 in^2

For isentropic flow ($A^*_2 = A^*_1$)

$A^*_2 =$	0.9857	in^2
$A_2/A^*_2 =$	1.0304	

Therefore the exit area is:

$A_2 =$ 1.016 in^2

Problem 13.10

[Difficulty: 3]

13.10 Air flows isentropically through a converging-diverging nozzle from a large tank containing air at 250°C. At two locations where the area is 1 cm^2, the static pressures are 200 kPa and 50 kPa. Find the mass flow rate, the throat area, and the Mach numbers at the two locations.

Given: Data on flow in a nozzle

Find: Mass flow rate; Throat area; Mach numbers

Solution:

The given or available data is:

$R =$	286.9	J/kg·K			
$k =$	1.4				
$T_0 =$	523	K			
$p_1 =$	200	kPa	$p_2 =$	50	kPa
$A =$	1	cm^2			

Equations and Computations:

We don't know the two Mach numbers. We do know for each that Eq. 13.7a applies:

$$\frac{p_0}{p} = \left[1 + \frac{k-1}{2}M^2\right]^{k/(k-1)}$$

Hence we can write two equations, but have three unknowns (M_1, M_2, and p_0)!

We also know that states 1 and 2 have the same area. Hence we can write Eq. 13.7d twice:

$$\frac{A}{A^*} = \frac{1}{M}\left[\frac{1 + \frac{k-1}{2}M^2}{\frac{k+1}{2}}\right]^{(k+1)/2(k-1)}$$

We now have four equations for four unknowns (A^*, M_1, M_2, and p_0)!
We make guesses (using Solver) for M_1 and M_2, and make the errors in computed A^* and p_0 zero.

							Errors
For:	$M_1 =$	0.512		$M_2 =$	1.68		
from Eq. 13.7a:	$p_0 =$	239	kPa	$p_0 =$	239	kPa	0.00%
and from Eq. 13.7d:	$A^* =$	0.759	cm^2	$A^* =$	0.759	cm^2	0.00%
Note that the throat area is the critical area						**Sum**	0.00%

The stagnation density is then obtained from the ideal gas equation

$\rho_0 =$ 1.59 kg/m^3

The density at critical state is obtained from Eq. 13.7a (or 12.22c)

$\rho^* =$ 1.01 kg/m^3

The velocity at critical state can be obtained from Eq. 12.23)

$$V^* = c^* = \sqrt{\frac{2k}{k+1}RT_0}$$

$V^* =$ 418 m/s

The mass flow rate is $\rho^* V^* A^*$

$m_{rate} =$ 0.0321 kg/s

Problem 13.11

[Difficulty: 2]

13.11 Air flows steadily and isentropically through a passage. At section ①, where the cross-sectional area is 0.02 m², the air is at 40.0 kPa (abs), 60°C, and $M = 2.0$. At section ② downstream, the speed is 519 m/s. Calculate the Mach number at section ②. Sketch the shape of the passage between sections ① and ②.

Solution:

Basic equations: $h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2}$

Assumptions: (1) steady flow
(2) isentropic flow
(3) uniform flow at a section
(4) $\Delta z = 0$
(5) ideal gas

$M_2 = \frac{V_2}{c_2}$ where $c_2 = (kRT_2)^{1/2}$. Hence T_2 must be found

$h_2 = h_1 + \frac{1}{2}(V_1^2 - V_2^2)$

$V_1 = M_1 c_1 = M_1 (kRT_1)^{1/2} = 2.0 \left(1.4 \times 287 \frac{N \cdot m}{kg \cdot K} \times 333 K \times \frac{kg \cdot m}{N \cdot s^2}\right)^{1/2} = 732 \ m/s$

$T_2 = T_1 + \frac{1}{2c_p}(V_1^2 - V_2^2)$

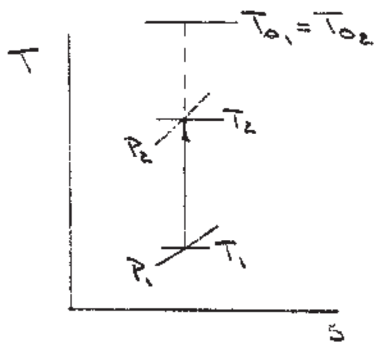
$= 333 K + [(732)^2 - (519)^2] \frac{m^2}{s^2} \cdot \frac{1}{2} \times \frac{kg \cdot K}{10^3 N \cdot m} \times \frac{N \cdot s^2}{kg \cdot m}$

$T_2 = 466 K$

$c_2 = (kRT_2)^{1/2} = \left(1.4 \times 287 \frac{N \cdot m}{kg \cdot K} \times 466 K \times \frac{kg \cdot m}{N \cdot s^2}\right)^{1/2} = 433 \ m/s$

$M_2 = \frac{V_2}{c_2} = \frac{519}{433} = 1.20$ ← M_2

Since $M_2 < M_1$ and $M_2 > 1.0$, then passage from ① to ② is a supersonic diffuser as shown above



Problem 13.12

[Difficulty: 3]

13.12 Air flows steadily and isentropically through a passage at 150 lbm/s. At the section where the diameter is $D = 3$ ft, $M = 1.75$, $T = 32°$F, and $p = 25$ psia. Determine the speed and cross-sectional area downstream where $T = 225°$F. Sketch the flow passage.

Given: Air flow in a passage

Find: Speed and area downstream; Sketch flow passage

Solution:

Basic equations:

$$\frac{T_0}{T} = 1 + \frac{k-1}{2} \cdot M^2 \qquad c = \sqrt{k \cdot R \cdot T} \qquad \frac{A}{A_{crit}} = \frac{1}{M} \cdot \left(\frac{1 + \frac{k-1}{2} \cdot M^2}{\frac{k+1}{2}} \right)^{\frac{k+1}{2 \cdot (k-1)}}$$

Given or available data

$$T_1 = (32 + 460) \cdot R \qquad p_1 = 25 \cdot psi \qquad M_1 = 1.75$$

$$T_2 = (225 + 460) \cdot R \qquad k = 1.4 \qquad R_{air} = 53.33 \cdot \frac{ft \cdot lbf}{lbm \cdot R}$$

$$D_1 = 3 \cdot ft \qquad A_1 = \frac{\pi \cdot D_1^{\ 2}}{4} \qquad A_1 = 7.07 \cdot ft^2$$

Hence

$$T_0 = T_1 \cdot \left(1 + \frac{k-1}{2} \cdot M_1^{\ 2} \right) \qquad T_0 = 793 \cdot R \qquad T_0 = 334 \cdot °F$$

For isentropic flow stagnation conditions are constant. Hence

$$M_2 = \sqrt{\frac{2}{k-1} \cdot \left(\frac{T_0}{T_2} - 1 \right)} \qquad M_2 = 0.889$$

We also have

$$c_2 = \sqrt{k \cdot R_{air} \cdot T_2} \qquad c_2 = 1283 \cdot \frac{ft}{s}$$

Hence

$$V_2 = M_2 \cdot c_2 \qquad V_2 = 1141 \cdot \frac{ft}{s}$$

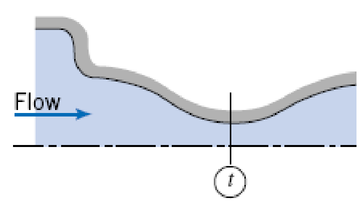
From state 1

$$A_{crit} = \frac{A_1 \cdot M_1}{\left(\frac{1 + \frac{k-1}{2} \cdot M_1^{\ 2}}{\frac{k+1}{2}} \right)^{\frac{k+1}{2 \cdot (k-1)}}} \qquad A_{crit} = 5.10 \cdot ft^2$$

Hence at state 2

$$A_2 = \frac{A_{crit}}{M_2} \cdot \left(\frac{1 + \frac{k-1}{2} \cdot M_2^{\ 2}}{\frac{k+1}{2}} \right)^{\frac{k+1}{2 \cdot (k-1)}} \qquad A_2 = 5.15 \cdot ft^2$$

Hence, as we go from supersonic to subsonic we must have a converging-diverging diffuser



Problem 13.13

[Difficulty: 2]

13.13 Air, at an absolute pressure of 60.0 kPa and 27°C, enters a passage at 486 m/s, where $A = 0.02\ \text{m}^2$. At section ② downstream, $p = 78.8$ kPa (abs). Assuming isentropic flow, calculate the Mach number at section ②. Sketch the flow passage.

Solution:

Computing equations: $\frac{p_0}{p} = \left[1 + \frac{k-1}{2} M^2\right]^{k/k-1}$ $\qquad c = \sqrt{kRT}$

Assumptions: 1) steady flow 2) isentropic flow 3) uniform flow at a section 4) ideal gas

For isentropic flow, $p_{0_1} = p_{0_2} = p_0 = \text{constant}$

$M_1 = \frac{V_1}{c_1} \qquad c_1 = (kRT_1)^{1/2} = \left(1.4 \times 287 \frac{\text{N}\cdot\text{m}}{\text{kg}\cdot\text{K}} \times 300\text{K} \times \frac{\text{kg}\cdot\text{m}}{\text{N}\cdot\text{s}^2}\right)^{1/2} = 347\ \text{m/s}$

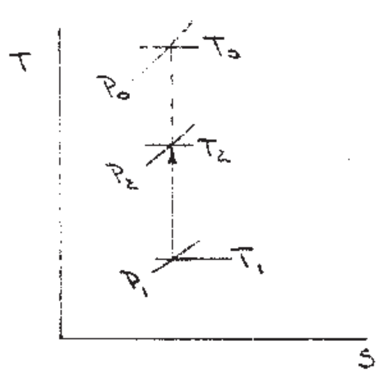
$M_1 = \frac{V_1}{c_1} = \frac{486}{347} = 1.40 \qquad p_{0_1} = p_1\left[1 + \frac{k-1}{2} M_1^2\right]^{k/k-1} = 60\ \text{kPa}\left[1 + 0.2(1.40)^2\right]^{3.5}$

$p_{0_1} = 191\ \text{kPa}$

$\frac{p_{0_2}}{p_2} = \left[1 + \frac{k-1}{2} M_2^2\right]^{k/k-1} \qquad p_{0_2} = p_{0_1}$

$M_2 = \left\{\frac{2}{k-1}\left[\left(\frac{p_{0_1}}{p_2}\right)^{k-1/k} - 1\right]\right\}^{1/2} = \left\{\frac{2}{0.4}\left[\left(\frac{191}{78.8}\right)^{0.286} - 1\right]\right\}^{1/2} = 1.20$ $\qquad M_2$

Since $M_2 < M_1$ and $M_2 > 1.0$, then passage from ① to ② is a supersonic diffuser as shown above



Problem 13.14

[Difficulty: 3]

13.14 Air flows adiabatically through a duct. At the entrance, the static temperature and pressure are 310 K and 200 kPa, respectively. At the exit, the static and stagnation temperatures are 294 K and 316 K, respectively, and the static pressure is 125 kPa. Find (a) the Mach numbers of the flow at the entrance and exit and (b) the area ratio A_2/A_1.

Given: Data on flow in a passage

Find: Mach numbers at entrance and exit; area ratio of duct

Solution:

The given or available data is:

$R =$	286.9	J/kg-K
$k =$	1.4	
$T_1 =$	310	K
$p_1 =$	200	kPa
$T_2 =$	294	K
$T_{02} =$	316	K
$p_2 =$	125	kPa

Equations and Computations:

Since the flow is adiabatic, the stagnation temperature is constant:

$T_{01} =$ 316 K

Solving for the Mach numbers at 1 and 2 using Eq. 13.7b
(using built-in function *IsenMfromT*(*Tratio*,*k*))

$$\frac{T_0}{T} = 1 + \frac{k-1}{2}M^2$$

Then

$M_1 =$ 0.311

$M_2 =$ 0.612

Using the ideal gas equation of state, we can calculate the densities of the gas:

$\rho_1 =$ 2.249 kg/m^3

$\rho_2 =$ 1.482 kg/m^3

From static temperatures and Eq. 12.18 $c = \sqrt{kRT}$

$c_1 =$	352.9	m/s
$c_2 =$	343.6	m/s
$V_1 =$	109.8	m/s
$V_2 =$	210.2	m/s

Since flow is steady, the mass flow rate must be equal at 1 and 2.
So the area ratio may be calculated from the densities and velocities:

$A_2/A_1 =$ 0.792

Note that we can not assume isentropic flow in this problem. While the flow is adiabatic, it is not reversible. There is a drop in stagnation pressure from state 1 to 2 which would invalidate the assumption of isentropic flow.

Problem 13.15

[Difficulty: 3]

13.15 Atmospheric air (101 kPa and 20°C) is drawn into a receiving pipe via a converging nozzle. The throat cross-section diameter is 1 cm. Plot the mass flow rate delivered for the receiving pipe pressure ranging from 100 kPa down to 5 kPa.

Given: Flow in a converging nozzle to a pipe

Find: Plot of mass flow rate

Solution:

The given or available data is

$R =$ 287 J/kg·K
$k =$ 1.4
$T_0 =$ 293 K
$p_0 =$ 101 kPa
$D_t =$ 1 cm
$A_t =$ 0.785 cm^2

Equations and Computations:

The critical pressure is given by

$$\frac{p_0}{p^*} = \left[\frac{k+1}{2}\right]^{k/(k-1)} \qquad (12.22a)$$

$p^* =$ 53.4 kPa

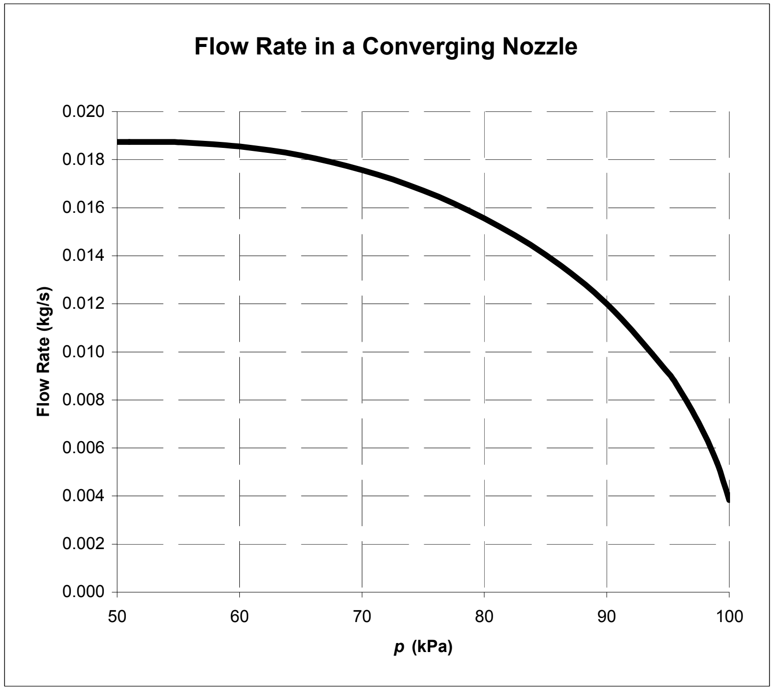
Hence for p = 100 kPa down to this pressure the flow gradually increases; then it is constant

p (kPa)	M (Eq. 13.7a)	T (K) (Eq. 13.7b)	c (m/s)	$V = M \cdot c$ (m/s)	$\rho = p/RT$ (kg/m^3)	Flow Rate (kg/s)
100	0.119	292	343	41	1.19	0.00383
99	0.169	291	342	58	1.18	0.00539
98	0.208	290	342	71	1.18	0.00656
97	0.241	290	341	82	1.17	0.00753
96	0.270	289	341	92	1.16	0.00838
95	0.297	288	340	101	1.15	0.0091
90	0.409	284	337	138	1.11	0.0120
85	0.503	279	335	168	1.06	0.0140
80	0.587	274	332	195	1.02	0.0156
75	0.666	269	329	219	0.971	0.0167
70	0.743	264	326	242	0.925	0.0176
65	0.819	258	322	264	0.877	0.0182
60	0.896	252	318	285	0.828	0.0186
55	0.974	246	315	306	0.778	0.0187
53.4	1.000	244	313	313	0.762	0.0187
53	1.000	244	313	313	0.762	0.0187
52	1.000	244	313	313	0.762	0.0187
51	1.000	244	313	313	0.762	0.0187
50	1.000	244	313	313	0.762	0.0187

Using critical conditions, and Eq. 13.9 for mass flow rate:

53.4	1.000	244	313	313	0.762	0.0185

(Note: discrepancy in mass flow rate is due to round-off error)



Problem 13.16

[Difficulty: 2]

13.16 For isentropic flow of air, at a section in a passage, A = 0.25 m^2, p = 15 kPa (abs), T = 10°C, and V = 590 m/s. Find the Mach number and the mass flow rate. At a section downstream the temperature is 137°C and the Mach number is 0.75. Determine the cross-sectional area and pressure at this downstream location. Sketch the passage shape.

Given: Data on flow in a passage

Find: Flow rate; area and pressure at downstream location; sketch passage shape

Solution:

The given or available data is:

R =	286.9	J/kg.K
k =	1.4	
A_1 =	0.25	m^2
T_1 =	283	K
p_1 =	15	kPa
V_1 =	590	m/s
T_2 =	410	
M_2 =	0.75	

Equations and Computations:

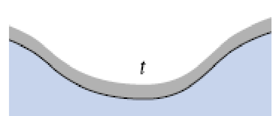
From T_1 and Eq. 12.18

$$c = \sqrt{kRT} \qquad (12.18)$$

c_1 = 337 m/s

Then M_1 = 1.75

Because the flow decreases isentropically from supersonic to subsonic the passage shape must be convergent-divergent



From p_1 and T_1 and the ideal gas equation

ρ_1 = 0.185 kg/m^3

The mass flow rate is $m_{rate} = \rho_1 A_1 V_1$

m_{rate} = 27.2 kg/s

From M_1 and A_1, and Eq. 13.7d
(using built-in function *IsenA* (M, k))

$$\frac{A}{A^*} = \frac{1}{M}\left[\frac{1+\frac{k-1}{2}M^2}{\frac{k+1}{2}}\right]^{(k+1)/2(k-1)} \qquad (13.7d)$$

$A^* =$ 0.180 m^2

From M_2 and A^*, and Eq. 13.7d
(using built-in function *IsenA* (M, k))

$A_2 =$ 0.192 m^2

From M_1 and p_1, and Eq. 13.7a
(using built-in function *Isenp* (M, k))

$$\frac{p_0}{p} = \left[1+\frac{k-1}{2}M^2\right]^{k/(k-1)} \qquad (13.7a)$$

$p_{01} =$ 79.9 kPa

For isentropic flow ($p_{01} = p_{02}$)

$p_{02} =$ 79.9 kPa

From M_2 and p_{02}, and Eq. 13.7a
(using built-in function *Isenp* (M, k))

$p_2 =$ 55.0 kPa

Problem 13.17

[Difficulty: 3]

13.17 A passage is designed to expand air isentropically to atmospheric pressure from a large tank in which properties are held constant at 40°F and 45 psia. The desired flow rate is 2.25 lbm/s. Assuming the passage is 20 ft long, and that the Mach number increases linearly with position in the passage, plot the cross-sectional area and pressure as functions of position.

Given: Data on tank conditions; isentropic flow

Find: Plot cross-section area and pressure distributions

Solution:

The given or available data is:

$R =$	53.33	ft·lbf/lbm·°R
$k =$	1.4	
$T_0 =$	500	°R
$p_0 =$	45	psia
$p_e =$	14.7	psia
$m_{rate} =$	2.25	lbm/s

Equations and Computations:

From p_0, p_e and Eq. 13.7a (using built-in function *IsenMfromp* (M,k))

$$\frac{p_0}{p} = \left[1 + \frac{k-1}{2}M^2\right]^{k/(k-1)} \qquad (13.7a)$$

$M_e =$ 1.37

Because the exit flow is supersonic, the passage must be a CD nozzle
We need a scale for the area.
From p_0, T_0, m_{flow}, and Eq. 13.10c

$$\dot{m}_{\text{choked}} = 76.6\,\frac{A_t p_0}{\sqrt{T_0}} \qquad (13.10c)$$

Then $A_t = A^* =$ 0.0146 ft^2

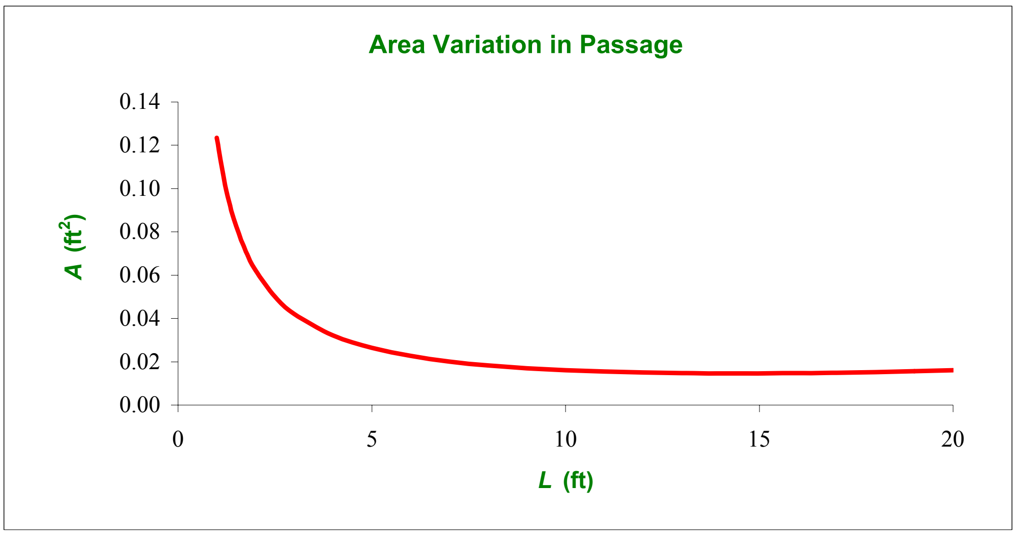
For each M, and A^*, and Eq. 13.7d
(using built-in function *IsenA* (M,k)

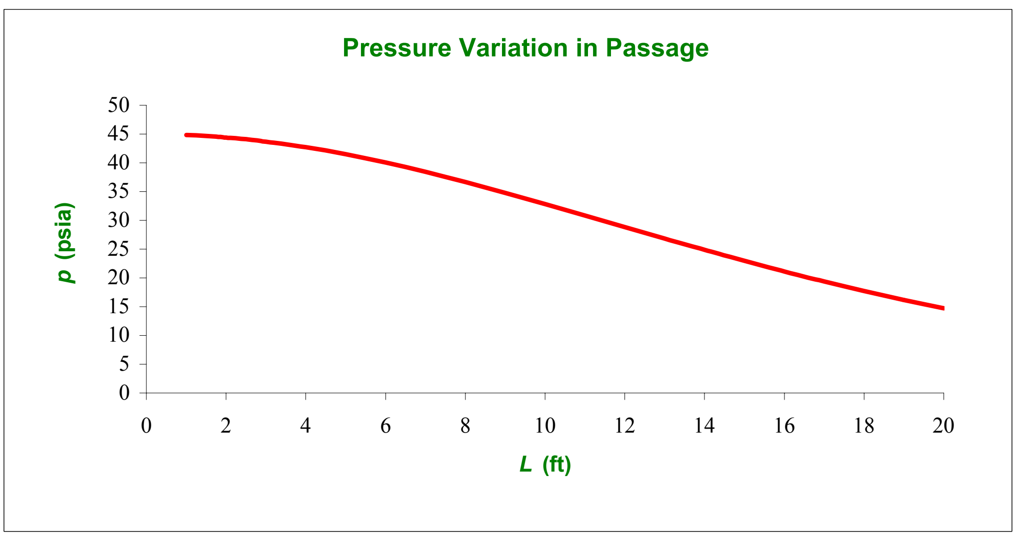
$$\frac{A}{A^*} = \frac{1}{M}\left[\frac{1 + \frac{k-1}{2}M^2}{\frac{k+1}{2}}\right]^{(k+1)/2(k-1)} \qquad (13.7d)$$

we can compute each area A.

From each M, and p_0, and Eq. 13.7a
(using built-in function *Isenp* (M,k)
we can compute each pressure p.

L (ft)	M	A (ft^2)	p (psia)
1.00	0.069	0.1234	44.9
1.25	0.086	0.0989	44.8
1.50	0.103	0.0826	44.7
1.75	0.120	0.0710	44.5
2.00	0.137	0.0622	44.4
2.50	0.172	0.0501	44.1
3.00	0.206	0.0421	43.7
4.00	0.274	0.0322	42.7
5.00	0.343	0.0264	41.5
6.00	0.412	0.0227	40.0
7.00	0.480	0.0201	38.4
8.00	0.549	0.0183	36.7
9.00	0.618	0.0171	34.8
10.00	0.686	0.0161	32.8
11.00	0.755	0.0155	30.8
12.00	0.823	0.0150	28.8
13.00	0.892	0.0147	26.8
14.00	0.961	0.0146	24.9
14.6	**1.000**	**0.0146**	**23.8**
16.00	1.098	0.0147	21.1
17.00	1.166	0.0149	19.4
18.00	1.235	0.0152	17.7
19.00	1.304	0.0156	16.2
20.00	1.372	0.0161	14.7





Problem 13.18

[Difficulty: 2]

13.18 Repeat Problem 13.15 if the converging nozzle is replaced with a converging-diverging nozzle with an exit diameter of 2.5 cm (same throat area).

Given: Flow in a converging-diverging nozzle to a pipe

Find: Plot of mass flow rate

Solution:

The given or available data is

$R =$ 286.9 J/kg·K
$k =$ 1.4
$T_0 =$ 293 K
$p_0 =$ 101 kPa
$D_t =$ 1 cm $\quad De =$ 2.5 cm
$A_t =$ 0.785 cm^2 $\quad A_e =$ 4.909 cm^2

Equations and Computations:

The critical pressure is given by

$$\frac{p_0}{p^*} = \left[\frac{k+1}{2}\right]^{k/(k-1)} \qquad (12.22a)$$

$p^* =$ 53.4 kPa This is the minimum throat pressure

For the CD nozzle, we can compute the pressure at the exit required for this to happen

$A^* =$ 0.785 cm^2 $(= A_t)$
$A_e/A^* =$ 6.25
$M_e =$ 0.0931 or 3.41 (Eq. 13.7d)
$p_e =$ 100.4 or 67.2 kPa (Eq. 13.7a)

Hence we conclude flow occurs in regimes *iii* down to *v* (Fig. 13.8); the flow is ALWAYS choked!

p^* (kPa)	M (Eq. 13.7a)	T^* (K) (Eq. 13.7b)	c^* (m/s)	$V^* = c^*$ (m/s)	$\rho = p/RT$ (kg/m^3)	Flow Rate (kg/s)
53.4	1.000	244	313	313	0.762	0.0187
(Note: discrepancy in mass flow rate is due to round-off error)						0.0185

(Using Eq. 13.9)

Problem 13.19

[Difficulty: 2]

13.19 Air flows isentropically through a converging nozzle into a receiver where the pressure is 250 kPa (abs). If the pressure is 350 kPa (abs) and the speed is 150 m/s at the nozzle location where the Mach number is 0.5, determine the pressure, speed, and Mach number at the nozzle throat.

Given: Isentropic air flow in converging nozzle

Find: Pressure, speed and Mach number at throat

Solution:

Basic equations: $\frac{T_0}{T} = 1 + \frac{k-1}{2} \cdot M^2$ $\qquad \frac{p_0}{p} = \left(1 + \frac{k-1}{2} \cdot M^2\right)^{\frac{k}{k-1}}$

Given or available data $\quad p_1 = 350 \cdot kPa \quad V_1 = 150 \cdot \frac{m}{s} \quad M_1 = 0.5 \quad p_b = 250 \cdot kPa$

$k = 1.4 \qquad R = 286.9 \cdot \frac{J}{kg \cdot K}$

The flow will be choked if $p_b/p_0 < 0.528$

$$p_0 = p_1 \cdot \left(1 + \frac{k-1}{2} \cdot M_1^2\right)^{\frac{k}{k-1}} \qquad p_0 = 415 \cdot kPa \qquad \frac{p_b}{p_0} = 0.602 \qquad \text{(Not choked)}$$

Hence $\quad \frac{p_0}{p_t} = \left(1 + \frac{k-1}{2} \cdot M_t^2\right)^{\frac{k}{k-1}}$ where $\quad p_t = p_b \qquad p_t = 250 \cdot kPa$

so $\quad M_t = \sqrt{\frac{2}{k-1} \cdot \left[\left(\frac{p_0}{p_t}\right)^{\frac{k-1}{k}} - 1\right]} \qquad M_t = 0.883$

Also $\quad V_1 = M_1 \cdot c_1 = M_1 \cdot \sqrt{k \cdot R \cdot T_1}$ or $\quad T_1 = \frac{1}{k \cdot R} \cdot \left(\frac{V_1}{M_1}\right)^2 \qquad T_1 = 224\,K \qquad T_1 = -49.1 \cdot °C$

Then $\quad T_0 = T_1 \cdot \left(1 + \frac{k-1}{2} \cdot M_1^2\right) \qquad T_0 = 235\,K \qquad T_0 = -37.9 \cdot °C$

Hence $\quad T_t = \frac{T_0}{1 + \frac{k-1}{2} \cdot M_t^2} \qquad T_t = 204\,K \qquad T_t = -69.6 \cdot °C$

Then $\quad c_t = \sqrt{k \cdot R \cdot T_t} \qquad c_t = 286 \frac{m}{s}$

Finally $\quad V_t = M_t \cdot c_t \qquad V_t = 252 \frac{m}{s}$

Problem 13.20

[Difficulty: 2]

13.20 Air flows isentropically through a converging nozzle into a receiver in which the absolute pressure is 35 psia. The air enters the nozzle with negligible speed at a pressure of 60 psia and a temperature of 200°F. Determine the mass flow rate through the nozzle for a throat diameter of 4 in.

Given: Air flow in a converging nozzle

Find: Mass flow rate

Solution:

Basic equations: $m_{rate} = \rho \cdot V \cdot A$ $\quad p = \rho \cdot R \cdot T$ $\quad \dfrac{T_0}{T} = 1 + \dfrac{k-1}{2} \cdot M^2$ $\quad \dfrac{p_0}{p} = \left(1 + \dfrac{k-1}{2} \cdot M^2\right)^{\frac{k}{k-1}}$

Given or available data $\quad p_b = 35 \cdot psi$ $\quad p_0 = 60 \cdot psi$ $\quad T_0 = (200 + 460) \cdot R$ $\quad D_t = 4 \cdot in$

$k = 1.4$ $\quad R_{air} = 53.33 \cdot \dfrac{ft \cdot lbf}{lbm \cdot R}$ $\quad A_t = \dfrac{\pi}{4} \cdot D_t^2$ $\quad A_t = 0.0873 \cdot ft^2$

Since $\dfrac{p_b}{p_0} = 0.583$ is greater than 0.528, the nozzle is not choked and $\quad p_t = p_b$

Hence $\quad M_t = \sqrt{\dfrac{2}{k-1} \cdot \left[\left(\dfrac{p_0}{p_t}\right)^{\frac{k-1}{k}} - 1\right]}$ $\quad M_t = 0.912$

and $\quad T_t = \dfrac{T_0}{1 + \dfrac{k-1}{2} \cdot M_t^2}$ $\quad T_t = 566 \cdot R$ $\quad T_t = 106 \cdot °F$

$c_t = \sqrt{k \cdot R_{air} \cdot T_t}$ $\quad V_t = c_t$ $\quad V_t = 1166 \cdot \dfrac{ft}{s}$

$\rho_t = \dfrac{p_t}{R_{air} \cdot T_t}$ $\quad \rho_t = 5.19 \times 10^{-3} \cdot \dfrac{slug}{ft^3}$

$m_{rate} = \rho_t \cdot A_t \cdot V_t$ $\quad m_{rate} = 0.528 \cdot \dfrac{slug}{s}$ $\quad m_{rate} = 17.0 \cdot \dfrac{lbm}{s}$

Problem 13.21

[Difficulty: 2]

13.21 Air flows through a diverging duct. At the entrance to the duct, the Mach number is 1 and the area is 0.2 m^2. At the exit to the duct, the area is 0.5 m^2. What are the two possible exit Mach numbers for this duct?

Given: Data on flow in a passage

Find: Possible Mach numbers at downstream location

Solution:

The given or available data is:

$R =$	286.9	J/kg-K
$k =$	1.4	
$M_1 =$	1	
$A_1 =$	0.2	m^2
$A_2 =$	0.5	m^2

Equations and Computations:

Since the flow is sonic at the entrance:

$A^*_1 =$ 0.2 m^2

For isentropic flow ($A^*_2 = A^*_1$)

$A^*_2 =$ 0.2 m^2

$A_2/A^*_2 =$ 2.5

Now there are two Mach numbers which could result from this area change, one subsonic and one supersonic.

From A_2/A^*_2, and Eq. 13.7d

(using built-in functions)

$M_{2sub} =$ 0.2395

$M_{2sup} =$ 2.4428

Problem 13.22

[Difficulty: 3]

13.22 Air is flowing steadily through a series of three tanks. The first very large tank contains air at 650 kPa and 35°C. Air flows from it to a second tank through a converging nozzle with exit area 1 cm². Finally the air flows from the second tank to a third very large tank through an identical nozzle. The flow rate through the two nozzles is the same, and the flow in them is isentropic. The pressure in the third tank is 65 kPa. Find the mass flow rate, and the pressure in the second tank.

Given: Data on three tanks

Find: Mass flow rate; Pressure in second tank

Solution:

The given or available data is:

	$R =$	286.9	J/kg.K
	$k =$	1.4	
	$A_t =$	1	cm^2

We need to establish whether each nozzle is choked. There is a large total pressure drop so this is likely. However, BOTH cannot be choked and have the same flow rate. This is because Eq. 13.9a, below

$$\dot{m}_{\text{choked}} = 0.04\frac{A_e p_0}{\sqrt{T_0}} \qquad (13.9b)$$

indicates that the choked flow rate depends on stagnation temperature (which is constant) but also stagnation pressure, which drops because of turbulent mixing in the middle chamber. Hence BOTH nozzles cannot be choked. We assume the second one only is choked (why?) and verify later.

Temperature and pressure in tank 1:	$T_{01} =$	308	K
	$p_{01} =$	650	kPa
We make a guess at the pressure at the first nozzle exit:	$p_{e1} =$	527	kPa

NOTE: The value shown is the final answer! It was obtained using *Solver* !

This will also be tank 2 stagnation pressure:	$p_{02} =$	527	kPa
Pressure in tank 3:	$p_3 =$	65	kPa

Equations and Computations:

From the p_{e1} guess and Eq. 13.17a:	$M_{e1} =$	0.556		
Then at the first throat (Eq.13.7b):	$T_{e1} =$	290	K	
The density at the first throat (Ideal Gas) is:	$\rho_{e1} =$	6.33	kg/m^3	
Then c at the first throat (Eq. 12.18) is:	$c_{e1} =$	341	m/s	
Then V at the first throat is:	$V_{e1} =$	190	m/s	
Finally the mass flow rate is:	$m_{rate} =$	0.120	kg/s	First Nozzle!

For the presumed choked flow at the second nozzle we use Eq. 13.9a, with $T_{01} = T_{02}$ and p_{02}:

$m_{rate} =$ 0.120 kg/s Second Nozzle!

For the guess value for p_{e1} we compute the error between the two flow rates:

$\Delta m_{rate} =$ 0.000 kg/s

Use Solver to vary the guess value for p_{e1} to make this error zero!

Note that this could also be done manually.

Problem 13.23

[Difficulty: 2]

13.23 Air flowing isentropically through a converging nozzle discharges to the atmosphere. At the section where the absolute pressure is 250 kPa, the temperature is 20°C and the air speed is 200 m/s. Determine the nozzle throat pressure.

Solution:

Computing equations: $\frac{p_0}{p} = \left[1 + \frac{k-1}{2}M^2\right]^{k/k-1}$ $\qquad c = \sqrt{kRT}$

Assumptions: (1) steady flow
(2) isentropic flow in the nozzle
(3) uniform flow at a section
(4) ideal gas

The nozzle will be choked, ie $M_t = 1.0$ if $\frac{p_b}{p_0} \leq 0.528$

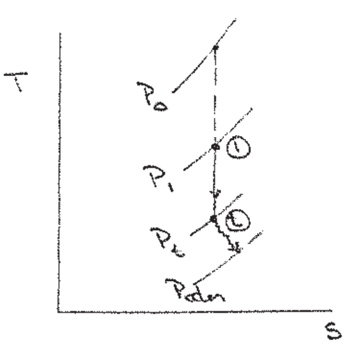
$M_1 = \frac{V_1}{c_1}$ $\quad c_1 = \sqrt{kRT_1} = \left(1.4 \times 287 \frac{N \cdot m}{kg \cdot K} \times 293K \times \frac{kg \cdot m}{N \cdot s^2}\right)^{1/2} = 343 \text{ m/s}$

$M_1 = \frac{V_1}{c_1} = \frac{200}{343} = 0.583$

$\frac{p_0}{p} = \left[1 + \frac{k-1}{2}M^2\right]^{k/k-1}$ $\quad p_0 = p_1\left[1 + \frac{k-1}{2}M_1^2\right]^{k/k-1} = 250\,kPa\left[1 + 0.2(0.583)^2\right]^{3.5} = 315 \text{ kPa}$

Then, $\frac{p_b}{p_0} = \frac{101}{315} = 0.321 < 0.528$ $\quad \therefore M_t = 1.0$

For $M_t = 1.0$, $\frac{p_t}{p_0} = 0.528$ $\quad p_t = 0.528\,p_0 = 0.528 \times 315\,kPa = 166\,kPa$ $\longleftarrow p_t$



Problem 13.24

[Difficulty: 2]

13.24 Air flows from a large tank (p = 650 kPa (abs), T = 550°C) through a converging nozzle, with a throat area of 600 mm^2, and discharges to the atmosphere. Determine the mass rate of flow for isentropic flow through the nozzle.

Solution:

Basic equations $\dot{m} = \rho V A = const$ $\quad p = \rho R T$

Computing equations: $\frac{T_0}{T} = 1 + \frac{k-1}{2}M^2$ $\quad \frac{p_0}{p} = \left[1 + \frac{k-1}{2}M^2\right]^{k/k-1}$ $\quad c = \sqrt{kRT}$

Assumptions: (1) steady flow (2) isentropic flow in nozzle (3) uniform flow at a section (4) ideal gas

Since $p_b/p_0 = \frac{101}{650} = 0.155 < 0.528$, the nozzle is choked and $M_t = 1.0$

From the continuity equation $\dot{m} = \rho V A$ and hence we need to determine ρ_t and V_t

$\frac{T_0}{T} = 1 + \frac{k-1}{2}M^2$; $T_t = \frac{T_0}{1 + \frac{k-1}{2}M_t^2} = \frac{823\text{ K}}{1 + 0.2(1.0)^2} = 686\text{ K}$

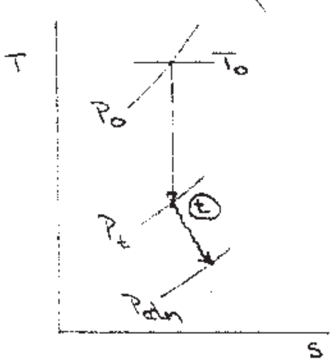
$V_t = M_t c_t = M_t (kRT_t)^{1/2} = 1.0\left(1.4 \times 287 \frac{N \cdot m}{kg \cdot K} \times 686 K \times \frac{kg \cdot m}{N \cdot s^2}\right)^{1/2} = 525 \text{ m/s}$

$\frac{p_0}{p} = \left[1 + \frac{k-1}{2}M^2\right]^{k/k-1}$; $p_t = \frac{p_0}{\left[1 + \frac{k-1}{2}M_t^2\right]^{k/k-1}} = \frac{650\text{ kPa}}{\left[1 + 0.2(1.0)^2\right]^{3.5}} = 343\text{ kPa}$

$\rho_t = \frac{p_t}{RT_t} = 343 \times 10^3 \frac{N}{m^2} \times \frac{kg \cdot K}{287 N \cdot m} \times \frac{1}{686 K} = 1.74 \text{ kg/m}^3$

Finally

$\dot{m} = \rho_t V_t A_t = 1.74 \frac{kg}{m^3} \times 525 \frac{m}{s} \times 6 \times 10^{-4} m^2 = 0.548 \text{ kg/s}$ $\qquad \dot{m}$



Problem 13.25

[Difficulty: 2]

13.25 Air flowing isentropically through a converging nozzle discharges to the atmosphere. At a section the area is $A = 0.05\ \text{m}^2$, $T = 3.3°\text{C}$, and $V = 200$ m/s. If the flow is just choked, find the pressure and the Mach number at this location. What is the throat area? What is the mass flow rate?

Given: Data on converging nozzle; isentropic flow

Find: Pressure and Mach number; throat area; mass flow rate

Solution:

The given or available data is:

$R =$	286.9	J/kg.K
$k =$	1.4	
$A_1 =$	0.05	m^2
$T_1 =$	276.3	K
$V_1 =$	200	m/s
$p_{atm} =$	101	kPa

Equations and Computations:

From T_1 and Eq. 12.18

$$c = \sqrt{kRT} \qquad (12.18)$$

$c_1 =$ 333 m/s

Then $M_1 =$ 0.60

To find the pressure, we first need the stagnation pressure.
If the flow is just choked

$p_e =$ $p_{atm} =$ $p^* =$ 101 kPa

From $p_e = p^*$ and Eq. 12.22a

$$\frac{p_0}{p^*} = \left[\frac{k+1}{2}\right]^{k/(k-1)} \qquad (12.22a)$$

$p_0 =$ 191 kPa

From M_1 and p_0, and Eq. 13.7a
(using built-in function *Isenp* (M, k))

$$\frac{p_0}{p} = \left[1 + \frac{k-1}{2}M^2\right]^{k/(k-1)} \qquad (13.7a)$$

Then $p_1 =$ 150 kPa

The mass flow rate is $m_{rate} = \rho_1 A_1 V_1$

Hence, we need ρ_1 from the ideal gas equation.

$$\rho_1 = 1.89 \quad \text{kg/m}^3$$

The mass flow rate m_{rate} is then

$$m_{rate} = 18.9 \quad \text{kg/s}$$

The throat area $A_t = A^*$ because the flow is choked.
From M_1 and A_1, and Eq. 13.7d
(using built-in function *IsenA* (M, k)

$$\frac{A}{A^*} = \frac{1}{M}\left[\frac{1 + \frac{k-1}{2}M^2}{\frac{k+1}{2}}\right]^{(k+1)/2(k-1)} \tag{13.7d}$$

$$A^* = 0.0421 \quad \text{m}^2$$

Hence $$A_t = 0.0421 \quad \text{m}^2$$

Problem 13.26

[Difficulty: 2]

13.26 A converging nozzle is connected to a large tank that contains compressed air at 15°C. The nozzle exit area is 0.001 m^2. The exhaust is discharged to the atmosphere. To obtain a satisfactory shadow photograph of the flow pattern leaving the nozzle exit, the pressure in the exit plane must be greater than 325 kPa (gage). What pressure is required in the tank? What mass flow rate of air must be supplied if the system is to run continuously? Show static and stagnation state points on a *Ts* diagram.

Solution:

Basic equations: $\dot{m} = \rho A V \qquad p = \rho R T$

Computing equations: $\frac{p_0}{p} = \left[1 + \frac{k-1}{2} M^2\right]^{k/k-1} \qquad \frac{T_0}{T} = 1 + \frac{k-1}{2} M^2$

Assumptions: (1) steady flow (2) isentropic flow (3) uniform flow at a section (4) ideal gas behavior

Since $p_e > p_b$, nozzle is choked and $M_e = 1.0$

$$p_0 = p_e \left[1 + \frac{k-1}{2} M_e^2\right]^{k/k-1} = 426 \text{ psia} \left[1 + 0.2\right]^{3.5} = 806 \text{ kPa} \quad \longleftarrow p_0$$

$$\frac{T_0}{T_e} = 1 + \frac{k-1}{2} M_e^2 \quad \therefore \quad T_e = \frac{T_0}{1.2} = \frac{288\text{ K}}{1.2} = 240\text{ K}$$

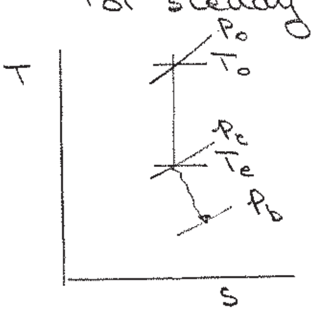
$$V_e = c_e = (kRT_e)^{1/2} = \left[1.4 \times 287 \frac{J}{kg \cdot K} \times 240\text{ K} \times \frac{N \cdot m}{J} \times \frac{kg \cdot m}{N \cdot s^2}\right]^{1/2} = 311 \text{ m/s}$$

$$\rho_e = \frac{p_e}{RT_e} = 426 \times 10^3 \frac{N}{m^2} \times \frac{kg \cdot K}{287\text{ J}} \times \frac{1}{240\text{ K}} \times \frac{J}{N \cdot m} = 6.18 \text{ kg/m}^3$$

Then

$$\dot{m} = \rho_e V_e A_e = 6.18 \frac{kg}{m^3} \times 311 \frac{m}{s} \times 0.001 m^2 = 1.92 \text{ kg/s}$$

For steady flow, $\dot{m} = 1.92$ kg/s must be supplied to the tank $\longleftarrow \dot{m}$



The corresponding volume flow rate of standard air is

$$Q = \frac{\dot{m}}{\rho_{std}} = 1.92 \frac{kg}{s} \times \frac{m^3}{6.18\text{ kg}}$$

$$Q = 0.31 \text{ m}^3/\text{s}$$

Problem 13.27

[Difficulty: 2]

13.27 Air, with $p_0 = 650$ kPa (abs) and $T_0 = 350$ K, flows isentropically through a converging nozzle. At the section in the nozzle where the area is 2.6×10^{-3} m^2, the Mach number is 0.5. The nozzle discharges to a back pressure of 270 kPa (abs). Determine the exit area of the nozzle.

Solution:

Computing equations:

$$p_0/p = \left[1 + \frac{k-1}{2}M^2\right]^{k/k-1}$$

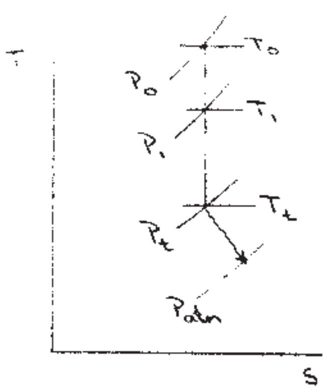
$$\frac{A}{A^*} = \frac{1}{M}\left[\frac{1+\frac{k-1}{2}M^2}{1+\frac{k-1}{2}}\right]^{(k+1)/2(k-1)} \qquad (12.6)$$

Assumptions:
(1) steady flow
(2) isentropic flow in nozzle
(3) uniform flow at a section
(4) ideal gas

Since $p_b/p_0 = \dfrac{270 \text{ kPa}}{650 \text{ kPa}} = 0.415 < 0.528$, the nozzle is choked and $M_t = 1.0$

From Eq 12.6 with $M_1 = 0.5$, $A_1/A^* = 1.340$

then $A_t = A^* = A_1/1.340 = \dfrac{2.6 \times 10^{-3} \text{ m}^2}{1.340} = 1.94 \times 10^{-3} \text{ m}^2 \longleftarrow A_t$



Problem 13.28

[Difficulty: 3]

13.28 Air flows through a converging duct. At the entrance, the static temperature is 450°R, the static pressure is 45 psia, the stagnation pressure is 51 psia, and the area is 4 ft^2. At the exit, the area is 3 ft^2. Assuming isentropic flow through the duct, what are the exit temperature and the mass flow rate of air through the duct?

Given: Data on flow in a passage

Find: Exit temperature and mass flow rate of air assuming isentropic flow

Solution:

The given or available data is:

$R =$	53.33	ft-lbf/lbm-°R
$k =$	1.4	
$T_1 =$	450	°R
$p_1 =$	45	psia
$p_{01} =$	51	psia
$A_1 =$	4	ft^2
$A_2 =$	3	ft^2

Equations and Computations:

From the static and stagnation pressures we can calculate M_1:

$$\frac{p_0}{p} = \left[1 + \frac{k-1}{2}M^2\right]^{k/(k-1)}$$

$M_1 =$ 0.427

From the M_1 and T_1 we can get T_{01}:

$T_{01} =$ 466.38 °R

From M_1, and Eq. 13.7d
(using built-in function $IsenA(M,k)$)

$$\frac{A}{A^*} = \frac{1}{M}\left[\frac{1 + \frac{k-1}{2}M^2}{\frac{k+1}{2}}\right]^{(k+1)/2(k-1)}$$

$A^*_1 =$ 2.649 ft^2

For isentropic flow ($p_{02} = p_{01}$, $T_{02} = T_{01}$, $A^*_2 = A^*_1$)

$p_{02} =$	51	psia
$T_{02} =$	466.38	°R
$A^*_2 =$	2.649	ft^2
$A_2/A^*_2 =$	1.1325	

Given subsonic flow in the duct, we can find the exit Mach number using Equation 13.7d

$M_2 =$ 0.653

From the Mach number and stagnation state we can calculate the static pressure and temperature:

$p_2 =$ 38.28 psia

$T_2 =$ 430 °R

From T_2 and Eq. 12.18 $c = \sqrt{kRT}$

$c_2 =$ 1016.38 ft/s

$V_2 =$ 664.11 ft/s

Using the ideal gas law we calculate the density at station 2:

$\rho_2 =$ 0.2406 lbm/ft^3

Now we can use the area, density, and velocity to calculate the mass flow rate:

$m =$ 479 lbm/s

Problem 13.29

[Difficulty: 2]

13.29 Air at 0°C is contained in a large tank on the space shuttle. A converging section with exit area 1×10^{-3} m^2 is attached to the tank, through which the air exits to space at a rate of 2 kg/s. What are the pressure in the tank, and the pressure, temperature, and speed at the exit?

Given: Temperature in and mass flow rate from a tank

Find: Tank pressure; pressure, temperature and speed at exit

Solution:

The given or available data is:

	$R =$	286.9	J/kg.K
	$k =$	1.4	
	$T_0 =$	273	K
	$A_t =$	0.001	m^2
	$m_{rate} =$	2	kg/s

Equations and Computations:

Because $p_b = 0$ $\quad p_e = p^*$

Hence the flow is choked!

Hence $\quad T_e = T^*$

From T_0, and Eq. 12.22b

$$\frac{T_0}{T^*} = \frac{k+1}{2} \qquad (12.22b)$$

$T^* =$ 228 K

$T_e =$ 228 K

-45.5 °C

Also	$M_e =$	1	
Hence	$V_e =$	$V^* =$	c_e

From T_e and Eq. 12.18

$$c = \sqrt{kRT} \qquad (12.18)$$

$c_e =$ 302 m/s

Then $V_e =$ 302 m/s

To find the exit pressure we use the ideal gas equation
after first finding the exit density.
The mass flow rate is $m_{rate} = \rho_e A_e V_e$

Hence $\rho_e =$ 6.62 kg/m^3

From the ideal gas equation $p_e = \rho_e R T_e$

$p_e =$ 432 kPa

From $p_e = p^*$ and Eq. 12.22a

$$\frac{p_0}{p^*} = \left[\frac{k+1}{2}\right]^{k/(k-1)} \qquad (12.22a)$$

$p_0 =$ 817 kPa

We can check our results:
From p_0, T_0, A_t, and Eq. 13.9a

$$\dot{m}_{choked} = A_e p_0 \sqrt{\frac{k}{RT_0}} \left(\frac{2}{k+1}\right)^{(k+1)/2(k-1)} \qquad (13.9a)$$

Then $m_{choked} =$ 2.00 kg/s

$m_{choked} =$ m_{rate} Correct!

Problem 13.30

[Difficulty: 2]

13.30 A large tank supplies air to a converging nozzle that discharges to atmospheric pressure. Assume the flow is reversible and adiabatic. For what range of tank pressures will the flow at the nozzle exit be sonic? If the tank pressure is 600 kPa (abs) and the temperature is 600 K, determine the mass flow rate through the nozzle, if the exit area is $1.29 \times 10^{-3}\ \text{m}^2$.

Solution:

Basic equations: $\dot{m} = \rho V A = \text{const}$ $\qquad p = \rho R T$

Computing equations: $\frac{T_0}{T} = 1 + \frac{k-1}{2} M^2$ $\qquad \frac{p_0}{p} = \left[1 + \frac{k-1}{2} M^2\right]^{k/k-1}$

Assumptions: (1) steady flow (2) isentropic flow in nozzle (3) uniform flow at a section (4) ideal gas

The nozzle will be choked, i.e. $M_t = 1.0$ for $p_b/p_0 \le 0.528$

Since $p_b = 101$ kPa, nozzle is choked for

$$p_0 \ge \frac{p_b}{0.528} = \frac{101\ \text{kPa}}{0.528} = 191\ \text{kPa}$$ p_0

Thus for $p_0 = 600$ kPa, $M_t = 1.0$

$$\frac{T_0}{T} = 1 + \frac{k-1}{2} M^2 \ ; \quad T_t = \frac{T_0}{1 + \frac{k-1}{2} M_t^2} = \frac{600\ \text{K}}{1 + 0.2(1.0)^2} = 500\ \text{K}$$

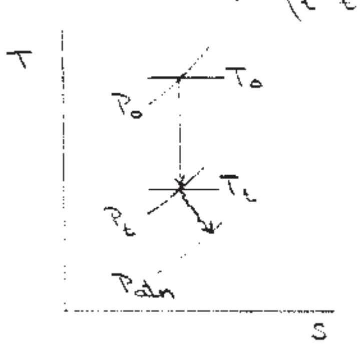
$$V_t = M_t c_t = M_t (kRT_t)^{1/2} = 1.0\left(1.4 \times 287 \frac{\text{N}\cdot\text{m}}{\text{kg}\cdot\text{K}} \times 500\,\text{K} \times \frac{\text{kg}\cdot\text{m}}{\text{N}\cdot\text{s}^2}\right)^{1/2} = 448\ \text{m/s}$$

$$\frac{p_0}{p} = \left[1 + \frac{k-1}{2} M^2\right]^{k/k-1} ; \quad p_t = \frac{p_0}{\left[1 + \frac{k-1}{2} M_t^2\right]^{k/k-1}} = \frac{600\ \text{kPa}}{\left[1 + 0.2(1.0)^2\right]^{3.5}} = 317\ \text{kPa}$$

$$\rho_t = \frac{p_t}{RT_t} = 317 \times 10^3 \frac{\text{N}}{\text{m}^2} \times \frac{\text{kg}\cdot\text{K}}{287\ \text{N}\cdot\text{m}} \times \frac{1}{500\,\text{K}} = 2.21\ \text{kg/m}^3$$

Finally,

$$\dot{m} = \rho_t V_t A_t = 2.21 \frac{\text{kg}}{\text{m}^3} \times 448 \frac{\text{m}}{\text{s}} \times 1.29 \times 10^{-3}\ \text{m}^2 = 1.28\ \text{kg/s}$$ $\dot{m}$



Problem 13.31

[Difficulty: 3]

13.31 Nitrogen is stored in a large chamber at 450 K and 150 kPa. The gas leaves the chamber through a converging-only nozzle with an outlet area of 30 cm^2. The ambient room pressure is 100 kPa, and the flow through the nozzle is isentropic. What is the mass flow rate of the nitrogen? If the room pressure could be lowered, what is the maximum possible mass flow rate for the nitrogen?

Given: Temperature and pressure in a tank; nozzle with specific area

Find: Mass flow rate of gas; maximum possible flow rate

Solution:

The given or available data is:

$R =$	296.8	J/kg.K
$k =$	1.4	
$T_0 =$	450	K
$p_0 =$	150	kPa
$A_t =$	30	cm^2
$A_t =$	0.003	m^2
$p_b =$	100	kPa

Equations and Computations:

Assuming that the nozzle exit pressure is the back pressure:

$p_e =$ 100 kPa

Then the nozzle exit Mach number is:

$M_e =$ 0.7837

This nozzle is not choked. The exit temperature is:

$T_e =$ 400.78 K

From T_e and Eq. 12.18

$$c = \sqrt{kRT} \qquad (12.18)$$

$c_e =$ 408.08 m/s

Then $V_e =$ 319.80 m/s

From the ideal gas equation of state, we can calculate the density:

$\rho_e =$ 0.8407 kg/m^3

Therefore the mass flow rate is:

$m =$ 0.807 kg/s

When the room pressure can be lowered, we can choke the nozzle.

$p_e = p^*$

$T_e = T^*$

From T_0, and Eq. 12.22b

$$\frac{T_0}{T^*} = \frac{k+1}{2} \qquad (12.22b)$$

$T^* =$ 375 K

$p^* =$ 79.24 kPa

$T_e =$ 375 K

Also $M_e =$ 1

Hence $V_e =$ $V^* =$ c_e

From T_e and Eq. 12.18 $c = \sqrt{kRT}$ (12.18)

$c_e =$ 395 m/s

Then $V_e =$ 395 m/s

To find the mass flow rate we calculate the density from the ideal gas equation of state:

Hence $\rho_e =$ 0.7120 kg/m^3

Therefore the mass flow rate is:

$m_{max} =$ 0.843 kg/s

We can check our results:
From p_0, T_0, A_t, and Eq. 13.9a

$$\dot{m}_{choked} = A_e p_0 \sqrt{\frac{k}{RT_0}} \left(\frac{2}{k+1} \right)^{(k+1)/2(k-1)} \qquad (13.9a)$$

Then $m_{choked} =$ 0.843 kg/s

$m_{choked} =$ m_{rate} Correct!

Problem 13.32

[Difficulty: 2]

13.32 A large tank initially is evacuated to −10 kPa (gage). (Ambient conditions are 101 kPa at 20°C.) At $t = 0$, an orifice of 5 mm diameter is opened in the tank wall; the vena contracta area is 65 percent of the geometric area. Calculate the mass flow rate at which air initially enters the tank. Show the process on a Ts diagram. Make a schematic plot of mass flow rate as a function of time. Explain why the plot is nonlinear.

Given: Isentropic air flow into a tank

Find: Initial mass flow rate; Ts process; explain nonlinear mass flow rate

Solution:

Basic equations: $\frac{T_0}{T} = 1 + \frac{k-1}{2}\cdot M^2$ $\qquad \frac{p_0}{p} = \left(1 + \frac{k-1}{2}\cdot M^2\right)^{\frac{k}{k-1}}$ $\qquad m_{rate} = \rho\cdot A\cdot V$

Given or available data $\quad p_0 = 101\cdot kPa \qquad p_b = p_0 - 10\cdot kPa \qquad p_b = 91\cdot kPa \qquad T_0 = (20 + 273)\cdot K$

$k = 1.4 \qquad R = 286.9\cdot\frac{J}{kg\cdot K} \qquad D = 5\cdot mm$

Then $\quad A = \frac{\pi}{4}\cdot D^2 \qquad A_{vena} = 65\cdot\%\cdot A \qquad A_{vena} = 12.8\cdot mm^2$

The flow will be choked if $p_b/p_0 < 0.528$ $\qquad \frac{p_b}{p_0} = 0.901 \qquad$ (Not choked)

Hence $\quad \frac{p_0}{p_{vena}} = \left(1 + \frac{k-1}{2}\cdot M^2\right)^{\frac{k}{k-1}}$ $\quad$ where $\quad p_{vena} = p_b \qquad p_{vena} = 91\cdot kPa$

so $\quad M_{vena} = \sqrt{\frac{2}{k-1}\cdot\left[\left(\frac{p_0}{p_{vena}}\right)^{\frac{k-1}{k}} - 1\right]} \qquad M_{vena} = 0.389$

Then $\quad T_{vena} = \frac{T_0}{1 + \frac{k-1}{2}\cdot M_{vena}^2} \qquad T_{vena} = 284\,K \qquad T_{vena} = 11.3\cdot °C$

Then $\quad c_{vena} = \sqrt{k\cdot R\cdot T_{vena}} \qquad c_{vena} = 338\frac{m}{s}$

and $\quad V_{vena} = M_{vena}\cdot c_{vena} \qquad V_{vena} = 131\frac{m}{s}$

Also $\quad \rho_{vena} = \frac{p_{vena}}{R\cdot T_{vena}} \qquad \rho_{vena} = 1.12\frac{kg}{m^3}$

Finally $\quad m_{rate} = \rho_{vena}\cdot A_{vena}\cdot V_{vena} \qquad m_{rate} = 1.87\times 10^{-3}\frac{kg}{s}$

The Ts diagram will be a vertical line (T decreases and s = const). After entering the tank there will be turbulent mixing (s increases) and the flow comes to rest (T increases). The mass flow rate versus time will look like the curved part of Fig. 13.6b; it is nonlinear because V AND ρ vary

Problem 13.33

[Difficulty: 3]

13.33 A 50-cm-diameter spherical cavity initially is evacuated. The cavity is to be filled with air for a combustion experiment. The pressure is to be 45 kPa (abs), measured after its temperature reaches T_{atm}. Assume the valve on the cavity is a converging nozzle with throat diameter of 1 mm, and the surrounding air is at standard conditions. For how long should the valve be opened to achieve the desired final pressure in the cavity? Calculate the entropy change for the air in the cavity.

Given:
Spherical cavity with valve

Find:
Time to reach desired pressure; Entropy change

Solution:

Basic equations:

$$\frac{T_0}{T} = 1 + \frac{k-1}{2}\cdot M^2 \qquad \frac{p_0}{p} = \left(1 + \frac{k-1}{2}\cdot M^2\right)^{\frac{k}{k-1}} \qquad \Delta s = c_p\cdot\ln\left(\frac{T_2}{T_1}\right) - R\cdot\ln\left(\frac{p_2}{p_1}\right)$$

$$p = \rho\cdot R\cdot T \qquad c = \sqrt{k\cdot R\cdot T} \qquad m_{rate} = \rho\cdot A\cdot V \qquad m_{choked} = A_t\cdot p_0\cdot\sqrt{\frac{k}{R\cdot T_0}}\cdot\left(\frac{2}{k+1}\right)^{\frac{k+1}{2\cdot(k-1)}}$$

Given or available data

$$p_0 = 101\cdot kPa \qquad T_{atm} = (20+273)\cdot K \qquad T_0 = T_{atm} \qquad d = 1\cdot mm \qquad D = 50\cdot cm$$

$$p_f = 45\cdot kPa \qquad T_f = T_{atm} \qquad k = 1.4 \qquad R = 286.9\cdot\frac{J}{kg\cdot K} \qquad c_p = 1004\cdot\frac{J}{kg\cdot K}$$

Then the inlet area is $A_t = \frac{\pi}{4}\cdot d^2$ $\quad A_t = 0.785\cdot mm^2$ $\quad$ and tank volume is $V = \frac{\pi}{3}\cdot D^3$ $\quad V = 0.131\cdot m^3$

The flow will be choked if $p_b/p_0 < 0.528$; the MAXIMUM back pressure is $p_b = p_f$ $\quad$ so $\quad \frac{p_b}{p_0} = 0.446$ $\quad$ (Choked)

The final density is $\rho_f = \frac{p_f}{R\cdot T_f}$ $\quad \rho_f = 0.535\,\frac{kg}{m^3}$ $\quad$ and final mass is $M = \rho_f\cdot V$ $\quad M = 0.0701\,kg$

Since the mass flow rate is constant (flow is always choked) $\quad M = m_{rate}\cdot\Delta t$ $\quad$ or $\quad \Delta t = \frac{M}{m_{rate}}$

We have choked flow so $\quad m_{rate} = A_t\cdot p_0\cdot\sqrt{\frac{k}{R\cdot T_0}}\cdot\left(\frac{2}{k+1}\right)^{\frac{k+1}{2\cdot(k-1)}}$ $\quad m_{rate} = 1.873\times 10^{-4}\,\frac{kg}{s}$

Hence $\quad \Delta t = \frac{M}{m_{rate}}$ $\quad \Delta t = 374\,s$ $\quad \Delta t = 6.23\cdot min$

The air in the tank will be cold when the valve is closed. Because $\rho = M/V$ is constant, $p = \rho RT = \text{const} \times T$, so as the temperature rises to ambient, the pressure will rise too.

For the entropy change during the charging process is given by $\Delta s = c_p\cdot\ln\left(\frac{T_2}{T_1}\right) - R\cdot\ln\left(\frac{p_2}{p_1}\right)$ where $\quad T_1 = T_{atm} \quad T_2 = T_{atm}$

and $\quad p_1 = p_0 \quad p_2 = p_f$ $\quad$ Hence $\quad \Delta s = c_p\cdot\ln\left(\frac{T_2}{T_1}\right) - R\cdot\ln\left(\frac{p_2}{p_1}\right)$ $\quad \Delta s = 232\cdot\frac{J}{kg\cdot K}$

Problem 13.34

[Difficulty: 3]

13.34 Air flows isentropically through a converging nozzle attached to a large tank, where the absolute pressure is 171 kPa and the temperature is 27°C. At the inlet section the Mach number is 0.2. The nozzle discharges to the atmosphere; the discharge area is 0.015 m². Determine the magnitude and direction of the force that must be applied to hold the nozzle in place.

Solution:

Basic equations: $F_{Sx} = P_1A_1 - P_2A_2 - P_{atm}(A_1 - A_2) - R_x = \dot{m}(V_2 - V_1)$

$\dot{m} = \rho V A = \text{const} \qquad P = \rho R T$

Computing equations: $\frac{T_0}{T} = 1 + \frac{k-1}{2}M^2 \qquad \frac{P_0}{P} = \left[1 + \frac{k-1}{2}M^2\right]^{k/k-1}$

Assumptions: (1) steady flow (2) isentropic flow (3) uniform flow at a section (4) ideal gas

$$M_2 = \left\{\frac{2}{k-1}\left[\left(\frac{P_0}{P_2}\right)^{k-1/k} - 1\right]\right\}^{1/2} = \left\{\frac{2}{0.4}\left[\left(\frac{171}{101}\right)^{0.286} - 1\right]\right\}^{1/2} = 0.901 \quad \text{Hence flow not choked}$$

$$T_2 = T_0 / \left[1 + \frac{k-1}{2}M_2^2\right] = 300\,K / [1 + 0.2(0.901)^2] = 258\,K$$

$$V_2 = M_2 c_2 = M_2 (kRT_2)^{1/2} = 0.901\left(1.4 \times 287\,\frac{N\cdot m}{kg\cdot K} \times 258\,K \times \frac{kg\cdot m}{N\cdot s^2}\right)^{1/2} = 290\,m/s$$

$$\rho_2 = \frac{P_2}{RT_2} = 101\times10^3\,\frac{N}{m^2} \times \frac{kg\cdot K}{287\,N\cdot m} \times \frac{1}{258\,K} = 1.36\,kg/m^3$$

$$\dot{m} = \rho_2 V_2 A_2 = 1.36\,\frac{kg}{m^3} \times 290\,\frac{m}{s} \times 0.015\,m^2 = 5.92\,kg/s$$

$$T_1 = T_0 / \left[1 + \frac{k-1}{2}M_1^2\right] = 300\,K / [1 + 0.2(0.2)^2] = 298\,K$$

$$V_1 = M_1 c_1 = M_1 (kRT_1) = 0.2\left(1.4 \times 287\,\frac{N\cdot m}{kg\cdot K} \times 298\,K \times \frac{kg\cdot m}{N\cdot s^2}\right)^{1/2} = 69.2\,m/s$$

$$P_1 = P_0 / \left[1 + \frac{k-1}{2}M_1^2\right]^{k/k-1} = 171\,kPa / [1 + 0.2(0.2)^2]^{3.5} = 166\,kPa$$

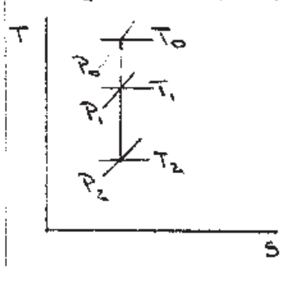
$$\rho_1 = \frac{P_1}{RT_1} = 166\times10^3\,\frac{N}{m^2} \times \frac{kg\cdot K}{287\,N\cdot m} \times \frac{1}{298\,K} = 1.94\,kg/m^3$$

$$A_1 = \dot{m}/\rho_1 V_1 = 5.92\,\frac{kg}{s} \times \frac{m^3}{1.94\,kg} \times \frac{s}{69.2\,m} = 0.0441\,m^2$$

$$R_x = P_1A_1 - P_2A_2 - P_{atm}(A_1 - A_2) - \dot{m}(V_2 - V_1) = P_{1g}A_1 - \cancel{P_{2g}}^{\,0}A_2 - \dot{m}(V_2 - V_1)$$

$$= (166 - 101)\times10^3\,\frac{N}{m^2} \times 0.0441\,m^2 - 5.92\,\frac{kg}{s}(290 - 69.2)\,\frac{m}{s} \times \frac{N\cdot s^2}{kg\cdot m}$$

$R_x = 1560\,N$ (to the left) R_x



13.35 Consider a "rocket cart" propelled by a jet supplied from a tank of compressed air on the cart. Initially, air in the tank is at 1.3 MPa (abs) and 20°C, and the mass of the cart and tank is $M_0 = 25$ kg. The air exhausts through a converging nozzle with exit area $A_e = 30$ mm^2. Rolling resistance of the cart is $F_R = 6$N; aerodynamic resistance is negligible. For the instant after air begins to flow through the nozzle: (a) compute the pressure in the nozzle exit plane, (b) evaluate the mass flow rate of air through the nozzle, and (c) calculate the acceleration of the tank and cart assembly.

Solution: Assume steady, one-dimensional flow of an ideal gas.

Computing equations: $\frac{T_0}{T} = 1 + \frac{k-1}{2}M^2$; $\frac{p_0}{p} = \left(1+\frac{k-1}{2}M^2\right)^{\frac{k}{k-1}}$; $c = \sqrt{kRT}$; $p = \rho RT$

Check for choking: $\frac{p_{atm}}{p_0} = \frac{101 \times 10^3}{1.3 \times 10^6} = 0.0777 < 0.528$, so Choked. $M_e = 1$

Thus
$$T_e = \frac{T_0}{1+\frac{k-1}{2}M_e^2} = \frac{(273+20)\,K}{1+\frac{k-1}{2}} = 244\ K$$

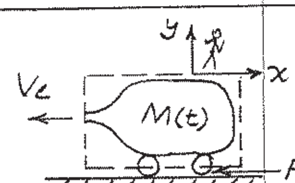
$$p_e = \frac{p_0}{\left(1+\frac{k-1}{2}M_e^2\right)^{k/k-1}} = \frac{1.3 \times 10^6\ Pa}{\left(1+\frac{k-1}{2}\right)^{k/k-1}} = \frac{1.3\times 10^6\ Pa}{(1.2)^{3.5}} = 687\ kPa\ (abs)$$ p_e

$\dot{m} = \rho_e V_e A_e$; $\rho_e = \frac{p_e}{RT_e} = 687 \times 10^3 \frac{N}{m^2} \times \frac{kg \cdot K}{287\, N \cdot m} \times \frac{1}{244\, K} = 9.81\ kg/m^3$

$$V_e = M_e c_e = c_e = \sqrt{kRT_e} = \left[1.4 \times 287 \frac{N \cdot m}{kg \cdot K} \times 244 K \times \frac{kg \cdot m}{N \cdot s^2}\right]^{1/2} = 313\ m/s$$

$$\dot{m} = 9.81 \frac{kg}{m^3} \times 313 \frac{m}{s} \times 30 \times 10^{-6} m^2 = 0.0921\ kg/s$$ $\dot{m}$

Apply momentum to find acceleration of assembly.



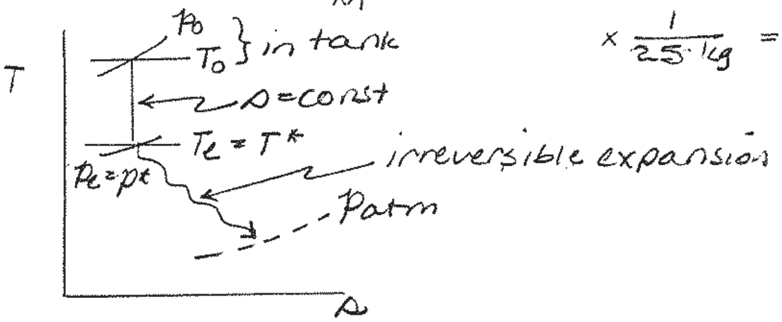
Basic equation: $\overset{=0(4)}{F_{Sx} + F_{Bx}} - \int_{CV} a_{rfx}\,\rho\, d\forall = \overset{\approx 0(5)}{\frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall} + \int_{CS} u \rho \vec{V}\cdot d\vec{A}$

Assume: (4) Horizontal; (5) $u \approx 0$ within CV; (6) Uniform flow at exit.

Then $-F_R + (p_e - p_{atm})A_e - M a_{rfx} = u_e\{+\dot{m}\} = -V_e \dot{m}$

$$u_e = -V_e$$

$$a_{rfx} = \frac{V_e \dot{m} - F_R + (p_e - p_{atm})A_e}{M} = \left[313 \frac{m}{s} \times 0.0921 \frac{kg}{s} - 6\ N + (687-101)10^3 \frac{N}{m^2} \times 30 \times 10^{-6} m^2\right] \times \frac{1}{25\ kg} = 1.62\ m/s^2$$ a_{rfx}



Problem 13.36

[Difficulty: 3]

13.36 A stream of air flowing in a duct ($A = 5 \times 10^{-4}$ m²) is at p = 300 kPa (abs), has M = 0.5, and flows at $\dot{m}$ = 0.25 kg/s. Determine the local isentropic stagnation temperature. If the cross-sectional area of the passage were reduced downstream, determine the maximum percentage reduction of area allowable without reducing the flow rate (assume isentropic flow). Determine the speed and pressure at the minimum area location.

Solution:

Basic equations: $\dot{m} = \rho V A \qquad p = \rho R T$

Computing equations: $\dfrac{T_0}{T} = 1 + \dfrac{k-1}{2}M^2 \qquad \dfrac{p_0}{p} = \left[1 + \dfrac{k-1}{2}M^2\right]^{k/(k-1)}$

$$\frac{A}{A^*} = \frac{1}{M}\left[\frac{1+\frac{k-1}{2}M^2}{1+\frac{k-1}{2}}\right]^{(k+1)/2(k-1)} \qquad (12.7d)$$

Assumptions: (1) steady flow (2) isentropic flow (3) uniform flow at a section (4) ideal gas

To determine T_0, we first need to find T_1

$$\dot{m} = \rho_1 V_1 A_1 = \frac{p_1}{RT_1} M_1 c_1 A_1 = \frac{p_1}{RT_1} M_1 (kRT_1)^{1/2} A_1 = p_1 M_1 A_1 \left(\frac{k}{RT_1}\right)^{1/2}$$

Solving for T_1,

$$T_1 = \frac{k}{R}\left(\frac{p_1 M_1 A_1}{\dot{m}}\right)^2 = 1.4 \times \frac{kg \cdot K}{287\,J}\left[300\times10^3 \frac{N}{m^2} \times 0.5 \times 5\times10^{-4}\,m^2 \times \frac{s}{0.25\,kg} \times \frac{kg\cdot m}{N\cdot s^2}\right]^2 \frac{N\cdot s^2}{kg\cdot m}$$

$T_1 = 439\,K$

$T_{0_1} = T_1\left[1 + \frac{k-1}{2}M_1^2\right] = 439\,K\left[1 + 0.2(0.5)^2\right] = 461\,K$ —— T_0

Maximum area reduction occurs where $M_2 = 1$.

For $M_1 = 0.5$, from Eq 12.7d $A/A^* = 1.34 \quad \therefore A^* = A_2 = \frac{A_1}{1.34} = 3.73\times10^{-4}\,m^2$

$\%\Delta A = \frac{A_2 - A_1}{A_1}\times100 = \frac{3.73-5}{5}\times100 = -25.4\%$ —— $\%\Delta A$

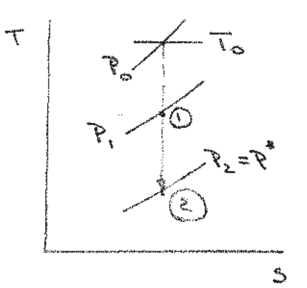
$T_2 = T_0 / \left[1 + \frac{k-1}{2}M_2^2\right] = 461\,K / \left[1 + 0.2(1)^2\right] = 384\,K$

$V_2 = M_2 c_2 = M_2 (kRT_2)^{1/2} = 1.0\left(1.4 \times 287 \frac{J}{kg\cdot K} \times 384\,K \times \frac{kg\cdot m}{N\cdot s^2}\right)^{1/2}$

$V_2 = 393\,m/s$ —— V_2

$p_{0_1} = p_{0_2} = p_1\left[1 + \frac{k-1}{2}M_1^2\right]^{k/(k-1)} = 300\,kPa\left[1 + 0.2(0.5)^2\right]^{3.5} = 356\,kPa$

$p_2 = p_{0_2} / \left[1 + \frac{k-1}{2}M_2^2\right]^{k/(k-1)} = 356\,kPa / \left[1 + 0.2(1)^2\right]^{3.5} = 188\,kPa$ —— p_2



Problem 13.37

[Difficulty: 3]

13.37 An air-jet-driven experimental rocket of 25 kg mass is to be launched from the space shuttle into space. The temperature of the air in the rocket's tank is 125°C. A converging section with exit area 25 mm^2 is attached to the tank, through which the air exits to space at a rate of 0.05 kg/s. What is the pressure in the tank, and the pressure, temperature, and air speed at the exit when the rocket is first released? What is the initial acceleration of the rocket?

Given: Air-driven rocket in space

Find: Tank pressure; pressure, temperature and speed at exit; initial acceleration

Solution:

The given or available data is:

$R =$	286.9	J/kg.K
$k =$	1.4	
$T_0 =$	398	K
$A_t =$	25	mm^2
$M =$	25	kg
$m_{rate} =$	0.05	kg/s

Equations and Computations:

Because $p_b = 0$ $\quad p_e = p^*$

Hence the flow is choked!

Hence $\quad T_e = T^*$

From T_0, and Eq. 12.22b

$$\frac{T_0}{T^*} = \frac{k+1}{2} \qquad (12.22b)$$

$T^* =$ 332 K

$T_e =$ 332 K

58.7 °C

Also $\quad M_e = 1$

Hence $\quad V_e = V^* = c_e$

From T_e and Eq. 12.18

$$c = \sqrt{kRT} \qquad (12.18)$$

$c_e =$ 365 m/s

Then $\quad V_e =$ 365 m/s

To find the exit pressure we use the ideal gas equation
after first finding the exit density.
The mass flow rate is $m_{rate} = \rho_e A_e V_e$

Hence $\rho_e =$ 0.0548 kg/m^3

From the ideal gas equation $p_e = \rho_e R T_e$

$p_e =$ 5.21 kPa

From $p_e = p^*$ and Eq. 12.22a

$$\frac{p_0}{p^*} = \left[\frac{k+1}{2}\right]^{k/(k-1)} \quad (12.22a)$$

$p_0 =$ 9.87 kPa

We can check our results:
From p_0, T_0, A_t, and Eq. 13.9a

$$\dot{m}_{\text{choked}} = A_e p_0 \sqrt{\frac{k}{RT_0}} \left(\frac{2}{k+1}\right)^{(k+1)/2(k-1)} \quad (13.9a)$$

Then $m_{choked} =$ 0.050 kg/s

$m_{choked} = m_{rate}$ Correct!

The initial acceleration is given by:

$$\vec{F} - \int_{CV} \vec{a}_{rf}\, \rho\, d\forall = \frac{\partial}{\partial t}\int_{CV} \vec{V}_{xyz}\, \rho\, d\forall + \int_{CS} \vec{V}_{xyz}\, \rho \vec{V}_{xyz} \cdot d\vec{A} \quad (4.33)$$

which simplifies to: $p_e A_t - M a_x = m_{rate} V$ or: $a_x = \dfrac{m_{rate} V + p_e A_t}{M}$

$a_x =$ 1.25 m/s^2

Problem 13.38

[Difficulty: 3]

13.38 Air enters a converging-diverging nozzle at 2 MPa (abs) and 313 K. At the exit of the nozzle, the pressure is 200 kPa (abs). Assume adiabatic, frictionless flow through the nozzle. The throat area is 20 cm^2. What is the area at the nozzle exit? What is the mass flow rate of the air?

Given: Air flow through a converging-diverging nozzle

Find: Nozzle exit area and mass flow rate

Solution:

The given or available data is:

	$R =$	286.9	J/kg-K
	$k =$	1.4	
	$p_0 =$	2	MPa
	$T_0 =$	313	K
	$p_e =$	200	kPa
	$A_t =$	20	cm^2

Equations and Computations:

Using the stagnation to exit static pressure ratio, we can find the exit Mach number:
(using built-in function *Isenp* (M, k))

$$\frac{p_0}{p} = \left[1 + \frac{k-1}{2} M^2\right]^{k/(k-1)}$$

$M_e =$ 2.1572

From M_e, and Eq. 13.7d
(using built-in function *IsenA* (M, k))

$$\frac{A}{A^*} = \frac{1}{M}\left[\frac{1 + \frac{k-1}{2} M^2}{\frac{k+1}{2}}\right]^{(k+1)/2(k-1)}$$

$A_e/A^* =$ 1.9307

At the throat the flow is sonic, so At = A*. Therefore:

$A_e =$ 38.6 cm^2

To find the mass flow rate at the exit, we will use the choked flow equation:
From p_0, T_0, A_t, and Eq. 13.9a

$$\dot{m}_{\text{choked}} = A_e p_0 \sqrt{\frac{k}{RT_0}} \left(\frac{2}{k+1}\right)^{(k+1)/2(k-1)} \qquad (13.9a)$$

$m =$ 17.646 kg/s

Problem 13.39

[Difficulty: 1]

13.39 Hydrogen is expanded adiabatically, without friction from 100 psia, at 540°F, and at negligible velocity to 20 psia via a converging-diverging nozzle. What is the exit Mach number?

Given: Hydrogen flow through a converging-diverging nozzle

Find: Nozzle exit Mach number

Solution:

The given or available data is:

$R =$	766.5	ft-lbf/lbm-°R
$k =$	1.41	
$p_0 =$	100	psia
$T_0 =$	540	°F
$T_0 =$	1000	°R
$p_e =$	20	psia

Equations and Computations:

Using the stagnation to exit static pressure ratio, we can find the exit Mach number:
(using built-in function *Isenp* (M, k))

$$\frac{p_0}{p} = \left[1 + \frac{k-1}{2} M^2\right]^{k/(k-1)}$$

$M_e =$ 1.706

Problem 13.40

[Difficulty: 3]

13.40 A cylinder of gas used for welding contains helium at 20 MPa (gage) and room temperature. The cylinder is knocked over, its valve is broken off, and gas escapes through a converging passage. The minimum flow diameter is 10 mm at the outlet section where the gas flow is uniform. Find (a) the mass flow rate at which gas leaves the cylinder and (b) the instantaneous acceleration of the cylinder (assume the cylinder axis is horizontal and its mass is 65 kg). Show static and stagnation states and the process path on a *Ts* diagram.

Given: Gas cylinder with broken valve

Find: Mass flow rate; acceleration of cylinder

Solution:

Basic equations:

$$\frac{T_0}{T} = 1 + \frac{k-1}{2}\cdot M^2 \qquad \frac{p_0}{p} = \left(1 + \frac{k-1}{2}\cdot M^2\right)^{\frac{k}{k-1}} \qquad p = \rho\cdot R\cdot T \qquad c = \sqrt{k\cdot R\cdot T} \qquad m_{rate} = \rho\cdot A\cdot V$$

$$\vec{F}_S + \vec{F}_B - \int_{CV} \vec{a}_{rf}\,\rho\, d\forall = \frac{\partial}{\partial t}\int_{CV} \vec{V}_{xyz}\,\rho\, d\forall + \int_{CS} \vec{V}_{xyz}\,\rho\,\vec{V}_{xyz}\cdot d\vec{A} \qquad (4.33)$$

Given or available data $p_{atm} = 101\cdot kPa$ $\quad p_0 = 20\cdot MPa + p_{atm} = 20.101\,MPa$ $\quad T_0 = (20 + 273)\cdot K$

$k = 1.66$ $\quad R = 2077\cdot\frac{J}{kg\cdot K}$ $\quad d = 10\cdot mm$ so the nozzle area is $A_e = \frac{\pi}{4}\cdot d^2$ $\quad A_e = 78.5\cdot mm^2$ $\quad M_{CV} = 65\cdot kg$

The flow will be choked if $p_b/p_0 < 0.528$: $\quad p_b = p_{atm}$ so $\frac{p_b}{p_0} = 5.025\times 10^{-3}$ (Choked: Critical conditions)

The exit temperature is $T_e = \frac{T_0}{\left(1 + \frac{k-1}{2}\right)}$ $\quad T_e = 220\,K$ $\quad T_e = -52.8\cdot °C$ $\quad c_e = \sqrt{k\cdot R\cdot T_e}$

The exit speed is $V_e = c_e$ $\quad V_e = 872\,\frac{m}{s}$

The exit pressure is $p_e = \frac{p_0}{\left(1 + \frac{k-1}{2}\right)^{\frac{k}{k-1}}}$ $\quad p_e = 9.8\cdot MPa$ and exit density is $\rho_e = \frac{p_e}{R\cdot T_e}$ $\quad \rho_e = 21\,\frac{kg}{m^3}$

Then $m_{rate} = \rho_e\cdot A_e\cdot V_e$ $\quad m_{rate} = 1.468\,\frac{kg}{s}$

The momentum equation (Eq. 4.33) simplifies to $\left(p_e - p_{atm}\right)\cdot A_e - M_{CV}\cdot a_x = -V_e\cdot m_{rate}$

Hence $a_x = \frac{\left(p_e - p_{atm}\right)\cdot A_e + V_e\cdot m_{rate}}{M_{CV}}$ $\quad a_x = 31.4\,\frac{m}{s^2}$

The process is isentropic, followed by nonisentropic expansion to atmospheric pressure

Problem 13.41

[Difficulty: 3]

13.41 A converging nozzle is bolted to the side of a large tank. Air inside the tank is maintained at a constant 50 psia and 100°F. The inlet area of the nozzle is 10 in.^{2} and the exit area is 1 in.^{2} The nozzle discharges to the atmosphere. For isentropic flow in the nozzle, determine the total force on the bolts, and indicate whether the bolts are in tension or compression.

Solution:

Basic equations: $F_{Sx} = P_1A_1 - P_2A_2 - P_{atm}(A_1 - A_2) - R_x = \dot{m}(V_2 - V_1)$

$\dot{m} = \rho V A = \text{const}$ $\quad P = \rho R T$

Computing equations: $\frac{T_0}{T} = 1 + \frac{k-1}{2}M^2$ $\quad \frac{P_0}{P} = \left[1 + \frac{k-1}{2}M^2\right]^{k/k-1}$

Assumptions: (1) steady flow
(2) isentropic flow in nozzle
(3) uniform flow at a section
(4) $F_{Bx} = 0$
(5) $V_1 \approx 0$

First check for choking

$\frac{P_b}{P_0} = \frac{14.7}{50} = 0.294 < 0.528$ and hence the nozzle is choked

$M_2 = 1.0$ and $P_2 = 0.528\ P_0 = 0.528(50\ \text{psia}) = 26.4$ psia

$T_0/T = 1 + \frac{k-1}{2}M^2$; $T_2 = T_0 / [1 + \frac{k-1}{2}M_2^2] = 560°R / [1 + 0.2(1.0)^2] = 467°R$

$\rho_2 = \frac{P_2}{RT_2} = 26.4\frac{lb}{in^2} \times \frac{lbm \cdot °R}{53.3\ ft \cdot lbf} \times \frac{1}{467°R} \times \frac{144\ in^2}{ft^2} = 0.153\ lbm/ft^3$

$V_2 = M_2 c_2 = M_2(kRT_2)^{1/2} = 1.0\left(1.4 \times 53.3\frac{ft \cdot lbf}{lbm \cdot °R} \times 467°R \times 32.2\frac{lbm}{slug} \times \frac{slug \cdot ft}{lbf \cdot s^2}\right)^{1/2} = 1060\ ft/s$

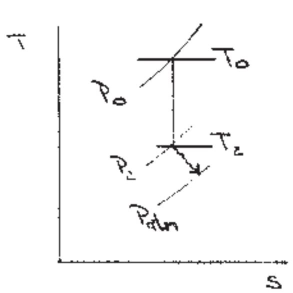
$\dot{m} = \rho_2 V_2 A_2 = 0.153\frac{lbm}{ft^3} \times 1060\frac{ft}{s} \times 1.0\ in^2 \times \frac{ft^2}{144\ in^2} = 1.13\ lbm/s$

$R_x = P_1A_1 - P_2A_2 - P_{atm}(A_1 - A_2) - \dot{m}(V_2 - \overset{\approx 0}{V_1}) = P_{1g}A_1 - P_{2g}A_2 - \dot{m}V_2$

$R_x = (50 - 14.7)\frac{lbf}{in^2} \times 10\ in^2 - (26.4 - 14.7)\frac{lbf}{in^2} \times 1.0\ in^2 - 1.13\frac{lbm}{s} \times 1060\frac{ft}{s} \times \frac{slug}{32.2\ lbm} \times \frac{lbf \cdot s^2}{ft \cdot slug}$

$R_x = 304$ lbf Since R_x acts to the left on CV, bolts are in tension

R_x



Problem 13.42

[Difficulty: 4]

13.42 An insulated spherical air tank with diameter $D = 2$ m is used in a blowdown installation. Initially the tank is charged to 2.75 MPa (abs) at 450 K. The mass flow rate of air from the tank is a function of time; during the first 30 s of blowdown 30 kg of air leaves the tank. Determine the air temperature in the tank after 30 s of blowdown. Estimate the nozzle throat area.

Given: Spherical air tank

Find: Air temperature after 30s; estimate throat area

Solution:

Basic equations: $\frac{T_0}{T} = 1 + \frac{k-1}{2}\cdot M^2$ $\quad \frac{p}{\rho^k} = \text{const}$ $\quad \frac{\partial}{\partial t}\int \rho\, dV_{CV} + \int \rho\cdot\vec{V}\, d\vec{A}_{CS} = 0$ (4.12)

Assumptions: 1) Large tank (stagnation conditions) 2) isentropic 3) uniform flow

Given or available data $\quad p_{atm} = 101\cdot kPa \quad p_1 = 2.75\cdot MPa \quad T_1 = 450\cdot K \quad D = 2\cdot m \quad V = \frac{\pi}{6}\cdot D^3 \quad V = 4.19\cdot m^3$

$\Delta M = 30\cdot kg \quad \Delta t = 30\cdot s \quad k = 1.4 \quad R = 286.9\frac{J}{kg\cdot K}$

The flow will be choked if $p_b/p_1 < 0.528$: $\quad p_b = p_{atm}$ so $\frac{p_b}{p_1} = 0.037$ (Initially choked: Critical conditions)

We need to see if the flow is still choked after 30s

The initial (State 1) density and mass are $\quad \rho_1 = \frac{p_1}{R\cdot T_1} \quad \rho_1 = 21.3\frac{kg}{m^3} \quad M_1 = \rho_1\cdot V \quad M_1 = 89.2\,kg$

The final (State 2) mass and density are then $\quad M_2 = M_1 - \Delta M \quad M_2 = 59.2\,kg \quad \rho_2 = \frac{M_2}{V} \quad \rho_2 = 14.1\frac{kg}{m^3}$

For an isentropic process $\quad \frac{p}{\rho^k} = \text{const}$ so $\quad p_2 = p_1\cdot\left(\frac{\rho_2}{\rho_1}\right)^k \quad p_2 = 1.55\cdot MPa \quad \frac{p_b}{p_2} = 0.0652$ (Still choked)

The final temperature is $\quad T_2 = \frac{p_2}{\rho_2\cdot R} \quad T_2 = 382\,K \quad T_2 = 109\cdot°C$

To estimate the throat area we use $\quad \frac{\Delta M}{\Delta t} = m_{tave} = \rho_{tave}\cdot A_t\cdot V_{tave}$ or $\quad A_t = \frac{\Delta M}{\Delta t\cdot\rho_{tave}\cdot V_{tave}}$

where we use average values of density and speed at the throat.

The average stagnation temperature is $\quad T_{0ave} = \frac{T_1 + T_2}{2} \quad T_{0ave} = 416\,K$

The average stagnation pressure is $\quad p_{0ave} = \frac{p_1 + p_2}{2} \quad p_{0ave} = 2.15\cdot MPa$

Hence the average temperature and pressure (critical) at the throat are

$$T_{tave} = \frac{T_{0ave}}{\left(1 + \frac{k-1}{2}\right)} \qquad T_{tave} = 347\,K \qquad \text{and} \qquad p_{tave} = \frac{p_{0ave}}{\left(1 + \frac{k-1}{2}\right)^{\frac{k}{k-1}}} \qquad p_{tave} = 1.14 \cdot MPa$$

Hence $\quad V_{tave} = \sqrt{k \cdot R \cdot T_{tave}} \qquad V_{tave} = 373\,\frac{m}{s} \qquad \rho_{tave} = \frac{p_{tave}}{R \cdot T_{tave}} \qquad \rho_{tave} = 11.4\,\frac{kg}{m^3}$

Finally $\quad A_t = \frac{\Delta M}{\Delta t \cdot \rho_{tave} \cdot V_{tave}} \qquad A_t = 2.35 \times 10^{-4}\,m^2 \qquad A_t = 235 \cdot mm^2$

This corresponds to a diameter

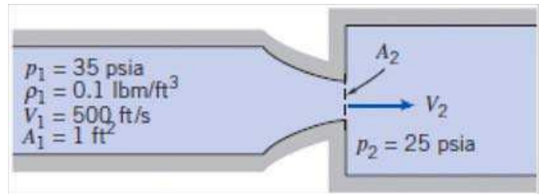
$$D_t = \sqrt{\frac{4 \cdot A_t}{\pi}} \qquad D_t = 0.0173\,m \qquad D_t = 17.3 \cdot mm$$

The process is isentropic, followed by nonisentropic expansion to atmospheric pressure

Problem 13.43

[Difficulty: 4]

13.43 An ideal gas, with $k = 1.25$, flows isentropically through the converging nozzle shown and discharges into a large duct where the pressure is $p_2 = 25$ psia. The gas is *not* air and the gas constant, R, is unknown. Flow is steady and uniform at all cross sections. Find the exit area of the nozzle, A_2, and the exit speed, V_2.



Given: Ideal gas flow in a converging nozzle

Find: Exit area and speed

Solution:

Basic equations:
$$\frac{T_0}{T} = 1 + \frac{k-1}{2}\cdot M^2 \qquad \frac{p_0}{p} = \left(1 + \frac{k-1}{2}\cdot M^2\right)^{\frac{k}{k-1}} \qquad \frac{A}{A_{crit}} = \frac{1}{M}\cdot\left(\frac{1 + \frac{k-1}{2}\cdot M^2}{\frac{k+1}{2}}\right)^{\frac{k+1}{2\cdot(k-1)}}$$

Given or available data
$$p_1 = 35\cdot psi \qquad \rho_1 = 0.1\cdot\frac{lbm}{ft^3} \qquad V_1 = 500\cdot\frac{ft}{s} \qquad A_1 = 1\cdot ft^2 \qquad p_2 = 25\cdot psi \qquad k = 1.25$$

Check for choking: $c_1 = \sqrt{k\cdot R\cdot T_1}$ or, replacing R using the ideal gas equation $c_1 = \sqrt{k\cdot\frac{p_1}{\rho_1}}$ $\quad c_1 = 1424\cdot\frac{ft}{s}$

Hence
$$M_1 = \frac{V_1}{c_1} \qquad M_1 = 0.351$$

Then
$$p_0 = p_1\cdot\left(1 + \frac{k-1}{2}\cdot M_1^{\,2}\right)^{\frac{k}{k-1}} \qquad p_0 = 37.8\cdot psi$$

The critical pressure is then
$$p_{crit} = \frac{p_0}{\left(\frac{k+1}{2}\right)^{\frac{k}{k-1}}} \qquad p_{crit} = 21.0\cdot psi$$
Hence $p_2 > p_{crit}$, so NOT choked

Then we have
$$M_2 = \sqrt{\frac{2}{k-1}\cdot\left[\left(\frac{p_0}{p_2}\right)^{\frac{k-1}{k}} - 1\right]} \qquad M_2 = 0.830$$

From M_1 we find
$$A_{crit} = \frac{M_1\cdot A_1}{\left(\frac{1 + \frac{k-1}{2}\cdot M_1^{\,2}}{\frac{k+1}{2}}\right)^{\frac{k+1}{2\cdot(k-1)}}} \qquad A_{crit} = 0.557\cdot ft^2 \qquad A_2 = \frac{A_{crit}}{M_2}\cdot\left(\frac{1 + \frac{k-1}{2}\cdot M_2^{\,2}}{\frac{k+1}{2}}\right)^{\frac{k+1}{2\cdot(k-1)}} \qquad A_2 = 0.573\cdot ft^2$$

For isentropic flow $\quad p\cdot\rho^k = const \quad$ so
$$\rho_2 = \rho_1\cdot\left(\frac{p_1}{p_2}\right)^{\frac{1}{k}} \qquad \rho_2 = 0.131\cdot\frac{lbm}{ft^3}$$

Finally from continuity $\quad \rho\cdot A\cdot V = const \quad$ so
$$V_2 = V_1\cdot\frac{A_1\cdot\rho_1}{A_2\cdot\rho_2} \qquad V_2 = 667\cdot\frac{ft}{s}$$

13.44 A jet transport aircraft, with pressurized cabin, cruises at 11 km altitude. The cabin temperature and pressure initially are at 25°C and equivalent to 2.5 km altitude. The interior volume of the cabin is 25 m^3. Air escapes through a small hole with effective flow area of 0.002 m^2. Calculate the time required for the cabin pressure to decrease by 40 percent. Plot the cabin pressure as a function of time.

Solution:

Basic equations: $\frac{\partial}{\partial t}\int_{cv} \rho\, d\forall + \int_{cs} \rho \vec{V} \cdot d\vec{A} = 0$

$\frac{p}{\rho^k} = \text{constant} \qquad p = \rho R T$

Assumptions: (1) model flow as isentropic flow through a converging nozzle
(2) assume uniform properties within the cabin, isentropic expansion
(3) ideal gas behavior.

Stagnation conditions within the cabin are

$T_i = 298\ K \qquad p_i = p_{atm}$ at 2.5 km $= 74.7$ kPa (Table A.3)

$p_f = 0.60\, p_i = 44.8$ kPa

Back pressure $p_b = p_{atm}$ at 11 km $= 22.7$ kPa

Then $p_b / p_i = 0.304$ and $p_b / p_f = 0.507$. Flow is choked.

Note: conditions in cabin are stagnation conditions.

From continuity.

$$\frac{\partial}{\partial t}\int_{cv} \rho\, d\forall = -\int_{cs} \rho \vec{V} \cdot d\vec{A} = -\rho_e V_e A_e$$

$$\forall \frac{d\rho}{dt} = -\rho_e V_e A_e.$$

For choked flow, $M_e = 1.0$

$$\frac{\rho}{\rho_e} = \left[1 + \frac{k-1}{2} M_e^2\right]^{\frac{1}{k-1}} = (1.20)^{2.5} = 1.5774 \qquad \therefore \rho_e = 0.6339\, \rho$$

$$\frac{T}{T_e} = 1 + \frac{k-1}{2} M_e^2 = 1.2 \qquad \therefore T_e = 0.8333\, T$$

$$V_e = (kRT_e)^{1/2} = (kR)^{1/2} (0.8333\, T)^{1/2} = 0.9129\, (kR)^{1/2}\, T^{1/2}$$

Then

$$\forall \frac{d\rho}{dt} = -\rho_e V_e A_e = -0.6339\rho\, (0.9129)(kR)^{1/2}\, T^{1/2}\, A_e$$

$$\forall \frac{d\rho}{dt} = -0.5787\, (kR)^{1/2} A_e\, \rho\, T^{1/2}$$

For an isentropic expansion ρ and T can be related

$$\frac{p}{\rho^k} = \text{const} = \frac{\rho R T}{\rho^k} \qquad \therefore \rho^{(1-k)}\, T = \text{constant}$$

Then, $\rho^{(1-k)} T = \rho_i^{(1-k)} T_i$ or $T = T_i \left(\frac{\rho_i}{\rho}\right)^{(1-k)}$ and $T^{1/2} = \frac{T_i^{1/2}}{\rho_i^{(k-1)/2}} \rho^{(k-1)/2}$

Substituting we obtain

$$\forall \frac{d\rho}{dt} = -0.5787 (kR)^{1/2} A_e \rho \frac{T_i^{1/2}}{\rho_i^{(k-1)/2}} \rho^{(k-1)/2}$$

$$\frac{d\rho}{dt} = -0.5787 \frac{(kR)^{1/2}}{\forall} A_e \frac{T_i^{1/2}}{\rho_i^{(k-1)/2}} \rho^{\frac{k+1}{2}} = c_1 \rho^{(k+1)/2}$$

$$\frac{d\rho}{\rho^{(k+1)/2}} = -c_1\, dt \quad \text{where } c_1 = 0.5787 (kR)^{1/2} \frac{A_e}{\forall} \frac{T_i^{1/2}}{\rho_i^{(k-1)/2}}$$

To integrate, we write

$$-c_1 t = \int \rho^{-\frac{(k+1)}{2}} d\rho = \frac{1}{1-\frac{(k+1)}{2}} \rho^{1-\frac{(k+1)}{2}} \Big]_{\rho_i}^{\rho_f} = \frac{2}{(1-k)} \rho^{\frac{(1-k)}{2}} \Big]_{\rho_i}^{\rho_f}$$

$$-c_1 t = \frac{2}{(k-1)} \left[\rho_i^{(1-k)/2} - \rho_f^{(1-k)/2} \right] = \frac{2}{(k-1)} \rho_i^{(1-k)/2} \left[1 - \left(\frac{\rho_f}{\rho_i}\right)^{(1-k)/2} \right]$$

$$c_1 t = \frac{2}{(k-1)} \rho_i^{(1-k)/2} \left[\left(\frac{\rho_i}{\rho_f}\right)^{(k-1)/2} - 1 \right]$$

$$0.5787 (kR)^{1/2} \frac{A_e}{\forall} \frac{T_i^{1/2}}{\rho_i^{(k-1)/2}} t = \frac{2}{(k-1)} \rho_i^{(1-k)/2} \left[\left(\frac{\rho_i}{\rho_f}\right)^{(k-1)/2} - 1 \right]$$

$$0.5787 (kR)^{1/2} \frac{A_e}{\forall} T_i^{1/2} t = \frac{2}{(k-1)} \left[\left(\frac{\rho_i}{\rho_f}\right)^{\frac{(k-1)}{2}} - 1 \right]$$

Since $p/\rho^k = \text{const}$, $\frac{\rho_i}{\rho_f} = \left(\frac{p_i}{p_f}\right)^{1/k}$

and
$$0.5787 (kR)^{1/2} \frac{A_e}{\forall} T_i^{1/2} t = \frac{2}{(k-1)} \left[\left(\frac{p_i}{p_f}\right)^{\frac{(k-1)}{2k}} - 1 \right] \qquad (1)$$

Substituting numerical values

$$0.5787 \left[1.4 \times 287 \frac{\text{N.m}}{\text{kg.K}} \times 298\,\text{K} \times \frac{\text{kg.m}}{\text{N.s}^2} \right]^{1/2} 0.002\,\text{m}^2 \times \frac{1}{25\,\text{m}^3} t$$
$$= \frac{2}{0.4} \left[\left(\frac{1}{0.6}\right)^{0.1429} - 1 \right]$$

$$0.01602\, t = 0.3786$$

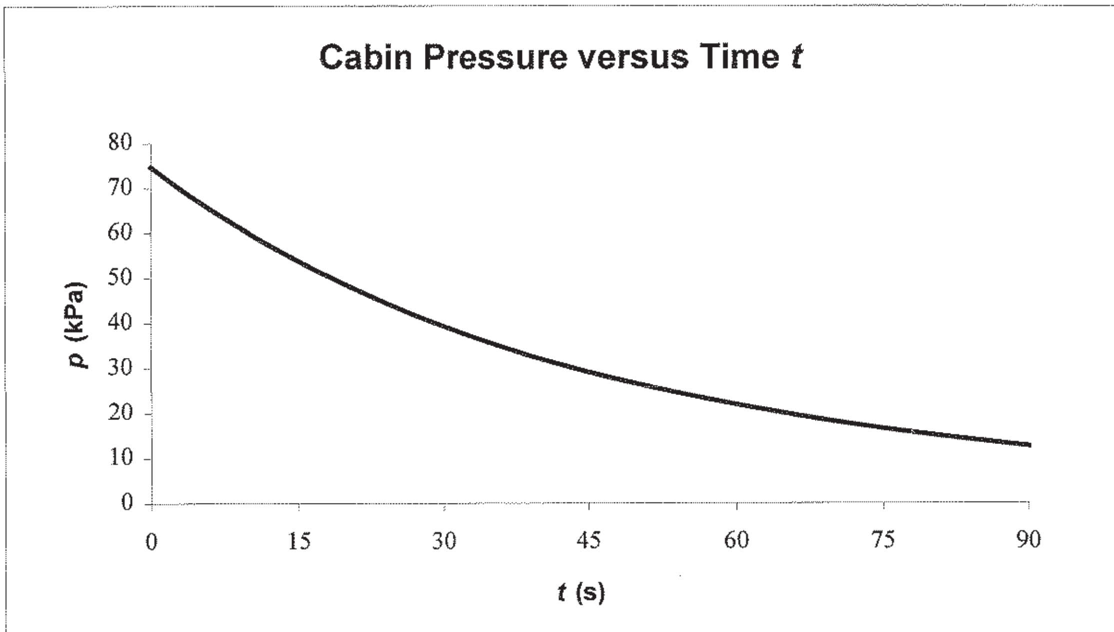
$$t = 23.6\,\text{s.} \longleftarrow t$$

Problem 13.44

[Difficulty: 4] Part 3/3

Equation 1 is plotted using *Excel*
Note that it's easier to compute *t* from *p* values!

t (s)	*p* (kPa)
0.000	74.7
1.03	73
2.27	71
3.56	69
4.89	67
6.26	65
7.69	63
9.17	61
10.7	59
12.3	57
14.0	55
15.7	53
17.5	51
19.4	49
21.4	47
23.6	**44.8**
25.6	43
27.9	41
30.4	39
33.0	37
35.7	35
38.6	33
41.8	31
45.2	29
48.8	27
52.8	25
57.2	23
62.0	21
67.4	19
73.5	17
80.5	15
90.0	12.7



Problem 13.45

[Difficulty: 3]

13.45 At some point upstream of the throat of a converging-diverging duct, air flows at a speed of 50 ft/s, with pressure and temperature of 15 psia and 70°F, respectively. If the throat area is 1 ft^2, and the discharge from the duct is supersonic, find the mass flow rate of air, assuming frictionless, adiabatic flow.

Given: Air flow through a converging-diverging nozzle

Find: Nozzle mass flow rate

Solution:

The given or available data is:

$R =$	53.33	ft-lbf/lbm-°R
$k =$	1.4	
$V_1 =$	50	ft/s
$p_1 =$	15	psia
$T_1 =$	70	°F
$T_1 =$	530	°R
$A_t =$	1	ft^2

Equations and Computations:

At station 1 the local sound speed is:

$c_1 =$ 1128.80 ft/s

So the upstream Mach number is:

$M_1 =$ 0.0443

So now we can calculate the stagnation temperature and pressure:

$$\frac{p_0}{p} = \left[1 + \frac{k-1}{2}M^2\right]^{k/(k-1)}$$

$p_0 =$ 15.021 psia

$T_0 =$ 530.21 °R

To find the mass flow rate, we will use the choked flow equation:

From p_0, T_0, A_t, and Eq. 13.10a

$$\dot{m}_{\text{choked}} = A_t p_0 \sqrt{\frac{k}{RT_0}}\left(\frac{2}{k+1}\right)^{(k+1)/2(k-1)} \qquad (13.10a)$$

$m =$ 50.0 lbm/s

Problem 13.46

[Difficulty: 2]

13.46 A converging-diverging nozzle is attached to a very large tank of air in which the pressure is 150 kPa and the temperature is 35°C. The nozzle exhausts to the atmosphere where the pressure is 101 kPa. The exit diameter of the nozzle is 2.75 cm. What is the flow rate through the nozzle? Assume the flow is isentropic.

Given: CD nozzle attached to large tank

Find: Flow rate

Solution:

Basic equations: $\frac{T_0}{T} = 1 + \frac{k-1}{2}\cdot M^2$ $\qquad \frac{p_0}{p} = \left(1 + \frac{k-1}{2}\cdot M^2\right)^{\frac{k}{k-1}}$ $\qquad m_{rate} = \rho\cdot V\cdot A$

Given or available data $\quad p_0 = 150\cdot kPa \qquad T_0 = (35 + 273)\cdot K \qquad p_e = 101\cdot kPa \qquad D = 2.75\cdot cm$

$k = 1.4 \qquad R = 286.9\cdot\frac{J}{kg\cdot K} \qquad A_e = \frac{\pi}{4}\cdot D^2 \qquad A_e = 5.94\cdot cm^2$

For isentropic flow $\quad M_e = \sqrt{\frac{2}{k-1}\cdot\left[\left(\frac{p_0}{p_e}\right)^{\frac{k-1}{k}} - 1\right]} \qquad M_e = 0.773$

Then $\quad T_e = \frac{T_0}{\left(1 + \frac{k-1}{2}\cdot M_e^{\,2}\right)} \qquad T_e = 275\,K \qquad T_e = 1.94\cdot°C$

Also $\quad c_e = \sqrt{k\cdot R\cdot T_e} \qquad c_e = 332\,\frac{m}{s} \qquad V_e = M_e\cdot c_e \qquad V_e = 257\,\frac{m}{s}$

$\rho_e = \frac{p_e}{R\cdot T_e} \qquad \rho_e = 1.28\,\frac{kg}{m^3}$

Finally $\quad m_{rate} = \rho_e\cdot V_e\cdot A_e \qquad m_{rate} = 0.195\,\frac{kg}{s}$

Problem 13.47

[Difficulty: 4]

13.47 A large insulated tank, pressurized to 620 kPa (gage), supplies air to a converging nozzle which discharges to atmosphere. The initial temperature in the tank is 127°C. When flow through the nozzle is initiated, what is the Mach number in the exit plane of the nozzle? What is the pressure in the exit plane when the flow is initiated? At what condition will the exit-plane Mach number change? How will the exit-plane pressure vary with time? How will flow rate through the nozzle vary with time? What would you estimate the air temperature in the tank to be when flow through the nozzle approaches zero?

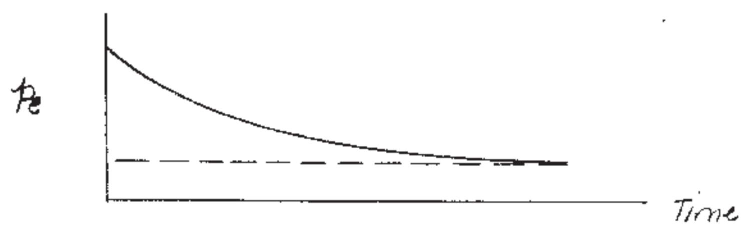
Solution: Assume stagnation conditions in tank, $p_b = p_{atm}$. Then

$$\frac{p_b}{p_0} = \frac{101.3\ \text{kPa}}{(620+101.3)\ \text{kPa (abs)}} = 0.140 \ll 0.528,$$ so flow is choked! $M_e = 1$ — M_e

At exit plane, $M_e = 1$, so

$$p_e = p_0\left(\frac{1}{1+\frac{k-1}{2}M_e^2}\right)^{\frac{k}{k-1}} = (620+101.3)\ \text{kPa (abs)}\left(\frac{1}{1.2}\right)^{3.5} = 381\ \text{kPa (abs)}$$ — p_e

Exit plane pressure decreases with time, asymptotically approaching p_{atm}.



Flow rate varies similarly.

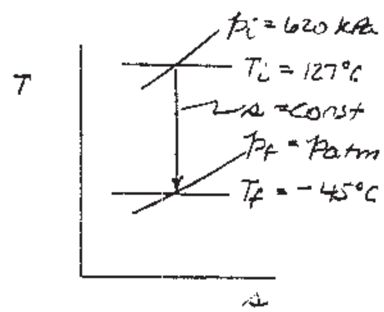
When flow rate approaches zero, $p_0 \to p_{atm}$. Assume tank behaves as a reversible adiabatic process. Thus

$$\frac{p_f}{p_0} = \left(\frac{T_f}{T_0}\right)^{\frac{k}{k-1}}$$

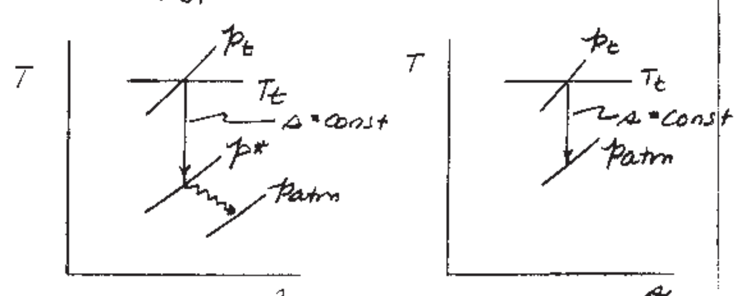
Thus $$T_f = T_0\left(\frac{p_f}{p_0}\right)^{\frac{k-1}{k}} = (273+127)\ \text{K}\left(\frac{101.3}{620+101.3}\right)^{0.286} = 228\ \text{K}$$

$$T_f = (228-273)\,°\text{C} = -45\,°\text{C}$$ — T_f

The Ts diagram for air in the tank is:



The Ts diagrams for air flowing from the tank are:



While choked
($p_t > p_{atm}/0.528$)

No longer choked
($p_t \le p_{atm}/0.528$)

Problem 13.48

13.48 Air escapes from a high-pressure bicycle tire through a hole with diameter $d = 0.254$ mm. The initial pressure in the tire is $p_1 = 620$ kPa (gage). (Assume the temperature remains constant at 27°C.) The internal volume of the tire is approximately 4.26×10^{-4} m^3, and is constant. Estimate the time needed for the pressure in the tire to drop to 310 kPa (gage). Compute the change in specific entropy of the air in the tire during this process. Plot the tire pressure as a function of time.

__Solution:__ Apply continuity equation, isentropic relationships.

Basic equations: $0 = \frac{\partial}{\partial t}\int_{CV} \rho \, dV + \int_{CS} \rho \vec{V} \cdot d\vec{A}$ $\qquad \frac{T_0}{T} = \left(1 + \frac{k-1}{2}M^2\right)$; $\frac{\rho_0}{\rho} = \left(\frac{T_0}{T}\right)^{\frac{1}{k-1}}$

Check for choking: $\frac{p_{atm}}{p_{min}} = \frac{101}{310+101} = 0.246 < 0.528$ so __always__ choked.

Thus $\dot{m} = \rho^* V^* A^*$. Assume: (1) Uniform density in tire: $\int_{CV} = \rho V$

(2) Uniform flow at throat

(3) Isentropic process to throat.

Then

$$0 = V \frac{d\rho}{dt} + \rho^* V^* A_e$$

But $\rho^* = \dfrac{\rho}{\left(1 + \frac{k-1}{2}M_*^2\right)^{1/k-1}} = \dfrac{\rho}{(1.2)^{2.5}} = 0.634\rho$

so $\dfrac{d\rho}{\rho} = -0.634 \dfrac{V^* A_e}{V} dt$

Integrating, $\ln\dfrac{\rho_2}{\rho_1} = -0.634\dfrac{V^* A_e}{V} t = \ln\dfrac{p_2}{p_1}$ since $T = $ constant

Thus

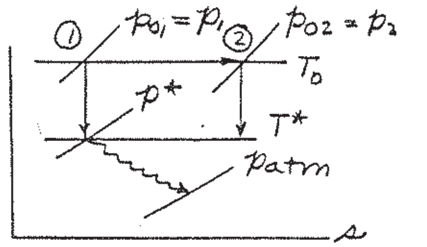
$$t = -\frac{V}{0.634 V^* A_e} \ln\frac{p_2}{p_1} \qquad (1)$$

$$V^* = c^* = \sqrt{kRT^*} = \left[1.4 \times 287 \frac{N \cdot m}{kg \cdot K} \times \frac{273+27}{1.2} K \times \frac{kg \cdot m}{N \cdot s^2}\right]^{1/2} = 317 \text{ m/s}$$

$$A^* = \frac{\pi D^2}{4} = \frac{\pi}{4}(0.000254)^2 m^2 = 5.07 \times 10^{-8} m^2$$

$$t = -\frac{1}{0.634} \times 4.26 \times 10^{-4} m^3 \times \frac{s}{317\, m} \times \frac{1}{5.07 \times 10^{-8} m^2} \times \ln\left(\frac{310+101}{620+101}\right) = 23.5 \text{ s} \qquad t$$

Ts diagram:



Process ① → ② in tire

Process (isentropic) ① → ⊛ (moving to) ② → ⊛ in converging passage.

In tire,

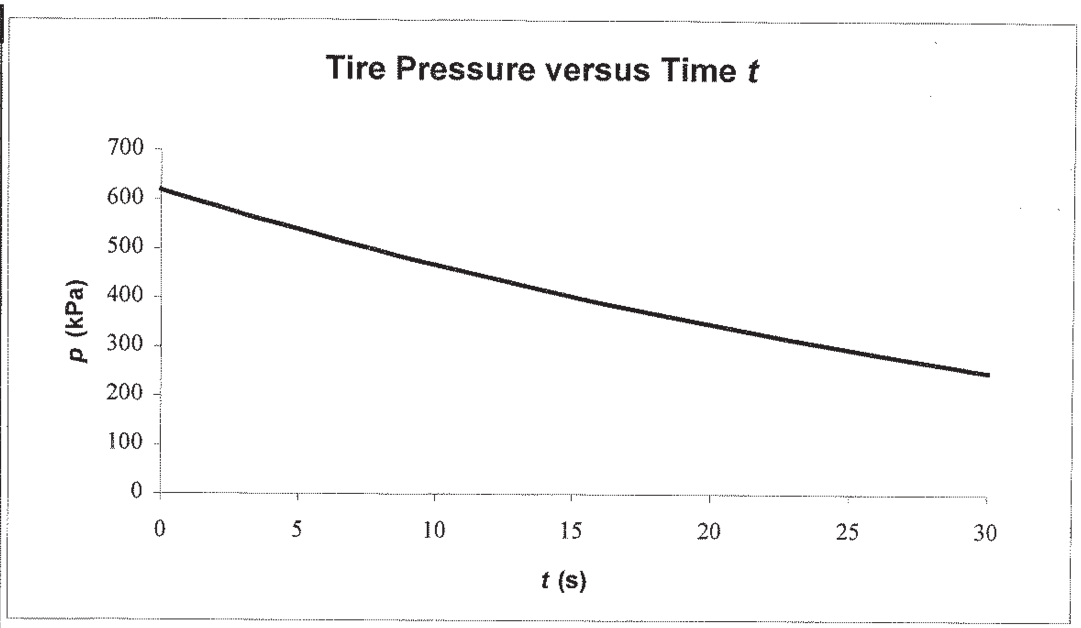
$$\Delta s = c_p \ln\frac{T_2}{T_1} - R\ln\frac{p_2}{p_1} = -287 \frac{N \cdot m}{kg \cdot K} \times \ln\frac{(310+101)}{(620+101)} = 161 \text{ J/(kg·K)} \qquad \Delta s$$

Problem 13.48

[Difficulty: 4] Part 2/2

Equation 1 is plotted using *Excel*
Note that it's easier to compute *t* from *p* values!

t (s)	*p* (kPa)
0.000	620
0.584	610
1.18	600
1.78	590
2.39	580
3.00	570
3.63	560
4.27	550
4.92	540
5.57	530
6.24	520
6.92	510
7.61	500
8.31	490
9.03	480
9.75	470
10.5	460
11.2	450
12.0	440
12.8	430
13.6	420
14.4	410
15.2	400
16.1	390
16.9	380
17.8	370
18.7	360
19.6	350
20.6	340
21.5	330
22.5	320
23.5	310
24.5	300
25.6	290
26.7	280
27.8	270
28.9	260
30.0	251



Problem 13.49

[Difficulty: 2]

13.49 At the design condition of the system of Problem 13.46, the exit Mach number is $M_e = 2.0$. Find the pressure in the tank of Problem 13.46 (keeping the temperature constant) for this condition. What is the flow rate? What is the throat area?

Given: Design condition in a converging-diverging nozzle

Find: Tank pressure; flow rate; throat area

Solution:

The given or available data is:

$R =$	53.33	ft.lbf/lbm.^{o}R
$k =$	1.4	
$T_0 =$	560	^{o}R
$A_e =$	1	in^2
$p_b =$	14.7	psia
$M_e =$	2	

Equations and Computations:

At design condition $p_e = p_b$

$p_e =$ 14.7 psia

From M_e and p_e, and Eq. 13.7a
(using built-in function *Isenp* (M,k)

$$\frac{p_0}{p} = \left[1 + \frac{k-1}{2}M^2\right]^{k/(k-1)} \tag{13.7a}$$

$p_0 =$ 115 psia

From M_e and A_e, and Eq. 13.7d
(using built-in function *IsenA* (M,k)

$$\frac{A}{A^*} = \frac{1}{M}\left[\frac{1 + \frac{k-1}{2}M^2}{\frac{k+1}{2}}\right]^{(k+1)/2(k-1)} \tag{13.7d}$$

$A^* =$ 0.593 in^2

Hence $A_t =$ 0.593 in^2

From p_0, T_0, A_t, and Eq. 13.10a

$$\dot{m}_{\text{choked}} = A_t p_0 \sqrt{\frac{k}{RT_0}}\left(\frac{2}{k+1}\right)^{(k+1)/2(k-1)} \tag{13.10a}$$

$m_{\text{choked}} =$ 1.53 lb/s

Problem 13.50

[Difficulty: 3]

13.50 When performing tests in a wind tunnel at conditions near Mach 1, the effects of model blockage become very important. Consider a wind tunnel with a test section of 1 ft^2 cross section. If the test section conditions are $M = 1.20$ and $T = 70°F$, how much area blockage could be tolerated before the flow choked in the test section? If a model with 3 $in.^2$ projected frontal area were inserted in the tunnel, what would the air velocity be in the test section?

Given: Wind tunnel test section with blockage

Find: Maximum blockage that can be tolerated; air speed given a fixed blockage

Solution:

The given or available data is:

$R =$ 53.33 ft-lbf/lbm-°R

$k =$ 1.4

$M_1 =$ 1.2

$T_1 =$ 70 °F

$T_1 =$ 530 °R

$A_t =$ 1 ft^2

Equations and Computations:

The test section will choke if the blockage decreases the area to A^*. In the test section:

$A_1/A^* =$ 1.0304

So the minimum area would be

$A^* =$ 0.9705 ft^2

And the blockage would be the difference between this and the test section area:

$A_1 - A^* =$ 0.0295 ft^2

$A_1 - A^* =$ 4.25 in^2

If we have a blockage of:

$A_1 - A =$ 3.0000 in^2

Then the actual area would be:

$A_{actual} =$ 0.9792 ft^2

The resulting isentropic area ratio is:

$A_{actual}/A^* =$ 1.0090

and the actual Mach number is:

$M_{actual} =$ 1.1066

(remember that since we're already supersonic, we should use the supersonic solution)

The stagnation temperature for the wind tunnel is (based on test section conditions)

$T_0 =$ 682.64 °R

So the actual static temperature in the tunnel is:

$T_{actual} =$ 548.35 °R

The sound speed would then be:

$c_{actual} =$ 1148.17 ft/s

And so the speed in the test section is:

$V_{actual} =$ 1270.5 ft/s

Problem 13.51

[Difficulty: 3]

13.51 A pitot static probe is placed in a converging-diverging duct through which air flows. The duct is fed by a reservoir kept at 20°C. If the probe reads a static pressure of 75 kPa and a stagnation pressure of 100 kPa at a location where the area is $0.00645\ \mathrm{m}^2$, what is the local velocity and the mass flow rate of air?

Given: Air flow through a converging-diverging nozzle equipped with pitot-static probe

Find: Nozzle velocity and mass flow rate

Solution:

The given or available data is:

$R =$	286.9	J/kg-K
$k =$	1.4	
$p_1 =$	75	kPa
$p_{01} =$	100	kPa
$T_1 =$	20	°C
$T_1 =$	293	K
$A_1 =$	10	in^2
$A_1 =$	0.006452	m^2

Equations and Computations:

At station 1 the local sound speed is:

$c_1 =$ 343.05 m/s

Based on the static and pitot pressures, the Mach number is:

$M_1 =$ 0.6545

Therefore the velocity is:

$V_1 =$ 225 m/s

The local density can be calculated using the ideal gas equation of state:

$\rho_1 =$ 0.8922 $\mathrm{kg/m^3}$

So the mass flow rate is:

$m =$ 1.292 kg/s

Problem 13.52

[Difficulty: 2]

13.52 A converging-diverging nozzle, with a throat area of 2 in.^{2}, is connected to a large tank in which air is kept at a pressure of 80 psia and a temperature of 60°F. If the nozzle is to operate at design conditions (flow is isentropic) and the ambient pressure outside the nozzle is 12.9 psia, calculate the exit area of the nozzle and the mass flow rate.

Solution

Computing equations: $\frac{T_0}{T} = 1 + \frac{k-1}{2}M^2 \qquad \frac{p_0}{p} = \left[1 + \frac{k-1}{2}M^2\right]^{k/k-1}$

$$A/A^* = \frac{1}{M}\left[\frac{1 + \frac{k-1}{2}M^2}{1 + \frac{k-1}{2}}\right]^{(k+1)/2(k-1)} \qquad (12.6)$$

Assumptions: (1) steady flow (2) isentropic flow (3) uniform flow at a section (4) ideal gas

$p_0/p = \left[1 + \frac{k-1}{2}M^2\right]^{k/k-1}$; $M_1 = \left\{\frac{2}{k-1}\left[\left(\frac{p_0}{p_1}\right)^{k-1/k} - 1\right]\right\}^{1/2} = \left\{\frac{2}{0.4}\left[\left(\frac{80}{12.9}\right)^{0.286} - 1\right]\right\}^{1/2} = 1.85$

$T_0/T = 1 + \frac{k-1}{2}M^2$; $T_1 = \frac{T_0}{1 + \frac{k-1}{2}M_1^2} = \frac{520°R}{1 + 0.2(1.85)^2} = 309°R$

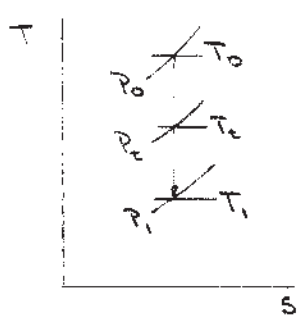
$V_1 = M_1 c_1 = M_1 (kRT_1)^{1/2} = 1.85\left(1.4 \times 53.3 \frac{ft\cdot lbf}{lbm\cdot°R} \times 309°R \times 32.2 \frac{lbm}{slug} \times \frac{slug\cdot ft}{lbf\cdot s^2}\right)^{1/2} = 1594\ ft/s$

$\rho_1 = \frac{p_1}{RT_1} = 12.9 \frac{lbf}{in^2} \times \frac{lbm\cdot°R}{53.3\,ft\cdot lbf} \times \frac{1}{309°R} \times \frac{144\,in^2}{ft^2} = 0.113\ lbm/ft^3$

Since $M_1 = 1.85$, nozzle must be choked and $M_t = 1.0$; $A_t = A^*$

For $M_1 = 1.85$, from Eq. 12.6 (and Fig E.1), $A_1/A^* = 1.496$; $\therefore A_1 = 2.99\,in^2$ ← A_1

$\dot{m} = \rho_1 V_1 A_1 = 0.113 \frac{lbm}{ft^3} \times 1594 \frac{ft}{s} \times 2.99\,in^2 \times \frac{ft^2}{144\,in^2} = 3.74\ lbm/s$ ← $\dot{m}$



Problem 13.53

[Difficulty: 2]

13.53 A converging-diverging nozzle, designed to expand air to $M = 3.0$, has a 250 mm^2 exit area. The nozzle is bolted to the side of a large tank and discharges to standard atmosphere. Air in the tank is pressurized to 4.5 MPa (gage) at 750 K. Assume flow within the nozzle is isentropic. Evaluate the pressure in the nozzle exit plane. Calculate the mass flow rate of air through the nozzle.

Solution: Assume stagnation conditions in tank, isentropic flow in nozzle.

Computing equations: $\frac{p_0}{p} = \left(1+\frac{k-1}{2}M^2\right)^{\frac{k}{k-1}}$; $\frac{T_0}{T} = 1+\frac{k-1}{2}M^2$; $c = \sqrt{kRT}$

For $M_e = 3.0$, $\frac{p_0}{p_e} = \left[1+\frac{k-1}{2}(3.0)^2\right]^{\frac{k}{k-1}} = 36.7$

$$p_e = \frac{p_0}{36.7} = \frac{(4.5\times10^6 + 101\times10^3)\,Pa}{36.7} = 125\ kPa\ (abs)\ \text{or}\ 24\ kPa\ (gage)$$ p_e

Thus the nozzle is underexpanded. For steady, 1-D flow, $\dot{m} = \rho_e V_e A_e$.

$$T_e = \frac{T_0}{1+\frac{k-1}{2}(3.0)^2} = \frac{750\ K}{1+0.2(3)^2} = 268\ K$$

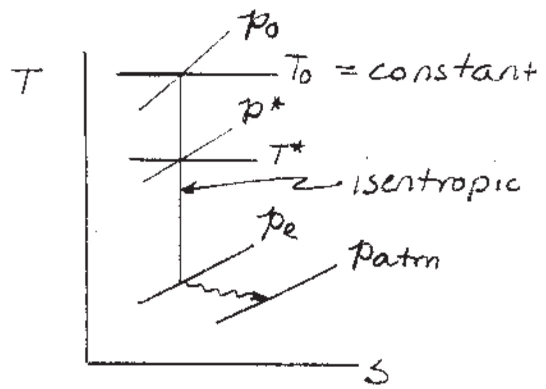
$$c_e = \sqrt{kRT_e} = \left[1.4 \times 287\frac{N\cdot m}{kg\cdot K}\times 268\ K \times \frac{kg\cdot m}{N\cdot s^2}\right]^{1/2} = 328\ m/s$$

$$\rho_e = \frac{p_e}{RT_e} = 125\times10^3\frac{N}{m^2}\times\frac{kg\cdot K}{287\ N\cdot m}\times\frac{1}{268\ K} = 1.63\ kg/m^3$$

Finally, $V_e = M_e c_e = 3.0 \times 328\ m/s = 984\ m/s$

$$\dot{m} = \rho_e V_e A_e = 1.63\frac{kg}{m^3}\times 984\frac{m}{s}\times 250\ mm^2 \times \frac{m^2}{10^6\ mm^2} = 0.401\ kg/s$$ $\dot{m}$

Ts diagram:



0 = tank conditions

* = throat conditions

e = exit plane conditions

Problem 13.54

[Difficulty: 3]

13.54 Methane is stored in a tank at 75 psia and 80°F. It discharges to another tank via a converging-only nozzle, with exit area 1 in.^{2}. What is the initial mass flow rate of methane when the discharge tank is at a pressure of (a) 15 psia, and (b) 60 psia?

Given: Methane discharging from one tank to another via a converging nozzle

Find: Mass flow rate at two different back pressures

Solution:

The given or available data is:

	$R =$	96.32	ft-lbf/lbm-°R
	$k =$	1.31	
	$p_0 =$	75	psia
	$T_0 =$	80	°F
	$T_0 =$	540	°R
	$A_e =$	1	in^2

Equations and Computations:

If the nozzle were choked, the exit Mach number is 1 and the pressure would be:

$p^* =$ 40.79 psia

Therefore, in part a, when

$p_e =$ 15 psia

The nozzle is choked, and we can use the choked flow equation:

From p_0, T_0, A_t, and Eq. 13.9a

$$\dot{m}_{\text{choked}} = A_e p_0 \sqrt{\frac{k}{RT_0}} \left(\frac{2}{k+1} \right)^{(k+1)/2(k-1)} \qquad (13.9a)$$

$m =$ 1.249 lbm/s

In part b, when

$p_e =$ 60 psia

The nozzle is not choked. The exit Mach number is:

$M_e =$ 0.5915

The exit temperature can be found from the Mach number:

$T_e =$ 512.2 °R

The sound speed at the exit is:

$c_e =$ 1442.6 ft/s

And so the exit flow speed is:

$V_e =$ 853.3 ft/s

The density can be calculated using the ideal gas equation of state:

$\rho_e =$ 0.1751 lbm/ft^3

The mass flow rate can then be calculated directly from continuity:

$m =$ 1.038 lbm/s

Problem 13.55

[Difficulty: 2]

13.55 Air, at a stagnation pressure of 7.20 MPa (abs) and a stagnation temperature of 1100 K, flows isentropically through a converging-diverging nozzle having a throat area of 0.01 m^2. Determine the speed and the mass flow rate at the downstream section where the Mach number is 4.0.

Solution:

Basic equations: $\dot{m} = \rho V A \qquad p = \rho R T$

Computing equations: $\frac{T_0}{T} = 1 + \frac{k-1}{2} M^2 \qquad \frac{p_0}{p} = \left[1 + \frac{k-1}{2} M^2\right]^{k/k-1}$

Assumptions: (1) steady flow (3) uniform flow at a section
(2) isentropic flow (4) ideal gas

$$\frac{T_0}{T} = 1 + \frac{k-1}{2} M^2 \; ; \; T_1 = \frac{T_0}{1 + \frac{k-1}{2} M^2} = \frac{1100\text{ K}}{1 + 0.2(4.0)^2} = 262\text{ K}.$$

$$V_1 = M_1 c_1 = M_1 (kRT_1)^{1/2} = 4.0\left(1.4 \times 287 \frac{N \cdot m}{kg \cdot K} \times 262 K \times \frac{kg \cdot m}{N \cdot s^2}\right)^{1/2} = 1300 \text{ m/s}$$ V_1

Since $M_1 = 4.0$, nozzle must be choked and $M_t = 1.0$

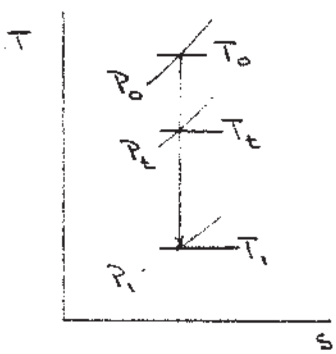
$$p_t = \frac{p_0}{\left[1 + \frac{k-1}{2} M_t^2\right]^{k/k-1}} = \frac{7.2 \times 10^6 \text{ Pa}}{\left[1 + 0.2(1.0)^2\right]^{3.5}} = 3.80 \text{ MPa}$$

$$T_t = \frac{T_0}{1 + \frac{k-1}{2} M_t^2} = \frac{1100\text{ K}}{1 + 0.2(1.0)^2} = 917\text{ K}$$

$$\rho_t = \frac{p_t}{RT_t} = 3.80 \times 10^6 \frac{N}{m^2} \times \frac{kg \cdot K}{287\, N \cdot m} \times \frac{1}{917 K} = 14.4 \text{ kg/m}^3$$

$$V_t = M_t c_t = M_t (kRT_t)^{1/2} = 1.0\left(1.4 \times 287 \frac{N \cdot m}{kg \cdot K} \times 917 K \times \frac{kg \cdot m}{N \cdot s^2}\right)^{1/2} = 607 \text{ m/s}$$

$$\dot{m} = \rho_t V_t A_t = 14.4 \frac{kg}{m^3} \times 607 \frac{m}{s} \times 0.01 m^2 = 87.4 \text{ kg/s}$$ $\dot{m}$



Problem 13.56

[Difficulty: 2]

13.56 Air is to be expanded through a converging-diverging nozzle by a frictionless adiabatic process, from a pressure of 1.10 MPa (abs) and a temperature of 115°C, to a pressure of 141 kPa (abs). Determine the throat and exit areas for a well-designed shockless nozzle, if the mass flow rate is 2 kg/s.

Solution:

Computing equations: $\frac{T_o}{T} = 1 + \frac{k-1}{2}M^2$, $\frac{p_o}{p} = \left[1 + \frac{k-1}{2}M^2\right]^{k/k-1}$

$$\frac{A}{A^*} = \frac{1}{M}\left[\frac{1 + \frac{k-1}{2}M^2}{1 + \frac{k-1}{2}}\right]^{(k+1)/2(k-1)} \quad (12.6)$$

Assumptions: (1) steady flow (3) uniform flow at a section
(2) isentropic flow (4) ideal gas

$p_o/p = \left[1 + \frac{k-1}{2}M^2\right]^{k/k-1}$; $M_1 = \left\{\frac{2}{k-1}\left[\left(\frac{p_o}{p_1}\right)^{k-1/k} - 1\right]\right\}^{1/2} = \left\{\frac{2}{0.4}\left[\left(\frac{1.1}{0.141}\right)^{0.286} - 1\right]\right\}^{1/2} = 2.0$

$T_o/T = 1 + \frac{k-1}{2}M^2$; $T_1 = \frac{T_o}{1 + \frac{k-1}{2}M_1^2} = \frac{388\,K}{1 + 0.2(2.0)^2} = 216\,K$

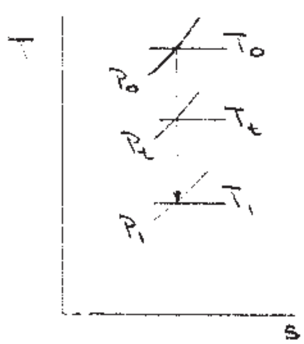
$V_1 = M_1 c_1 = M_1 (kRT_1)^{1/2} = 2.0\left(1.4 \times 287\frac{N \cdot m}{kg \cdot K} \times 216\,K \times \frac{kg \cdot m}{N \cdot s^2}\right)^{1/2} = 589\ m/s$

$\rho_1 = \frac{p_1}{RT_1} = 141 \times 10^3 \frac{N}{m^2} \times \frac{kg \cdot K}{287\,N \cdot m} \times \frac{1}{216\,K} = 2.27\ kg/m^3$

$A_1 = \frac{\dot{m}}{\rho_1 V_1} = 2.0\frac{kg}{s} \times \frac{m^3}{2.27\,kg} \times \frac{s}{589\,m} = 1.50 \times 10^{-3}\ m^2$ ——— A_1

For $M_1 = 2.0$, from Eq. 12.6 and Fig. E.1, $A_1/A^* = 1.688$

Then $A_t = A^* = A_1/1.688 = 8.89 \times 10^{-4}\ m^2$ ——— A_t



13.57 Air flows isentropically through a converging-diverging nozzle attached to a large tank, in which the pressure is 251 psia and the temperature is 500°R. The nozzle is operating at design conditions for which the nozzle exit pressure, p_e, is equal to the surrounding atmospheric pressure, p_a. The exit area of the nozzle is $A_e = 1.575\,\text{in.}^2$. Calculate the flow rate through the nozzle. Plot the mass flow rate as the temperature of the tank is progressively increased to 2000°R (all pressures remaining the same). Explain this result (e.g., compare the mass flow rates at 500°R and 2000°R).

Solution:

Basic equations: $\dot{m} = \rho V A \qquad p = \rho R T$

Computing equations: $\dfrac{T_0}{T} = 1 + \dfrac{k-1}{2}M^2 \qquad \dfrac{p_0}{p} = \left[1 + \dfrac{k-1}{2}M^2\right]^{k/k-1}$

Assumptions: (1) steady flow (2) isentropic flow (3) uniform flow at a section (4) ideal gas

$$\frac{p_0}{p} = \left[1+\frac{k-1}{2}M^2\right]^{k/k-1} \; ; \; M_e = \left\{\frac{2}{k-1}\left[\left(\frac{p_0}{p_e}\right)^{k-1/k} - 1\right]\right\}^{1/2} = \left\{\frac{2}{0.4}\left[\left(\frac{251}{14.7}\right)^{0.286} - 1\right]\right\}^{1/2} = 2.50$$

$$\frac{T_0}{T} = 1 + \frac{k-1}{2}M^2 \; ; \; T_e = \frac{T_0}{1+\frac{k-1}{2}M_e^2} = \frac{500°R}{1+0.2\,(2.50)^2} = 222\,R$$

$$V_e = M_e c_e = M_e (kRT_e)^{1/2} = 2.5\left(1.4 \times 53.3\,\frac{ft\cdot lbf}{lbm\cdot°R} \times 222°R \times 32.2\,\frac{lbm}{slug} \times \frac{slug\cdot ft}{lbf\cdot s^2}\right)^{1/2} = 1826\,ft/s$$

$$\rho_e = \frac{p_e}{RT_e} = 14.7\,\frac{lbf}{in^2} \times \frac{lbm\cdot°R}{53.3\,ft\cdot lbf} \times \frac{1}{222°R} \times 144\,\frac{in^2}{ft^2} = 0.179\,lbm/ft^3$$

$$\dot{m} = \rho_e V_e A_e = 0.179\,\frac{lbm}{ft^3} \times 1827\,\frac{ft}{s} \times 1.575\,in^2 \times \frac{ft^2}{144\,in^2} = 3.57\,lbm/s \longleftarrow \dot{m}$$

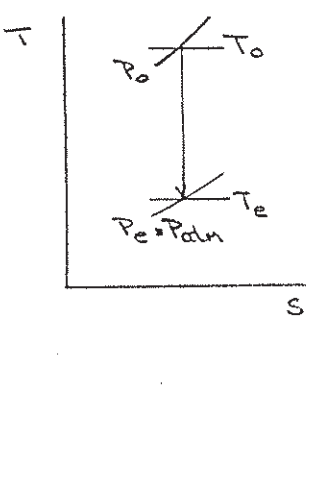
$\dot{m} = \rho_e V_e A_e$ If T_0 is increased by a factor of 4 (holding pressures constant)

(1) T_e will increase by a factor of 4 (since T_0/T = const)

(2) V_e " " " " " 2 (since $V_e \propto T_e^{1/2}$)

(3) ρ_e " decrease " " " 4 (since $\rho_e \propto 1/T_e$)

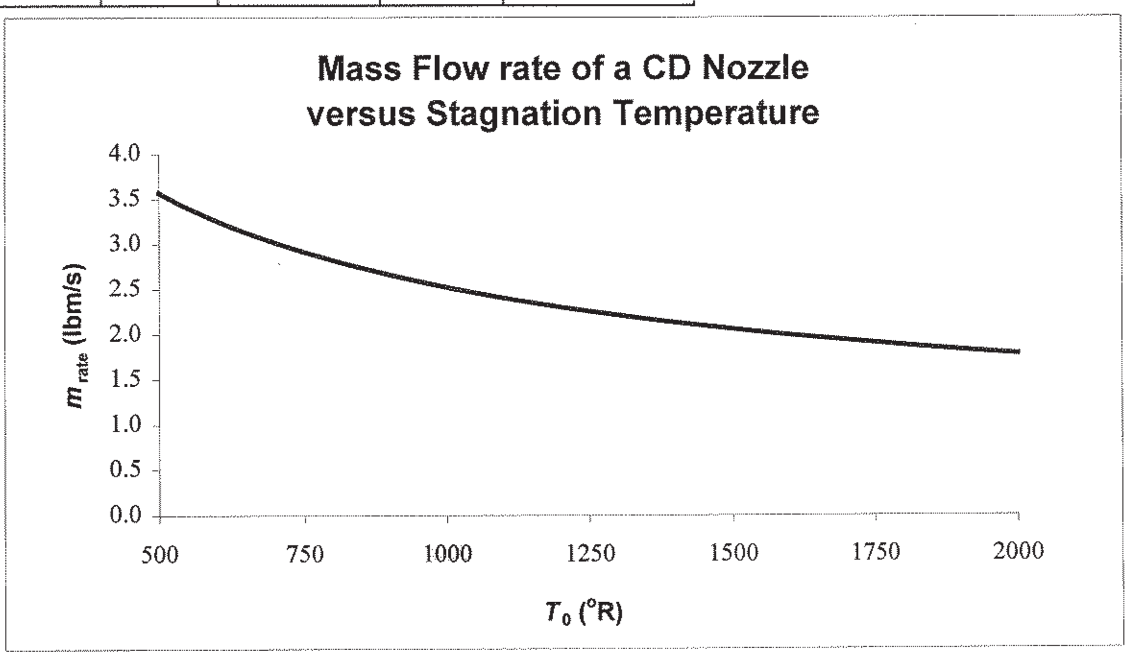
Thus the mass flow rate, $\dot{m}$, will decrease by a factor of 2 ⟵



Problem 13.57

The calculations from page 1 are repeated for various T_0 values and plotted using *Excel*

T_0 (°R)	T_e (°R)	ρ_e (lbm/ft^3)	V_e (ft/s)	m_{rate} (lbm/s)
500	222	0.179	1827	3.57
550	244	0.162	1916	3.40
600	267	0.149	2001	3.26
650	289	0.137	2083	3.13
700	311	0.128	2161	3.02
750	333	0.119	2237	2.92
800	356	0.112	2311	2.82
850	378	0.105	2382	2.74
900	400	0.0993	2451	2.66
950	422	0.0941	2518	2.59
1000	444	0.0894	2583	2.52
1050	467	0.0851	2647	2.46
1100	489	0.0812	2710	2.41
1150	511	0.0777	2770	2.35
1200	533	0.0745	2830	2.30
1250	556	0.0715	2888	2.26
1300	578	0.0687	2946	2.21
1350	600	0.0662	3002	2.17
1400	622	0.0638	3057	2.13
1450	644	0.0616	3111	2.10
1500	667	0.0596	3164	2.06
1550	689	0.0577	3216	2.03
1600	711	0.0558	3268	2.00
1650	733	0.0542	3319	1.97
1700	756	0.0526	3368	1.94
1750	778	0.0511	3418	1.91
1800	800	0.0496	3466	1.88
1850	822	0.0483	3514	1.86
1900	844	0.0470	3561	1.83
1950	867	0.0458	3608	1.81
2000	889	0.0447	3654	1.79



Problem 13.58

[Difficulty: 3]

13.58 A small, solid fuel rocket motor is tested on a thrust stand. The chamber pressure and temperature are 4 MPa and 3250 K. The propulsion nozzle is designed to expand the exhaust gases isentropically to a pressure of 75 kPa. The nozzle exit diameter is 25 cm. Treat the gas as ideal with $k = 1.25$ and $R = 300\ \mathrm{J/(kg\cdot K)}$. Determine the mass flow rate of propellant gas and the thrust force exerted against the test stand.

Given: Rocket motor on test stand

Find: Mass flow rate; thrust force

Solution:

Basic equations: $\dfrac{T_0}{T} = 1 + \dfrac{k-1}{2}\cdot M^2$ $\quad \dfrac{p_0}{p} = \left(1 + \dfrac{k-1}{2}\cdot M^2\right)^{\frac{k}{k-1}}$ $\quad p = \rho\cdot R\cdot T$ $\quad c = \sqrt{k\cdot R\cdot T}$ $\quad m_{rate} = \rho\cdot A\cdot V$

$\left(p_{atm} - p_e\right)\cdot A_e + R_x = m_{rate}\cdot V_e$ Momentum for pressure p_e and velocity V_e at exit; R_x is the reaction force

Given or available data $p_e = 75\cdot\mathrm{kPa}$ $\quad p_{atm} = 101\cdot\mathrm{kPa}$ $\quad p_0 = 4\cdot\mathrm{MPa}$ $\quad T_0 = 3250\cdot\mathrm{K}$ $\quad k = 1.25$ $\quad R = 300\cdot\dfrac{\mathrm{J}}{\mathrm{kg\cdot K}}$

$d = 25\cdot\mathrm{cm}$ so the nozzle exit area is $A_e = \dfrac{\pi}{4}\cdot d^2$ $\quad A_e = 491\cdot\mathrm{cm}^2$

From the pressures $M_e = \sqrt{\dfrac{2}{k-1}\cdot\left[\left(\dfrac{p_0}{p_e}\right)^{\frac{k-1}{k}} - 1\right]}$ $\quad M_e = 3.12$

The exit temperature is $T_e = \dfrac{T_0}{\left(1 + \dfrac{k-1}{2}\cdot M_e^{\,2}\right)}$ $\quad T_e = 1467\,\mathrm{K}$ $\quad c_e = \sqrt{k\cdot R\cdot T_e}$ $\quad c_e = 742\dfrac{\mathrm{m}}{\mathrm{s}}$

The exit speed is $V_e = M_e\cdot c_e$ $\quad V_e = 2313\dfrac{\mathrm{m}}{\mathrm{s}}$ and $\rho_e = \dfrac{p_e}{R\cdot T_e}$ $\quad \rho_e = 0.170\dfrac{\mathrm{kg}}{\mathrm{m}^3}$

Then $m_{rate} = \rho_e\cdot A_e\cdot V_e$ $\quad m_{rate} = 19.3\dfrac{\mathrm{kg}}{\mathrm{s}}$

The momentum equation (Eq. 4.33) simplifies to $\left(p_e - p_{atm}\right)\cdot A_e - M_{CV}\cdot a_x = -V_e\cdot m_{rate}$

Hence $R_x = \left(p_e - p_{atm}\right)\cdot A_e + V_e\cdot m_{rate}$ $\quad R_x = 43.5\cdot\mathrm{kN}$

Problem 13.59

[Difficulty: 3]

13.59 Nitrogen, at a pressure and temperature of 371 kPa (abs) and 400 K, enters a nozzle with negligible speed. The exhaust jet is directed against a large flat plate that is perpendicular to the jet axis. The flow leaves the nozzle at atmospheric pressure. The exit area is 0.003 m². Find the force required to hold the plate.

Solution:

Basic equations: $F_{S_x} = -R_x = -\dot{m}V_1$ (momentum eq^n for CV shown)

$\dot{m} = \rho V A = \text{const}$ $\qquad p = \rho R T$

Computing equations: $\frac{T_0}{T} = 1 + \frac{k-1}{2}M^2$ $\qquad \frac{p_0}{p} = \left[1 + \frac{k-1}{2}M^2\right]^{k/k-1}$

Assumptions: (1) steady flow
(2) isentropic flow in nozzle
(3) uniform flow at a section
(4) p_{atm} over entire CS
(5) $F_{B_x} = 0$
(6) ideal gas $k = 1.40$

$$\frac{p_0}{p} = \left[1 + \frac{k-1}{2}M^2\right]^{k/k-1} \;; \quad M_1 = \left\{\frac{2}{k-1}\left[\left(\frac{p_0}{p_1}\right)^{k-1/k} - 1\right]\right\}^{1/2} = \left\{\frac{2}{0.4}\left[\left(\frac{371}{101}\right)^{0.286} - 1\right]\right\}^{1/2} = 1.50$$

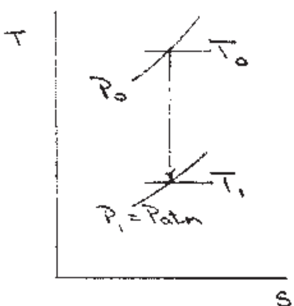
$$\frac{T_0}{T} = 1 + \frac{k-1}{2}M^2 \;; \quad T_1 = \frac{T_0}{1 + \frac{k-1}{2}M_1^2} = \frac{400\,K}{1 + 0.2(1.50)^2} = 276\,K$$

$$V_1 = M_1 c_1 = M_1 (kRT_1)^{1/2} = 1.5\left(1.4 \times 297\,\frac{N\cdot m}{kg\cdot K} \times 276\,K \times \frac{kg\cdot m}{N\cdot s^2}\right)^{1/2} = 508\ m/s$$

$$\rho_1 = \frac{p_1}{RT_1} = 101\times 10^3\,\frac{N}{m^2} \times \frac{kg\cdot K}{297\,N\cdot m} \times \frac{1}{276\,K} = 1.23\ kg/m^3$$

$$\dot{m} = \rho_1 V_1 A_1 = 1.23\,\frac{kg}{m^3} \times 508\,\frac{m}{s} \times 0.003\,m^2 = 1.87\ kg/s$$

$$R_x = \dot{m}V_1 = 1.87\,\frac{kg}{s} \times 508\,\frac{m}{s} \times \frac{N\cdot s^2}{kg\cdot m} = 950\,N \quad \text{(to the left as shown).} \quad R_x$$



Problem 13.60

[Difficulty: 3]

13.60 A liquid rocket motor is fueled with hydrogen and oxygen. The chamber temperature and absolute pressure are 3300 K and 6.90 MPa. The nozzle is designed to expand the exhaust gases isentropically to a design back pressure corresponding to an altitude of 10 km on a standard day. The thrust produced by the motor is to be 100 kN at design conditions. Treat the exhaust gases as water vapor and assume ideal gas behavior. Determine the propellant mass flow rate needed to produce the desired thrust, the nozzle exit area, and the area ratio, A_e/A_t.

Solution

Basic equations: $F_{S_x} = R_x = \dot{m} V_e$ $\quad \dot{m} = \rho V A \quad p = \rho R T$

Computing equations: $\frac{T_0}{T} = 1 + \frac{k-1}{2} M^2 \quad \frac{p_0}{p} = \left[1 + \frac{k-1}{2} M^2\right]^{k/k-1}$

$$\frac{A}{A^*} = \frac{1}{M}\left[\frac{1 + \frac{k-1}{2} M^2}{1 + \frac{k-1}{2}}\right]^{(k+1)/2(k-1)} \quad (12.6)$$

Assumptions: (1) steady flow (4) thrust specified at altitude
(2) isentropic flow (5) $F_{B_x} = 0$
(3) uniform flow at a section (6) ideal gas, $k = 1.3$, $R = 461 \frac{N \cdot m}{kg \cdot K}$

$p_0/p = \left[1 + \frac{k-1}{2} M^2\right]^{k/k-1}$; $M_e = \left\{\frac{2}{k-1}\left[\left(\frac{p_0}{p}\right)^{\frac{k-1}{k}} - 1\right]\right\}^{1/2} = \left\{\frac{2}{0.30}\left[\left(\frac{690}{2.64}\right)^{0.231} - 1\right]\right\}^{1/2} = 4.18$

$T_0/T = 1 + \frac{k-1}{2} M^2$; $T_e = \frac{T_0}{1 + \frac{k-1}{2} M_e^2} = \frac{3300 K}{1 + 0.15(4.18)^2} = 911 K$

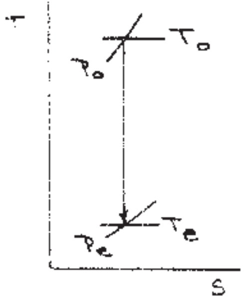
$V_e = M_e c_e = M_e (k R T_e)^{1/2} = 4.18\left[1.3 \times 461 \frac{N \cdot m}{kg \cdot K} \times 911 K \times \frac{kg \cdot m}{N \cdot s^2}\right]^{1/2} = 3090 m/s$

$R_x = \dot{m} V_e$; $\dot{m} = \frac{R_x}{V_e} = 10^5 N \times \frac{s}{3090 m} \times \frac{kg \cdot m}{N \cdot s^2} = 32.4 kg/s$ ← $\dot{m}$

$\rho_e = \frac{p_e}{R T_e} = 26.4 \times 10^3 \frac{N}{m^2} \times \frac{kg \cdot K}{461 N \cdot m} \times \frac{1}{911 K} = 6.29 \times 10^{-2} kg/m^3$

$\dot{m} = \rho_e V_e A_e$; $A_e = \frac{\dot{m}}{\rho_e V_e} = 32.4 \frac{kg}{s} \times \frac{m^3}{6.29 \times 10^{-2} kg} \times \frac{s}{3090 m} = 0.167 m^2$ ← A_e

For $M_e = 4.18$, from Eq 12.6, $A_e/A^* = A_e/A_t = 19.4$ ← A_e/A_t



Problem 13.61

[Difficulty: 3]

13.61 A small rocket motor, fueled with hydrogen and oxygen, is tested on a thrust stand at a simulated altitude of 10 km. The motor is operated at chamber stagnation conditions of 1500 K and 8.0 MPa (gage). The combustion product is water vapor, which may be treated as an ideal gas. Expansion occurs through a converging-diverging nozzle with design Mach number of 3.5 and exit area of 700 mm^2. Evaluate the pressure at the nozzle exit plane. Calculate the mass flow rate of exhaust gas. Determine the force exerted by the rocket motor on the thrust stand.

Solution:

Basic equations: $\dot{m} = \rho V A$, $p = \rho R T$

Assumptions: (1) steady flow (2) isentropic flow (3) ideal gas behavior $k = 1.3$, $R = 461 \text{ J/kg}\cdot\text{K}$

At 10 km altitude, $p_b = 26.5$ kPa (Table A.3)

Evaluate design pressure at exit

$$\frac{p_0}{p} = \left[1 + \frac{(k-1)}{2} M^2\right]^{k/k-1} \quad \therefore \; p_d = \frac{8.10 \times 10^6 \text{ Pa}}{\left[1 + 0.15(3.5)^2\right]^{4.333}} = 88.3 \text{ kPa (abs)}$$

Since $p_b < p_d$, $p_e = p_d = 88.3$ kPa $\quad p_e$

$$\frac{T_0}{T_e} = 1 + \frac{k-1}{2} M_e^2 \quad \therefore \; T_e = \frac{T_0}{1 + \frac{(k-1)}{2} M^2} = \frac{1500 \text{ K}}{1 + 0.15(3.5)^2} = 529 \text{ K}$$

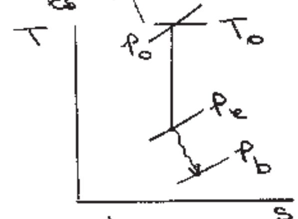
$$\rho_e = \frac{p_e}{R T_e} = 88.3 \times 10^3 \frac{\text{N}}{\text{m}^2} \times \frac{\text{kg}\cdot\text{K}}{461 \text{ N}\cdot\text{m}} \times \frac{1}{529 \text{ K}} = 0.362 \text{ kg/m}^3$$

$$V_e = M c_e = 3.5\left[1.30 \times 461 \frac{\text{N}\cdot\text{m}}{\text{kg}\cdot\text{K}} \times 529 \text{ K} \times \frac{\text{kg}\cdot\text{m}}{\text{N}\cdot\text{s}^2}\right]^{1/2} = 1970 \text{ m/s}$$

$$\dot{m} = \rho_e V_e A_e = 0.362 \frac{\text{kg}}{\text{m}^3} \times 1970 \frac{\text{m}}{\text{s}} \times 700 \text{ mm}^2 \times \frac{\text{m}^2}{10^6 \text{ mm}^2} = 0.499 \text{ kg/s} \quad \dot{m}$$

To determine force on test stand apply x momentum equation to CV shown

Basic equation: $$F_{S_x} + \cancel{F_{B_x}}^{\,0} = \frac{\partial}{\partial t}\int_{CV} V_x \rho \, d\forall + \int_{CS} V_x \rho \vec{V} \cdot d\vec{A}$$



$$R_x + p_b A_e - p_e A_e = \dot{m} V_e$$

Force on test stand is $K_x = -R_x$

$$\therefore K_x = -R_x = -\dot{m} V_e - A_e(p_e - p_b)$$

$$K_x = -0.499 \frac{\text{kg}}{\text{s}} \times 1970 \frac{\text{m}}{\text{s}} \times \frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}} - 700 \text{ mm}^2 \times \frac{\text{m}^2}{10^6 \text{ mm}^2} (88.3 - 26.5) \times 10^3 \frac{\text{N}}{\text{m}^2}$$

$K_x = -1{,}026$ N (to left) $\quad K_x$

Problem 13.62

[Difficulty: 4]

13.62 A CO_2 cartridge is used to propel a small rocket cart. Compressed gas, stored at 35 MPa and 20°C, is expanded through a smoothly contoured converging nozzle with 0.5 mm throat diameter. The back pressure is atmospheric. Calculate the pressure at the nozzle throat. Evaluate the mass flow rate of carbon dioxide through the nozzle. Determine the thrust available to propel the cart. How much would the thrust increase if a diverging section were added to the nozzle to expand the gas to atmospheric pressure? What is the exit area? Show stagnation states, static states, and the processes on a *Ts* diagram.

Given: Compressed CO_2 in a cartridge expanding through a nozzle

Find: Throat pressure; Mass flow rate; Thrust; Thrust increase with diverging section; Exit area

Solution:

Basic equations: $$F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\,d\forall + \int_{CS} u\,\rho\,\vec{V}\cdot d\vec{A}$$

Assumptions: 1) Isentropic flow 2) Stagnation in cartridge 3) Ideal gas 4) Uniform flow

Given or available data: $k = 1.29$ $\quad R = 188.9\cdot\frac{J}{kg\cdot K}$ $\quad p_{atm} = 101\cdot kPa$

$p_0 = 35\cdot MPa$ $\quad T_0 = (20 + 273)\cdot K$ $\quad d_t = 0.5\cdot mm$

From isentropic relations $$p_{crit} = \frac{p_0}{\left(1 + \frac{k-1}{2}\right)^{\frac{k}{k-1}}} \qquad p_{crit} = 19.2\cdot MPa$$

Since $p_b << p_{crit}$, then $$p_t = p_{crit} \qquad p_t = 19.2\cdot MPa$$

Throat is critical so $$m_{rate} = \rho_t\cdot V_t\cdot A_t$$

$$T_t = \frac{T_0}{1 + \frac{k-1}{2}} \qquad T_t = 256\,K$$

$$V_t = \sqrt{k\cdot R\cdot T_t} \qquad V_t = 250\,\frac{m}{s}$$

$$A_t = \frac{\pi\cdot d_t^{\,2}}{4} \qquad A_t = 1.963\times 10^{-7}\,m^2$$

$$\rho_t = \frac{p_t}{R\cdot T_t} \qquad \rho_t = 396\,\frac{kg}{m^3}$$

$$m_{rate} = \rho_t\cdot V_t\cdot A_t \qquad m_{rate} = 0.0194\,\frac{kg}{s}$$

For 1D flow with no body force the momentum equation reduces to $R_x - p_{tgage} \cdot A_t = m_{rate} \cdot V_t$ $\qquad p_{tgage} = p_t - p_{atm}$

$$R_x = m_{rate} \cdot V_t + p_{tgage} \cdot A_t \qquad R_x = 8.60\,\text{N}$$

When a diverging section is added the nozzle can exit to atmospheric pressure $p_e = p_{atm}$

Hence the Mach number at exit is

$$M_e = \left[\frac{2}{k-1} \cdot \left[\left(\frac{p_0}{p_e}\right)^{\frac{k-1}{k}} - 1\right]\right]^{\frac{1}{2}} \qquad M_e = 4.334$$

$$T_e = \frac{T_0}{1 + \frac{k-1}{2} \cdot M_e^2} \qquad T_e = 78.7\,\text{K}$$

$$c_e = \sqrt{k \cdot R \cdot T_e} \qquad c_e = 138\,\frac{\text{m}}{\text{s}}$$

$$V_e = M_e \cdot c_e \qquad V_e = 600\,\frac{\text{m}}{\text{s}}$$

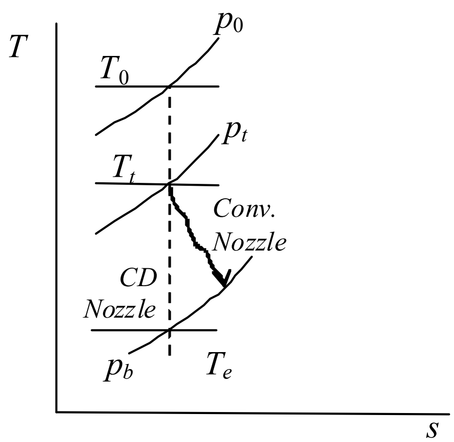
The mass flow rate is unchanged (choked flow)

From the momentum equation $R_x = m_{rate} \cdot V_e$ $\qquad R_x = 11.67\,\text{N}$

The percentage increase in thrust is $\dfrac{11.67 \cdot \text{N} - 8.60 \cdot \text{N}}{8.60 \cdot \text{N}} = 35.7 \cdot \%$

The exit area is obtained from $m_{rate} = \rho_e \cdot V_e \cdot A_e$ and $\rho_e = \dfrac{p_e}{R \cdot T_e}$ $\qquad \rho_e = 6.79\,\dfrac{\text{kg}}{\text{m}^3}$

$$A_e = \frac{m_{rate}}{\rho_e \cdot V_e} \qquad A_e = 4.77 \times 10^{-6}\,\text{m}^2 \qquad A_e = 4.77 \cdot \text{mm}^2$$



Problem 13.63

[Difficulty: 3]

13.63 A rocket motor is being tested at sea level where the pressure is 14.7 psia. The chamber pressure is 175 psia, the chamber temperature is 5400°R, and the nozzle has a throat area of 1 in^2. The exhaust gas has a ratio of specific heats of $k = 1.25$ and a gas constant $R = 70.6$ ft·lbf/lbm·°R. Assuming adiabatic, frictionless flow in the nozzle, determine (a) the nozzle exit area and velocity and (b) the thrust generated.

Given: Rocket motor

Find: Nozzle exit area, velocity, and thrust generated

Solution:

The given or available data is:

$R =$	70.6	ft-lbf/lbm-°R
$k =$	1.25	
$p_0 =$	175	psia
$T_0 =$	5400	°R
$A_t =$	1	in^2
$p_e =$	14.7	psia

Equations and Computations:

The exit Mach number can be calculated based on the pressure ratio:

$M_e =$ 2.2647

The isentropic area ratio at this Mach number is:

$A_e/A^* =$ 2.4151

So the nozzle exit area is:

$A_e =$ 2.42 in^2

The exit temperature can be found from the Mach number:

$T_e =$ 3290.4 °R

The sound speed at the exit is:

$c_e =$ 3057.8 ft/s

And so the exit flow speed is:

$V_e =$ 6925.2 ft/s

The density can be calculated using the ideal gas equation of state:

$\rho_e =$ 0.009112 lbm/ft^3

The nozzle is choked, and we can use the choked flow equation:

From p_0, T_0, A_t, and Eq. 13.10a

$$\dot{m}_{\text{choked}} = A_t p_0 \sqrt{\frac{k}{RT_0}} \left(\frac{2}{k+1}\right)^{(k+1)/2(k-1)} \qquad (13.10a)$$

$m =$ 1.058 lbm/s

Based on the momentum equation, we can calculate the thrust generated:

$R_x =$ 228 lbf

Note that since the flow expanded perfectly (the nozzle exit pressure is equal to the ambient pressure), the pressure terms drop out of the thrust calculation.

Problem 13.64

[Difficulty: 4]

13.64 If the rocket motor of Problem 13.63 is modified by cutting off the diverging portion of the nozzle, what will be the exit pressure and thrust?

Given: Rocket motor with converging-only nozzle

Find: Nozzle exit pressure and thrust

Solution:

The given or available data is:

$R =$	70.6	ft-lbf/lbm-°R
$k =$	1.25	
$p_0 =$	175	psia
$T_0 =$	5400	°R
$A_t =$	1	in^2
$p_b =$	14.7	psia

Equations and Computations:

If the diverging portion of the nozzle is removed, the exit Mach number is 1:

The exit Mach number can be calculated based on the pressure ratio:

$M_e =$ 1.0000

The isentropic area ratio at this Mach number is:

$A_e/A^* =$ 1.0000

So the nozzle exit area is:

$A_t =$ 1.00 in^2

The exit temperature and pressure can be found from the Mach number:

$T_e =$ 4800.0 °R

$p_e =$ 97.1 psia

The sound speed at the exit is:

$c_e =$ 3693.2 ft/s

And so the exit flow speed is:

$V_e =$ 3693.2 ft/s

The nozzle is choked, and we can use the choked flow equation:

From p_0, T_0, A_t, and Eq. 13.9a

$$\dot{m}_{\text{choked}} = A_e p_0 \sqrt{\frac{k}{RT_0}}\left(\frac{2}{k+1}\right)^{(k+1)/2(k-1)} \qquad (13.9a)$$

$m =$ 1.058 lbm/s

Based on the momentum equation, we can calculate the thrust generated:

$F =$ 204 lbf

Problem 13.65

[Difficulty: 3]

13.65 Consider the converging-diverging option of Problem 13.62. To what pressure would the compressed gas need to be raised (keeping the temperature at 20°C) to develop a thrust of 15N? (Assume isentropic flow.)

Given: CO_2 cartridge and convergent nozzle

Find: Tank pressure to develop thrust of 15 N

Solution:

The given or available data is:

$R =$	188.9	J/kg·K
$k =$	1.29	
$T_0 =$	293	K
$p_b =$	101	kPa
$D_t =$	0.5	mm

Equations and Computations:

$A_t =$ 0.196 mm^2

The momentum equation gives

$$R_x = m_{flow} V_e$$

Hence, we need m_{flow} and V_e

For isentropic flow

$p_e = p_b$

$p_e =$ 101 kPa

If we knew p_0 we could use it and p_e, and Eq. 13.7a, to find M_e.

Once M_e is known, the other exit conditions can be found.

Make a guess for p_0, and eventually use *Goal Seek* (see below).

$p_0 =$ 44.6 MPa

From p_0 and p_e, and Eq. 13.7a
(using built-in function *IsenMfromp* (M,k))

$$\frac{p_0}{p} = \left[1 + \frac{k-1}{2}M^2\right]^{k/(k-1)} \qquad (13.7a)$$

$M_e =$ 4.5

From M_e and T_0 and Eq. 13.7b
(using built-in function $IsenT(M, k)$)

$$\frac{T_0}{T} = 1 + \frac{k-1}{2}M^2 \qquad (13.7b)$$

$T_e =$ 74.5 K

From T_e and Eq. 12.18

$$c = \sqrt{kRT} \qquad (12.18)$$

$c_e =$ 134.8 m/s

Then $V_e =$ 606 m/s

The mass flow rate is obtained from p_0, T_0, A_t, and Eq. 13.10a

$$\dot{m}_{\text{choked}} = A_t p_0 \sqrt{\frac{k}{RT_0}} \left(\frac{2}{k+1}\right)^{(k+1)/2(k-1)} \qquad (13.10a)$$

$m_{\text{choked}} =$ 0.0248 kg/s

Finally, the momentum equation gives

$R_x = m_{\text{flow}} V_e$
$=$ 15.0 N

We need to set R_x to 15 N. To do this use *Goal Seek*
to vary p_0 to obtain the result!

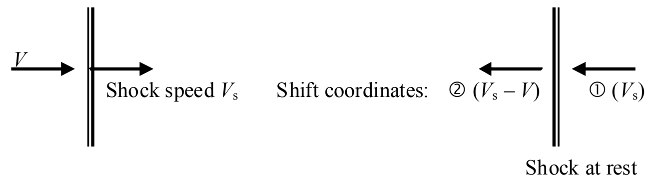
Problem 13.66

[Difficulty: 3]

13.66 Testing of a demolition explosion is to be evaluated. Sensors indicate that the shock wave generated at the instant of explosion is 30 MPa (abs). If the explosion occurs in air at 20°C and 101 kPa, find the speed of the shock wave, and the temperature and speed of the air just after the shock passes. As an approximation assume $k = 1.4$. (Why is this an approximation?)

Given: Normal shock due to explosion

Find: Shock speed; temperature and speed after shock



Solution:

Basic equations:

$$M_2^2 = \frac{M_1^2 + \frac{2}{k-1}}{\left(\frac{2 \cdot k}{k-1}\right) \cdot M_1^2 - 1} \qquad V = M \cdot c = M \cdot \sqrt{k \cdot R \cdot T}$$

$$\frac{p_2}{p_1} = \frac{2 \cdot k}{k+1} \cdot M_1^2 - \frac{k-1}{k+1} \qquad \frac{T_2}{T_1} = \frac{\left(1 + \frac{k-1}{2} \cdot M_1^2\right) \cdot \left(k \cdot M_1^2 - \frac{k-1}{2}\right)}{\left(\frac{k+1}{2}\right)^2 \cdot M_1^2}$$

Given or available data $\quad k = 1.4 \quad R = 286.9 \cdot \frac{J}{kg \cdot K} \quad p_2 = 30 \cdot MPa \quad p_1 = 101 \cdot kPa \quad T_1 = (20 + 273) \cdot K$

From the pressure ratio

$$M_1 = \sqrt{\left(\frac{k+1}{2 \cdot k}\right) \cdot \left(\frac{p_2}{p_1} + \frac{k-1}{k+1}\right)} \qquad M_1 = 16.0$$

Then we have

$$T_2 = T_1 \cdot \frac{\left(1 + \frac{k-1}{2} \cdot M_1^2\right) \cdot \left(k \cdot M_1^2 - \frac{k-1}{2}\right)}{\left(\frac{k+1}{2}\right)^2 \cdot M_1^2} \qquad T_2 = 14790\,K \qquad T_2 = 14517 \cdot °C$$

$$M_2 = \sqrt{\frac{M_1^2 + \frac{2}{k-1}}{\left(\frac{2 \cdot k}{k-1}\right) \cdot M_1^2 - 1}} \qquad M_2 = 0.382$$

Then the speed of the shock ($V_s = V_1$) is $\quad V_1 = M_1 \cdot \sqrt{k \cdot R \cdot T_1} \quad V_1 = 5475 \frac{m}{s} \quad V_s = V_1 \quad V_s = 5475 \frac{m}{s}$

After the shock (V_2) the speed is $\quad V_2 = M_2 \cdot \sqrt{k \cdot R \cdot T_2} \quad V_2 = 930 \frac{m}{s}$

But we have $\quad V_2 = V_s - V \quad V = V_s - V_2 \quad V = 4545 \frac{m}{s}$

These results are unrealistic because at the very high post-shock temperatures experienced, the specific heat ratio will NOT be constant! The extremely high initial air velocity and temperature will rapidly decrease as the shock wave expands in a spherical manner and thus weakens.

Problem 13.67

[Difficulty: 2]

13.67 A standing normal shock occurs in air which is flowing at a Mach number of 1.75. What are the pressure and temperature ratios across the shock? What is the increase in entropy across the shock?

Given: Standing normal shock

Find: Pressure and temperature ratios; entropy increase

Solution:

The given or available data is:

	$R =$	286.9	J/kg-K
	$c_p =$	1004	J/kg-K
	$k =$	1.4	
	$M_1 =$	1.75	

Equations and Computations:

The pressure ratio is:

$p_2/p_1 =$ 3.41

The tempeature ratio is:

$T_2/T_1 =$ 1.495

The entropy increase across the shock is:

$\Delta s =$ 51.8 J/kg-K

Problem 13.68

[Difficulty: 3]

13.68 Air flows into a converging duct, and a normal shock stands at the exit of the duct. Downstream of the shock, the Mach number is 0.54. If $p_2/p_1 = 2$, compute the Mach number at the entrance of the duct and the area ratio A_1/A_2.

Given: Air flowing into converging duct, normal shock standing at duct exit

Find: Mach number at duct entrance, duct area ratio

Solution:

The given or available data is:

$R =$ 286.9 J/kg-K
$c_p =$ 1004 J/kg-K
$k =$ 1.4
$M_3 =$ 0.54
$p_2/p_1 =$ 2

Equations and Computations:

For the given post-shock Mach number, there can be only one Mach number upstream of the shock wave:

$M_2 =$ 2.254
$M_3 =$ 0.5400

(We used Solver to match the post-shock Mach number by varying M_2.)

The stagnation pressure is constant in the duct:

$p_0/p_2 =$ 11.643
$p_0/p_1 =$ 23.285

So the duct entrance Mach number is:

$M_1 =$ 2.70

The isentropic area ratios at stations 1 and 2 are:

$A_1/A^* =$ 3.1832
$A_2/A^* =$ 2.1047

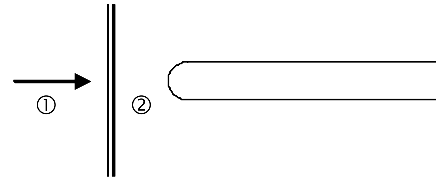
So the duct area ratio is:

$A_1/A_2 =$ 1.512

Problem 13.69

[Difficulty: 2]

13.69 A normal shock occurs when a pitot-static tube is inserted into a supersonic wind tunnel. Pressures measured by the tube are $p_{0_2} = 10$ psia and $p_2 = 8$ psia. Before the shock, $T_1 = 285°$R and $p_1 = 1.75$ psia. Calculate the air speed in the wind tunnel.



Given: Normal shock near pitot tube

Find: Air speed

Solution:

Basic equations: $p_1 - p_2 = \rho_1 \cdot V_1 \cdot (V_2 - V_1)$ (Momentum) $\qquad \dfrac{p_0}{p} = \left(1 + \dfrac{k-1}{2} \cdot M^2\right)^{\frac{k}{k-1}}$

Given or available data $T_1 = 285 \cdot R$ $\qquad p_1 = 1.75 \cdot psi$ $\qquad p_{02} = 10 \cdot psi$ $\qquad p_2 = 8 \cdot psi$

$k = 1.4$ $\qquad R_{air} = 53.33 \cdot \dfrac{ft \cdot lbf}{lbm \cdot R}$

At state 2 $\qquad M_2 = \sqrt{\dfrac{2}{k-1} \cdot \left[\left(\dfrac{p_{02}}{p_2}\right)^{\frac{k-1}{k}} - 1\right]}$ $\qquad M_2 = 0.574$

From momentum $\qquad p_1 - p_2 = \rho_2 \cdot V_2^2 - \rho_1 \cdot V_1^2$ $\qquad$ but $\qquad \rho \cdot V^2 = \rho \cdot c^2 \cdot M^2 = \dfrac{p}{R \cdot T} \cdot k \cdot R \cdot T \cdot M^2 = k \cdot p \cdot M^2$

$p_1 - p_2 = k \cdot p_2 \cdot M_2^2 - k \cdot p_1 \cdot M_1^2$ $\qquad$ or $\qquad p_1 \cdot \left(1 + k \cdot M_1^2\right) = p_2 \cdot \left(1 + k \cdot M_2^2\right)$

Hence $\qquad M_1 = \sqrt{\dfrac{1}{k} \cdot \left[\dfrac{p_2}{p_1} \cdot \left(1 + k \cdot M_2^2\right) - 1\right]}$ $\qquad M_1 = 2.01$

Also $\qquad c_1 = \sqrt{k \cdot R_{air} \cdot T_1}$ $\qquad c_1 = 827 \cdot \dfrac{ft}{s}$

Then $\qquad V_1 = M_1 \cdot c_1$ $\qquad V_1 = 1666 \cdot \dfrac{ft}{s}$

Note: With $p_1 = 1.5$ psi we obtain $\qquad V_1 = 1822 \cdot \dfrac{ft}{s}$

(Using normal shock functions, for $\dfrac{p_2}{p_1} = 4.571$ we find $\qquad M_1 = 2.02$ $\qquad M_2 = 0.573$ Check!)

Problem 13.70

[Difficulty: 3]

13.70 A large tank containing air at 125 psia and 175°F is attached to a converging-diverging nozzle that has a throat area of 1.5 in.² through which the air is exiting. A normal shock sits at a point in the nozzle where the area is 2.5 in.². The nozzle exit area is 3.5 in.². What are the Mach numbers just after the shock and at the exit? What are the stagnation and static pressures before and after the shock?

Given: C-D nozzle with normal shock

Find: Mach numbers at the shock and at exit; Stagnation and static pressures before and after the shock

Solution:

Basic equations: Isentropic flow
$$\frac{A}{A_{crit}} = \frac{1}{M}\cdot\left(\frac{1 + \frac{k-1}{2}\cdot M^2}{\frac{k+1}{2}}\right)^{\frac{k+1}{2\cdot(k-1)}} \qquad \frac{p_0}{p} = \left(1 + \frac{k-1}{2}\cdot M^2\right)^{\frac{k}{k-1}}$$

Normal shock
$$M_2^{\,2} = \frac{M_1^{\,2} + \frac{2}{k-1}}{\left(\frac{2\cdot k}{k-1}\right)\cdot M_1^{\,2} - 1} \qquad \frac{p_2}{p_1} = \frac{2\cdot k}{k+1}\cdot M_1^{\,2} - \frac{k-1}{k+1} \qquad \frac{p_{02}}{p_{01}} = \frac{\left(\frac{\frac{k+1}{2}\cdot M_1^{\,2}}{1 + \frac{k-1}{2}\cdot M_1^{\,2}}\right)^{\frac{k}{k-1}}}{\left(\frac{2\cdot k}{k+1}\cdot M_1^{\,2} - \frac{k-1}{k+1}\right)^{\frac{1}{k-1}}}$$

Given or available data $k = 1.4$ $\quad R_{air} = 53.33\cdot\frac{ft\cdot lbf}{lbm\cdot R}$ $\quad p_{01} = 125\cdot psi$ $\quad T_0 = (175 + 460)\cdot R$

$A_t = 1.5\cdot in^2$ $\quad A_s = 2.5\cdot in^2$ (Shock area) $\quad A_e = 3.5\cdot in^2$

Because we have a normal shock the CD must be accelerating the flow to supersonic so the throat is at critical state.

$$A_{crit} = A_t$$

At the shock we have $\frac{A_s}{A_{crit}} = 1.667$

At this area ratio we can find the Mach number before the shock from the isentropic relation
$$\frac{A_s}{A_{crit}} = \frac{1}{M_1}\cdot\left(\frac{1 + \frac{k-1}{2}\cdot M_1^{\,2}}{\frac{k+1}{2}}\right)^{\frac{k+1}{2\cdot(k-1)}}$$

Solving iteratively (or using *Excel*'s *Solver*, or even better the function *isenMsupfromA* from the Web site!) $\quad M_1 = 1.985$

The stagnation pressure before the shock was given: $\quad p_{01} = 125\cdot psi$

The static pressure is then
$$p_1 = \frac{p_{01}}{\left(1 + \frac{k-1}{2}\cdot M_1^{\,2}\right)^{\frac{k}{k-1}}} \qquad p_1 = 16.4\cdot psi$$

After the shock we have $$M_2 = \sqrt{\frac{M_1^2 + \frac{2}{k-1}}{\left(\frac{2\cdot k}{k-1}\right)\cdot M_1^2 - 1}}$$ $M_2 = 0.580$

Also $$p_{02} = p_{01}\cdot\frac{\left(\frac{\frac{k+1}{2}\cdot M_1^2}{1 + \frac{k-1}{2}\cdot M_1^2}\right)^{\frac{k}{k-1}}}{\left(\frac{2\cdot k}{k+1}\cdot M_1^2 - \frac{k-1}{k+1}\right)^{\frac{1}{k-1}}}$$ $p_{02} = 91.0\cdot\text{psi}$

and $$p_2 = p_1\cdot\left(\frac{2\cdot k}{k+1}\cdot M_1^2 - \frac{k-1}{k+1}\right)$$ $p_2 = 72.4\cdot\text{psi}$

Finally, for the Mach number at the exit, we could find the critical area change across the shock; instead we find the new critical area from isentropic conditions at state 2.

$$A_{crit2} = A_s\cdot M_2\cdot\left(\frac{1 + \frac{k-1}{2}\cdot M_2^2}{\frac{k+1}{2}}\right)^{-\frac{k+1}{2\cdot(k-1)}}$$ $A_{crit2} = 2.06\cdot\text{in}^2$

At the exit we have $$\frac{A_e}{A_{crit2}} = 1.698$$

At this area ratio we can find the Mach number before the shock from the isentropic relation $$\frac{A_e}{A_{crit2}} = \frac{1}{M_e}\cdot\left(\frac{1 + \frac{k-1}{2}\cdot M_e^2}{\frac{k+1}{2}}\right)^{\frac{k+1}{2\cdot(k-1)}}$$

Solving iteratively (or using *Excel*'s *Solver*, or even better the function *isenMsubfromA* from the Web site!) $M_e = 0.369$

These calculations are obviously a LOT easier using the *Excel* functions available on the Web site!

Problem 13.71

[Difficulty: 2]

13.71 A total-pressure probe is placed in a supersonic wind tunnel where $T = 530°R$ and $M = 2.0$. A normal shock stands in front of the probe. Behind the shock, $M_2 = 0.577$ and $p_2 = 5.76$ psia. Find (a) the downstream stagnation pressure and stagnation temperature and (b) all fluid properties upstream from the shock. Show static and stagnation state points and the process path on a Ts diagram.

Solution:

Computing equations: $p(1+kM^2) = \text{const.}$ (across shock)

$$\frac{T_0}{T} = 1 + \frac{k-1}{2}M^2 \qquad \frac{p_0}{p} = \left(\frac{T_0}{T}\right)^{k/k-1}$$

Assumptions: (1) steady flow (3) uniform flow at a section
(2) ideal gas

Across the shock $p(1+kM^2) = \text{const}$

$$\therefore p_1 = p_2\frac{(1+kM_2^2)}{(1+kM_1^2)} = 5.76\ \text{psia}\ \frac{[1+1.4(0.5774)^2]}{[1+1.4(2)^2]} = 1.28\ \text{psia} \longleftarrow p_1$$

$$\rho_1 = \frac{p_1}{RT_1} = 1.28\ \frac{\text{lbf}}{\text{in}^2} \times \frac{\text{lbm}\cdot{}^\circ\text{R}}{53.3\ \text{ft}\cdot\text{lbf}} \times \frac{1}{530^\circ\text{R}} \times \frac{144\ \text{in}^2}{\text{ft}^2} = 0.00653\ \text{lbm/ft}^3 \longleftarrow \rho_1$$

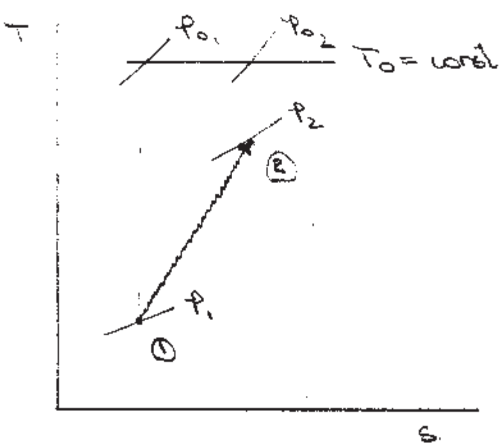
$$V_1 = M_1c_1 = M_1(kRT_1)^{1/2} = 2.0\left[1.4 \times 53.3\ \frac{\text{ft}\cdot\text{lbf}}{\text{lbm}\cdot{}^\circ\text{R}} \times 530^\circ\text{R} \times 32.2\ \frac{\text{lbm}}{\text{slug}} \times \frac{\text{slug}\cdot\text{ft}}{\text{lbf}\cdot\text{s}^2}\right]^{1/2} = 2260\ \text{ft/s} \longleftarrow V_1$$

$$T_{0_1} = T_1\left(1+\frac{k-1}{2}M_1^2\right) = 530^\circ\text{R}[1+0.2(2)^2] = 954^\circ\text{R} \longleftarrow T_{0_1}$$

$$p_{0_1} = p_1\left(1+\frac{k-1}{2}M_1^2\right)^{k/k-1} = 1.28\ \text{psia}\,(1.8)^{3.5} = 10.0\ \text{psia} \longleftarrow p_{0_1}$$

$$p_{0_2} = p_2\left(1+\frac{k-1}{2}M_2^2\right)^{k/k-1} = 5.76\ \text{psia}\,[1+0.2(0.5774)^2]^{3.5} = 7.22\ \text{psia} \longleftarrow p_{0_2}$$

$$T_{0_2} = T_{0_1} = 954\ \text{R} \longleftarrow$$



Compressible-Flow Functions (Appendix E)

For $M_1 = 2.0$, from App. E.4

$M_2 = 0.577$ (12.34b)

$p_2/p_1 = 4.50$ (12.36a) $\therefore p_1 = 1.28$ psia

$p_{0_2}/p_{0_1} = 0.721$ (12.37)

For $M_1 = 2.0$, from App. E.1

$p_1/p_{0_1} = 0.1278$ (11.17a) $\therefore p_{0_1} = 10.0$ psia

and $p_{0_2} = 0.721\,p_{0_1} = 7.21$ psia

$T_1/T_{0_1} = 0.556$ (11.17b) $\therefore T_{0_1} = 954^\circ$R

Note: In using the tables it is not necessary to know the downstream Mach number.

Problem 13.72

[Difficulty: 2]

13.72 Air flows steadily through a long, insulated constant-area pipe. At section ①, $M_1 = 2.0$, $T_1 = 140°F$, and $p_1 = 35.9$ psia. At section ②, downstream from a normal shock, $V_2 = 1080$ ft/s. Determine the density and Mach number at section ②. Make a qualitative sketch of the pressure distribution along the pipe.

Solution:

Basic equation: $h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2}$

Assumptions: (1) steady flow (2) uniform flow at a section (3) $\dot{Q} = \dot{W}_s = \dot{W}_{shear} = 0$ (4) $\Delta z = 0$ (5) ideal gas (6) $A_1 = A_2 = A$

$$V_1 = M_1 c_1 = M_1 (kRT_1)^{1/2} = 2.0\left(1.4 \times 53.3 \frac{ft \cdot lbf}{lbm \cdot °R} \times 600°R \times 32.2 \frac{lbm}{slug} \times \frac{slug \cdot ft}{lbf \cdot s^2}\right)^{1/2} = 2400 \text{ ft/s}$$

$$h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2} \qquad T_2 = T_1 + \frac{1}{2c_p}(V_1^2 - V_2^2)$$

$$T_2 = 600°R + \frac{1}{2}\left[(2.4)^2 - 1.08^2\right] \times 10^6 \frac{ft^2}{s^2} \times \frac{lbm \cdot °R}{0.24\, Btu} \times \frac{Btu}{778\, ft \cdot lbf} \times \frac{slug}{32.2\, lbm} \times \frac{lbf \cdot s^2}{slug \cdot ft}$$

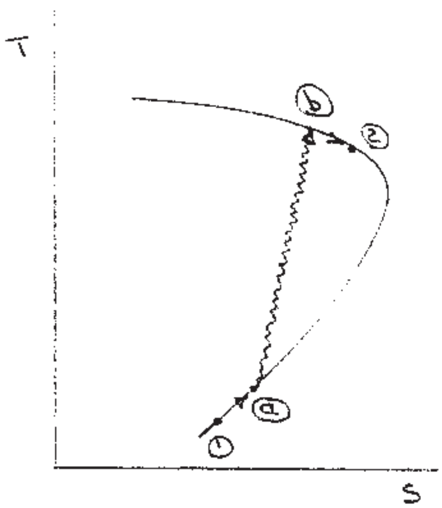
$$T_2 = 982°R$$

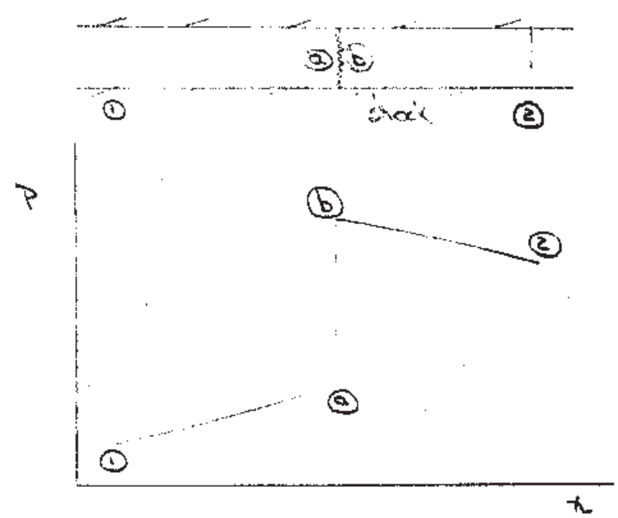
$$M_2 = \frac{V_2}{c_2} \qquad c_2 = (kRT_2)^{1/2} = \left(1.4 \times 53.3 \frac{ft \cdot lbf}{lbm \cdot °R} \times 982°R \times 32.2 \frac{lbm}{slug} \times \frac{slug \cdot ft}{lbf \cdot s^2}\right)^{1/2} = 1540 \text{ ft/s}$$

$$M_2 = \frac{1080}{1540} = 0.701$$ M_2

From continuity $\rho_1 V_1 = \rho_2 V_2$ $\qquad \rho_2 = \frac{V_1}{V_2}\rho_1 = \frac{V_1}{V_2}\frac{p_1}{RT_1}$

$$\rho_2 = \frac{2400}{1080} \times 35.9 \frac{lbf}{in^2} \times \frac{lbm \cdot °R}{53.3\, ft \cdot lbf} \times \frac{1}{600°R} \times \frac{144\, in^2}{ft^2} = 0.359 \text{ lbm/ft}^3$$ ρ_2





Problem 13.73

[Difficulty: 3]

13.73 A wind tunnel nozzle is designed to operate at a Mach number of 5. To check the flow velocity, a pitot probe is placed at the nozzle exit. Since the probe tip is blunt, a normal shock stands off the tip of the probe. If the nozzle exit static pressure is 10 kPa, what absolute pressure should the pitot probe measure? If the stagnation temperature before the nozzle is 1450 K, what is the nozzle exit velocity?

Given: Pitot probe used in supersonic wind tunnel nozzle

Find: Pressure measured by pitot probe; nozzle exit velocity

Solution:

The given or available data is:

$R =$	286.9	J/kg-K
$k =$	1.4	
$M_1 =$	5	
$p_1 =$	10	kPa
$T_0 =$	1450	K

Equations and Computations:

Downstream of the normal shock wave, the Mach number is:

$M_2 =$ 0.4152

The static and stagnation pressure ratios are:

$p_2/p_1 =$ 29.000

$p_{02}/p_{01} =$ 0.06172

So the static pressure after the shock is:

$p_2 =$ 290 kPa

The pitot pressure, however, is the stagnation pressure:

$p_{02}/p_2 =$ 1.12598

$p_{02} =$ 327 kPa

The static temperature at the nozzle exit can be calculated:

$T_{01}/T_1 =$ 6.000

$T_1 =$ 241.67 K

At the nozzle exit the sound speed is:

$c_2 =$ 311.56 m/s

Therefore the flow velocity at the nozzle exit is:

$V_2 =$ 1558 m/s

Problem 13.74

[Difficulty: 3]

13.74 Air approaches a normal shock at $V_1 = 900$ m/s, $p_1 = 50$ kPa, and $T_1 = 220$ K. What are the velocity and pressure after the shock? What would the velocity and pressure be if the flow were decelerated isentropically to the same Mach number?

Given: Air approaching a normal shock

Find: Pressure and velocity after the shock; pressure and velocity if flow were decelerated isentropically

Solution:

The given or available data is:

$R =$	286.9	J/kg-K
$k =$	1.4	
$V_1 =$	900	m/s
$p_1 =$	50	kPa
$T_1 =$	220	K

Equations and Computations:

The sonic velocity at station 1 is:

$c_1 =$ 297.26 m/s

So the Mach number at 1 is:

$M_1 =$ 3.028

Downstream of the normal shock wave, the Mach number is:

$M_2 =$ 0.4736

The static pressure and temperature ratios are:

$p_2/p_1 =$ 10.528

$T_2/T_1 =$ 2.712

So the exit temperature and pressure are:

$p_2 =$ 526 kPa

$T_2 =$ 596.6 K

At station 2 the sound speed is:

$c_2 =$ 489.51 m/s

Therefore the flow velocity is:

$V_2 =$ 232 m/s

If we decelerate the flow isentropically to

$M_{2s} =$ 0.4736

The isentropic pressure ratios at station 1 and 2s are:

$p_0/p_1 =$ 38.285

$p_0/p_{2s} =$ 1.166

$p_{2s}/p_1 =$ 32.834

So the final pressure is:

$p_{2s} =$ 1642 kPa

The temperature ratios are:

$T_0/T_1 =$ 2.833

$T_0/T_{2s} =$ 1.045

$T_{2s}/T_1 =$ 2.712

So the final temperature is:

$T_{2s} =$ 596.6 K

The sonic velocity at station 2s is:

$c_{2s} =$ 489.51 m/s

Therefore the flow velocity is:

$V_{2s} =$ 232 m/s

Problem 13.75 [Difficulty: 3]

13.75 Air with stagnation conditions of 150 psia and 400°F accelerates through a converging-diverging nozzle with throat area 3 in.^{2}. A normal shock is located where the area is 6 in.^{2}. What is the Mach number before and after the shock? What is the rate of entropy generation through the nozzle, if there is negligible friction between the flow and the nozzle walls?

Given: Air accelerating through a converging-diverging nozzle, passes through a normal shock

Find: Mach number before and after shock; entropy generation

Solution:

The given or available data is:

R =	53.33	ft-lbf/lbm-°R
k =	1.4	
p_{01} =	150	psia
T_{01} =	400	°F
T_{01} =	860	°R
A_t =	3	in^2
$A_1 = A_2$ =	6	in^2

Equations and Computations:

The isentropic area ratio at the station of interest is:

$$A_1/A_1^* = 2.00$$

So the Mach number at 1 is:

$$M_1 = 2.20$$

Downstream of the normal shock wave, the Mach number is:

$$M_2 = 0.547$$

The total pressure ratio across the normal shock is:

$$p_{02}/p_{01} = 0.6294$$

Since stagnation temperature does not change across a normal shock,
the increase in entropy is related to the stagnation pressure loss only:

$$\Delta s_{1\text{-}2} = 24.7 \text{ ft-lbf/lbm-°R}$$

$$\Delta s_{1\text{-}2} = 0.0317 \text{ Btu/lbm-°R}$$

Problem 13.76

[Difficulty: 2]

13.76 Air approaches a normal shock at $M_1 = 2.5$, with $T_{0_1} = 1250°\text{R}$ and $p_1 = 20$ psia. Determine the speed and temperature of the air leaving the shock and the entropy change across the shock.

Given: Normal shock

Find: Speed and temperature after shock; Entropy change

Solution:

The given or available data is:

$R =$	53.33	ft·lbf/lbm·R	0.0685	Btu/lbm·R
$k =$	1.4			
$c_p =$	0.2399	Btu/lbm·R		
$T_{01} =$	1250	^{o}R		
$p_1 =$	20	psi		
$M_1 =$	2.5			

Equations and Computations:

From $p_1 = \rho_1 R T_1$ $\quad \rho_1 =$ 0.0432 slug/ft^3

$V_1 =$ 4334 ft/s

Using built-in function *IsenT* (M,k):

$T_{01}/T_1 =$ 2.25 $\quad T_1 =$ 556 ^{o}R

96 ^{o}F

Using built-in function *NormM2fromM* (M,k):

$M_2 =$ 0.513

Using built-in function *NormTfromM* (M,k):

$T_2/T_1 =$ 2.14 $\quad T_2 =$ 1188 ^{o}R

728 ^{o}F

Using built-in function *NormpfromM* (M,k):

$p_2/p_1 =$ 7.13 $\quad p_2 =$ 143 psi

From $V_2 = M_2\sqrt{kRT_2}$ $\quad V_2 =$ 867 ft/s

From $$\Delta s = c_p \ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{p_2}{p_1}\right)$$

$\Delta s =$ 0.0476 Btu/lbm·R

37.1 ft·lbf/lbm·R

Problem 13.77

[Difficulty: 2]

13.77 Air undergoes a normal shock. Upstream, $T_1 = 35°C$, $p_1 = 229$ kPa (abs), and $V_1 = 704$ m/s. Determine the temperature and stagnation pressure of the air stream leaving the shock.

Solution:

Compressible flow functions (Appendix E) to be used in solution

Assumptions:
1) steady flow
2) uniform flow at a section
3) $\dot{Q} = \dot{W}_s = \dot{W}_{shear} = 0$
4) $\Delta z = 0$
5) $F_{B_x} = 0$
6) no friction forces
7) ideal gas
8) $A_1 = A_2 = A$

$M_1 = \frac{V_1}{c_1}$ $\quad c_1 = (kRT_1)^{1/2} = \left(1.4 \times 287 \frac{N \cdot m}{kg \cdot K} \times 308\,K \times \frac{kg \cdot m}{N \cdot s^2}\right)^{1/2} = 352$ m/s

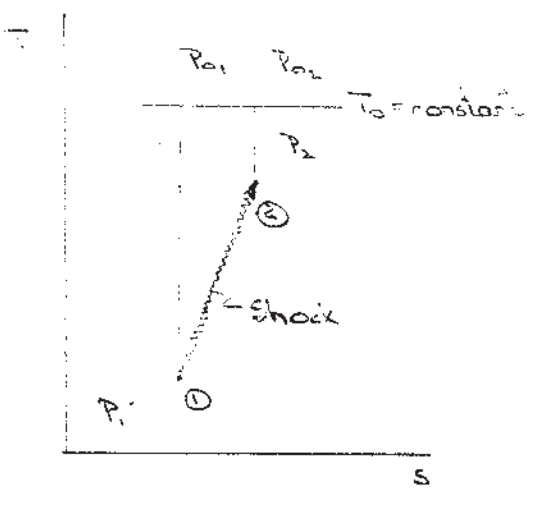
$M_1 = \frac{704}{352} = 2.0$

From App. E.1 $\quad p_1/p_{0_1} = 0.1278$

From App. E.4 $\quad p_{0_2}/p_{0_1} = 0.7209 \quad T_2/T_1 = 1.687$

$T_2 = \frac{T_2}{T_1} T_1 = 1.687 \times 308\,K = 520\,K$ $\quad T_2$

$p_{0_2} = \frac{p_{0_2}}{p_{0_1}} \frac{p_{0_1}}{p_1} p_1 = 0.7209 \times \frac{1}{0.1278} \times 229$ kPa $= 1.29$ MPa (abs) $\quad p_{0_2}$



Problem 13.78

[Difficulty: 2]

13.78 A normal shock stands in a constant-area duct. Air approaches the shock with $T_{0_1} = 550\,\text{K}$, $p_{0_1} = 650\,\text{kPa (abs)}$, and $M_1 = 2.5$. Determine the static pressure downstream from the shock. Compare the downstream pressure with that reached by decelerating isentropically to the same subsonic Mach number.

Given: Normal shock

Find: Pressure after shock; Compare to isentropic deceleration

Solution:

The given or available data is:

$R =$	286.9	J/kg K
$k =$	1.4	
$T_{01} =$	550	K
$p_{01} =$	650	kPa
$M_1 =$	2.5	

Equations and Computations:

Using built-in function *Isenp* (M,k):

$p_{01}/p_1 =$ 17.09 $\qquad p_1 =$ 38 kPa

Using built-in function *NormM2fromM* (M,k):

$M_2 =$ 0.513

Using built-in function *NormpfromM* (M,k):

$p_2/p_1 =$ 7.13 $\qquad p_2 =$ 271 kPa

Using built-in function *Isenp* (M,k) at M_2:

$p_{02}/p_2 =$ 1.20

But for the isentropic case: $p_{02} = p_{01}$

Hence for isentropic deceleration: $p_2 =$ 543 kPa

Problem 13.79

[Difficulty: 2]

13.79 A normal shock occurs in air at a section where $V_1 = 2000$ mph, $T_1 = -15°F$, and $p_1 = 5$ psia. Determine the speed and Mach number downstream from the shock, and the change in stagnation pressure across the shock.

Given: Normal shock

Find: Speed and Mach number after shock; Change in stagnation pressure

Solution:

The given or available data is:

$R =$	53.33	ft·lbf/lbm·R	0.0685	Btu/lbm·R
$k =$	1.4			
$T_1 =$	445	^{o}R		
$p_1 =$	5	psi		
$V_1 =$	2000	mph	2933	ft/s

Equations and Computations:

From $c_1 = \sqrt{kRT_1}$ $\quad c_1 =$ 1034 ft/s

Then $\quad M_1 =$ 2.84

Using built-in function *NormM2fromM* (M,k):

$M_2 =$ 0.486

Using built-in function *NormdfromM* (M,k):

$\rho_2/\rho_1 =$ 3.70

Using built-in function *Normp0fromM* (M,k):

$p_{02}/p_{01} =$ 0.378

Then $V_2 = \dfrac{\rho_1}{\rho_2} V_1$ $\quad V_2 =$ 541 mph $\quad$ 793 ft/s

Using built-in function *Isenp* (M,k) at M_1:

$p_{01}/p_1 =$ 28.7

From the above ratios and given p_1:

$p_{01} =$ 143 psi

$p_{02} =$ 54.2 psi

$p_{01} - p_{02} =$ 89.2 psi

Problem 13.80

[Difficulty: 2]

13.80 Air approaches a normal shock with $T_1 = -7.5°F$, $p_1 = 14.7$ psia, and $V_1 = 1750$ mph. Determine the speed immediately downstream from the shock and the pressure change across the shock. Calculate the corresponding pressure change for a frictionless, shockless deceleration between the same speeds.

Given: Normal shock

Find: Speed; Change in pressure; Compare to shockless deceleration

Solution:

The given or available data is:

$R =$	53.33	ft·lbf/lbm·R	0.0685	Btu/lbm·R
$k =$	1.4			
$T_1 =$	452.5	^oR		
$p_1 =$	14.7	psi		
$V_1 =$	1750	mph	2567	ft/s

Equations and Computations:

From $c_1 = \sqrt{kRT_1}$ $\quad c_1 = 1043$ ft/s

Then $\quad M_1 = 2.46$

Using built-in function *NormM2fromM* (M,k):

$M_2 = 0.517$

Using built-in function *NormdfromM* (M,k):

$\rho_2/\rho_1 = 3.29$

Using built-in function *NormpfromM* (M,k):

$p_2/p_1 = 6.90$ $\quad p_2 = 101$ psi

$p_2 - p_1 = 86.7$ psi

Then $V_2 = \dfrac{\rho_1}{\rho_2}V_1$ $\quad V_2 = 532$ mph $\quad 781$ ft/s

Using built-in function *Isenp* (M,k) at M_1:

$p_{01}/p_1 = 16.1$

Using built-in function *Isenp* (M,k) at M_2:

$p_{02}/p_2 = 1.20$

From above ratios and p_1, for isentropic flow (p_0 = const): $\quad p_2 = 197$ psi

$p_2 - p_1 = 182$ psi

Problem 13.81

[Difficulty: 2]

13.81 A supersonic aircraft cruises at $M = 2.2$ at 12 km altitude. A pitot tube is used to sense pressure for calculating air speed. A normal shock stands in front of the tube. Evaluate the local isentropic stagnation conditions in front of the shock. Estimate the stagnation pressure sensed by the pitot tube. Show static and stagnation state points and the process path on a *Ts* diagram.

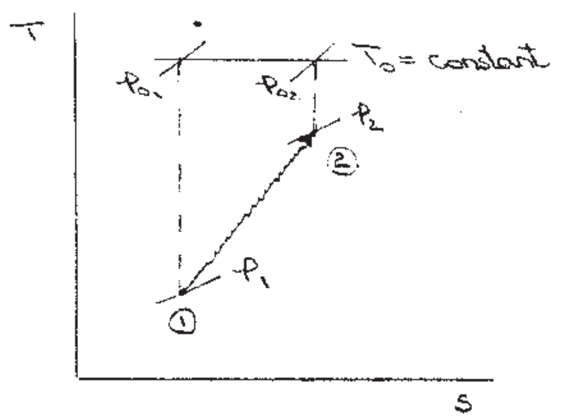
Solution: Compressible flow functions (Appendix E) to be used in solution

Assumptions: (1) steady flow (2) uniform flow at a section
(3) thin shock
(4) ideal gas

Use table A.3 to determine properties at state ① At 12 km altitude,
$T_1 = 216.7\ K$ $\quad p_1 = 19.4$ kPa

From App. E.1, for $M_1 = 2.2$,
$T_1/T_{0_1} = 0.5081$ $\quad \therefore T_{0_1} = 426\ K$ ← T_{01}
$p_1/p_{0_1} = 0.09352$ $\quad \therefore p_{0_1} = 207$ kPa (abs) ← p_{01}

From App. E.4, for $M_1 = 2.2$, $M_2 = 0.5471$
$p_{0_2}/p_{0_1} = 0.6281$ $\quad \therefore p_{0_2} = 130$ kPa (abs) ← p_{02}



Problem 13.82

[Difficulty: 2]

13.82 The Concorde supersonic transport flew at $M = 2.2$ at 20 km altitude. Air is decelerated isentropically by the engine inlet system to a local Mach number of 1.3. The air passed through a normal shock and was decelerated further to $M = 0.4$ at the engine compressor section. Assume, as a first approximation, that this subsonic diffusion process was isentropic and use standard atmosphere data for freestream conditions. Determine the temperature, pressure, and stagnation pressure of the air entering the engine compressor.

Solution:

Compressible flow functions (Appendix E) to be used in solution

Assumptions: 1) Steady flow
2) Ideal gas
3) Flow is isentropic except across the shock.

At 20 km altitude $T_\infty = 217\ \text{K}$, $P_\infty = 5.53\ \text{kPa}$ (Table A.3)

$M_\infty = 2.2$ From App. E.1, $T_\infty/T_0 = 0.5081$ $\quad P_\infty/P_0 = 0.09352$

$\therefore T_0 = 427\ \text{K}$ $\quad P_0 = 59.1\ \text{kPa}$

$M_1 = 1.3$ From App. E.1, $T_1/T_0 = 0.7474$ $\quad P_1/P_0 = 0.3609$

$\therefore T_1 = 319\ \text{K}$ $\quad P_1 = 21.3\ \text{kPa}$

From App. E.4, $M_2 = 0.786$ $\quad P_{02}/P_{01} = 0.9794$ $\quad \dfrac{T_2}{T_1} = 1.191$ $\quad \dfrac{P_2}{P_1} = 1.805$

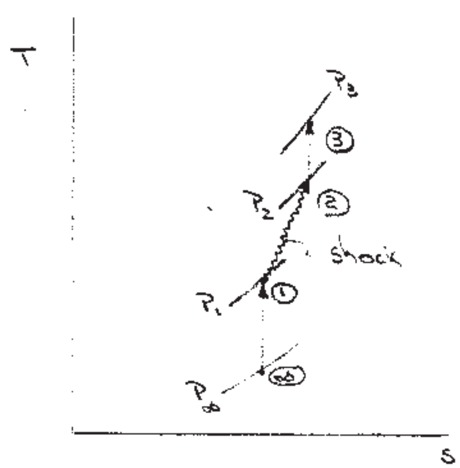
$\therefore P_{02} = 57.9\ \text{kPa}$, $T_2 = 380\ \text{K}$, $P_2 = 38.4\ \text{kPa}$.

$P_{03} = P_{02} = 57.9\ \text{kPa (abs)}$ $\qquad P_{03}$

$M_3 = 0.4$ From App. E.1, $T_3/T_0 = 0.9690$ $\quad P_3/P_{03} = 0.8956$

$\therefore T_3 = 414\ \text{K}$ $\qquad T_3$

$P_3 = 0.8956\, P_{03} = 0.8956\, P_{02} = 51.9\ \text{kPa (abs)}$ $\qquad P_3$



Problem 13.83

[Difficulty: 2]

13.83 Stagnation pressure and temperature probes are located on the nose of a supersonic aircraft. At 35,000 ft altitude a normal shock stands in front of the probes. The temperature probe indicates $T_0 = 420°F$ behind the shock. Calculate the Mach number and air speed of the plane. Find the static and stagnation pressures behind the shock. Show the process and the static and stagnation state points on a Ts diagram.

Solution:

Compressible flow functions (Appendix E) to be used in solution

Assumptions: (1) steady flow (2) uniform flow at a section
(3) thin shock, $\delta_x = 0$
(4) ideal gas

Use table A.3 to determine properties at state 1.

Altitude $= 35000\,ft \times 0.3048\,\frac{m}{ft} = 10{,}670\,m$

From Table A.3 $\quad T_1 = 219\,K = -54°C = -65°F$

$$p_1 = 23.9\times10^3\,\frac{N}{m^2}\times\frac{lbf}{4.448\,N}\times\left(\frac{0.3048\,m}{ft}\right)^2 = 499\ psfa$$

At state 1, $\quad T_1/T_{0_1} = 395/880 = 0.4489$

From App. E.1, for $T_1/T_{0_1} = 0.4489$, $\quad M_1 = 2.48$ ——— M_1

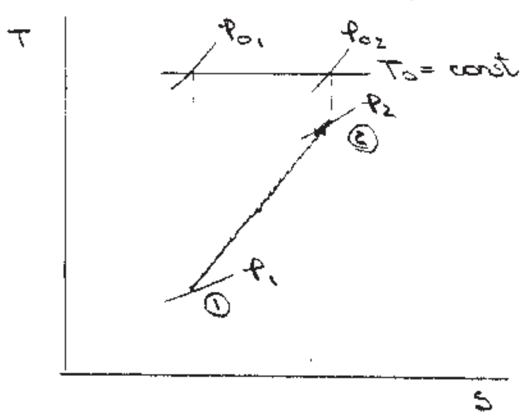
$$V_1 = M_1 c_1 = M_1(kRT_1)^{1/2} = 2.48\left[1.4\times53.3\,\frac{ft\cdot lbf}{lbm\cdot°R}\times395°R\times32.2\,\frac{lbm}{slug}\times\frac{slug\cdot ft}{lbf\cdot s^2}\right]^{1/2} = 2420\,ft/s$$ ——— V_1

From App. E.1, for $M_1 = 2.48$, $\quad p_1/p_{0_1} = 0.06038 \quad \therefore p_{0_1} = 57.4\ psia$

From App. E.4, for $M_1 = 2.48$, $\quad M_2 = 0.5149$

$p_{0_2}/p_{0_1} = 0.5071 \quad \therefore p_{0_2} = 29.1\ psia$ ——— p_{0_2}

$p_2/p_1 = 7.009 \quad \therefore p_2 = 24.3\ psia$ ——— p_2



Problem 13.84

[Difficulty: 3]

13.84 The NASA X-43A *Hyper-X* experimental hypersonic vehicle flew at Mach 9.68 at an altitude of 110,000 ft. Stagnation pressure and temperature probes were located on the nose of the aircraft. A normal shock wave stood in front of these probes. Estimate the stagnation pressure and temperature measured by the probes.

Given: Stagnation pressure and temperature probes on the nose of the Hyper-X

Find: Pressure and temperature read by those probes

Solution:

The given or available data is:

$R =$	53.33	ft-lbf/lbm-°R
$k =$	1.4	
$M_1 =$	9.68	
$z =$	110000	ft
$z =$	33528	m
$p_{SL} =$	14.696	psia
$T_{SL} =$	518.76	°R

Equations and Computations:

At this altitude the local pressure and temperature are:

$p_1/p_{SL} =$	0.008643	
$p_1 =$	0.12702	psia
$T_1 =$	422.88	°R

The stagnation pressure and temperature at these conditions are:

$p_{01}/p_1 =$	34178.42	
$p_{01} =$	4341.36	psia
$T_{01}/T_1 =$	19.74	
$T_{01} =$	8347.81	°R

Downstream of the normal shock wave, the Mach number is:

$M_2 =$ 0.3882

The total pressure ratio across the normal shock is:

$p_{02}/p_{01} =$ 0.003543

So the pressure read by the probe is:

$p_{02} =$ 15.38 psia

Since stagnation temperature is constant across the shock, the probe reads:

$T_{02} =$ 8348 °R

Problem 13.85

[Difficulty: 4]

13.85 Equations 13.20 are a useful set of equations for analyzing flow through a normal shock. Derive another useful equation, the Rankine-Hugoniot relation,

$$\frac{p_2}{p_1} = \frac{(k+1)\dfrac{\rho_2}{\rho_1} - (k-1)}{(k+1) - (k-1)\dfrac{\rho_2}{\rho_1}}$$

and use it to find the density ratio for air as $p_2/p_1 \rightarrow \infty$.

Given: Normal shock

Find: Rankine-Hugoniot relation

Solution:

Basic equations: Momentum: $p_1 + \rho_1 \cdot V_1^{\,2} = p_2 + \rho_2 \cdot V_2^{\,2}$ Mass: $\rho_1 \cdot V_1 = \rho_2 \cdot V_2$

Energy: $h_1 + \frac{1}{2} \cdot V_1^{\,2} = h_2 + \frac{1}{2} \cdot V_2^{\,2}$ Ideal Gas: $p = \rho \cdot R \cdot T$

From the energy equation $2 \cdot \left(h_2 - h_1\right) = 2 \cdot c_p \cdot \left(T_2 - T_1\right) = V_1^{\,2} - V_2^{\,2} = \left(V_1 - V_1\right) \cdot \left(V_1 + V_2\right)$ (1)

From the momentum equation $p_2 - p_1 = \rho_1 \cdot V_1^{\,2} - \rho_2 \cdot V_2^{\,2} = \rho_1 \cdot V_1 \cdot \left(V_1 - V_2\right)$ where we have used the mass equation

Hence $V_1 - V_2 = \dfrac{p_2 - p_1}{\rho_1 \cdot V_1}$

Using this in Eq 1 $2 \cdot c_p \cdot \left(T_2 - T_1\right) = \dfrac{p_2 - p_1}{\rho_1 \cdot V_1} \cdot \left(V_1 + V_2\right) = \dfrac{p_2 - p_1}{\rho_1} \cdot \left(1 + \dfrac{V_2}{V_1}\right) = \dfrac{p_2 - p_1}{\rho_1} \cdot \left(1 + \dfrac{\rho_1}{\rho_2}\right) = \left(p_2 - p_1\right) \cdot \left(\dfrac{1}{\rho_1} + \dfrac{1}{\rho_2}\right)$

where we again used the mass equation

Using the ideal gas equation $2 \cdot c_p \cdot \left(\dfrac{p_2}{\rho_2 \cdot R} - \dfrac{p_1}{\rho_1 \cdot R}\right) = \left(p_2 - p_1\right) \cdot \left(\dfrac{1}{\rho_1} + \dfrac{1}{\rho_2}\right)$

Dividing by p_1 and multiplying by ρ_2, and using $R = c_p - c_v$, $k = c_p/c_v$

$$2 \cdot \frac{c_p}{R} \cdot \left(\frac{p_2}{p_1} - \frac{\rho_2}{\rho_1}\right) = 2 \cdot \frac{k}{k-1} \cdot \left(\frac{p_2}{p_1} - \frac{\rho_2}{\rho_1}\right) = \left(\frac{p_2}{p_1} - 1\right) \cdot \left(\frac{\rho_2}{\rho_1} + 1\right)$$

Collecting terms $\dfrac{p_2}{p_1} \cdot \left(\dfrac{2 \cdot k}{k-1} - 1 - \dfrac{\rho_2}{\rho_1}\right) = \dfrac{2 \cdot k}{k-1} \cdot \dfrac{\rho_2}{\rho_1} - \dfrac{\rho_2}{\rho_1} - 1$

$$\frac{p_2}{p_1} = \frac{\dfrac{2 \cdot k}{k-1} \cdot \dfrac{\rho_2}{\rho_1} - \dfrac{\rho_2}{\rho_1} - 1}{\left(\dfrac{2 \cdot k}{k-1} - 1 - \dfrac{\rho_2}{\rho_1}\right)} = \frac{\dfrac{(k+1)}{(k-1)} \cdot \dfrac{\rho_2}{\rho_1} - 1}{\dfrac{(k+1)}{(k-1)} - \dfrac{\rho_2}{\rho_1}} \quad \text{or} \quad \frac{p_2}{p_1} = \frac{(k+1) \cdot \dfrac{\rho_2}{\rho_1} - (k-1)}{(k+1) - (k-1) \cdot \dfrac{\rho_2}{\rho_1}}$$

For an infinite pressure ratio $(k+1) - (k-1) \cdot \dfrac{\rho_2}{\rho_1} = 0$ or $\dfrac{\rho_2}{\rho_1} = \dfrac{k+1}{k-1}$ (= 6 for air)

Problem 13.86

[Difficulty: 3]

13.86 A supersonic aircraft cruises at $M = 2.7$ at 60,000 ft altitude. A normal shock stands in front of a pitot tube on the aircraft; the tube senses a stagnation pressure of 10.4 psia. Calculate the static pressure and temperature behind the shock. Evaluate the loss in stagnation pressure through the shock. Determine the change in specific entropy across the shock. Show static and stagnation states and the process path on a *Ts* diagram.

Solution.

Compressible flow functions (Appendix E) to be used in solution

Basic equation: $Tds = dh - v\,dp$

Assumptions: (1) steady flow (2) uniform flow at a section
(3) thin shock, $A_x = 0$
(4) ideal gas

Use table A.3 to determine properties at state ①

Altitude $= 60{,}000\,\text{ft} \times 0.3048\,\frac{m}{ft} = 18{,}290\,m$

From table A.3 $\quad T_1 = 216.7\,K = -56.3°C = -69°F$

$$p_1 = 7.25 \times 10^3\,\frac{N}{m^2} \times \frac{lbf}{4.448\,N} \times \left(\frac{0.3048\,m}{ft}\right)^2 = 151\,lbf/ft^2$$

From App. E.1 for $M_1 = 2.7$ $\quad p_1/p_{o_1} = 0.04295 \quad \therefore p_{o_1} = 24.4\,lbf/in^2$

From App. E.4 for $M_1 = 2.7$ $\quad M_2 = 0.4956$

$p_{o_2}/p_{o_1} = 0.4236 \quad \therefore p_{o_2} = 10.3$ psia

$T_2/T_1 = 2.343 \quad \therefore T_2 = 916\,R$ ——— T_2

$p_2/p_1 = 8.338 \quad \therefore p_2 = 8.74$ psia ——— p_2

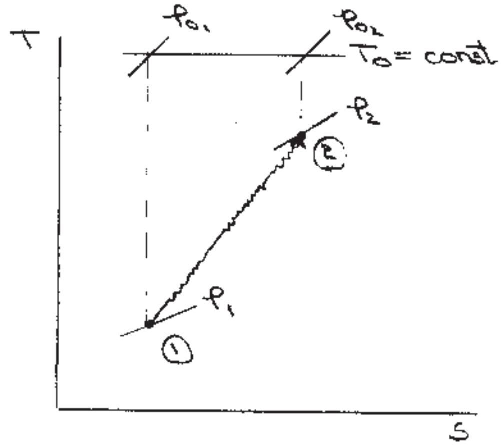
$p_{o_2} - p_{o_1} = 10.3\,\text{psi} - 24.4\,\text{psi} = -14.1\,\text{psi}$ ——— $p_{o_2} - p_{o_1}$

From the Tds equation, $Tds = dh - v\,dp = c_p\,dT - RT\,\frac{dp}{p}$

$$\therefore ds = c_p\,\frac{dT}{T} - R\,\frac{dp}{p}$$

Then $s_2 - s_1 = s_{o_2} - s_{o_1} = \cancel{c_p \ln \frac{T_{o_2}}{T_{o_1}}} - R \ln \frac{p_{o_2}}{p_{o_1}}$

$$s_2 - s_1 = -53.3\,\frac{ft\cdot lbf}{lbm\cdot R}\,\ln\frac{10.3}{24.4} \times \frac{Btu}{778\,ft\cdot lbf} = 0.0591\,\frac{Btu}{lbm\cdot R}$$ ——— $s_2 - s_1$



Problem 13.87

[Difficulty: 3]

13.87 An aircraft is in supersonic flight at 10 km altitude on a standard day. The true air speed of the plane is 659 m/s. Calculate the flight Mach number of the aircraft. A total-head tube attached to the plane is used to sense stagnation pressure which is converted to flight Mach number by an on-board computer. However, the computer programmer has ignored the normal shock that stands in front of the total-head tube and has assumed isentropic flow. Evaluate the pressure sensed by the total-head tube. Determine the erroneous air speed calculated by the computer program.

Solution: (using compressible flow functions - Appendix E)

From Table A.3 at $z = 10$ km, $T = 223.3$ K, $p/p_{SL} = 0.2615$

Thus $T = 223$ K, $p = 0.2615\, p_{SL} = 0.2615 \times 101.3$ kPa $= 26.5$ kPa

$c_1 = (kRT_1)^{1/2} = \left[1.4 \times 287 \frac{N \cdot m}{kg \cdot K} \times 223\,K \times \frac{kg \cdot m}{N \cdot s^2}\right]^{1/2} = 299$ m/s

shock

$\overleftarrow{M_2}$ $\overleftarrow{M_1}$ $\quad M_1 = \frac{V_1}{c_1} = \frac{659}{299} = 2.20$ $\quad M_1$

From App. E.1 at $M_1 = 2.20$, $p/p_0 = 0.09352$ $\therefore p_{0_1} = 283$ kPa

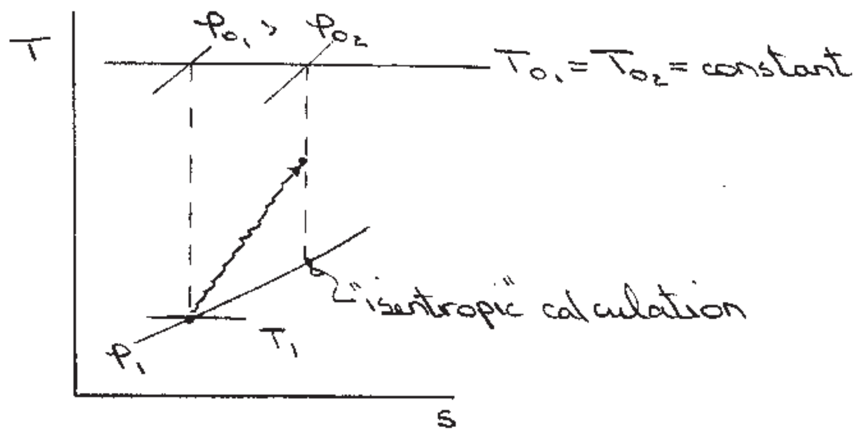
From App E.4 at $M_1 = 2.20$, $p_{0_2}/p_{0_1} = 0.6281$, $p_2/p_1 = 5.480$

$\therefore p_{0_2} = 0.6281\, p_{0_1} = 0.6281 \times 283$ kPa $= 178$ kPa (abs) $\quad p_{0_2}$

If Mach number is calculated neglecting the shock then M is calculated with $p = 26.5$ kPa and $p_0 = 178$ kPa.

For $p/p_0 = 26.5/178 = 0.149$, from App. E.1, $M_s = 1.90$

$V_{1_s} = M_s c_1 = 1.90 \times 299$ m/s $= 568$ m/s $\quad V_{1_s}$



Problem 13.88

[Difficulty: 3]

13.88 A supersonic aircraft flies at $M_1 = 2.7$ at 20 km altitude on a standard day. Air enters the engine inlet system, where it is slowed isentropically to $M_2 = 1.3$. A normal shock occurs at that location. The resulting subsonic flow is decelerated further to $M_4 = 0.40$. The subsonic diffusion is adiabatic but not isentropic; the final pressure is 104 kPa (abs). Evaluate (a) the stagnation temperature for the flow, (b) the pressure change across the shock, (c) the entropy change, $s_4 - s_1$, and (d) the final stagnation pressure. Sketch the process path on a Ts diagram, indicating all static and stagnation states.

Solution:

Compressible flow functions (Appendix E) to be used in solution

Assumptions: (1) steady flow (2) uniform flow at a section
(3) Δ in shock, $A_2 = A_3$
(4) ideal gas

For $M_1 = 2.7$, from App. E.1, $T_{0_1}/T_1 = 2.458$ $\therefore T_{0_1} = 533$ K — T_0
$p_{0_1}/p_1 = 23.283$ $\therefore p_{0_1} = 128.8$ kPa

Note: T_0 = constant, $p_{0_2} = p_{0_1}$

For $M_2 = 1.3$, from App. E.1, $T_{0_2}/T_2 = 1.338$ $\therefore T_2 = 398$ K
$p_{0_2}/p_2 = 2.771$ $p_2 = 46.48$ kPa

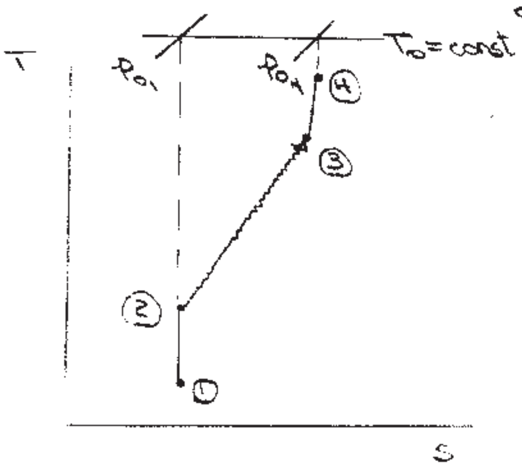
For $M_2 = 1.3$, from App E.4, $M_3 = 0.786$
$p_3/p_2 = 1.805$ $\therefore p_3 = 83.9$ kPa
$p_{0_3}/p_{0_2} = 0.9794$ $p_{0_3} = 126$ kPa
$T_3/T_2 = 1.191$ $T_3 = 474$ K

$p_3 - p_2 = (83.9 - 46.5)$ kPa $= 37.4$ kPa — $p_3 - p_2$

For $M_4 = 0.4$, from App. E.1, $p_{0_4}/p_4 = 1.117$ $\therefore p_{0_4} = 116$ kPa

$$s_4 - s_1 = s_{0_4} - s_{0_1} = c_p \ln \frac{T_{0_4}}{T_{0_1}} - R \ln \frac{p_{0_4}}{p_{0_1}} = -287 \frac{J}{kg \cdot K} \ln \frac{116}{129}$$

$s_4 - s_1 = -30.5$ J/kg·K — $s_4 - s_1$



Problem 13.89

[Difficulty: 4]

13.89 A blast wave propagates outward from an explosion. At large radii, curvature is small and the wave may be treated as a strong normal shock. (The pressure and temperature rise associated with the blast wave decrease as the wave travels outward.) At one instant, a blast wave front travels at $M = 1.60$ with respect to undisturbed air at standard conditions. Find (a) the speed of the air behind the blast wave with respect to the wave and (b) the speed of the air behind the blast wave as seen by an observer on the ground. Draw a Ts diagram for the process as seen by an observer on the wave, indicating static and stagnation state points and property values.

Solution:

Compressible flow functions (Appendix E) to be used in solution

Assumptions: (1) steady flow as seen by an observer on the wave
(2) uniform flow at a section
(3) thin shock, $R_x = 0$
(4) ideal gas

At state ①

$$V_1 = M_1 c_1 = M_1 (kRT_1)^{1/2} = 1.6\left[1.4 \times 287\,\frac{N\cdot m}{kg\cdot K} \times 288\,K \times \frac{kg\cdot m}{N\cdot s^2}\right]^{1/2} = 544\ m/s$$

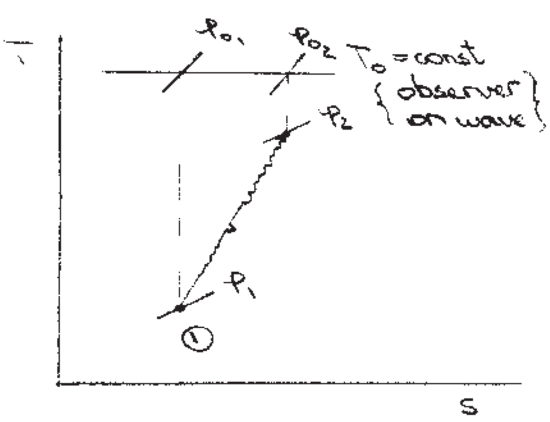
For $M_1 = 1.6$, from Appendix E.4, $V_1/V_2 = \rho_2/\rho_1 = 2.032$

$$\therefore \vec{V}_2 = 268\,\hat{i}\ m/s \qquad \vec{V}_{2\,rel.\,wave}$$

The wave moves to the right at $V_1 = 544$ m/s. Air moves to the left with respect to the wave at $V_{2\,rel} = 268$ m/s.

$$\vec{V}_{2\,abs} = \vec{V}_{2\,wave} + \vec{V}_{2\,rel} = -544\,\hat{i} + 268\,\hat{i} = -276\,\hat{i}\ m/s. \qquad \vec{V}_{2\,abs}$$

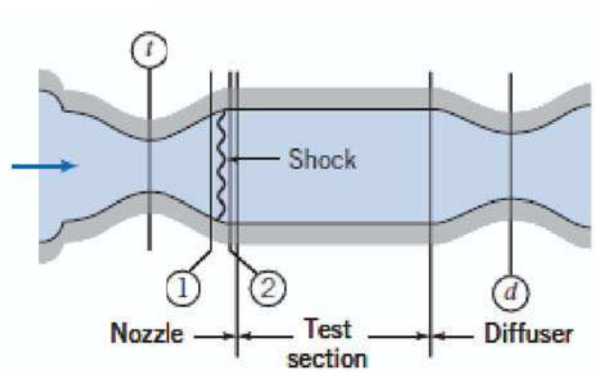
{relative to ground, air behind wave moves to right}.



Problem 13.90

[Difficulty: 3]

13.90 Consider a supersonic wind tunnel starting as shown. The nozzle throat area is 1.25 ft^2, and the test section design Mach number is 2.50. As the tunnel starts, a normal shock stands in the divergence of the nozzle where the area is 3.05 ft^2. Upstream stagnation conditions are T_0 = 1080°R and p_0 = 115 psia. Find the minimum theoretically possible diffuser throat area at this instant. Calculate the entropy increase across the shock.



Solution: Use functions for steady, one-dimensional compressible flow.

Computing equations: A/A^* vs. M from isentropic flow functions (App. E.1)

p_{02}/p_{01} vs. M from shock flow functions (App. E.4)

Assumptions: (1) Steady flow
(2) Uniform flow at each section
(3) Ideal gas
(4) Isentropic except across shock
(5) Adiabatic flow
(6) $F_{B_x} = 0$
(7) $\Delta z = 0$

Then from App. E.1, $M_1 = 2.416$ at $\frac{A_1}{A^*} = \frac{A_1}{A_t} = \frac{3.05\,ft^2}{1.25\,ft^2} = 2.44$.

From App. E.4, at $M_1 = 2.416$, $\frac{p_{02}}{p_{01}} = 0.5395$. Thus $p_{0d} = 0.5395\,p_{01} = 62.0$ psia.

For adiabatic flow, T_0 = constant and $T^* = \frac{T_0}{1.2} = \frac{1080\,R}{1.2} = 900\,R$ = constant.

From continuity, $\dot m = \rho_t A_t V_t = \rho_d V_d A_d$. Substituting $\rho = \frac{p}{RT}$ and $V = M\sqrt{kRT}$,

$$\frac{p_t}{RT_t}\sqrt{kRT_t}\,A_t = \frac{p_d}{RT_d}\sqrt{kRT_d}\,A_d\ ;\quad A_d = A_t\frac{p_t}{p_d} = A_t\frac{p_{0t}}{p_{0d}} = \frac{A_t}{0.5395}$$

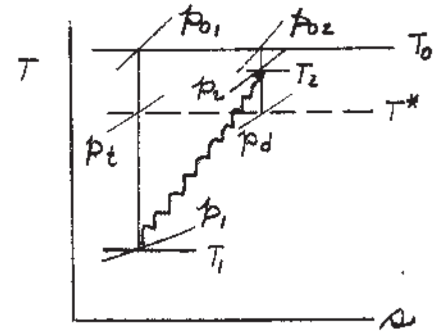
$$A_d = \frac{1.25\,ft^2}{0.5395} = 2.32\,ft^2$$ A_d

From the Gibbs equation,

$$T\,ds = dh - v\,dp\ ;\quad ds = c_p\frac{dT}{T} - R\frac{dp}{p}$$

$$\Delta s = c_p \ln\frac{T_{02}}{T_{01}} - R\ln\frac{p_{02}}{p_{01}}$$

Since T_0 = constant, $\ln(T_{02}/T_{01}) = 0$, and



$$\Delta s = -53.3\,\frac{ft\cdot lbf}{lbm\cdot °R}\times \ln(0.5395)\times\frac{Btu}{778\,ft\cdot lbf} = 0.0423\ Btu/lbm\cdot°R$$ Δs

{ Note a slightly stronger shock could occur at $M = 2.50$; a larger diffuser throat ($A_d \simeq 2.51\,ft^2$) would be needed to start this tunnel. }

Problem 13.91

[Difficulty: 4]

13.91 Air enters a wind tunnel with stagnation conditions of 14.7 psia and 75°F. The test section has a cross-sectional area of 1 ft^2 and a Mach number of 2.3. Find (a) the throat area of the nozzle, (b) the mass flow rate, (c) the pressure and temperature in the test section, and (d) the minimum possible throat area for the diffuser to ensure starting.

Given: Air flowing through a wind tunnel, stagnation and test section conditions known

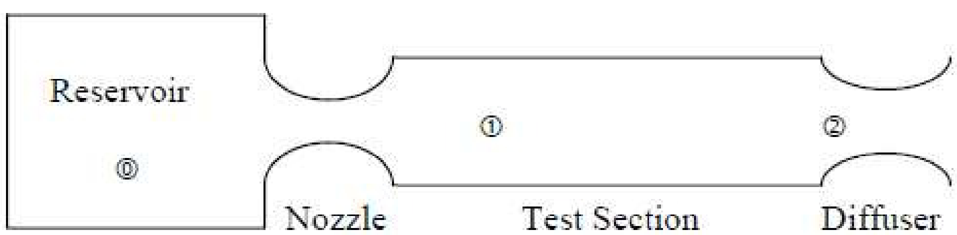
Find: Throat area, mass flow rate, static conditions in test section, minumum diffuser area

Solution:

The given or available data is:

$R =$	53.33	ft-lbf/lbm-°R
$k =$	1.4	
$p_{01} =$	14.7	psia
$T_{01} =$	75	°F
$T_{01} =$	535	°R
$A_1 =$	1	ft^2
$M_1 =$	2.3	

A schematic of this wind tunnel is shown here:



Equations and Computations:

For the Mach number in the test section, the corresponding area ratio is:

$A_1/A_1^* =$ 2.193

So the throat area is:

$A_t =$ 0.456 ft^2

The mass flow rate can be calculated using the choked flow equation:

$m =$ 22.2 lbm/s

The static conditions in the test section are:

$p_{01}/p_1 =$	12.5043	
$T_{01}/T_1 =$	2.0580	
$p_1 =$	1.176	psia
$T_1 =$	260	°R

The strongest possible shock that can occur downstream of the first throat is when the shock wave is in the test section. The post-shock Mach number is then

$M_2 =$ 0.5344

The area ratio corresponding to this Mach number is:

$A_2/A_2^* =$ 1.2792

Therefore, the minimum diffuser throat area is

$A_2^* =$ 0.782 ft^2

Problem 13.92

[Difficulty: 2]

13.92 Air flows through a converging-diverging nozzle with $A_e/A_t = 3.5$. The upstream stagnation conditions are atmospheric; the back pressure is maintained by a vacuum pump. Determine the back pressure required to cause a normal shock to stand in the nozzle exit plane and the flow speed leaving the shock.

Solution:

* Compressible flow tables to be used in the solution

Assumptions: (1) steady flow (2) uniform flow at a section (3) ideal gas (4) isentropic flow except across the shock

$A_1/A^* = 3.5$

From Fig. E.1 and Eq. 12.6, $M_1 = 2.80$; then $T_1/T_0 = 0.3894$ $p_1/p_{0_1} = 0.03685$

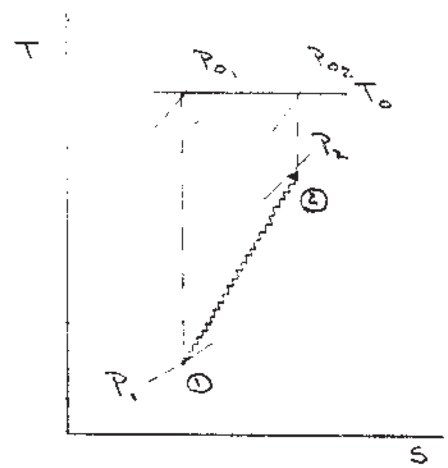
$\therefore T_1 = 112\ K$ $p_1 = 3.72\ kPa\ (abs)$

For $M_1 = 2.80$, from App. E.4, $M_2 = 0.488$ $T_2/T_1 = 2.451$ $p_2/p_1 = 8.980$

$\therefore p_2 = 8.98\,p_1 = 8.98 \times 3.72\ kPa = 33.4\ kPa\ (abs)$ p_2

$T_2 = 2.451\,T_1 = 275\ K$

$V_2 = M_2 c_2 = M_2 (kRT_2)^{1/2} = 0.488 \left(1.4 \times 287\ \frac{N \cdot m}{kg \cdot K} \times 275\ K \times \frac{kg \cdot m}{N \cdot s^2}\right)^{1/2} = 162\ m/s$ V_2



Problem 13.93

[Difficulty: 3]

13.93 A supersonic wind tunnel is to be operated at $M = 2.2$ in the test section. Upstream from the test section, the nozzle throat area is $0.07\ m^2$. Air is supplied at stagnation conditions of 500 K and 1.0 MPa (abs). At one flow condition, while the tunnel is being brought up to speed, a normal shock stands at the nozzle exit plane. The flow is steady. For this *starting* condition, immediately downstream from the shock find (a) the Mach number, (b) the static pressure, (c) the stagnation pressure, and (d) the minimum area theoretically possible for the second throat downstream from the test section. On a *Ts* diagram show static and stagnation state points and the process path.

Solution:

Compressible flow functions (Appendix E) to be used in solution.

Assumptions: (1) steady flow (2) uniform flow at a section
(3) isentropic flow in nozzles, adiabatic flow across shock
(4) ideal gas.

At $M_{2u} = 2.2$, from App E.1, $p_{2u}/p_0 = 0.09352$ $\therefore p_{2u} = 93.5\ kPa$

At $M_{2u} = 2.2$, from App E.4, $M_{2d} = 0.547$ (12.34b) $\quad M_{2d}$

$p_2/p_1 = 5.480$ $\therefore p_{2d} = 512\ kPa$ $\quad p_{2d}$

$p_{0_2}/p_{0_1} = 0.6281$ $\therefore p_{02d} = 628\ kPa$ $\quad p_{02d}$

At $M_{2u} = 2.2$, from App E.1, $A_2/A^* = A_2/A_1 = 2.005$ $\therefore A_2 = 0.1404\ m^2$

At $M_{2d} = 0.547$, from App E.1, $A_2/A^* = A_2/A^*_s = 1.259$ $\therefore A^*_s = 0.111\ m^2$ $\quad A^*_s$

Problem 13.94

[Difficulty: 2]

13.94 A converging-diverging nozzle is attached to a large tank of air, in which $T_{0_1} = 300$ K and $p_{0_1} = 250$ kPa (abs). At the nozzle throat the pressure is 132 kPa (abs). In the diverging section, the pressure falls to 68.1 kPa before rising suddenly across a normal shock. At the nozzle exit the pressure is 180 kPa. Find the Mach number immediately behind the shock. Determine the pressure immediately downstream from the shock. Calculate the entropy change across the shock. Sketch the Ts diagram for this flow, indicating static and stagnation state points for conditions at the nozzle throat, both sides of the shock, and the exit plane.

Solution: Compressible flow functions (Appendix E) to be used in solution

Assumptions: (1) steady flow (3) uniform flow at each section
(2) ideal gas (4) isentropic flow, except across shock

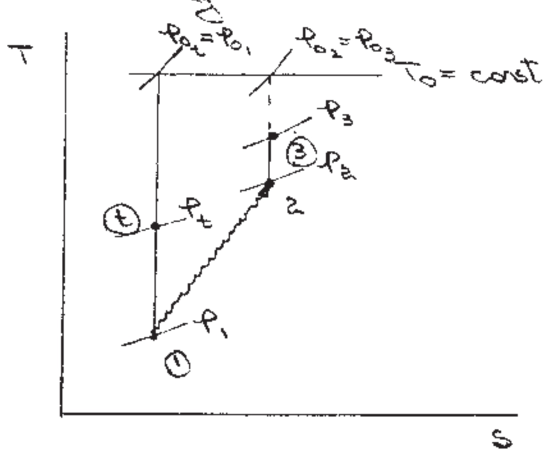
At section ① $p_0 = 250$ kPa (abs), $p_1 = 68.1$ kPa (abs) $\therefore p_1/p_0 = 0.2724$
From App. E.1 $M_1 = 1.50$

From App. E.4 with $M_1 = 1.50$, $M_2 = 0.701$ ——— M_2
$p_2/p_1 = 2.458$ $\therefore p_2 = 167$ kPa (abs) ——— p_2
$p_{0_2}/p_{0_1} = 0.9298$

From the Tds equation, $T\,ds = dh - v\,dp = c_p\,dT - RT\,\frac{dp}{p}$
$\therefore ds = c_p\,\frac{dT}{T} - R\,\frac{dp}{p}$

$$s_2 - s_1 = s_{0_2} - s_{0_1} = c_p \ln \frac{T_{0_2}}{T_{0_1}} - R \ln \frac{p_{0_2}}{p_{0_1}}$$

$$s_2 - s_1 = -287\ \frac{\text{J}}{\text{kg}\cdot\text{K}} \ln 0.9298 = 20.9\ \text{J/kg}\cdot\text{K}$$ ——— $s_2 - s_1$



Problem 13.95

[Difficulty: 2]

13.95 A converging-diverging nozzle expands air from 250°F and 50.5 psia to 14.7 psia. The throat and exit plane areas are 0.801 and 0.917 in.^{2}, respectively. Calculate the exit Mach number. Evaluate the mass flow rate through the nozzle.

Solution: Use functions for steady, one-dimensional compressible flow.

Computing equations: A/A^*, p/p_0, and T/T_0 from isentropic (Appendix E.1)

Assumptions:
(1) Steady flow
(2) Uniform flow at each section
(3) Isentropic if no shock
(4) Ideal gas
(5) $F_{Bx} = 0$
(6) $\Delta z = 0$

Check the exit condition: $\frac{A_e}{A^*} = \frac{A_e}{A_t} = \frac{0.917}{0.801} = 1.145 \rightarrow M_e = 1.452$ (App. E.1, Eq. 12.6)

Also from App. E.1, at $M = 1.452$, $\frac{p}{p_0} = 0.2919$; $p_e = 0.2919\, p_0 = 14.74$ psia.

Thus p_e is just slightly above $p_{atm} = 14.7$ psia; flow at exit is supersonic, so $M_t = 1.0$. From continuity

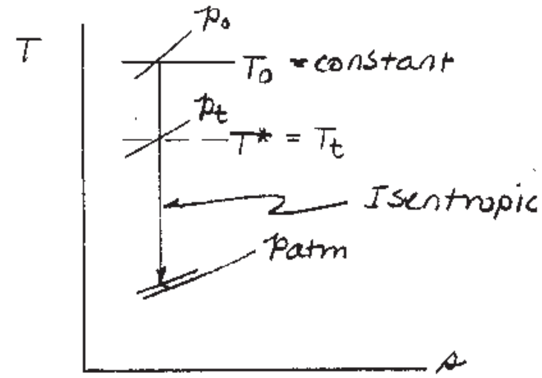
$$\dot{m} = \rho_t V_t A_t = \frac{p_t}{RT_t} M_t \sqrt{kRT_t}\, A_t = p_t \sqrt{\frac{k}{RT_t}}\, A_t$$

From App. E.1, at $M = 1$, $T/T_0 = 0.8333$ and $p/p_0 = 0.5283$, so

$$T_t = 0.8333 \times (460 + 250)^\circ R = 592^\circ R \quad \text{and} \quad p_t = 0.5283 \times 50.5 \text{ psia} = 26.7 \text{ psia}$$

$$\dot{m} = 26.7 \frac{\text{lbf}}{\text{in.}^2}\left[1.4 \times \frac{\text{lbm} \cdot {}^\circ R}{53.3 \text{ ft} \cdot \text{lbf}} \times \frac{1}{592^\circ R} \times \frac{\text{slug}}{32.2 \text{ lbm}} \times \frac{\text{lbf} \cdot \text{s}^2}{\text{slug} \cdot \text{ft}}\right]^{1/2} 0.801 \text{ in.}^2 \times \frac{\text{slug} \cdot \text{ft}}{\text{lbf} \cdot \text{s}^2} \times 32.2 \frac{\text{lbm}}{\text{slug}}$$

$\dot{m} = 0.808$ lbm/s ← $\dot{m}$

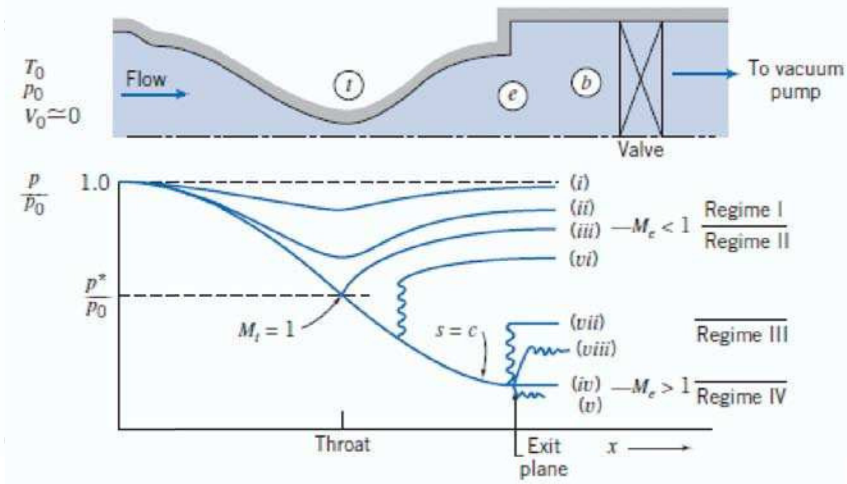


{ Flow in the nozzle is just slightly underexpanded, since $p_{exit} > p_{back}$. }

Problem 13.96

[Difficulty: 2]

13.96 A converging-diverging nozzle, with throat area $A_t = 1.0$ in.^{2}, is attached to a large tank in which the pressure and temperature are maintained at 100 psia and 600°R. The nozzle exit area is 1.58 in.^{2} Determine the exit Mach number at design conditions. Referring to Fig. 13.12, determine the back pressures corresponding to the boundaries of Regimes I, II, III, and IV. Sketch the corresponding plot for this nozzle.



Find: M_e at design conditions; P_b corresponding to regime boundaries of Fig 12.20, sketch P(x)

Solution:

Compressible flow functions (Appendix E) to be used in solution

Assumptions: (1) steady flow (2) ideal gas (3) uniform flow at a section (4) isentropic flow except across a shock

For $A_e/A^* = 1.58$,

from Fig. E.1 and Eq 12.6, $M_e = 0.402$ $P_e/P_0 = 0.8940$

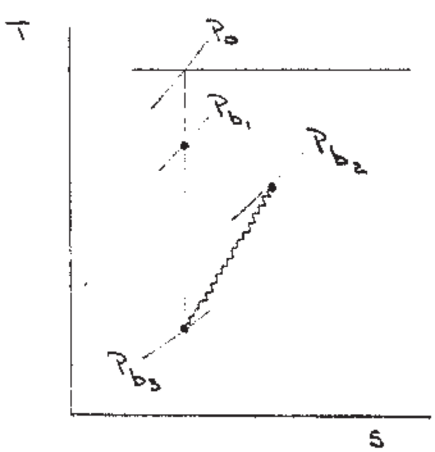
or $M_e = 1.92$ $P_e/P_0 = 0.1447$ M_e

Then $P_{b_1} = \frac{P_{e_1}}{P_0} \cdot P_0 = 0.8940 \times 100$ psia $= 89.4$ psia P_{b_1}

$P_{b_3} = \frac{P_{e_3}}{P_0} \times P_0 = 0.1447 \times 100$ psia $= 14.5$ psia P_{b_3}

For $M_e = 1.92$, from App. E.4, $\frac{P_2}{P_1} = 4.045$

$P_{b_2} = \frac{P_2}{P_1} \times P_{e_3} = 4.045 \times 14.5$ psia $= 58.6$ psia P_{b_2}



Problem 13.97

[Difficulty: 3]

13.97 A converging-diverging nozzle is designed to produce a Mach number of 2.5 with air. What operating pressure ratios ($p_b/p_{t\,inlet}$) will cause this nozzle to operate with isentropic flow throughout and supersonic flow at the exit (the so-called "third critical point"), with isentropic flow throughout and subsonic flow at the exit (the "first critical point"), and with a normal shock at the nozzle exit (the "second critical point")?

Given: Air accelerating through a converging-diverging nozzle

Find: Pressure ratios needed to operate with isentropic flow throughout, supersonic flow at exit (third critical); isentropic flow throughout, subsonic flow at exit (first critical point); and isentropic flow throughout, supersonic flow in the diverging portion, and a normal shock at the exit (second critical point).

Solution:

The given or available data is:

$k =$ 1.4

$M_d =$ 2.5

Equations and Computations:

The pressure ratio for the third critical can be found from the design point Mach number:

$p_{0inlet}/p_{b,3rd} =$ 17.0859

$p_{b,3rd}/p_{0inlet} =$ 0.0585

The area ratio for this nozzle is:

$A/A^* =$ 2.637

So to operate at first critical the exit Mach number would be:

$M_{1st} =$ 0.226

Since at first critical the flow is isentropic, the pressure ratio is:

$p_{0inlet}/p_{b,1st} =$ 1.0363

$p_{b,1st}/p_{0inlet} =$ 0.9650

At second critical, the flow is isentropic to the exit, followed by a normal shock.

At the design Mach number, the pressure ratio is:

$p_{b,2nd}/p_{b,3rd} =$ 7.125

Therefore, the back pressure ratio at the second critical is:

$p_{b,2nd}/p_{0inlet} =$ 0.4170

$p_{b,1st}/p_{0inlet} =$ 0.9650

$p_{b,2nd}/p_{0inlet} =$ 0.4170

$p_{b,3rd}/p_{0inlet} =$ 0.0585

Problem 13.98

[Difficulty: 3]

13.98 Oxygen flows through a converging-diverging nozzle with a exit-to-throat area ratio of 3.0. The stagnation pressure at the inlet is 120 psia, and the back pressure is 50 psia. Compute the pressure ratios for the nozzle and demonstrate that a normal shock wave should be located within the diverging portion of the nozzle. Compute the area ratio at which the shock occurs, the pre- and post-shock Mach numbers, and the Mach number at the nozzle exit.

Given: Oxygen accelerating through a converging-diverging nozzle

Find: Pressure ratios for critical points, show that a shock forms in the nozzle, pre- and post-shock Mach numbers, exit Mach number

Solution:

The given or available data is:

$R =$ 48.29 ft-lbf/lbm-°R

$k =$ 1.4

$p_{0inlet} =$ 120 psia

$p_b =$ 50 psia

$A_e/A_t =$ 3

Equations and Computations:

Based on the area ratio, the design Mach number is:

$$M_d = 2.637$$

The pressure ratio for the third critical can be found from the design point Mach number:

$$p_{0inlet}/p_{b,3rd} = 21.1422$$

$$p_{b,3rd}/p_{0inlet} = 0.04730$$

If a normal shock exists in the nozzle, the pressure ratio should be between the first and second critical points. At the first critical point the exit Mach number is

$$M_{1st} = 0.197$$

Since at first critical the flow is isentropic, the pressure ratio is:

$$p_{0inlet}/p_{b,1st} = 1.0276$$

$$p_{b,1st}/p_{0inlet} = 0.9732$$

At second critical, the flow is isentropic to the exit, followed by a normal shock. At the design Mach number, the pressure ratio is:

$$p_{b,2nd}/p_{b,3rd} = 7.949$$

Therefore, the back pressure ratio at the second critical is:

$$p_{b,2nd}/p_{0inlet} = 0.3760$$

The actual back pressure ratio is

$$p_b/p_{0inlet} = 0.4167$$

This pressure ratio is between those for the first and second critical points, so a shock exists in the nozzle. We need to use an iterative solution to find the exact location of the shock wave. Specifically, we iterate on the pre-shock Mach number until we match the exit pressure to the given back pressure:

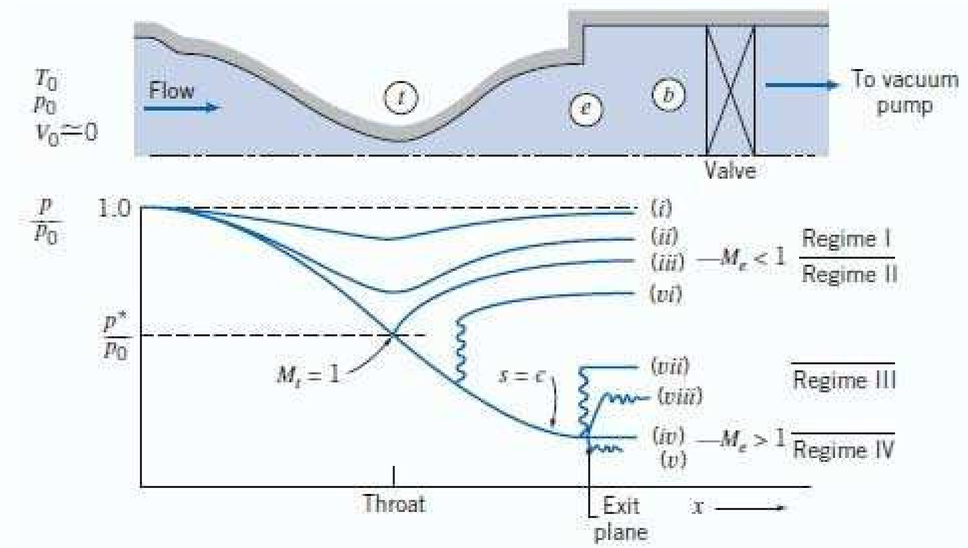
$M_1 =$	2.55	
$A_1/A_t =$	2.759	
$p_{0inlet}/p_1 =$	18.4233	
$p_1 =$	6.513	psia
$M_2 =$	0.508	
$p_2/p_1 =$	7.4107	
$p_2 =$	48.269	psia
$A_e/A_2 =$	1.0873	
$A_2/A_2^* =$	1.324	
$A_e/A_2^* =$	1.440	
$M_e =$	0.454	
$p_{02}/p_2 =$	1.193	
$p_{02}/p_e =$	1.152	
$p_e =$	50.000	psia

(We used *Goal Seek* in *Excel* for this solution.)

Problem 13.99

[Difficulty: 2]

13.99 A converging-diverging nozzle, with $A_e/A_t = 4.0$, is designed to expand air isentropically to atmospheric pressure. Determine the exit Mach number at design conditions and the required inlet stagnation pressure. Referring to Fig. 13.20, determine the back pressures that correspond to the boundaries of Regimes I, II, III, and IV. Sketch the plot of pressure ratio versus axial distance for this nozzle.



Solution:

Compressible flow functions (Appendix E) to be used in solution

Assumptions: (1) steady flow (2) ideal gas
(3) uniform flow at a section
(4) isentropic flow except across a shock

For $A_e/A^* = 4.00$,
from Fig. E.1 and Eq. 12.6, $M_e = 2.94$ $p/p_0 = 0.0298$
or $M_e = 0.147$ $p/p_0 = 0.9887$

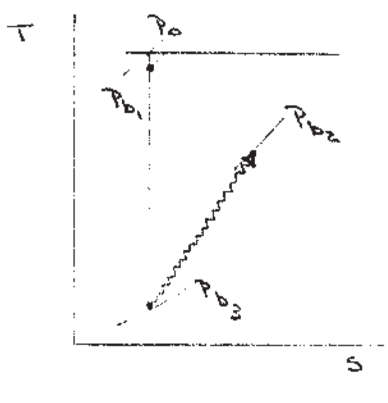
Then
$M_{ed} = 2.94$ $p_{0d} = \frac{p_e}{0.0298} = \frac{101\text{ kPa}}{0.0298} = 3.39\text{ MPa (abs)}$ M_e, p_0

$p_{b_1} = \frac{p_{e_1}}{p_0} \times p_0 = 0.9887 \times 3.39\text{ MPa} = 3.35\text{ MPa (abs)}$ p_{b_1}

$p_{b_3} = 101\text{ kPa (abs)}$ p_{b_3}

For $M_e = 2.94$ from App. E.4, $p_2/p_1 = 9.918$

$p_{b_2} = \frac{p_2}{p_1} \times p_{e_3} = 9.918 \times 101\text{ kPa} = 1.00\text{ MPa (abs)}$ p_{b_2}



Problem 13.100

[Difficulty: 3]

13.100 A normal shock occurs in the diverging section of a converging-diverging nozzle where $A = 25\ \text{cm}^2$ and $M = 2.75$. Upstream, $T_0 = 550$ K and $p_0 = 700$ kPa (abs). The nozzle exit area is $40\ \text{cm}^2$. Assume the flow is isentropic except across the shock. Determine the nozzle exit pressure, throat area, and mass flow rate.

Given: Normal shock in CD nozzle

Find: Exit pressure; Throat area; Mass flow rate

Solution:

The given or available data is:

$R =$	286.9	J/kg·K
$k =$	1.4	
$T_{01} =$	550	K
$p_{01} =$	700	kPa
$M_1 =$	2.75	
$A_1 =$	25	cm^2
$A_e =$	40	cm^2

Equations and Computations (assuming State 1 and 2 before and after the shock):

Using built-in function *Isenp* (M,k):

$p_{01}/p_1 = 25.14$ $\qquad p_1 = 28$ kPa

Using built-in function *IsenT* (M,k):

$T_{01}/T_1 = 2.51$ $\qquad T_1 = 219$ K

Using built-in function *IsenA* (M,k):

$A_1/A_1^* = 3.34$ $\qquad A_1^* = A_t = 7.49\ \text{cm}^2$

Then from the Ideal Gas equation:

$\rho_1 = 0.4433\ \text{kg/m}^3$

Also: $\qquad c_1 = 297$ m/s

So: $\qquad V_1 = 815$ m/s

Then the mass flow rate is: $\qquad m_{rate} = \rho_1 V_1 A_1$

$m_{rate} = 0.904$ kg/s

For the normal shock:

Using built-in function *NormM2fromM* (M,k):

$M_2 = 0.492$

Using built-in function *Normp0fromM* (M,k) at M_1:

$p_{02}/p_{01} = 0.41$ $\qquad p_{02} = 284$ kPa

For isentropic flow after the shock:

Using built-in function *IsenA* (M,k):

	$A_2/A_2^* =$	1.356	
But:	$A_2 =$	A_1	
Hence:	$A_2^* =$	18.44	cm^2

Using built-in function *IsenAMsubfromA* (Aratio,k):

For:	$A_e/A_2^* =$	2.17	$M_e =$	0.279

Using built-in function *Isenp* (M,k):

$p_{02}/p_e =$	1.06	$p_e =$	269	kPa

Problem 13.101

[Difficulty: 2]

13.101 Air flows adiabatically from a reservoir, where $T = 60°C$ and $p = 600$ kPa (abs), through a converging-diverging nozzle with $A_e/A_t = 4.0$. A normal shock occurs where $M = 2.42$. Assuming isentropic flow before and after the shock, determine the back pressure downstream from the nozzle. Sketch the pressure distribution.

Solution:

Compressible flow functions (Appendix E) to be used in solution

Assumptions: (1) steady flow
(2) uniform flow at a section
(3) ideal gas
(4) isentropic flow except across the shock

$M_1 = 2.42$ From App. E.1, $p/p_0 = 0.06630$ $A_1/A_1^* = A_1/A_t = 2.448$

$\therefore p_1 = 39.8$ kPa (abs)

From App. E.4, $M_2 = 0.521$ $p_2/p_1 = 6.666$ $p_{02}/p_{01} = 0.5318$

$\therefore p_2 = 265$ kPa (abs) $p_{02} = 319$ kPa (abs)

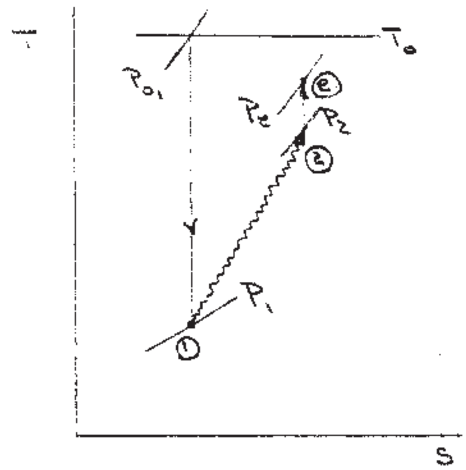
$M_2 = 0.521$ From App. E.1, $A_2/A_2^* = 1.301 = A_1/A_2^*$

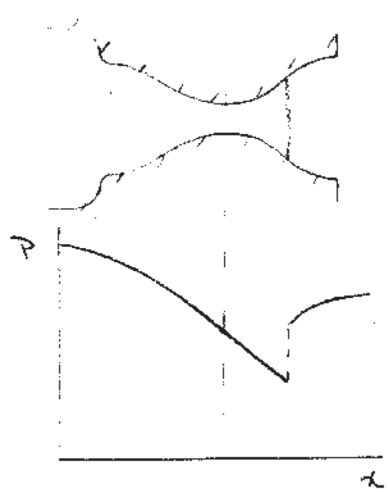
Then $\dfrac{A_e}{A_2^*} = \dfrac{A_e}{A_1^*} \times \dfrac{A_1^*}{A_1} \times \dfrac{A_1}{A_2^*} = 4.0 \times \dfrac{1}{2.448} \times 1.301 = 2.126$

For $A_e/A^* = 2.126$, from Fig. E.1 and Eq. 12.6, $M_e = 0.286$, $p_e/p_0 = 0.9447$

Then $p_b = p_e = 0.9447\, p_0 = 0.9447 \times 319$ kPa $= 301$ kPa

p_e





Problem 13.102

[Difficulty: 3]

13.102 A converging-diverging nozzle is designed to expand air isentropically to atmospheric pressure from a large tank, where $T_0 = 150°C$ and $p_0 = 790$ kPa (abs). A normal shock stands in the diverging section, where $p = 160$ kPa (abs) and $A = 600\ \text{mm}^2$. Determine the nozzle back pressure, exit area, and throat area.

Solution:

Compressible flow functions (Appendix E) to be used in solution

Assumptions: (1) steady flow (2) uniform flow at a section (3) ideal gas (4) isentropic flow except across the shock

At design conditions,

$p_3/p_0 = 101/790 = 0.1278$, From App. E.1 $M_{3\,design} = 2.00$ $\frac{A_3}{A_t} = 1.688$

At section ①, $\frac{p_1}{p_0} = \frac{160}{790} = 0.2025$ From App. E.1, $M_1 = 1.70$ $A_1/A_1^* = 1.338$

$A_t = A_1^* = \frac{A_1}{1.338} = \frac{600\ mm^2}{1.338} = 448\ mm^2$ —— A_t

Then $A_3 = 1.688\ A_t = 1.688\ (448\ mm^2) = 756\ mm^2$ —— A_3

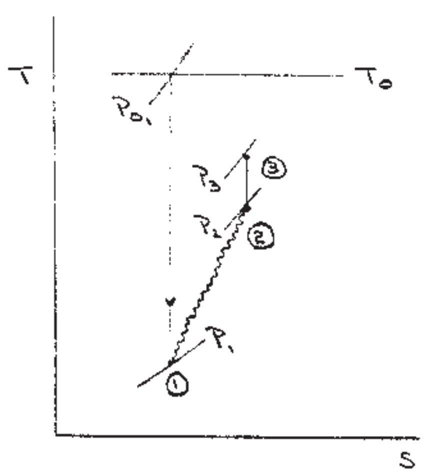
$M_1 = 1.70$ From App. E.4, $M_2 = 0.641$ $p_{0_2}/p_{0_1} = 0.8557$ $p_2/p_1 = 3.205$

$\therefore p_{0_2} = 676$ kPa (abs) $p_2 = 513$ kPa (abs)

$M_2 = 0.641$ From App. E.1, $A_2/A_2^* = 1.145$ $\therefore A_2^* = \frac{A_2}{1.145} = 524\ mm^2$

$A_3/A_2^* = \frac{756}{524} = 1.443$. From Fig. E.1 and Eq 12.6, $M_3 = 0.453$; $\frac{p_3}{p_{0_3}} = 0.8688$

$\therefore p_b = p_3 = 0.8688\ p_{0_3} = 0.8688\ p_{0_2} = 587$ kPa (abs) —— p_b



Problem 13.103

[Difficulty: 3]

13.103 A converging-diverging nozzle, with design pressure ratio $p_e/p_0 = 0.128$, is operated with a back pressure condition such that $p_b/p_0 = 0.830$, causing a normal shock to stand in the diverging section. Determine the Mach number at which the shock occurs.

Solution:

Compressible flow functions (Appendix E) to be used in solution

Assumptions: 1) steady flow 2) uniform flow at a section 3) ideal gas 4) isentropic flow except across the shock

At design conditions, $p_e/p_{0_1} = 0.1278$

From App. E.1, $M_d = 2.0$ and $A_e/A_1^* = 1.688$

Now $\dfrac{p_e}{p_{0_1}} = \dfrac{p_e}{p_{0_2}} \cdot \dfrac{p_{0_2}}{p_{0_1}}$ and $\dfrac{A_e}{A_1^*} = \dfrac{A_e}{A_2^*} \cdot \dfrac{A_2^*}{A_1^*}$ $\qquad \dfrac{A_2^*}{A_1^*} = \dfrac{A_2^*}{A_2} \cdot \dfrac{A_1}{A_1^*}$

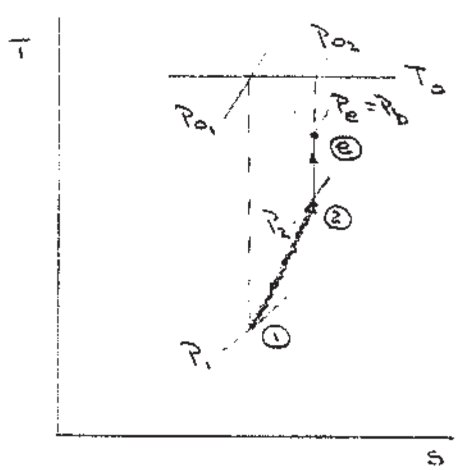
$\dfrac{p_e}{p_{0_2}}$: $f(M_e)$; $\dfrac{p_{0_2}}{p_{0_1}}$: $f(M_1)$; $\dfrac{A_e}{A_2^*}$: $f(M_e)$; $\dfrac{A_2^*}{A_1^*}$: $f(M_1)$

We thus have a trial and error solution to determine M_1.

M_1	$\frac{A_1}{A_1^*}$ (a)	M_2 (b)	$\frac{p_{0_2}}{p_{0_1}}$ (c)	$\frac{A_2}{A_2^*}$ (d)	$\frac{A_2^*}{A_1^*}$	$\frac{A_e}{A_2^*}$	M_e (e)	$\frac{p_e}{p_{0_2}}$ (f)	$\frac{p_e}{p_{0_1}} \overset{?}{=} 0.8300$
1.60	1.25	0.67	0.8952	1.119	1.117	1.511	0.42	0.8857	0.7927
1.40	1.150	0.74	0.9582	1.068	1.077	1.567	0.41	0.8907	0.8534
1.50	1.176	0.70	0.9298	1.094	1.075	1.570	0.403	0.8923	0.830 ✓

$\therefore M_1 = 1.50$ ← M_1

a	App. E.1	(12.6)
b.	App E.4	(12.34b).
c.	App E.4	(12.37)
d.	App. E.1	(12.6).
e.	App. E.1	Fig. E.1 & Eq. 12.6
f.	App E.1	(11.17a)



Problem 13.104

[Difficulty: 3]

13.104 Air flows through a converging-diverging nozzle, with $A_e/A_t = 3.5$. The upstream stagnation conditions are atmospheric; the back pressure is maintained by a vacuum system. Determine the range of back pressures for which a normal shock will occur within the nozzle and the corresponding mass flow rate, if $A_t = 500\ \text{mm}^2$.

Solution: Use compressible flow functions in solution.

Computing equation: $\dot{m} = \rho V A$

Assumptions: (1) Steady flow (2) Ideal gas (3) Uniform flow at a section (4) Isentropic, except across shock

A normal shock will occur within the nozzle for back pressure conditions in Regime III of Fig. 12.20. For isentropic flow, with $A_e/A_t = 3.5$, from Fig. E.1 and Eq. 12.6,

M	p/p_0	p_b
0.169	0.9858	99.6 kPa
2.80	0.03685	

From Appendix E.4,

M_1	M_2	p_2/p_1
2.80	0.4882	8.980

Thus $$p_b = p_2 = p_0 \frac{p_1}{p_0}\frac{p_2}{p_1} = 101\ \text{kPa}\,(0.03685)(8.980) = 33.4\ \text{kPa}$$

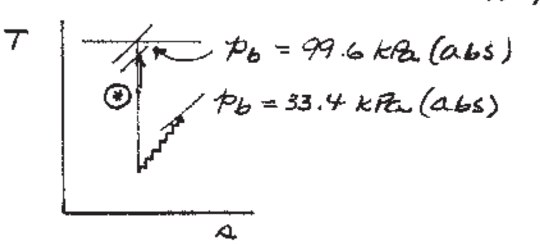
$$33.4\ \text{kPa} < p_b < 99.6\ \text{kPa (abs)}\ \text{(for normal shock } \underline{\text{in}} \text{ nozzle)}$$ p_b

Flow is choked throughout this regime. Thus

$$\dot{m} = \rho_t V_t A_t \qquad \rho_t = \frac{p_t}{RT_t} = (0.5283)\,1.01\times10^5\frac{\text{N}}{\text{m}^2}\times\frac{\text{kg}\cdot\text{K}}{287\ \text{N}\cdot\text{m}}\times\frac{1}{(0.8333)\,288\ \text{K}} = 0.775\ \frac{\text{kg}}{\text{m}^3}$$

$$V_t = c_t = \sqrt{kRT_t} = \left[1.4\times 287\frac{\text{N}\cdot\text{m}}{\text{kg}\cdot\text{K}}\times(0.8333)\,288\ \text{K}\times\frac{\text{kg}\cdot\text{m}}{\text{N}\cdot\text{s}^2}\right]^{\frac{1}{2}} = 311\ \text{m/s}$$

$$\dot{m} = 0.775\ \frac{\text{kg}}{\text{m}^3}\times 311\ \frac{\text{m}}{\text{s}}\times 500\ \text{mm}^2\times\frac{\text{m}^2}{10^6\ \text{mm}^2} = 0.121\ \frac{\text{kg}}{\text{s}}$$ $\dot{m}$



Problem 13.105

[Difficulty: 3]

13.105 A converging-diverging nozzle, with $A_e/A_t = 1.633$, is designed to operate with atmospheric pressure at the exit plane. Determine the range(s) of inlet stagnation pressures for which the nozzle will be free from normal shocks.

Solution: Use compressible flow tables in solution.

Assume flow in nozzle is isentropic when shock-free. From Appendix E.1 and Eq. 12.6,

M	p/p_0	A/A^*
1.96	0.1360	1.633
0.38	0.9052	1.659
0.40	0.8956	1.590

At design conditions, $M_e = 1.96$, and

$$p_0 \geq \frac{p_e}{(p/p_0)_e} = \frac{101\ \text{kPa}}{0.1360} = 743\ \text{kPa (abs)}$$

By iteration, then the given area ratio corresponds to isentropic choked flow with $M_e = 0.388$ and $(p/p_0)_e = 0.9014$. The corresponding stagnation pressure is

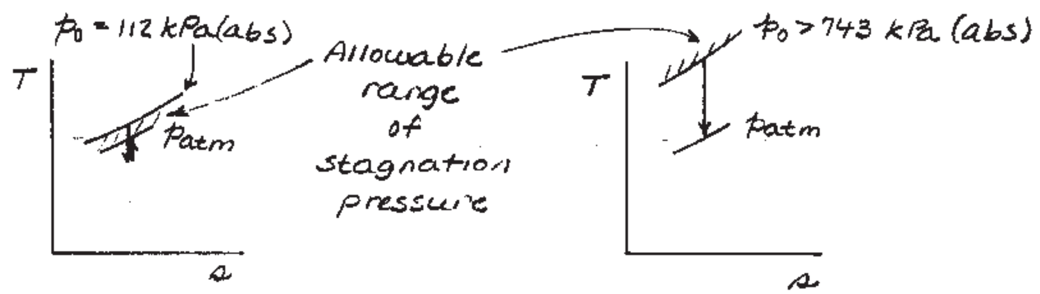
$$p_0 = \frac{p_e}{(p/p_0)_e} = \frac{101\ \text{kPa}}{0.9014} = 112\ \text{kPa (abs)}$$

Flow will be isentropic and shock-free for

(a) $p_{atm} < p_0 < 112$ kPa (abs) $(0 < M_e < 0.388)$ p_0

(b) $p_0 > 743$ kPa (abs) $(M_e = 1.96)$

The corresponding Ts diagrams are:



Problem 13.106

[Difficulty: 3]

13.106 Air flows through a converging-diverging nozzle with $A_e/A_t = 1.87$. Upstream, $T_{0_1} = 240°F$ and $p_{0_1} = 100$ psia. The back pressure is maintained at 40 psia. Determine the Mach number and flow speed in the nozzle exit plane.

Solution: Use compressible flow functions in solution. Assume ideal gas.

For isentropic flow through the nozzle to $A_e/A_t = 1.87$, from Fig. E.1 and Eq. 12.6,

M	T/T_0	p/p_0	p_d (psia)
2.12	0.5266	0.1060	10.6
~0.33	~0.98	~0.92	~92.0

Neither of these conditions matches the back pressure. Check the case of a shock (at $M = 2.12$) in the exit plane. From Appendix E.4,

M_1	M_2	p_2/p_1
2.12	0.5583	5.077

Then $p_e = 5.077\,p_d = 5.077\,(10.6\text{ psia}) = 53.8$ psia.

The back pressure of 40 psia is therefore between the design pressure and the pressure that would exist downstream from a normal shock in the exit plane. The flow is in regime III of Fig. 12.20; supersonic in the exit plane with external compression. Thus

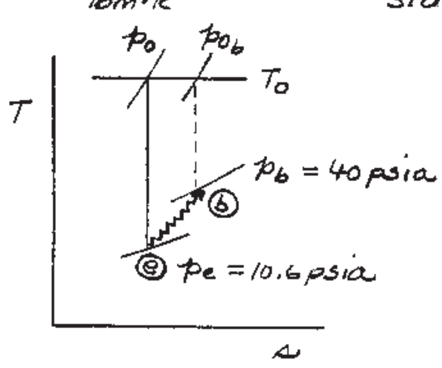
$$M_e = M_d = 2.12$$ Me

$$V_e = M_e c_e = M_e\sqrt{kRT_e}$$

$$T_e = \frac{T}{T_0}\,T_0 = 0.5266\,(460+240)°R \approx 369°R$$

$$V_e = 2.12\left[1.4 \times 53.3\,\frac{ft\cdot lbf}{lbm\,°R} \times 369°R \times 32.2\,\frac{lbm}{slug} \times \frac{slug\cdot ft}{lbf\cdot s^2}\right]^{\frac{1}{2}} = 2000\,\frac{ft}{s}$$ Ve

The Ts diagram is



Problem 13.107

[Difficulty: 3]

13.107 A normal shock occurs in the diverging section of a converging-diverging nozzle where $A = 4.0\ \text{in.}^2$ and $M = 2.00$. Upstream, $T_{0_1} = 1000°\text{R}$ and $p_{0_1} = 100\,\text{psia}$. The nozzle exit area is $6.0\ \text{in.}^2$ Assume that flow is isentropic except across the shock. Find the nozzle exit pressure. Show the processes on a Ts diagram, and indicate the static and stagnation state points.

Solution:

Compressible flow functions (Appendix E) to be used in solution

Assumptions: (1) steady flow (2) ideal gas (3) uniform flow at each section (4) isentropic flow, except across shock.

For $M_1 = 2.0$, from App. E.1, $p_1/p_{0_1} = 0.1278$ $\therefore p_1 = 12.78$ psia.

For $M_1 = 2.0$, from App. E.4, $M_2 = 0.5774$

$p_2/p_1 = 4.50$ $\therefore p_2 = 57.5$ psia

$p_{0_2}/p_{0_1} = 0.7209$ $\therefore p_{0_2} = 72.1$ psia

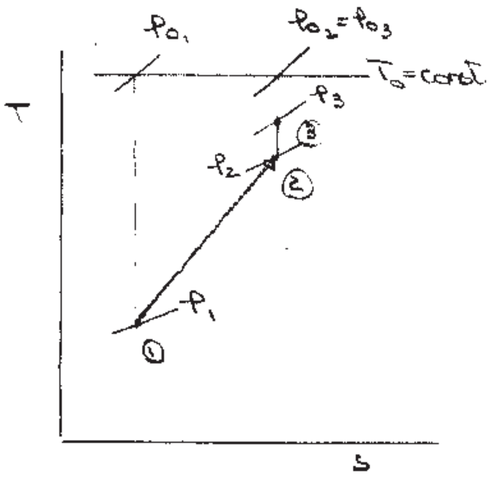
For $M_2 = 0.5774$, from App. E.1, $A_2/A_2^* = 1.216$ $\therefore A_2^* = 3.288\ \text{in}^2$

Then $A_3/A_2^* = 6.0/3.288 = 1.825$

With $A_3/A^* = 1.825$, from App. E.1 (Fig. E.1 and Eq. 12.6) $M_3 = 0.340$

With $M_3 = 0.340$, from App. E.1, $p_{0_3}/p_3 = 1.083$

$$p_3 = \frac{p_{0_3}}{1.083} = \frac{p_{0_2}}{1.083} = \frac{72.1\ \text{psia}}{1.083} = 66.6\ \text{psia}$$ p_3

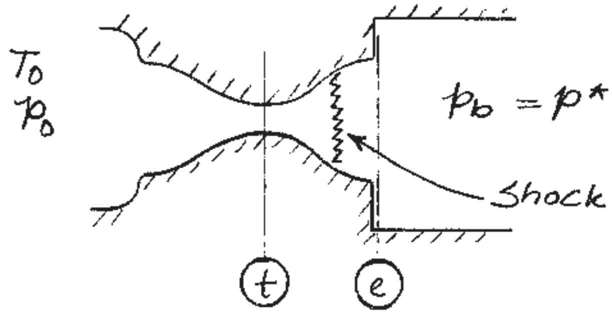


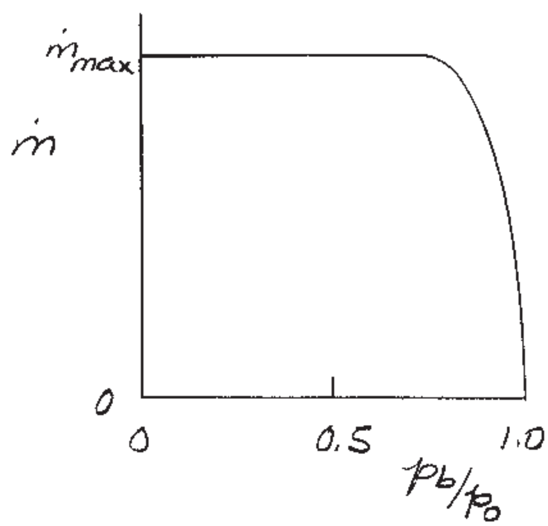
Problem 13.108

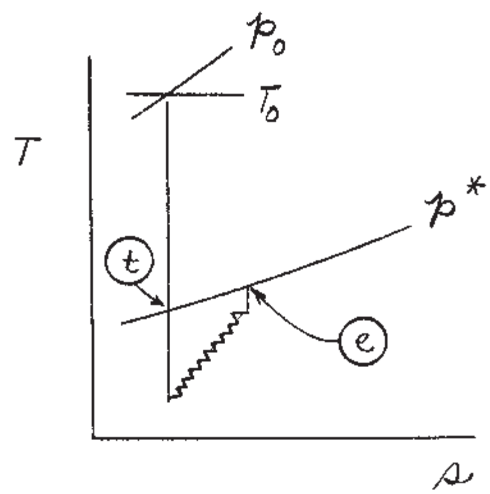
[Difficulty: 4]

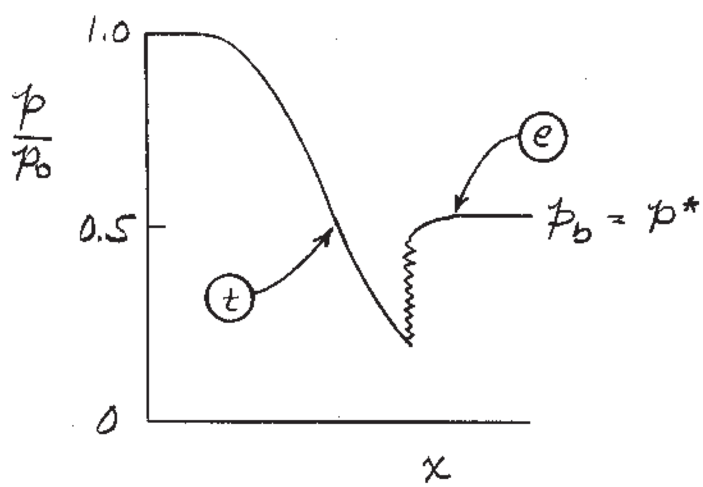
13.108 Consider flow of air through a converging-diverging nozzle. Sketch the approximate behavior of the mass flow rate versus back pressure ratio, p_b/p_0. Sketch the variation of pressure with distance along the nozzle, and the Ts diagram for the nozzle flow, when the back pressure is p^*.

<u>Solution</u>: When $p_b = p^*$, flow in the nozzle will be <u>choked</u> and a shock will stand in the diverging section.









Problem 13.109

[Difficulty: 3]

13.109 Air enters a converging-diverging nozzle with an area ratio of 1.76. Entrance stagnation conditions are 150 psia and 200°F. A normal shock stands at a location where the area is 1.2 times the throat area. Determine the exit Mach number and static pressure. What is the design point exit pressure?

Given: Air flowing through a converging-diverging nozzle with standing normal shock

Find: Exit Mach number and static pressure; design point pressure

Solution:

The given or available data is:

$R =$	53.33	ft-lbf/lbm-°R
$k =$	1.4	
$p_{0inlet} =$	150	psia
$T_{01} =$	200	°F
$T_{01} =$	660	°R
$A_e/A_t =$	1.76	
$A_1/A_t =$	1.2	

Equations and Computations:

The pre-shock Mach number can be found based on the area ratio:

$M_1 =$ 1.5341

The static pressure before the shock wave is:

$p_{0inlet}/p_1 =$ 3.8580

$p_1 =$ 38.881 psia

The Mach number and static pressure after the shock wave are:

$M_2 =$ 0.689

$p_2/p_1 =$ 2.5792

$p_2 =$ 100.282 psia

The area ratio for the remainder of the nozzle is:

$A_e/A_2 =$ 1.4667

Based on this and the post-shock Mach number, we can determine the exit Mach number:

$A_2/A_2^* =$ 1.102

$A_e/A_2^* =$ 1.617

$M_e =$ 0.392

Therefore the exit pressure is:

$p_{02}/p_2 =$ 1.374

$p_{02}/p_e =$ 1.112

$p_e =$ 123.9 psia

Based on the area ratio, the design Mach number is:

$M_d =$ 2.050

The pressure ratio for the third critical can be found from the design point Mach number:

$p_{0inlet}/p_{b,3rd} =$ 8.4583

$p_{b,3rd}/p_{0inlet} =$ 0.1182

So the design pressure is:

$p_d =$ 17.73 psia

Problem 13.110

[Difficulty: 4]

13.110 A stationary normal shock stands in the diverging section of a converging-diverging nozzle. The Mach number ahead of the shock is 3.0. The nozzle area at the shock is 500 mm^2. The nozzle is fed from a large tank where the pressure is 1000 kPa (gage) and the temperature is 400 K. Find the Mach number, stagnation pressure, and static pressure after the shock. Calculate the nozzle throat area. Evaluate the entropy change across the shock. Finally, if the nozzle exit area is 600 mm^2, estimate the exit Mach number. Would the actual exit Mach number be higher, lower, or the same as your estimate? Why?

Solution:

Compressible flow functions (Appendix E) to be used in solution

Assumptions: (1) steady flow (2) ideal gas (3) uniform flow at each section (4) isentropic flow, except across shock

For $M_1 = 3.0$, from Appendix E.1, $p_1/p_0 = 0.02722$ $\therefore p_1 = 29.97$ kPa

$A_1/A^* = 4.235$ $\therefore A^* = A_t = 118$ mm^2 — A_t

For $M_1 = 3.0$, from Appendix E.4, $M_2 = 0.475$ — M_2

$p_{02}/p_{01} = 0.3283$ $\therefore p_{02} = 361$ kPa (abs) — p_{02}

$p_2/p_1 = 10.33$ $\therefore p_2 = 310$ kPa (abs) — p_2

Since $T_{02} = T_{01}$, then

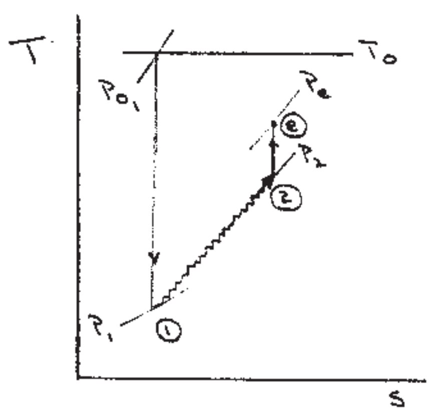
$$s_2 - s_1 = s_{02} - s_{01} = \cancel{c_p \ln \frac{T_{02}}{T_{01}}} - R \ln \frac{p_{02}}{p_{01}} = -287 \frac{N \cdot m}{kg \cdot K} \ln 0.3283$$

$$s_2 - s_1 = -0.320 \text{ kJ/kg} \cdot K$$ — $s_2 - s_1$

At $M_2 = 0.475$, from App. E.1 $A_2/A_2^* = 1.391$ $\therefore A_2^* = 359.5$ mm^2

At exit $A_e/A_2^* = \frac{600}{359.5} = 1.669$. From App E.1 (Fig E.1) & Eq 12.6

$M_e = 0.377$ — M_e



The actual exit Mach number would be higher than the estimate based on isentropic flow downstream from the shock.

Flow downstream from the shock is subsonic. Flow slows in the diverging passage, which acts as a subsonic diffuser, causing pressure to increase in the direction of flow.

The result will be rapid growth of boundary layers on the channel walls. The boundary layers reduce the effective flow area of the passage. Because the boundary layers thicken rapidly, the area ratio for slowing the flow will be less than for isentropic flow. Therefore the actual flow will not slow as much as the isentropic model predicts.

The actual exit Mach number will be higher than the estimate based on isentropic flow.

Problem 13.111

[Difficulty: 3]

13.111 Air flows adiabatically from a reservoir, where $T_{0_1} = 60°C$ and $p_{0_1} = 600$ kPa (abs), through a converging-diverging nozzle. The design Mach number of the nozzle is 2.94. A normal shock occurs at the location in the nozzle where $M = 2.42$. Assuming isentropic flow before and after the shock, determine the back pressure downstream from the nozzle. Sketch the pressure distribution.

Solution:

Compressible flow tables (Appendix E) to be used in solution.

Assumptions: (1) steady flow (3) uniform flow at each section
(2) ideal gas (4) isentropic flow, except across shock.

At design, $M_e = 2.94$, from Appendix E.1, $A_e/A_1^* = 3.999$

For $M_1 = 2.42$, from Appendix E.1, $p_1/p_{0_1} = 0.06630$
$A_1/A_1^* = 2.448$

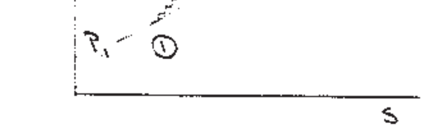
For $M_1 = 2.42$, from Appendix E.4, $M_2 = 0.521$
$p_{0_2}/p_{0_1} = 0.5318$
$p_2/p_1 = 6.666$

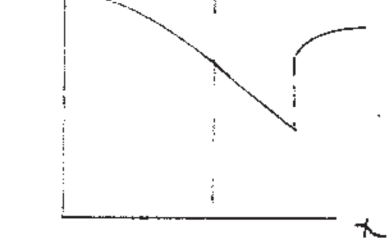
For $M_2 = 0.521$, from Appendix E.1, $A_2/A_2^* = A_1/A_2^* = 1.302$

Then, $\dfrac{A_e}{A_2^*} = \dfrac{A_e}{A_1^*} \times \dfrac{A_1^*}{A_1} \times \dfrac{A_1}{A_2^*} = 3.999 \times \dfrac{1}{2.448} \times 1.302 = 2.127$

For $A_e/A_2^* = 2.127$, from App. E.1 (Fig E.1 & Eq 12.7b), $M_e = 0.286$, $\dfrac{p_e}{p_{0_2}} = 0.9447$

$\therefore p_b = p_e = \dfrac{p_e}{p_{0_2}} \times \dfrac{p_{0_2}}{p_{0_1}} \times p_{0_1} = 0.9447 \times 0.5318 \times 600 \text{ kPa} = 301 \text{ kPa}$ ← p_b





Problem 13.112

[Difficulty: 5]

***13.112** Air flows through a converging-diverging nozzle with an area ratio of 2.5. Stagnation conditions at the inlet are 1 MPa and 320 K. A constant-area, adiabatic duct with $L/D = 10$ and $f = 0.03$ is attached to the nozzle outlet. (a) Compute the back pressure that would place a normal shock at the nozzle exit. (b) What back pressure would place the normal shock at the duct exit? (c) What back pressure would result in shock-free flow?

Given: Air flowing through a converging-diverging nozzle followed by duct with friction

Find: Back pressure needed for (a) normal shock at nozzle exit, (b) normal shock at duct exit, (c) back pressure for shock-free flow

Solution:

The given or available data is:

$R =$	286.9	J/kg-K
$k =$	1.4	
$p_{0inlet} =$	1	MPa
$T_{0inlet} =$	320	K
$A_e/A_t =$	2.5	
$L/D =$	10	
$f =$	0.03	

Equations and Computations:

(a) For a shock wave at the nozzle exit:

The pre-shock Mach number can be found based on the area ratio:

$M_1 =$ 2.4428

The static pressure before the shock wave is:

$p_{0inlet}/p_1 =$ 15.6288

$p_1 =$ 63.984 kPa

The Mach number and static pressure after the shock wave are:

$M_2 =$ 0.5187

$p_2/p_1 =$ 6.7950

$p_2 =$ 434.770 kPa

The friction length and critical pressure ratio after the shock wave are:

$fL/D_2 =$ 0.9269

$p_2/p_2^* =$ 2.0575

The friction length for the duct is:

$fL/D_{2\text{-}3} =$ 0.3000

Therefore, the friction length at the duct exit is:

$fL/D_3 =$ 0.6269

Iterating on Mach number with *Solver* to match this friction length yields:

$M_3 =$ 0.5692

$fL/D_3 =$ 0.6269

The critical pressure ratio for this condition is:

$p_3/p_3^* =$ 1.8652

Since the critical pressure at 2 and 3 are equal, the back pressure is:

$p_b = p_3 =$ 394 kPa

(b) For a shock wave at the duct exit:
We use the same nozzle exit Mach number and pressure:

$M_1 =$ 2.443

$p_1 =$ 63.984 kPa

The friction length and critical pressure ratio at this condition are:

$fL/D_1 =$ 0.4195

$p_1/p_1^* =$ 0.3028

The friction length for the duct is:

$fL/D_{1\text{-}2} =$ 0.3000

Therefore, the friction length at the duct exit is:

$fL/D_2 =$ 0.1195

Iterating on Mach number with *Solver* to match this friction length yields:

$M_2 =$ 1.4547

$fL/D_2 =$ 0.1195

The critical pressure ratio for this condition is:

$p_2/p_2^* =$ 0.6312

Since the critical pressure at 1 and 2 are equal, the pressure is:

$p_2 =$ 133.388 kPa

The Mach number and static pressure after the shock wave are:

$M_3 =$ 0.7178

$p_3/p_2 =$ 2.3021

$p_b = p_3 =$ 307 kPa

(c) For shock-free flow, we use the conditions from part b before the shock wave:

$p_b = p_3 =$ 133.4 kPa

Problem 13.113

[Difficulty: 4]

***13.113** Consider the setup of Problem 13.112, except that the constant-area duct is frictionless and no longer adiabatic. A normal shock stands at the duct exit, after which the temperature is 350 K. Calculate the Mach number after the shock wave, and the heat addition in the constant-area duct.

Given: Air flowing through a converging-diverging nozzle followed by diabatic duct

Find: Mach number at duct exit and heat addition in duct

Solution:

The given or available data is:

$R =$	286.9	J/kg-K
$c_p =$	1004	J/kg-K
$k =$	1.4	
$p_{0inlet} =$	1	MPa
$T_{0inlet} =$	320	K
$A_1/A_t =$	2.5	
$T_e =$	350	K

Equations and Computations:

The Mach number at the nozzle exit can be found based on the area ratio:

$M_1 =$ 2.4428

The static temperature is:

$T_{0inlet}/T_1 =$ 2.1934

$T_1 =$ 145.891 K

The Rayliegh flow critical ratios at this condition are:

$T_1/T_1^* =$ 0.39282

$T_{01}/T_{01}^* =$ 0.71802

Since all we know is the static temperature at the exit, we need to iterate on a solution. We can guess at a pre-shock Mach number at the duct exit, and iterate on that value until we match the exit temperature:

$M_2 =$	1.753	
$T_2/T_2^* =$	0.62964	
$T_{02}/T_{02}^* =$	0.84716	
$T_2 =$	233.844	K
$T_{02} =$	377.553	K
$M_3 =$	0.6274	
$T_3/T_2 =$	1.4967	
$T_3 =$	350.000	K

In this case we used *Solver* to match the exit temperature.

Therefore, the exit Mach number is:

$M_3 =$ 0.627

The rate of heat addition is calculated from the rise in stagnation temperature:

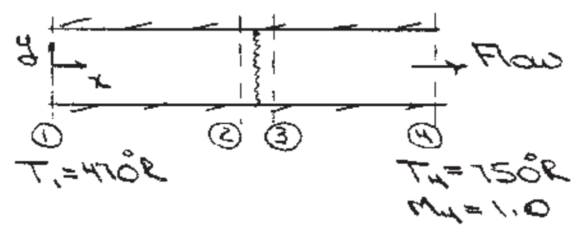
$q_{1\text{-}2} =$ 57.78 kJ/kg

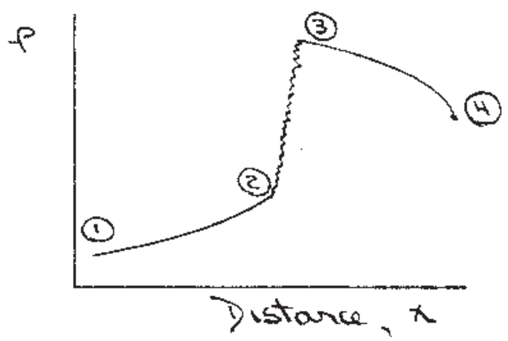
Problem 13.114

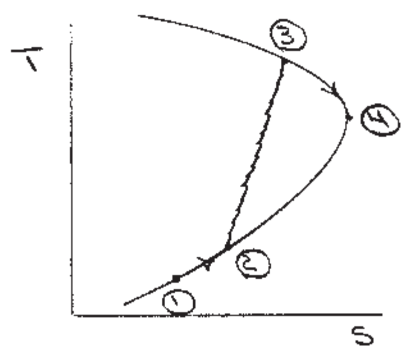
[Difficulty: 2]

*13.114 A normal shock stands in a section of insulated constant-area duct. The flow is frictional. At section ①, some distance upstream from the shock, $T_1 = 470°R$. At section ④, some distance downstream from the shock, $T_4 = 750°R$ and $M_4 = 1.0$. Denote conditions immediately upstream and downstream from the shock by subscripts ② and ③, respectively. Sketch the pressure distribution along the duct, indicating clearly the locations of sections ① through ④. Sketch a Ts diagram for the flow. Determine the Mach number at section ①.

Solution:







For adiabatic flow, T_0 = constant (from energy eq.).

Thus, $T_{0_1} = T_{0_4} = T_4\left[1+\frac{k-1}{2}M_4^2\right] = 750°R\,[1.2] = 900°R$

Since $T_{0_1} = T_1\left[1+\frac{k-1}{2}M_1^2\right]$, then

$$M_1^2 = \frac{2}{k-1}\left[\frac{T_{0_1}}{T_1}-1\right] = \frac{2}{0.4}\left[\frac{900R}{470R}-1\right] = 4.57$$

$M_1 = \sqrt{4.57} = 2.14$ ——— M_1

Problem 13.115

[Difficulty: 5]

*13.115 A supersonic wind tunnel must have two throats, with the second throat larger than the first. Explain why this must be so.

Discussion: The first throat is located in the supersonic nozzle from which flow enters the test section. The second throat is located in the supersonic diffuser that slows flow leaving the test section to subsonic speed for re-compression and re-circulation.

The second throat must be larger than the first for two reasons. First, it is impossible to slow a real flow to a Mach number of exactly one in a supersonic diffuser. The minimum Mach number that can be achieved with stable flow is about $M = 1.3$. Therefore even if the flow were isentropic everywhere the second throat would have to be larger than the first by the area ratio A/A^* corresponding to $M = 1.3$ at the throat.

The second reason is that flow is not isentropic through the tunnel. Some friction exists, which must reduce the stagnation pressure of the flow stream. This also reduces the stagnation density. Therefore a larger area is needed to carry the mass flow at any given flow speed.

For these reasons the second throat area must be larger than the first throat area.

Problem 13.116

[Difficulty: 4]

*13.116 A normal shock stands in a section of insulated constant-area duct. The flow is frictional. At section ①, some distance upstream from the shock, $T_1 = 668°R$, $p_{0_1} = 78.2$ psia, and $M_1 = 2.05$. At section ④, some distance downstream from the shock, $M_4 = 1.00$. Calculate the air speed, V_2, immediately ahead of the shock, where $T_2 = 388°F$. Evaluate the entropy change, $s_4 - s_1$.

Solution: Use functions for steady, one-dimensional compressible flow.

Computing equations: $\frac{T}{T_0}$ from isentropic functions (Appendix E.1)

$\frac{p}{p^*}$ from Fanno-line functions (Appendix E.2)

Assumptions: (1) Steady flow (2) Uniform flow at each cross-section (3) Ideal gas (4) Fanno line flow (5) $F_{B_x} = 0$ (6) $\Delta z = 0$

For adiabatic flow on Fanno line and across shock, T_0 = constant. At $M_1 = 2.05$, $T/T_0 = 0.5433$ (Eq. 11.17b). Thus

$$T_0 = T_{0_1} = \frac{T_1}{(T/T_0)_1} = \frac{668°R}{0.5433} = 1230°R$$

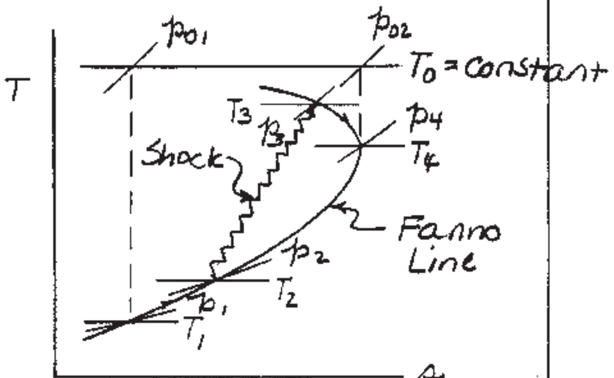
Using $T_2 = 388$ F ($848°R$), $\frac{T}{T_0} = \frac{848°R}{1230°R} = 0.6894$ and $M_2 = 1.50$ (Table E.1).

$$V_2 = M_2 c_2 = M_2\sqrt{kRT_2} = 1.50\left[1.4 \times 53.3\frac{ft\cdot lbf}{lbm\cdot °R} \times 848°R \times 32.2\frac{lbm}{slug} \times \frac{slug\cdot ft}{lbf\cdot s^2}\right]^{\frac{1}{2}}$$

$V_2 = 2140$ ft/s $\quad V_2$

Flow must stay on the same Fanno line. Thus $(p_0/p_0^*)_1 = 1.760$ at $M_1 = 2.05$ (Eq. 12.18e). Thus

$$p_0^* = \frac{p_{01}}{(p_0/p_0^*)_1} = \frac{78.2 \text{ psia}}{1.760} = 44.4 \text{ psia}$$



From the Gibbs equation

$$Tds = dh - v\,dp\ ;\ ds = c_p\frac{dT}{T} - R\frac{dp}{p}$$

$$\Delta s = \cancel{c_p \ln\frac{T_{04}}{T_{01}}}^{\,0} - R\ln\frac{p_{04}}{p_{01}}$$

$$\Delta s = -53.3\frac{ft\cdot lbf}{lbm\cdot °R} \times \ln\left(\frac{44.4}{78.2}\right) \times \frac{Btu}{778\ ft\cdot lbf} = 0.0388\ Btu/lbm\cdot °R$$

Δs

{This problem could be solved without using functions, but the solution would require more calculations.}

Problem 13.117

[Difficulty: 2]

13.117 Nitrogen is discharged from a 30-cm-diameter duct at $M_2 = 0.85$, $T_2 = 300$ K, and $p_2 = 200$ kPa. The temperature at the inlet of the duct is $T_1 = 330$ K. Compute the pressure at the inlet and the mass flow rate.

Given: Nitrogen traveling through duct

Find: Inlet pressure and mass flow rate

Solution:

The given or available data is:

$R =$	296.8	J/kg-K
$k =$	1.4	
$D =$	30	cm
$M_2 =$	0.85	
$T_2 =$	300	K
$p_2 =$	200	kPa
$T_1 =$	330	K

Equations and Computations:

We can find the critical temperature and pressure for choking at station 2:

$T_2/T^* =$ 1.0485

$T^* =$ 286.1 K

$p_2/p^* =$ 1.2047

$p^* =$ 166.0 kPa

Knowing the critical state, the Mach number at station 1 can be found:
(we will use *Goal Seek* to match the Mach number)

$T_1/T^* =$ 1.1533

$M_1 =$ 0.4497

$T_1/T^* =$ 1.1533

The static to critical pressure ratio is a function of Mach number. Therefore:

$p_1 =$ 396 kPa

The sound speed at station 1 is:

$c_1 =$ 370.30 m/s

So the velocity at 1 is:

$V_1 =$ 166.54 m/s

The density at 1 can be calculated from the ideal gas equation of state:

$\rho_1 =$ 4.0476 kg/m^3

The area of the duct is:

$A =$ 0.0707 m^2

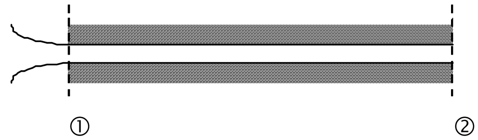
So the mass flow rate is:

$m =$ 47.6 kg/s

Problem 13.118

[Difficulty: 2]

13.118 Room air is drawn into an insulated duct of constant area through a smoothly contoured converging nozzle. Room conditions are $T = 80°\text{F}$ and $p = 14.7$ psia. The duct diameter is $D = 1$ in. The pressure at the duct inlet (nozzle outlet) is $p_1 = 13$ psia. Find (a) the mass flow rate in the duct and (b) the range of exit pressures for which the duct exit flow is choked.



Given: Air flow in an insulated duct

Find: Mass flow rate; Range of choked exit pressures

Solution:

Basic equations: $\dfrac{T_0}{T} = 1 + \dfrac{k-1}{2}\cdot M^2$ $\quad c = \sqrt{k\cdot R\cdot T}$ $\quad \dfrac{A}{A_{crit}} = \dfrac{1}{M}\cdot\left(\dfrac{1 + \dfrac{k-1}{2}\cdot M^2}{\dfrac{k+1}{2}}\right)^{\frac{k+1}{2\cdot(k-1)}}$

Given or available data $\quad T_0 = (80 + 460)\cdot R \quad p_0 = 14.7\cdot psi \quad p_1 = 13\cdot psi \quad D = 1\cdot in$

$k = 1.4 \quad R_{air} = 53.33\cdot\dfrac{ft\cdot lbf}{lbm\cdot R} \quad A = \dfrac{\pi\cdot D^2}{4} \quad A = 0.785\cdot in^2$

Assuming isentropic flow, stagnation conditions are constant. Hence

$$M_1 = \sqrt{\frac{2}{k-1}\cdot\left[\left(\frac{p_0}{p_1}\right)^{\frac{k-1}{k}} - 1\right]} \quad M_1 = 0.423 \quad T_1 = \frac{T_0}{1 + \frac{k-1}{2}\cdot M_1^2} \quad T_1 = 521\cdot R \quad T_1 = 61.7\cdot °F$$

$$c_1 = \sqrt{k\cdot R_{air}\cdot T_1} \quad c_1 = 341\frac{m}{s} \quad V_1 = M_1\cdot c_1 \quad V_1 = 144\frac{m}{s}$$

Also $\quad \rho_1 = \dfrac{p_1}{R_{air}\cdot T_1} \quad \rho_1 = 0.0673\cdot\dfrac{lbm}{ft^3}$

Hence $\quad m_{rate} = \rho_1\cdot V_1\cdot A \quad m_{rate} = 0.174\cdot\dfrac{lbm}{s}$

When flow is choked $\quad M_2 = 1$ hence $\quad T_2 = \dfrac{T_0}{1 + \dfrac{k-1}{2}} \quad T_2 = 450\cdot R \quad T_2 = -9.7\cdot °F$

We also have $\quad c_2 = \sqrt{k\cdot R_{air}\cdot T_2} \quad c_2 = 1040\cdot\dfrac{ft}{s} \quad V_2 = c_2 \quad V_2 = 1040\cdot\dfrac{ft}{s}$

From continuity $\quad \rho_1\cdot V_1 = \rho_2\cdot V_2 \quad \rho_2 = \rho_1\cdot\dfrac{V_1}{V_2} \quad \rho_2 = 0.0306\cdot\dfrac{lbm}{ft^3}$

Hence $\quad p_2 = \rho_2\cdot R_{air}\cdot T_2 \quad p_2 = 5.11\cdot psi$

The flow will therefore choke for any back pressure (pressure at the exit) less than or equal to this pressure

(From Fanno line function $\quad \dfrac{p_1}{p_{crit}} = 2.545 \quad$ at $\quad M_1 = 0.423 \quad$ so $\quad p_{crit} = \dfrac{p_1}{2.545} \quad p_{crit} = 5.11\cdot psi \quad$ Check!)

Problem 13.119 [Difficulty: 4]

13.119 Air from a large reservoir at 25 psia and 250°F flows isentropically through a converging nozzle into an insulated pipe at 24 psia. The pipe flow experiences friction effects. Obtain a plot of the Ts diagram for this flow, until $M = 1$. Also plot the pressure and speed distributions from the entrance to the location at which $M = 1$.

Given: Air flow from converging nozzle into pipe

Find: Plot Ts diagram and pressure and speed curves

Solution:

The given or available data is:

$R =$	53.33	ft·lbf/lbm·°R
$k =$	1.4	
$c_p =$	0.2399	Btu/lbm·°R
	187	ft·lbf/lbm·°R
$T_0 =$	710	°R
$p_0 =$	25	psi
$p_e =$	24	psi

Equations and Computations:

From p_0 and p_e, and Eq. 13.7a (using built-in function *IsenMfromp*(M,k)) $M_e =$ 0.242

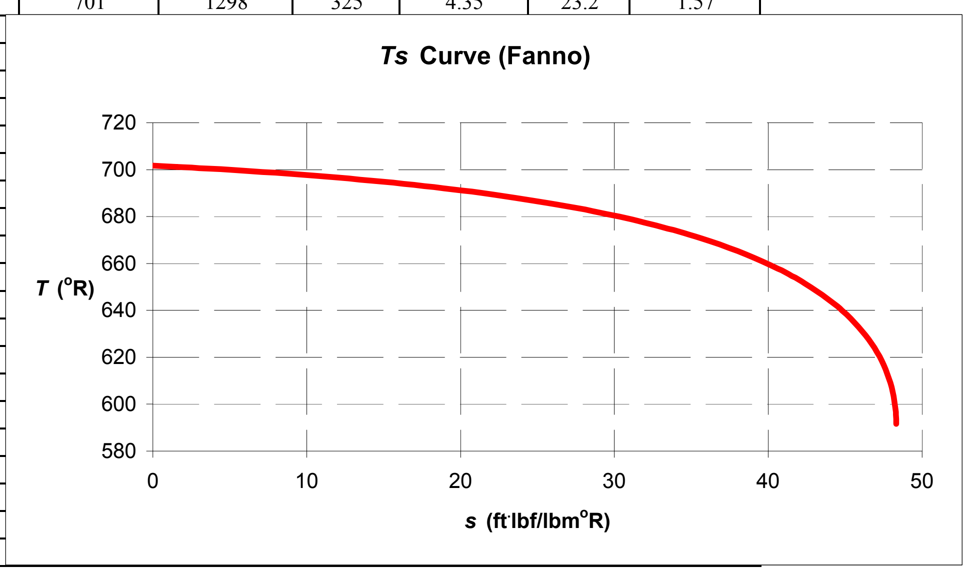
Using built-in function *IsenT*(M,k) $T_e =$ 702 °R

Using p_e, M_e, and function *Fannop*(M,k) $p^* =$ 5.34 psi

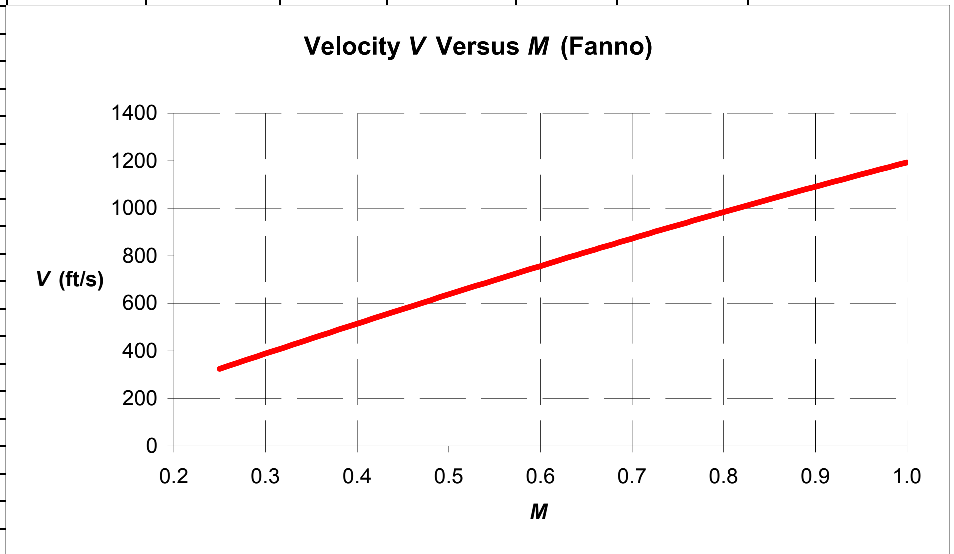
Using T_e, M_e, and function *FannoT*(M,k) $T^* =$ 592 °R

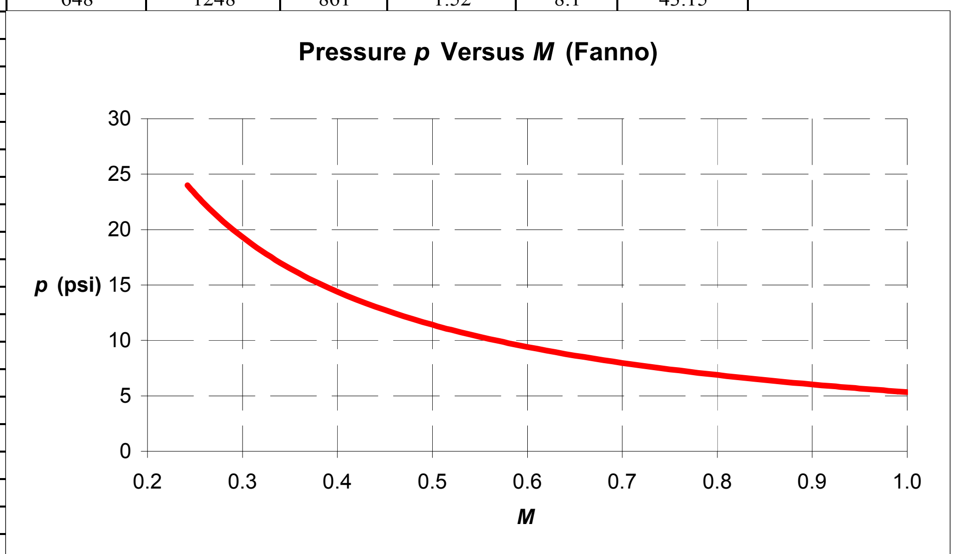
We can now use Fanno-line relations to compute values for a range of Mach numbers:

M	T/T^*	T (°R)	c (ft/s)	V (ft/s)	p/p^*	p (psi)	Δs (ft·lbf/lbm·°R) Eq. (12.11b)
0.242	1.186	702	1299	315	4.50	24.0	0.00
0.25	1.185	701	1298	325	4.35	23.2	1.57
0.26	1.184						
0.27	1.183						
0.28	1.181						
0.29	1.180						
0.3	1.179						
0.31	1.177						
0.32	1.176						
0.33	1.174						
0.34	1.173						
0.35	1.171						
0.36	1.170						
0.37	1.168						
0.38	1.166						
0.39	1.165						
0.4	1.163						
0.41	1.161						
0.42	1.159						
0.43	1.157						
0.44	1.155						
0.45	1.153						



0.46	1.151	681	1280	589	2.33	12.4	29.44
0.47	1.149	680	1279	601	2.28	12.2	30.31
0.48	1.147						
0.49	1.145						
0.5	1.143						
0.51	1.141						
0.52	1.138						
0.53	1.136						
0.54	1.134						
0.55	1.132						
0.56	1.129						
0.57	1.127						
0.58	1.124						
0.59	1.122						
0.6	1.119						
0.61	1.117						
0.62	1.114						
0.63	1.112						
0.64	1.109						
0.65	1.107						
0.66	1.104						
0.67	1.101						
0.68	1.098	650	1250	850	1.54	8.2	42.77
0.69	1.096	648	1248	861	1.52	8.1	43.15
0.7	1.093						
0.71	1.090						
0.72	1.087						
0.73	1.084						
0.74	1.082						
0.75	1.079						
0.76	1.076						
0.77	1.073						
0.78	1.070						
0.79	1.067						
0.8	1.064						
0.81	1.061						
0.82	1.058						
0.83	1.055						
0.84	1.052						
0.85	1.048						
0.86	1.045						
0.87	1.042						
0.88	1.039						
0.89	1.036						
0.9	1.033	611	1212	1091	1.13	6.0	47.84
0.91	1.029	609	1210	1101	1.11	6.0	47.94
0.92	1.026	607	1208	1112	1.10	5.9	48.02
0.93	1.023	605	1206	1122	1.09	5.8	48.09
0.94	1.020	603	1204	1132	1.07	5.7	48.15
0.95	1.017	601	1202	1142	1.06	5.7	48.20
0.96	1.013	600	1201	1153	1.05	5.6	48.24
0.97	1.010	598	1199	1163	1.04	5.5	48.27
0.98	1.007	596	1197	1173	1.02	5.5	48.30
0.99	1.003	594	1195	1183	1.01	5.4	48.31
1	1.000	592	1193	1193	1.00	5.3	48.31





Problem 13.120 [Difficulty: 4]

13.120 Repeat Problem 13.119 except the nozzle is now a converging-diverging nozzle delivering the air to the pipe at 2.5 psia.

Given: Air flow from converging-diverging nozzle into pipe

Find: Plot Ts diagram and pressure and speed curves

Solution:

The given or available data is:

$R =$	53.33	ft·lbf/lbm·°R
$k =$	1.4	
$c_p =$	0.2399	Btu/lbm·°R
	187	ft·lbf/lbm·°R
$T_0 =$	710	°R
$p_0 =$	25	psi
$p_e =$	2.5	psi

Equations and Computations:

From p_0 and p_e, and Eq. 13.7a (using built-in function *IsenMfromp*(M,k)) $M_e =$ 2.16

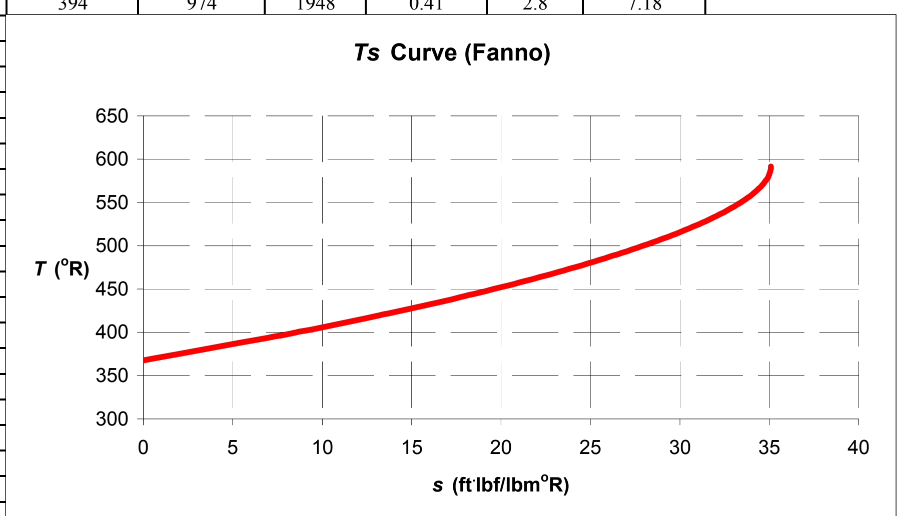
Using built-in function *IsenT*(M,k) $T_e =$ 368 °R

Using p_e, M_e, and function *Fannop*(M,k) $p^* =$ 6.84 psi

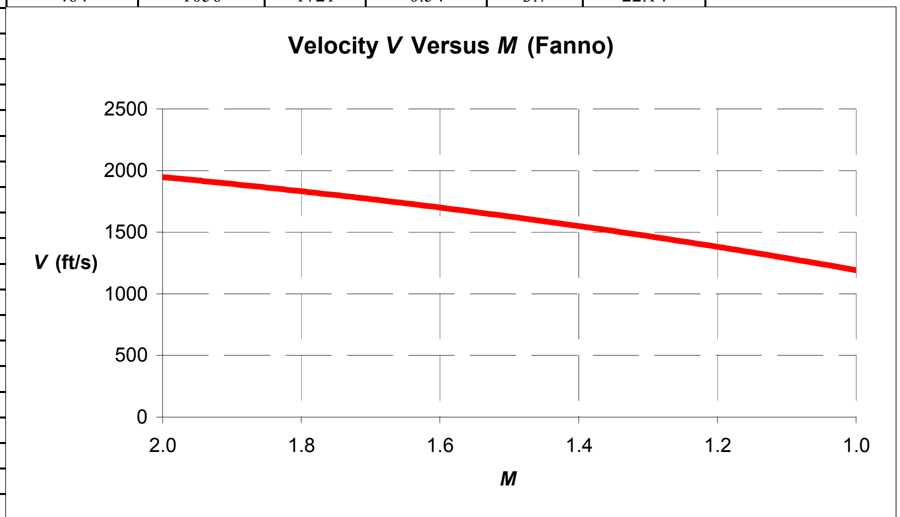
Using T_e, M_e, and function *FannoT*(M,k) $T^* =$ 592 °R

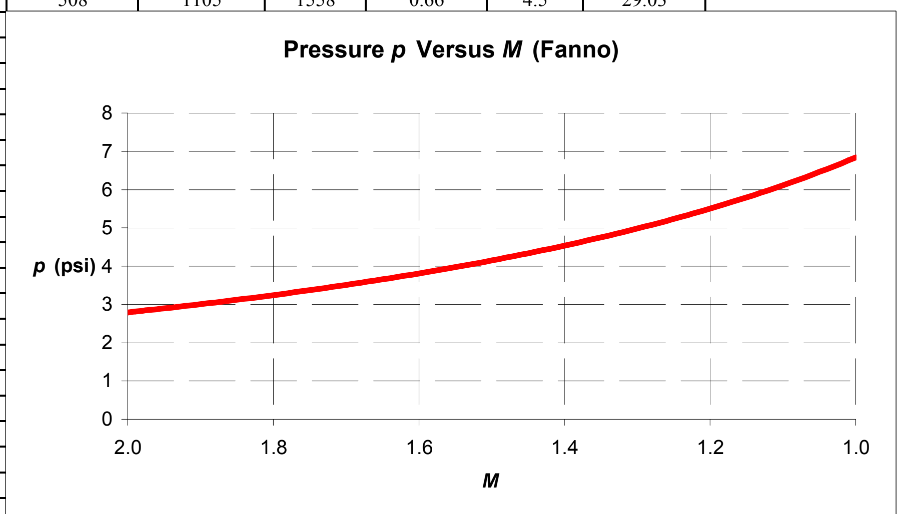
We can now use Fanno-line relations to compute values for a range of Mach numbers:

M	T/T^*	T (°R)	c (ft/s)	V (ft/s)	p/p^*	p (psi)	Δs (ft·lbf/lbm·°R) Eq. (12.11b)
2.157	0.622	368	940	2028	0.37	2.5	0.00
2	0.667	394	974	1948	0.41	2.8	7.18
1.99	0.670						
1.98	0.673						
1.97	0.676						
1.96	0.679						
1.95	0.682						
1.94	0.685						
1.93	0.688						
1.92	0.691						
1.91	0.694						
1.9	0.697						
1.89	0.700						
1.88	0.703						
1.87	0.706						
1.86	0.709						
1.85	0.712						
1.84	0.716						
1.83	0.719						
1.82	0.722						
1.81	0.725						
1.8	0.728						
1.79	0.731	433	1020	1826	0.48	3.3	16.08
1.78	0.735	435	1022	1819	0.48	3.3	16.48
1.77	0.738	436	1024	1813	0.49	3.3	16.88
1.76	0.741	438	1027	1807	0.49	3.3	17.27
1.75	0.744	440	1029	1801	0.49	3.4	17.66
1.74	0.747	442	1031	1794	0.50	3.4	18.05
1.73	0.751	444	1033	1788	0.50	3.4	18.44
1.72	0.754	446	1036	1781	0.50	3.5	18.82
1.71	0.757	448	1038	1775	0.51	3.5	19.20



1.7	0.760	450	1040	1768	0.51	3.5	19.58
1.69	0.764	452	1042	1761	0.52	3.5	19.95
1.68	0.767	454	1045	1755	0.52	3.6	20.32
1.67	0.770	456	1047	1748	0.53	3.6	20.69
1.66	0.774	458	1049	1741	0.53	3.6	21.06
1.65	0.777	460	1051	1735	0.53	3.7	21.42
1.64	0.780	462	1054	1728	0.54	3.7	21.78
1.63	0.784	464	1056	1721	0.54	3.7	22.14
1.62	0.787						
1.61	0.790						
1.6	0.794						
1.59	0.797						
1.58	0.800						
1.57	0.804						
1.56	0.807						
1.55	0.811						
1.54	0.814						
1.53	0.817						
1.52	0.821						
1.51	0.824						
1.5	0.828						
1.49	0.831						
1.48	0.834						
1.47	0.838						
1.46	0.841						
1.45	0.845						
1.44	0.848						
1.43	0.852						
1.42	0.855	506	1103	1566	0.65	4.5	28.76
1.41	0.859	508	1105	1558	0.66	4.5	29.03
1.4	0.862						
1.39	0.866						
1.38	0.869						
1.37	0.872						
1.36	0.876						
1.35	0.879						
1.34	0.883						
1.33	0.886						
1.32	0.890						
1.31	0.893						
1.3	0.897						
1.29	0.900						
1.28	0.904						
1.27	0.907						
1.26	0.911						
1.25	0.914						
1.24	0.918						
1.23	0.921						
1.22	0.925						
1.21	0.928						
1.2	0.932	551	1151	1381	0.80	5.5	33.50
1.19	0.935	553	1153	1372	0.81	5.6	33.65
1.18	0.939	555	1155	1363	0.82	5.6	33.79
1.17	0.942	557	1158	1354	0.83	5.7	33.93
1.16	0.946	559	1160	1345	0.84	5.7	34.05
1.15	0.949	561	1162	1336	0.85	5.8	34.18
1.14	0.952	564	1164	1327	0.86	5.9	34.29
1.13	0.956	566	1166	1318	0.87	5.9	34.40
1.12	0.959	568	1168	1308	0.87	6.0	34.50
1.11	0.963	570	1170	1299	0.88	6.0	34.59
1.1	0.966	572	1172	1290	0.89	6.1	34.68
1.09	0.970	574	1174	1280	0.90	6.2	34.76
1.08	0.973	576	1176	1271	0.91	6.2	34.83
1.07	0.976	578	1179	1261	0.92	6.3	34.89
1.06	0.980	580	1181	1251	0.93	6.4	34.95
1.05	0.983	582	1183	1242	0.94	6.5	34.99
1.04	0.987	584	1185	1232	0.96	6.5	35.03
1.03	0.990	586	1187	1222	0.97	6.6	35.06
1.02	0.993	588	1189	1212	0.98	6.7	35.08
1.01	0.997	590	1191	1203	0.99	6.8	35.10
1	1.000	592	1193	1193	1.00	6.8	35.10





Problem 13.121

[Difficulty: 3]

13.121 A 5-m duct 35 cm in diameter contains oxygen flowing at the rate of 40 kg/s. The inlet conditions are $p_1 = 200$ kPa and $T_1 = 450$ K. The exit pressure is $p_2 = 160$ kPa. Calculate the inlet and exit Mach number, and the exit stagnation pressure and temperature. Determine the friction factor, and estimate the absolute roughness of the duct material.

Given: Oxygen traveling through duct

Find: Inlet and exit Mach numbers, exit stagnation conditions, friction factor and absolute roughness

Solution:

The given or available data is:

$R =$	259.8	J/kg-K
$k =$	1.4	
$D =$	35	cm
$L =$	5	m
$m =$	40.0	kg/s
$p_1 =$	200	kPa
$T_1 =$	450	K
$p_2 =$	160	kPa

Equations and Computations:

The area of the duct is:

$A =$ 0.0962 m^2

The sound speed at station 1 is:

$c_1 =$ 404.57 m/s

The density at 1 can be calculated from the ideal gas equation of state:

$\rho_1 =$ 1.7107 kg/m^3

So the velocity at 1 is:

$V_1 =$ 243.03 m/s

and the Mach number at 1 is:

$M_1 =$ 0.601

The critical temperature and pressure may then be calculated:

$p_1/p^* =$ 1.7611

$p^* =$ 113.6 kPa

$T_1/T^* =$ 1.1192

$T^* =$ 402.1 K

Since the critical pressure is equal at 1 and 2, we can find the pressure ratio at 2:

$p_2/p^* =$ 1.4089

The static to critical pressure ratio is a function of Mach number. Therefore:

$M_2 =$ 0.738

$p_2/p^* =$ 1.4089

(we used *Solver* to find the correct Mach number to match the pressure ratio)

The exit temperature is:

$T_2/T^* =$ 1.0820

$T_2 =$ 435.0 K

Based on the exit Mach number, pressure, and temperature, stagnation conditions are:

$p_{02} =$ 230 kPa

$T_{02} =$ 482 K

The maximum friction lengths at stations 1 and 2 are:

$fL_1/D =$ 0.48802

$fL_2/D =$ 0.14357

So the friction length for this duct is:

$fL/D =$ 0.34445

and the friction factor is:

$f =$ 0.02411

Now to find the roughness of the pipe, we need the Reynolds number.
From the LMNO Engineering website, we can find the viscosities of oxygen:

$\mu_1 =$ 2.688E-05 N-s/m^2

$\mu_2 =$ 2.802E-05 N-s/m^2

Therefore the Reynolds number at station 1 is:

$Re_1 =$ 5.413E+06

At station 2, we will need to find density and velocity first. From ideal gas equation:

$\rho_2 =$ 1.4156 kg/m^3

The sound speed at 2 is:

$c_2 =$ 397.79 m/s

So the velocity at 2 is:

$V_2 =$ 293.69 m/s

and the Reynolds number is:

$Re_2 =$ 5.193E+06

So the Reynolds number does not change significantly over the length of duct.
We will use an average of the two to find the relative roughness:

$Re =$ 5.303E+06

The relative roughness for this pipe is:

$e/D =$ 0.00222

$f =$ 0.02411

(we used *Solver* to find the correct roughness to match the friction factor.)

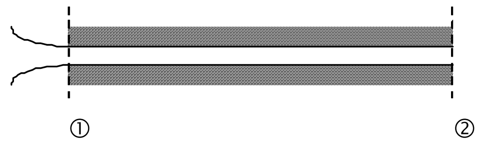
Therefore, the roughness of the duct material is:

$e =$ 0.0776 cm

Problem 13.122

[Difficulty: 3]

13.122 Air flows steadily and adiabatically from a large tank through a converging nozzle connected to an insulated constant-area duct. The nozzle may be considered frictionless. Air in the tank is at $p = 145$ psia and $T = 250°F$. The absolute pressure at the nozzle exit (duct inlet) is 125 psia. Determine the pressure at the end of the duct, if the temperature there is 150°F. Find the entropy increase.



Given: Air flow in a converging nozzle and insulated duct

Find: Pressure at end of duct; Entropy increase

Solution:

Basic equations:
$$\frac{T_0}{T} = 1 + \frac{k-1}{2}\cdot M^2 \qquad \frac{p_0}{p} = \left(1 + \frac{k-1}{2}\cdot M^2\right)^{\frac{k}{k-1}} \qquad \Delta s = c_p\cdot\ln\left(\frac{T_2}{T_1}\right) - R_{air}\cdot\ln\left(\frac{p_2}{p_1}\right) \qquad c = \sqrt{k\cdot R\cdot T}$$

Given or available data
$$T_0 = (250+460)\cdot R \qquad p_0 = 145\cdot psi \qquad p_1 = 125\cdot psi \qquad T_2 = (150+460)\cdot R$$
$$k = 1.4 \qquad c_p = 0.2399\cdot\frac{BTU}{lbm\cdot R} \qquad R_{air} = 53.33\cdot\frac{ft\cdot lbf}{lbm\cdot R}$$

Assuming isentropic flow in the nozzle
$$M_1 = \sqrt{\frac{2}{k-1}\cdot\left[\left(\frac{p_0}{p_1}\right)^{\frac{k-1}{k}} - 1\right]} \qquad M_1 = 0.465 \qquad T_1 = \frac{T_0}{1 + \frac{k-1}{2}\cdot M_1^{\,2}} \qquad T_1 = 681\cdot R \qquad T_1 = 221\cdot °F$$

In the duct T_0 (a measure of total energy) is constant, s
$$M_2 = \sqrt{\frac{2}{k-1}\cdot\left[\left(\frac{T_0}{T_2}\right) - 1\right]} \qquad M_2 = 0.905$$

At each location
$$c_1 = \sqrt{k\cdot R_{air}\cdot T_1} \qquad c_1 = 1279\cdot\frac{ft}{s} \qquad V_1 = M_1\cdot c_1 \qquad V_1 = 595\cdot\frac{ft}{s}$$
$$c_2 = \sqrt{k\cdot R_{air}\cdot T_2} \qquad c_2 = 1211\cdot\frac{ft}{s} \qquad V_2 = M_2\cdot c_2 \qquad V_2 = 1096\cdot\frac{ft}{s}$$

Also
$$\rho_1 = \frac{p_1}{R_{air}\cdot T_1} \qquad \rho_1 = 0.4960\cdot\frac{lbm}{ft^3}$$

Hence
$$m_{rate} = \rho_1\cdot V_1\cdot A = \rho_2\cdot V_2\cdot A \quad \text{so} \quad \rho_2 = \rho_1\cdot\frac{V_1}{V_2} \qquad \rho_2 = 0.269\cdot\frac{lbm}{ft^3}$$

Then
$$p_2 = \rho_2\cdot R_{air}\cdot T_2 \qquad p_2 = 60.8\cdot psi$$

Finall y
$$\Delta s = c_p\cdot\ln\left(\frac{T_2}{T_1}\right) - R_{air}\cdot\ln\left(\frac{p_2}{p_1}\right) \qquad \Delta s = 0.0231\cdot\frac{BTU}{lbm\cdot R}$$

(Note: Using Fanno line relations, at $M_1 = 0.465$
$$\frac{T_1}{T_{crit}} = 1.150 \qquad T_{crit} = \frac{T_1}{1.150} \qquad T_{crit} = 329\,K$$
$$\frac{p_1}{p_{crit}} = 2.306 \qquad p_{crit} = \frac{p_1}{2.3060} \qquad p_{crit} = 54.2\cdot psi$$

Then $\frac{T_2}{T_{crit}} = 1.031$ so $M_2 = 0.907$
$$\frac{p_2}{p_{crit}} = 1.119 \qquad p_2 = 1.119\cdot p_{crit} \qquad p_2 = 60.7\cdot psi$$
Check!)

Problem 13.123

[Difficulty: 2]

13.123 A Fanno-line flow apparatus in an undergraduate fluid mechanics laboratory consists of a smooth brass tube of 7.16 mm inside diameter, fed by a converging nozzle. The lab temperature and uncorrected barometer reading are 23.5°C and 755.1 mm of mercury. The pressure at the exit from the converging nozzle (entrance to the constant-area duct) is −20.8 mm of mercury (gage). Compute the Mach number at the entrance to the constant-area tube. Calculate the mass flow rate in the tube. Evaluate the pressure at the location in the tube where the Mach number is 0.4.

Solution: Apply equations for steady, 1-D compressible flow:

Computing equations: $T_0 = T\left(1+\frac{k-1}{2}M^2\right)$ {Entire flow adiabatic}

$p_0 = p\left(1+\frac{k-1}{2}M^2\right)^{\frac{k}{k-1}}$ {Isentropic in nozzle}

Assume: (1) Stagnation conditions in laboratory
(2) Ideal gas

Then $\frac{p_0}{p_1} = \left(1+\frac{k-1}{2}M_1^2\right)^{\frac{k}{k-1}} = \frac{755.1 \text{ mm Hg}}{(755.1-20.8)\text{ mm Hg}} = 1.03$

$$M_1 = \left\{\frac{2}{k-1}\left[\left(\frac{p_0}{p_1}\right)^{\frac{k-1}{k}}-1\right]\right\}^{\frac{1}{2}} = \left\{5\left[\left(\frac{755.1}{734.3}\right)^{0.286}-1\right]\right\}^{\frac{1}{2}} = 0.200 \qquad M_1$$

From continuity, $\dot{m} = \rho_1 V_1 A_1$; $\rho_1 = \frac{p_1}{RT_1}$

$$p_1 = \rho_{Hg} g h_1 = (13.5)\,1000\,\frac{kg}{m^3} \times 9.81\,\frac{m}{s^2} \times 0.734\,m \times \frac{N\cdot s^2}{kg\cdot m} = 97.2 \text{ kPa (abs)}$$

$$T_1 = \frac{T_0}{1+\frac{k-1}{2}M_1^2} = \frac{(273+23)K}{1+0.2(0.200)^2} = 294 \text{ K} \;;\; c_1 = \sqrt{kRT_1} = 344 \text{ m/s}$$

$$\rho_1 = 97.2\times10^3\,\frac{N}{m^2} \times \frac{kg\cdot K}{287\,N\cdot m} \times \frac{1}{294\,K} = 1.15 \text{ kg/m}^3$$

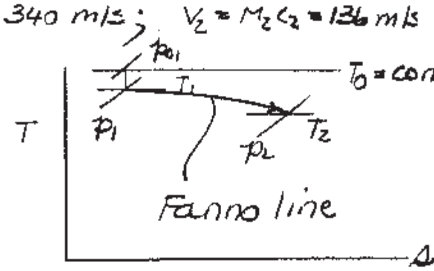
$$V_1 = M_1 c_1 = 0.200 \times 344 \text{ m/s} = 68.8 \text{ m/s}$$

$$A_1 = \frac{\pi D^2}{4} = \frac{\pi}{4}(0.00716)^2\,m^2 = 4.03\times10^{-5}\,m^2$$

$$\dot{m} = 1.15\,\frac{kg}{m^3} \times 68.8\,\frac{m}{s} \times 4.03\times10^{-5}\,m^2 = 3.19\times10^{-3}\text{ kg/s} \qquad \dot{m}$$

Since T_0 = constant, $T_2 = T_0\Big/\left(1+\frac{k-1}{2}M_2^2\right) = 287$ K ; $c_2 = 340$ m/s ; $V_2 = M_2 c_2 = 136$ m/s

$$\rho_2 = \rho_1 \frac{V_1}{V_2} = 1.15\,\frac{kg}{m^3} \times \frac{68.8}{136} = 0.582 \text{ kg/m}^3$$



Then $p_2 = \rho_2 R T_2$

$$p_2 = 0.582\,\frac{kg}{m^3} \times 287\,\frac{N\cdot m}{kg\cdot K} \times 287\,K = 47.9 \text{ kPa (abs)} \qquad p_2$$

Problem 13.124

[Difficulty: 3]

13.124 Measurements are made of compressible flow in a long smooth 7.16 mm i.d. tube. Air is drawn from the surroundings (20°C and 101 kPa) by a vacuum pump downstream. Pressure readings along the tube become steady when the downstream pressure is reduced to 626 mm Hg (vacuum) or below. For these conditions, determine (a) the maximum mass flow rate possible through the tube, (b) the stagnation pressure of the air leaving the tube, and (c) the entropy change of the air in the tube. Show static and stagnation state points and the process path on a *Ts* diagram.

Solution:

Basic equations: $\dot{m} = \rho V A$ $\quad p = \rho R T$ $\quad T\,ds = dh - v\,dp$

Computing equations: $\frac{T_0}{T} = 1 + \frac{k-1}{2} M^2$ $\quad \frac{p_0}{p} = \left(\frac{T_0}{T}\right)^{k/k-1}$

Assumptions: (1) steady flow (2) ideal gas
(3) adiabatic flow through tube
(4) uniform flow at a section

Since pressure distribution does not change when p_b reaches 626 mmHg (vacuum), flow is choked, $M_e = 1.0$ and $p_e = p_b$

$$p_e = p_{atm} - \rho g \Delta h = 101\times10^3 \frac{N}{m^2} - 13.55\times 999 \frac{kg}{m^3}\times 9.81 \frac{m}{s^2} \times 0.626\,m \times \frac{N\cdot s^2}{kg\cdot m} = 17.87\times10^3 \frac{N}{m^2}$$

$$T_e = \frac{T_0}{1+\frac{k-1}{2}M_e^2} = \frac{(273+20)\,K}{1+0.2} = 244\,K$$

$$\rho_e = \frac{p_e}{RT_e} = 17.87\times10^3 \frac{N}{m^2} \times \frac{kg\cdot K}{287\,N\cdot m} \times \frac{1}{244\,K} = 0.255\ kg/m^3$$

$$V_e = (kRT_e)^{1/2} = \left(1.4\times 287 \frac{N\cdot m}{kg\cdot K}\times 244\,K \times \frac{kg\cdot m}{N\cdot s^2}\right)^{1/2} = 313\ m/s$$

$$\dot{m} = \rho_e V_e A = 0.255 \frac{kg}{m^3} \times 313 \frac{m}{s} \times \frac{\pi}{4}(0.00716\,m)^2 = 0.00321\ kg/s \qquad \dot{m}$$

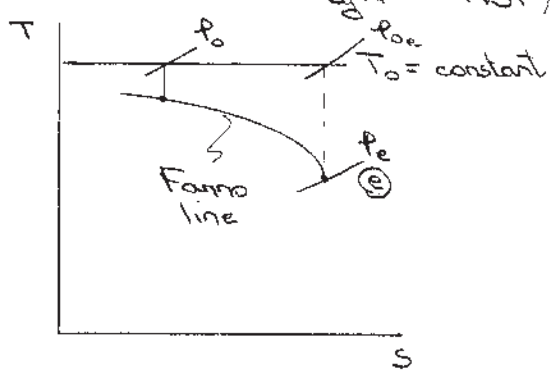
$$p_{0e} = p_e\left(1+\frac{k-1}{2}M_e^2\right)^{k/k-1} = 17.87\,kPa\,(1+0.2)^{3.5} = 33.8\ kPa\ (abs) \qquad p_{0e}$$

For an ideal gas, the Tds equation can be written as

$$T\,ds = dh - v\,dp = c_p\,dT - RT\frac{dp}{p} \qquad \therefore\ ds = c_p \frac{dT}{T} - R\frac{dp}{p}$$

Then, $\quad s_e - s_1 = s_{0e} - s_{01} = c_p \ln \cancel{\frac{T_{0e}}{T_{01}}} - R \ln \frac{p_{0e}}{p_{01}}$

$$s_e - s_1 = -287 \frac{J}{kg\cdot K} \ln\left(\frac{33.8}{101}\right) = 314\ J/kg\cdot K \qquad s_e - s_1$$



Problem 13.125

[Difficulty: 3]

13.125 Air flows through a smooth well-insulated 4-in.-diameter pipe at 600 lbm/min. At one section the air is at 100 psia and 80°F. Determine the minimum pressure and the maximum speed that can occur in the pipe.

Solution:

Basic equations: $\dot{m} = \rho V A \qquad p = \rho R T$

Computing equations: $T_0/T = 1 + \frac{k-1}{2}M^2 \qquad T_0 = \text{constant}$

Assumptions: (1) steady flow (2) adiabatic flow (3) uniform flow at a section (4) ideal gas

$$\rho_1 = \frac{p_1}{RT_1} = 100\,\frac{\text{lbf}}{\text{in}^2} \times \frac{\text{lbm}\cdot{}^\circ R}{53.3\ \text{ft}\cdot\text{lbf}} \times \frac{1}{540\,R} \times \frac{144\ \text{in}^2}{\text{ft}^2} = 0.500\ \text{lbm}/\text{ft}^3$$

$$\dot{m} = \rho V_1 A \quad ; \quad A = \pi D^2/4$$

$$V_1 = \frac{4\dot{m}}{\pi D^2 \rho_1} = \frac{4}{\pi} \times 600\,\frac{\text{lbm}}{\text{min}} \times \left(\frac{12}{4}\right)^2 \frac{1}{\text{ft}^2} \times \frac{\text{ft}^3}{0.500\ \text{lbm}} \times \frac{\text{min}}{60\ \text{s}} = 229\ \text{ft/s}$$

$$M_1 = \frac{V_1}{c_1} \qquad c_1 = (kRT_1)^{1/2} = \left(1.4 \times 53.3\,\frac{\text{ft}\cdot\text{lbf}}{\text{lbm}\cdot{}^\circ R} \times 540\,R \times 32.2\,\frac{\text{lbm}}{\text{slug}} \times \frac{\text{slug}\cdot\text{ft}}{\text{lbf}\cdot\text{s}^2}\right)^{1/2} = 1140\ \text{ft/s}$$

$$M_1 = \frac{229}{1140} = 0.201$$

Since $M_1 < 1.0$, the minimum pressure and max velocity occur for $M_2 = 1.0$

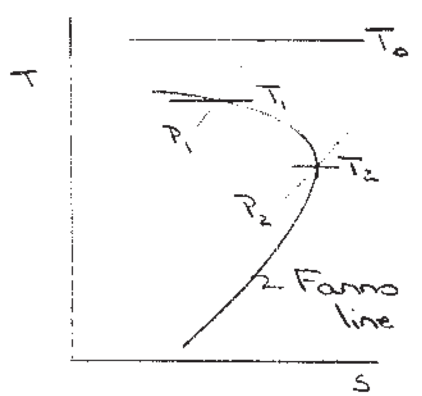
$$T_0/T = 1 + \frac{k-1}{2}M^2 \quad ; \quad \frac{T_2}{T_1} = \frac{1 + \frac{k-1}{2}M_1^2}{1 + \frac{k-1}{2}M_2^2} = \frac{1 + 0.2(0.201)^2}{1 + 0.2(1.0)^2} = 0.840$$

$$T_2 = 0.840\,(T_1) = 0.840\,(540\,R) = 454\,R$$

$$V_2 = M_2 c_2 = M_2 (kRT_2)^{1/2} = 1.0\left(1.4 \times 53.3\,\frac{\text{ft}\cdot\text{lbf}}{\text{lbm}\cdot{}^\circ R} \times 454\,R \times 32.2\,\frac{\text{lbm}}{\text{slug}} \times \frac{\text{slug}\cdot\text{ft}}{\text{lbf}\cdot\text{s}^2}\right)^{1/2} = 1040\ \text{ft/s} \qquad V_{max}$$

$$\dot{m} = \rho_1 V_1 A = \rho_2 V_2 A \qquad \rho_2 = \frac{V_1}{V_2}\rho_1 = \frac{229}{1040} \times 0.500\,\frac{\text{lbm}}{\text{ft}^3} = 0.110\ \text{lbm}/\text{ft}^3$$

$$p_2 = \rho_2 R T_2 = 0.110\,\frac{\text{lbm}}{\text{ft}^3} \times 53.3\,\frac{\text{ft}\cdot\text{lbf}}{\text{lbm}\cdot{}^\circ R} \times 454\,R \times \frac{\text{ft}^2}{144\ \text{in}^2} = 18.5\ \text{psia} \qquad p_{min}$$



*Fanno-Line Flow Functions

From Appendix E with $M_1 = 0.20$,

$T_1/T^* = 1.191$ (12.18a) $\quad \therefore T_2 = T^* = 453\,^\circ R$

$p_1/p^* = 5.456$ (12.18d) $\quad \therefore p_2 = p^* = 18.3$ psia

$V_1/V^* = 0.2182$ (12.18c) $\quad \therefore V_2 = V^* = 1050$ ft/s

Problem 13.126

[Difficulty: 3]

13.126 Nitrogen at stagnation conditions of 105 psia and 100°F flows through an insulated converging-diverging nozzle without friction. The nozzle, which has an exit-to-throat area ratio of 4, discharges supersonically into a constant area duct, which has a friction length $fL/D = 0.355$. Determine the temperature and pressure at the exit of the duct.

Given: Nitrogen traveling through C-D nozzle and constant-area duct with friction

Find: Exit temperature and pressure

Solution:

The given or available data is:

$R =$	55.16	ft-lbf/lbm-°R
$k =$	1.4	
$p_{01} =$	105	psia
$T_{01} =$	100	°F
$T_{01} =$	560	°R
$A_e/A_t =$	4	
$fL/D =$	0.355	

Equations and Computations:

Based on the area ratio of the nozzle, we can find the nozzle exit Mach number:

$M_1 =$ 2.940

The pressure and temperature at station 1 are therefore:

$p_1 =$	3.128	psia
$T_1 =$	205.2	°R

The critical temperature, pressure, and maximum friction length at 1 are:

$p_1/p^* =$	0.2255	
$p^* =$	13.867	psia
$T_1/T^* =$	0.4397	
$T^* =$	466.7	°R
$fL_1/D =$	0.51293	

Based on the maximum and actual friction lengths, the maximum friction length at station 2 is:

$fL_2/D =$ 0.15793

So the exit Mach number is:

$M_2 =$	1.560
$fL_2/D =$	0.15793

(we used *Solver* to find the correct Mach number to match the friction length)

The critical pressure and temperature ratios at station 2 are:

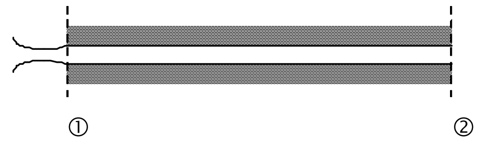
$p_2/p^* =$	0.5759
$T_2/T^* =$	0.8071

So the exit temperature and pressure are:

$p_2 =$	7.99	psia
$T_2 =$	377	°R

Problem 13.127 [Difficulty: 3]

13.127 A converging-diverging nozzle discharges air into an insulated pipe with area $A = 1\ \text{in}^2$. At the pipe inlet, $p = 18.5$ psia, $T = 100°\text{F}$, and $M = 2.0$. For shockless flow to a Mach number of unity at the pipe exit, calculate the exit temperature, the net force of the fluid on the pipe, and the entropy change.



Given: Air flow in a CD nozzle and insulated duct

Find: Temperature at end of duct; Force on duct; Entropy increase

Solution:

Basic equations: $F_s = p_1 \cdot A - p_2 \cdot A + R_x = m_{rate} \cdot (V_2 - V_1)$ $\quad \dfrac{T_0}{T} = 1 + \dfrac{k-1}{2} \cdot M^2$ $\quad \Delta s = c_p \cdot \ln\left(\dfrac{T_2}{T_1}\right) - R_{air} \cdot \ln\left(\dfrac{p_2}{p_1}\right)$

Given or available data $T_1 = (100 + 460) \cdot R$ $\quad p_1 = 18.5 \cdot psi$ $\quad M_1 = 2$ $\quad M_2 = 1$ $\quad A = 1 \cdot in^2$

$k = 1.4$ $\quad c_p = 0.2399 \cdot \dfrac{BTU}{lbm \cdot R}$ $\quad R_{air} = 53.33 \cdot \dfrac{ft \cdot lbf}{lbm \cdot R}$

Assuming isentropic flow in the nozzle

$$\frac{T_0}{T_1} \cdot \frac{T_2}{T_0} = \frac{1 + \frac{k-1}{2} \cdot M_1^2}{1 + \frac{k-1}{2} \cdot M_2^2} \quad \text{so} \quad T_2 = T_1 \cdot \frac{1 + \frac{k-1}{2} \cdot M_1^2}{1 + \frac{k-1}{2} \cdot M_2^2} \qquad T_2 = 840 \cdot R \qquad T_2 = 380 \cdot °F$$

Also $c_1 = \sqrt{k \cdot R_{air} \cdot T_1}$ $\quad V_1 = M_1 \cdot c_1$ $\quad V_1 = 2320 \cdot \dfrac{ft}{s}$ $\quad c_2 = \sqrt{k \cdot R_{air} \cdot T_2}$ $\quad V_2 = M_2 \cdot c_2$ $\quad V_2 = 1421 \cdot \dfrac{ft}{s}$

$\rho_1 = \dfrac{p_1}{R_{air} \cdot T_1}$ $\quad \rho_1 = 0.0892 \cdot \dfrac{lbm}{ft^3}$ $\quad m_{rate} = \rho_1 \cdot V_1 \cdot A = \rho_2 \cdot V_2 \cdot A_2$ $\quad$ so $\quad \rho_2 = \rho_1 \cdot \dfrac{V_1}{V_2}$ $\quad \rho_2 = 0.146 \cdot \dfrac{lbm}{ft^3}$

$m_{rate} = \rho_1 \cdot V_1 \cdot A$ $\quad m_{rate} = 1.44 \cdot \dfrac{lbm}{s}$ $\quad p_2 = \rho_2 \cdot R_{air} \cdot T_2$ $\quad p_2 = 45.3 \cdot psi$

Hence $\quad R_x = (p_2 - p_1) \cdot A + m_{rate} \cdot (V_2 - V_1)$ $\quad R_x = -13.3 \cdot lbf$ $\quad$ (Force is to the right)

Finally $\quad \Delta s = c_p \cdot \ln\left(\dfrac{T_2}{T_1}\right) - R_{air} \cdot \ln\left(\dfrac{p_2}{p_1}\right)$ $\quad \Delta s = 0.0359 \cdot \dfrac{BTU}{lbm \cdot R}$

(Note: Using Fanno line relations, at $M_1 = 2$ $\quad \dfrac{T_1}{T_{crit}} = \dfrac{T_1}{T_2} = 0.6667$ $\quad T_2 = \dfrac{T_1}{0.667}$ $\quad T_2 = 840 \cdot R$

$\dfrac{p_1}{p_{crit}} = \dfrac{p_1}{p_2} = 0.4083$ $\quad p_2 = \dfrac{p_1}{0.4083}$ $\quad p_2 = 45.3 \cdot psi$ $\quad$ Check!)

Problem 13.128

[Difficulty: 3]

13.128 Air is drawn from the atmosphere (20°C and 101 kPa) through a converging nozzle into a long insulated 20-mm-diameter tube of constant area. Flow in the nozzle is isentropic. The pressure at the inlet to the constant-area tube is $p_1 = 99.4$ kPa. Evaluate the mass flow rate through the tube. Calculate T^* and p^* for the isentropic process. Calculate T^* and p^* for flow leaving the constant-area tube. Show the corresponding static and stagnation state points on a Ts diagram.

Solution:

Basic equations: $p = \rho R T$ $\quad \dot{m} = \rho V A$

Computing equations: $\dfrac{T_0}{T} = 1 + \dfrac{k-1}{2} M^2 \qquad \dfrac{p_0}{p} = \left(\dfrac{T_0}{T}\right)^{k/k-1}$

Assumptions: (1) steady flow (2) ideal gas
(3) uniform flow at a section
(4) isentropic flow in nozzle, adiabatic flow in tube.

Since flow in nozzle is isentropic, $T_{0_1} = T_0$ and $p_{0_1} = p_0$

$\therefore T_1 = \dfrac{T_{0_1}}{1+\frac{k-1}{2}M_1^2} = \dfrac{293\,K}{1+0.2(0.151)^2} = 291.7\,K \; ; \; M_1 = \left\{\dfrac{2}{k-1}\left[\left(\dfrac{p_0}{p_1}\right)^{k-1/k} - 1\right]\right\}^{1/2} = 0.151$

$V_1 = M_1 c_1 = M_1 (kRT_1)^{1/2} = 0.151\left[1.4 \times 287 \dfrac{N\cdot m}{kg\cdot K} \times 291.7\,K \times \dfrac{kg\cdot m}{N\cdot s^2}\right]^{1/2} = 51.7\,m/s$

$\rho_1 = \dfrac{p_1}{RT_1} = 99.4\times10^3 \dfrac{N}{m^2} \times \dfrac{kg\cdot K}{287\,N\cdot m} \times \dfrac{1}{291.7\,K} = 1.187\,kg/m^3$

$\dot{m} = \rho_1 V_1 A_1 = 1.187 \dfrac{kg}{m^3} \times 51.7 \dfrac{m}{s} \times \dfrac{\pi}{4}(0.020\,m)^2 = 0.0193\,kg/s$ $\quad \dot{m}$

(b) For isentropic flow, $\dfrac{T_0}{T^*} = 1 + \dfrac{k-1}{2} = 1.20 \qquad \therefore T^* = \dfrac{T_0}{1.20} = \dfrac{293\,K}{1.20} = 244\,K$ $\quad T^*_{s=c}$

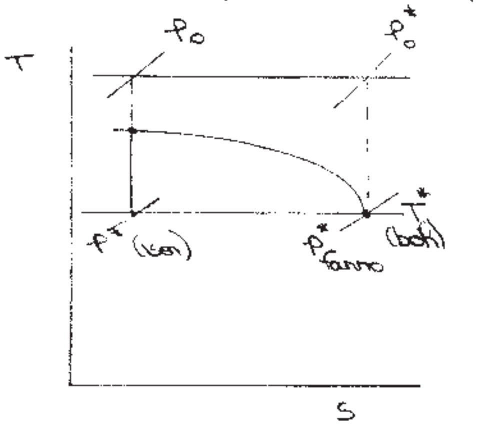
$\dfrac{p_0}{p^*} = \left(\dfrac{T_0}{T^*}\right)^{k/k-1} \qquad \therefore p^* = \dfrac{p_0}{(T_0/T^*)^{k/k-1}} = \dfrac{101\,kPa}{(1.2)^{3.5}} = 53.4\,kPa\,(abs)$ $\quad p^*_{s=c}$

(c) For Fanno line flow $T_0^* = T_{0_1}$ $\qquad \therefore T^* = T^*_{isen} = 244\,K$ $\quad T^*$

At *, $M=1$, $\therefore V = c = (kRT^*)^{1/2} = \left[1.4 \times 287 \dfrac{N\cdot m}{kg\cdot K} \times 244\,K \times \dfrac{kg\cdot m}{N\cdot s^2}\right]^{1/2} = 313\,m/s$

From continuity $\rho_1 V_1 = \rho^* V^* \quad \therefore \rho^* = \dfrac{V_1}{V^*}\rho_1 = \dfrac{51.4}{313} \times 1.187 \dfrac{kg}{m^3} = 0.1949\,kg/m^3$

$p^* = \rho^* R T^* = 0.1949 \dfrac{kg}{m^3} \times 287 \dfrac{N\cdot m}{kg\cdot K} \times 244\,K = 13.6\,kPa\,(abs)$ $\quad p^*$



* Fanno-Line Flow Functions (Appendix E.2)

For $M_1 = 0.151$, Eq 12.18d gives $p_1/p^* = 7.238$

$\therefore p^* = 13.7\,kPa$ ✓

Problem 13.129

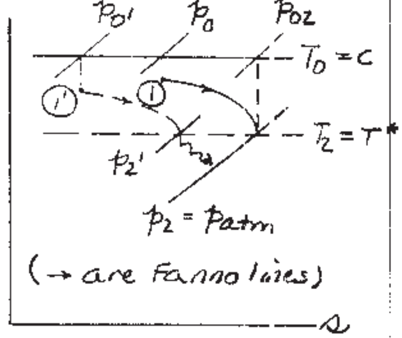
[Difficulty: 4]

13.129 Air flows through a converging nozzle and then a length of insulated duct. The air is supplied from a tank where the temperature is constant at 59°F and the pressure is variable. The outlet end of the duct exhausts to atmosphere. When the exit flow is just choked, pressure measurements show the duct inlet pressure and Mach number are 53.2 psia and 0.30. Determine the pressure in the tank and the temperature, stagnation pressure, and mass flow rate of the outlet flow, if the tube diameter is 0.249 in. Show on a *Ts* diagram the effect of raising the tank pressure to 100 psia. Sketch the pressure distribution versus distance along the channel for this new flow condition.

<u>Solution</u>: Assume conditions in tank are stagnation properties; flow is isentropic in nozzle, Fanno flow in duct.

Basic equations: $T_0 = T\left(1+\frac{k-1}{2}M^2\right) = \text{const (energy)}$

$$p_0 = p\left(1+\frac{k-1}{2}M^2\right)^{\frac{k}{k-1}} \quad \text{(isentropic)}$$



Assume steady, 1-D compressible flow, with constant stagnation temperature, $T_0 = 15°C$ (59°F)

$T_0 = (460+59)°R = 519°R$

$p_{0_t} = p_1\left(1+\frac{k-1}{2}M_1^2\right)^{\frac{k}{k-1}} = 53.2 \text{ psia}\left(1+0.2(0.3)^2\right)^{3.5} = 56.6 \text{ psia}$ ← p_{0t}

At exit plane, $M_2 = 1$

$$T_2 = \frac{T_0}{\left(1+\frac{k-1}{2}M_2^2\right)} = \frac{519°R}{1+0.2(1)^2} = 433°R$$ ← T_2

$p_{02} = p_2\left(1+\frac{k-1}{2}M_2^2\right)^{\frac{k}{k-1}} = 14.7 \text{ psia}\left(1+0.2(1)^2\right)^{3.5} = 27.8 \text{ psia}$ ← p_{02}

At exit plane, $V = V^* = c^* = \sqrt{kRT^*} = 1020 \text{ ft/s}$

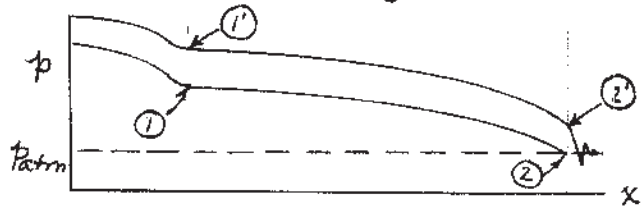
$$\rho_2 = \frac{p_2}{RT_2} = 14.7\frac{\text{lbf}}{\text{in.}^2}\times\frac{\text{lbm}\cdot°R}{53.3 \text{ ft}\cdot\text{lbf}}\times\frac{1}{433°R}\times 144\frac{\text{in.}^2}{\text{ft}^2} = 0.0917 \text{ lbm/ft}^3$$

$$\dot{m} = \rho_2 V_2 A = 0.0917\frac{\text{lbm}}{\text{ft}^3}\times 1020\frac{\text{ft}}{\text{s}}\times\frac{\pi}{4}\left(\frac{0.249}{12}\right)^2\text{ft}^2 = 0.0316 \text{ lbm/s}$$ ← $\dot{m}$

When p_t is increased, density and mass flow rate increase. Flow will shift to a new Fanno line as shown dashed in the Ts diagram above.

T_0 is fixed, so T^* remains fixed.

{Since ρ and $\dot{m}$ increase, $p_2' > p_2 = p_{atm}$. The plot of p vs. x is as shown.}



Problem 13.130

[Difficulty: 4]

13.130 A constant-area duct is fed by a converging-only nozzle. The nozzle receives air from a large chamber at $p_1 = 600$ kPa and $T_1 = 550$ K. The duct has a friction length of 5.3, and it is choked at the exit. What is the pressure at the end of the duct? If 80 percent of the duct is removed, and the conditions at station 1 and the friction factor remain constant, what is the new exit pressure and Mach number? Sketch both of these processes on a Ts diagram.

Given: Air traveling through converging nozzle and constant-area duct with friction; choked flow at duct exit.

Find: Pressure at end of duct; exit conditions if 80% of duct were removed

Solution:

The given or available data is:

$R =$	286.9	J/kg-K
$k =$	1.4	
$p_1 =$	600	kPa
$T_1 =$	550	K

Equations and Computations:

Station 1 is a stagnation state, station 2 is between the nozzle and friction duct, and station 3 is at the duct exit.

For part (a) we know:

$fL_{2\text{-}3}/D =$ 5.3

$M_3 =$ 1

Therefore, we can make the following statements:

$fL_3/D =$ 0

$fL_2/D =$ 5.300

So the Mach number at the duct entrance is:

$M_2 =$ 0.300

$fL_2/D =$ 5.300

(we used *Solver* to find the correct Mach number to match the friction length)

The pressure at station 2 can be found from the Mach number and stagnation state:

$p_1/p_2 =$ 1.0644

$p_2 =$ 563.69 kPa

Since state 3 is the critical state, we can find the pressure at state 3:

$p_2/p^* =$ 3.6193

$p^* =$ 155.75 kPa

$p_3 =$ 155.7 kPa

For part (a) we know that if we remove 80% of the duct:

$fL_{2\text{-}3}/D =$ 1.06

$M_2 =$ 0.300

$fL_2/D =$ 5.300

$p_2 =$ 563.69 kPa

Since we know state 2 and the friction length of the duct, we can find state 3:

$fL_3/D =$ 4.240

So the Mach number at the duct exit is:

$$M_3 = \quad 0.326$$

$$fL_2/D = \quad 4.240$$

(we used *Solver* to find the correct Mach number to match the friction length)
To find the exit pressure:

$$p_2/p^* = \quad 3.6193$$

$$p^* = \quad 155.75 \quad \text{kPa}$$

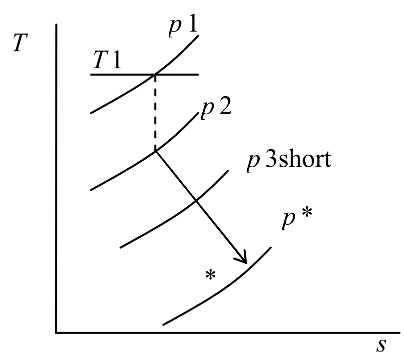
At state 3 the pressure ratio is:

$$p_3/p^* = \quad 3.3299$$

So the pressure is:

$$p_3 = \quad 519 \quad \text{kPa}$$

These processes are plotted in the *Ts* diagram below:



Problem 13.131

[Difficulty: 4]

13.131 We wish to build a supersonic wind tunnel using an insulated nozzle and constant-area duct assembly. Shock-free operation is desired, with $M_1 = 2.1$ at the test section inlet and $M_2 = 1.1$ at the test section outlet. Stagnation conditions are $T_0 = 295$ K and $p_0 = 101$ kPa (abs). Calculate the outlet pressure and temperature and the entropy change through the test section.

Solution: Consider steady, 1-D, comp. flow, ideal gas.

Computing equations: $T_0 = T\left(1+\frac{k-1}{2}M^2\right)$; $p_0 = p\left(1+\frac{k-1}{2}M^2\right)^{k/k-1}$ (s = const)

$T\,ds = dh - v\,dp \qquad \rho VA = \text{constant}$

Assumptions: (1) Isentropic flow in nozzle
(2) Insulated, so T_0 = constant

Then $T_2 = \dfrac{T_0}{1+\frac{k-1}{2}M_2^2} = \dfrac{295\text{ K}}{1+0.2(1.1)^2} = 238\text{ K}$ $\quad T_2$

From continuity, $\rho_1 V_1 = \rho_2 V_2$, since $A_1 = A_2$. But $V = Mc = M\sqrt{kRT}$ and $\rho = p/RT$, so

$$\frac{p_1}{RT_1} M_1\sqrt{kRT_1} = \frac{p_2}{RT_2} M_2\sqrt{kRT_2} \;;\; \frac{p_1}{\sqrt{T_1}} M_1 = \frac{p_2}{\sqrt{T_2}} M_2 \;;\; p_2 = p_1 \frac{M_1}{M_2}\sqrt{\frac{T_2}{T_1}}$$

From isentropic relations,

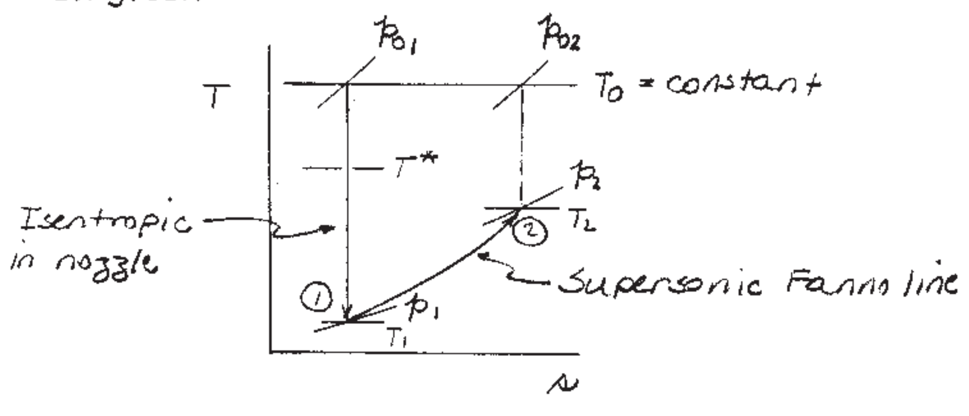
$$T_1 = \frac{T_0}{1+\frac{k-1}{2}M_1^2} = \frac{295\text{ K}}{1+0.2(2.1)^2} = 157\text{ K} \;;\; p_1 = \frac{p_{01}}{\left(1+\frac{k-1}{2}M_1^2\right)^{k/k-1}} = \frac{101\text{ kPa}}{(1.88)^{3.5}} = 11.1\text{ kPa}$$

Thus $p_2 = \dfrac{2.1}{1.1}\sqrt{\dfrac{238\text{ K}}{157\text{ K}}}\; 11.1\text{ kPa} = 26.1\text{ kPa (abs)}$ $\quad p_2$

$$ds = \frac{dh}{T} - v\frac{dp}{T} = c_p\frac{dT}{T} - \frac{RT}{p}\frac{dp}{T} = c_p\frac{dT}{T} - R\frac{dp}{p}$$

$$\Delta s = c_p \ln\frac{T_2}{T_1} - R\ln\frac{p_2}{p_1} = 1004\frac{\text{J}}{\text{kg}\cdot\text{K}}\ln\left(\frac{238}{157}\right) - 287\frac{\text{J}}{\text{kg}\cdot\text{K}}\ln\left(\frac{26.1}{11.1}\right) = 172\text{ J/kg}\cdot\text{K}$$ $\quad \Delta s$

The Ts diagram is



* Fanno-Line Flow Functions (Appendix E.2):

For $M_1 = 2.1$, Eq. 12.18d gives $p_1/p^* = 0.380$, so $p_2 = 26.1$ kPa (abs)
$M_2 = 1.1$ $\qquad p_2/p^* = 0.894$

Problem 13.132

[Difficulty: 3]

13.132 Consider adiabatic flow of air in a constant-area pipe with friction. At one section of the pipe, $p_0 = 100$ psia, $T_0 = 500°R$, and $M = 0.70$. If the cross-sectional area is 1 ft² and the Mach number at the exit is $M_2 = 1$, find the friction force exerted on the fluid by the pipe.

Solution:

Basic equations: $F_{S_x} = P_1A - P_2A - F_f = \dot{m}(V_2 - V_1)$ $\qquad \dot{m} = \rho V A$

Computing equations: $T_0/T = 1 + \frac{k-1}{2}M^2$ $\qquad P_0/P = \left[1 + \frac{k-1}{2}M^2\right]^{k/k-1}$

Assumptions (1) steady flow
(2) adiabatic flow, $T_0 = $ const
(3) uniform flow at a section
(4) $F_{B_x} = 0$
(5) ideal gas

$P_0/P = \left[1 + \frac{k-1}{2}M^2\right]^{k/k-1}$ $\qquad P_1 = \dfrac{P_{0_1}}{\left[1 + \frac{k-1}{2}M_1^2\right]^{k/k-1}} = \dfrac{100 \text{ psia}}{\left[1 + 0.2(0.70)^2\right]^{3.5}} = 72.1 \text{ psia}$

$T_0/T = 1 + \frac{k-1}{2}M^2$ $\qquad T_1 = \dfrac{T_0}{1 + \frac{k-1}{2}M_1^2} = \dfrac{500°R}{1 + 0.2(0.70)^2} = 455°R$

$\rho_1 = \dfrac{P_1}{RT_1} = 72.1 \dfrac{\text{lbf}}{\text{in}^2} \times \dfrac{\text{lbm} \cdot °R}{53.3 \text{ ft} \cdot \text{lbf}} \times \dfrac{1}{455°R} \times \dfrac{144 \text{ in}^2}{\text{ft}^2} = 0.428 \text{ lbm/ft}^3$

$V_1 = M_1 c_1 = M_1 (kRT_1)^{1/2} = 0.70\left(1.4 \times 53.3 \dfrac{\text{ft} \cdot \text{lbf}}{\text{lbm} \cdot °R} \times 455°R \times 32.2 \dfrac{\text{lbm}}{\text{slug}} \cdot \dfrac{\text{slug} \cdot \text{ft}}{\text{lbf} \cdot \text{s}^2}\right)^{1/2} = 732 \text{ ft/s}$

$\dot{m} = \rho_1 V_1 A = 0.428 \dfrac{\text{lbm}}{\text{ft}^3} \times 732 \dfrac{\text{ft}}{\text{s}} \times 1.0 \text{ ft}^2 = 313 \text{ lbm/s}$

$T_0/T = 1 + \frac{k-1}{2}M^2$, $T_{0_2} = T_{0_1}$, $\quad T_2 = \dfrac{T_0}{1 + \frac{k-1}{2}M_2^2} = \dfrac{500°R}{1 + 0.2(1.0)^2} = 417°R$

$V_2 = M_2 c_2 = M_2 (kRT_2)^{1/2} = 1.0\left(1.4 \times 53.3 \dfrac{\text{ft} \cdot \text{lbf}}{\text{lbm} \cdot °R} \times 417°R \times 32.2 \dfrac{\text{lbm}}{\text{slug}} \cdot \dfrac{\text{slug} \cdot \text{ft}}{\text{lbf} \cdot \text{s}^2}\right)^{1/2} = 1000 \text{ ft/s}$

$\dot{m} = \rho_1 V_1 A = \rho_2 V_2 A$ $\qquad \rho_2 = \dfrac{V_1}{V_2}\rho_1 = \dfrac{732}{1000} \times 0.428 \text{ lbm/ft}^3 = 0.313 \text{ lbm/ft}^3$

$P_2 = \rho_2 R T_2 = 0.313 \dfrac{\text{lbm}}{\text{ft}^3} \times 53.3 \dfrac{\text{ft} \cdot \text{lbf}}{\text{lbm} \cdot °R} \times 417°R \times \dfrac{\text{ft}^2}{144 \text{ in}^2} = 48.3 \text{ psia}$

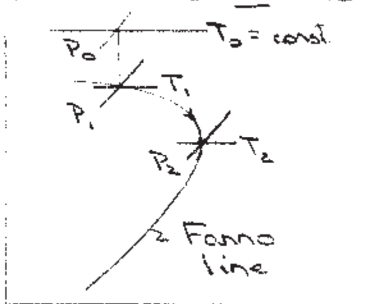
Solving the momentum equation for F_f

$F_f = (P_1 - P_2)A - \dot{m}(V_2 - V_1)$

$= (72.1 - 48.3) \dfrac{\text{lbf}}{\text{in}^2} \times 1 \text{ ft}^2 \times \dfrac{144 \text{ in}^2}{\text{ft}^2} - 313 \dfrac{\text{lbm}}{\text{s}}(1000 - 732)\dfrac{\text{ft}}{\text{s}} \times \dfrac{\text{slug}}{32.2 \text{ lbm}} \times \dfrac{\text{lbf} \cdot \text{s}^2}{\text{slug} \cdot \text{ft}}$

$F_f = 822 \text{ lbf}$ $\qquad F_f$

F_f is the force on the control volume from the surroundings. Consequently, F_f is the force on the fluid from the pipe: F_f opposes the motion

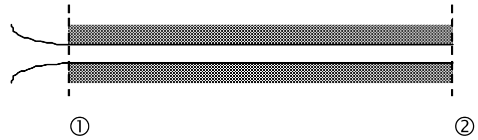


* Fanno-Line Flow Functions
From Appendix E.2 with $M_1 = 0.70$,
$P_1/P^* = 1.493$ (12.18d) $\quad \therefore P^* = P_2 = 48.3$ psia
$V_1/V^* = 0.732$ (12.18c) $\quad \therefore V^* = V_2 = 1000$ ft/s

Problem 13.133

[Difficulty: 2]

13.133 For the conditions of Problem 13.122, find the length, L, of commercial steel pipe of 2 in. diameter between sections ① and ②.



Given: Air flow in a converging nozzle and insulated duct

Find: Length of pipe

Solution:

Basic equations: Fanno-line flow equations, and friction factor

Given or available data

$T_0 = (250 + 460)\cdot R \qquad p_0 = 145\cdot psi \qquad p_1 = 125\cdot psi \qquad T_2 = (150 + 460)\cdot R$

$D = 2\cdot in \qquad k = 1.4 \qquad c_p = 0.2399\cdot\frac{BTU}{lbm\cdot R} \qquad R_{air} = 53.33\cdot\frac{ft\cdot lbf}{lbm\cdot R}$

From isentropic relations

$$M_1 = \left[\frac{2}{k-1}\cdot\left[\left(\frac{p_0}{p_1}\right)^{\frac{k-1}{k}} - 1\right]\right]^{\frac{1}{2}} \qquad M_1 = 0.465$$

$$\frac{T_0}{T_1} = 1 + \frac{k-1}{2}\cdot M_1^{\,2} \quad \text{so} \quad T_1 = \frac{T_0}{\left(1 + \frac{k-1}{2}\cdot M_1^{\,2}\right)} \qquad T_1 = 681\cdot R \qquad T_1 = 221\cdot°F$$

Then for Fanno-line flow

$$\frac{f_{ave}\cdot L_{max1}}{D_h} = \frac{1 - M_1^{\,2}}{k\cdot M_1^{\,2}} + \frac{k+1}{2\cdot k}\cdot\ln\left[\frac{(k+1)\cdot M_1^{\,2}}{2\cdot\left(1 + \frac{k-1}{2}\cdot M_1^{\,2}\right)}\right] = 1.3923$$

$$\frac{p_1}{p_{crit}} = \frac{p_1}{p_2} = \frac{1}{M_1}\cdot\left(\frac{\frac{k+1}{2}}{1 + \frac{k-1}{2}\cdot M_1^{\,2}}\right)^{\frac{1}{2}} = 2.3044 \qquad \frac{T_1}{T_{crit}} = \frac{\frac{k+1}{2}}{1 + \frac{k-1}{2}\cdot M_1^{\,2}} = 1.150 \qquad T_{crit} = \frac{T_1}{1.150}$$

$$p_{crit} = \frac{p_1}{2.3044} \qquad p_{crit} = 54.2\cdot psi \qquad T_{crit} = 592\cdot R \qquad T_{crit} = 132\cdot°F$$

Also, for $\frac{T_2}{T_{crit}} = 1.031$

$$\frac{T_2}{T_{crit}} = \frac{\frac{k+1}{2}}{1 + \frac{k-1}{2}\cdot M_2^{\,2}} \quad \text{leads to} \quad M_2 = \sqrt{\frac{2}{k-1}\cdot\left(\frac{k+1}{2}\cdot\frac{T_{crit}}{T_2} - 1\right)} \qquad M_2 = 0.906$$

Then

$$\frac{f_{ave}\cdot L_{max2}}{D_h} = \frac{1 - M_2^{\,2}}{k\cdot M_2^{\,2}} + \frac{k+1}{2\cdot k}\cdot\ln\left[\frac{(k+1)\cdot M_2^{\,2}}{2\cdot\left(1 + \frac{k-1}{2}\cdot M_2^{\,2}\right)}\right] = 0.01271$$

Also

$$\rho_1 = \frac{p_1}{R_{air} \cdot T_1} \qquad \rho_1 = 0.496 \cdot \frac{lbm}{ft^3} \qquad V_1 = M_1 \cdot \sqrt{k \cdot R_{air} \cdot T_1} \qquad V_1 = 595 \cdot \frac{ft}{s}$$

For air at $T_1 = 221 \cdot °F$, from Table A.9 (approximately) $\mu = 4.48 \times 10^{-7} \cdot \frac{lbf \cdot s}{ft^2}$ so $Re_1 = \frac{\rho_1 \cdot V_1 \cdot D}{\mu}$

For commercial steel pipe (Table 8.1) $e = 0.00015 \cdot ft$ $\frac{e}{D} = 9 \times 10^{-4}$ and $Re_1 = 3.41 \times 10^6$

Hence at this Reynolds number and roughness (Eq. 8.37) $f = 0.01924$

Combining results $$L_{12} = \frac{D}{f} \cdot \left(\frac{f_{ave} \cdot L_{max2}}{D_h} - \frac{f_{ave} \cdot L_{max1}}{D_h} \right) = \frac{\frac{2}{12} \cdot ft}{.01924} \cdot (1.3923 - 0.01271) \qquad L_{12} = 12.0 \cdot ft$$

These calculations are a LOT easier using the *Excel* Add-ins!

Problem 13.134

[Difficulty: 2]

13.134 Consider the laboratory Fanno-line flow channel of Problem 13.123. Assume laboratory conditions are 22.5°C and 760 mm of mercury (uncorrected). The manometer reading at a pressure tap at the end of the converging nozzle is −11.8 mm of mercury (gage). Calculate the Mach number at this location. Determine the duct length required to attain choked flow. Calculate the temperature and stagnation pressure at the choked state in the constant-area duct.

Solution: * Compressible flow functions to be used

Assumptions: (1) steady flow (5) $\Delta z = 0$
(2) isentropic flow in nozzle (6) ideal gas, air
(3) adiabatic " in duct (7) uniform flow at a section
(4) $W_s = W_{shear} = 0$

$$\frac{p_1}{p_0} = \frac{p_1}{p_{atm}} = \frac{\rho g \Delta h_1}{\rho g \Delta h_{atm}} = \frac{(760 - 11.8)\,mm}{760\,mm} = 0.9845$$

From App. E.1 (Eq. 11.17a) $M_1 = 0.150$ ——— M_1

$$p_1 = \rho_{Hg}\, g h_1 = (13.5)\,1000\,\frac{kg}{m^3} \times 9.81\,\frac{m}{sec^2} \times 0.748\,m \times \frac{N \cdot sec^2}{kg \cdot m} = 99.1\,kPa\,(abs)$$

From App. E.1 (Eq. 11.17b), $T_1/T_0 = 0.9555$ $\therefore T_1 = 294\,K$

From App. E.2, with $M_1 = 0.150$

$$p_1/p^* = 7.287, \quad T_1/T^* = 1.195, \quad \frac{f L_{max}}{D} = 27.93$$

$$p_2 = p^* = 13.60 \text{ and } p_{0_2} = \frac{p_2}{0.5283} = 25.7\,kPa, \quad T^* = T_2 = 247\,K$$ ——— p_{0_2}, T_2

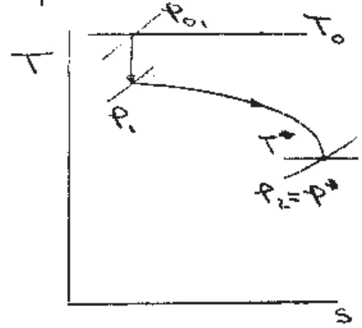
$$f = f(Re), \quad Re = \frac{\rho V D}{\mu}$$

$$\rho_1 = \frac{p_1}{RT_1} = 99.1 \times 10^3\,\frac{N}{m^2} \times \frac{kg \cdot K}{287\,N \cdot m} \times \frac{1}{294\,K} = 1.17\,kg/m^3$$

$$V_1 = M_1 c_1 = M_1 (kRT_1)^{1/2} = 0.149 \left(1.4 \times 287\,\frac{N \cdot m}{kg \cdot K} \times 294\,K \times \frac{kg \cdot m}{N \cdot s^2}\right)^{1/2} = 51.2\,m/s$$

From Table A.10 @ $T_1 = 21°C$, $\mu = 1.82 \times 10^{-5}\,N \cdot sec/m^2$

$$Re_1 = \frac{\rho_1 V_1 D}{\mu} = 1.17\,\frac{kg}{m^3} \times 51.2\,\frac{m}{s} \times 7.16 \times 10^{-3}\,m \times \frac{m^2}{1.82 \times 10^{-5}\,N \cdot s} \times \frac{N \cdot s^2}{kg \cdot m}$$



$$Re_1 = 2.36 \times 10^4$$

From Fig. 8.13, friction factor $f = 0.0245$

$$\therefore L_{12} = 27.9\,\frac{D}{f} = 27.9 \times 7.16 \times 10^{-3}\,m \cdot \frac{1}{0.0245}$$

$$L_{12} = 8.2\,m$$ ——— L_{12}

Problem 13.135

[Difficulty: 3]

13.135 A 2 ft × 2 ft duct is 40 ft long. Air enters at $M_1 = 3.0$ and leaves at $M_2 = 1.7$, with $T_2 = 500°R$ and $p_2 = 110$ psia. Find the static and stagnation conditions at the entrance. What is the friction factor for the duct?

Given: Air traveling through a square duct

Find: Entrance static and stagnation conditions; friction factor

Solution:

The given or available data is:

$R =$	53.33	ft-lbf/lbm-°R
$k =$	1.4	
$s =$	2	ft
$L =$	40	ft
$M_1 =$	3	
$M_2 =$	1.7	
$T_2 =$	500	°R
$p_2 =$	110	psia

Equations and Computations:

From the entrance Mach number we can calculate:

$p_{01}/p_1 =$ 36.7327

$T_{01}/T_1 =$ 2.8000

$p_1/p^* =$ 0.2182

$T_1/T^* =$ 0.4286

$fL_1/D =$ 0.52216

From the exit Mach number we can calculate:

$p_2/p^* =$ 0.5130

$T_2/T^* =$ 0.7605

$fL_2/D =$ 0.20780

Since we know static conditions at 2, we can find the critical pressure and temperature:

$p^* =$ 214.4 psia

$T^* =$ 657.5 °R

Therefore, the static conditions at the duct entrance are:

$p_1 =$ 46.8 psia

$T_1 =$ 282 °R

and from the isentropic relations we can find stagnation conditions:

$p_{01} =$ 1719 psia

$T_{01} =$ 789 °R

To find the friction factor of the duct, first we need to friction length:

$fL_{1\text{-}2}/D =$ 0.31436

The area and perimeter of the duct are:

$A =$ 4.0 ft^2

$P =$ 8.0 ft

Therefore the hydraulic diameter of the duct is:

$D_H =$ 2.0 ft

From the hydraulic diameter, length, and friction length, the friction factor is:

$f =$ 0.01572

Problem 13.136

[Difficulty: 3]

13.136 Air flows in a 3-in. (nominal) i.d. pipe that is 10 ft long. The air enters with a Mach number of 0.5 and a temperature of 70°F. What friction factor would cause the flow to be sonic at the exit? If the exit pressure is 14.7 psia and the pipe is made of cast iron, estimate the inlet pressure.

Given: Air traveling through a cast iron pipe

Find: Friction factor needed for sonic flow at exit; inlet pressure

Solution:

The given or available data is:

$R =$	53.33	ft-lbf/lbm-°R
$k =$	1.4	
$D =$	3.068	in
$L =$	10	ft
$M_1 =$	0.5	
$T_1 =$	70	°F
$T_1 =$	530	°R
$M_2 =$	1	
$p_2 =$	14.7	psia

Equations and Computations:

From the entrance Mach number we can calculate:

$p_1/p^* =$ 2.1381

$fL_1/D =$ 1.06906

From the exit Mach number we can calculate:

$p_2/p^* =$ 1.0000

$fL_2/D =$ 0.00000

To find the friction factor of the duct, first we need to friction length:

$fL_{1\text{-}2}/D =$ 1.06906

Based on this, and the pipe length and diameter, the friction factor is:

$f =$ 0.0273

We can calculate the critical pressure from the exit pressure:

$p^* =$ 14.7 psia

Therefore, the static pressure at the duct entrance is:

$p_1 =$ 31.4 psia

13.137 For the conditions of Problem 13.132, determine the duct length. Assume the duct is circular and made from commercial steel. Plot the variations of pressure and Mach number versus distance along the duct.

Solution:

Compressible flow functions to be used in the solution

Assumptions: (1) steady flow (2) adiabatic flow, T_0 = const (3) uniform flow at a section (4) ideal gas

For $M_1 = 0.70$, from Appendix E.1, $T_0/T_1 = 1.098$ (11.17b) $\therefore T_1 = 455°R$

$p_0/p_1 = 1.387$ (11.17a) $\therefore p_1 = 72.1$ psia

For $M_1 = 0.70$, from Appendix E.2, $p_1/p^* = 1.493$ (12.18d) $\therefore p^* = 48.3$ psia

$\bar{f}L_{max}/D_h = 0.2081$ (12.17)

For $M_2 = 1.0$, $\bar{f}L_{max}/D_h = 0$ $\therefore \bar{f}L_{12}/D_h = 0.2081$

$\bar{f} = \bar{f}(Re, e/D)$

$A = \frac{\pi D^2}{4}$ $\therefore D = \left(\frac{4A}{\pi}\right)^{1/2} = \left[\frac{4}{\pi} \times 1.0\,ft^2\right]^{1/2} = 1.13$ ft

For commercial steel from Table 8.1, $e = 0.00015$ ft $\therefore \frac{e}{D} = 0.00013$

$$\rho_1 = \frac{p_1}{RT_1} = 72.1\,\frac{lbf}{in^2} \times \frac{lbm \cdot °R}{53.3\,ft \cdot lbf} \times \frac{1}{455°R} \times \frac{144\,in^2}{ft^2} = 0.428\,lbm/ft^3$$

$$V_1 = M_1 c_1 = M_1 (kRT_1)^{1/2} = 0.7\left(1.4 \times 53.3\,\frac{ft \cdot lbf}{lbm \cdot °R} \times 455°R \times 32.2\,\frac{lbm}{slug} \times \frac{slug \cdot ft}{lbf \cdot s^2}\right)^{1/2} = 732\,ft/s$$

$T_1 = 455\,°R = -5°F = -20°C = 253\,K$.

From the Sutherland correlation, $\mu = \frac{bT^{1/2}}{1 + S/T}$ (A.1)

$$\mu = 1.458 \times 10^{-6}\,\frac{kg}{m \cdot s \cdot K^{1/2}} \times \frac{(253\,K)^{1/2}}{1 + 110/253} = 1.62 \times 10^{-5}\,\frac{kg}{m \cdot s} \times \frac{N \cdot s^2}{kg \cdot m} = 1.62 \times 10^{-5}\,\frac{N \cdot s}{m^2}$$

$$\mu = 1.62 \times 10^{-5}\,\frac{N \cdot s}{m^2} \times 0.0209\,\frac{lbf \cdot s/ft^2}{N \cdot s/m^2} = 3.39 \times 10^{-7}\,lbf \cdot s/ft^2$$

$$Re = \frac{\rho_1 V_1 D}{\mu} = 0.428\,\frac{lbm}{ft^3} \times 732\,\frac{ft}{s} \times 1.13\,ft \times \frac{ft^2}{3.39 \times 10^{-7}\,lbf \cdot s} \times \frac{slug}{32.2\,lbm} \times \frac{lbf \cdot s^2}{ft \cdot slug}$$

$Re = 3.24 \times 10^7$.

From Fig 8.13, friction factor $f = 0.0125$

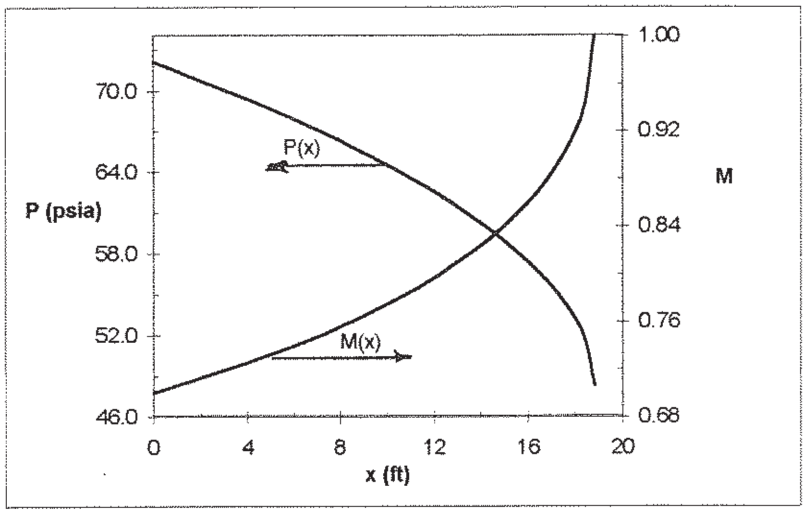
$\bar{f}\frac{L_{12}}{D} = 0.2081$ $\therefore L_{12} = \frac{0.2081\,D}{\bar{f}} = \frac{0.2081}{0.0125} \times 1.13\,ft = 18.8\,ft$ $\longleftarrow L_{12}$

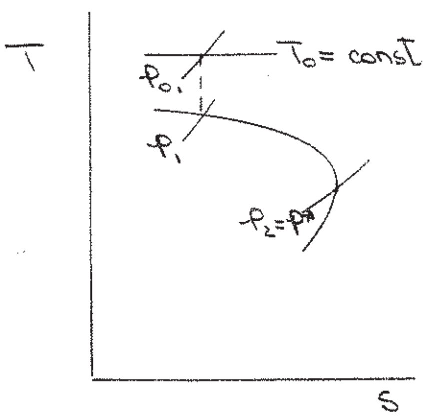
Problem 13.137

To plot $p(x)$, $M(x)$

- assume values of M, $0.70 \le M \le 1.00$
  - calculate corresponding $\bar{f}L/D$ from Eq. 12.17
    - solve for corresponding $\bar{f}\Delta L/D$ where $\Delta L = x$, assuming constant $\bar{f}$
  - calculate corresponding p/p^* from Eq. 12.18d

M	$(fL/D)_M$	$\Delta(fL/D)$	x(ft)	P/P*	P(psia)
0.70	0.2081	0	0	1.4935	72.1
0.74	0.1411	0.0670	6.1	1.4054	67.9
0.78	0.0917	0.1165	10.5	1.3261	64.0
0.82	0.0559	0.1522	13.8	1.2542	60.6
0.86	0.0310	0.1772	16.0	1.1889	57.4
0.90	0.0145	0.1936	17.5	1.1291	54.5
0.94	0.0048	0.2033	18.4	1.0743	51.9
1.0	0.0000	0.2081	18.8	1	48.3





Problem 13.138

[Difficulty: 3]

13.138 Using coordinates T/T_0 and $(s - s*)/c_p$, where $s*$ is the entropy at $M = 1$, plot the Fanno line starting from the inlet conditions specified in Example 13.8. Proceed to $M = 1$.

Solution:

- Compressible flow functions to be used in the solution

Basic equations: $Tds = dh - v\,dP$, $P = \rho RT$

$$Tds = dh - v\,dP = c_p\,dT - \frac{1}{\rho}\,dP \qquad ds = c_p \frac{dT}{T} - \frac{dP}{P}$$

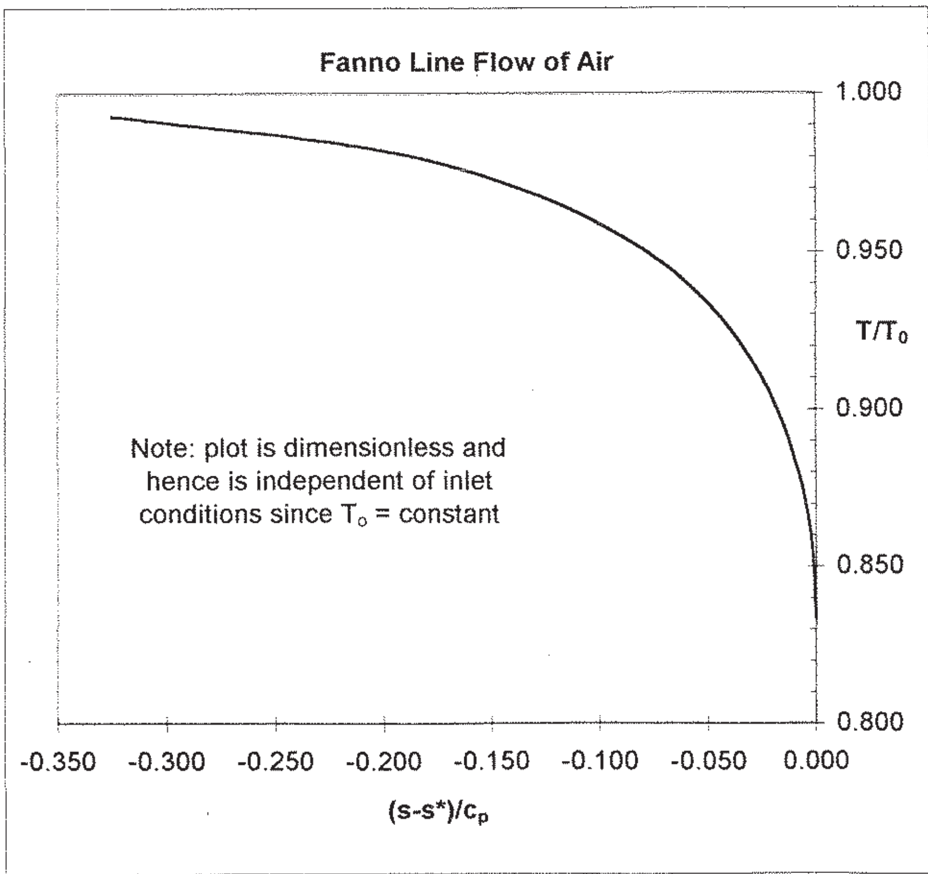
$$s - s^* = s_0 - s_0^* = c_p \int_{T_0^*}^{T_0} \frac{dT}{T} - R \int_{P_0^*}^{P_0} \frac{dP}{P} = \cancelto{0}{c_p \ln \frac{T_0}{T_0^*}} - R \ln \frac{P_0}{P_0^*} = -(c_p - c_v) \ln \frac{P_0}{P_0^*}$$

$$\frac{s - s^*}{c_p} = -\frac{(c_p - c_v)}{c_p} \ln \frac{P_0}{P_0^*} = -\frac{(k-1)}{k} \ln \frac{P_0}{P_0^*} = -\frac{0.4}{1.4} \ln \frac{P_0}{P_0^*} = -0.286 \ln \frac{P_0}{P_0^*}$$

For a given value of M, T/T_0 is found from Appendix E.1 (11.17b)

P_0/P_0^* " " " " E.2 (12.18e)

M	T/T_0	P_0/P_0^*	$(s-s^*)/c_p$
0.19	0.993	3.112	-0.325
0.30	0.982	2.035	-0.203
0.40	0.969	1.590	-0.133
0.50	0.952	1.340	-0.084
0.60	0.933	1.188	-0.049
0.70	0.911	1.094	-0.026
0.80	0.887	1.038	-0.011
0.90	0.861	1.009	-0.003
1.00	0.833	1.000	0.000



Problem 13.139

[Difficulty: 3]

13.139 Consider the flow described in Example 13.8. Using the flow functions for Fanno-line flow of an ideal gas, plot static pressure, temperature, and Mach number versus L/D measured from the tube inlet; continue until the choked state is reached.

Solution:

* Compressible flow functions to be used in the solution

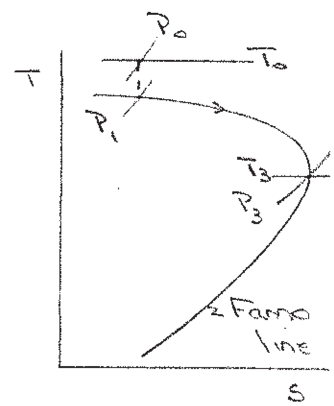
Assumptions: (1) steady flow (2) isentropic flow in nozzle (3) $\dot{Q}=0$ in duct (4) uniform flow at a section (5) ideal gas

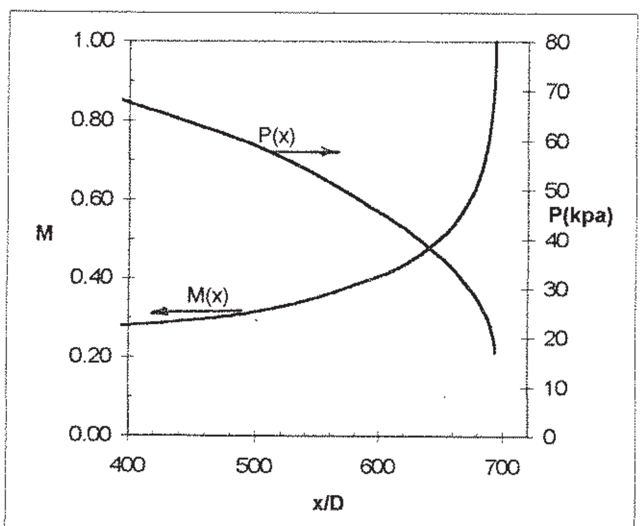
From the solution of Example 13.8 we know that

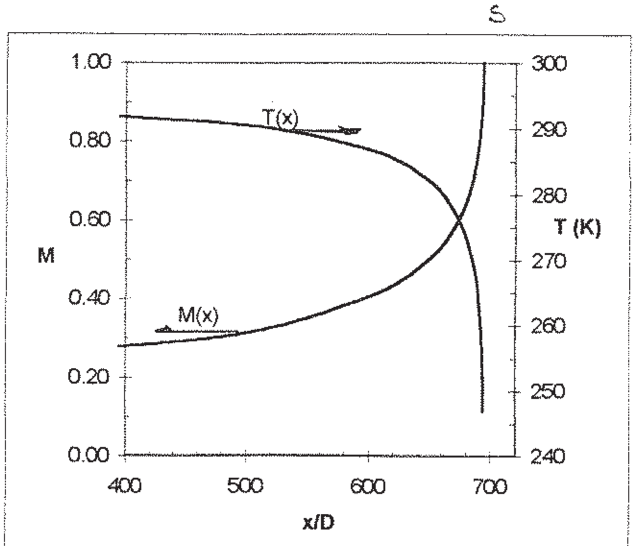
$M_1 = 0.190$, $T_1 = 294$ K, $P_1 = 98.8$ kPa, $P_{0_1} = 101$ kPa, $\bar{f} = 0.0235$

To determine $P(x)$, $T(x)$, $M(x)$, use Appendix E.2

M	$(fL/D)_M$	$\Delta(fL/D)$	x/D	T/T*	T(K)	P/P*	P(kPa)
0.19	16.38	0	0	1.191	294	5.745	98
0.30	5.30	11.08	469	1.179	291	3.619	62
0.40	2.31	14.07	596	1.163	287	2.696	46
0.50	1.07	15.31	649	1.143	282	2.138	37
0.60	0.49	15.88	673	1.119	276	1.763	30
0.70	0.21	16.17	685	1.093	270	1.493	26
0.80	0.07	16.30	691	1.064	263	1.289	22
0.90	0.01	16.36	693	1.033	255	1.129	19
1.00	0.00	16.38	694	1.000	247	1.000	17







Problem 13.140

[Difficulty: 3]

13.140 Using coordinates T/T^* and $(s - s*)/c_p$, where s^* is the entropy at $M = 1$, plot the Fanno line for air flow for $0.1 < M < 3.0$.

Solution:

* Compressible flow functions to be used in the solution.

Basic equation: $T\,ds = dh - v\,dp$, $P = \rho R T$

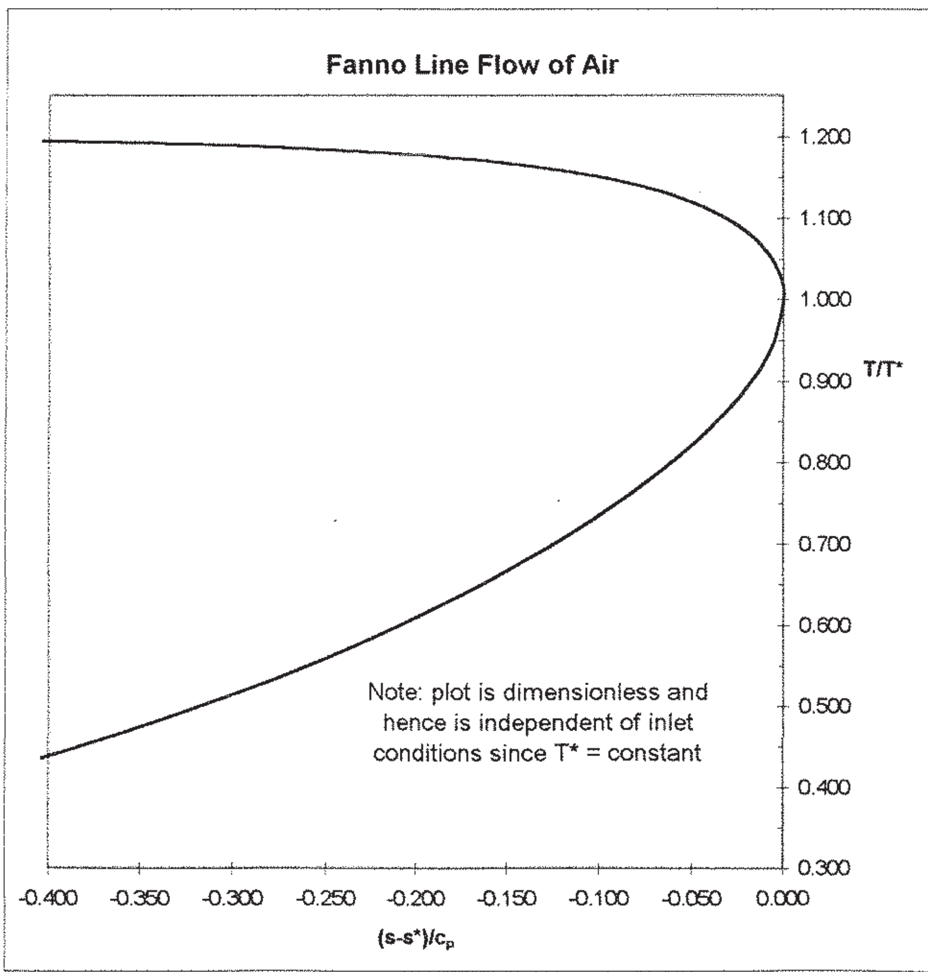
$$T\,ds = dh - v\,dP = c_p\,dT - \frac{1}{\rho}dP \quad ; \quad ds = c_p\frac{dT}{T} - \frac{dP}{\rho}$$

$$s - s^* = s_0 - s_0^* = c_p\int_{T_0^*}^{T_0}\frac{dT}{T} - R\int_{P_0^*}^{P_0}\frac{dP}{P} = c_p\ln\underbrace{\cancel{\frac{T_0}{T_0^*}}}_{0} - R\ln\frac{P_0}{P_0^*} = -(c_p - c_v)\ln\frac{P_0}{P_0^*}$$

$$\frac{s-s^*}{c_p} = -\frac{(c_p-c_v)}{c_p}\ln\frac{P_0}{P_0^*} = -\frac{(k-1)}{k}\ln\frac{P_0}{P_0^*} = -\frac{(0.4)}{1.4}\ln\frac{P_0}{P_0^*} = -0.286\ln\frac{P_0}{P_0^*}$$

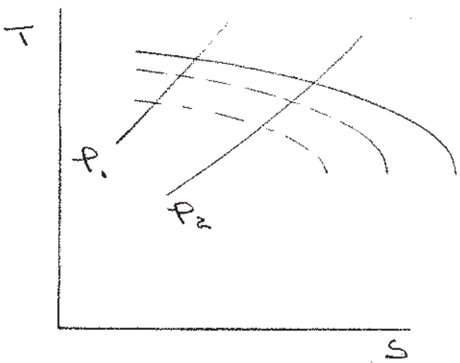
For a given value of M, use Appendix E.2 to determine T/T^* (12.18a), P_0/P_0^* (12.18e).

M	T/T^*	P_0/P_0^*	$(s-s^*)/c_p$
0.10	1.198	5.822	-0.504
0.14	1.195	4.182	-0.409
0.20	1.190	2.964	-0.311
0.30	1.179	2.035	-0.203
0.40	1.163	1.590	-0.133
0.50	1.143	1.340	-0.084
0.60	1.119	1.188	-0.049
0.70	1.093	1.094	-0.026
0.80	1.064	1.038	-0.011
0.90	1.033	1.009	-0.003
1.00	1.000	1.000	0.000
1.20	0.932	1.030	-0.009
1.40	0.862	1.115	-0.031
1.60	0.794	1.250	-0.064
1.80	0.728	1.439	-0.104
2.00	0.667	1.688	-0.150
2.20	0.610	2.005	-0.199
2.40	0.558	2.403	-0.251
2.60	0.510	2.896	-0.304
2.80	0.467	3.500	-0.358
3.00	0.429	4.235	-0.413



13.141 Air flows through a 40 ft length of insulated constant-area duct with $D = 2.12$ ft. The relative roughness is $e/D = 0.002$. At the duct inlet, $T_1 = 100°$F and $p_1 = 17.0$ psia. At a location downstream, $p_2 = 14.7$ psia, and the flow is subsonic. Is sufficient information given to solve for M_1 and M_2? Prove your answer graphically. Find the mass flow rate in the duct and T_2.

Solution:



(a) It is possible to solve for M_1 and M_2. There is a different fanno line for each different flow rate (or M_1)

- need to find the value of M_1 that gives the pressure drop $p_1 - p_2$ over the length L_{12}

(b) Procedure for trial and error solution is to assume M_1 and calculate p_2

Use Fanno-line flow functions of Appendix E-2

Assume M_1

- determine p^*, Appendix E (12.18 d)
  $\bar{f}L/D$, " " (12.17)
  $\bar{f} = \bar{f}\,(Re)$
- calculate $\bar{f}L_{12}/D$ and $\bar{f}L/D)_2 = \bar{f}L/D)_1 - \bar{f}L_{12}/D$
- knowing $\bar{f}L/D)_2$, iterate (12.17) to determine M_2
- determine p_2 from M_2 (12.18d) and check vs known p_2

Repeat with another assumed value of M_1

Additional computing equations:

$$V_1 = M_1 c_1 = M_1 (kRT_1)^{1/2} = M_1\left(1.4 \times 53.3\,\frac{ft\cdot lbf}{lbm\cdot °R} \times 560°R \times 32.2\,\frac{lbm}{slug} \times \frac{slug\cdot ft}{lbf\cdot s^2}\right)^{1/2} = 1160\,M_1$$

$$\rho_1 = \frac{p_1}{RT_1} = 17\,\frac{lbf}{in^2} \times 144\,\frac{in^2}{ft^2} \times \frac{lbm\cdot °R}{53.3\,ft\cdot lbf} \times \frac{1}{560°R} \times \frac{slug}{32.2\,lbm} = 2.55 \times 10^{-3}\,\frac{slug}{ft^3}$$

From Table A.9, $\mu = 3.97 \times 10^{-7}$ lbf·s/ft²

$$Re = \frac{\rho_1 V_1 D}{\mu} = 2.55\times10^{-3}\,\frac{slug}{ft^3} \times 1160\,M_1\,\frac{ft}{s} \times 2.12\,ft \times \frac{ft^2}{3.97\times10^{-7}\,lbf\cdot s} \times \frac{lbf\cdot s^2}{ft\cdot slug} = 7.45\times10^{6}\,M_1$$

For $M_1 = 0.4$, $Re = 3.0 \times 10^6$. With $e/D = 0.002$, Fig. 8.13 gives $f = 0.023$

Assume f is constant.

Problem 13.141

[Difficulty: 4] Part 2/2

M_1	T_1/T^*	P_1/P^*	P^*(psia)		$(fL_m/D)_1$	V_1 (ft/s)	Re(10^6)	f	fL_{12}/D	$(fL_m/D)_2$	M_2	T_2/T^*	P_2/P^*	P_2(psia)
0.400	1.163	2.6958	6.30606	0.186	2.31	464	2.98	0.0230	0.434	1.875	0.427	1.158	2.520	15.9
0.450	1.153	2.3865	7.12347	0.234	1.57	522	3.35	0.0230	0.434	1.132	0.493	1.144	2.170	15.5
0.500	1.143	2.1381	7.95102	0.286	1.07	580	3.73	0.0230	0.434	0.635	0.568	1.127	1.869	14.9
0.510	1.141	2.0942	8.118	0.297	0.99	592	3.80	0.023	0.434	0.556	0.585	1.123	1.812	14.7

Iterate to determine M_2 for known M_1

M_1	$(fL_m/D)_2$	$(M_2)_{guess}$		$(fL_m/D)_2$
0.400	1.875	0.427	0.211	1.870
0.450	1.132	0.493	0.278	1.128
0.500	0.635	0.568	0.364	0.633
0.510	0.556	0.585	0.384	0.553

$M_1 = 0.510 \quad \frac{T_1}{T^*} = 1.141 \,, \quad \frac{T_2}{T^*} = 1.123 \,, \quad \therefore \frac{T_2}{T_1} = 0.984$

$T_2 = 551°R$ ← T_2

$\dot{m} = \rho_1 V_1 A = 2.55 \times 10^{-3} \frac{slug}{ft^3} \times 592 \frac{ft}{s} \times \frac{\pi}{4}\left(\frac{2}{12}\right)^2 ft^2 = 5.33 \ slug/s$ ← $\dot{m}$

Problem 13.142

[Difficulty: 2]

13.142 Air brought into a tube through a converging-diverging nozzle initially has stagnation temperature and pressure of 550 K and 1.35 MPa (abs). Flow in the nozzle is isentropic; flow in the tube is adiabatic. At the junction between the nozzle and tube the pressure is 15 kPa. The tube is 1.5 m long and 2.5 cm in diameter. If the outlet Mach number is unity, find the average friction factor over the tube length. Calculate the change in pressure between the tube inlet and discharge.

Given: Air flow through a CD nozzle and tube.

Find: Average friction factor; Pressure drop in tube

Solution:

Assumptions: 1) Isentropic flow in nozzle 2) Adiabatic flow in tube 3) Ideal gas 4) Uniform flow

Given or available data: $k = 1.40$ $\quad R = 286.9\cdot\frac{J}{kg\cdot K}$ $\quad p_1 = 15\cdot kPa$ $\quad$ where State 1 is the nozzle exit

$p_0 = 1.35\cdot MPa$ $\quad T_0 = 550\cdot K$ $\quad D = 2.5\cdot cm$ $\quad L = 1.5\cdot m$

From isentropic relations $M_1 = \left[\frac{2}{k-1}\cdot\left[\left(\frac{p_0}{p_1}\right)^{\frac{k-1}{k}} - 1\right]\right]^{\frac{1}{2}}$ $\quad M_1 = 3.617$

Then for Fanno-line flow (for choking at the exit)

$$\frac{f_{ave}\cdot L_{max}}{D_h} = \frac{1 - M_1^2}{k\cdot M_1^2} + \frac{k+1}{2\cdot k}\cdot\ln\left[\frac{(k+1)\cdot M_1^2}{2\cdot\left(1 + \frac{k-1}{2}\cdot M_1^2\right)}\right] = 0.599$$

Hence $\quad f_{ave} = \frac{D}{L}\cdot\left[\frac{1 - M_1^2}{k\cdot M_1^2} + \frac{k+1}{2\cdot k}\cdot\ln\left[\frac{(k+1)\cdot M_1^2}{2\cdot\left(1 + \frac{k-1}{2}\cdot M_1^2\right)}\right]\right]$ $\quad f_{ave} = 0.0100$

$$\frac{p_1}{p_{crit}} = \frac{p_1}{p_2} = \frac{1}{M_1}\cdot\left(\frac{\frac{k+1}{2}}{1 + \frac{k-1}{2}\cdot M_1^2}\right)^{\frac{1}{2}} = 0.159$$

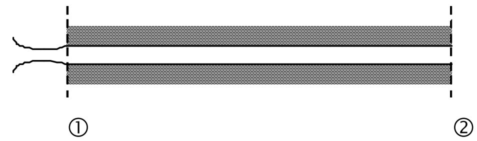
$$p_2 = \frac{p_1}{\left[\frac{1}{M_1}\cdot\left(\frac{\frac{k+1}{2}}{1 + \frac{k-1}{2}\cdot M_1^2}\right)^{\frac{1}{2}}\right]} \qquad p_2 = 94.2\cdot kPa$$

$$\Delta p = p_1 - p_2 \qquad \Delta p = -79.2\cdot kPa$$

These calculations are a LOT easier using the Excel Add-ins!

Problem 13.143 [Difficulty: 3]

13.143 For the conditions of Problem 13.127, determine the duct length. Assume the duct is circular and made from commercial steel. Plot the variations of pressure and Mach number versus distance along the duct.



Given:
Air flow in a CD nozzle and insulated duct

Find:
Duct length; Plot of M and p

Solution:

Basic equations: Fanno-line flow equations, and friction factor

Given or available data $T_1 = (100 + 460)\cdot R$ $\quad p_1 = 18.5\cdot psi$ $\quad M_1 = 2$ $\quad M_2 = 1$ $\quad A = 1\cdot in^2$

$k = 1.4$ $\quad c_p = 0.2399\cdot\frac{BTU}{lbm\cdot R}$ $\quad R_{air} = 53.33\cdot\frac{ft\cdot lbf}{lbm\cdot R}$

Then for Fanno-line flow at $M_1 = 2$

$$\frac{p_1}{p_{crit}} = \frac{p_1}{p_2} = \frac{1}{M_1}\cdot\left(\frac{\frac{k+1}{2}}{1 + \frac{k-1}{2}\cdot M_1^2}\right)^{\frac{1}{2}} = 0.4082 \qquad \frac{f_{ave}\cdot L_{max1}}{D_h} = \frac{1 - M_1^2}{k\cdot M_1^2} + \frac{k+1}{2\cdot k}\cdot\ln\left[\frac{(k+1)\cdot M_1^2}{2\cdot\left(1 + \frac{k-1}{2}\cdot M_1^2\right)}\right] = 0.305$$

so $\quad p_{crit} = \frac{p_1}{0.4082}$ $\quad p_{crit} = 45.3\cdot psi$

and at $M_2 = 1$

$$\frac{f_{ave}\cdot L_{max2}}{D_h} = \frac{1 - M_2^2}{k\cdot M_2^2} + \frac{k+1}{2\cdot k}\cdot\ln\left[\frac{(k+1)\cdot M_2^2}{2\cdot\left(1 + \frac{k-1}{2}\cdot M_2^2\right)}\right] = 0$$

Also $\quad \rho_1 = \frac{p_1}{R_{air}\cdot T}$ $\quad \rho_1 = 0.089\cdot\frac{lbm}{ft^3}$ $\quad V_1 = M_1\cdot\sqrt{k\cdot R_{air}\cdot T_1}$ $\quad V_1 = 2320\cdot\frac{ft}{s}$ $\quad D = \sqrt{\frac{4\cdot A}{\pi}}$ $\quad D = 1.13\cdot in$

For air at $T_1 = 100\cdot°F$, from Table A.9 $\quad \mu = 3.96\times 10^{-7}\cdot\frac{lbf\cdot s}{ft^2}$ $\quad$ so $\quad Re_1 = \frac{\rho_1\cdot V_1\cdot D}{\mu}$

For commercial steel pipe (Table 8.1) $\quad e = 0.00015\cdot ft$ $\quad \frac{e}{D} = 1.595\times 10^{-3}$ $\quad$ and $\quad Re_1 = 1.53\times 10^6$

Hence at this Reynolds number and roughness (Eq. 8.37) $\quad f = .02222$

Combining results

$$L_{12} = \frac{D}{f}\cdot\left(\frac{f_{ave}\cdot L_{max2}}{D_h} - \frac{f_{ave}\cdot L_{max1}}{D_h}\right) = \frac{\frac{1.13}{12}\cdot ft}{.02222}\cdot(0.3050 - 0) \qquad L_{12} = 1.29\cdot ft \qquad L_{12} = 15.5\cdot in$$

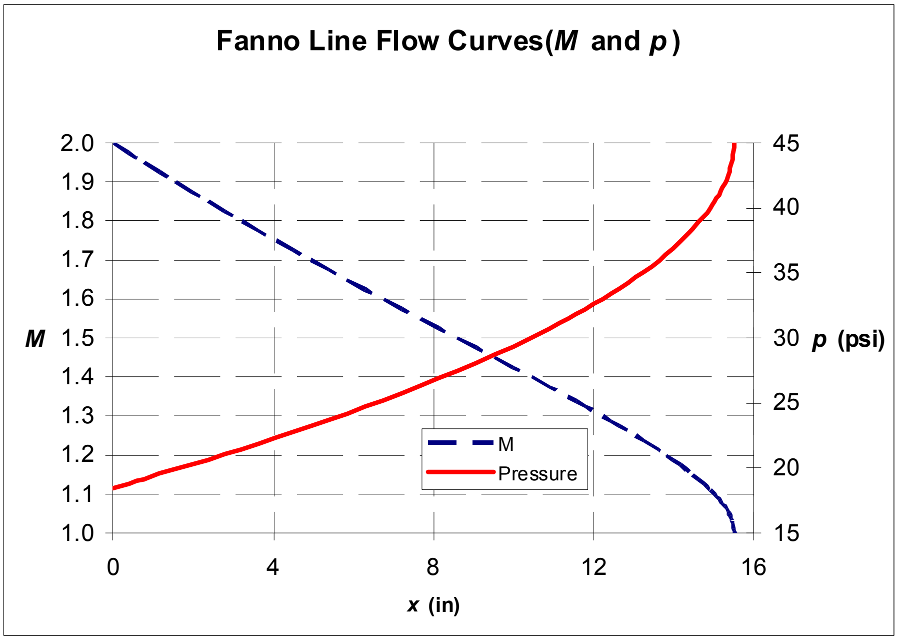
These calculations are a LOT easier using the *Excel* Add-ins! The *M* and *p* plots are shown in the *Excel* spreadsheet on the next page.

The given or available data is: $f =$ 0.0222

$p^* =$ 45.3 kPa

$D =$ 1.13 in

M	fL_{max}/D	$\Delta fL_{max}/D$	x (in)	p/p^*	p (psi)
2.00	0.305	0.000	0	0.408	18.49
1.95	0.290	0.015	0.8	0.423	19.18
1.90	0.274	0.031	1.6	0.439	19.90
1.85	0.258	0.047	2.4	0.456	20.67
1.80	0.242	0.063	3.2	0.474	21.48
1.75	0.225	0.080	4.1	0.493	22.33
1.70	0.208	0.097	4.9	0.513	23.24
1.65	0.190	0.115	5.8	0.534	24.20
1.60	0.172	0.133	6.7	0.557	25.22
1.55	0.154	0.151	7.7	0.581	26.31
1.50	0.136	0.169	8.6	0.606	27.47
1.45	0.118	0.187	9.5	0.634	28.71
1.40	0.100	0.205	10.4	0.663	30.04
1.35	0.082	0.223	11.3	0.695	31.47
1.30	0.065	0.240	12.2	0.728	33.00
1.25	0.049	0.256	13.0	0.765	34.65
1.20	0.034	0.271	13.8	0.804	36.44
1.15	0.021	0.284	14.5	0.847	38.37
1.10	0.010	0.295	15.0	0.894	40.48
1.05	0.003	0.302	15.4	0.944	42.78
1.00	0.000	0.305	15.5	1.000	45.30



Problem 13.144

[Difficulty: 3]

13.144 A smooth constant-area duct assembly (D = 150 mm) is to be fed by a converging-diverging nozzle from a tank containing air at 295 K and 1.0 MPa (abs). Shock-free operation is desired. The Mach number at the duct inlet is to be 2.1 and the Mach number at the duct outlet is to be 1.4. The entire assembly will be insulated. Find (a) the pressure required at the duct outlet, (b) the duct length required, and (c) the change in specific entropy. Show the static and stagnation state points and the process path on a Ts diagram.

Solution: • Compressible flow functions to be used in solution.

Assumptions: (1) steady flow (2) uniform flow at a section (3) isentropic flow in nozzle, adiabatic flow in duct (4) ideal gas

(a)

From App. E.1 for $M_1 = 2.1$ $\quad T/T_0 = 0.5314 \quad \therefore T_1 = 156.7\,K$

$p/p_0 = 0.1094 \quad \therefore p_1 = 109.4\,kPa.$

$V_1 = M_1 c_1 = M_1(kRT_1)^{1/2} = 2.1\left[1.4 \times 287 \frac{N.m}{kg.K} \times 156.7K \times \frac{kg.m}{N.s^2}\right]^{1/2} = 527\,m/s$

For App. E.2 for $M = 2.1$ $\quad p_{01}/p_0^* = 1.837 \quad \therefore p_0^* = 544.4\,kPa$

$p_1/p^* = 0.3802 \quad \therefore p^* = 287.7\,kPa.$

for $M_2 = 1.4$ $\quad p_2/p^* = 0.6632 \quad \therefore p_2 = 190.8\,kPa$ — p_2

$p_{02}/p_0^* = 1.115 \quad \therefore p_{02} = 320.8\,kPa$

(b) From App. E.2 for $M_1 = 2.1$, $\bar{f}L_{max}/D_h = 0.3339$

$M_2 = 1.4$, $\bar{f}L_{max}/D_h = 0.09974$

$\therefore \bar{f}L_{12}/D_h = 0.3339 - 0.09974 = 0.2342$

To obtain L_{12}, we need f; $f = f(Re)$; $Re = \frac{\rho V D}{\mu}$ {use conditions at ①}

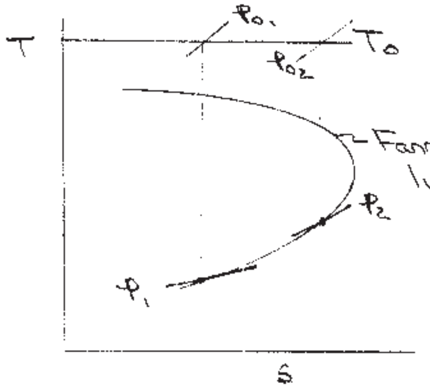
To obtain μ at 156.7K, use Sutherland correlation (Appendix A)

$$\mu = \frac{bT^{1/2}}{1+S/T} = 1.458\times10^{-6}\frac{kg}{m.s.K^{1/2}} \times (156.7K)^{1/2} \times \frac{1}{1+\frac{110.4}{156.7}} = 1.071\times10^{-5}\,\frac{N.s}{m^2}$$

$$Re = \frac{\rho_1 V_1 D}{\mu} = \frac{p_1 V_1 D}{RT_1\mu} = 109.4\times10^3\frac{N}{m^2}\times\frac{527m}{s}\times(0.15m)\times\frac{kg.K}{287\,N.m}\times\frac{1}{156.7K}\times\frac{1}{1.071\times10^{-5}}\frac{m^2}{N.s}\times\frac{N.s^2}{kg.m}$$

$Re = 1.80\times10^7$. For smooth pipe from Fig 8.13 friction factor, $f \approx 0.007$

$\therefore 0.007\,L_{12}/D_h = 0.2342 \qquad L_{12} = \frac{0.2342}{0.007}(D) = 5.02\,m$ — L_{12}



From the Tds eq.

$$s_2 - s_1 = s_{02} - s_{01} = c_p \ln\frac{T_{02}}{T_{01}} - R\ln\frac{p_{02}}{p_{01}}$$

$$s_2 - s_1 = -287\frac{N.m}{kg.K}\ln\frac{320.8}{1000} = 0.326\,\frac{kJ}{kg.K}$$ — $s_2 - s_1$

Problem 13.145

[Difficulty: 4]

*13.145 Natural gas is to be pumped through 60 mi of 30-in.-diameter pipe with an average friction factor of 0.025. The temperature of the gas remains constant at 140°F, and the mass flow rate is 40 lbm/s. The downstream pressure of the gas is 150 kPa. Estimate the required entrance pressure, and the power needed to pump the gas through the pipe.

Given: Natural gas pumped through a pipe

Find: Required entrance pressure and power needed to pump gas through the pipe

Solution:

The given or available data is:

$R =$	96.32	ft-lbf/lbm-°R
$c_p =$	0.5231	Btu/lbm-°R
$k =$	1.31	
$D =$	30	in
$L =$	60	mi
$f =$	0.025	
$T_1 =$	140	°F
$T_1 =$	600	°R
$T_2 =$	600	°R
$m =$	40	lbm/s
$p_2 =$	150	kPa

Equations and Computations:

At the exit of the pipe we can calculate the density:

$p_2 =$ 21.756 psia

$\rho_2 =$ 0.05421 lbm/ft^3

The pipe area is:

$A =$ 4.909 ft^2

Therefore, the flow velocity is:

$V_2 =$ 150.32 ft/s

The local sound speed is:

$c_2 =$ 1561.3 ft/s

So the Mach number is:

$M_2 =$ 0.09628

From the exit Mach number we can calculate:

$T_{02}/T_2 =$ 1.0014

$fL_2/D =$ 76.94219

Given the length, diameter, and friction factor, we know:

$fL_{1\text{-}2}/D =$ 3168.0

Therefore: $fL_1/D =$ 3244.9

So from this information we can calculate the entrance Mach number:

$$M_1 = 0.01532$$

$$fL_1/D = 3244.9$$

(We use *Solver* to calculate the Mach number based on the friction length)

The entrance sound speed is the same as that at the exit:

$$c_1 = 1561.3 \quad \text{ft/s}$$

So the flow velocity is:

$$V_1 = 23.91 \quad \text{ft/s}$$

We can calculate the pressure ratio from the velocity ratio:

$$p_1 = 136.8 \quad \text{psi}$$

From the entrance Mach number we can calculate:

$$T_{01}/T_1 = 1.0000$$

So the entrance and exit stagnation temperatures are:

$$T_{01} = 600.02 \quad {}^\circ\text{R}$$

$$T_{02} = 600.86 \quad {}^\circ\text{R}$$

The work needed to pump the gas through the pipeline would be:

$$W = 17.5810 \quad \text{Btu/s}$$

$$W = 24.9 \quad \text{hp}$$

Problem 13.146

[Difficulty: 5]

*13.146 Air flows through a 1-in.-diameter, 10-ft-long tube. The friction factor of the tube is 0.03. If the entrance conditions are 15 psia and 530°R, calculate the mass flow rate for (a) incompressible flow (using the methods of Chapter 8), (b) adiabatic (Fanno) flow, and (c) isothermal flow. Assume for parts (b) and (c) that the exit pressure is 14.7 psia.

Given: Air flowing through a tube

Find: Mass flow rate assuming incompressible, adiabatic, and isothermal flow

Solution:

The given or available data is:

$R =$	53.33	ft-lbf/lbm-°R
$k =$	1.4	
$\nu =$	0.000163	ft^2/s
$D =$	1	in
$L =$	10	ft
$f =$	0.03	
$p_1 =$	15	psia
$T_1 =$	530	°R
$p_2 =$	14.7	psia

Equations and Computations:

The tube flow area is:

$A =$ 0.005454 ft^2

For incompressible flow, the density is:

$\rho_1 =$ 0.07642 lbm/ft^3

The velocity of the flow is:

$V_1 =$ 100.56 ft/s

The mass flow rate is:

$m_{incomp} =$ 0.0419 lbm/s

For Fanno flow, the duct friction length is:

$fL_{1\text{-}2}/D =$ 3.600

and the pressure ratio across the duct is:

$p_1/p_2 =$ 1.0204

To solve this problem, we have to guess M_1. Based on this and the friction length, we can determine a corresponding M_2. The pressure ratios for M_1 and M_2 will be used to check the validity of our guess.

M_1	M_2	fL_1/D	fL_2/D	$fL_{1\text{-}2}/D$	p_1/p_2
0.0800	0.0813	106.72	103.12	3.600	1.0167
0.0900	0.0919	83.50	79.90	3.600	1.0213
0.1000	0.1027	66.92	63.32	3.600	1.0266
0.1100	0.1136	54.69	51.09	3.600	1.0326

Here we used Solver to match the friction length. When both the friction length and the pressure ratios match the constraints set above, we have our solution.

Therefore our entrance and exit Mach numbers are:

$M_1 =$ 0.0900

$M_2 =$ 0.0919

The density at 1 was already determined. The sound speed at 1 is:

$c_1 =$ 1128.8 ft/s

so the velocity at 1 is:

$V_1 =$ 101.59 ft/s

and the mass flow rate is:

$m_{\text{Fanno}} =$ 0.0423 lbm/s

To solve this problem for isothermal flow, we perform a calculation similar to that done above for the Fanno flow. The only difference is that we use the friction length relation and pressure ratio relation for isothermal flow:

M_1	M_2	fL_1/D	fL_2/D	$fL_{1\text{-}2}/D$	p_1/p_2
0.0800	0.0813	105.89216	102.29216	3.600	1.0167
0.0900	0.0919	82.70400	79.10400	3.600	1.0213
0.1000	0.1027	66.15987	62.55987	3.600	1.0266
0.1100	0.1136	53.95380	50.35380	3.600	1.0326

Here we used Solver to match the friction length. When both the friction length and the pressure ratios match the constraints set above, we have our solution.
Therefore our entrance and exit Mach numbers are:

$M_1 =$ 0.0900

$M_2 =$ 0.0919

The density and sound speed at 1 were already determined. The velocity at 1 is:

$V_1 =$ 101.59 ft/s

and the mass flow rate is:

$m_{\text{Isothermal}} =$ 0.0423 lbm/s

Note that in this situation, since the Mach number was low, the assumption of incompressible flow was a good one. Also, since the Fanno flow solution shows a very small change in Mach number, the temperature does not change much, and so the isothermal solution gives almost identical results.

Problem 13.147

[Difficulty: 4]

***13.147** A 15-m umbilical line for an astronaut on a space walk is held at a constant temperature of 20°C. Oxygen is supplied to the astronaut at a rate of 10 L/min, through a 1-cm tube in the umbilical line with an average friction factor of 0.01. If the oxygen pressure at the downstream end is 30 kPa, what does the upstream pressure need to be? How much power is needed to feed the oxygen to the astronaut?

Given: Oxygen supplied to astronaut via umbilical

Find: Required entrance pressure and power needed to pump gas through the tube

Solution:

The given or available data is:

$R =$	259.8	J/kg-K
$c_p =$	909.4	J/kg-K
$k =$	1.4	
$Q =$	10	L/min
$D =$	1	cm
$L =$	15	m
$f =$	0.01	
$T_1 =$	20	°C
$T_1 =$	293	K
$T_2 =$	293	K
$p_2 =$	30	kPa

Equations and Computations:

At the exit of the pipe we can calculate the density:

$\rho_2 =$ 0.39411 kg/m^3

so the mass flow rate is:

$m =$ 6.568E-05 kg/s

The pipe area is:

$A =$ 7.854E-05 m^2

Therefore, the flow velocity is:

$V_2 =$ 2.12 m/s

The local sound speed is:

$c_2 =$ 326.5 m/s

So the Mach number is:

$M_2 =$ 0.006500

From the exit Mach number we can calculate:

$T_{02}/T_2 =$ 1.0000

$fL_2/D =$ 16893.2

Given the length, diameter, and friction factor, we know:

$fL_{1\text{-}2}/D =$ 15.0

Therefore: $fL_1/D =$ 16908.2

So from this information we can calculate the entrance Mach number:

$$M_1 = \quad 0.006498$$

$$fL_1/D = \quad 16908.2$$

(We use *Solver* to calculate the Mach number based on the friction length)

The entrance sound speed is the same as that at the exit:

$$c_1 = \quad 326.5 \quad \text{m/s}$$

So the flow velocity is:

$$V_1 = \quad 2.12 \quad \text{m/s}$$

We can calculate the pressure ratio from the velocity ratio:

$$p_1 = \quad 30.0 \quad \text{kPa}$$

From the entrance Mach number we can calculate:

$$T_{01}/T_1 = \quad 1.0000$$

So the entrance and exit stagnation temperatures are:

$$T_{01} = \quad 293.00 \quad \text{K}$$

$$T_{02} = \quad 293.00 \quad \text{K}$$

The work needed to pump the gas through the pipeline would be:

$$W = \quad 1.3073\text{E-}07 \quad \text{W}$$

$$W = \quad 0.1307 \quad \text{microwatts}$$

Problem 13.148

[Difficulty: 5]

*13.148 Air enters a 15-cm-diameter pipe at 15°C, 1.5 MPa, and 60 m/s. The average friction factor is 0.013. Flow is isothermal. Calculate the local Mach number and the distance from the entrance of the channel, at the point where the pressure reaches 500 kPa.

Given: Isothermal air flow in a pipe

Find: Mach number and location at which pressure is 500 kPa

Solution:

Basic equations: $m_{rate} = \rho \cdot V \cdot A$ $\quad p = \rho \cdot R \cdot T$ $\quad \dfrac{f \cdot L_{max}}{D} = \dfrac{1 - k \cdot M^2}{k \cdot M^2} + \ln\left(k \cdot M^2\right)$

Given or available data $\quad T_1 = (15 + 273) \cdot K$ $\quad p_1 = 1.5 \cdot MPa$ $\quad V_1 = 60 \cdot \dfrac{m}{s}$ $\quad f = 0.013$ $\quad p_2 = 500 \cdot kPa$

$D = 15 \cdot cm$ $\quad k = 1.4$ $\quad R = 286.9 \cdot \dfrac{J}{kg \cdot K}$

From continuity $\quad \rho_1 \cdot V_1 = \rho_2 \cdot V_2$ $\quad$ or $\quad \dfrac{p_1}{T_1} \cdot V_1 = \dfrac{p_2}{T_2} \cdot V_2$

Since $\quad T_1 = T_2$ $\quad$ and $\quad V = M \cdot c = M \cdot \sqrt{k \cdot R \cdot T}$ $\quad M_2 = M_1 \cdot \dfrac{p_1}{p_2}$

$c_1 = \sqrt{k \cdot R \cdot T_1}$ $\quad c_1 = 340 \dfrac{m}{s}$ $\quad M_1 = \dfrac{V_1}{c_1}$ $\quad M_1 = 0.176$

Then $\quad M_2 = M_1 \cdot \dfrac{p_1}{p_2}$ $\quad M_2 = 0.529$

At $M_1 = 0.176$ $\quad \dfrac{f \cdot L_{max1}}{D} = \dfrac{1 - k \cdot M_1^{\,2}}{k \cdot M_1^{\,2}} + \ln\left(k \cdot M_1^{\,2}\right) = 18.819$

At $M_2 = 0.529$ $\quad \dfrac{f \cdot L_{max2}}{D} = \dfrac{1 - k \cdot M_2^{\,2}}{k \cdot M_2^{\,2}} + \ln\left(k \cdot M_2^{\,2}\right) = 0.614$

Hence $\quad \dfrac{f \cdot L_{12}}{D} = \dfrac{f \cdot L_{max2}}{D} - \dfrac{f \cdot L_{max1}}{D} = 18.819 - 0.614 = 18.2$

$L_{12} = 18.2 \cdot \dfrac{D}{f}$ $\quad L_{12} = 210\,m$

Problem 13.149

[Difficulty: 2]

*13.149 In long, constant-area pipelines, as used for natural gas, temperature is constant. Assume gas leaves a pumping station at 350 kPa and 20°C at $M = 0.10$. At the section along the pipe where the pressure has dropped to 150 kPa, calculate the Mach number of the flow. Is heat added to or removed from the gas over the length between the pressure taps? Justify your answer: Sketch the process on a Ts diagram. Indicate (qualitatively) T_{0_1}, T_{0_2}, and p_{0_2}.

Given: Isothermal air flow in a duct

Find: Downstream Mach number; Direction of heat transfer; Plot of Ts diagram

Solution:

Basic equations:

$$h_1 + \frac{V_1^2}{2} + \frac{\delta Q}{dm} = h_2 + \frac{V_2^2}{2} \qquad \frac{T_0}{T} = 1 + \frac{k-1}{2}\cdot M^2 \qquad m_{rate} = \rho\cdot V\cdot A$$

Given or available data

$$T_1 = (20 + 273)\cdot K \qquad p_1 = 350\cdot kPa \qquad M_1 = 0.1 \qquad p_2 = 150\cdot kPa$$

From continuity $\quad m_{rate} = \rho_1\cdot V_1\cdot A = \rho_2\cdot V_2\cdot A \quad$ so $\quad \rho_1\cdot V_1 = \rho_2\cdot V_2$

Also $\quad p = \rho\cdot R\cdot T \quad$ and $\quad M = \frac{V}{c} \quad$ or $\quad V = M\cdot c$

Hence continuity becomes

$$\frac{p_1}{R\cdot T_1}\cdot M_1\cdot c_1 = \frac{p_2}{R\cdot T_2}\cdot M_2\cdot c_2$$

Since $\quad T_1 = T_2 \qquad c_1 = c_2 \quad$ so $\quad p_1\cdot M_1 = p_2\cdot M_2$

Hence $\quad M_2 = \frac{p_1}{p_2}\cdot M_1 \qquad M_2 = 0.233$

From energy

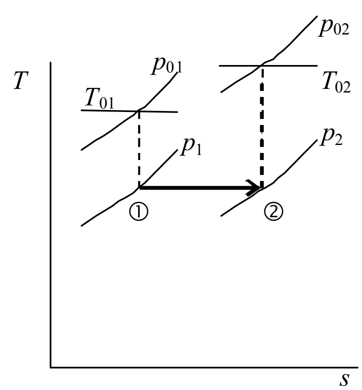
$$\frac{\delta Q}{dm} = \left(h_2 + \frac{V_2^2}{2}\right) - \left(h_1 + \frac{V_1^2}{2}\right) = h_{02} - h_{01} = c_p\cdot\left(T_{02} - T_{01}\right)$$

But at each state

$$\frac{T_0}{T} = 1 + \frac{k-1}{2}\cdot M^2 \qquad \text{or} \qquad T_0 = T\cdot\left(1 + \frac{k-1}{2}\cdot M^2\right)$$

Since T = const, but $M_2 > M_1$, then $T_{02} > T_{01}$, and

$\frac{\delta Q}{dm} > 0 \quad$ so energy is ADDED to the system



*13.150 A clean steel pipe is 950 ft long and 5.25 in. inside diameter. Air at 80°F, 120 psia, and 80 ft/s enters the pipe. Calculate and compare the pressure drops through the pipe for (a) incompressible, (b) isothermal, and (c) adiabatic flows.

Solution:

Basic equations: $\dot{m} = \rho V A \qquad p = \rho R T$

Computing equations: $p_1 - p_2 = \rho f \frac{L}{D}\frac{\bar{V}^2}{2}$ (ρ = constant)

$$f\frac{L_{max}}{D} = \frac{1-kM^2}{kM^2} + \ln kM^2 \quad (T = \text{constant})$$

$$\frac{\bar{f} L_{max}}{D} = \frac{1-M^2}{kM^2} + \frac{k+1}{2k}\ln\left[\frac{(k+1)M^2}{2\left(1+\frac{k-1}{2}M^2\right)}\right] \quad (Q=0)$$

$$\rho_1 = \frac{p_1}{RT_1} = 120\frac{\text{lbf}}{\text{in}^2} \times 144\frac{\text{in}^2}{\text{ft}^2} \times \frac{\text{lbm}\cdot°\text{R}}{53.3\ \text{ft}\cdot\text{lbf}} \times \frac{1}{540°\text{R}} = 0.600\ \text{lbm}/\text{ft}^3$$

From Table A.9 {at T = 80°F} $\mu = 3.86\times10^{-7}\ \text{lbf}\cdot\text{s}/\text{ft}^2$

$$Re_1 = \frac{\rho D V}{\mu} = 0.600\frac{\text{lbm}}{\text{ft}^3} \times \frac{5.25}{12}\text{ft} \times 80\frac{\text{ft}}{\text{s}} \times \frac{1}{3.86\times10^{-7}}\frac{\text{ft}^2}{\text{lbf}\cdot\text{s}} \times \frac{\text{slug}}{32.2\ \text{lbm}} \times \frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}}$$

$Re_1 = 1.69 \cdot 10^6$

For commercial steel (Table 8.1), $e = 0.00015$ ft. $\therefore e/D = 0.00034$

From Fig. 8.13, $f = 0.0155$

(a) For incompressible flow

$$p_1 - p_2 = \rho f \frac{L}{D}\frac{\bar{V}^2}{2} = 0.60\frac{\text{lbm}}{\text{ft}^3} \times 0.0155 \times \frac{950\times12}{5.25} \times \frac{(80)^2}{2}\frac{\text{ft}^2}{\text{s}^2} \times \frac{\text{slug}}{32.2\ \text{lbm}} \times \frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}} \times \frac{\text{ft}^2}{144\ \text{in}^2}$$

$p_1 - p_2 = 13.9$ psia $\qquad (p_1 - p_2)_{\rho = c}$

(b) For isothermal flow

$$c_1 = (kRT_1)^{1/2} = \left(1.4 \times 53.3\frac{\text{ft}\cdot\text{lbf}}{\text{lbm}\cdot°\text{R}} \times 540°\text{R} \times 32.2\frac{\text{lbm}}{\text{slug}} \times \frac{\text{slug}\cdot\text{ft}}{\text{lbf}\cdot\text{s}^2}\right)^{1/2} = 1140\ \text{ft/s}$$

$$M_1 = \frac{V_1}{c_1} = \frac{80}{1140} = 0.0702$$

At state ①
$$\left(f\frac{L_{max}}{D}\right)_1 = \frac{1-kM_1^2}{kM_1^2} + \ln kM_1^2$$

$$= \frac{1-1.4(0.0702)^2}{1.4(0.0702)^2} + \ln\left[1.4(0.0702)^2\right]$$

$$\left(f\frac{L_{max}}{D}\right)_1 = 139$$

$$f\frac{L_{12}}{D} = 0.0155 \times \frac{950 \times 12}{5.25} = 33.6$$

$$\therefore \quad f\frac{L_{max}}{D}\Big)_2 = 139 - 33.6 = 105 = \frac{1-kM_2^2}{kM_2^2} + \ln kM_2^2$$

Trial and error solution for M_2

M_2	$f\frac{L_{max}}{D}\Big)_2$
0.10	66.2
0.08	106
0.081	103
0.0805	105

$V_2 = M_2 c_2 = M_2 c_1 = 0.0805 \times 1140$ ft/s $= 91.8$ ft/s

$\rho_1 V_1 = \rho_2 V_2$ or $\frac{p_1}{T_1} V_1 = \frac{p_2}{T_2} V_2$ ......(1)

Since $T_2 = T_1$, $p_2 = p_1 \frac{V_1}{V_2} = 120 \text{ psia} \times \frac{80}{91.8} = 105$ psia

$p_1 - p_2 = 15.0$ psia ______ $(p_2 - p_1)_{T=c}$

(c) For adiabatic flow, $M_1 = 0.0702$

$$f\frac{L_{max}}{D}\Big)_1 = \frac{1-(0.0702)^2}{1.4(0.0702)^2} + \frac{1.4+1}{2(1.4)} \ln\left[\frac{(2.4)(0.0702)^2}{2(1+0.2(0.0702)^2)}\right] = 139.8$$

$$\therefore \quad f\frac{L_{max}}{D}\Big)_2 = 139.8 - 33.6 = 106.2 = \frac{1-M_2^2}{1.4M_2^2} + \frac{2.4}{2.8}\ln\left[\frac{2.4M_2^2}{2(1+0.2M_2^2)}\right]$$

Trial and error solution for M_2

M_2	$f\frac{L_{max}}{D}\Big)_2$
0.085	94.1
0.080	106.7
0.0802	106.2 ✓

For adiabatic flow, T_0 = constant

$$\therefore \quad \frac{T_2}{T_1} = \frac{1+\frac{k-1}{2}M_1^2}{1+\frac{k-1}{2}M_2^2} = \frac{1+0.2(0.0702)^2}{1+0.2(0.0802)^2} = 1.00$$

$\therefore V_2 = M_2 c_2 = M_2 c_1 = 0.0802 \times 1140$ ft/sec $= 91.4$ ft/s

From continuity (Eq. 1)

$$p_2 = p_1 \frac{V_1}{V_2} = 120 \text{ psia} \times \frac{80}{91.4} = 105 \text{ psia}$$

$\therefore p_1 - p_2 = 15.0$ psia ______ $(p_2 - p_1)_{Q=0}$

Note: p_2 is essentially the same for isothermal and adiabatic flow; the value is higher that for incompressible flow

Problem 13.151

[Difficulty: 2]

*13.151 Air enters a horizontal channel of constant area at 200°F, 600 psia, and 350 ft/s. Determine the limiting pressure for isothermal flow. Compare with the limiting pressure for frictional adiabatic flow.

Solution:

Basic equations: $h_1 + \frac{V_1^2}{2} + \frac{\delta Q}{dm} = h_2 + \frac{V_2^2}{2}$ $\qquad \dot{m} = \rho V A$

Computing equation: $T_0/T = 1 + \frac{k-1}{2}M^2$

Assumptions: (1) steady flow (2) ideal gas
(3) uniform flow at a section (4) $\dot{W}_s = \dot{W}_{shear} = 0$

$c_1 = (kRT_1)^{1/2} = \left(1.4 \times 53.3 \frac{ft \cdot lbf}{lbm \cdot °R} \times 660°R \times 32.2 \frac{lbm}{slug} \times \frac{slug \cdot ft}{lbf \cdot s^2}\right)^{1/2} = 1260\ ft/s$

$M_1 = \frac{V_1}{c_1} = \frac{350}{1260} = 0.278$

$V_2 = M_2 c_2 = M_2 c_1 = \frac{1}{\sqrt{1.4}} \times 1260\ ft/s = 1060\ ft/s$

$\rho_1 V_1 = \rho_2 V_2$ or $\frac{p_1}{RT_1} V_1 = \frac{p_2}{RT_2} V_2$ ------(1)

Since $T_1 = T_2$, $p_2 = p_1 \frac{V_1}{V_2} = 600\ psia \times \frac{350}{1060} = 198\ psia$ ← $p_2)_{T=C}$

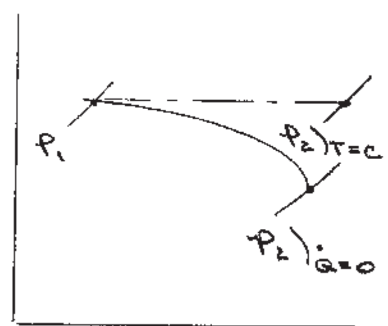
For adiabatic flow, T_0 = constant and $M_2 = 1.0$

$T_{0_2} = T_{0_1} = T_1\left(1 + \frac{k-1}{2}M_1^2\right) = T_2\left(1 + \frac{k-1}{2}M_2^2\right)$

$\frac{T_2}{T_1} = \frac{1 + 0.2(0.278)^2}{1 + 0.2} = 0.846$

$T_2 = 558°R$

$V_2 = c_2 = (kRT_2)^{1/2} = (1.4 \times 53.3 \times 558 \times 32.2)^{1/2} = 1160\ ft/s$



From continuity (Eq. 1)

$p_2 = p_1 \frac{V_1}{V_2} \frac{T_2}{T_1} = 600\ psia \times \frac{350}{1160} \times 0.846$

$p_2 = 153\ psia$ ← $p_2)_{\dot{Q}=0}$

***13.152** Natural gas (molecular mass $M_m = 18$ and $k = 1.3$) is to be pumped through a 36 in. i.d. pipe connecting two compressor stations 40 miles apart. At the upstream station the pressure is not to exceed 90 psig, and at the downstream station it is to be at least 10 psig. Calculate the maximum allowable rate of flow (ft^3/day at 70°F and 1 atm) assuming sufficient heat exchange through the pipe to maintain the gas at 70°F.

Solution:

Basic equations: $\dot{m} = \rho V A$ $\quad p = \rho R T$

Computing equation: $f\frac{L_{max}}{D} = \frac{1-kM^2}{kM^2} + \ln kM^2$

Then $f\frac{L_{12}}{D} = f\frac{L_{max}}{D}\Big)_1 - f\frac{L_{max}}{D}\Big)_2 = \frac{1-kM_1^2}{kM_1^2} - \frac{1-kM_2^2}{kM_2^2} + \ln\frac{M_1^2}{M_2^2}$

Since information on pressures is known, we relate p and M

From the ideal gas equation of state, for T = constant $\frac{p_1}{p_2} = \frac{\rho_1}{\rho_2}$

From the continuity equation $\frac{\rho_1}{\rho_2} = \frac{V_2}{V_1}$ and for T = constant, $\frac{V_2}{V_1} = \frac{M_2}{M_1}$

Hence $\frac{p_1}{p_2} = \frac{M_2}{M_1}$ and $M_2 = \frac{p_1}{p_2}M_1$

Substituting for M_2 into the equation for $f\frac{L_{12}}{D_h}$ and rearranging we obtain

$$f\frac{L_{12}}{D_h} = \frac{1-(p_2/p_1)^2}{kM_1^2} - \ln\left(\frac{p_1}{p_2}\right)^2$$

Solving this equation for M_1, then $M_1 = \left\{\frac{1}{k}\left[\frac{1-\left(\frac{p_2}{p_1}\right)^2}{f\frac{L_{12}}{D_h} + \ln\left(\frac{p_1}{p_2}\right)^2}\right]\right\}^{1/2}$

$\frac{L}{D_h} = 40\ mi \times 5280\frac{ft}{mi} \times \frac{1}{3\ ft} = 70{,}400$

Assume pipe is commercial steel. From Table 8.1, $e = 0.00015$ ft and hence $e/D = 0.00005$. Assume $Re > 3.0\times10^7$, then $f = 0.0105$. Solving for M_1,

$$M_1 = \left\{\frac{1}{1.3}\left[\frac{1-\left(\frac{24.7}{104.7}\right)^2}{0.0105(70{,}400) + \ln\left(\frac{104.7}{24.7}\right)^2}\right]\right\}^{1/2} = 0.0313$$

$R_{Nat\,gas} = \frac{R_u}{M} = 1544\frac{ft\cdot lbf}{lbmole\cdot°R}\frac{lbmole}{18\ lbm} = 85.8\frac{ft\cdot lbf}{lbm\cdot°R}$

$c_1 = (kRT_1)^{1/2} = \left(1.3\times85.8\frac{ft\cdot lbf}{lbm\cdot°R}\times530°R\times\frac{32.2\ lbm}{slug}\times\frac{slug\cdot ft}{lbf\cdot s^2}\right)^{1/2} = 1380\ ft/s$

$V_1 = M_1c_1 = 0.0313\times1380\ ft/s = 43.2\ ft/s$

$\rho_1 = \frac{p_1}{RT_1} = 104.7\frac{lbf}{in^2}\times\frac{lbm\cdot°R}{85.8\ ft\cdot lbf}\times\frac{1}{530°R}\times144\frac{in^2}{ft^2} = 0.332\ lbm/ft^3$

Check assumption on Re: from Fig A.2 (for methane), $\mu = 1.08\times10^{-5}\times2.09\times10^{-2}\ lbf\cdot s/ft^2$

$Re = \frac{\rho V D}{\mu} = 0.332\frac{lbm}{ft^3}\times43.2\frac{ft}{s}\times3\ ft\times\frac{ft^2}{2.26\times10^{-7}\ lbf\cdot s}\times\frac{slug}{32.2\ lbm}\times\frac{lbf\cdot s^2}{ft\cdot slug} = 5.91\times10^6$

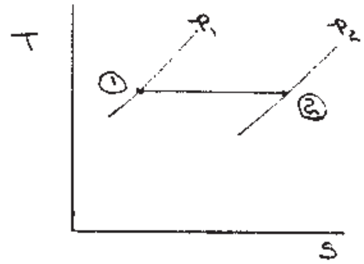
Then $f = 0.011$ and $\frac{f L_{max}}{D} = 774$ $M_1 = 0.0205$ and $V_1 = 42.1\ \mathrm{ft/sec}$

$$\dot{m} = \rho_1 V_1 A = 0.332\ \frac{\mathrm{lbm}}{\mathrm{ft}^3} \times 42.1\ \frac{\mathrm{ft}}{\mathrm{sec}} \times \frac{\pi}{4}(3)^2\ \mathrm{ft}^2 = 98.8\ \mathrm{lbm/sec}$$

$$\rho_{atm} = \frac{p_{atm}}{RT_1} = 14.7\ \frac{\mathrm{lbf}}{\mathrm{in}^2} \times \frac{\mathrm{lbm \cdot R}}{85.8\ \mathrm{ft \cdot lbf}} \times \frac{1}{530\mathrm{R}} \times \frac{144\ \mathrm{in}^2}{\mathrm{ft}^2} = 4.65 \times 10^{-2}\ \mathrm{lbm/ft^3}$$

Volume flowrate at atmospheric pressure

$$Q = \frac{\dot{m}}{\rho_{atm}} = 98.8\ \frac{\mathrm{lbm}}{\mathrm{sec}} \times \frac{\mathrm{ft}^3}{4.65 \times 10^{-2}\ \mathrm{lbm}} \times \frac{3600\ \mathrm{sec}}{\mathrm{hr}} \times \frac{24\ \mathrm{hr}}{\mathrm{day}} = 1.84 \times 10^{8}\ \mathrm{ft^3/day} \leftarrow Q$$



Problem 13.153 [Difficulty: 4]

13.153 Air from a large reservoir at 25 psia and 250°F flows isentropically through a converging nozzle into a frictionless pipe at 24 psia. The flow is heated as it flows along the pipe. Obtain a plot of the *Ts* diagram for this flow, until $M = 1$. Also plot the pressure and speed distributions from the entrance to the location at which $M = 1$.

Given: Air flow from converging nozzle into heated pipe

Find: Plot Ts diagram and pressure and speed curves

Solution:

The given or available data is:

$R =$	53.33	ft·lbf/lbm·°R
$k =$	1.4	
$c_p =$	0.2399	Btu/lbm·°R
	187	ft·lbf/lbm·°R
$T_0 =$	710	°R
$p_0 =$	25	psi
$p_e =$	24	psi

Equations and Computations:

From p_0 and p_e, and Eq. 13.7a
(using built-in function *IsenMfromp* (M,k)) $M_e =$ 0.242

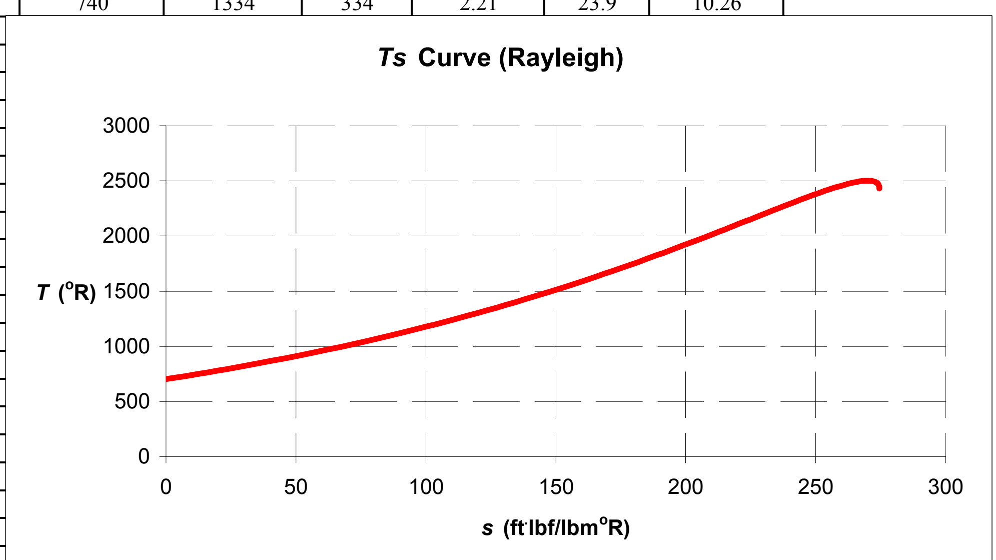
Using built-in function *IsenT* (M,k) $T_e =$ 702 °R

Using p_e, M_e, and function *Rayp* (M,k) $p^* =$ 10.82 psi

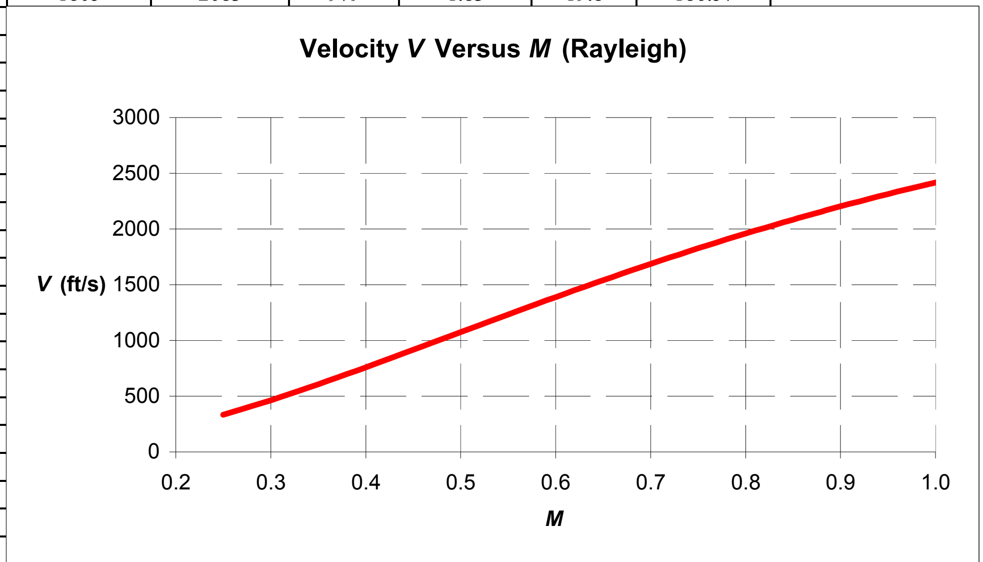
Using T_e, M_e, and function *RayT* (M,k) $T^* =$ 2432 °R

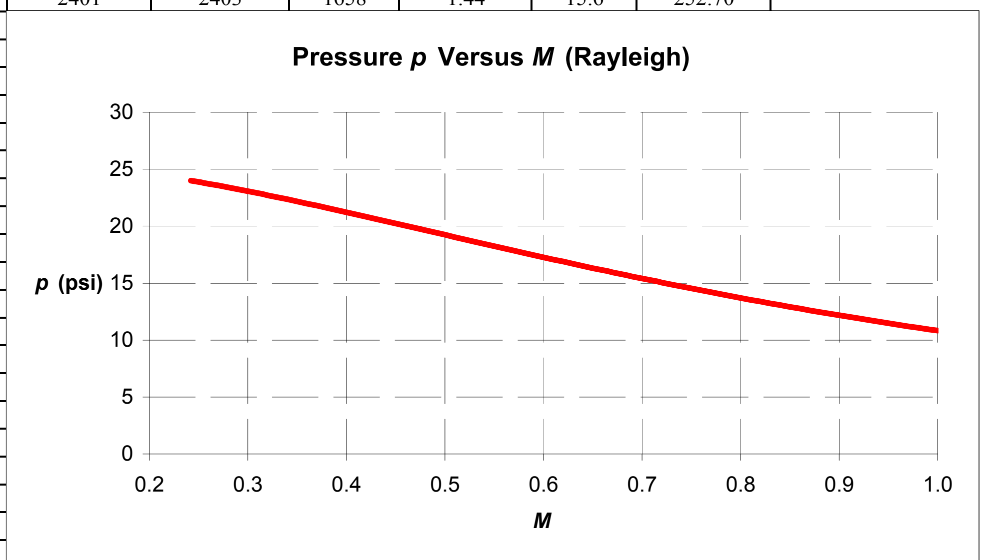
We can now use Rayleigh-line relations to compute values for a range of Mach numbers:

M	T/T^*	T (°R)	c (ft/s)	V (ft/s)	p/p^*	p (psi)	Δs (ft·lbf/lbm·°R) Eq. (12.11b)
0.242	0.289	702	1299	315	2.22	24.0	0.00
0.25	0.304	740	1334	334	2.21	23.9	10.26
0.26	0.325						
0.27	0.346						
0.28	0.367						
0.29	0.388						
0.3	0.409						
0.31	0.430						
0.32	0.451						
0.33	0.472						
0.34	0.493						
0.35	0.514						
0.36	0.535						
0.37	0.555						
0.38	0.576						
0.39	0.595						
0.4	0.615						
0.41	0.634						
0.42	0.653						
0.43	0.672						
0.44	0.690						
0.45	0.708						
0.46	0.725	1764	2059	947	1.85	20.0	181.73



0.47	0.742	1805	2083	979	1.83	19.8	186.57
0.48	0.759						
0.49	0.775						
0.5	0.790						
0.51	0.805						
0.52	0.820						
0.53	0.834						
0.54	0.847						
0.55	0.860						
0.56	0.872						
0.57	0.884						
0.58	0.896						
0.59	0.906						
0.6	0.917						
0.61	0.927						
0.62	0.936						
0.63	0.945						
0.64	0.953						
0.65	0.961						
0.66	0.968						
0.67	0.975						
0.68	0.981	2387	2396	1629	1.46	15.8	250.96
0.69	0.987	2401	2403	1658	1.44	15.6	252.70
0.7	0.993						
0.71	0.998						
0.72	1.003						
0.73	1.007						
0.74	1.011						
0.75	1.014						
0.76	1.017						
0.77	1.020						
0.78	1.022						
0.79	1.024						
0.8	1.025						
0.81	1.027						
0.82	1.028						
0.83	1.028						
0.84	1.029						
0.85	1.029						
0.86	1.028						
0.87	1.028						
0.88	1.027						
0.89	1.026						
0.9	1.025	2492	2448	2203	1.12	12.2	272.78
0.91	1.023	2488	2446	2226	1.11	12.0	273.13
0.92	1.021	2484	2444	2248	1.10	11.9	273.43
0.93	1.019	2479	2441	2270	1.09	11.7	273.70
0.94	1.017	2474	2439	2292	1.07	11.6	273.92
0.95	1.015	2468	2436	2314	1.06	11.5	274.11
0.96	1.012	2461	2433	2335	1.05	11.3	274.26
0.97	1.009	2455	2429	2356	1.04	11.2	274.38
0.98	1.006	2448	2426	2377	1.02	11.1	274.46
0.99	1.003	2440	2422	2398	1.01	10.9	274.51
1	1.000	2432	2418	2418	1.00	10.8	274.52





Problem 13.154

[Difficulty: 2]

13.154 Air enters a constant-area duct with $M_1 = 3.0$ and $T_1 = 250$ K. Heat transfer decreases the outlet Mach number to $M_2 = 1.60$. Compute the exit static and stagnation temperatures, and find the magnitude and direction of the heat transfer.

Given: Air flow through a duct with heat transfer

Find: Exit static and stagnation temperatures; magnitude and direction of heat transfer

Solution:

The given or available data is:

$R =$	286.9	J/kg-K
$c_p =$	1004	J/kg-K
$k =$	1.4	
$M_1 =$	3	
$T_1 =$	250	K
$M_2 =$	1.6	

Equations and Computations:

We can determine the stagnation temperature at the entrance:

$T_{01}/T_1 =$ 2.8000

So the entrance stagnation temperature is:

$T_{01} =$ 700.00 K

The reference stagnation temperature for Rayliegh flow can be calculated:

$T_{01}/T_0^* =$ 0.6540

$T_0^* =$ 1070.4 K

Since the reference state is the same at stations 1 and 2, state 2 is:

$T_{02}/T_0^* =$ 0.8842

$T_{02} =$ 946 K

$T_{02}/T_2 =$ 1.5120

$T_2 =$ 626 K

The heat transfer is related to the change in stagnation temperature:

$q_{1\text{-}2} =$ 247 kJ/kg

Problem 13.155

[Difficulty: 4]

13.155 Repeat Problem 13.153 except the nozzle is now a converging-diverging nozzle delivering the air to the pipe at 2.5 psia.

Given: Air flow from converging-diverging nozzle into heated pipe

Find: Plot Ts diagram and pressure and speed curves

Solution:

The given or available data is:

R =	53.33	ft·lbf/lbm·°R
k =	1.4	
c_p =	0.2399	Btu/lbm·°R
	187	ft·lbf/lbm·°R
T_0 =	710	°R
p_0 =	25	psi
p_e =	2.5	psi

Equations and Computations:

From p_0 and p_e, and Eq. 13.7a (using built-in function *IsenMfromp* (M,k)) $\quad M_e$ = 2.16

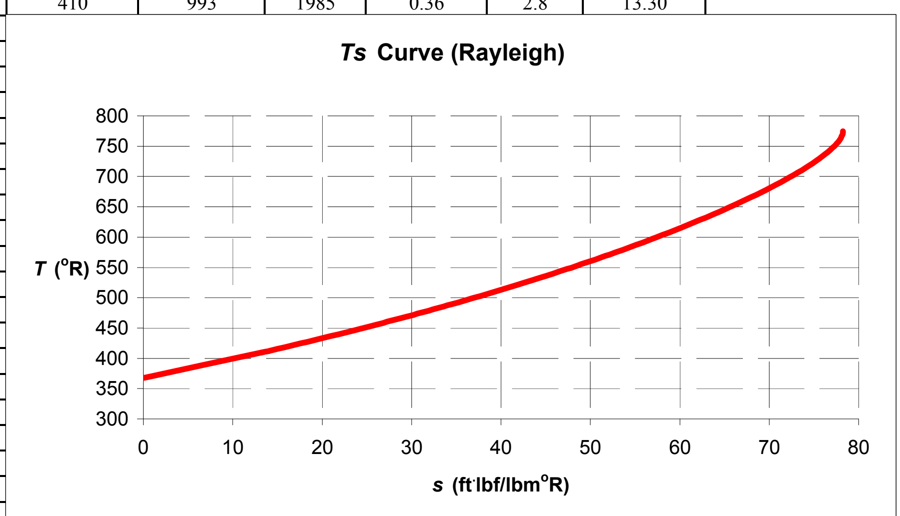
Using built-in function *IsenT* (M,k) $\quad T_e$ = 368 °R

Using p_e, M_e, and function *Rayp* (M,k) $\quad p^*$ = 7.83 psi

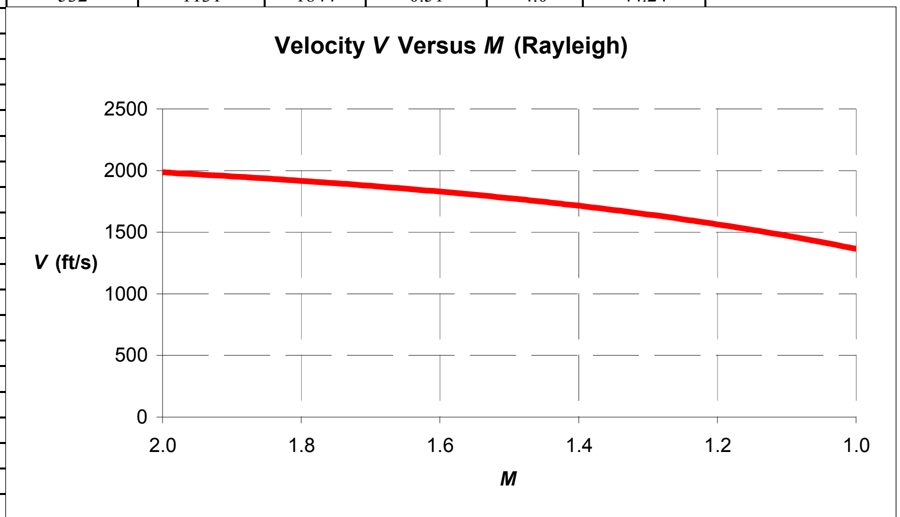
Using T_e, M_e, and function *RayT* (M,k) $\quad T^*$ = 775 °R

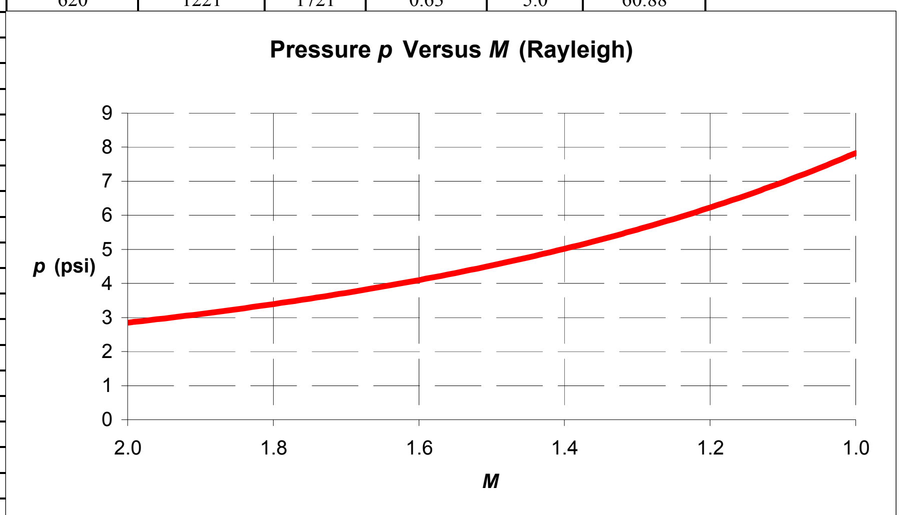
We can now use Rayleigh-line relations to compute values for a range of Mach numbers:

M	T/T^*	T (°R)	c (ft/s)	V (ft/s)	p/p^*	p (psi)	Δs (ft·lbf/lbm·°R) Eq. (12.11b)
2.157	0.475	368	940	2028	0.32	2.5	0.00
2	0.529	410	993	1985	0.36	2.8	13.30
1.99	0.533						
1.98	0.536						
1.97	0.540						
1.96	0.544						
1.95	0.548						
1.94	0.552						
1.93	0.555						
1.92	0.559						
1.91	0.563						
1.9	0.567						
1.89	0.571						
1.88	0.575						
1.87	0.579						
1.86	0.584						
1.85	0.588						
1.84	0.592						
1.83	0.596						
1.82	0.600						
1.81	0.605						
1.8	0.609						
1.79	0.613	475	1069	1913	0.44	3.4	31.06
1.78	0.618	479	1073	1909	0.44	3.5	31.90
1.77	0.622	482	1076	1905	0.45	3.5	32.73
1.76	0.626	485	1080	1901	0.45	3.5	33.57
1.75	0.631	489	1084	1897	0.45	3.6	34.40
1.74	0.635	492	1088	1893	0.46	3.6	35.23
1.73	0.640	496	1092	1889	0.46	3.6	36.06
1.72	0.645	499	1096	1885	0.47	3.7	36.89
1.71	0.649	503	1100	1880	0.47	3.7	37.72



1.7	0.654	507	1104	1876	0.48	3.7	38.54
1.69	0.658	510	1107	1872	0.48	3.8	39.36
1.68	0.663	514	1111	1867	0.48	3.8	40.18
1.67	0.668	517	1115	1863	0.49	3.8	41.00
1.66	0.673	521	1119	1858	0.49	3.9	41.81
1.65	0.677	525	1123	1853	0.50	3.9	42.62
1.64	0.682	529	1127	1849	0.50	3.9	43.43
1.63	0.687	532	1131	1844	0.51	4.0	44.24
1.62	0.692						
1.61	0.697						
1.6	0.702						
1.59	0.707						
1.58	0.712						
1.57	0.717						
1.56	0.722						
1.55	0.727						
1.54	0.732						
1.53	0.737						
1.52	0.742						
1.51	0.747						
1.5	0.753						
1.49	0.758						
1.48	0.763						
1.47	0.768						
1.46	0.773						
1.45	0.779						
1.44	0.784						
1.43	0.789						
1.42	0.795	616	1217	1728	0.63	4.9	60.18
1.41	0.800	620	1221	1721	0.63	5.0	60.88
1.4	0.805						
1.39	0.811						
1.38	0.816						
1.37	0.822						
1.36	0.827						
1.35	0.832						
1.34	0.838						
1.33	0.843						
1.32	0.848						
1.31	0.854						
1.3	0.859						
1.29	0.865						
1.28	0.870						
1.27	0.875						
1.26	0.881						
1.25	0.886						
1.24	0.891						
1.23	0.896						
1.22	0.902						
1.21	0.907						
1.2	0.912	706	1303	1564	0.80	6.2	73.21
1.19	0.917	710	1307	1555	0.80	6.3	73.65
1.18	0.922	714	1310	1546	0.81	6.4	74.08
1.17	0.927	718	1314	1537	0.82	6.4	74.50
1.16	0.932	722	1318	1528	0.83	6.5	74.89
1.15	0.937	726	1321	1519	0.84	6.6	75.27
1.14	0.942	730	1324	1510	0.85	6.7	75.63
1.13	0.946	733	1328	1500	0.86	6.7	75.96
1.12	0.951	737	1331	1491	0.87	6.8	76.28
1.11	0.956	741	1334	1481	0.88	6.9	76.58
1.1	0.960	744	1337	1471	0.89	7.0	76.86
1.09	0.965	747	1341	1461	0.90	7.1	77.11
1.08	0.969	751	1344	1451	0.91	7.1	77.34
1.07	0.973	754	1347	1441	0.92	7.2	77.55
1.06	0.978	757	1349	1430	0.93	7.3	77.73
1.05	0.982	761	1352	1420	0.94	7.4	77.88
1.04	0.986	764	1355	1409	0.95	7.5	78.01
1.03	0.989	767	1358	1398	0.97	7.6	78.12
1.02	0.993	769	1360	1387	0.98	7.6	78.19
1.01	0.997	772	1362	1376	0.99	7.7	78.24
1	1.000	775	1365	1365	1.00	7.8	78.25





Problem 13.156

[Difficulty: 2]

13.156 Consider frictionless flow of air in a constant-area duct. At section ①, $M_1 = 0.50$, $p_1 = 1.10$ MPa (abs), and $T_{0_1} = 333$ K. Through the effect of heat exchange, the Mach number at section ② is $M_2 = 0.90$ and the stagnation temperature is $T_{0_2} = 478$ K. Determine the amount of heat exchange per unit mass to or from the fluid between sections ① and ② and the pressure difference, $p_1 - p_2$.

Solution:

Basic equations: $h_1 + \frac{V_1^2}{2} + \frac{\delta Q}{dm} = h_2 + \frac{V_2^2}{2}$ $\qquad p_1A - p_2A = \dot{m}(V_2 - V_1)$

Computing equation: $T_0/T = 1 + \frac{k-1}{2}M^2$

Assumptions: (1) steady flow (5) $F_{B_x} = 0$
(2) frictionless flow (6) $\dot{W}_s = \dot{W}_{shear} = 0$
(3) uniform flow at a section (7) $\Delta z = 0$
(4) ideal gas

$$h_1 + \frac{V_1^2}{2} + \frac{\delta Q}{dm} = h_2 + \frac{V_2^2}{2} \qquad \frac{\delta Q}{dm} = h_{0_2} - h_{0_1} = c_p(T_{0_2} - T_{0_1}) = 10^3 \frac{J}{kg \cdot K}(478 - 333)K = 145 \frac{kJ}{kg}$$

$\frac{\delta Q}{dm}$

$$p_1A - p_2A = \dot{m}(V_2 - V_1) = \rho_1 V_1 A(V_2 - V_1) \qquad \text{and} \qquad p_1 - p_2 = \rho_1 V_1 (V_2 - V_1)$$

$$T_0/T = 1 + \frac{k-1}{2}M^2 \qquad T_1 = \frac{T_{0_1}}{1 + \frac{k-1}{2}M_1^2} = \frac{333\,K}{1 + 0.2(0.50)^2} = 317\,K$$

$$T_2 = \frac{T_{0_2}}{1 + \frac{k-1}{2}M_2^2} = \frac{478\,K}{1 + 0.2(0.90)^2} = 411\,K$$

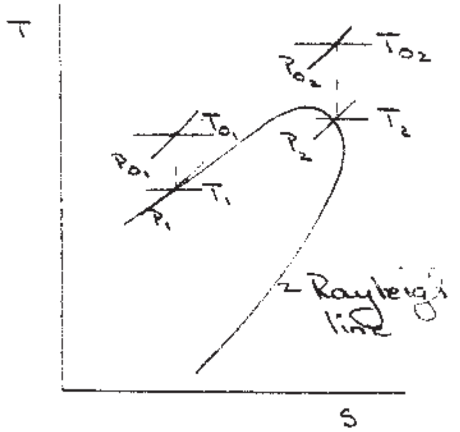
$$V_1 = M_1c_1 = M_1(kRT_1)^{1/2} = 0.50\left(1.4 \times 287 \frac{N \cdot m}{kg \cdot K} \times 317\,K \times \frac{kg \cdot m}{N \cdot s^2}\right)^{1/2} = 178\ m/s$$

$$V_2 = M_2c_2 = M_2(kRT_2)^{1/2} = 0.90\left(1.4 \times 287 \frac{N \cdot m}{kg \cdot K} \times 411\,K \times \frac{kg \cdot m}{N \cdot s^2}\right)^{1/2} = 366\ m/s$$

$$\rho_1 = \frac{p_1}{RT_1} = 1.10 \times 10^6 \frac{N}{m^2} \times \frac{kg \cdot K}{287\,N \cdot m} \times \frac{1}{317\,K} = 12.1\ kg/m^3$$

$$p_1 - p_2 = \rho_1 V_1 (V_2 - V_1) = 12.1 \frac{kg}{m^3} \times 178 \frac{m}{s} \times (366 - 178)\frac{m}{s} \times \frac{N \cdot s^2}{kg \cdot m} = 405\ kPa$$

$p_1 - p_2$



* Rayleigh-Line Flow Functions

From Appendix E.3

- for $M_1 = 0.5$, $p_1/p^* = 1.778$ (12.30a)
  $\therefore p^* = 619$ kPa
- for $M_2 = 0.90$, $p_2/p^* = 1.125$ (12.30a)
  $\therefore p_2 = 696$ kPa $\Rightarrow p_1 - p_2 = 404$ kPa.

Problem 13.157

[Difficulty: 2]

13.157 Air flows without friction through a short duct of constant area. At the duct entrance, $M_1 = 0.30$, $T_1 = 50°C$, and $\rho_1 = 2.16\ \text{kg/m}^3$. As a result of heating, the Mach number and pressure at the tube outlet are $M_2 = 0.60$ and $p_2 = 150$ kPa. Determine the heat addition per unit mass and the entropy change for the process.

Solution:

Basic equations: $h_1 + \frac{V_1^2}{2} + \frac{\delta Q}{dm} = h_2 + \frac{V_2^2}{2}$ $\qquad$ $Tds = dh - v\,dp$

Computing equations: $T_0/T = 1 + \frac{k-1}{2}M^2$

Assumptions: (1) steady flow $\qquad$ (4) ideal gas

(2) frictionless flow $\qquad$ (5) $\dot W_s = \dot W_{shear} = 0$

(3) uniform flow at a section $\qquad$ (6) $\Delta z = 0$

$$V_1 = M_1 c_1 = M_1 (kRT_1)^{1/2} = 0.3\left(1.4 \times 287\ \frac{N\cdot m}{kg\cdot K} \times 323K \times \frac{kg\cdot m}{N\cdot s^2}\right)^{1/2} = 108\ m/sec$$

$$p_1 = \rho_1 R T_1 = 2.16\ \frac{kg}{m^3} \times 287\ \frac{N\cdot m}{kg\cdot K} \times 323K = 200\ kPa$$

$$\rho_1 V_1 = \rho_2 V_2 \qquad \rho_2 = \frac{p_2}{RT_2} \qquad ; \quad V_2 = M_2 c_2 = M(kRT_2)^{1/2}$$

$$\therefore \rho_1 V_1 = \frac{p_2}{RT_2} M (kRT_2)^{1/2} = p_2 M_2 \left(\frac{k}{RT_2}\right)^{1/2} \quad . \text{ Solving for } T_2,$$

$$T_2 = \frac{k}{R}\left(\frac{p_2 M_2}{\rho_1 V_1}\right)^2 = 1.4 \times \frac{kg\cdot K}{287\ N\cdot m}\left[150\times 10^3\ \frac{N}{m^2} \times 0.60 \times \frac{m^3}{2.16\ kg} \times \frac{s}{108\ m} \times \frac{kg\cdot m}{N\cdot s^2}\right]^2 \times \frac{N\cdot s^2}{kg\cdot m}$$

$$T_2 = 726\ K.$$

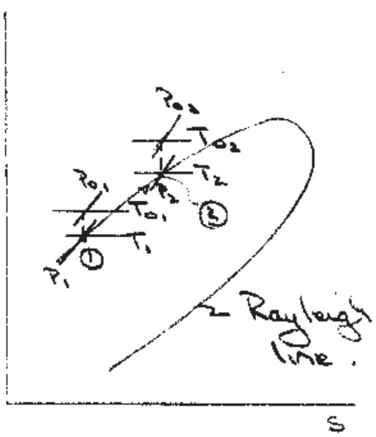
$$s_2 - s_1 = \int_1^2 ds = \int_{T_1}^{T_2} c_p \frac{dT}{T} - \int_{p_1}^{p_2} R\frac{dp}{p} = c_p \ln\frac{T_2}{T_1} - R\ln\frac{p_2}{p_1}$$

$$s_2 - s_1 = 1.0\ \frac{kJ}{kg\cdot K}\ln\frac{726}{323} - 0.287\ \frac{kJ}{kg\cdot K}\ln\frac{150}{200} = 0.892\ kJ/kg\cdot K$$ $\qquad s_2 - s_1$

$$T_{0_1} = T_1\left[1 + \frac{k-1}{2}M_1^2\right] = 323K\left[1 + 0.2(0.3)^2\right] = 329K$$

$$T_{0_2} = T_2\left[1 + \frac{k-1}{2}M_2^2\right] = 726K\left[1 + 0.2(0.6)^2\right] = 778K$$

$$\frac{\delta Q}{dm} = h_{0_2} - h_{0_1} = c_p(T_{0_2} - T_{0_1}) = 1.0\ \frac{kJ}{kg\cdot K}(778 - 329)K = 449\ kJ/kg$$ $\qquad \frac{\delta Q}{dm}$



* Rayleigh-Line Flow Functions (App. E.3)

For $M_1 = 0.30$ $\quad T_0/T_0^* = 0.3469 \quad \therefore T_0^* = 948K$

$T_1/T^* = 0.4089 \quad \therefore T^* = 790K$

$p_1/p^* = 2.131 \quad \therefore p^* = 93.9\ kPa$

For $M_2 = 0.60$ $\quad T_{0_2}/T_0^* = 0.8189 \quad \therefore T_{0_2} = 776K$

$T_2/T^* = 0.9167 \quad \therefore T_2 = 724K$

$p_2/p^* = 1.596 \quad \therefore p_2 = 150\ kPa$

Note: In using the flow functions, it is not necessary to know $p_2 = 150\ kPa$

Problem 13.158

[Difficulty: 3]

13.158 Air enters a 6-in.-diameter duct with a velocity of 300 ft/s. The entrance conditions area 14.7 psia and 200°F. How much heat must be added to the flow to yield (a) maximum static temperature at the exit, and (b) sonic flow at the exit?

Given: Air flow through a duct with heat transfer

Find: Heat addition needed to yield maximum static temperature and choked flow

Solution:

The given or available data is:

$R =$	53.33	ft-lbf/lbm-°R
$c_p =$	0.2399	Btu/lbm-°R
$k =$	1.4	
$D =$	6	in
$V_1 =$	300	ft/s
$p_1 =$	14.7	psia
$T_1 =$	200	°F
$T_1 =$	660	°R

Equations and Computations:

The sound speed at station 1 is:

$c_1 =$ 1259.65 ft/s

So the Mach number is:

$M_1 =$ 0.2382

We can determine the stagnation temperature at the entrance:

$T_{01}/T_1 =$ 1.0113

So the entrance stagnation temperature is:

$T_{01} =$ 667.49 °R

The reference stagnation temperature for Rayliegh flow can be calculated:

$T_{01}/T_0^* =$ 0.2363

$T_0^* =$ 2824.4 °R

For the maximum static temperature, the corresponding Mach number is:

$M_2 =$ 0.8452

Since the reference state is the same at stations 1 and 2, state 2 is:

$T_{02}/T_0^* =$ 0.9796

$T_{02} =$ 2767 °R

The heat transfer is related to the change in stagnation temperature:

$q_{1\text{-}2} =$ 504 Btu/lb

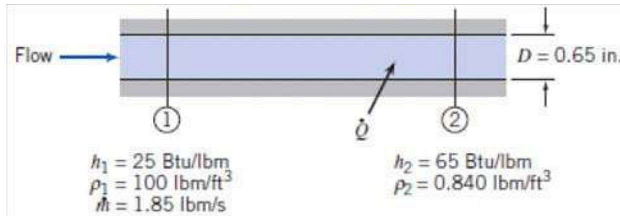
For acceleration to sonic flow the exit state is the * state:

$q_{1\text{-}*} =$ 517 Btu/lb

Problem 13.159

[Difficulty: 2]

13.159 Liquid Freon, used to cool electronic components, flows steadily into a horizontal tube of constant diameter, $D = 0.65$ in. Heat is transferred to the flow, and the liquid boils and leaves the tube as vapor. The effects of friction are negligible compared with the effects of heat addition. Flow conditions are shown. Find (a) the rate of heat transfer and (b) the pressure difference, $p_1 - p_2$.



Given: Frictionless flow of Freon in a tube

Find: Heat transfer; Pressure drop

NOTE: ρ_2 is NOT as stated; see below

Solution:

Basic equations: $m_{rate} = \rho \cdot V \cdot A$ $\quad p = \rho \cdot R \cdot T$ $\quad Q = m_{rate} \cdot \left(h_{02} - h_{01}\right)$ $\quad h_0 = h + \dfrac{V^2}{2}$ $\quad p_1 - p_2 = \rho_1 \cdot V_1 \cdot \left(V_2 - V_1\right)$

Given or available data $\quad h_1 = 25 \cdot \dfrac{BTU}{lbm}$ $\quad \rho_1 = 100 \cdot \dfrac{lbm}{ft^3}$ $\quad h_2 = 65 \cdot \dfrac{BTU}{lbm}$ $\quad \rho_2 = 0.850 \cdot \dfrac{lbm}{ft^3}$

$D = 0.65 \cdot in$ $\quad A = \dfrac{\pi}{4} \cdot D^2$ $\quad A = 0.332 \, in^2$ $\quad m_{rate} = 1.85 \cdot \dfrac{lbm}{s}$

Then $\quad V_1 = \dfrac{m_{rate}}{\rho_1 \cdot A}$ $\quad V_1 = 8.03 \cdot \dfrac{ft}{s}$ $\quad h_{01} = h_1 + \dfrac{V_1^{\,2}}{2}$ $\quad h_{01} = 25.0 \cdot \dfrac{BTU}{lbm}$

$V_2 = \dfrac{m_{rate}}{\rho_2 \cdot A}$ $\quad V_2 = 944 \cdot \dfrac{ft}{s}$ $\quad h_{02} = h_2 + \dfrac{V_2^{\,2}}{2}$ $\quad h_{02} = 82.8 \cdot \dfrac{BTU}{lbm}$

The heat transfer is $\quad Q = m_{rate} \cdot \left(h_{02} - h_{01}\right)$ $\quad Q = 107 \cdot \dfrac{BTU}{s}$ $\quad$ (74 Btu/s with the wrong ρ_2!)

The pressure drop is $\quad \Delta p = \rho_1 \cdot V_1 \cdot \left(V_2 - V_1\right)$ $\quad \Delta p = 162 \cdot psi$ $\quad$ (-1 psi with the wrong ρ_2!)

Problem 13.160

[Difficulty: 2]

13.160 Air flows through a 5-cm inside diameter pipe with negligible friction. Inlet conditions are $T_1 = 15°C$, $p_1 = 1$ MPa (abs), and $M_1 = 0.35$. Determine the heat exchange per kg of air required to produce $M_2 = 1.0$ at the pipe exit, where $p_2 = 500$ kPa.

Given: Frictionless air flow in a pipe

Find: Heat exchange per lb (or kg) at exit, where 500 kPa

Solution:

Basic equations: $m_{rate} = \rho \cdot V \cdot A$ $\quad p = \rho \cdot R \cdot T$ $\quad \frac{\delta Q}{dm} = c_p \cdot (T_{02} - T_{01})$ (Energy) $\quad p_1 - p_2 = \rho_1 \cdot V_1 \cdot (V_2 - V_1)$ (Momentum)

Given or available data $T_1 = (15 + 273) \cdot K$ $\quad p_1 = 1 \cdot MPa$ $\quad M_1 = 0.35$ $\quad p_2 = 500 \cdot kPa$ $\quad M_2 = 1$

$D = 5 \cdot cm$ $\quad k = 1.4$ $\quad c_p = 1004 \cdot \frac{J}{kg \cdot K}$ $\quad R = 286.9 \cdot \frac{J}{kg \cdot K}$

At section 1 $\quad \rho_1 = \frac{p_1}{R \cdot T_1}$ $\quad \rho_1 = 12.1 \frac{kg}{m^3}$ $\quad c_1 = \sqrt{k \cdot R \cdot T_1}$ $\quad c_1 = 340 \frac{m}{s}$

$V_1 = M_1 \cdot c_1$ $\quad V_1 = 119 \frac{m}{s}$

From momentum $\quad V_2 = \frac{p_1 - p_2}{\rho_1 \cdot V_1} + V_1$ $\quad V_2 = 466 \frac{m}{s}$

From continuity $\quad \rho_1 \cdot V_1 = \rho_2 \cdot V_2$ $\quad \rho_2 = \rho_1 \cdot \frac{V_1}{V_2}$ $\quad \rho_2 = 3.09 \frac{kg}{m^3}$

Hence $\quad T_2 = \frac{p_2}{\rho_2 \cdot R}$ $\quad T_2 = 564\,K$ $\quad T_2 = 291 \cdot °C$

and $\quad T_{02} = T_2 \cdot \left(1 + \frac{k-1}{2} \cdot M_2^{\,2}\right)$ $\quad T_{02} = 677\,K$ $\quad T_{02} = 403 \cdot °C$

with $\quad T_{01} = T_1 \cdot \left(1 + \frac{k-1}{2} \cdot M_1^{\,2}\right)$ $\quad T_{01} = 295\,K$ $\quad T_{01} = 21.9 \cdot °C$

Then $\quad \frac{\delta Q}{dm} = c_p \cdot (T_{02} - T_{01}) = 164 \cdot \frac{Btu}{lbm} = 383 \cdot \frac{kJ}{kg}$

(Note: Using Rayleigh line functions, for $M_1 = 0.3 \quad \frac{T_0}{T_{0crit}} = 0.4389$

so $\quad T_{0crit} = \frac{T_{01}}{0.4389}$ $\quad T_{0crit} = 672K$ close to T_2 ... Check!)

Problem 13.161

[Difficulty: 3]

13.161 Air flows at 1.42 kg/s through a 100-mm-diameter duct. At the inlet section, the temperature and absolute pressure are 52°C and 60.0 kPa. At the section downstream where the flow is choked, $T_2 = 45°C$. Determine the heat addition per unit mass, the entropy change, and the change in stagnation pressure for the process, assuming frictionless flow.

Solution:

Basic equations: $h_1 + \frac{V_1^2}{2} + \frac{\delta Q}{dm} = h_2 + \frac{V_2^2}{2}$ $\qquad Tds = dh - vdP$

Computing equations: $T_0/T = 1 + \frac{k-1}{2}M^2$ $\qquad P_0/p = \left[1 + \frac{k-1}{2}M^2\right]^{k/k-1}$

Assumptions: (1) steady flow (2) frictionless flow (3) uniform flow at a section (4) ideal gas (5) $\dot{W}_s = \dot{W}_{shear} = 0$ (6) $\Delta z = 0$

$A = \frac{\pi D^2}{4} = \frac{\pi}{4}(0.1)^2 m^2 = 7.85 \times 10^{-3} m^2$

$P = \rho RT \qquad \rho_1 = \frac{P_1}{RT_1} = 60 \times 10^3 \frac{N}{m^2} \times \frac{kg \cdot K}{287 N \cdot m} \times \frac{1}{325 K} = 0.643 \; kg/m^3$

$\dot{m} = \rho VA \qquad V_1 = \frac{\dot{m}}{\rho A} = 1.42 \frac{kg}{s} \times \frac{m^3}{0.643 \; kg} \times \frac{1}{7.85 \times 10^{-3} m^2} = 281 \; m/s$

$V_2 = M_2 c_2 = M_2 (kRT_2)^{1/2} = 1.0 \left(1.4 \times 287 \frac{N \cdot m}{kg \cdot K} \times 318 K \times \frac{kg \cdot m}{N \cdot s^2}\right)^{1/2} = 357 \; m/s$

$\dot{m} = \rho_1 V_1 A = \rho_2 V_2 A \; ; \quad \rho_2 = \frac{V_1}{V_2}\rho_1 = \frac{281}{357} \times 0.643 \; kg/m^3 = 0.506 \; kg/m^3$

$P = \rho RT \qquad P_2 = \rho_2 RT_2 = 0.506 \frac{kg}{m^3} \times 287 \frac{N \cdot m}{kg \cdot K} \times 318 K = 46.2 \; kPa$

$Tds = dh - vdP = c_p dT - \frac{1}{\rho}dP \qquad ds = c_p \frac{dT}{T} - R\frac{dP}{P}$

$s_2 - s_1 = \int_{s_1}^{s_2} ds = \int_{T_1}^{T_2} c_p \frac{dT}{T} - \int_{P_1}^{P_2} R\frac{dP}{P} = c_p \ln\frac{T_2}{T_1} - R\ln\frac{P_2}{P_1} = 1.0 \frac{kJ}{kg \cdot K} \ln\frac{318}{325} - 0.287 \frac{kJ}{kg \cdot K} \ln\frac{46.2}{60}$

$s_2 - s_1 = 0.0532 \; kJ/kg \cdot K$ $\qquad s_2 - s_1$

$M_1 = \frac{V_1}{c_1} \qquad c_1 = (kRT_1)^{1/2} = \left(1.4 \times 287 \frac{N \cdot m}{kg \cdot K} \times 325 K \times \frac{kg \cdot m}{N \cdot s^2}\right)^{1/2} = 361 \; m/s \; , \quad M_1 = \frac{V_1}{c_1} = \frac{281}{361} = 0.778$

$T_0/T = 1 + \frac{k-1}{2}M^2 \qquad T_{0_1} = T_1\left[1 + \frac{k-1}{2}M_1^2\right] = 325 K\left[1 + 0.2(0.778)^2\right] = 364 K$

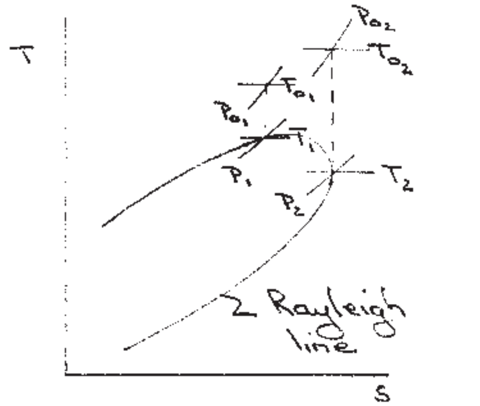
$T_{0_2} = T_2\left[1 + \frac{k-1}{2}M_2^2\right] = 318 K\left[1 + 0.2(1.0)^2\right] = 382 K$

$h_1 + \frac{V_1^2}{2} + \frac{\delta Q}{dm} = h_2 + \frac{V_2^2}{2} \qquad \frac{\delta Q}{dm} = h_{0_2} - h_{0_1} = c_p(T_{0_2} - T_{0_1}) = 1.0 \frac{kJ}{kg \cdot K}(382 - 364)K = 18 \; kJ/kg$ $\qquad \delta Q/dm$

$P_0/P = \left[1 + \frac{k-1}{2}M^2\right]^{k/k-1} \qquad P_{0_1} = P_1\left[1 + \frac{k-1}{2}M_1^2\right]^{k/k-1} = 60 \; kPa\left[1 + 0.2(0.778)^2\right]^{3.5} = 89.5 \; kPa$

$P_{0_2} = P_2\left[1 + \frac{k-1}{2}M_2^2\right]^{k/k-1} = 46.2 \; kPa\left[1 + 0.2(1.0)^2\right]^{3.5} = 87.5 \; kPa$

$P_{0_1} - P_{0_2} = (89.5 - 87.5)\; kPa = 2.0 \; kPa$ $\qquad P_{0_1} - P_{0_2}$



* Rayleigh-Line Flow Functions
From Appendix E.3 for $M_1 = 0.778$

$T_{0_1}/T_0^* = 0.954$ (12.30d) $\therefore T_0^* = T_{0_2} = 381 K$

$P_{0_1}/P_0^* = 1.024$ (12.30e) $\therefore P_0^* = P_{0_2} = 87.4 \; kPa$

$P_1/p^* = 1.299$ (12.30a) $\therefore p^* = P_2 = 46.2 \; kPa$

Problem 13.162

[Difficulty: 3]

13.162 Consider frictionless flow of air in a duct of constant area, $A = 0.087\ \text{ft}^2$. At one section, the static properties are 500°R and 15.0 psia and the Mach number is 0.2. At a section downstream, the static pressure is 10.0 psia. Draw a Ts diagram showing the static and stagnation states. Calculate the flow speed and temperature at the downstream location. Evaluate the rate of heat exchange for the process.

Solution:

Basic equations: $h_1 + \frac{V_1^2}{2} + \frac{\delta Q}{dm} = h_2 + \frac{V_2^2}{2}$ $\qquad p_1A - p_2A = \dot{m}(V_2 - V_1)$

Computing equation: $\frac{T_0}{T} = 1 + \frac{k-1}{2}M^2$

Assumptions: (1) steady flow (2) uniform flow at a section (3) ideal gas (4) frictionless flow (5) $F_{B_x} = 0$ (6) $\dot{W}_s = \dot{W}_{shear} = 0$

$$\rho_1 = \frac{p_1}{RT_1} = 15\,\frac{\text{lbf}}{\text{in}^2} \times \frac{\text{lbm}\cdot°\text{R}}{53.3\ \text{ft}\cdot\text{lbf}} \times \frac{1}{500°\text{R}} \times \frac{144\ \text{in}^2}{\text{ft}^2} = 0.0811\ \text{lbm/ft}^3$$

$$V_1 = M_1c_1 = M_1(kRT_1)^{1/2} = 0.2\left[1.4 \times 53.3\,\frac{\text{ft}\cdot\text{lbf}}{\text{lbm}\cdot°\text{R}} \times 500°\text{R} \times 32.2\,\frac{\text{lbm}}{\text{slug}} \times \frac{\text{slug}\cdot\text{ft}}{\text{lbf}\cdot\text{s}^2}\right]^{1/2} = 219\ \text{ft/s}$$

From the momentum eq. $(p_1 - p_2)A = \dot{m}(V_2 - V_1) = \rho_1V_1A(V_2 - V_1)$

$$\therefore V_2 = \frac{(p_1 - p_2)}{\rho_1V_1} + V_1 = (15-10)\frac{\text{lbf}}{\text{in}^2} \times \frac{\text{ft}^3}{0.0811\ \text{lbm}} \times \frac{\text{s}}{219\ \text{ft}} \times \frac{144\ \text{in}^2}{\text{ft}^2} \times 32.2\,\frac{\text{lbm}}{\text{slug}} \times \frac{\text{slug}\cdot\text{ft}}{\text{lbf}\cdot\text{s}^2} + 219\ \text{ft/s}$$

$V_2 = 1520$ ft/s $\qquad V_2$

From continuity $\rho_2 = \frac{V_1}{V_2}\rho_1 = \frac{219}{1520} \times 0.0811\,\frac{\text{lbm}}{\text{ft}^3} = 0.0117\ \text{lbm/ft}^3$

$$T_2 = \frac{p_2}{\rho_2R} = 10\,\frac{\text{lbf}}{\text{in}^2} \times \frac{\text{ft}^3}{0.0117\ \text{lbm}} \times \frac{\text{lbm}\cdot°\text{R}}{53.3\ \text{ft}\cdot\text{lbf}} \times \frac{144\ \text{in}^2}{\text{ft}^2} = 2310°\text{R}$$ $\qquad T_2$

$$T_{0_1} = T_1\left(1 + \frac{k-1}{2}M_1^2\right) = 500\left[1 + 0.2(0.2)^2\right] = 504°\text{R}$$

$$c_2 = (kRT_2)^{1/2} = \left[1.4 \times 53.3\,\frac{\text{ft}\cdot\text{lbf}}{\text{lbm}\cdot°\text{R}} \times 2310°\text{R} \times 32.2\,\frac{\text{lbm}}{\text{slug}} \times \frac{\text{slug}\cdot\text{ft}}{\text{lbf}\cdot\text{s}^2}\right]^{1/2} = 2360\ \text{ft/s}$$

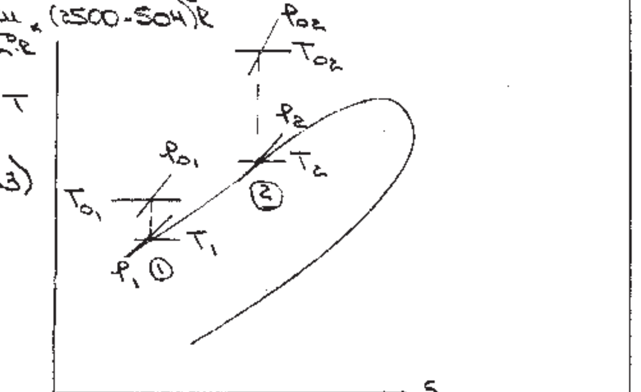
$M_2 = \frac{V_2}{c_2} = \frac{1520}{2360} = 0.644 \qquad T_{0_2} = T_2\left[1 + \frac{k-1}{2}M_2^2\right] = 2310°\text{R}\left[1 + 0.2(0.644)^2\right] = 2500°\text{R}$

Since $h_0 = h + \frac{V^2}{2}$, the energy eq. can be written as $\frac{\delta Q}{dm} = h_{0_2} - h_{0_1}$

$$\therefore \dot{Q} = \dot{m}\frac{\delta Q}{dm} = \dot{m}(h_{0_2} - h_{0_1}) = \dot{m}c_p(T_{0_2} - T_{0_1}) = \rho_1V_1Ac_p(T_{0_2} - T_{0_1})$$

$$\dot{Q} = 0.0811\,\frac{\text{lbm}}{\text{ft}^3} \times 219\,\frac{\text{ft}}{\text{s}} \times 0.087\ \text{ft}^2 \times 0.24\,\frac{\text{Btu}}{\text{lbm}\cdot°\text{R}} \times (2500 - 504)°\text{R}$$

$\dot{Q} = 740$ Btu/s



* Rayleigh-Line Flow Functions (App. E.3)

At $M_1 = 0.2$

$p_1/p^* = 2.273$ (12.30a) $\quad \therefore p^* = 6.60$ psia

$T_1/T^* = 0.2066$ (12.30b) $\quad \therefore T^* = 2420°\text{R}$

$T_{0_1}/T_0^* = 0.1736$ (12.30d) $\quad \therefore T_0^* = 2900°\text{R}$

At (2) $p_2/p^* = 1.515 \quad \therefore M_2 = 0.646$

Also $T_{0_2}/T_0^* = 0.8644$, $\therefore T_{0_2} = 2510°\text{R}$ and $T_2/T^* = 0.9408 \quad \therefore T_2 = 2280°\text{R}$

Problem 13.163

[Difficulty: 3]

13.163 Nitrogen flows through a frictionless duct. At the entrance of the duct, the conditions are $M_1 = 0.75$, $T_{0_1} = 500°$R, and $p_1 = 24$ psia. At the exit of the duct the pressure is $p_2 = 40$ psia. Determine the direction and the amount of the heat transfer with the nitrogen.

Given: Nitrogen flow through a duct with heat transfer

Find: Heat transfer

Solution:

The given or available data is:

$R =$	55.16	ft-lbf/lbm-°R
$c_p =$	0.2481	Btu/lbm-°R
$k =$	1.4	
$M_1 =$	0.75	
$T_{01} =$	500	°R
$p_1 =$	24	psia
$p_2 =$	40	psia

Equations and Computations:

We can find the pressure and stagnation temperature at the reference state:

$p_1/p^* =$ 1.3427

$T_{01}/T_0^* =$ 0.9401

So the reference pressure and stagnation temperature are:

$p^* =$ 17.875 psia

$T_0^* =$ 531.9 °R

We can now find the exit Mach number through the reference pressure:

$p_2/p^* =$ 2.2378

$M_2 =$ 0.2276

$p_2/p^* =$ 2.2378

(We used Solver to match the reference pressure ratio by varying M_2.)

Since the reference state is the same at stations 1 and 2, state 2 is:

$T_{02}/T_0^* =$ 0.2183

$T_{02} =$ 116 °R

The heat transfer is related to the change in stagnation temperature:

$q_{1\text{-}2} =$ -95.2 Btu/lb

(The negative number indicates heat loss from the nitrogen)

Problem 13.164

[Difficulty: 3]

13.164 A combustor from a JT8D jet engine (as used on the Douglas DC-9 aircraft) has an air flow rate of 15 lbm/s. The area is constant and frictional effects are negligible. Properties at the combustor inlet are 1260°R, 235 psia, and 609 ft/s. At the combustor outlet, T = 1840°R and M = 0.476. The heating value of the fuel is 18,000 Btu/lbm; the air-fuel ratio is large enough so properties are those of air. Calculate the pressure at the combustor outlet. Determine the rate of energy addition to the air stream. Find the mass flow rate of fuel required; compare it to the air flow rate. Show the process on a Ts diagram, indicating static and stagnation states and the process path.

Solution:

Basic equations: $h_1 + \frac{V_1^2}{2} + \frac{\delta Q}{dm} = h_2 + \frac{V_2^2}{2}$ $\qquad p_1A - p_2A = \dot{m}(V_2 - V_1)$

Computing equation: $\frac{T_0}{T} = 1 + \frac{k-1}{2}M^2$

Assumptions: (1) steady flow (2) frictionless flow
(3) ideal gas, properties are those of air
(4) uniform flow at a section
(5) $W_s = W_{shear} = 0$

$\rho_1 = \frac{p_1}{RT_1} = 235\frac{lbf}{in^2} \times \frac{lbm\,°R}{53.3\,ft\cdot lbf} \times \frac{1}{1260°R} \times \frac{144\,in^2}{ft^2} = 0.504\ lbm/ft^3$

$c_1 = (kRT_1)^{1/2} = \left[1.4 \times 53.3\frac{ft\cdot lbf}{lbm\cdot R} \times 1260R \times 32.2\frac{lbm}{slug} \times \frac{slug\cdot ft}{lbf\cdot s^2}\right]^{1/2} = 1740\ ft/s$

$M_1 = V_1/c_1 = 609/1740 = 0.350 \qquad T_{01} = T_1\left[1 + \frac{k-1}{2}M_1^2\right] = 1260°R\left[1 + 0.2(0.35)^2\right] = 1290°R$

$T_{02} = T_2\left[1 + \frac{k-1}{2}M_2^2\right] = 1840\left[1 + 0.2(0.476)^2\right] = 1920°R$

From energy eq. $\dot{Q} = \dot{m}(h_{02} - h_{01}) = \dot{m}c_p(T_{02} - T_{01}) = 15\frac{lbm}{s} \times 0.24\frac{Btu}{lbm\cdot R} \times (1920-1290)R$

$\dot{Q} = 2270\ Btu/s$ $\qquad \dot{Q}$

But $\dot{Q} = \dot{m}_{fuel}h_{fuel} \quad \therefore \dot{m}_f = \frac{\dot{Q}}{h_f} = 2270\frac{Btu}{s} \times \frac{lbm}{18{,}000\,Btu} = 0.126\ lbm/s$ $\qquad \dot{m}_f$

$\dot{m}_f/\dot{m}_{air} = 0.126/15 = 0.0084$ $\qquad \dot{m}_f/\dot{m}_{air}$

$V_2 = M_2c_2 = M_2(kRT_2)^{1/2} = 0.476\left[1.4 \times 53.3\frac{ft\cdot lbf}{lbm\cdot R} \times 1840R \times 32.2\frac{lbm}{slug} \times \frac{slug\cdot ft}{lbf\cdot s^2}\right]^{1/2} = 1000\ ft/s$

From momentum

$p_2 = p_1 - \frac{\dot{m}}{A}(V_2 - V_1) = p_1 - \rho_1V_1(V_2 - V_1)$

$p_2 = 235\frac{lbf}{in^2} - 0.504\frac{lbm}{ft^3} \times 609\frac{ft}{s} \times (1000-609)\frac{ft}{s} \times \frac{slug}{32.2\,lbm} \times \frac{lbf\cdot s^2}{ft\cdot slug} \times \frac{ft^2}{144\,in^2} = 209\ psia$ $\qquad p_2$

* Rayleigh-Line Flow Functions (App. E.3)

For $M_1 = 0.35$,

$T_0/T_0^* = 0.4389$ (12.30d) $\therefore T_0^* = 2940°R$

$T_1/T^* = 0.5141$ (12.30b) $\therefore T^* = 2450°R$

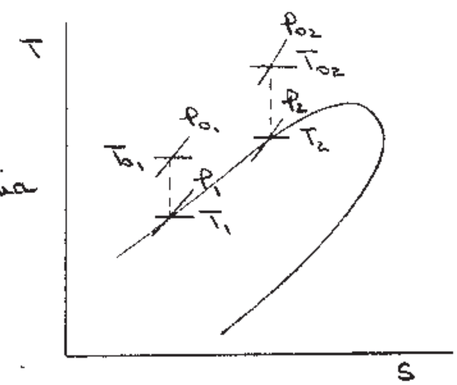
$p_1/p^* = 2.0487$ (12.30a) $\therefore p^* = 114.7$ psia

For $M_2 = 0.476$

$T_{02}/T_0^* = 0.6551 \quad \therefore T_{02} = 1930°R$

$T_2/T^* = 0.7522 \quad \therefore T_2 = 1840°R$

$p_2/p^* = 1.822 \quad \therefore p_2 = 209$ psia



Problem 13.165

[Difficulty: 3]

13.165 Consider frictionless flow of air in a duct with $D = 10\,\text{cm}$. At section ①, the temperature and pressure are 0°C and 70 kPa; the mass flow rate is 0.5 kg/s. How much heat may be added without choking the flow? Evaluate the resulting change in stagnation pressure.

Given: Frictionless flow of air in a duct

Find: Heat transfer without choking flow; change in stagnation pressure

Solution:

Basic equations:
$$\frac{T_0}{T} = 1 + \frac{k-1}{2}\cdot M^2 \qquad \frac{p_0}{p} = \left(1 + \frac{k-1}{2}\cdot M^2\right)^{\frac{k}{k-1}} \qquad p = \rho\cdot R\cdot T \qquad m_{rate} = \rho\cdot A\cdot V$$

$$p_1 - p_2 = \frac{m_{rate}}{A}\cdot\left(V_2 - V_1\right) \qquad \frac{\delta Q}{dm} = c_p\cdot\left(T_{02} - T_{01}\right)$$

Given or available data
$$T_1 = (0 + 273)\cdot K \qquad p_1 = 70\cdot kPa \qquad m_{rate} = 0.5\cdot\frac{kg}{s} \qquad D = 10\cdot cm$$

$$A = \frac{\pi}{4}\cdot D^2 \qquad A = 78.54\cdot cm^2 \qquad k = 1.4 \qquad M_2 = 1 \qquad c_p = 1004\cdot\frac{J}{kg\cdot K} \qquad R = 286.9\cdot\frac{J}{kg\cdot K}$$

At state 1
$$\rho_1 = \frac{p_1}{R\cdot T_1} \qquad \rho_1 = 0.894\,\frac{kg}{m^3} \qquad c_1 = \sqrt{k\cdot R\cdot T_1} \qquad c_1 = 331\,\frac{m}{s}$$

From continuity
$$V_1 = \frac{m_{rate}}{\rho_1\cdot A} \qquad V_1 = 71.2\,\frac{m}{s} \qquad \text{then} \qquad M_1 = \frac{V_1}{c_1} \qquad M_1 = 0.215$$

From momentum
$$p_1 - p_2 = \frac{m_{rate}}{A}\cdot\left(V_2 - V_1\right) = \rho_2\cdot V_2^2 - \rho_1\cdot V_1^2 \quad \text{but} \quad \rho\cdot V^2 = \rho\cdot c^2\cdot M^2 = \frac{p}{R\cdot T}\cdot k\cdot R\cdot T\cdot M^2 = k\cdot p\cdot M^2$$

Hence
$$p_1 - p_2 = k\cdot p_2\cdot M_2^2 - k\cdot p_1\cdot M_1^2 \qquad \text{or} \qquad p_2 = p_1\cdot\left(\frac{1 + k\cdot M_1^2}{1 + k\cdot M_2^2}\right) \qquad p_2 = 31.1\cdot kPa$$

From continuity
$$\rho_1\cdot V_1 = \frac{p_1}{R\cdot T_1}\cdot M_1\cdot c_1 = \frac{p_1}{R\cdot T_1}\cdot M_1\cdot\sqrt{k\cdot R\cdot T_1} = \sqrt{\frac{k}{R}}\cdot\frac{p_1\cdot M_1}{\sqrt{T_1}} = \rho_2\cdot V_2 = \sqrt{\frac{k}{R}}\cdot\frac{p_2\cdot M_2}{\sqrt{T_2}}$$

Hence
$$\frac{p_1\cdot M_1}{\sqrt{T_1}} = \frac{p_2\cdot M_2}{\sqrt{T_2}} \qquad T_2 = T_1\cdot\left(\frac{p_2}{p_1}\cdot\frac{M_2}{M_1}\right)^2 \qquad T_2 = 1161\,K \qquad T_2 = 888\cdot{}^\circ C$$

Then
$$T_{02} = T_2\cdot\left(1 + \frac{k-1}{2}\cdot M_2^2\right) \qquad T_{02} = 1394\,K \qquad T_{01} = T_1\cdot\left(1 + \frac{k-1}{2}\cdot M_1^2\right) \qquad T_{01} = 276\,K$$

$$p_{02} = p_2\cdot\left(1 + \frac{k-1}{2}\cdot M_2^2\right)^{\frac{k}{k-1}} \qquad p_{02} = 58.8\cdot kPa \qquad p_{01} = p_1\cdot\left(1 + \frac{k-1}{2}\cdot M_1^2\right)^{\frac{k}{k-1}} \qquad p_{01} = 72.3\cdot kPa$$

Finally
$$\frac{\delta Q}{dm} = c_p\cdot\left(T_{02} - T_{01}\right) = 1.12\cdot\frac{MJ}{kg} \qquad \Delta p_0 = p_{02} - p_{01} \qquad \Delta p_0 = -13.5\cdot kPa$$

(Using Rayleigh functions, at $M_1 = 0.215$ $\quad \frac{T_{01}}{T_{0crit}} = \frac{T_{01}}{T_{02}} = 0.1975 \quad T_{02} = \frac{T_{01}}{0.1975} \quad T_{02} = 1395\,K$ and ditto for p_{02} ...Check!)

Problem 13.166

[Difficulty: 3]

13.166 A constant-area duct is fed with air from a converging-diverging nozzle. At the entrance to the duct, the following properties are known: $p_{0_1} = 800$ kPa (abs), $T_{0_1} = 700$ K, and $M_1 = 3.0$. A short distance down the duct (at section ②) $p_2 = 46.4$ kPa. Assuming frictionless flow, determine the speed and Mach number at section ②, and the heat exchange between the inlet and section ②.

Solution:

Basic equations: $h_1 + \frac{V_1^2}{2} + \frac{\delta Q}{dm} = h_2 + \frac{V_2^2}{2}$ $\quad p_1A - p_2A = \dot{m}(V_2 - V_1)$

Computing equations: $T_0/T = 1 + \frac{k-1}{2}M^2$ $\quad p_0/p = \left[1 + \frac{k-1}{2}M^2\right]^{k/k-1}$

Assumptions:
(1) steady flow
(2) frictionless flow
(3) uniform flow at a section
(4) ideal gas
(5) $F_{B_x} = 0$
(6) $\dot{W}_s = \dot{W}_{shear} = 0$
(7) $\Delta z = 0$

$T_0/T = 1 + \frac{k-1}{2}M^2 \qquad T_1 = \frac{T_{0_1}}{1 + \frac{k-1}{2}M_1^2} = \frac{700\text{ K}}{1 + 0.2(3.0)^2} = 250\text{ K}$

$p_0/p = \left[1 + \frac{k-1}{2}M^2\right]^{k/k-1} \qquad p_1 = \frac{p_{0_1}}{\left[1 + \frac{k-1}{2}M_1^2\right]^{k/k-1}} = \frac{800\text{ kPa}}{\left[1 + 0.2(3.0)^2\right]^{3.5}} = 21.8\text{ kPa}$

$V_1 = M_1c_1 = M_1(kRT_1)^{1/2} = 3.0\left(1.4 \times 287\,\frac{N\cdot m}{kg\cdot K} \times 250\text{ K} \times \frac{kg\cdot m}{N\cdot s^2}\right)^{1/2} = 951\text{ m/s}$

$p = \rho RT \qquad \rho_1 = \frac{p_1}{RT_1} = 21.8 \times 10^3\,\frac{N}{m^2} \times \frac{kg\cdot K}{287\,N\cdot m} \times \frac{1}{250\text{ K}} = 0.304\text{ kg/m}^3$

$(p_1 - p_2)A = \rho_1V_1A(V_2 - V_1) \quad \therefore \quad V_2 = V_1 + \frac{(p_1 - p_2)}{\rho_1V_1}$. Solving for V_2,

$V_2 = 951\text{ m/s} + (21.8 - 46.4)10^3\,\frac{N}{m^2} \times \frac{m^3}{0.304\text{ kg}} \times \frac{s}{951\text{ m}} \times \frac{kg\cdot m}{N\cdot s^2} = 866\text{ m/s}$ $\quad V_2$

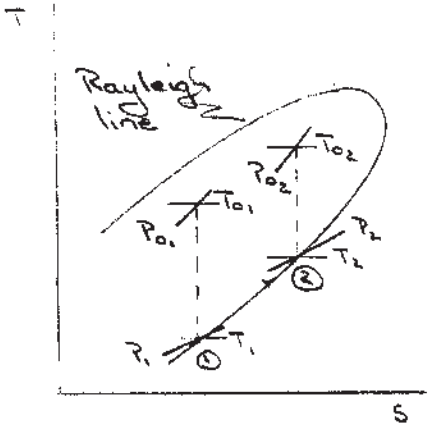
From continuity, $\rho_2 = \rho_1\frac{V_1}{V_2} = 0.304\,\frac{kg}{m^3} \times \frac{951}{866} = 0.334\text{ kg/m}^3$

$T_2 = \frac{p_2}{\rho_2R} = 46.4 \times 10^3\,\frac{N}{m^2} \times \frac{m^3}{0.334\text{ kg}} \times \frac{kg\cdot K}{287\,N\cdot m} = 484\text{ K}$

$c_2 = (kRT_2)^{1/2} = \left[1.4 \times 287\,\frac{N\cdot m}{kg\cdot K} \times 484\text{ K} \times \frac{kg\cdot m}{N\cdot s^2}\right]^{1/2} = 441\text{ m/s}\,; \quad M_2 = \frac{V_2}{c_2} = \frac{866}{441} = 1.96$ $\quad M_2$

$T_{0_2} = T_2\left[1 + \frac{k-1}{2}M_2^2\right] = 484\left[1 + 0.2(1.96)^2\right] = 856\text{ K}$

$\delta Q/dm = h_{0_2} - h_{0_1} = c_p(T_{0_2} - T_{0_1}) = 1\,\frac{kJ}{kg\cdot K}(856\text{ K} - 700\text{ K}) = 156\text{ kJ/kg}$ $\quad \frac{\delta Q}{dm}$



• Rayleigh-Line Flow Functions (Appendix E.3)

For $M_1 = 3.0$, $T_{0_1}/T_0^* = 0.6540$, $\therefore T_0^* = 1070\text{ K}$

$T_1/T^* = 0.2803 \quad \therefore T^* = 892\text{ K}$

$p_1/p^* = 0.1765 \quad \therefore p^* = 124\text{ kPa}$

$V_1/V^* = 1.588 \quad \therefore V^* = 599\text{ m/s}$

At section ② for $p_2/p^* = 0.3742$

$M_2 = 1.966$ ✓

$T_{0_2}/T_0^* = 0.800 \quad \therefore T_{0_2} = 856\text{ K}$ ✓

$T_2/T^* = 0.542 \quad \therefore T_2 = 483\text{ K}$

$V_2/V^* = 1.447 \quad \therefore V_2 = 867\text{ m/s}$

Problem 13.167

[Difficulty: 2]

13.167 Air flows steadily and without friction at 1.83 kg/s through a duct with cross-sectional area of 0.02 m². At the duct inlet, the temperature and absolute pressure are 260°C and 126 kPa. The exit flow discharges subsonically to atmospheric pressure. Determine the Mach number, temperature, and stagnation temperature at the duct outlet and the heat exchange rate.

Solution:

* Compressible flow functions (Appendix E) to be used in solution

Basic equation: $h_1 + \frac{V_1^2}{2} + \frac{\delta Q}{dm} = h_2 + \frac{V_2^2}{2}$

Assumptions:
1) steady flow
2) frictionless flow
3) uniform flow at a section
4) ideal gas
5) $F_{Bx} = 0$
6) $\dot{W}_s = \dot{W}_{shear} = 0$
7) $\Delta z = 0$

$\rho_1 = \frac{p_1}{RT_1} = 126 \times 10^3 \frac{N}{m^2} \times \frac{kg \cdot K}{287\, N \cdot m} \times \frac{1}{533\, K} = 0.8237\ kg/m^3$

$\dot{m} = \rho_1 V_1 A \qquad V_1 = \frac{\dot{m}}{\rho_1 A} = 1.83 \frac{kg}{s} \times \frac{m^3}{0.8237\, kg} \times \frac{1}{0.02\, m^2} = 111\ m/s$

$M_1 = \frac{V_1}{c_1} \qquad c_1 = (kRT_1)^{1/2} = \left(1.4 \times 287 \frac{N \cdot m}{kg \cdot K} \times 533\, K \times \frac{kg \cdot m}{N \cdot s^2}\right)^{1/2} = 463\ m/s$

$M_1 = \frac{111}{463} = 0.240$

From App. E.1, $T_{0_1}/T_1 = 1.012$ (11.17b) $\therefore T_{0_1} = 539\ K$

From App E.3, $T_{0_1}/T_0^* = 0.2395$ (12.30d), $T_1/T^* = 0.2841$ (12.30b), $p_1/p^* = 2.221$ (12.30a)

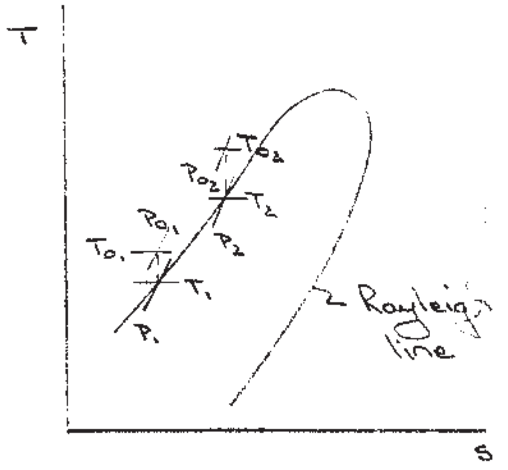
$\therefore T_0^* = 2250\,K \qquad T^* = 1876\,K \qquad p^* = 56.73\ kPa$

At section ② $\frac{p_2}{p^*} = \frac{101}{56.73} = 1.780$ From App E.3 (Eq. 12.30a), $M = 0.50$ $\qquad M_2$

Also, $T_{02}/T_0^* = 0.6914$ (12.30d), $T_2/T^* = 0.7901$. Therefore: $T_{0_2} = 1556\,K$, $T_2 = 1480\,K$ $\qquad T_{0_2}, T_2$

$\dot{Q} = \dot{m}\frac{\delta Q}{dm} = \dot{m}(h_{0_2} - h_{0_1}) = \dot{m}c_p(T_{0_2} - T_{0_1}) = 1.83 \frac{kg}{s} \times 1.0 \frac{kJ}{kg \cdot K} \times (1556 - 539)\,K$

$\dot{Q} = 1.86\ MJ/s \qquad \dot{Q}$



Problem 13.168

[Difficulty: 3]

13.168 20 kg/s of air enters a 0.06 m^2 duct at a pressure of 320 kPa, and a temperature of 350 K. Find the exit conditions (pressure, temperature, and Mach number) if heat is added to the duct at a rate of 650 kJ/kg of air.

Given: Air flow through a duct with heat transfer

Find: Exit conditions

Solution:

The given or available data is:

$R =$	286.9	J/kg-K
$c_p =$	1004	J/kg-K
$k =$	1.4	
$m =$	20	kg/s
$A =$	0.06	m^2
$p_1 =$	320	kPa
$T_1 =$	350	K
$q_{1\text{-}2} =$	650	kJ/kg

Equations and Computations:

The density at the entrance is:

$\rho_1 =$ 3.1868 kg/m^3

So the entrance velocity is:

$V_1 =$ 104.5990 m/s

The sonic velocity is:

$c_1 =$ 374.9413 m/s

So the Mach number is:

$M_1 =$ 0.2790

We can determine the stagnation temperature at the entrance:

$T_{01}/T_1 =$ 1.0156

So the entrance stagnation temperature is:

$T_{01} =$ 355.45 K

The reference conditions for Rayliegh flow can be calculated:

$T_{01}/T_0^* =$ 0.3085

$T_0^* =$ 1152.2 K

$T_1/T^* =$ 0.3645

$T^* =$ 960.2 K

$p_1/p^* =$ 2.1642

$p^* =$ 147.9 kPa

The heat transfer is related to the change in stagnation temperature:

$T_{02} =$ 1002.86 K

The stagnation temperature ratio at state 2 is:

$T_{02}/T_0^* =$ 0.8704

We can now find the exit Mach number:

$M_2 =$ 0.652

$T_{02}/T_0^* =$ 0.8704

(We used Solver to match the reference pressure ratio by varying M_2.)

We can now calculate the exit temperature and pressure:

$T_2/T^* =$ 0.9625

$T_2 =$ 924 K

$p_2/p^* =$ 1.5040

$T_2 =$ 222 kPa

Problem 13.169

[Difficulty: 3]

13.169 Air enters a frictionless, constant-area duct with $p_1 = 135$ kPa, $T_1 = 500$ K, and $V_1 = 540$ m/s. How much heat transfer is needed to choke the flow? Is the heat transfer into or out of the duct?

Given: Air flow through a duct with heat transfer

Find: Heat transfer needed to choke the flow

Solution:

The given or available data is:

$R =$	286.9	J/kg-K
$c_p =$	1004	J/kg-K
$k =$	1.4	
$p_1 =$	135	kPa
$T_1 =$	500	K
$V_1 =$	540	m/s

Equations and Computations:

The sonic velocity at state 1 is:

$c_1 =$ 448.1406 m/s

So the Mach number is:

$M_1 =$ 1.2050

We can determine the stagnation temperature at the entrance:

$T_{01}/T_1 =$ 1.2904

So the entrance stagnation temperature is:

$T_{01} =$ 645.20 K

The reference conditions for Rayliegh flow can be calculated:

$T_{01}/T_0^* =$ 0.9778

$T_0^* =$ 659.9 K

Since the flow is choked, state 2 is:

$M_2 =$ 1.000

$T_{02} =$ 659.85 K

The heat transfer is related to the change in stagnation temperature:

$q_{1\text{-}2} =$ 14.71 kJ/kg

To choke a flow, heat must *always* be *added* .

Problem 13.170

[Difficulty: 2]

13.170 In the frictionless flow of air through a 100-mm-diameter duct, 1.42 kg/s enters at 52°C and 60.0 kPa (abs). Determine the amount of heat that must be added to choke the flow, and the fluid properties at the choked state.

Solution:

* Compressible flow functions (Appendix E) to be used in solution

Basic equation: $h_1 + \frac{V_1^2}{2} + \frac{\delta Q}{dm} = h_2 + \frac{V_2^2}{2}$

Assumptions:
(1) steady flow
(2) frictionless flow
(3) uniform flow at a section
(4) ideal gas
(5) $\dot{W}_s = \dot{W}_{shear} = 0$
(6) $\Delta z = 0$

$A = \frac{\pi D^2}{4} = \frac{\pi}{4}(0.1)^2\ m^2 = 7.85 \times 10^{-3}\ m^2$

$P = \rho R T \qquad \rho_1 = \frac{P_1}{RT_1} = 60 \times 10^3 \frac{N}{m^2} \times \frac{kg \cdot K}{287\ N \cdot m} \times \frac{1}{325\ K} = 0.643\ kg/m^3$

$\dot{m} = \rho V A \qquad V_1 = \frac{\dot{m}}{\rho A} = 1.42 \frac{kg}{s} \div 0.643 \frac{kg}{m^3} \times \frac{1}{7.85 \times 10^{-3}\ m^2} = 281\ m/s$

$M_1 = \frac{V_1}{c_1} \qquad c_1 = (kRT_1)^{1/2} = \left(1.4 \times 287 \frac{N \cdot m}{kg \cdot K} \times 325\ K \times \frac{kg \cdot m}{N \cdot s^2}\right)^{1/2} = 362\ m/s$

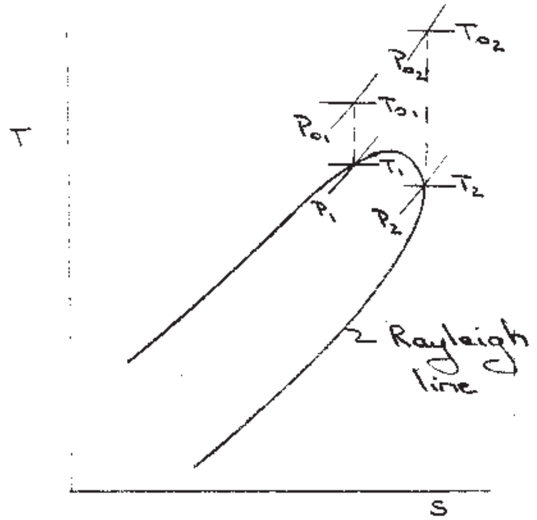
$M_1 = \frac{V_1}{c_1} = \frac{281}{362} = 0.776$

From App E.1, $T_1/T_{0_1} = 0.8925 \quad P_1/P_{0_1} = 0.6717 \quad \therefore T_{0_1} = 364\ K,\ P_{0_1} = 89.3\ kPa$

From App E.3, $\frac{T_{0_1}}{T_0^*} = 0.9535 \quad \frac{P_{0_1}}{P_0^*} = 1.024 \quad \frac{T_1}{T^*} = 1.022 \quad \frac{P_1}{P^*} = 1.302 \quad \frac{V}{V^*} = \frac{\rho^*}{\rho} = 0.7844$

$\therefore T_{0_2} = 382\ K,\ P_{0_2} = 87.2\ kPa,\ T_2 = 318\ K,\ P_2 = 46.1\ kPa\ \ \rho_2 = 0.504 \frac{kg}{m^3},\ V_2 = 358 \frac{m}{s}$ ← $()_2$

$\frac{\delta Q}{dm} = h_{0_2} - h_{0_1} = c_p(T_{0_2} - T_{0_1}) = 1.0 \frac{kJ}{kg \cdot K}(382 - 364)K = 18.0\ kJ/kg$ ← $\delta Q/dm$



Problem 13.171

[Difficulty: 2]

13.171 Air flows without friction in a short section of constant-area duct. At the duct inlet, $M_1 = 0.30$, $T_1 = 50°C$, and $\rho_1 = 2.16\ kg/m^3$. At the duct outlet, $M_2 = 0.60$. Determine the heat addition per unit mass, the entropy change, and the change in stagnation pressure for the process.

Solution:

* Compressible flow functions (Appendix E) to be used in solution

Basic equations: $h_1 + \frac{V_1^2}{2} + \frac{\delta Q}{dm} = h_2 + \frac{V_2^2}{2}$ $\qquad T\,ds = dh - v\,dP$

Assumptions: (1) steady flow (4) ideal gas
(2) frictionless flow (5) $\dot{W}_s = \dot{W}_{shear} = 0$
(3) uniform flow at a section (6) $\Delta z = 0$

$p_1 = \rho_1 R T_1 = 2.16\ \frac{kg}{m^3} \cdot 287\ \frac{N \cdot m}{kg \cdot K} \times 323\ K = 200\ kPa$

$M_1 = 0.30$ From App. E.1 $T_1/T_{01} = 0.9823$ (11.17b), $p_1/p_{01} = 0.9395$ (11.17a)

$\therefore T_{01} = 329\ K \qquad p_{01} = 213\ kPa$

From App. E.3 $T_{01}/T_0^* = 0.3469$ (12.30d), $p_{01}/p_0^* = 1.199$ (12.30e)

$\therefore T_0^* = 948\ K \qquad p_0^* = 178\ kPa$

$M_2 = 0.60$ From App. E.3 $T_{02}/T_0^* = 0.8189$ (12.30d), $p_{02}/p_0^* = 1.075$ (12.30e)

$\therefore T_{02} = 776\ K \qquad p_{02} = 191\ kPa$

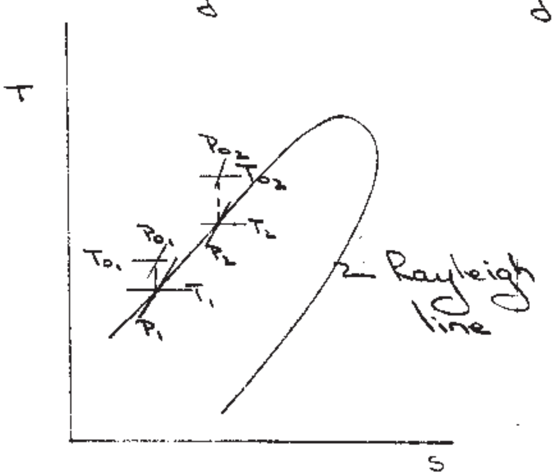
$p_{01} - p_{02} = (213 - 191)\ kPa = 22\ kPa$ $\qquad p_{01} - p_0$

$\delta Q/dm = h_{02} - h_{01} = c_p(T_{02} - T_{01}) = 1.0\ \frac{kJ}{kg \cdot K}(776 - 329)K = 447\ kJ/kg$ $\qquad \delta Q/dm$

$T\,ds = dh - v\,dP = c_p\,dT - \frac{1}{\rho}dP \qquad ds = c_p \frac{dT}{T} - R\frac{dP}{P}$

$s_2 - s_1 = s_{02} - s_{01} = \int_{s_{01}}^{s_{02}} ds = \int_{T_{01}}^{T_{02}} c_p \frac{dT}{T} - \int_{p_{01}}^{p_{02}} R\frac{dP}{P} = c_p \ln\frac{T_{02}}{T_{01}} - R\ln\frac{p_{02}}{p_{01}}$

$s_2 - s_1 = 1.0\ \frac{kJ}{kg \cdot K} \ln\frac{776}{329} - 0.287\ \frac{kJ}{kg \cdot K}\ln\frac{191}{213} = 0.889\ kJ/kg \cdot K$ $\qquad s_2 - s_1$



Problem 13.172

[Difficulty: 2]

13.172 Air, from an aircraft inlet system, enters the engine combustion chamber, where heat is added during a frictionless process in a tube with constant area of 0.01 m^2. The local isentropic stagnation temperature and Mach number entering the combustor are 427 K and 0.3. The mass flow rate is 0.5 kg/s. When the rate of heat addition is set at 404 kW, flow leaves the combustor at 1026 K and 22.9 kPa (abs). Determine for this process (a) the Mach number at the combustor outlet, (b) the static pressure at the combustor inlet, and (c) the change in local isentropic stagnation pressure during the heat addition process. Show static and stagnation state points and indicate the process path on a *Ts* diagram.

Solution: * Compressible flow functions (Appendix E) to be used in solution

Basic equation: $h_{01} + \frac{\delta Q}{dm} = h_{02}$ $\quad \dot{m} = \rho V A \quad p = \rho R T$

Assumptions: (1) steady flow (2) frictionless flow
(3) ideal gas (4) uniform flow at a section
(5) $\dot{W}_s = \dot{W}_{shear} = 0$

From the energy equation. $\dot{Q} = \dot{m}(h_{02} - h_{01}) = \dot{m} c_p (T_{02} - T_{01})$

$\therefore T_{02} = T_{01} + \frac{\dot{Q}}{\dot{m} c_p} = 427\,K + 404\times 10^3\,W \times \frac{s}{0.5\,kg} \times \frac{kg\cdot K}{1004\,J} \times \frac{J}{W\cdot s} = 1232\,K$

At $M_1 = 0.3$, from App. E.3, $T_{01}/T_0^* = 0.3469$ $\quad \therefore T_0^* = 1231\,K$

Since $T_{02} \approx T_0^*$, then $M_2 = 1.0$ ← M_2

$\therefore p_2 = p^* = 22.9$ kPa

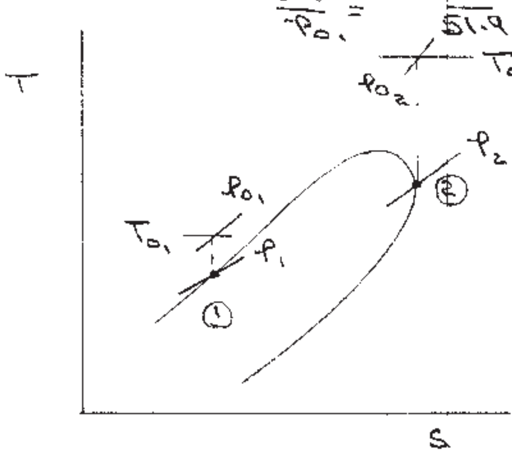
From Appendix E.3 for $M_1 = 0.3$, $p_1/p^* = 2.131$ $\quad \therefore p_1 = 48.8$ kPa ← p_1

" " E.1 " $M_1 = 0.3$, $p_1/p_{01} = 0.9395$ $\quad \therefore p_{01} = 51.9$ kPa

From Appendix E.1 for $M_2 = 1.0$, $p_2/p_{02} = 0.5283$ $\quad \therefore p_{02} = 43.3$ kPa

$\therefore p_{02} - p_{01} = (43.3 - 51.9)\,kPa = -8.6\,kPa$

$\frac{\Delta p_0}{p_{01}} = \frac{-8.6}{51.9} = -0.166$ (or -16.6%) ← Δp_0



Problem 13.173 [Difficulty: 4]

13.173 Air enters a frictionless, constant-area duct with $M_1 = 2.0$, $T_1 = 300°R$, and $p_1 = 70$ psia. Heat transfer occurs as the air travels down the duct. A converging section ($A_2/A_3 = 1.5$) is placed at the end of the constant area duct and $M_3 = 1.0$. Assuming isentropic flow (aside from the heat transfer through the duct), calculate the amount and direction of heat transfer.

Given: Air flow through a duct with heat transfer followed by converging duct, sonic at exit

Find: Magnitude and direction of heat transfer

Solution:

The given or available data is:

$R =$	53.33	ft-lbf/lbm-°R
$c_p =$	0.2399	Btu/lbm-°R
$k =$	1.4	
$M_1 =$	2	
$T_1 =$	300	°R
$p_1 =$	70	psia
$A_2/A_3 =$	1.5	
$M_3 =$	1	

Equations and Computations:

We can determine the stagnation temperature at the entrance:

$T_{01}/T_1 =$ 1.8000

So the entrance stagnation temperature is:

$T_{01} =$ 540.00 °R

The reference stagnation temperature ratio at state 1 is:

$T_{01}/T_0^* =$ 0.7934

The reference conditions for Rayliegh flow can be calculated:

$T_0^* =$ 680.6 °R

Since the flow is sonic at state 3, we can find the Mach number at state 2:

$M_2 =$ 1.8541

We know that the flow must be supersonic at 2 since the flow at $M_1 > 1$.

The reference stagnation temperature ratio at state 2 is:

$T_{02}/T_0^* =$ 0.8241

Since the reference stagnation temperature at 1 and 2 are the same:

$T_{02} =$ 560.92 °R

The heat transfer is related to the change in stagnation temperature:

$q_{1\text{-}2} =$ 5.02 Btu/lbm

The heat is being added to the flow.

Problem 13.174

[Difficulty: 3]

13.174 Consider steady, one-dimensional flow of air in a combustor with constant area of 0.5 ft², where hydrocarbon fuel, added to the air stream, burns. The process is equivalent to simple heating because the amount of fuel is small compared to the amount of air; heating occurs over a short distance so that friction is negligible. Properties at the combustor inlet are 818°R, 200 psia, and $M = 0.3$. The speed at the combustor outlet must not exceed 2000 ft/s. Find the properties at the combustor outlet and the heat addition rate. Show the process path on a Ts diagram, indicating static and stagnation state points before and after the heat addition.

Solution: (using compressible flow functions - Appendix E)

Basic equations: $h_1 + \frac{V_1^2}{2} + \frac{\delta Q}{dm} = h_2 + \frac{V_2^2}{2}$ $\quad p = \rho R T \quad \dot{m} = \rho V A$

Assumptions: (1) steady flow (2) frictionless flow
(3) ideal gas, properties are those of air
(4) uniform flow at a section
(5) $\dot{W}_s = \dot{W}_{shear} = 0$

$\rho_1 = \frac{p_1}{RT_1} = 200\frac{lbf}{in^2} \times \frac{lbm\,°R}{53.3\,ft\,lbf} \times \frac{1}{818\,R} \times 144\frac{in^2}{ft^2} = 0.6606\ lbm/ft^3$

$V_1 = M_1 c_1 = M_1 (kRT_1)^{1/2} = 0.3\left[1.4 \times 53.3\frac{ft\,lbf}{lbm\,°R} \times 818\,°R \times 32.2\frac{lbm}{slug} \times \frac{slug\,ft}{lbf\,s^2}\right]^{1/2} = 421\ ft/s$

From App. E.1 $\quad M_1 = 0.3 \quad T/T_0 = 0.9823 \quad \therefore T_{01} = 832.7\,°R$
$\quad p/p_0 = 0.9395 \quad \therefore p_{01} = 212.9$ psia

From continuity, $\rho_1 V_1 = \rho_2 V_2$
$\therefore \rho_2 = \frac{V_1}{V_2}\rho_1 = \frac{421}{2000} \times 0.6606\ lbm/ft^3 = 0.139\ lbm/ft^3$ $\quad \rho_2$

From App E.3
$M_1 = 0.3 \quad \frac{T_{01}}{T_0^*} = 0.3469 \quad \frac{p_{01}}{p_0^*} = 1.199 \quad \frac{T_1}{T^*} = 0.4089 \quad \frac{p_1}{p^*} = 2.131 \quad \frac{V_1}{V^*} = 0.1918$

$\therefore T_0^* = 2400\,R \quad p_0^* = 177.6$ psia $\quad T^* = 2000\,R \quad p^* = 93.85$ psia $\quad V^* = 2193\ \frac{ft}{sec}$

At section ② $\quad V_2/V^* = 2000/2193 = 0.9120$

From App. E.3 $\quad M_2 = 0.90$ $\quad M_2$

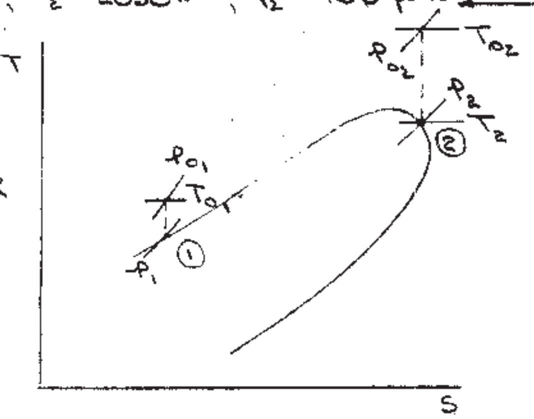
$T_{02}/T_0^* = 0.9921, \quad p_{02}/p_0^* = 1.005, \quad T_2/T^* = 1.025, \quad p_2/p^* = 1.125$
$\therefore T_{02} = 2380\,R, \quad p_{02} = 178$ psia, $\quad T_2 = 2050\,R, \quad p_2 = 106$ psia $\quad T_{02}, p_{02}, T_2, p_2$

From the energy equation,

$\dot{Q} = \dot{m}\frac{\delta Q}{dm} = \rho_1 V_1 A (h_{02} - h_{01}) = \rho_1 V_1 A c_p (T_{02} - T_{01})$

$\dot{Q} = 0.6606\frac{lbm}{ft^3} \times 420.6\frac{ft}{s} \times 0.5\,ft^2 \times 0.24\frac{BTU}{lbm\,°R} \times (2380 - 833)\,°R$

$\dot{Q} = 5.16 \times 10^4$ Btu/s $\quad \dot{Q}$



Problem 13.175

[Difficulty: 3]

13.175 Flow in a gas turbine combustor is modeled as steady, one-dimensional, frictionless heating of air in a channel of constant area. For a certain process, the inlet conditions are 500°C, 1.5 MPa (abs), and $M = 0.5$. Calculate the maximum possible heat addition. Find all fluid properties at the outlet section and the reduction in stagnation pressure. Show the process path on a Ts diagram, indicating all static and stagnation state points.

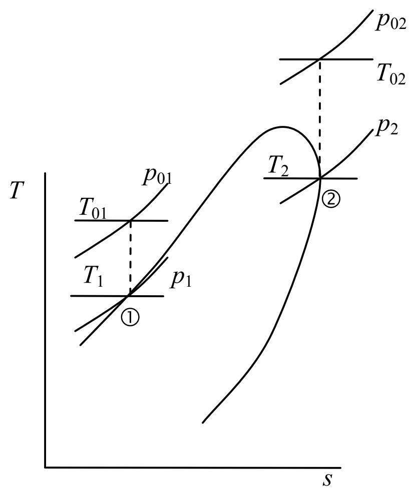
Given: Data on flow through gas turbine combustor

Find: Maximum heat addition; Outlet conditions; Reduction in stagnation pressure; Plot of process

Solution:

The given or available data is:

$R =$	286.9	J/kg·K
$k =$	1.4	
$c_p =$	1004	J/kg·K
$T_1 =$	773	K
$p_1 =$	1.5	MPa
$M_1 =$	0.5	



Equations and Computations:

From $p_1 = \rho_1 R T_1$ $\rho_1 =$ 6.76 kg/m^3

From $V_1 = M_1\sqrt{kRT_1}$ $V_1 =$ 279 m/s

Using built-in function *IsenT* (M,k):

$T_{01}/T_1 =$ 1.05 $T_{01} =$ 812 K

Using built-in function *Isenp* (M,k):

$p_{01}/p_1 =$ 1.19 $p_{01} =$ 1.78 MPa

For maximum heat transfer: $M_2 =$ 1

Using built-in function *rayT0* (M,k), *rayp0* (M,k), *rayT* (M,k), *rayp* (M,k), *rayV* (M,k):

$T_{01}/T_0^* =$	0.691	$T_0^* =$	1174	K	$(= T_{02})$
$p_{01}/p_0^* =$	1.114	$p_0^* =$	1.60	MPa	$(= p_{02})$
$T/T^* =$	0.790	$T^* =$	978	K	$(= T_{02})$
$p/p^* =$	1.778	$p^* =$	0.844	MPa	$(= p_2)$
$\rho^*/\rho =$	0.444	$\rho^* =$	3.01	kg/m^3	$(= \rho_2)$

Note that at state 2 we have critical conditions!

Hence: $p_{012} - p_{01} =$ -0.182 MPa -182 kPa

From the energy equation: $$\frac{\delta Q}{dm} = c_p\left(T_{02} - T_{01}\right)$$

$\delta Q/dm =$ 364 kJ/kg

Problem 13.176

[Difficulty: 3]

13.176 A supersonic wind tunnel is supplied from a high-pressure tank of air at 25°C. The test section temperature is to be maintained above 0°C to prevent formation of ice particles. To accomplish this, air from the tank is heated before it flows into a converging-diverging nozzle which feeds the test section. The heating is done in a short section with constant area. The heater output is $\dot{Q} = 10$ kW. The design Mach number in the wind tunnel test section is to be 3.0. Evaluate the stagnation temperature required at the heater exit. Calculate the maximum mass flow rate at which air can be supplied to the wind tunnel test section. Determine the area ratio, A_e/A_t.

Solution:

(using compressible flow functions)

Basic equations: $h_1 + \frac{V_1^2}{2} + \frac{\delta Q}{dm} = h_2 + \frac{V_2^2}{2}$ $\quad \dot{m} = \rho V A \quad p = \rho R T$

Assumptions: (1) steady flow (2) uniform flow at a section
(3) frictionless flow in the heater
(4) isentropic flow through the nozzle
(5) ideal gas (6) $\dot{W}_s = \dot{W}_{shear} = 0$

$T_{01} = T = 298$ K

From Appendix E.1 at $M=3$, $T/T_0 = 0.3571$, $A_e/A^* = A_e/A_t = 4.23$ $\qquad \frac{A}{A_t}$

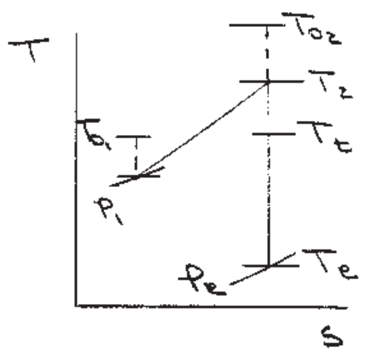
With $T_e = 273$ K, $T_{0e} = T_{02} = 273\text{K}/0.3571 = 764$ K $\qquad T_{02}$

From the energy equation $\frac{\delta Q}{dm} = h_{02} - h_{01} = c_p(T_{02} - T_{01})$

Since $\dot{Q} = \dot{m}\frac{\delta Q}{dm}$, then $\dot{m} = \frac{\dot{Q}}{c_p(T_{02} - T_{01})}$

and

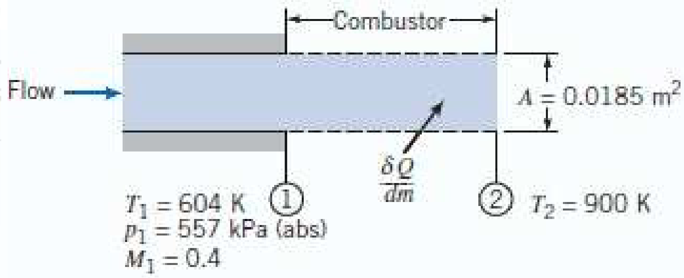
$$\dot{m} = 10\,\text{kW} \times \frac{10^3\,\text{N}\cdot\text{m}}{\text{s}\cdot\text{kW}} \times \frac{\text{kg}\cdot\text{K}}{1000\,\text{N}\cdot\text{m}} \times \frac{1}{(764-298)\text{K}} = 0.0215\ \text{kg/s} \qquad \dot{m}$$



Problem 13.177

[Difficulty: 3]

13.177 Consider steady flow of air in a combustor where thermal energy is added by burning fuel. Neglect friction. Assume thermodynamic properties are constant and equal to those of pure air. Calculate the stagnation temperature at the burner exit. Compute the Mach number at the burner exit. Evaluate the heat addition per unit mass and the heat exchange rate. Express the rate of heat addition as a fraction of the maximum rate of heat addition possible with this inlet Mach number.



Solution:

(using compressible flow functions - Appendix E)

Basic equations: $h_1 + \frac{V_1^2}{2} + \frac{\delta Q}{dm} = h_2 + \frac{V_2^2}{2}$ $\qquad \dot{m} = \rho V A \qquad p = \rho R T$

Assumptions: (1) steady flow (2) ideal gas (3) uniform flow at a section (4) $\dot{W}_s = \dot{W}_{shear} = 0$

$$\rho_1 = \frac{p_1}{RT_1} = 557\times10^3 \frac{N}{m^2} \times \frac{kg\cdot K}{287\,N\cdot m} \times \frac{1}{604\,K} = 3.21\ kg/m^3$$

$$V_1 = M_1 c_1 = M_1 (kRT_1)^{1/2} = 0.4\left(1.4\times287\frac{N\cdot m}{kg\cdot K}\times604\,K\cdot\frac{kg\cdot m}{N\cdot s^2}\right)^{1/2} = 197\ m/s$$

$$\dot{m} = \rho_1 V_1 A = 3.21\frac{kg}{m^3}\times197\frac{m}{s}\times0.0185\,m^2 = 11.7\ kg/s$$

From App. E.1 at $M_1 = 0.4$, $T_1/T_{0_1} = 0.969$ $\quad\therefore T_{0_1} = 623$ K

From App. E.3 at $M_1 = 0.4$, $T/T^* = 0.6152$, $T_0/T_0^* = 0.5290$

$\therefore T^* = 982$ and $T_0^* = 1180$ K

At section ② $T_2 = 900$ K $\quad\therefore (T/T^*)_2 = 0.916$

From App. E.3, $M_2 = 0.60$, $(T_0/T_0^*)_2 = 0.8189$

$\therefore T_{0_2} = 966$ K, $M_2 = 0.60$ $\longleftarrow$ M_2, T_{0_2}

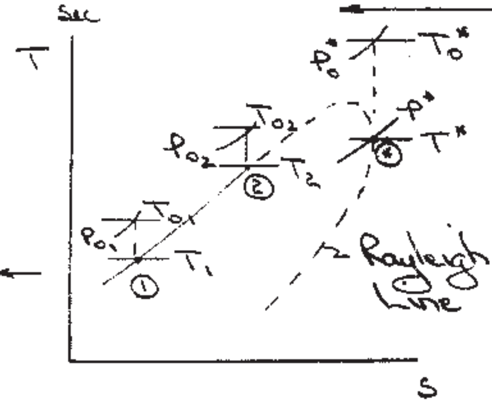
From the energy equation,

$$\frac{\delta Q}{dm} = h_{0_2} - h_{0_1} = c_p(T_{0_2} - T_{0_1}) = 1.00\frac{kJ}{kg\cdot K}(966-623)K = 343\frac{kJ}{kg} \longleftarrow \frac{\delta Q}{dm}$$

$$\dot{Q} = \dot{m}\frac{\delta Q}{dm} = 11.7\frac{kg}{sec}\times343\frac{kJ}{kg} = 4010\frac{kJ}{sec} = 4010\ kW \longleftarrow \dot{Q}$$

The heat transfer may be expressed as a fraction of the maximum possible heat addition

$$\frac{\delta Q/dm}{(\delta Q/dm)_{max}} = \frac{T_{0_2} - T_{0_1}}{T_0^* - T_{0_1}} = \frac{966-623}{1180-623} = 0.616 \longleftarrow$$



13.178 A jet transport aircraft cruises at $M = 0.85$ at an altitude of 40,000 ft. Air for the cabin pressurization system is taken aboard through an inlet duct and slowed isentropically to 100 ft/s relative to the aircraft. Then it enters a compressor, where its pressure is raised adiabatically to provide a cabin pressure equivalent to 8000 ft altitude. The air temperature increase across the compressor is 170°F. Finally, the air is cooled to 70°F (in a heat exchanger with negligible friction) before it is added to the cabin air. Sketch a diagram of the system, labeling all components and numbering appropriate cross sections. Determine the stagnation and static temperature and pressure at each cross section. Sketch to scale and label a Ts diagram showing the static and stagnation state points and indicating the process paths. Evaluate the work added in the compressor and the energy rejected in the heat exchanger.

Solution: From the standard atmosphere, at $z_1 = 40{,}000$ ft, $T_1 = -70°F$ and $p_1 = 2.73$ psia

$$c_1 = \sqrt{kRT_1} = \left[1.4 \times 53.3 \frac{ft \cdot lbf}{lbm \cdot °R} \times (460-70)°R \times 32.2 \frac{lbm}{slug} \times \frac{slug \cdot ft}{lbf \cdot s^2}\right]^{\frac{1}{2}} = 968 \frac{ft}{s}$$

$$V_1 = M_1 c_1 = 0.85 \times 968 \frac{ft}{s} = 823 \text{ ft/s}$$

For isentropic deceleration $T_{01} = T_1 + \dfrac{V_1^2}{2c_p} = T_{02} = T_2 + \dfrac{V_2^2}{2c_p}$

$$T_2 = T_1 + \frac{1}{2c_p}(V_1^2 - V_2^2)$$

$$T_2 = 390°R + \frac{1}{2} \times \frac{lbm \cdot °R}{0.240\ Btu}\left[(823)^2 - (100)^2\right]\frac{ft^2}{s^2} \times \frac{Btu}{778\ ft \cdot lbf} \times \frac{lbf \cdot s^2}{slug \cdot ft} \times \frac{slug}{32.2\ lbm} = 445°R$$

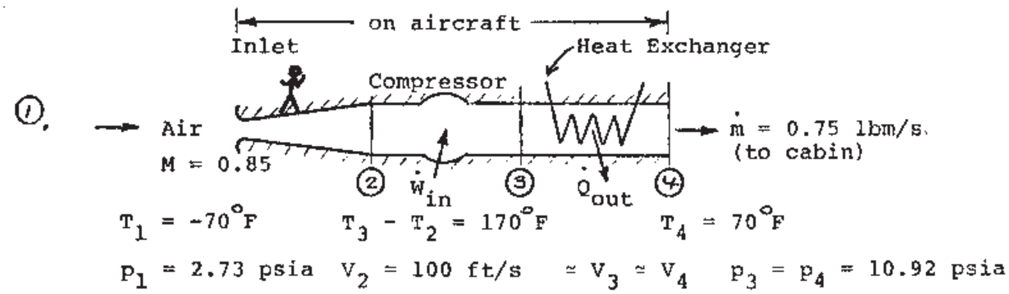
For isentropic deceleration,

$$p_2 = p_1\left(\frac{T_2}{T_1}\right)^{k/k-1} = 2.73\ psia\left(\frac{445}{390}\right)^{3.5} = 4.33\ psia$$

The compressor raises the air the equivalent of $z_3 = z_4 = 8{,}000$ ft; from the Standard Atmosphere

$$p_3 = p_4 = 10.92\ psia\ ; \quad p_{03} = p_3\left(1 + \frac{k-1}{2}M_3^2\right)^{\frac{k}{k-1}} = 10.92\left(1 + 0.2(0.0823)^2\right)^{3.5} = 10.97\ psia$$

Assume $V_2 \simeq V_3 \simeq V_4$ since no other data are known. The system is



Problem 13.178

Evaluating properties: $T_{01} = T_1\left(1+\frac{k-1}{2}M_1^2\right) = 390°R\left(1+0.2(0.85)^2\right) = 446°R = T_{02}$

$p_{01} = p_1\left(1+\frac{k-1}{2}M_1^2\right)^{\frac{k}{k-1}} = 2.73\ psia\left(1+0.2(0.85)^2\right)^{3.5} = 4.38\ psia = p_{02}$

$T_3 = T_2 + 170°R = 445 + 170°R = 615°R$; $T_{03} = T_{02} + 170°R = 616°R$ (since $V_2 \simeq$ const)

From the energy equation

$\dot{W}_{in} = \dot{m}(h_3 - h_2) = \dot{m}c_p(T_3 - T_2)$

$\dot{W}_{in} = 0.75\ \frac{lbm}{s} \times 0.240\ \frac{Btu}{lbm\,°R}(170)°R \times 778\ \frac{ft\cdot lbf}{Btu} \times \frac{hp\cdot s}{550\ ft\cdot lbf} = 43.3\ hp$ $\quad \dot{W}_{in}$

$\dot{Q}_{in} = \dot{m}(h_4 - h_3) = \dot{m}c_p(T_4 - T_3)$

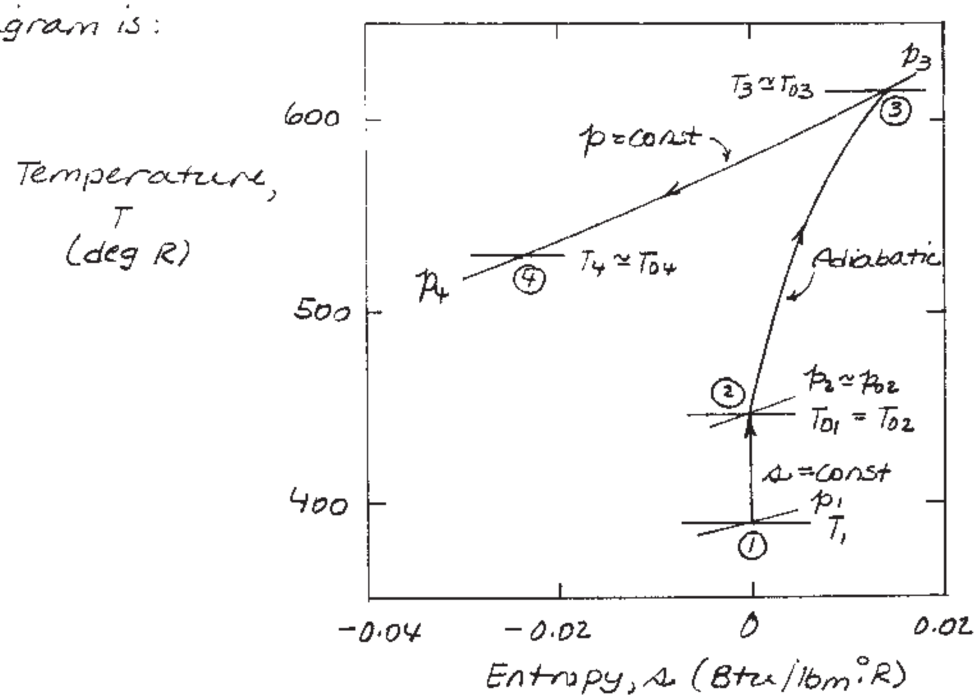
$\dot{Q}_{in} = 0.75\ \frac{lbm}{s} \cdot 0.240\ \frac{Btu}{lbm\,°R}(530 - 615)°R = -15.3\ Btu/s$ (out) $\quad \dot{Q}_{in}$

The entropy changes are computed using $Tds = dh - vdp$, as

$\Delta s_{23} = c_p \ln\frac{T_{03}}{T_{02}} - R\ln\frac{p_{03}}{p_{02}} = 0.240\ \frac{Btu}{lbm\,°R}\ln\left(\frac{616}{446}\right) - 53.3\ \frac{ft\cdot lbf}{lbm\,°R}\ln\left(\frac{10.97}{4.38}\right)\frac{Btu}{778\ ft\cdot lbf} = 0.0146\ \frac{Btu}{lbm\,°R}$

$\Delta s_{34} = c_p \ln\frac{T_{04}}{T_{03}} - R\ln\frac{p_{04}}{p_{03}} = 0.240\ \frac{Btu}{lbm\,°R}\ln\left(\frac{530}{616}\right) - \frac{53.3}{778}\ \frac{Btu}{lbm\,°R}\ln\left(\frac{10.97}{10.97}\right) = -0.0361\ Btu/lbm\,°R$

The Ts diagram is:



Problem 13.179

13.179 Frictionless flow of air in a constant-area duct discharges to atmospheric pressure at section ②. Upstream at section ①, $M_1 = 3.0$, $T_1 = 215°R$, and $p_1 = 1.73$ psia. Between sections ① and ②, 48.5 Btu/lbm of air is added to the flow. Determine M_2 and p_2. In addition to a Ts diagram, sketch the pressure distribution versus distance along the channel, labeling sections ① and ②.

Solution: Apply equations for Rayleigh line flow of an ideal gas.

Basic equations: $c_p T_{01} + \frac{\delta Q}{dm} = c_p T_{02}$ $\qquad p_1 A - p_2 A = \dot{m}(V_2 - V_1)$

Computing equation: $T_0 = T\left(1 + \frac{k-1}{2}M^2\right)$

Assumptions: (1) Steady flow (2) Frictionless flow (3) Uniform flow at each section (4) Ideal gas (5) $F_{Bx} = 0$ (6) $\dot{W}_s = \dot{W}_{shear} = 0$ (7) $\Delta z = 0$

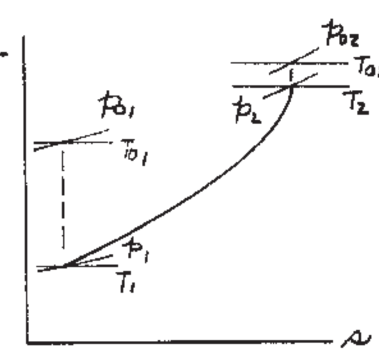
The minimum possible Mach number for supersonic flow with heating is $M_2 = 1$.

$$T_{01} = T_1\left(1 + \frac{k-1}{2}M^2\right) = 215°R\left(1 + 0.2(3.0)^2\right) = 613°R$$

$$T_{02} = T_{01} + \frac{1}{c_p}\frac{\delta Q}{dm} = 613°R + \frac{\text{lbm}\cdot°R}{0.240\ \text{Btu}} \times 48.5\ \frac{\text{Btu}}{\text{lbm}} = 815°R$$

Check for $M_2 = 1.0$: $T_2 = \frac{T_{02}}{1 + 0.2(1)^2} = \frac{815°R}{1.2} = 679°R$

$$V_2 = c_2 = \sqrt{kRT_2} = 1{,}280\ \text{ft/s}$$



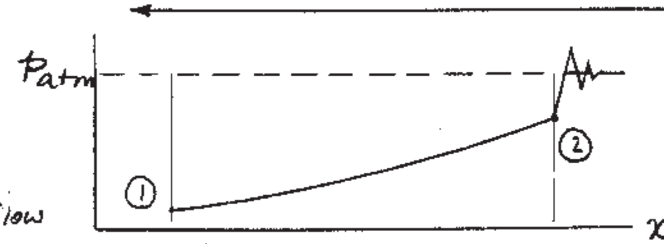
Thus $p_2 = p_1 + \frac{\dot{m}}{A}(V_1 - V_2) = p_1 + \rho_1 V_1 (V_1 - V_2)$

$V_1 = M_1 c_1 = M_1\sqrt{kRT_1} = 3.0 \times 719\ \frac{\text{ft}}{\text{s}} = 2{,}160\ \text{ft/s}$; $\rho_1 = \frac{p_1}{RT_1} = 0.0217\ \text{lbm/ft}^3$

$$p_2 = 1.73\ \frac{\text{lbf}}{\text{in.}^2} + 0.0217\ \frac{\text{lbm}}{\text{ft}^3} \times 2160\ \frac{\text{ft}}{\text{s}}\,(2160 - 1280)\frac{\text{ft}}{\text{s}} \times \frac{\text{slug}}{32.2\ \text{lbm}} \times \frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}} \times \frac{\text{ft}^2}{144\ \text{in}^2} = 10.6\ \text{psia}$$

Thus if $M_2 = 1.0$, the $p_2 < p_{atm}$, which is not possible for sonic flow. Therefore $p_2 < p_{atm}$ and $M_2 > 1.0$ for this flow.

The pressure vs. distance plot is:



{This problem could be solved quantitatively, but only by either iteration or by use of compressible flow functions; that solution is presented on the next page.}

For $M_1 = 3.0$, from App. E.3: $\frac{T_0}{T_0^*} = 0.6540$, and $\frac{p}{p^*} = 0.1765$

Thus $T_0^* = \frac{T_{01}}{(T_0/T_0^*)_1} = \frac{613°R}{0.6540} = 937°R$; $p^* = \frac{p_1}{(p/p^*)_1} = \frac{1.73 \text{ psia}}{0.1765} = 9.80 \text{ psia}$

At section 2, $T_{02} = 815\ R$ and $\frac{T_{02}}{T_0^*} = \frac{815°R}{937°R} = 0.870$

From App. E.3 this corresponds to $M_2 = 1.74$. At this Mach number, $\frac{p}{p^*} = 0.4581$.

Thus $p_2 = p^*\left(\frac{p}{p^*}\right)_2 = 9.80 \text{ psia} \times 0.4581 = 4.49 \text{ psia}$ $\qquad p_2$

{ These calculations confirm that $M_2 > 1$ and $p_2 < p_{atm}$. }

Problem 13.180

[Difficulty: 3]

13.180 Show that as the upstream Mach number approaches infinity, the Mach number after an oblique shock becomes

$$M_2 \approx \sqrt{\frac{k-1}{2k\sin^2(\beta-\theta)}}$$

Given: Normal shock

Find: Approximation for downstream Mach number as upstream one approaches infinity

Solution:

Basic equations:

$$M_{2n}^2 = \frac{M_{1n}^2 + \frac{2}{k-1}}{\left(\frac{2 \cdot k}{k-1}\right) \cdot M_{1n}^2 - 1} \qquad (13.48a) \qquad M_{2n} = M_2 \cdot \sin(\beta - \theta) \qquad (13.47b)$$

Combining the two equations

$$M_2 = \frac{M_{2n}}{\sin(\beta-\theta)} = \frac{\sqrt{\frac{M_{1n}^2 + \frac{2}{k-1}}{\left(\frac{2 \cdot k}{k-1}\right) \cdot M_{1n}^2 - 1}}}{\sin(\beta-\theta)} = \sqrt{\frac{M_{1n}^2 + \frac{2}{k-1}}{\left[\left(\frac{2 \cdot k}{k-1}\right) \cdot M_{1n}^2 - 1\right] \cdot \sin(\beta-\theta)^2}}$$

$$M_2 = \sqrt{\frac{1 + \frac{2}{(k-1) \cdot M_{1n}^2}}{\left[\left(\frac{2 \cdot k}{k-1}\right) - \frac{1}{M_{1n}^2}\right] \cdot \sin(\beta-\theta)^2}}$$

As M_1 goes to infinity, so does M_{1n}, so

$$M_2 = \sqrt{\frac{1}{\left(\frac{2 \cdot k}{k-1}\right) \cdot \sin(\beta-\theta)^2}} \qquad M_2 = \sqrt{\frac{k-1}{2 \cdot k \cdot \sin(\beta-\theta)^2}}$$

Problem 13.181 [Difficulty: 3]

13.181 Air at 400 K and 100 kPa is flowing at a Mach number of 1.8 and is deflected through a 14° angle. The directional change is accompanied by an oblique shock. What are the possible shock angles? For each of these shock angles, what is the pressure and temperature after the shock?

Given: Air deflected at an angle, causing an oblique shock

Find: Possible shock angles; pressure and temperature corresponding to those angles

Solution:

The given or available data is:

$$R = 286.9 \text{ J/kg-K}$$
$$k = 1.4$$
$$M_1 = 1.8$$
$$T_1 = 400 \text{ K}$$
$$p_1 = 100 \text{ kPa}$$
$$\theta = 14\ °$$

Equations and Computations:

There are two possible shock angles for a given deflection, corresponding to the weak and strong shock solutions. To find the shock angle, we have to iterate on the shock angle until we match the deflection angle, which is a function of Mach number, specific heat ratio, and shock angle.

The weak shock solution is:

$$\beta_{weak} = 49.7\ °$$
$$\theta = 14.0000\ °$$

The strong shock solution is:

$$\beta_{strong} = 78.0\ °$$
$$\theta = 14.0000\ °$$

We used *Solver* in *Excel* to iterate on the shock angles.

For the weak shock, the pre-shock Mach number normal to the wave is:

$$M_{1nweak} = 1.3720$$

The pressure and temperature ratios across the shock wave are:

$$p_2/p_{1weak} = 2.0295$$
$$T_2/T_{1weak} = 1.2367$$

Therefore, the post-shock temperature and pressure are:

$$p_{2weak} = 203 \text{ kPa}$$
$$T_{2weak} = 495 \text{ K}$$

For the weak shock, the pre-shock Mach number normal to the wave is:

$$M_{1nstrong} = 1.7608$$

The pressure and temperature ratios across the shock wave are:

$$p_2/p_{1strong} = 3.4505$$
$$T_2/T_{1strong} = 1.5025$$

Therefore, the post-shock temperature and pressure are:

$$p_{2strong} = 345 \text{ kPa}$$
$$T_{2strong} = 601 \text{ K}$$

Problem 13.182

[Difficulty: 3]

13.182 Consider supersonic flow of air at $M_1 = 3.0$. What is the range of possible values of the oblique shock angle β? For this range of β, plot the pressure ratio across the shock.

Given: Oblique shock in flow at $M = 3$

Find: Minimum and maximum β, plot of pressure rise across shock

Solution:

The given or available data is:

$R =$	286.9	J/kg.K
$k =$	1.4	
$M_1 =$	3	

Equations and Computations:

The smallest value of β is when the shock is a Mach wave (no deflection)

$$\beta = \sin^{-1}(1/M_1)$$

$$\beta = 19.5 \ ^{o}$$

The largest value is $\beta = 90.0 \ ^{o}$

The normal component of Mach number is

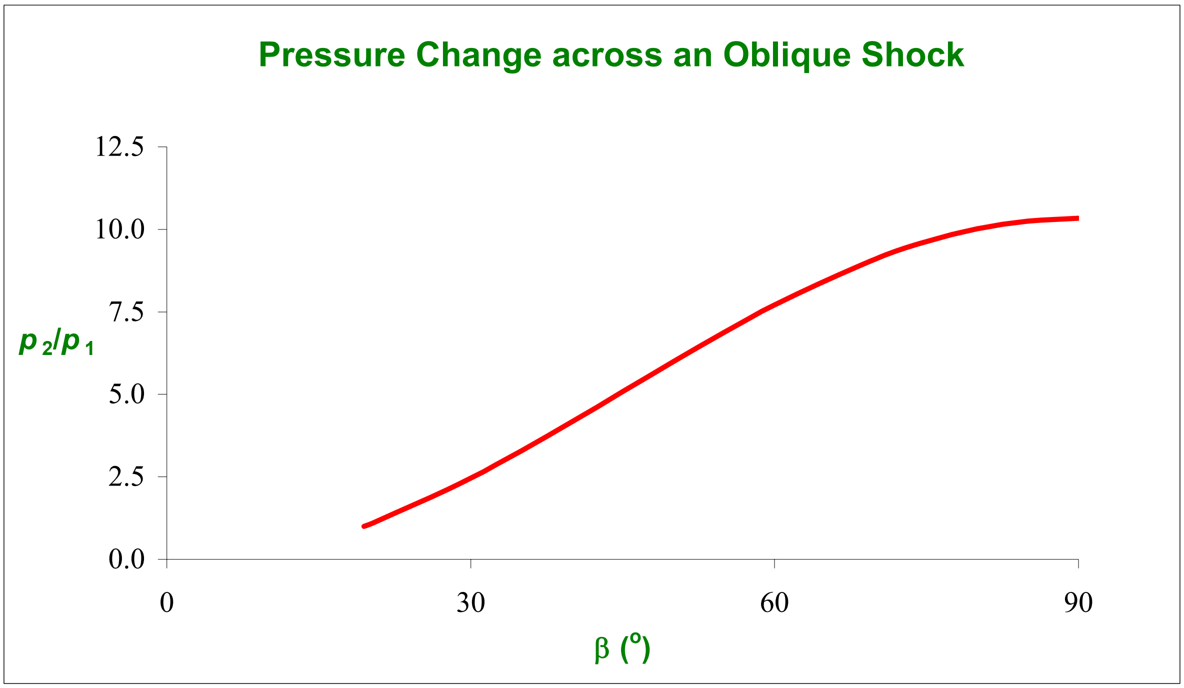
$$M_{1n} = M_1 \sin(\beta) \qquad (13.47a)$$

For each β, p_2/p_1 is obtained from M1n, and Eq. 13.48d
(using built-in function *NormpfromM*(M,k))

$$\frac{p_2}{p_1} = \frac{2k}{k+1} M_{1_n}^2 - \frac{k-1}{k+1} \qquad (13.48d)$$

Computed results:

β (°)	M_{1n}	p_2/p_1
19.5	1.00	1.00
20	1.03	1.06
30	1.50	2.46
40	1.93	4.17
50	2.30	5.99
60	2.60	7.71
70	2.82	9.11
75	2.90	9.63
80	2.95	10.0
85	2.99	10.3
90	3.00	10.3



Problem 13.183

[Difficulty: 3]

13.183 Supersonic air flow at $M_1 = 2.5$ and 80 kPa (abs) is deflected by an oblique shock with angle $\beta = 35°$. Find the Mach number and pressure after the shock, and the deflection angle. Compare these results to those obtained if instead the flow had experienced a normal shock. What is the smallest possible value of angle β for this upstream Mach number?

Given: Data on an oblique shock

Find: Mach number and pressure downstream; compare to normal shock

Solution:

The given or available data is:

$R =$	286.9	J/kg.K
$k =$	1.4	
$p_1 =$	80	kPa
$M_1 =$	2.5	
$\beta =$	35	°

Equations and Computations:

From M_1 and β

$M_{1n} =$ 1.43

$M_{1t} =$ 2.05

From M_{1n} and p_1, and Eq. 13.48d
(using built-in function *NormpfromM*(M, k))

$$\frac{p_2}{p_1} = \frac{2k}{k+1} M_{1_n}^2 - \frac{k-1}{k+1} \qquad (13.48d)$$

$p_2 =$ 178.6 kPa

The tangential velocity is unchanged

$$V_{t1} = V_{t2}$$

Hence

$$c_{t1} M_{t1} = c_{t2} M_{t2}$$

$$(T_1)^{1/2} M_{t1} = (T_2)^{1/2} M_{t2}$$

$$M_{2t} = (T_1/T_2)^{1/2} M_{t1}$$

From M_{1n}, and Eq. 13.48c
(using built-in function *NormTfromM*(M, k))

$T_2/T_1 =$ 1.28

Hence

$M_{2t} =$ 1.81

Also, from M_{1n}, and Eq. 13.48a
(using built-in function *NormM2fromM* (M, k))

$$M_{2_n}^2 = \frac{M_{1_n}^2 + \frac{2}{k-1}}{\frac{2k}{k-1} M_{1_n}^2 - 1} \qquad (13.48a)$$

$$M_{2n} = 0.726$$

The downstream Mach number is then

$$M_2 = (M_{2t}^{\ 2} + M_{2n}^{\ 2})^{1/2}$$

$$M_2 = 1.95$$

Finally, from geometry

$$V_{2n} = V_2 \sin(\beta - \theta)$$

Hence $$\theta = \beta - \sin^{-1}(V_{2n}/V_2)$$

or $$\theta = \beta - \sin^{-1}(M_{2n}/M_2)$$

$$\theta = 13.2 \quad ^{o}$$

For the normal shock:
From M_1 and p_1, and Eq. 13.48d
(using built-in function *NormpfromM* (M, k))

$$p_2 = 570 \quad \text{kPa}$$

Also, from M_1, and Eq. 13.48a
(using built-in function *NormM2fromM* (M, k))

$$M_2 = 0.513$$

For the minimum β:
The smallest value of β is when the shock is a Mach wave (no deflection)

$$\beta = \sin^{-1}(1/M_1)$$

$$\beta = 23.6 \quad ^{o}$$

Problem 13.184
[Difficulty: 3]

13.184 The temperature and Mach number before an oblique shock are $T_1 = 10°C$ and $M_1 = 3.25$, respectively, and the pressure ratio across the shock is 5. Find the deflection angle, θ, the shock angle, β, and the Mach number after the shock, M_2.

Given: Data on an oblique shock

Find: Deflection angle θ; shock angle β; Mach number after shock

Solution:

The given or available data is:

$R =$	286.9	J/kg.K
$k =$	1.4	
$M_1 =$	3.25	
$T_1 =$	283	K
$p_2/p_1 =$	5	

Equations and Computations:

From p_2/p_1, and Eq. 13.48d
(using built-in function *NormpfromM* (M,k)
and *Goal Seek* or *Solver*)

$$\frac{p_2}{p_1} = \frac{2k}{k+1} M_{1_n}^2 - \frac{k-1}{k+1} \qquad (13.48d)$$

For $p_2/p_1 =$ 5.00

$M_{1n} =$ 2.10

From M_1 and M_{1n}, and Eq 13.47a

$$M_{1n} = M_1 \sin(\beta) \qquad (13.47a)$$

$\beta =$ 40.4 °

From M_1 and β, and Eq. 13.49
(using built-in function *Theta* (M,β, k)

$$\tan\theta = \frac{2\cot\beta\left(M_1^2\sin^2\beta - 1\right)}{M_1^2(k+\cos 2\beta)+2} \qquad (13.49)$$

$\theta =$ 23.6 °

To find M_2 we need M_{2n}. From M_{1n}, and Eq. 13.48a
(using built-in function *NormM2fromM* (M,k))

$$M_{2_n}^2 = \frac{M_{1_n}^2 + \dfrac{2}{k-1}}{\dfrac{2k}{k-1}M_{1_n}^2 - 1} \qquad (13.48a)$$

$M_{2n} =$ 0.561

The downstream Mach number is then obtained from
from M_{2n}, θ and β, and Eq. 13.47b

$$M_{2n} = M_2\sin(\beta - \theta) \qquad (13.47b)$$

Hence $M_2 =$ 1.94

Problem 13.185

[Difficulty: 3]

13.185 The air velocities before and after an oblique shock are 1250 m/s and 650 m/s, respectively, and the deflection angle is $\theta = 35°$. Find the oblique shock angle β, and the pressure ratio across the shock.

Given: Velocities and deflection angle of an oblique shock

Find: Shock angle β; pressure ratio across shock

Solution:

The given or available data is:

$R =$	286.9	J/kg.K
$k =$	1.4	
$V_1 =$	1250	m/s
$V_2 =$	650	m/s
$\theta =$	35	o

Equations and Computations:

From geometry we can write two equations for tangential velocity:

For V_{1t} $\quad V_{1t} = V_1\cos(\beta)$ (1)

For V_{2t} $\quad V_{2t} = V_2\cos(\beta - \theta)$ (2)

For an oblique shock $V_{2t} = V_{1t}$, so Eqs. 1 and 2 give

$$V_1\cos(\beta) = V_2\cos(\beta - \theta) \quad (3)$$

Solving for β $\quad \beta = \tan^{-1}((V_1 - V_2\cos(\theta))/(V_2\sin(\theta)))$

$\beta =$ 62.5 o

(Alternatively, solve Eq. 3 using *Goal Seek*!)

For p_2/p_1, we need M_{1n} for use in Eq. 13.48d

$$\frac{p_2}{p_1} = \frac{2k}{k+1} M_{1_n}^2 - \frac{k-1}{k+1} \qquad (13.48d)$$

We can compute M_1 from θ and β, and Eq. 13.49
(using built-in function *Theta* (M,β, k))

$$\tan\theta = \frac{2\cot\beta\left(M_1^2\sin^2\beta - 1\right)}{M_1^2(k+\cos 2\beta)+2} \qquad (13.49)$$

For $\theta =$ 35.0 $^\circ$

$\beta =$ 62.5 $^\circ$

$M_1 =$ 3.19

This value of M_1 was obtained by using *Goal Seek* :
Vary M_1 so that θ becomes the required value.
(Alternatively, find M_1 from Eq. 13.49 by explicitly solving for it!)

We can now find M_{1n} from M_1. From M_1 and Eq. 13.47a

$$M_{1n} = M_1 \sin(\beta) \qquad (13.47a)$$

Hence $M_{1n} =$ 2.83

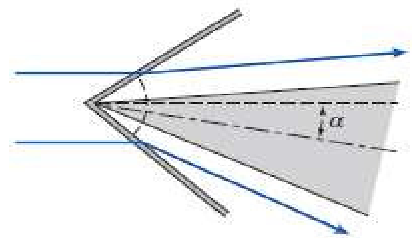
Finally, for p_2/p_1, we use M_{1n} in Eq. 13.48d
(using built-in function *NormpfromM* (M,k)

$p_2/p_1 =$ 9.15

Problem 13.186

[Difficulty: 4]

13.186 An airfoil has a sharp leading edge with an included angle of $\delta = 60°$. It is being tested in a wind tunnel running at 1200 m/s (the air pressure and temperature upstream are 75 kPa and 3.5°C). Plot the pressure and temperature in the region adjacent to the upper surface as functions of angle of attack, α, ranging from $\alpha = 0°$ to 30°. What are the maximum pressure and temperature? (Ignore the possibility of a detached shock developing if α is too large; see Problem 13.189.)



Given: Airfoil with included angle of 60°

Find: Plot of temperature and pressure as functions of angle of attack

Solution:

The given or available data is:

$R =$ 286.9 J/kg.K
$k =$ 1.4
$T_1 =$ 276.5 K
$p_1 =$ 75 kPa
$V_1 =$ 1200 m/s
$\delta =$ 60 °

Equations and Computations:

From T_1 $c_1 =$ 333 m/s

Then $M_1 =$ 3.60

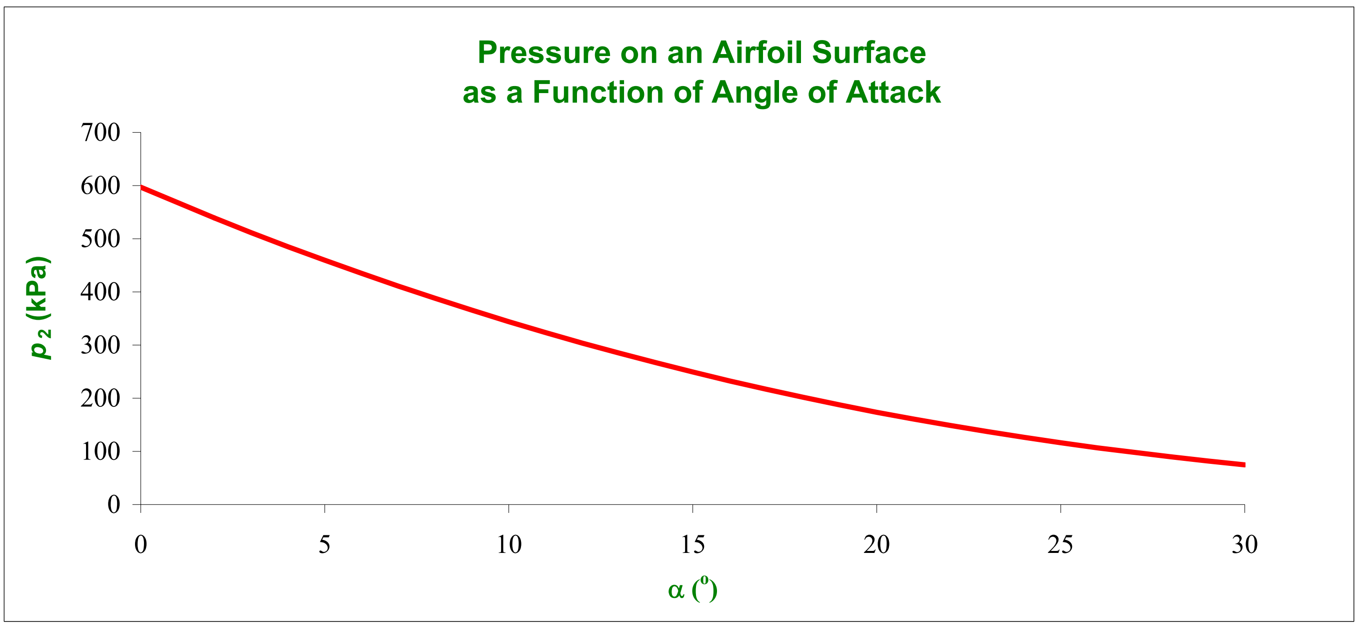
Computed results:

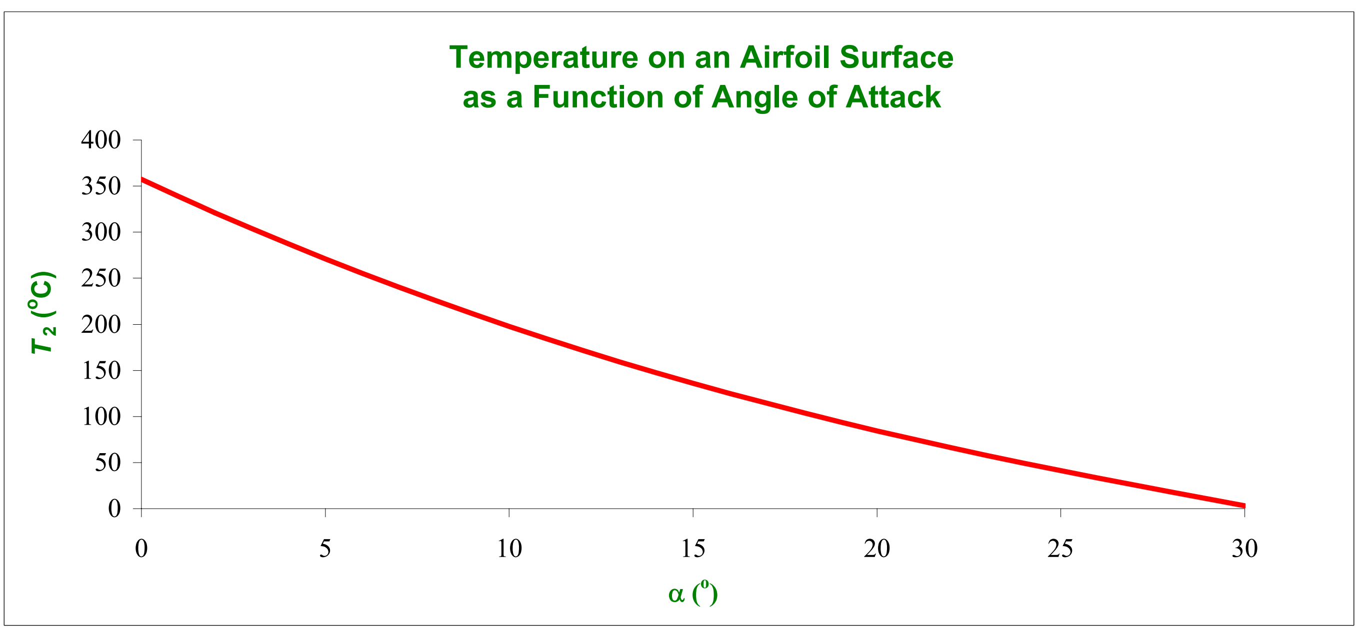
α (°)	β (°)	θ (°) Needed	θ (°)	Error	M_{1n}	p_2 (kPa)	T_2 (°C)
0.00	47.1	30.0	30.0	0.0%	2.64	597	357
2.00	44.2	28.0	28.0	0.0%	2.51	539	321
4.00	41.5	26.0	26.0	0.0%	2.38	485	287
6.00	38.9	24.0	24.0	0.0%	2.26	435	255
8.00	36.4	22.0	22.0	0.0%	2.14	388	226
10.00	34.1	20.0	20.0	0.0%	2.02	344	198
12.00	31.9	18.0	18.0	0.0%	1.90	304	172
14.00	29.7	16.0	16.0	0.0%	1.79	267	148
16.00	27.7	14.0	14.0	0.0%	1.67	233	125
18.00	25.7	12.0	12.0	0.0%	1.56	202	104
20.00	23.9	10.0	10.0	0.0%	1.46	174	84
22.00	22.1	8.0	8.0	0.0%	1.36	149	66
24.00	20.5	6.0	6.0	0.0%	1.26	126	49
26.00	18.9	4.0	4.0	0.0%	1.17	107	33
28.00	17.5	2.0	2.0	0.0%	1.08	90	18
30.00	16.1	0.0	-	0.0%	1.00	75	3

Sum: 0.0% Max: 597 357

To compute this table:

1) Type the range of α
2) Type in guess values for β
3) Compute θ_{Needed} from $\theta = \delta/2 - \alpha$
4) Compute θ from Eq. 13.49
   (using built-in function *Theta* (M, β, k)
5) Compute the absolute error between each θ and θ_{Needed}
6) Compute the sum of the errors
7) Use *Solver* to minimize the sum by varying the β values
   (Note: You may need to interactively type in new β values
   if *Solver* generates β values that lead to no θ)
8) For each α, M_{1n} is obtained from M_1, and Eq. 13.47a
9) For each α, p_2 is obtained from p_1, M_{1n}, and Eq. 13.48d
   (using built-in function *NormpfromM* (M, k))
10) For each α, T_2 is obtained from T_1, M_{1n}, and Eq. 13.48c
   (using built-in function *NormTfromM* (M, k))





Problem 13.187

[Difficulty: 4]

13.187 An airfoil at zero angle of attack has a sharp leading edge with an included angle of 20°. It is being tested over a range of speeds in a wind tunnel. The air temperature upstream is maintained at 15°C. Determine the Mach number and corresponding air speed at which a detached normal shock first attaches to the leading edge, and the angle of the resulting oblique shock. Plot the oblique shock angle β as a function of upstream Mach number M_1, from the minimum attached-shock value through $M_1 = 7$.

Given: Airfoil with included angle of 20^o

Find: Mach number and speed at which oblique shock forms

Solution:

The given or available data is:

$R =$	286.9	J/kg.K
$k =$	1.4	
$T_1 =$	288	K
$\theta =$	10	^{o}

Equations and Computations:

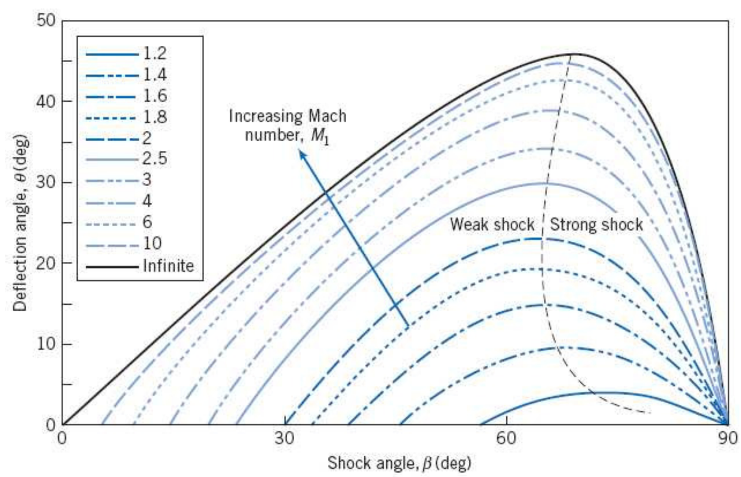


Fig. 13.29 Oblique shock deflection angle.

From Fig. 13.29 the smallest Mach number for which an oblique shock exists at a deflection $\theta = 10^o$ is approximately $M_1 = 1.4$.

By trial and error, a more precise answer is
(using built-in function *Theta* (M, β, k))

$M_1 =$	1.42	
$\beta =$	67.4	^{o}
$\theta =$	10.00	^{o}
$c_1 =$	340	m/s
$V_1 =$	483	m/s

A suggested procedure is:

1) Type in a guess value for M_1
2) Type in a guess value for β

3) Compute θ from Eq. 13.49
   (using built-in function *Theta* (M,β, k))

$$\tan\theta = \frac{2\cot\beta\left(M_1^2\sin^2\beta - 1\right)}{M_1^2(k+\cos 2\beta)+2} \qquad (13.49)$$

4) Use *Solver* to maximize θ by varying β
5) If θ is not 10°, make a new guess for M_1
6) Repeat steps 1 - 5 until θ = 10°

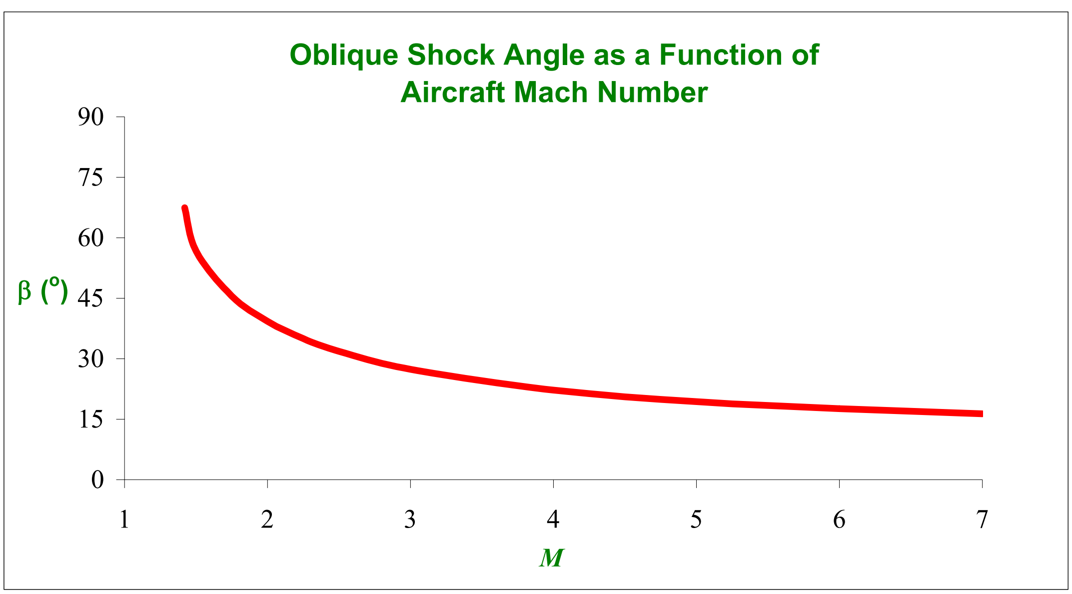
Computed results:

M_1	β (°)	θ (°)	Error
1.42	67.4	10.0	0.0%
1.50	56.7	10.0	0.0%
1.75	45.5	10.0	0.0%
2.00	39.3	10.0	0.0%
2.25	35.0	10.0	0.0%
2.50	31.9	10.0	0.0%
3.00	27.4	10.0	0.0%
4.00	22.2	10.0	0.0%
5.00	19.4	10.0	0.0%
6.00	17.6	10.0	0.0%
7.00	16.4	10.0	0.0%

Sum: 0.0%

To compute this table:

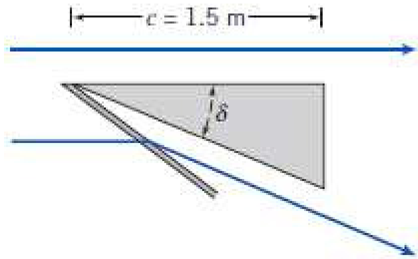
1) Type the range of M_1
2) Type in guess values for β
3) Compute θ from Eq. 13.49
   (using built-in function *Theta* (M,β, k)
4) Compute the absolute error between each θ and θ = 10°
5) Compute the sum of the errors
6) Use *Solver* to minimize the sum by varying the β values
   (Note: You may need to interactively type in new β values
   if *Solver* generates β values that lead to no θ, or to
   β values that correspond to a strong rather than weak shock)



Problem 13.188

[Difficulty: 3]

13.188 The wedge-shaped airfoil shown has chord $c = 1.5$ m and included angle $\delta = 7°$. Find the lift per unit span at a Mach number of 2.75 in air for which the static pressure is 70 kPa.



Given: Data on airfoil flight

Find: Lift per unit span

Solution:

The given or available data is:

$R =$	286.9	J/kg.K
$k =$	1.4	
$p_1 =$	70	kPa
$M_1 =$	2.75	
$\delta =$	7	°
$c =$	1.5	m

Equations and Computations:

The lift per unit span is

$$L = (p_L - p_U)c \qquad (1)$$

(Note that p_L acts on area $c/\cos(\delta)$, but its normal component is multiplied by $\cos(\delta)$)

For the upper surface:

$p_U = p_1$

$p_U =$ 70.0 kPa

For the lower surface:

We need to find M_{1n}

The deflection angle is $\theta = \delta$

$\theta = 7\ ^{o}$

From M_1 and θ, and Eq. 13.49
(using built-in function *Theta* (M, β,k))

$$\tan\theta = \frac{2\cot\beta\left(M_1^2\sin^2\beta - 1\right)}{M_1^2(k + \cos 2\beta) + 2} \qquad (13.49)$$

For $\theta = 7.0\ ^{o}$

$\beta = 26.7\ ^{o}$

(Use *Goal Seek* to vary β so that $\theta = \delta$)

From M_1 and β $\quad M_{1n} = 1.24$

From M_{1n} and p_1, and Eq. 13.48d
(using built-in function *NormpfromM* (M,k))

$$\frac{p_2}{p_1} = \frac{2k}{k+1}M_{1_n}^2 - \frac{k-1}{k+1} \qquad (13.48d)$$

$p_2 = 113$ kPa

$p_L = p_2$

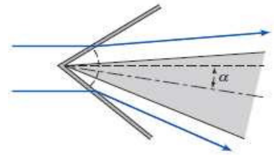
$p_L = 113$ kPa

From Eq 1 $\quad L = 64.7$ kN/m

Problem 13.189

[Difficulty: 4]

13.189 The airfoil of Problem 13.186 will develop a detached shock on the lower surface if the angle of attack, α, exceeds a certain value. What is this angle of attack? Plot the pressure and temperature in the region adjacent to the lower surface as functions of angle of attack, α, ranging from $\alpha = 0°$ to the angle at which the shock becomes detached. What are the maximum pressure and temperature?



Given: Airfoil with included angle of 60°

Find: Angle of attack at which oblique shock becomes detached

Solution:

The given or available data is:

	R =	286.9	J/kg.K
	k =	1.4	
	T_1 =	276.5	K
	p_1 =	75	kPa
	V_1 =	1200	m/s
	δ =	60	°

Equations and Computations:

From T_1 $\quad c_1 =$ 333 m/s

Then $\quad M_1 =$ 3.60

From Fig. 13.29, at this Mach number the smallest deflection angle for which an oblique shock exists is approximately $\theta = 35°$.

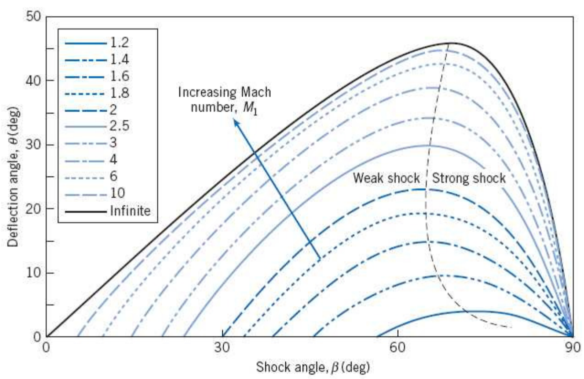


Fig. 13.29 Oblique shock deflection angle.

By using *Solver*, a more precise answer is (using built-in function *Theta*(M, β, k)

M_1 = 3.60

β = 65.8 °

θ = 37.3 °

A suggested procedure is:

1) Type in a guess value for β
2) Compute θ from Eq. 13.49 (using built-in function *Theta*(M, β, k))

$$\tan\theta = \frac{2\cot\beta\left(M_1^2\sin^2\beta - 1\right)}{M_1^2(k + \cos 2\beta) + 2} \qquad (13.49)$$

3) Use *Solver* to maximize θ by varying β

For a deflection angle θ the angle of attack α is

$\alpha = \theta - \delta/2$

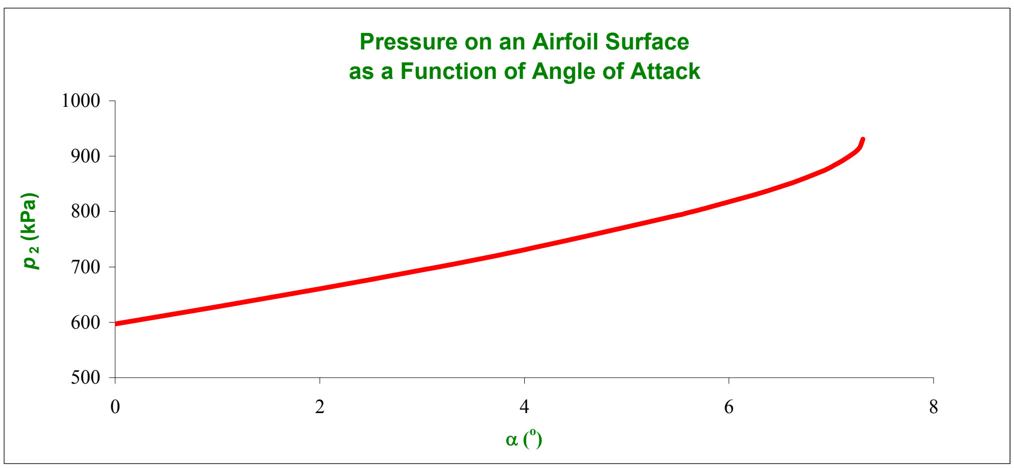
α = 7.31 °

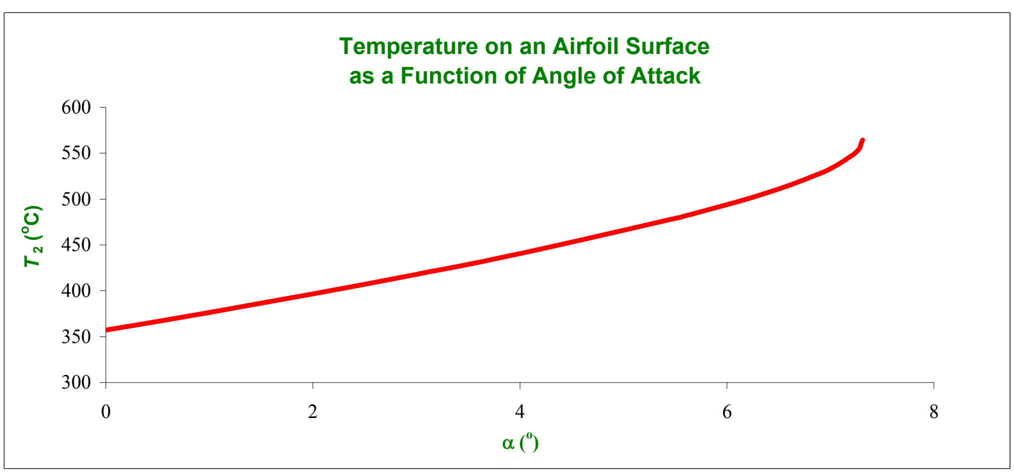
Computed results:

α (°)	β (°)	θ (°) Needed	θ (°)	Error	M_{1n}	p_2 (kPa)	T_2 (°C)
0.00	47.1	30.0	30.0	0.0%	2.64	597	357
1.00	48.7	31.0	31.0	0.0%	2.71	628	377
2.00	50.4	32.0	32.0	0.0%	2.77	660	397
3.00	52.1	33.0	33.0	0.0%	2.84	695	418
4.00	54.1	34.0	34.0	0.0%	2.92	731	441
5.50	57.4	35.5	35.5	0.0%	3.03	793	479
5.75	58.1	35.8	35.7	0.0%	3.06	805	486
6.00	58.8	36.0	36.0	0.0%	3.08	817	494
6.25	59.5	36.3	36.2	0.0%	3.10	831	502
6.50	60.4	36.5	36.5	0.0%	3.13	845	511
6.75	61.3	36.8	36.7	0.0%	3.16	861	521
7.00	62.5	37.0	37.0	0.0%	3.19	881	533
7.25	64.4	37.3	37.2	0.0%	3.25	910	551
7.31	65.8	37.3	37.3	0.0%	3.28	931	564
			Sum:	0.0%	Max:	931	564

To compute this table:

1) Type the range of α
2) Type in guess values for β
3) Compute θ_{Needed} from $\theta = \alpha + \delta/2$
4) Compute θ from Eq. 13.49
   (using built-in function *Theta* (M,β, k)
5) Compute the absolute error between each θ and θ_{Needed}
6) Compute the sum of the errors
7) Use *Solver* to minimize the sum by varying the β values
   (Note: You may need to interactively type in new β values
   if *Solver* generates β values that lead to no θ)
8) For each α, M_{1n} is obtained from M_1, and Eq. 13.47a
9) For each α, p_2 is obtained from p_1, M_{1n}, and Eq. 13.48d
   (using built-in function *NormpfromM* (M,k))
10) For each α, T_2 is obtained from T_1, M_{1n}, and Eq. 13.48c
   (using built-in function *NormTfromM* (M,k))





Problem 13.190

[Difficulty: 3]

13.190 An oblique shock causes a flow that was at $M = 4$ and a static pressure of 75 kPa to slow down to $M = 2.5$. Find the deflection angle and the static pressure after the shock.

Given: Oblique shock Mach numbers

Find: Deflection angle; Pressure after shock

Solution:

The given or available data is:

$k =$	1.4	
$p_1 =$	75	kPa
$M_1 =$	4	
$M_2 =$	2.5	

Equations and Computations:

We make a guess for β: $\beta = 33.6$ °

From M_1 and β, and Eq. 13.49 (using built-in function *Theta* (M, β,k))

$$\tan\theta = \frac{2\cot\beta\left(M_1^2\sin^2\beta - 1\right)}{M_1^2(k + \cos 2\beta) + 2} \qquad (13.49)$$

$\theta = 21.0$ °

From M_1 and β: $M_{1n} = 2.211$

From M_2, θ, and β: $M_{2n} = 0.546$ (1)

We can also obtain M_{2n} from Eq. 13.48a (using built-in function *normM2fromM* (M,k))

$$M_{2_n}^2 = \frac{M_{1_n}^2 + \frac{2}{k-1}}{\frac{2k}{k-1}M_{1_n}^2 - 1} \qquad (13.48a)$$

$M_{2n} = 0.546$ (2)

We need to manually change β so that Eqs. 1 and 2 give the same answer.

Alternatively, we can compute the difference between 1 and 2, and use *Solver* to vary β to make the difference zero

Error in $M_{2n} =$ 0.00%

Then p_2 is obtained from Eq. 13.48d (using built-in function *normpfromm* (M,k))

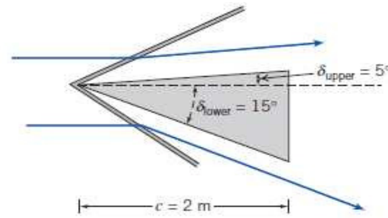
$$\frac{p_2}{p_1} = \frac{2k}{k+1}M_{1_n}^2 - \frac{k-1}{k+1} \qquad (13.48d)$$

$p_2 = 415$ kPa

Problem 13.191

[Difficulty: 3]

13.191 The wedge-shaped airfoil shown has chord $c = 2$ m and angles $\delta_{lower} = 15°$ and $\delta_{upper} = 5°$. Find the lift per unit span at a Mach number of 2.75 in air at a static pressure of 75 kPa.



Given: Data on airfoil flight

Find: Lift per unit span

Solution:

The given or available data is:

$R =$	286.9	J/kg.K
$k =$	1.4	
$p_1 =$	75	kPa
$M_1 =$	2.75	
$\delta_U =$	5	o
$\delta_L =$	15	o
$c =$	2	m

Equations and Computations:

The lift per unit span is

$$L = (p_L - p_U)c \qquad (1)$$

(Note that each p acts on area $c/\cos(\delta)$, but its normal component is multiplied by $\cos(\delta)$)

For the upper surface:

We need to find $M_{1n(U)}$

The deflection angle is $\theta_U = \delta_U$

$\theta_U =$ 5 o

From M_1 and θ_U, and Eq. 13.49
(using built-in function *Theta* (M, β, k))

$$\tan\theta = \frac{2\cot\beta\left(M_1^2\sin^2\beta - 1\right)}{M_1^2(k + \cos 2\beta) + 2} \qquad (13.49)$$

For $\theta_U =$ 5.00 o

$\beta_U =$ 25.1 o

(Use *Goal Seek* to vary β_U so that $\theta_U = \delta_U$)

From M_1 and β_U $M_{1n(U)} =$ 1.16

From $M_{1n(U)}$ and p_1, and Eq. 13.48d
(using built-in function *NormpfromM* (M, k))

$$\frac{p_2}{p_1} = \frac{2k}{k+1} M_{1_n}^2 - \frac{k-1}{k+1} \qquad (13.48d)$$

$p_2 =$ 106 kPa

$p_U =$ p_2

$p_U =$ 106 kPa

For the lower surface:

We need to find $M_{1n(L)}$

The deflection angle is $\theta_L =$ δ_L

$\theta_L =$ 15 °

From M_1 and θ_L, and Eq. 13.49
(using built-in function *Theta* (M, β, k))

For $\theta_L =$ 15.00 °
$\beta_L =$ 34.3 °

(Use *Goal Seek* to vary β_L so that $\theta_L = \delta_L$)

From M_1 and β_L $M_{1n(L)} =$ 1.55

From $M_{1n(L)}$ and p_1, and Eq. 13.48d
(using built-in function *NormpfromM* (M, k))

$p_2 =$ 198 kPa

$p_L =$ p_2

$p_L =$ 198 kPa

From Eq 1 $L =$ 183 kN/m

Problem 13.192

[Difficulty: 3]

13.192 Air flows at a Mach number of 3.3, with static conditions of 100°F and 20 psia. An oblique shock is observed at an angle of 45° relative to the flow. Calculate the post-shock conditions (pressure, temperature, Mach number). What is the deflection angle for the flow? Is this a strong or a weak shock?

Given:
Air deflected at an angle, causing an oblique shock

Find:
Post shock pressure, temperature, and Mach number, deflection angle, strong or weak

Solution:

The given or available data is:

$R =$	53.33	ft-lbf/lbm-°R
$k =$	1.4	
$M_1 =$	3.3	
$T_1 =$	100	°F
$T_1 =$	560	°R
$p_1 =$	20	psia
$\beta =$	45	°

Equations and Computations:

The pre-shock Mach numbers normal and parallel to the wave are:

$M_{1n} =$ 2.3335

$M_{1t} =$ 2.3335

The sound speed upstream of the shock is:

$c_1 =$ 1160.30 ft/s

Therefore, the speed of the flow parallel to the wave is:

$V_{1t} =$ 2707.51 ft/s

The post-shock Mach number normal to the wave is:

$M_{2n} =$ 0.5305

The pressure and temperature ratios across the shock wave are:

$p_2/p_1 =$ 6.1858

$T_2/T_1 =$ 1.9777

Therefore, the post-shock temperature and pressure are:

$p_2 =$ 124 psia

$T_2 =$ 1108 °R

$T_2 =$ 648 °F

The sound speed downstream of the shock is:

$c_2 =$ 1631.74 ft/s

So the speed of the flow normal to wave is:

$V_{2n} =$ 865.63 ft/s

The speed of the flow parallel to the wave is preserved through the shock:

$V_{2t} =$ 2707.51 ft/s

Therefore the flow speed after the shock is:

$V_2 =$ 2842.52 ft/s

and the Mach number is:

$M_2 =$ 1.742

Based on the Mach number and shock angle, the deflection angle is:

$\theta =$ 27.3 °

Since the Mach number at 2 is supersonic, this is a ***weak*** wave. This can be confirmed by inspecting Fig. 13.29 in the text.

Problem 13.193 [Difficulty: 3]

13.193 Air entering the inlet of a jet engine is turned through an angle of 8°, creating an oblique shock. If the freestream flow of air is at Mach 4 and 8 psia, what is the pressure after the oblique shock? What would the pressure be if the flow were through two separate 4° wedges instead of a single 8° wedge?

Given: Air passing through jet inlet

Find: Pressure after one oblique shock; pressure after two shocks totaling same overall turn

Solution:

The given or available data is:

$R =$	53.33	ft-lbf/lbm-°R
$k =$	1.4	
$M_1 =$	4	
$p_1 =$	8	psia
$\theta =$	8	°

Equations and Computations:

To find the shock angle, we have to iterate on the shock angle until we match the deflection angle, which is a function of Mach number, specific heat ratio, and shock angle.

$\beta =$ 20.472 °

$\theta =$ 8.0000 °

We used *Solver* in *Excel* to iterate on the shock angle.

The pre-shock Mach number normal to the wave is:

$M_{1n} =$ 1.3990

The pressure ratio across the shock wave is:

$p_2/p_1 =$ 2.1167

Therefore, the post-shock pressure is:

$p_2 =$ 16.93 psia

Now if we use two 4-degree turns, we perform two oblique-shock calculations.

For the first turn:

$\beta_{1\text{-}2a} =$ 17.258 °

$\theta =$ 4.0000 °

We used *Solver* in *Excel* to iterate on the shock angle.

The pre-shock Mach number normal to the wave is:

$M_{1n} =$ 1.1867

The post-shock Mach number normal to the wave is:

$M_{2an} =$ 0.8506

The pressure ratio across the shock wave is:

$p_{2a}/p_1 =$ 1.4763

Therefore, the post-shock pressure is:

$p_{2a} =$ 11.8100 psia

So the Mach number after the first shock wave is:

$$M_{2a} = 3.7089$$

For the second turn:

$$\beta_{2a\text{-}2b} = 18.438 \quad °$$

$$\theta = 4.0000 \quad °$$

We used *Solver* in *Excel* to iterate on the shock angle.

The pre-shock Mach number normal to the wave is:

$$M_{2an} = 1.1731$$

The post-shock Mach number normal to the wave is:

$$M_{2bn} = 0.8594$$

The pressure ratio across the shock wave is:

$$p_{2b}/p_{2a} = 1.4388$$

Therefore, the post-shock pressure is:

$$p_{2b} = 16.99 \quad \text{psia}$$

The pressure recovery is slightly better for two weaker shocks than a single stronger one!

Problem 13.194

[Difficulty: 4]

13.194 Air having an initial Mach number of 2.3 and static conditions of 14.7 psia and 80°F is turned through an angle of 10°. The resulting shock at the corner is reflected from the opposite wall, turning the flow back 10° to its original direction. Calculate the pressure, temperature, and Mach number after the initial and reflected shock waves.

Given:
Air turning through an incident and reflected shock wave

Find:
Pressure, temperature, and Mach number after each wave

Solution:

The given or available data is:

$R =$	53.33	ft-lbf/lbm-°R
$k =$	1.4	
$M_1 =$	2.3	
$p_1 =$	14.7	psia
$T_1 =$	80	°F
$T_1 =$	540	°R
$\theta =$	10	°

Equations and Computations:

To find the shock angle, we have to iterate on the shock angle until we match the deflection angle, which is a function of Mach number, specific heat ratio, and shock angle.

For the first turn:

$\beta_{1\text{-}2} =$ 34.326 °

$\theta =$ 10.0000 °

We used *Solver* in *Excel* to iterate on the shock angle.

The pre-shock Mach numbers normal and parallel to the wave are:

$M_{1n} =$ 1.2970

$M_{1t} =$ 1.8994

The post-shock Mach number normal to the wave is:

$M_{2n} =$ 0.7875

The pressure and temperature ratios across the shock wave are:

$p_2/p_1 =$ 1.7959

$T_2/T_1 =$ 1.1890

Therefore, the post-shock pressure and temperature are:

$p_2 =$ 26.4 psia

$T_2 =$ 642 °R

Since the parallel component of velocity is preserved across the shock and the Mach number is related to the square root of temperature, the new parallel component of Mach number is:

$M_{2t} =$ 1.7420

So the Mach number after the first shock wave is:

$M_2 =$ 1.912

For the second turn:

$\beta_{2\text{-}3} =$ 41.218 °

$\theta =$ 10.0000 °

We used *Solver* in *Excel* to iterate on the shock angle.

The pre-shock Mach numbers normal and parallel to the wave are:

$$M_{1n} = 1.2597$$

$$M_{1t} = 1.4380$$

The post-shock Mach number normal to the wave is:

$$M_{2an} = 0.8073$$

The pressure and temperature ratios across the shock wave are:

$$p_3/p_2 = 1.6845$$

$$T_2/T_1 = 1.1654$$

Therefore, the post-shock pressure is:

$$p_3 = 44.5 \text{ psia}$$

$$T_3 = 748 \text{ °R}$$

Since the parallel component of velocity is preserved across the shock and the Mach number is related to the square root of temperature, the new parallel component of Mach number is:

$$M_{2t} = 1.3320$$

So the Mach number after the second shock wave is:

$$M_2 = 1.558$$

Problem 13.195

[Difficulty: 3]

13.195 A wedge-shaped projectile (half angle is 10°) is launched through air at 1 psia and 10°F. If the static pressure measurement on the surface of the wedge is 3 psia, calculate the speed at which the projectile is moving through the air.

Given: Wedge-shaped projectile

Find: Speed at which projectile is traveling through the air

Solution:

The given or available data is:

$R =$	53.33	ft-lbf/lbm-°R
$k =$	1.4	
$p_1 =$	1	psia
$T_1 =$	10	°F
$T_1 =$	470	°R
$\theta =$	10	°
$p_2 =$	3	psia

Equations and Computations:

The pressure ratio across the shock wave is:

$p_2/p_1 =$ 3.0000

For this pressure ratio, we can iterate to find the Mach number of the flow normal to the shock wave:

$M_{1n} =$ 1.6475

$p_2/p_1 =$ 3.0000

We used *Solver* in *Excel* to iterate on the Mach number.

With the normal Mach number, we can iterate on the incident Mach number to find the right combination of Mach number and shock angle to match the turning angle of the flow and normal Mach number:

$M_1 =$ 4.9243

$\beta_{1\text{-}2} =$ 19.546 °

$\theta =$ 10.0000 °

The pre-shock Mach numbers normal and parallel to the wave are:

$M_{1n} =$ 1.6475

$M_{1t} =$ 4.6406

We used *Solver* in *Excel* to iterate on the Mach number and shock angle.

Now that we have the upstream Mach number, we can find the speed. The sound speed upstream of the shock wave is:

$c_1 =$ 1062.9839 ft/s

Therefore, the speed of the flow relative to the wedge is:

$V_1 =$ 5234 ft/s

Problem 13.196

[Difficulty: 4]

13.196 Air at Mach 2 and 1 atmosphere is turned through an expansion of 16°, followed by another turn of 16°, causing an oblique shock wave. Calculate the Mach number and pressure downstream of the oblique shock.

Given: Flow turned through an expansion followed by a oblique shock wave

Find: Mach number and pressure downstream of the shock wave

Solution:

The given or available data is:

$R =$	53.33	ft-lbf/lbm-°R
$k =$	1.4	
$M_1 =$	2	
$p_1 =$	1	atm
$\theta =$	16	°

Equations and Computations:

The Prandtl-Meyer function of the flow before the expansion is:

$\omega_1 =$ 26.380 °

Since we know the turning angle of the flow, we know the Prandtl-Meyer function after the expansion:

$\omega_2 =$ 42.380 °

We can iterate to find the Mach number after the expansion:

$M_2 =$ 2.6433

$\omega_2 =$ 42.380 °

The pressure ratio across the expansion wave is:

$p_2/p_1 =$ 0.3668

Therefore the pressure after the expansion is:

$p_2 =$ 0.3668 atm

We can iterate on the shock angle to find the conditions after the oblique shock:

$\beta_{2\text{-}3} =$ 36.438 °

$\theta =$ 16.0000 °

We used *Solver* in *Excel* to iterate on the shock angle.

The pre-shock Mach numbers normal and parallel to the wave are:

$M_{2n} =$ 1.5700

$M_{2t} =$ 2.1265

The post-shock Mach number normal to the wave is:

$M_{3n} =$ 0.6777

The pressure and tempreature ratios across the shock are:

$p_3/p_2 =$ 2.7090

$T_3/T_2 =$ 1.3674

The pressure after the shock wave is:

$p_3 =$ 0.994 atm

We can get the post-shock Mach number parallel to the shock from the temperature ratio:

$M_{3t} =$ 1.8185

So the post-shock Mach number is:

$M_3 =$ 1.941

Problem 13.197

[Difficulty: 4]

13.197 Air at Mach 2.0 and 5 psia static pressure is turned through an angle of 20°. Determine the resulting static pressure and stagnation pressure when the turning is achieved through (a) a single oblique shock, (b) two oblique shocks, each turning the flow 10°, and (c) an isentropic compression wave system.

Given: Air passing through jet inlet

Find: Pressure after one oblique shock; after two shocks totaling same overall turn, after isentropic compression

Solution:

The given or available data is:

$R =$	53.33	ft-lbf/lbm-°R
$k =$	1.4	
$M_1 =$	2	
$p_1 =$	5	psia
$\theta =$	20	°

Equations and Computations:

To find the shock angle, we have to iterate on the shock angle until we match the deflection angle, which is a function of Mach number, specific heat ratio, and shock angle.

$\beta =$ 53.423 °

$\theta =$ 20.0000 °

We used *Solver* in *Excel* to iterate on the shock angle.

The pre-shock Mach number normal to the wave is:

$M_{1n} =$ 1.6061

The pressure ratio across the shock wave is:

$p_2/p_1 =$ 2.8429

Therefore, the post-shock pressure is:

$p_2 =$ 14.21 psia

Now if we use two 10-degree turns, we perform two oblique-shock calculations.

For the first turn:

$\beta_{1\text{-}2a} =$ 39.314 °

$\theta =$ 10.0000 °

We used *Solver* in *Excel* to iterate on the shock angle.

The pre-shock Mach number normal to the wave is:

$M_{1n} =$ 1.2671

The post-shock Mach number normal to the wave is:

$M_{2an} =$ 0.8032

The pressure ratio across the shock wave is:

$p_{2a}/p_1 =$ 1.7066

Therefore, the post-shock pressure is:

$p_{2a} =$ 8.5329 psia

So the Mach number after the first shock wave is:

$M_{2a} =$ 1.6405

For the second turn:

$\beta_{2a\text{-}2b} =$ 49.384 °

$\theta =$ 10.0000 °

We used *Solver* in *Excel* to iterate on the shock angle.

The pre-shock Mach number normal to the wave is:

$$M_{2an} = 1.2453$$

The post-shock Mach number normal to the wave is:

$$M_{2bn} = 0.8153$$

The pressure ratio across the shock wave is:

$$p_{2b}/p_{2a} = 1.6426$$

Therefore, the post-shock pressure is:

$$p_{2b} = 14.02 \quad \text{psia}$$

For the isentropic compression, we need to calculate the Prandtl-Meyer function for the incident flow:

$$\omega_1 = 26.3798 \quad °$$

The flow out of the compression will have a Prandtl-Meyer function of:

$$\omega_{2i} = 6.3798 \quad °$$

To find the exit Mach number, we need to iterate on the Mach number to match the Prandtl-Meyer function:

$$M_{2i} = 1.3076$$

$$\omega_{2i} = 6.3798 \quad °$$

The pressure ratio across the compression wave is:

$$p_{2i}/p_1 = 2.7947$$

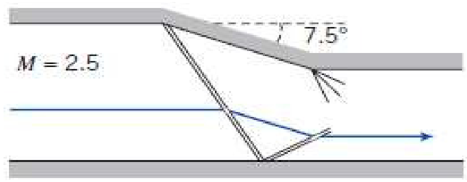
Therefore, the exit pressure is:

$$p_{2i} = 13.97 \quad \text{psia}$$

Problem 13.198

[Difficulty: 3]

13.198 Air flows isentropically at $M = 2.5$ in a duct. There is a 7.5° contraction that triggers an oblique shock, which in turn reflects off a wall generating a second oblique shock. This second shock is necessary so the flow ends up flowing parallel to the channel walls after the two shocks. Find the Mach number and pressure in the contraction and downstream of the contraction. (Note that the convex corner will have expansion waves to redirect the flow along the upper wall.)



Given: Air flow in a duct

Find: Mach number and pressure at contraction and downstream;

Solution:

The given or available data is:

$k =$ 1.4

$M_1 =$ 2.5

$\theta =$ 7.5 °

$p_1 =$ 50 kPa

Equations and Computations:

For the first oblique shock (1 to 2) we need to find β from Eq. 13.49

$$\tan\theta = \frac{2\cot\beta(M_1^2\sin^2\beta - 1)}{M_1^2(k+\cos 2\beta)+2} \qquad (13.49)$$

We choose β by iterating or by using *Goal Seek* to target θ (below) to equal the given θ
Using built-in function *theta* (*M*, β,*k*)

$\theta =$ 7.50 °

$\beta =$ 29.6 °

Then M_{1n} can be found from geometry (Eq. 13.47a)

$M_{1n} =$ 1.233

Then M_{2n} can be found from Eq. 13.48a)
Using built-in function *NormM2fromM* (*M*,*k*)

$$M_{2_n} = f(M_{1_n}) \qquad (13.48a)$$

$M_{2n} =$ 0.822

Then, from M_{2n} and geometry (Eq. 13.47b)

$M_2 =$ 2.19

From M_{1n} and Eq. 13.48d (using built-in function *NormpfromM* (M, k))

$$\frac{p_2}{p_1} = f(M_{1_n}) \qquad (13.48d)$$

$p_2/p_1 =$ 1.61 Pressure ratio
$p_2 =$ 80.40

We repeat the analysis of states 1 to 2 for states 2 to 3, to analyze the second oblique shock

We choose β for M_2 by iterating or by using *Goal Seek* to target θ (below) to equal the given θ
Using built-in function *theta* (M, β, k)

$\theta =$ 7.50 °
$\beta =$ 33.5 °

Then M_{2n} (normal to second shock!) can be found from geometry (Eq. 13.47a)

$M_{2n} =$ 1.209

Then M_{3n} can be found from Eq. 13.48a)
Using built-in function *NormM2fromM* (M, k)

$M_{3n} =$ 0.837

Then, from M_{3n} and geometry (Eq. 13.47b)

$M_3 =$ 1.91

From M_{2n} and Eq. 13.48d (using built-in function *NormpfromM* (M, k))

$p_3/p_2 =$ 1.54 Pressure ratio
$p_3 =$ 124

Problem 13.199

[Difficulty: 4]

13.199 The geometry of the fuselage and engine cowling near the inlet to the engine of a supersonic fighter aircraft is designed so that the incoming air at $M = 3$ is deflected 7.5 degrees, and then experiences a normal shock at the engine entrance. If the incoming air is at 50 kPa, what is the pressure of the air entering the engine? What would be the pressure if the incoming air was slowed down by only a normal shock?

Given: Air flow into engine

Find: Pressure of air in engine; Compare to normal shock

Solution:

The given or available data is:

	$k =$	1.4	
	$p_1 =$	50	kPa
	$M_1 =$	3	
	$\theta =$	7.5	°

Equations and Computations:

Assuming isentropic flow deflection

$$p_0 = \text{constant}$$

$$p_{02} = p_{01}$$

For p_{01} we use Eq. 13.7a (using built-in function *Isenp* (M, k))

$$\frac{p_0}{p} = \left[1 + \frac{k-1}{2}M^2\right]^{k/(k-1)} \qquad (13.7a)$$

$p_{01} =$ 1837 kPa

$p_{02} =$ 1837 kPa

For the deflection $\theta =$ 7.5 °

From M_1 and Eq. 13.55 (using built-in function *Omega* (M, k))

$$\omega = \sqrt{\frac{k+1}{k-1}}\tan^{-1}\left(\sqrt{\frac{k-1}{k+1}(M^2-1)}\right) - \tan^{-1}\left(\sqrt{M^2-1}\right) \qquad (13.55)$$

$\omega_1 =$ 49.8 °

Deflection = $\omega_2 - \omega_1 = \omega(M_2) - \omega(M_1)$ (1)

Applying Eq. 1 $\omega_2 = \omega_1 - \theta$ (Compression!)

$\omega_2 =$ 42.3 °

From ω_2, and Eq. 13.55 (using built-in function *Omega* (M, k))

For $\omega_2 =$ 42.3 °

$M_2 =$ 2.64

(Use *Goal Seek* to vary M_2 so that ω_2 is correct)

Hence for p_2 we use Eq. 13.7a
(using built-in function *Isenp* (M, k))

$$p_2 = p_{02}/(p_{02}/p_2)$$

$p_2 =$ 86.8 kPa

For the normal shock (2 to 3) $M_2 =$ 2.64

From M_2 and p_2, and Eq. 13.41d (using built-in function *NormpfromM* (M,k))

$$\frac{p_2}{p_1} = \frac{2k}{k+1} M_1^2 - \frac{k-1}{k+1} \qquad (13.41d)$$

$p_3 =$ 690 kPa

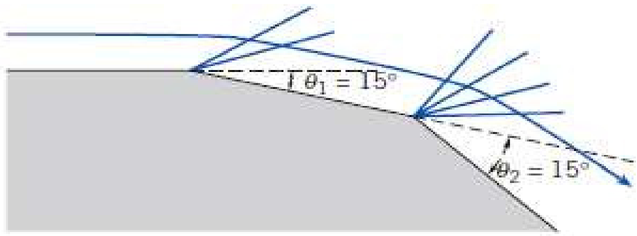
For slowing the flow down from M_1 with only a normal shock, using Eq. 13.41d

$p =$ 517 kPa

Problem 13.200

[Difficulty: 3]

13.200 Air flows at Mach number of 1.5, static pressure 95 kPa, and is expanded by angles $\theta_1 = 15°$ and $\theta_2 = 15°$, as shown. Find the pressure changes.



Given: Deflection of air flow

Find: Pressure changes

Solution:

The given or available data is:

$R =$	286.9	J/kg.K
$k =$	1.4	
$p =$	95	kPa
$M =$	1.5	
$\theta_1 =$	15	o
$\theta_2 =$	15	o

Equations and Computations:

We use Eq. 13.55

$$\omega = \sqrt{\frac{k+1}{k-1}}\tan^{-1}\left(\sqrt{\frac{k-1}{k+1}(M^2-1)}\right) - \tan^{-1}\left(\sqrt{M^2-1}\right) \qquad (13.55)$$

and

$$\text{Deflection} = \omega_a - \omega_b = \omega(M_a) - \omega(M_b) \qquad (1)$$

From M and Eq. 13.55 (using built-in function $Omega(M, k)$)

$\omega =$ 11.9 o

For the first deflection:

Applying Eq. 1

$\theta_1 = \omega_1 - \omega$

$\omega_1 = \theta_1 + \omega$

$\omega_1 =$ 26.9 o

From ω_1, and Eq. 13.55
(using built-in function $Omega(M, k)$)

For $\omega_1 =$ 26.9 o

$M_1 =$ 2.02

(Use *Goal Seek* to vary M_1 so that ω_1 is correct)

Hence for p_1 we use Eq. 13.7a

$$\frac{p_0}{p} = \left[1 + \frac{k-1}{2}M^2\right]^{k/(k-1)} \qquad (13.7a)$$

The approach is to apply Eq. 13.7a twice, so that
(using built-in function *Isenp* (M, k))

$p_1 = p\,(p_0/p)/(p_0/p_1)$

$p_1 =$ 43.3 kPa

For the second deflection:

We repeat the analysis of the first deflection

Applying Eq. 1

$\theta_2 + \theta_1 = \omega_2 - \omega$

$\omega_2 = \theta_2 + \theta_1 + \omega$

$\omega_2 =$ 41.9 $^{\circ}$

(Note that instead of working from the initial state to state 2 we could have worked from state 1 to state 2 because the entire flow is isentropic)

From ω_2, and Eq. 13.55
(using built-in function *Omega* (M, k))

For $\omega_2 =$ 41.9 $^{\circ}$
$M_2 =$ 2.62

(Use *Goal Seek* to vary M_2 so that ω_2 is correct)

Hence for p_2 we use Eq. 13.7a
(using built-in function *Isenp* (M, k))

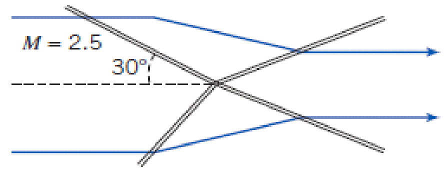
$p_2 = p\,(p_0/p)/(p_0/p_2)$

$p_2 =$ 16.9 kPa

Problem 13.201

[Difficulty: 3]

13.201 A flow at $M = 2.5$ is deflected by a combination of interacting oblique shocks as shown. The first shock pair is aligned at 30° to the flow. A second oblique shock pair deflects the flow again so it ends up parallel to the original flow. If the pressure before any deflections is 50 kPa, find the pressure after two deflections.



Given: Air flow in a duct

Find: Mach number and pressure at contraction and downstream;

Solution:

The given or available data is:

$k =$ 1.4

$M_1 =$ 2.5

$\beta =$ 30 °

$p_1 =$ 50 kPa

Equations and Computations:

For the first oblique shock (1 to 2) we find θ from Eq. 13.49

$$\tan\theta = \frac{2\cot\beta(M_1^2\sin^2\beta - 1)}{M_1^2(k + \cos 2\beta) + 2} \qquad (13.49)$$

Using built-in function *theta* (M, β, k)

$\theta =$ 7.99 °

Also, M_{1n} can be found from geometry (Eq. 13.47a)

$M_{1n} =$ 1.250

Then M_{2n} can be found from Eq. 13.48a)

Using built-in function *NormM2fromM* (M,k)

$$M_{2_n} = f(M_{1_n}) \qquad (13.48a)$$

$M_{2n} =$ 0.813

Then, from M_{2n} and geometry (Eq. 13.47b)

$M_2 =$ 2.17

From M_{1n} and Eq. 13.48d (using built-in function *NormpfromM* (M, k))

$$\frac{p_2}{p_1} = f(M_{1_n}) \qquad (13.48d)$$

$p_2/p_1 =$	1.66	Pressure ratio
$p_2 =$	82.8	

We repeat the analysis for states 1 to 2 for 2 to 3, for the second oblique shock

We choose β for M_2 by iterating or by using *Goal Seek* to target θ (below) to equal the previous θ, using built-in function *theta* (M, β, k)

$\theta =$	7.99	$^{\circ}$
$\beta =$	34.3	$^{\circ}$

Then M_{2n} (normal to second shock!) can be found from geometry (Eq. 13.47a)

$M_{2n} =$ 1.22

Then M_{3n} can be found from Eq. 13.48a)
Using built-in function *NormM2fromM* (M, k)

$M_{3n} =$ 0.829

Then, from M_{3n} and geometry (Eq. 13.47b)

$M_3 =$ 1.87

From M_{2n} and Eq. 13.48d (using built-in function *NormpfromM* (M, k))

$p_3/p_2 =$	1.58	Pressure ratio
$p_3 =$	130	

Problem 13.202

[Difficulty: 4]

13.202 Compare the static and stagnation pressures produced by (a) an oblique shock and (b) isentropic *compression* waves as they each deflect a flow at a Mach number of 3.5 through a deflection angle of 35° in air for which the static pressure is 50 kPa.

Given: Mach number and deflection angle

Find: Static and stagnation pressures due to: oblique shock; compression wave

Solution:

The given or available data is:

$R =$	286.9	J/kg.K
$k =$	1.4	
$p_1 =$	50	kPa
$M_1 =$	3.5	
$\theta =$	35	°

Equations and Computations:

For the oblique shock:

We need to find M_{1n}

The deflection angle is $\theta =$ 35 °

From M_1 and θ, and Eq. 13.49
(using built-in function *Theta* (M, β, k))

$$\tan\theta = \frac{2\cot\beta\left(M_1^2\sin^2\beta - 1\right)}{M_1^2(k + \cos 2\beta) + 2} \qquad (13.49)$$

For $\theta =$ 35.0 °
$\beta =$ 57.2 °

(Use *Goal Seek* to vary β so that $\theta = 35^o$)

From M_1 and β $M_{1n} =$ 2.94

From M_{1n} and p_1, and Eq. 13.48d
(using built-in function *NormpfromM* (M, k))

$$\frac{p_2}{p_1} = \frac{2k}{k+1}M_{1_n}^2 - \frac{k-1}{k+1} \qquad (13.48d)$$

$p_2 =$ 496 kPa

To find M_2 we need M_{2n}. From M_{1n}, and Eq. 13.48a
(using built-in function *NormM2fromM* (M, k))

$$M_{2_n}^2 = \frac{M_{1_n}^2 + \dfrac{2}{k-1}}{\dfrac{2k}{k-1}M_{1_n}^2 - 1} \qquad (13.48a)$$

$M_{2n} =$ 0.479

The downstream Mach number is then obtained from
from M_{2n}, θ and β, and Eq. 13.47b

$$M_{2n} = M_2 \sin(\beta - \theta) \qquad (13.47b)$$

Hence $M_2 =$ 1.27

For p_{02} we use Eq. 12.7a
(using built-in function *Isenp* (M, k))

$$\frac{p_0}{p} = \left[1 + \frac{k-1}{2}M^2\right]^{k/(k-1)} \qquad (13.7a)$$

$p_{02} = p_2/(p_{02}/p_2)$

$p_{02} =$ 1316 kPa

For the isentropic compression wave:

For isentropic flow $p_0 =$ constant

$p_{02} = p_{01}$

For p_{01} we use Eq. 13.7a
(using built-in function *Isenp* (M, k))

$p_{01} =$ 3814 kPa

$p_{02} =$ 3814 kPa

(Note that for the oblique shock, as required by Eq. 13.48b

$$\frac{p_{0_2}}{p_{0_1}} = \frac{\left[\dfrac{\dfrac{k+1}{2}M_{1_n}^2}{1 + \dfrac{k-1}{2}M_{1_n}^2}\right]^{k/(k-1)}}{\left[\dfrac{2k}{k+1}M_{1_n}^2 - \dfrac{k-1}{k+1}\right]^{1/(k-1)}} \qquad (13.48b)$$

$p_{02}/p_{01} =$ 0.345
(using built-in function *Normp0fromM* (M, k)

$p_{02}/p_{01} =$ 0.345

(using p_{02} from the shock and p_{01})

For the deflection $\theta = -\theta$ (*Compression*)

$\theta =$ -35.0 $^{\circ}$

We use Eq. 13.55

$$\omega = \sqrt{\frac{k+1}{k-1}}\tan^{-1}\left(\sqrt{\frac{k-1}{k+1}(M^2-1)}\right) - \tan^{-1}\left(\sqrt{M^2-1}\right) \quad (13.55)$$

and

$$\text{Deflection} = \omega_2 - \omega_1 = \omega(M_2) - \omega(M_1) \qquad (1)$$

From M_1 and Eq. 13.55 (using built-in function *Omega* (M, k))

$\omega_1 =$ 58.5 $^{\circ}$

Applying Eq. 1 $\omega_2 = \omega_1 + \theta$

$\omega_2 =$ 23.5 $^{\circ}$

From ω_2, and Eq. 13.55

(using built-in function *Omega* (M, k))

For $\omega_2 =$ 23.5 $^{\circ}$

$M_2 =$ 1.90

(Use *Goal Seek* to vary M_2 so that $\omega_2 = 23.5^{\circ}$)

Hence for p_2 we use Eq. 13.7a

(using built-in function *Isenp* (M, k))

$p_2 = p_{02}/(p_{02}/p_2)$

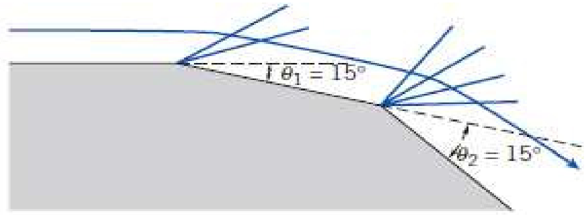
$p_2 =$ 572 kPa

Problem 13.203 [Difficulty: 3]

13.203 Find the incoming and intermediate Mach numbers and static pressures if, after two expansions of $\theta_1 = 15°$ and $\theta_2 = 15°$, the Mach number is 4, and static pressure is 10 kPa.

Given: Deflection of air flow

Find: Mach numbers and pressures



Solution

The given or available data is:

$R =$	286.9	J/kg.K
$k =$	1.4	
$p_2 =$	10	kPa
$M_2 =$	4	
$\theta_1 =$	15	o
$\theta_2 =$	15	o

Equations and Computations:

We use Eq. 13.55

$$\omega = \sqrt{\frac{k+1}{k-1}}\tan^{-1}\left(\sqrt{\frac{k-1}{k+1}(M^2-1)}\right) - \tan^{-1}\left(\sqrt{M^2-1}\right) \qquad (13.55)$$

and

$$\text{Deflection} = \omega_a - \omega_b = \omega(M_a) - \omega(M_b) \qquad (1)$$

From M and Eq. 13.55 (using built-in function *Omega* (M, k))

$\omega_2 =$ 65.8 o

For the second deflection:

Applying Eq. 1

$\omega_1 = \omega_2 - \theta_2$

$\omega_1 =$ 50.8 o

From ω_1, and Eq. 13.55
(using built-in function *Omega* (M, k))

For $\omega_1 =$ 50.8 o
$M_1 =$ 3.05

(Use *Goal Seek* to vary M_1 so that ω_1 is correct)

Hence for p_1 we use Eq. 13.7a

$$\frac{p_0}{p} = \left[1 + \frac{k-1}{2} M^2\right]^{k/(k-1)} \qquad (13.7a)$$

The approach is to apply Eq. 13.7a twice, so that
(using built-in function *Isenp* (M, k))

$$p_1 = p_2(p_0/p_2)/(p_0/p_1)$$

$p_1 =$ 38.1 kPa

For the first deflection:

We repeat the analysis of the second deflection

Applying Eq. 1

$$\theta_2 + \theta_1 = \omega_2 - \omega$$

$$\omega = \omega_2 - (\theta_2 + \theta_1)$$

$\omega =$ 35.8 $^{\text{o}}$

(Note that instead of working from state 2 to the initial state we could have worked from state 1 to the initial state because the entire flow is isentropic)

From ω, and Eq. 13.55
(using built-in function *Omega* (M, k))

For $\omega =$ 35.8 $^{\text{o}}$
$M =$ 2.36

(Use *Goal Seek* to vary M so that ω is correct)

Hence for p we use Eq. 13.7a
(using built-in function *Isenp* (M, k))

$$p = p_2(p_0/p_2)/(p_0/p)$$

$p =$ 110 kPa

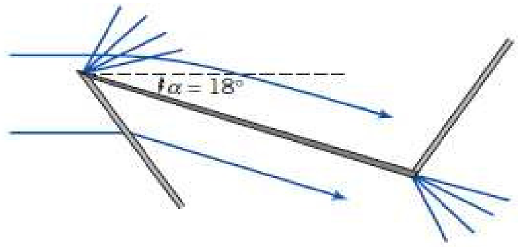
Problem 13.204

[Difficulty: 4]

13.204 Find the lift and drag per unit span on the airfoil shown for flight at a Mach number of 1.75 in air for which the static pressure is 50 kPa. The chord length is 1 m.

Given: Mach number and airfoil geometry

Find: Lift and drag per unit span



Solution:

The given or available data is:

	$R =$	286.9	J/kg.K
	$k =$	1.4	
	$p_1 =$	50	kPa
	$M_1 =$	1.75	
	$\alpha =$	18	°
	$c =$	1	m

Equations and Computations:

The net force per unit span is $F = (p_L - p_U)c$

Hence, the lift force per unit span is

$$L = (p_L - p_U)c\cos(\alpha) \qquad (1)$$

The drag force per unit span is

$$D = (p_L - p_U)c\sin(\alpha) \qquad (2)$$

For the lower surface (oblique shock):

We need to find M_{1n}

The deflection angle is $\theta = \alpha$

$\theta =$ 18 °

From M_1 and θ, and Eq. 13.49
(using built-in function *Theta* (M, β, k))

$$\tan\theta = \frac{2\cot\beta\left(M_1^2\sin^2\beta - 1\right)}{M_1^2(k + \cos 2\beta) + 2} \qquad (13.49)$$

For $\theta =$ 18.0 °
$\beta =$ 62.9 °

(Use *Goal Seek* to vary β so that θ is correct)

From M_1 and β $M_{1n} =$ 1.56

From M_{1n} and p_1, and Eq. 13.48d
(using built-in function *NormpfromM* (M, k))

$$\frac{p_2}{p_1} = \frac{2k}{k+1} M_{1_n}^2 - \frac{k-1}{k+1} \qquad (13.48d)$$

$p_2 =$ 133.2 kPa

$p_L = p_2$

$p_L =$ 133.2 kPa

For the upper surface (isentropic expansion wave):

For isentropic flow $p_0 =$ constant

$p_{02} = p_{01}$

For p_{01} we use Eq. 13.7a
(using built-in function *Isenp* (M, k))

$$\frac{p_0}{p} = \left[1 + \frac{k-1}{2} M^2\right]^{k/(k-1)} \qquad (13.7a)$$

$p_{01} =$ 266 kPa

$p_{02} =$ 266 kPa

For the deflection $\theta = \alpha$ (*Compression*)

$\theta =$ 18.0 $^{\circ}$

We use Eq. 13.55

$$\omega = \sqrt{\frac{k+1}{k-1}} \tan^{-1}\left(\sqrt{\frac{k-1}{k+1}(M^2-1)}\right) - \tan^{-1}\left(\sqrt{M^2-1}\right) \qquad (13.55)$$

and

Deflection = $\omega_2 - \omega_1 = \omega(M_2) - \omega(M_1)$ (3)

From M_1 and Eq. 13.55 (using built-in function *Omega* (M, k))

$\omega_1 =$ 19.3 $^{\circ}$

Applying Eq. 3 $\omega_2 = \omega_1 + \theta$

$\omega_2 =$ 37.3 $^{\circ}$

From ω_2, and Eq. 13.55 (using built-in function *Omega* (M, k))

For $\omega_2 =$ 37.3 ^{o}

$M_2 =$ 2.42

(Use *Goal Seek* to vary M_2 so that ω_2 is correct)

Hence for p_2 we use Eq. 13.7a
(using built-in function *Isenp* (M, k))

$$p_2 = p_{02}/(p_{02}/p_2)$$

$p_2 =$ 17.6 kPa

$p_U = p_2$

$p_U =$ 17.6 kPa

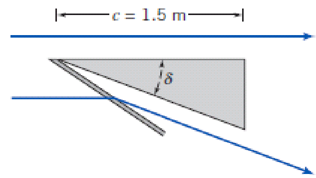
From Eq. 1 $L =$ 110.0 kN/m

From Eq. 2 $D =$ 35.7 kN/m

Problem 13.205

[Difficulty: 3]

13.205 Consider the wedge-shaped airfoil of Problem 13.188. Suppose the oblique shock could be replaced by isentropic *compression* waves. Find the lift per unit span at the Mach number of 2.75 in air for which the static pressure is 70 kPa.



Given:
Wedge-shaped airfoil

Find:
Lift per unit span assuming isentropic flow

Solution:

The given or available data is:

$R =$ 286.9 J/kg.K
$k =$ 1.4
$p =$ 70 kPa
$M =$ 2.75
$\delta =$ 7 $^{\circ}$
$c =$ 1.5 m

Equations and Computations:

The lift per unit span is

$$L = (p_L - p_U)c \qquad (1)$$

(Note that p_L acts on area $c/\cos(\delta)$, but its normal component is multiplied by $\cos(\delta)$)

For the upper surface:

$p_U = p$

$p_U =$ 70 kPa

For the lower surface:

$$\theta = -\delta$$

$$\theta = -7.0 \quad ^{o}$$

We use Eq. 13.55

$$\omega = \sqrt{\frac{k+1}{k-1}}\tan^{-1}\left(\sqrt{\frac{k-1}{k+1}(M^2-1)}\right) - \tan^{-1}\left(\sqrt{M^2-1}\right) \qquad (13.55)$$

and

$$\text{Deflection} = \omega_L - \omega = \omega(M_L) - \omega(M) \qquad (2)$$

From M and Eq. 13.55 (using built-in function *Omega* (M, k))

$$\omega = 44.7 \quad ^{o}$$

Applying Eq. 2

$$\theta = \omega_L - \omega$$

$$\omega_L = \theta + \omega$$

$$\omega_L = 37.7 \quad ^{o}$$

From ω_L, and Eq. 13.55
(using built-in function *Omega* (M, k))

For $\omega_L = 37.7 \quad ^{o}$
$M_L = 2.44$

(Use *Goal Seek* to vary M_L so that ω_L is correct)

Hence for p_L we use Eq. 13.7a

$$\frac{p_0}{p} = \left[1 + \frac{k-1}{2} M^2\right]^{k/(k-1)} \tag{13.7a}$$

The approach is to apply Eq. 13.7a twice, so that
(using built-in function $Isenp(M, k)$)

$$p_L = p\,(p_0/p)/(p_0/p_L)$$

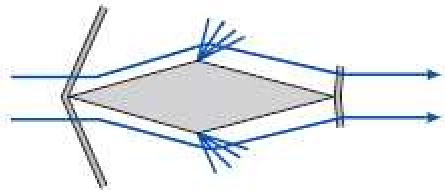
$$p_L = 113 \quad \text{kPa}$$

From Eq 1 $\quad L = 64.7 \quad$ kN/m

Problem 13.206

[Difficulty: 4]

13.206 Find the drag coefficient of the symmetric, zero angle of attack airfoil shown for a Mach number of 2.0 in air for which the static pressure is 95 kPa and temperature is 0°C. The included angles at the nose and tail are each 10°.



Given: Mach number and airfoil geometry

Find: Drag coefficient

Solution:

The given or available data is:

R =	286.9	J/kg.K
k =	1.4	
p_1 =	95	kPa
M_1 =	2	
α =	0	°
δ =	10	°

Equations and Computations:

The drag force is

$$D = (p_F - p_R)cs\tan(\delta/2) \qquad (1)$$

(s and c are the span and chord)

This is obtained from the following analysis

Airfoil thickness (frontal area) = $2s(c/2\tan(\delta/2))$

Pressure difference acting on frontal area = $(p_F - p_R)$

(p_F and p_R are the pressures on the front and rear surfaces)

The drag coefficient is $\quad C_D = D/(1/2\rho V^2 A) \qquad (2)$

But it can easily be shown that

$$\rho V^2 = pkM^2$$

Hence, from Eqs. 1 and 2

$$C_D = (p_F - p_R)\tan(\delta/2)/(1/2pkM^2) \qquad (3)$$

For the frontal surfaces (oblique shocks):

We need to find M_{1n}

The deflection angle is $\theta =$ $\delta/2$

$\theta =$ 5 $^\circ$

From M_1 and θ, and Eq. 13.49
(using built-in function *Theta* (M, β, k))

$$\tan\theta = \frac{2\cot\beta\left(M_1^2\sin^2\beta - 1\right)}{M_1^2(k+\cos 2\beta)+2} \qquad (13.49)$$

For $\theta =$ 5.0 $^\circ$
$\beta =$ 34.3 $^\circ$

(Use *Goal Seek* to vary β so that $\theta = 5^\circ$)

From M_1 and β $M_{1n} =$ 1.13

From M_{1n} and p_1, and Eq. 13.48d
(using built-in function *NormpfromM* (M, k))

$$\frac{p_2}{p_1} = \frac{2k}{k+1}M_{1_n}^2 - \frac{k-1}{k+1} \qquad (13.48d)$$

$p_2 =$ 125.0 kPa

$p_F =$ p_2

$p_F =$ 125.0 kPa

To find M_2 we need M_{2n}. From M_{1n}, and Eq. 13.48a
(using built-in function *NormM2fromM* (M, k))

$$M_{2_n}^2 = \frac{M_{1_n}^2 + \frac{2}{k-1}}{\frac{2k}{k-1} M_{1_n}^2 - 1} \qquad (13.48a)$$

$M_{2n} =$ 0.891

The downstream Mach number is then obtained from
from M_{2n}, θ and β, and Eq. 13.47b

$$M_{2n} = M_2 \sin(\beta - \theta) \qquad (13.47b)$$

Hence $M_2 =$ 1.82

For p_{02} we use Eq. 13.7a
(using built-in function *Isenp* (M, k))

$$\frac{p_0}{p} = \left[1 + \frac{k-1}{2} M^2\right]^{k/(k-1)} \qquad (13.7a)$$

$p_{02} =$ 742 kPa

For the rear surfaces (isentropic expansion waves):

Treating as a new problem

Here: M_1 is the Mach number after the shock
and M_2 is the Mach number after the expansion wave
p_{01} is the stagnation pressure after the shock
and p_{02} is the stagnation pressure after the expansion wave

$M_1 = M_2$ (shock)

$M_1 =$ 1.82

$p_{01} = p_{02}$ (shock)

$p_{01} =$ 742 kPa

For isentropic flow $p_0 =$ constant

$p_{02} = p_{01}$

$p_{02} =$ 742 kPa

For the deflection $\theta = \delta$

$\theta =$ 10.0 $^{\circ}$

We use Eq. 13.55

$$\omega = \sqrt{\frac{k+1}{k-1}}\tan^{-1}\left(\sqrt{\frac{k-1}{k+1}(M^2-1)}\right) - \tan^{-1}\left(\sqrt{M^2-1}\right) \qquad (13.55)$$

and

$$\text{Deflection} = \omega_2 - \omega_1 = \omega(M_2) - \omega(M_1) \qquad (3)$$

From M_1 and Eq. 13.55 (using built-in function *Omega* (M, k))

$\omega_1 =$ 21.3 $^{\circ}$

Applying Eq. 3 $\omega_2 = \omega_1 + \theta$

$\omega_2 =$ 31.3 $^{\circ}$

From ω_2, and Eq. 13.55 (using built-in function Omega(M, k))

For $\omega_2 =$ 31.3 $^{\circ}$

$M_2 =$ 2.18

(Use *Goal Seek* to vary M_2 so that $\omega_2 = 31.3^{\circ}$)

Hence for p_2 we use Eq. 13.7a
(using built-in function *Isenp* (M, k))

$$p_2 = p_{02}/(p_{02}/p_2)$$

$p_2 =$ 71.2 kPa

$p_R =$ p_2

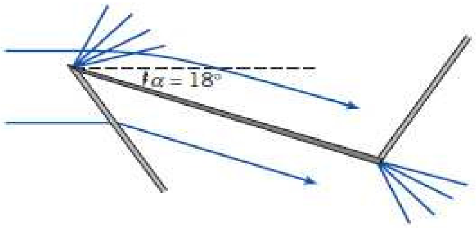
$p_R =$ 71.2 kPa

Finally, from Eq. 1 $C_D =$ 0.0177

Problem 13.207

[Difficulty: 4]

13.207 Plot the lift and drag per unit span, and the lift/drag ratio, as functions of angle of attack for $\alpha = 0°$ to $18°$, for the airfoil shown, for flight at a Mach number of 1.75 in air for which the static pressure is 50 kPa. The chord length is 1 m.



Given: Mach number and airfoil geometry

Find: Plot of lift and drag and lift/drag versus angle of attack

Solution:

The given or available data is:

k =	1.4	
p_1 =	50	kPa
M_1 =	1.75	
α =	12	°
c =	1	m

Equations and Computations:

The net force per unit span is

$$F = (p_L - p_U)c$$

Hence, the lift force per unit span is

$$L = (p_L - p_U)c\cos(\alpha) \qquad (1)$$

The drag force per unit span is

$$D = (p_L - p_U)c\sin(\alpha) \qquad (2)$$

For each angle of attack the following needs to be computed:

For the lower surface (oblique shock):

We need to find M_{1n}

Deflection $\theta =$ α

From M_1 and θ, and Eq. 13.49

(using built-in function *Theta* (M, β, k))

$$\tan\theta = \frac{2\cot\beta\left(M_1^2\sin^2\beta - 1\right)}{M_1^2(k+\cos 2\beta)+2} \quad (13.49)$$

find β

(Use *Goal Seek* to vary β so that θ is the correct value)

From M_1 and β find M_{1n}

From M_{1n} and p_1, and Eq. 13.48d

(using built-in function *NormpfromM* (M, k))

$$\frac{p_2}{p_1} = \frac{2k}{k+1}M_{1_n}^2 - \frac{k-1}{k+1} \quad (13.48d)$$

find p_2

and $p_L =$ p_2

For the upper surface (isentropic expansion wave):

For isentropic flow $p_0 =$ constant

$p_{02} = p_{01}$

For p_{01} we use Eq. 13.7a
(using built-in function *Isenp* (M, k))

$$\frac{p_0}{p} = \left[1 + \frac{k-1}{2} M^2\right]^{k/(k-1)} \qquad (13.7a)$$

find $p_{02} =$ 266 kPa

Deflection $\theta = \alpha$

we use Eq. 13.55

$$\omega = \sqrt{\frac{k+1}{k-1}} \tan^{-1}\left(\sqrt{\frac{k-1}{k+1}(M^2-1)}\right) - \tan^{-1}\left(\sqrt{M^2-1}\right) \qquad (13.55)$$

and

Deflection = $\omega_2 - \omega_1 = \omega(M_2) - \omega(M_1)$ (3)

From M_1 and Eq. 13.55 (using built-in function *Omega* (M, k))

find $\omega_1 =$ 19.3 °

Applying Eq. 3 $\omega_2 = \omega_1 + \theta$ (4)

From ω_2, and Eq. 12.55 (using built-in function *Omega* (M, k))

From ω_2 find M_2

(Use *Goal Seek* to vary M_2 so that ω_2 is the correct value)

Hence for p_2 we use Eq. 13.7a
(using built-in function *Isenp* (M, k))

$$p_2 = p_{02}/(p_{02}/p_2)$$

$$p_U = \quad p_2$$

Finally, from Eqs. 1 and 2, compute L and D

Computed results:

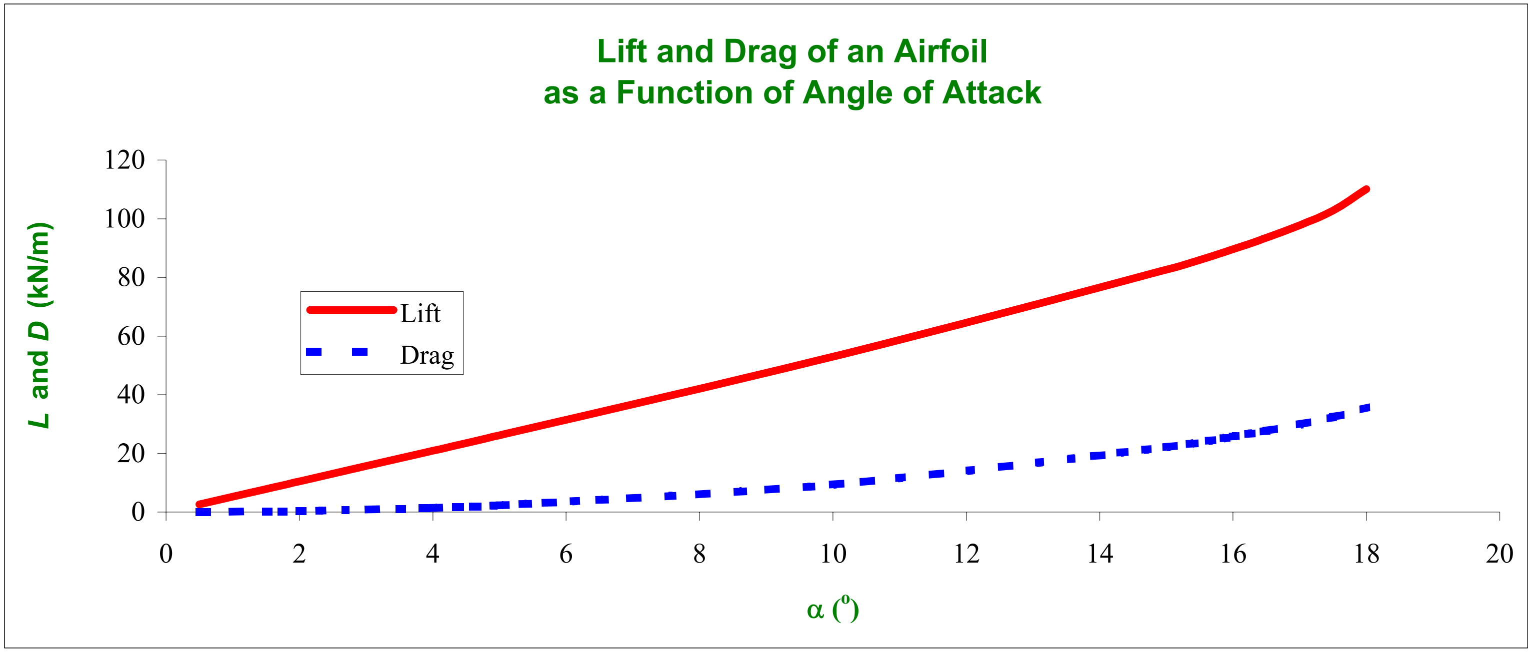
α (°)	β (°)	θ (°)	Error	M_{1n}	p_L (kPa)	ω_2 (°)	ω_2 from M_2 (°)	Error	M_2	p_U (kPa)	L (kN/m)	D (kN/m)	L/D
0.50	35.3	0.50	0.0%	1.01	51.3	19.8	19.8	0.0%	1.77	48.7	2.61	0.0227	115
1.00	35.8	1.00	0.0%	1.02	52.7	20.3	20.3	0.0%	1.78	47.4	5.21	0.091	57.3
1.50	36.2	1.50	0.0%	1.03	54.0	20.8	20.8	0.0%	1.80	46.2	7.82	0.205	38.2
2.00	36.7	2.00	0.0%	1.05	55.4	21.3	21.3	0.0%	1.82	45.0	10.4	0.364	28.6
4.00	38.7	4.00	0.0%	1.09	61.4	23.3	23.3	0.0%	1.89	40.4	20.9	1.46	14.3
5.00	39.7	5.00	0.0%	1.12	64.5	24.3	24.3	0.0%	1.92	38.3	26.1	2.29	11.4
10.00	45.5	10.0	0.0%	1.25	82.6	29.3	29.3	0.0%	2.11	28.8	53.0	9.35	5.67
15.00	53.4	15.0	0.0%	1.41	106.9	34.3	34.3	0.0%	2.30	21.3	82.7	22.1	3.73
16.00	55.6	16.0	0.0%	1.44	113.3	35.3	35.3	0.0%	2.34	20.0	89.6	25.7	3.49
16.50	56.8	16.5	0.0%	1.47	116.9	35.8	35.8	0.0%	2.36	19.4	93.5	27.7	3.38
17.00	58.3	17.0	0.0%	1.49	121.0	36.3	36.3	0.0%	2.38	18.8	97.7	29.9	3.27
17.50	60.1	17.5	0.0%	1.52	125.9	36.8	36.8	0.0%	2.40	18.2	102.7	32.4	3.17
18.00	62.9	18.0	0.0%	1.56	133.4	37.3	37.3	0.0%	2.42	17.6	110	35.8	3.08

Sum: 0.0%

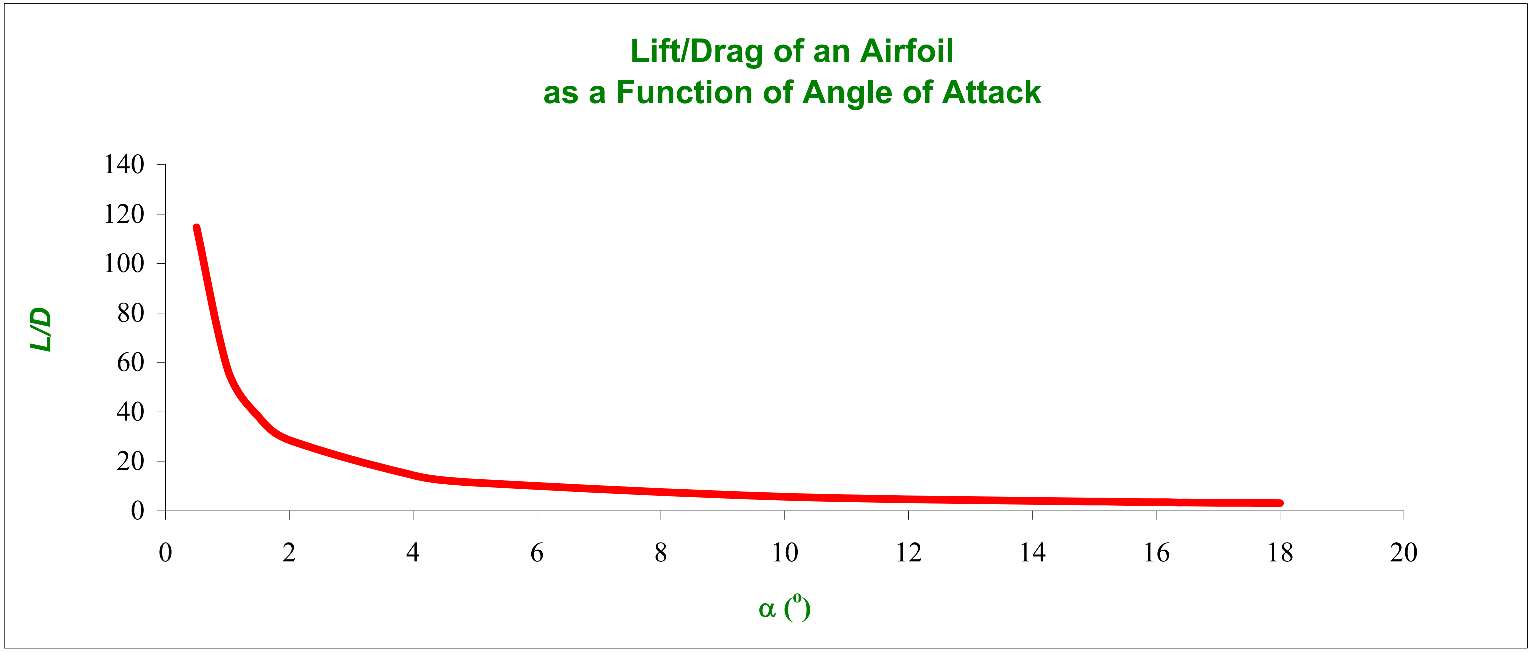
Sum: 0.0%

To compute this table:

1) Type the range of α
2) Type in guess values for β
3) Compute θ from Eq. 13.49
   (using built-in function *Theta* (M,β, k)
4) Compute the absolute error between each θ and α
5) Compute the sum of the errors
6) Use *Solver* to minimize the sum by varying the β values
   (Note: You may need to interactively type in new β values
   if *Solver* generates β values that lead to no θ)
7) For each α, M_{1n} is obtained from M_1, and Eq. 13.47a
8) For each α, p_L is obtained from p_1, M_{1n}, and Eq. 13.48d
   (using built-in function *NormpfromM* (M,k))
9) For each α, compute ω_2 from Eq. 4
10) For each α, compute ω_2 from M_2, and Eq. 13.55
    (using built-in function *Omega* (M,k))
11) Compute the absolute error between the two values of ω_2
12) Compute the sum of the errors
13) Use *Solver* to minimize the sum by varying the M_2 values
    (Note: You may need to interactively type in new M_2 values)
    if *Solver* generates β values that lead to no θ)
14) For each α, p_U is obtained from p_{02}, M_2, and Eq. 13.47a
    (using built-in function *Isenp* (M, k))
15) Compute L and D from Eqs. 1 and 2



Lift and Drag of an Airfoil
as a Function of Angle of Attack
120
100
80
60
40
20
0
0
2
4
6
8
10
12
14
16
18
20
L and D (kN/m)
α (°)
Lift
Drag



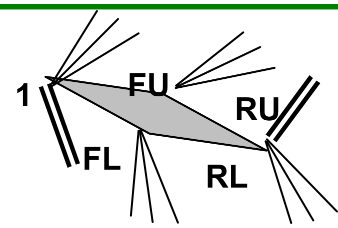
Lift/Drag of an Airfoil
as a Function of Angle of Attack
140
120
100
80
60
40
20
0
0
2
4
6
8
10
12
14
16
18
20
L/D
α (°)

Problem 13.208

[Difficulty: 4]

13.208 Find the lift and drag coefficients of the airfoil of Problem 13.206 if the airfoil now has an angle of attack of 12°.

Given: Mach number and airfoil geometry



Find: Lift and Drag coefficients

Solution:

The given or available data is:

R =	286.9	J/kg.K
k =	1.4	
p_1 =	95	kPa
M_1 =	2	
α =	12	°
δ =	10	°

Equations and Computations:

Following the analysis of Example 13.14
the force component perpendicular to the major axis, per area, is

$$F_V/sc = 1/2\{(p_{FL} + p_{RL}) - (p_{FU} + p_{RU})\} \qquad (1)$$

and the force component parallel to the major axis, per area, is

$$F_H/sc = 1/2\tan(\delta/2)\{(p_{FU} + p_{FL}) - (p_{RU} + p_{RL})\} \qquad (2)$$

using the notation of the figure above.
(s and c are the span and chord)

The lift force per area is

$$F_L/sc = (F_V\cos(\alpha) - F_H\sin(\alpha))/sc \qquad (3)$$

The drag force per area is

$$F_D/sc = (F_V\sin(\alpha) + F_H\cos(\alpha))/sc \qquad (4)$$

The lift coefficient is $\quad C_L = F_L/(1/2\rho V^2 A) \qquad (5)$

But it can be shown that

$$\rho V^2 = pkM^2 \qquad (6)$$

Hence, combining Eqs. 3, 4, 5 and 6

$$C_L = (F_V/sc\,\cos(\alpha) - F_H/sc\,\sin(\alpha))/(1/2pkM^2) \qquad (7)$$

Similarly, for the drag coefficient

$$C_D = (F_V/sc\sin(\alpha) + F_H/sc\cos(\alpha))/(1/2pkM^2) \qquad (8)$$

For surface FL (oblique shock):

We need to find M_{1n}

The deflection angle is $\theta = \alpha + \delta/2$

$\theta = 17$ °

From M_1 and θ, and Eq. 13.49
(using built-in function *Theta* (M, β,k))

$$\tan\theta = \frac{2\cot\beta\left(M_1^2\sin^2\beta - 1\right)}{M_1^2(k+\cos 2\beta)+2} \qquad (13.49)$$

For $\theta = 17.0$ °
$\beta = 48.2$ °

(Use *Goal Seek* to vary β so that $\theta = 17^o$)

From M_1 and β $M_{1n} = 1.49$

From M_{1n} and p_1, and Eq. 13.48d
(using built-in function *NormpfromM* (M,k))

$$\frac{p_2}{p_1} = \frac{2k}{k+1}M_{1_n}^2 - \frac{k-1}{k+1} \qquad (13.48d)$$

$p_2 =$ 230.6 kPa

$p_{FL} = p_2$

$p_{FL} =$ 230.6 kPa

To find M_2 we need M_{2n}. From M_{1n}, and Eq. 13.48a
(using built-in function *NormM2fromM* (M,k))

$$M_{2_n}^2 = \frac{M_{1_n}^2 + \frac{2}{k-1}}{\frac{2k}{k-1}M_{1_n}^2 - 1} \qquad (13.48a)$$

$M_{2n} =$ 0.704

The downstream Mach number is then obtained from
from M_{2n}, θ and β, and Eq. 13.47b

$$M_{2n} = M_2 \sin(\beta - \theta) \qquad (13.47b)$$

Hence $M_2 = 1.36$

For p_{02} we use Eq. 13.7a
(using built-in function *Isenp* (M, k))

$$\frac{p_0}{p} = \left[1 + \frac{k-1}{2} M^2\right]^{k/(k-1)} \qquad (13.7a)$$

$p_{02} =$ 693 kPa

For surface RL (isentropic expansion wave):

Treating as a new problem

Here: M_1 is the Mach number after the shock
and M_2 is the Mach number after the expansion wave
p_{01} is the stagnation pressure after the shock
and p_{02} is the stagnation pressure after the expansion wave

$M_1 = M_2$ (shock)

$M_1 = 1.36$

$p_{01} = p_{02}$ (shock)

$p_{01} =$ 693 kPa

For isentropic flow $p_0 =$ constant

$p_{02} = p_{01}$

$p_{02} =$ 693 kPa

For the deflection $\theta = \delta$

$\theta = 10.0$ °

We use Eq. 13.55

$$\omega = \sqrt{\frac{k+1}{k-1}} \tan^{-1}\left(\sqrt{\frac{k-1}{k+1}(M^2-1)}\right) - \tan^{-1}\left(\sqrt{M^2-1}\right) \qquad (13.55)$$

and

$$\text{Deflection} = \omega_2 - \omega_1 = \omega(M_2) - \omega(M_1) \qquad (3)$$

From M_1 and Eq. 13.55 (using built-in function *Omega* (M, k))

$$\omega_1 = 7.8 \quad ^o$$

Applying Eq. 3 $\quad \omega_2 = \omega_1 + \theta$

$$\omega_2 = 17.8 \quad ^o$$

From ω_2, and Eq. 13.55 (using built-in function *Omega* (M, k))

For $\quad \omega_2 = 17.8 \quad ^o$

$$M_2 = 1.70$$

(Use *Goal Seek* to vary M_2 so that $\omega_2 = 17.8^o$)

Hence for p_2 we use Eq. 13.7a
(using built-in function *Isenp* (M, k))

$$p_2 = p_{02}/(p_{02}/p_2)$$

$$p_2 = 141 \quad \text{kPa}$$

$$p_{RL} = p_2$$

$$p_{RL} = 141 \quad \text{kPa}$$

For surface FU (isentropic expansion wave):

$$M_1 = 2.0$$

For isentropic flow $\quad p_0 = \text{constant}$

$$p_{02} = p_{01}$$

For p_{01} we use Eq. 13.7a
(using built-in function *Isenp* (M, k))

$$p_{01} = 743$$

$$p_{02} = 743 \quad \text{kPa}$$

For the deflection $\quad \theta = \alpha - \delta/2$

$$\theta = 7.0 \quad ^o$$

We use Eq. 13.55

and

$$\text{Deflection} = \omega_2 - \omega_1 = \omega(M_2) - \omega(M_1) \qquad (3)$$

From M_1 and Eq. 13.55 (using built-in function *Omega* (M, k))

$\omega_1 =$ 26.4 ^{o}

Applying Eq. 3 $\omega_2 = \omega_1 + \theta$

$\omega_2 =$ 33.4 ^{o}

From ω_2, and Eq. 13.55 (using built-in function Omega(M, k))

For $\omega_2 =$ 33.4 ^{o}

$M_2 =$ 2.27

(Use *Goal Seek* to vary M_2 so that $\omega_2 = 33.4^{o}$)

Hence for p_2 we use Eq. 13.7a
(using built-in function *Isenp* (M, k))

$p_2 = p_{02}/(p_{02}/p_2)$

$p_2 =$ 62.8 kPa

$p_{FU} = p_2$

$p_{FU} =$ 62.8 kPa

For surface RU (isentropic expansion wave):

Treat as a new problem.

Flow is isentropic so we could analyse from region FU to RU
but instead analyse from region 1 to region RU.

$M_1 =$ 2.0

For isentropic flow $p_0 =$ constant

$p_{02} = p_{01}$

$p_{01} =$ 743 kPa

$p_{02} =$ 743 kPa

TOTAL deflection $\theta = \alpha + \delta/2$

$\theta =$ 17.0 ^{o}

We use Eq. 13.55

and

$$\text{Deflection} = \omega_2 - \omega_1 = \omega(M_2) - \omega(M_1) \qquad (3)$$

From M_1 and Eq. 13.55 (using built-in function *Omega* (M, k))

$\omega_1 =$ 26.4 ^{o}

Applying Eq. 3 $\omega_2 = \omega_1 + \theta$

$\omega_2 =$ 43.4 ^{o}

From ω_2, and Eq. 13.55 (using built-in function Omega(M, k))

For $\omega_2 =$ 43.4 ^{o}

$M_2 =$ 2.69

(Use *Goal Seek* to vary M_2 so that $\omega_2 = 43.4^o$)

Hence for p_2 we use Eq. 13.7a
(using built-in function *Isenp* (M, k))

$p_2 = p_{02}/(p_{02}/p_2)$

$p_2 =$ 32.4 kPa

$p_{RU} = p_2$

$p_{RU} =$ 32.4 kPa

The four pressures are:

$p_{FL} =$ 230.6 kPa
$p_{RL} =$ 140.5 kPa
$p_{FU} =$ 62.8 kPa
$p_{RU} =$ 32.4 kPa

From Eq 1 $F_V/sc =$ 138 kPa

From Eq 2 $F_H/sc =$ 5.3 kPa

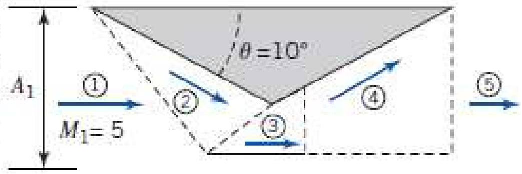
From Eq 7 $C_L =$ 0.503

From Eq 8 $C_D =$ 0.127

Problem 13.209

[Difficulty: 3]

13.209 An airplane is flying at Mach 5 at an altitude of 16,764 m, where $T_1 = 216.67$ K and $p_1 = 9.122$ kPa. The airplane uses a scramjet engine. Two oblique shocks are formed in the intake ② prior to entering the combustion chamber ③ at supersonic speed. The inlet and exit areas are equal, $A_1 = A_5 = 0.2\ \text{m}^2$. Calculate the stagnation temperature, T_2/T_1, and the Mach number in the intake ②.



Given: The gas dynamic relations for compressible flow

Find: The shock values and angles in each region

Solution: Begin with the 1-D gas dynamic relations for compressible flow

Governing equations:

$$V_1 = M_1\sqrt{kRT_1}\ ;\ T_{0_1} = T_1\left(1+\frac{k-1}{2}M_1^2\right)$$

Assumption: The flow is compressible and supersonic

$$V_1 = M_1\sqrt{kRT_a} = 5\sqrt{1.4\times 287\times 216.7} = 1475.4\,\frac{\text{m}}{\text{s}}$$

$$V_f = V_1 = 1475.4\,\frac{\text{m}}{\text{s}}$$

$$T_{0_1} = T_1\left(1+\frac{k-1}{2}M_1^2\right) = 1300\ \text{K}$$

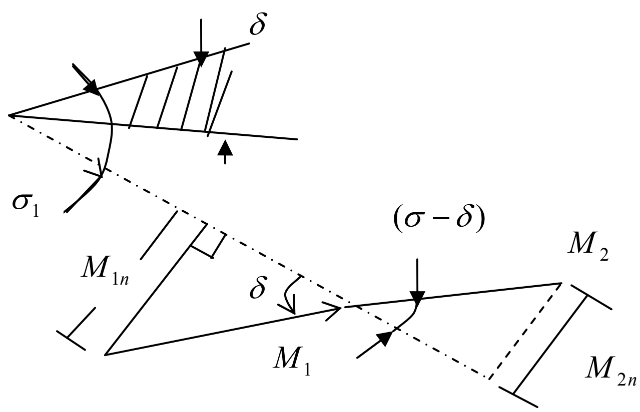
<u>From (1) to (2)</u> there is an oblique shock with M_1=5 and $\delta = 10^0$

From the oblique shock figure (or tables)

$$\therefore \sigma_1 = 19.38^\circ$$

$$M_{1n} = M_1\sin(\sigma)$$

$$M_{1n} = 1.659$$



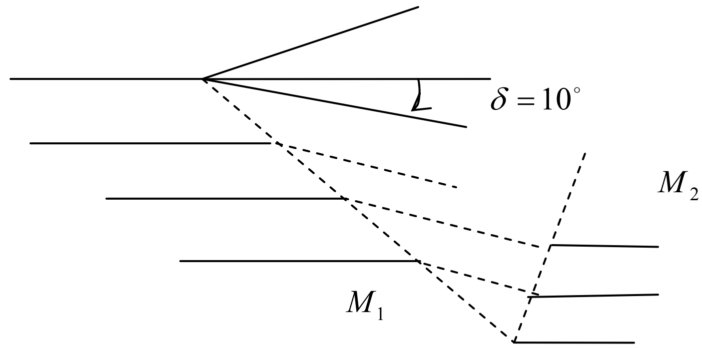
From Normal Shock Tables

$M_{1n} = 1.659$

$\therefore M_{2n} = 0.65119$

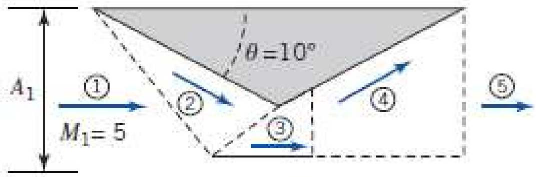
$\frac{T_2}{T_1} = 1.429$

$M_2 = \frac{M_{2n}}{\sin(\sigma - \delta)} = 4.0$



Problem 13.210 [Difficulty: 3]

13.210 Two oblique shocks are formed in a scramjet engine intake prior to entering the combustion chamber. The inlet Mach number is $M_1 = 5$, the incoming air temperature is $T_1 = 216.67$ K, $p_1 = 9.122$ kPa, and $A_1 = 0.2\,\text{m}^2$. Calculate M_3 in the combustion chamber ③ if $M_2 = 4.0$.



Given: The gas dynamic relations for compressible flow

Find: The shock values and angles in each region

Solution: Begin with the 1-D gas dynamic relations for compressible flow

Governing equations:

$$V_1 = M_1\sqrt{kRT_1}\ ;\ T_{0_1} = T_1\left(1+\frac{k-1}{2}M_1^2\right)$$

Assumption: The flow is compressible and supersonic

$$V_1 = M_1\sqrt{kRT_a} = 5\sqrt{1.4\times 287\times 216.7} = 1475.4\,\frac{\text{m}}{\text{s}}$$

$$V_f = V_1 = 1475.4\,\frac{\text{m}}{\text{s}}$$

$$T_{0_1} = T_1\left(1+\frac{k-1}{2}M_1^2\right) = 1300\ \text{K}$$

From (2) to (3) A second oblique shock with $M_2 = 4.0$ and $\delta = 10^0$

$\therefore$ From the oblique shock tables

$\therefore \sigma_2 = 22.23^0$

and

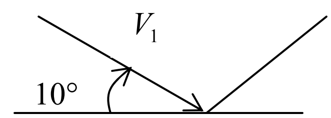
$M_{2_n} = M_2 \sin\sigma = 1.513$

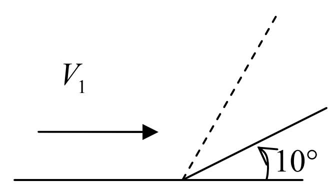
From normal shock tables

$\therefore \mathrm{M}_{3n} = 0.698$

$$\mathrm{M}_3 = \frac{\mathrm{M}_{3n}}{\sin(\sigma-\delta)} = \frac{0.698}{\sin 12.23}$$

$\mathrm{M}_3 = 3.295$

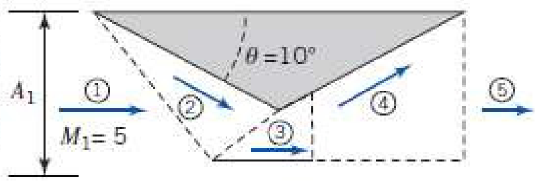




Problem 13.211

[Difficulty: 4]

13.211 An airplane is flying at Mach 5, where $T_1 = 216.67$ K. Oblique shocks form in the intake prior to entering the combustion chamber ③. The nozzle expansion ratio is $A_5/A_4 = 5$. The inlet and exit areas are equal, $A_1 = A_5 = 0.2\ \text{m}^2$. Assuming isentropic flow with $M_2 = 4$, $M_3 = 3.295$, and $M_4 = 1.26$, calculate the exit Mach number and the exhaust jet velocity ⑤. *Hint:* Calculate the temperature ratios in each section.



Given: The gas dynamic relations for compressible flow

Find: Exit Mach number and velocity

Solution: Begin with the 1-D gas dynamic relations for compressible flow

Governing equations:

$$V_1 = M_1\sqrt{kRT_1}\ ;\ T_{0_1} = T_1\left(1 + \frac{k-1}{2}M_1^2\right)$$

Assumption: The flow is compressible and supersonic

$$V_1 = M_1\sqrt{kRT_a} = 5\sqrt{1.4\times 287\times 216.7} = 1475.4\,\frac{\text{m}}{\text{s}}$$

Assuming $M_2 = 4.0$, $M_3 = 3.295$, and $M_4 = 1.26$

$$\therefore \frac{A_4}{A^*} = 1.05$$

$$and \quad \frac{T_{04}}{T_4} = 1.317$$

$$With \quad \frac{A_5}{A^*} = \frac{A_5}{A_4}\frac{A_4}{A^*} = 5\times 1.05 = 5.25$$

$$M_5 = 3.23$$

$$\frac{T_{0_5}}{T_5} = 1 + \frac{k-1}{2}M_5^2 = 3.11$$

To find the temperature at state 5, we need to express the temperature in terms of the entrance temperature and known temperature ratios:

$$T_5 = T_1\frac{T_2}{T_1}\frac{T_3}{T_2}\frac{T_4}{T_3}\frac{T_{0_4}}{T_4}\frac{T_{0_5}}{T_{0_4}}\frac{T_5}{T_{0_5}}$$

Now since the stagnation temperatures at 4 and 5 are equal (isentropic flow through the nozzle):

$$T_5 = 216.7\ \text{K}\times 1.429\times 1.333\times 3.744\times 1.317\times 1\times\frac{1}{3.11}$$

$$T_5 = 654.5\ \text{K}$$

Therefore, the exhaust velocity is:

$$V_5 = M_5\sqrt{kRT_5} = 3.23\sqrt{1.4\times 287\times 654.5} = 1656\,\frac{\text{m}}{\text{s}}$$